

# Math Camp Lesson 2

## Vectors and Matrices (Linear Algebra)

UW–Madison Political Science

August 18, 2020

# Linear Algebra

# Review/Overview

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Algebra is a fundamental basis for more advanced mathematical manipulation:

- Use to derive statistical estimators, and to understand their properties and the assumptions necessary to apply them.
- Use to evaluate the optimal choices of strategic actors.

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If no range is indicated ( $\sum x_i$ ), this implies all observations are included.

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$$\begin{aligned}\sum_{i=1}^3 (x_i^2 + 3) &= (x_1^2 + 3) + (x_2^2 + 3) + (x_3^2 + 3) \\ &= (3^2 + 3) + (4^2 + 3) + (1^2 + 3) \\ &= 35\end{aligned}$$

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$$\begin{aligned}\prod_{i=1}^3 (x_i^2 + 3) &= (x_1^2 + 3) \times (x_2^2 + 3) \times (x_3^2 + 3) \\ &= (3^2 + 3) \times (4^2 + 3) \times (1^2 + 3) \\ &= 912\end{aligned}$$

# Summations and Products

Given these data:  $x_1 = 3$ ,  $x_2 = 4$ ,  $x_3 = 1$ , and  $x_4 = 0$ ; and  $y_1 = 1$ ,  $y_2 = 2$ ,  $y_3 = 3$ , and  $y_4 = 4$ . Find these quantities:

- $\sum x_i + \sum y_i$
- $\sum (x_i + y_i)$
- $\prod x_i + \prod y_i$
- $\prod (x_i \times y_i)$

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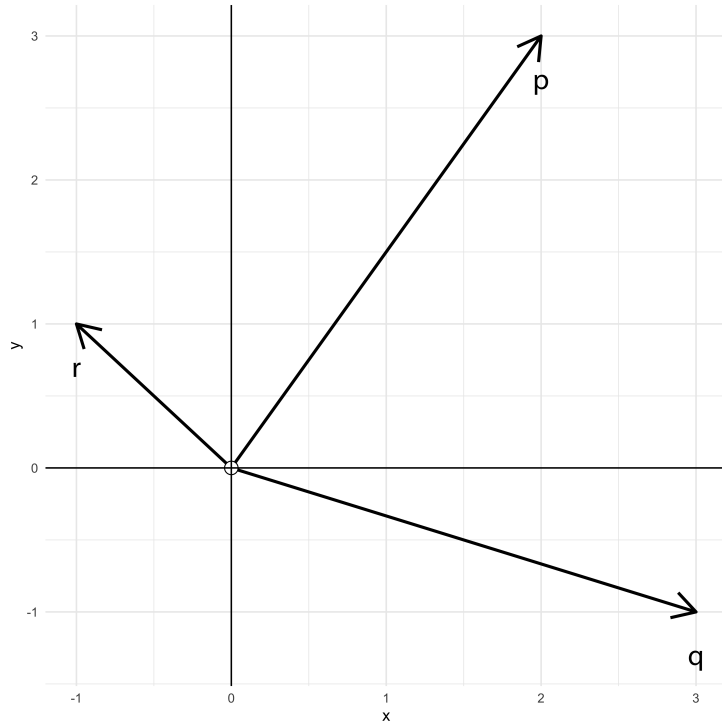
$$\mathbf{v} = [v_1, v_2, v_3, v_4]$$

or **column** vectors like

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Ordered simply means that  $[v_1, v_2, v_3, v_4] \neq [v_4, v_3, v_2, v_1]$

# Vectors in Space



Vectors can be thought of as lines from the origin in  $k$ -dimensional space (where  $k$  is the number of vector elements) going to a point with the coordinates of the elements of the vector.

$$\mathbf{p} = [2, 3]$$

$$\mathbf{q} = [3, -1]$$

$$\mathbf{r} = [-1, 1]$$

# Vector Operations: Addition and Subtraction

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$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \mathbf{w} \\ [1, 2, 3, 4] + [4, 8, 12, 16] &= \mathbf{w} \\ [1 + 4, 2 + 8, 3 + 12, 4 + 16] &= \mathbf{w} \\ [5, 10, 15, 20] &= \mathbf{w}\end{aligned}$$

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$$[5, 10, 15, 20] = \mathbf{w}$$

$$\mathbf{u} - \mathbf{v} = \mathbf{w}$$

$$[1, 2, 3, 4] - [4, 8, 12, 16] = \mathbf{w}$$

$$[1 - 4, 2 - 8, 3 - 12, 4 - 16] = \mathbf{w}$$

$$[-3, -6, -9, -12] = \mathbf{w}$$

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It is important to note that conformability does not matter for scalar multiplication and division.

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$$\mathbf{x} \cdot \mathbf{y} = [x_1 \times y_1 + x_2 \times y_2, \dots + x_{k-1} \times y_{k-1} + x_k \times y_k] = \sum_{i=1}^k x_i y_i$$

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The dot product will start with two vectors *and* result in a scalar.

