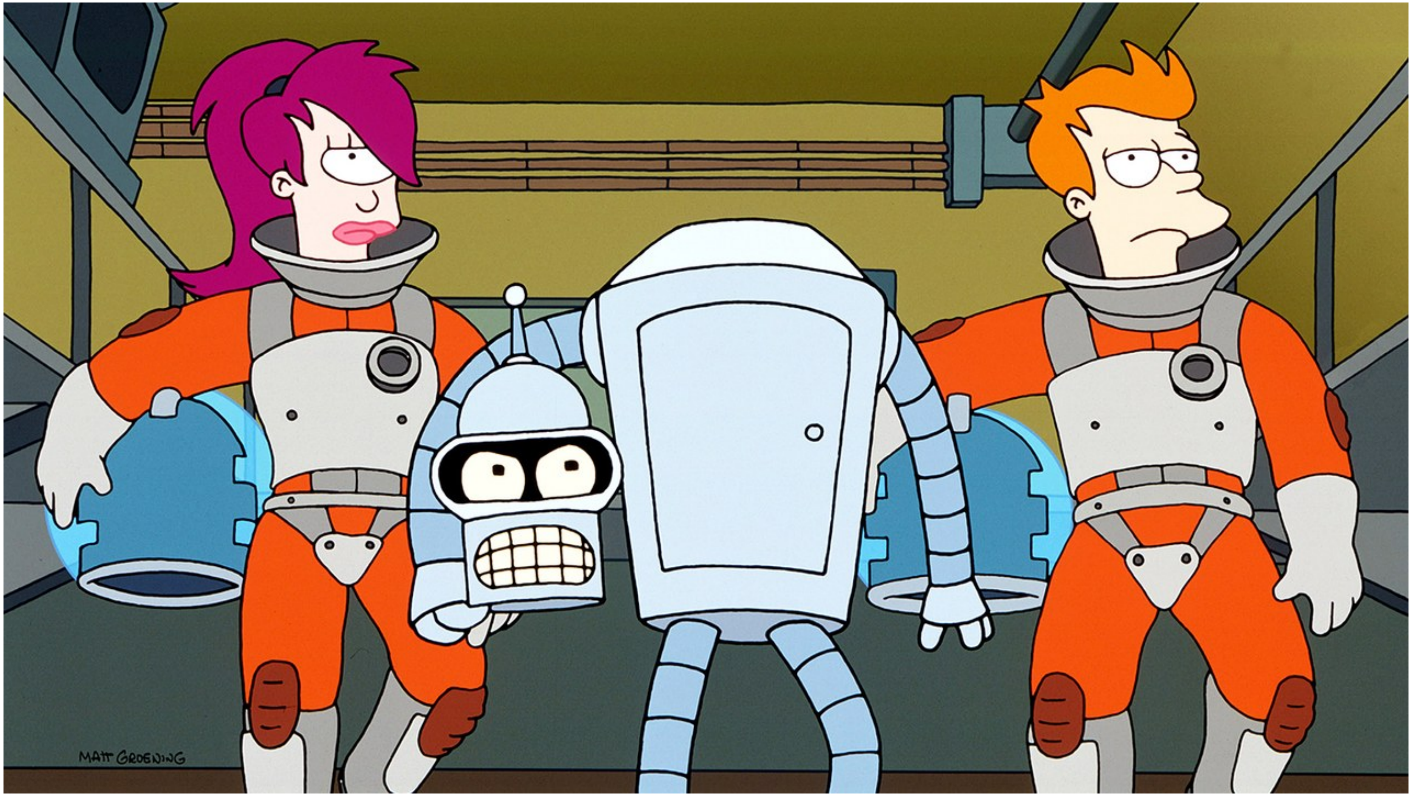
Math Camp: Lesson 4

Statistics and Probability

UW–Madison Political Science

August 21, 2020

Hang in there



Agenda

- Why do we need statistics?
- Counting
- Set theory
- Probability
- Independence, joint probability
- Bayes' Theorem
- Looking ahead

Why Statistics?

Let's say we want to know under what conditions democracies fail. Lots of factors may lead to such outcomes, and we want to learn which of these factors are more important. How do we do that?

Why Statistics?

Let's say we want to know under what conditions democracies fail. Lots of factors may lead to such outcomes, and we want to learn which of these factors are more important. How do we do that?

- We build a model that includes a number of variables (e.g., economy, conflict, etc.)
- In the model, we have parameters associated with these factors that tell us about their influence
- Statistics allows us:
    - to estimate these parameters, to learn which factors are systematically related to our outcome of interest
    - to estimate how (un)certain we should be in our estimates

Statistics

The mathematical study of data

Statistics

The mathematical study of data

- Data come from some underlying, unknown process (e.g., poor economy makes citizens more sympathetic to authoritarian leaders)

Statistics

The mathematical study of data

- Data come from some underlying, unknown process (e.g., poor economy makes citizens more sympathetic to authoritarian leaders)
- Descriptive statistics: describe the data (how often democracies fail, in what time periods they failed more)

Statistics

The mathematical study of data

- Data come from some underlying, unknown process (e.g., poor economy makes citizens more sympathetic to authoritarian leaders)
- Descriptive statistics: describe the data (how often democracies fail, in what time periods they failed more)
- Inferential statistics: describe the underlying process, argue about causality, test theories

Statistics

The mathematical study of data

- Data come from some underlying, unknown process (e.g., poor economy makes citizens more sympathetic to authoritarian leaders)
- **Descriptive statistics:** describe the data (how often democracies fail, in what time periods they failed more)
- **Inferential statistics:** describe the underlying process, argue about causality, test theories

In political science, we do both, but a lot of emphasis is on inference and causality

Statistics and Probability

To make inferences about data generating process, we use probability

Statistics and Probability

To make inferences about data generating process, we use probability

- We have some data: regime failures, GDP statistics, conflict occurrence, etc.
- But we don't know which process regime failures come from (e.g., do democracies fail because of poor economy, or because of violent conflict, or both?)

Statistics and Probability

To make inferences about **data generating process**, we use probability

- We have some data: regime failures, GDP statistics, conflict occurrence, etc.
- But we don't know which process regime failures come from (e.g., do democracies fail because of poor economy, or because of violent conflict, or both?)
- We entertain different **models** for the process and pick the best model based on the probability of data under each model

Statistics and Probability

To make inferences about **data generating process**, we use probability

- We have some data: regime failures, GDP statistics, conflict occurrence, etc.
- But we don't know which process regime failures come from (e.g., do democracies fail because of poor economy, or because of violent conflict, or both?)
- We entertain different **models** for the process and pick the best model based on the probability of data under each model
- Probability also helps us evaluate the level of uncertainty around our findings

Helpful Vocabulary

A **random variable** is a realization of a process that is at least partially random (i.e. unpredictable)

- e.g. coin flip, dice roll, regime failure (or absence of failure)
- probability enters statistics through the assumptions we make about the nature of randomness in a random variable (e.g., whether certain kinds of outcomes are more likely than others)

Helpful Vocabulary

A **random variable** is a realization of a process that is at least partially random (i.e. unpredictable)

- e.g. coin flip, dice roll, regime failure (or absence of failure)
- probability enters statistics through the assumptions we make about the nature of randomness in a random variable (e.g., whether certain kinds of outcomes are more likely than others)

A random variable could have a number of different **potential outcomes** (e.g., heads or tails, democratic survival or failure, the number of deaths in a conflict)

Helpful Vocabulary

A **random variable** is a realization of a process that is at least partially random (i.e. unpredictable)

- e.g. coin flip, dice roll, regime failure (or absence of failure)
- probability enters statistics through the assumptions we make about the nature of randomness in a random variable (e.g., whether certain kinds of outcomes are more likely than others)

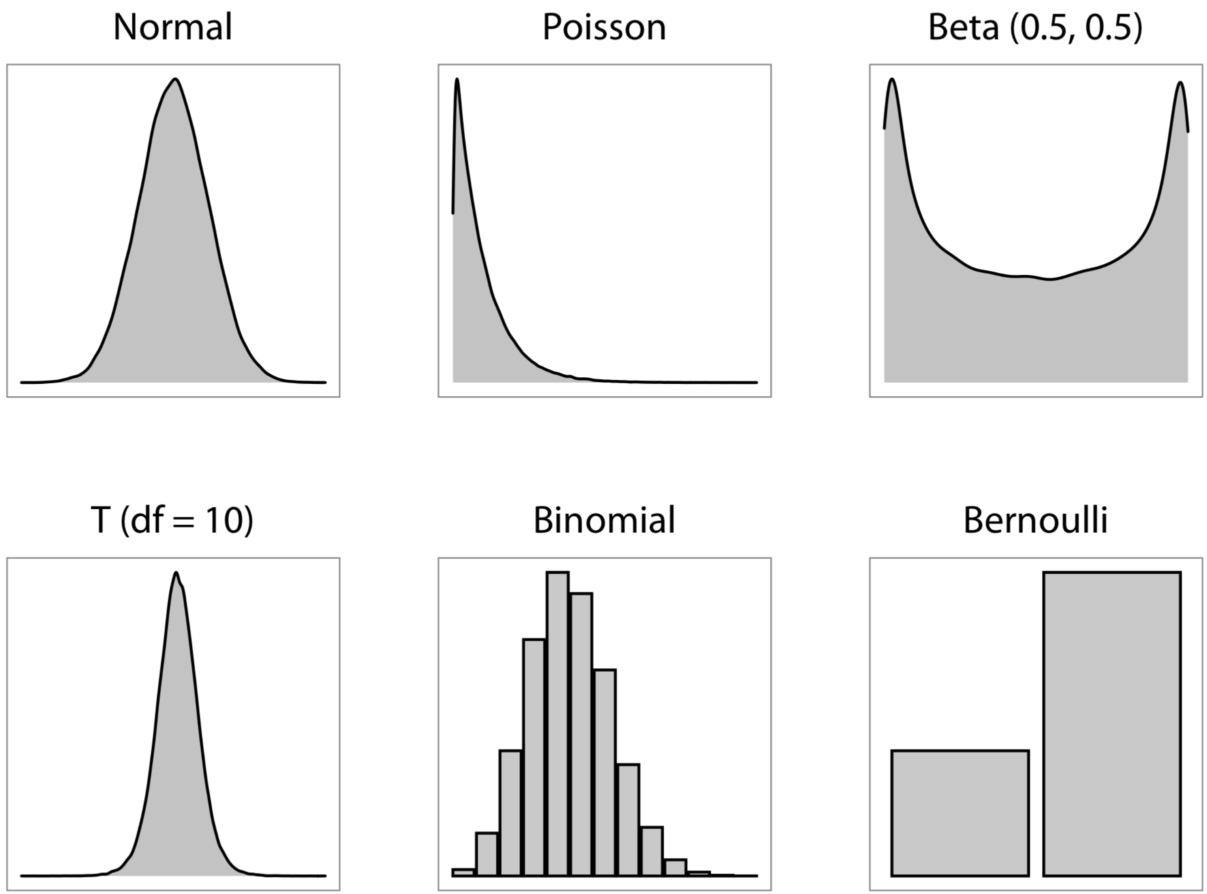
A random variable could have a number of different **potential outcomes** (e.g., heads or tails, democratic survival or failure, the number of deaths in a conflict)