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Distributive Property:  $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

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$$\mathbf{v}^T = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$



# Vectors

Given vectors  $\mathbf{x} = [1, 2, 0, 4]$  and  $\mathbf{y} = [5, 3, 2, 3]$ , find:

- $\mathbf{x}^T$
- $\mathbf{x} + \mathbf{y}$
- $\mathbf{x} \cdot \mathbf{y}$

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Like vectors, each value is referred to as an *element*. When referring to elements of a matrix, we will not bold the vector and add a subscript to denote their position. For example,  $x_{1,2}$  refers to the element in the first row, second column of  $\mathbf{X}$ .

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- The most basic special matrix is the *square* matrix. As the name implies, this is a matrix with the same number of rows and columns (e.g.  $2 \times 2$ ,  $3 \times 3$ , or generically,  $k \times k$ ).



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- A very general square matrix form is the *symmetric* matrix. This is a matrix that is symmetric across the diagonal from the upper left-hand corner to the lower right-hand corner (also called the **main diagonal**) .

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This switches the dimensions (here, from  $2 \times 3$  to  $3 \times 2$ ).

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$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix}$$

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Such that, e.g.  $a_{2,1} + b_{2,1} = c_{2,1}$

# Matrix Operations: Matrix Addition and Subtraction

Consider:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+4 & 3+6 \\ 4+8 & 5+10 & 6+12 \end{bmatrix} \\ = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

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or

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} = \begin{bmatrix} 1-2 & 2-4 & 3-6 \\ 4-8 & 5-10 & 6-12 \end{bmatrix} \\ = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{bmatrix}$$

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To multiply a matrix by another matrix, each element of the result is found by taking the corresponding row of the first matrix and multiplying each element by the corresponding column in the second matrix, and summing the result, or the *dot product*!

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Such that, e.g.  $c_{1,1} = a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1}$

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What do  $c$  and  $d$  equal?

$$c = 3 \times 5 + 4 \times 7 = 43$$

$$d = 3 \times 6 + 4 \times 8 = 50$$

# Matrix Multiplication

Therefore,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

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The result of matrix multiplication will have dimensions equal to the outer dimensions of the two matrices.

Unlike with scalars, order matters. Reversing the order may result in a different product, or may not even be possible depending on the dimensions of the matrices.

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Like vector multiplication, matrix multiplication has some special properties consider for conformable matrices.

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- Additive Distributive Property:  $(\mathbf{X} + \mathbf{Y})\mathbf{Z} = \mathbf{XZ} + \mathbf{YZ}$
- Identity Property:  $\mathbf{XI} = \mathbf{IX} = \mathbf{X}$

# Matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 7 & 2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 \\ 3 & 3 \\ 1 & 5 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

Given the matrices above, calculate

- $\mathbf{A} + \mathbf{B}$
- $\mathbf{C}^T$
- $\mathbf{DB}$
- $\mathbf{CD}^T$



# Matrix Inversion

The operation most closely analogous to division for matrices is inversion. The inverse of a matrix (denoted with the superscript  $^{-1}$ ) is the matrix that, when multiplied by the original, produces the identity matrix:

$$\mathbf{X}\mathbf{X}^{-1} = \mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$$

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Matrix inversion is only possible with **some square matrices**. If a square matrix is not invertible it is called a *singular* or *non-invertible* matrix.

# Matrix Inversion (cont'd)

A handy shortcut to find the inverse of a  $2 \times 2$  matrix, calculate the **determinant** (product of the main diagonal minus the product of the off diagonal) and adjust the elements as such:

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$$\begin{aligned} \mathbf{X}^{-1} &= \frac{1}{\det(\mathbf{X})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \end{aligned}$$

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If the determinant is zero, there is no inverse. Calculating inverses of larger (square) matrices is more complicated.

# Matrix Inversion

Consider:

$$\mathbf{X} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \implies \mathbf{X}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix} = \begin{bmatrix} 2 \times \frac{1}{2} + 0 \times -\frac{3}{2} & 2 \times 0 + 0 \times 1 \\ 3 \times \frac{1}{2} + 1 \times -\frac{3}{2} & 3 \times 0 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Let's call of these matrices and vector.

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$$

# Solving Systems of Equations (cont'd)

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$$\mathbf{A}^{-1} = \frac{1}{2 \times 4 - 6 \times 3} \begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$$

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Let's verify our results

$$2(-10) + 6(5) = 10$$

$$3(-10) + 4(5) = -10$$

**Where is linear algebra in political science?**

More like, where *isn't* it?



# Regression

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In matrix form:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \alpha + \beta \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

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or ...

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

# Matrix-form regression

We can also write the regression equation for an arbitrary number of  $x$  variables as...

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It so happens that when we do the *matrix calculus* to solve for  $\beta$ ...

$$\beta = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

Point being...

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These basic principles apply to all regression modeling



# Point being...

These basic principles apply to all regression modeling  
and *tons* of political science boils down to regression modeling

**End Day 2**