The probability of all potential outcomes is described with a **probability distribution**, a *function* that maps each potential outcome to a certain probability

- x = potential outcome
- $f(x)$ = probability of x

These also matter for formal (non-statistical) models (e.g., an actor learns certain information with some probability)

Common Probability Distributions



Based on simulated data

More About Probability Distributions

Probability distributions can describe discrete outcomes (regime survival or failure) or continuous outcomes (election turnout, vote margin)

More About Probability Distributions

Probability distributions can describe discrete outcomes (regime survival or failure) or continuous outcomes (election turnout, vote margin)

Probability distributions always sum to 1

- "The law of total probability" (allows to calculate probabilities of certain events based on probabilities of other events)
- Discrete distributions: sum the probabilities of all potential outcomes
- Continuous distributions: integrate over the continuous space of outcomes

More About Probability Distributions

Probability distributions can describe discrete outcomes (regime survival or failure) or continuous outcomes (election turnout, vote margin)

Probability distributions always sum to 1

- "The law of total probability" (allows to calculate probabilities of certain events based on probabilities of other events)
- Discrete distributions: sum the probabilities of all potential outcomes
- Continuous distributions: integrate over the continuous space of outcomes

Probability distributions are the basis for statistical inference

- z-scores, p-values
- Prior and posterior beliefs

Counting

Counting

First, what is probability?

Counting

First, what is probability?

- The ratio of an event's *expected frequency* to the number of possible events

Counting

First, what is probability?

- The ratio of an event's *expected frequency* to the number of possible events
- We need to count the events in question & total possible outcomes

Counting

First, what is probability?

- The ratio of an event's *expected frequency* to the number of possible events
- We need to count the events in question & total possible outcomes

Suppose an **event** is described by K different component parts. (E.g. we roll a die K many times.) Each component $k = \{1, 2, \ldots, K\}$ has n_k possible values. What is the number of distinct outcomes we could get?

Counting

First, what is probability?

- The ratio of an event's *expected frequency* to the number of possible events
- We need to count the events in question & total possible outcomes

Suppose an event is described by K different component parts. (E.g. we roll a die K many times.) Each component $k = \{1, 2, \ldots, K\}$ has n_k possible values. What is the number of distinct outcomes we could get?

$$\prod_{k=1}^{K} n_k$$

(multiply the n_k's)

Counting

First, what is probability?

- The ratio of an event's *expected frequency* to the number of possible events
- We need to count the events in question & total possible outcomes

Suppose an event is described by K different component parts. (E.g. we roll a die K many times.) Each component $k = \{1, 2, \ldots, K\}$ has n_k possible values. What is the number of distinct outcomes we could get?

$$\prod_{k=1}^{K} n_k$$

(multiply the n_k's)

I roll a 6-sided die 4 times. How many unique sets of 4 rolls can I obtain (assuming that different orderings of the same 4 numbers are different events)?

Complex Counting Considerations

Does the *order of selection* matter? (Is $\{1, 2\} = \{2, 1\}$?)

Complex Counting Considerations

Does the *order of selection* matter? (Is $\{1, 2\} = \{2, 1\}$?)

Are selected objects *replaced* (able to be selected again) or *not replaced*?

Ordering with Replacement

This is easiest because (a) no need to adjust for "double-counting" and (b) the number of possibilities is always constant.

The number of possible ways to select k elements from a larger pool of n is

$$n \times n \times n \times \ldots \times n = n^k$$

Intuition: in each draw, there are n possibilities. Each of n outcomes in one draw can be combined with the n outcomes in any (and all) other draws.

Example: rolling two dice several times

Order, No Replacement

Also called permutation.

The number of ways to select k objects from a pool of n possible objects, where order matters but replacement does not occur.

$$n * (n-1) * (n-2) * \ldots * (n-k-1) = \frac{n!}{(n-k)!}$$

Intuition: each draw *removes the object* from the larger pool. Subsequent draws have one less element to choose from.

For example: the number of possible ways to deal a card game, winning lottery numbers

Unordered, No Replacement

Also called combinations: The number of possible ways to select k objects from a pool of n possible objects, where *order does not matter* and *replacement does not occur*

Intuition: we have fewer possibilities than before. Substantively identical elements (A and then B; B and then A) are not double counted

$$\frac{n!}{k!(n-k)!} = \binom{n}{k}$$

For example: survey samples, raffles, possible groups of 2 in a classroom

Unordered, With Replacement

The number of possible ways to select k elements from a larger pool of n possible elements, where order does not matter and replacement does occur

$$\frac{(n+k-1)!}{(n-1)!k!} = \binom{n+k-1}{k}$$

Example: the number of heads if you flip a coin n times

Exercises

Imagine we rank the 3 top swimmers among us.

- Is this a situation where order matters?

Exercises

Imagine we rank the 3 top swimmers among us.

- Is this a situation where order matters?
- Is this a situation with replacement, or no replacement?

Exercises

Imagine we rank the 3 top swimmers among us.

- Is this a situation where order matters?
- Is this a situation with replacement, or no replacement?
- How many different possible rankings could there be?

Exercises

Imagine we rank the 3 top swimmers among us.

- Is this a situation where order matters?
- Is this a situation with replacement, or no replacement?
- How many different possible rankings could there be?

Imagine we have 4 different scholarships for students in this class. You can win more than one scholarship. How many different sets of winners can there be?

Exercises

Imagine we rank the 3 top swimmers among us.

- Is this a situation where order matters?
- Is this a situation with replacement, or no replacement?
- How many different possible rankings could there be?

Imagine we have 4 different scholarships for students in this class. You can win more than one scholarship. How many different sets of winners can there be?

Imagine we have 5 identical candies for students in this class. You can win more than 1 candy. How many different sets of winners can there be?

Exercises

Imagine we rank the 3 top swimmers among us.

- Is this a situation where order matters?
- Is this a situation with replacement, or no replacement?
- How many different possible rankings could there be?

Imagine we have 4 different scholarships for students in this class. You can win more than one scholarship. How many different sets of winners can there be?

Imagine we have 5 identical candies for students in this class. You can win more than 1 candy. How many different sets of winners can there be?

Imagine we have 2 identical bicycles for students in this class. You can only win 1 bicycle. How many sets of winners?

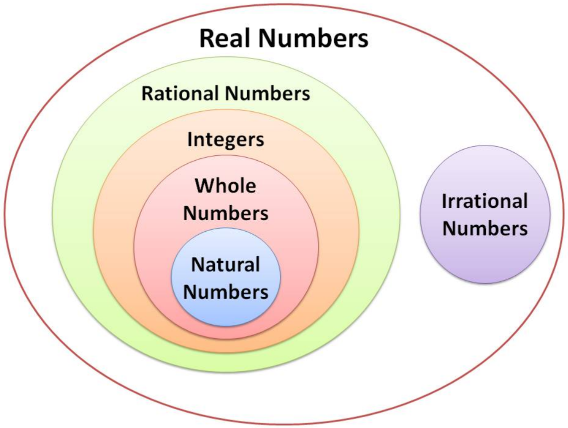
Set Theory

Sets

Remember: a set is a collection of elements. Could be numbers, units, areas in space...

- $F = \{1, 2, 3, 4\}$
- $G = \{1, 3, 5\}$
- $H = [0, 1] \cup (2, 3)$

What are unions? Intersections?
Disjoints? Subsets? Supersets?



- $P = \{\text{Reagan}, \text{Bush41}, \text{Clinton}, \text{Bush43}, \text{Obama}, \text{Trump}\}$
- $D = \{\text{Carter}, \text{Mondale}, \text{Dukakis}, \text{Clinton}, \text{Gore}, \text{Kerry}, \text{Obama}, \text{HRC}\}$
- $R = \{\text{Reagan}, \text{Bush41}, \text{Dole}, \text{Bush43}, \text{McCain}, \text{Romney}, \text{Trump}\}$
- $I = \{\text{Perot}, \text{Nader}\}$

The Sample Space

The **sample space** (denoted S or Ω) is the set that contains all elements in question.

Sometimes called the *universal set*

The Sample Space

The sample space (denoted S or Ω) is the set that contains all elements in question.

Sometimes called the *universal set*

Not the same as the set that contains *everything*. Only the relevant things for what we're currently talking about (e.g., a country's population is the universal set for any survey sample)

Complementary Sets

The complement of set A (denoted as A^C) is the set of all elements in the sample space that are *not contained* in A

$$A^C \equiv X \text{ such that } X \notin A$$

Complementary Sets

The complement of set A (denoted as A^C) is the set of all elements in the sample space that are *not contained* in A

$$A^C \equiv X \text{ such that } X \notin A$$

Example, $\Omega = [0, 1]$

- If $X = (0, 0.5]$, what is X^C?

Complementary Sets

The complement of set A (denoted as A^C) is the set of all elements in the sample space that are *not contained* in A

$$A^C \equiv X \text{ such that } X \notin A$$

Example, $\Omega = [0, 1]$

- If $X = (0, 0.5]$, what is X^C?
- $X^C = \{0\} \cup (0.5, 1]$

Complementary Sets

The complement of set A (denoted as A^C) is the set of all elements in the sample space that are *not contained* in A

$$A^C \equiv X \text{ such that } X \notin A$$

Example, $\Omega = [0, 1]$

- If $X = (0, 0.5]$, what is X^C?
- $X^C = \{0\} \cup (0.5, 1]$

What is Ω^C?

Complementary Sets

The complement of set A (denoted as A^C) is the set of all elements in the sample space that are *not contained* in A

$$A^C \equiv X \text{ such that } X \notin A$$

Example, $\Omega = [0, 1]$

- If $X = (0, 0.5]$, what is X^C?
- $X^C = \{0\} \cup (0.5, 1]$

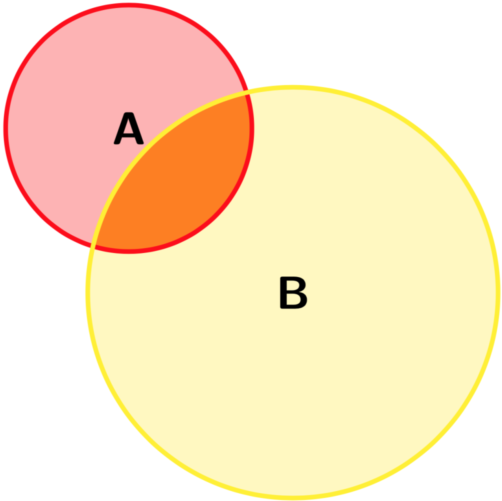
What is Ω^C?

- $\emptyset$

Probability

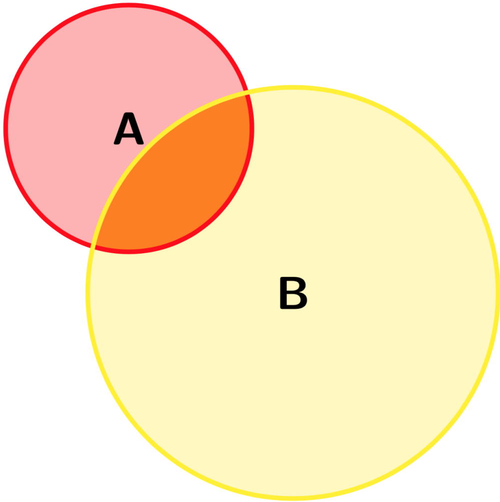
Probability as Sets

We can use sets to represent the probability of events. Total area represents total probability of all events (equal to 1).



Probability as Sets

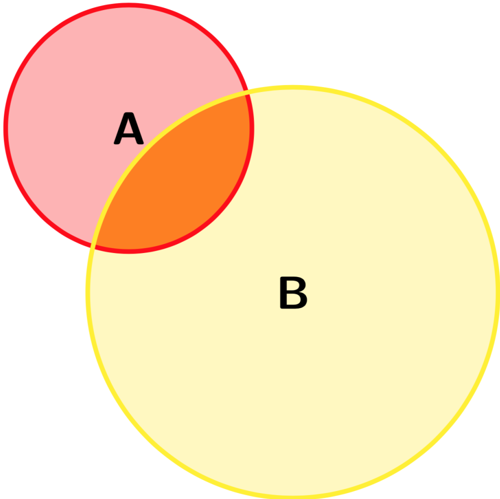
We can use sets to represent the probability of events. Total area represents total probability of all events (equal to 1).



A is an event, and its area is a subset of the total area.

Probability as Sets

We can use sets to represent the probability of events. Total area represents total probability of all events (equal to 1).



A is an event, and its area is a subset of the total area.

$\Pr(A)$?

$\Pr(A^C)$?

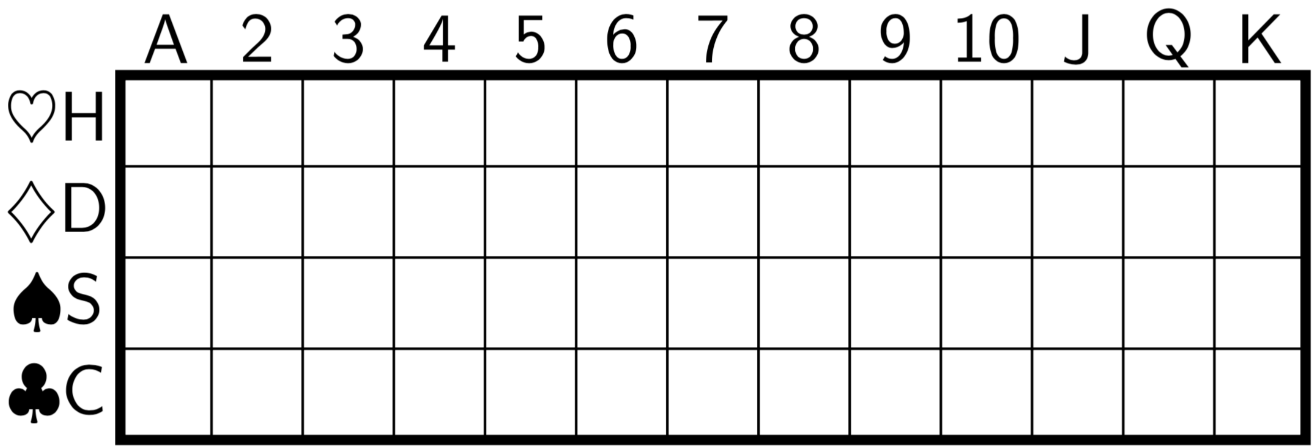
Let's play cards

We have 4 suits (hearts, diamonds, spades, clubs) and 13 card values (Ace, 2, 3, ..., Jack, Queen, King). Suits and values can both be sets.

	A	2	3	4	5	6	7	8	9	10	J	Q	K
♡H													
♢D													
♠S													
♣C													

Let's play cards

We have 4 suits (hearts, diamonds, spades, clubs) and 13 card values (Ace, 2, 3, ..., Jack, Queen, King). Suits and values can both be sets.



Total area = 1

Probability of an individual card: $\frac{1}{52}$

Properties of Probabilities

Probabilities are strictly bounded on the closed interval $[0, 1]$

- $p(A) \in [0, 1]$

Properties of Probabilities

Probabilities are strictly bounded on the closed interval $[0, 1]$

- $p(A) \in [0, 1]$
- A is either impossible ($p = 0$), certain ($p = 1$), or in between (possible, $p \in (0, 1)$)

Properties of Probabilities

Probabilities are strictly bounded on the closed interval $[0, 1]$

- $p(A) \in [0, 1]$
- A is either impossible ($p = 0$), certain ($p = 1$), or in between (possible, $p \in (0, 1)$)

If we had N many *collectively exhaustive* and *mutually exclusive* sets of potential outcomes, their probabilities sum to 1. Which is to say, *something must happen.*

$$\sum_{n=1}^{N} p(A_n) = 1$$

Probability of Complements

If Ω contains the set of all potential outcomes, and A is an event that is a subset of the outcome space that occurs with $p(A)$...

What is $p(A^C)$?

Probability of Complements

If Ω contains the set of all potential outcomes, and A is an event that is a subset of the outcome space that occurs with $p(A)$...

What is $p(A^C)$?

$p(A^C) = 1 - p(A)$

Probability of Complements

If Ω contains the set of all potential outcomes, and A is an event that is a subset of the outcome space that occurs with $p(A)$...

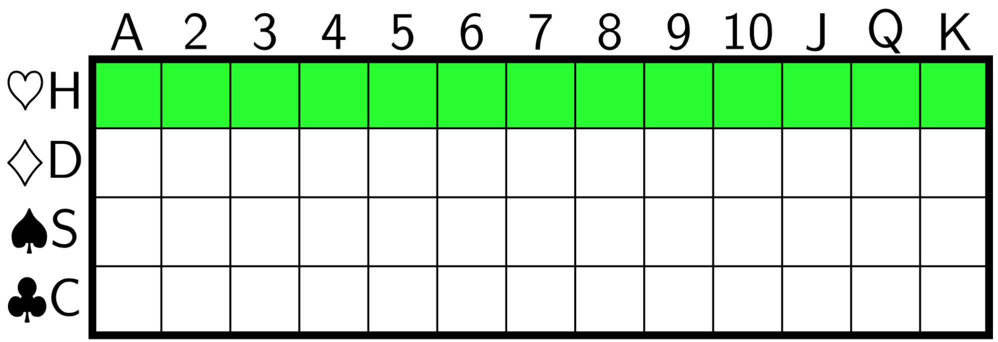
What is $p(A^C)$?

$p(A^C) = 1 - p(A)$

The intuition: *Something* must happen, either A or not A

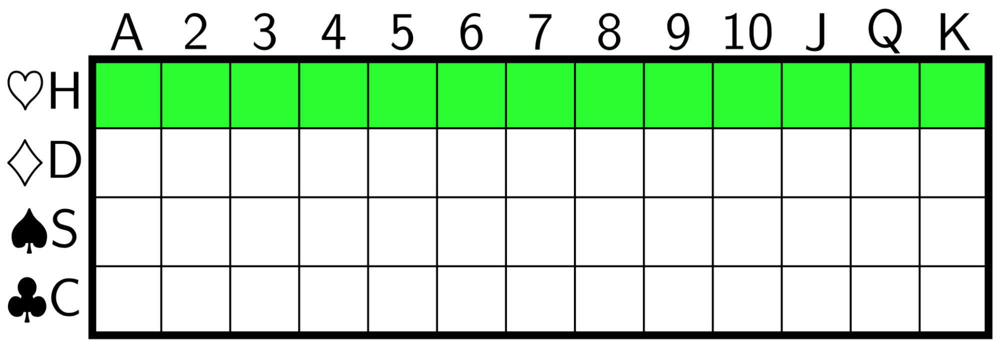
Example of Complements

Probability that a random card is a Heart? $p(H) = \frac{1}{4}$



Example of Complements

Probability that a random card is a Heart? $p(H) = \frac{1}{4}$

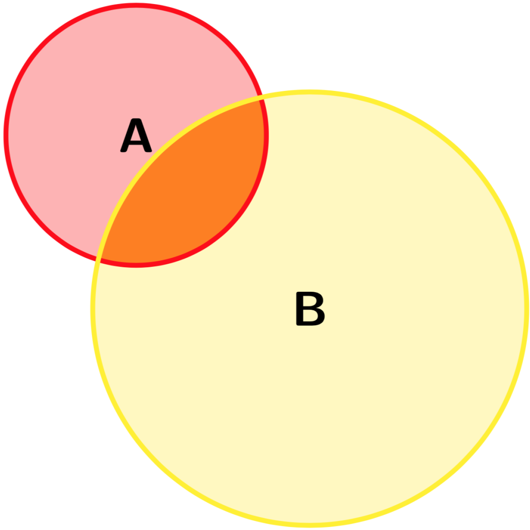


Probability that a card is not a Heart? $1 - p(H) = \frac{3}{4}$

Probability of Unions

The probability of $A \cup B$

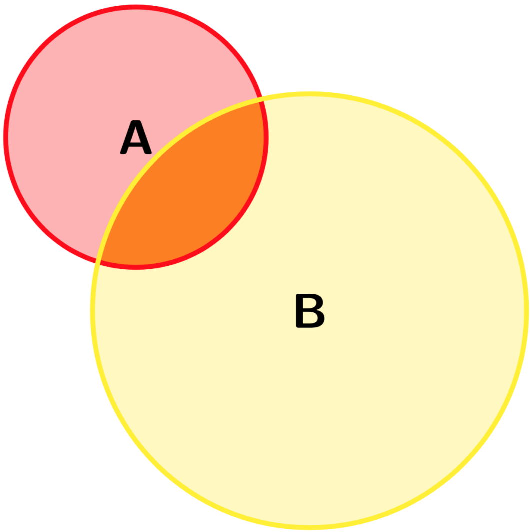
The probability that *either* A or B occurs



Probability of Unions

The probability of $A \cup B$

The probability that *either* A or B occurs

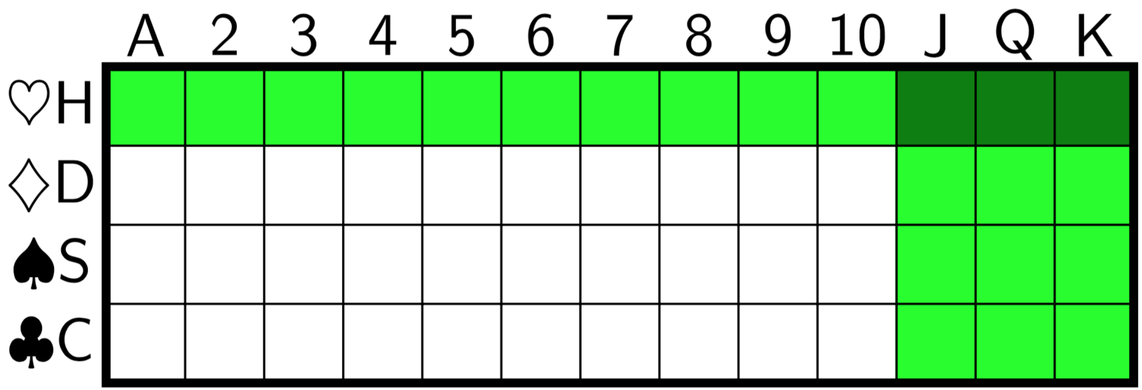


$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

The intuition: the sum of A and B will double count $A \cap B$, so we need to subtract one instance of $A \cap B$

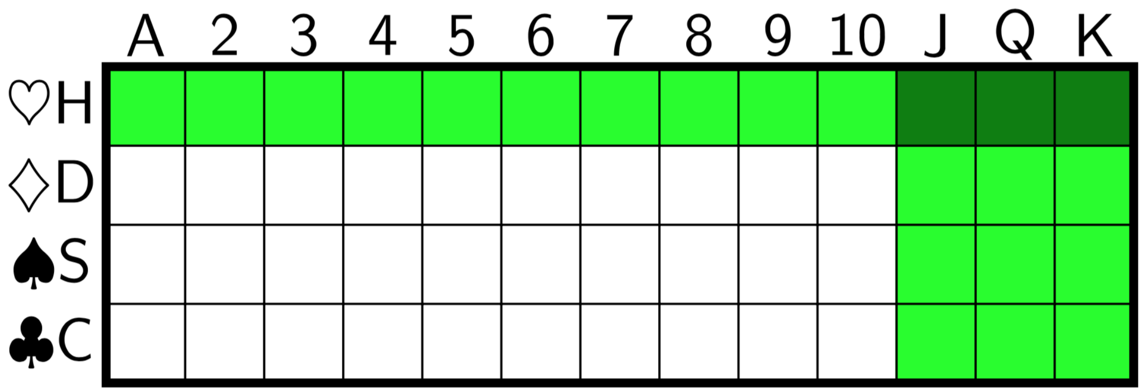
Probability of Unions

What is the probability that we draw a card that is *either* a Heart *or* a face card?



Probability of Unions

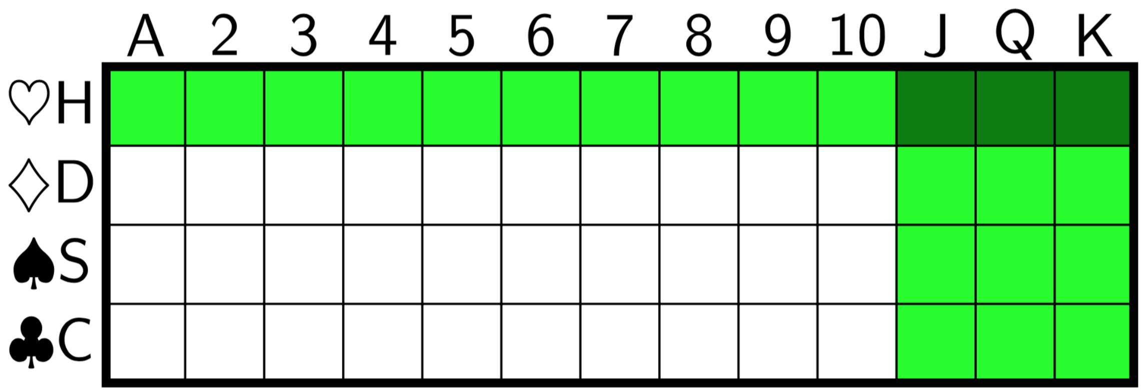
What is the probability that we draw a card that is *either* a Heart *or* a face card?



$p(H) = ?$

Probability of Unions

What is the probability that we draw a card that is *either* a Heart *or* a face card?

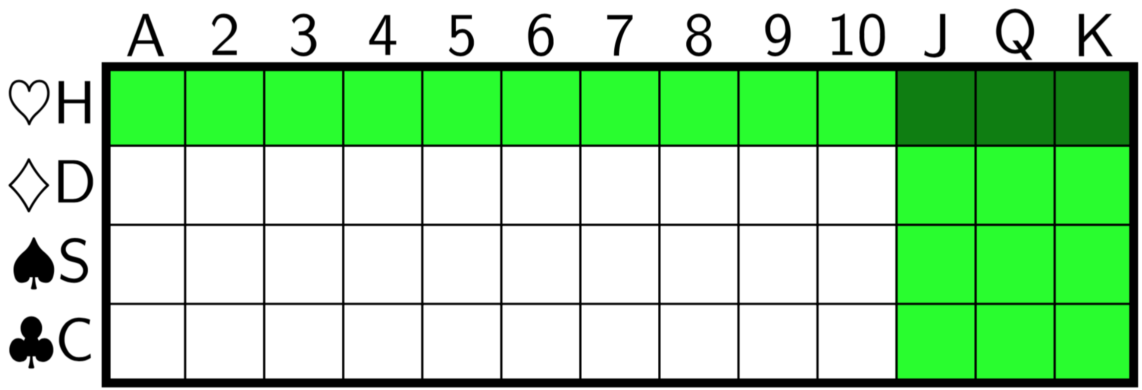


$p(H) = ?$

$p(F) = ?$

Probability of Unions

What is the probability that we draw a card that is *either* a Heart *or* a face card?



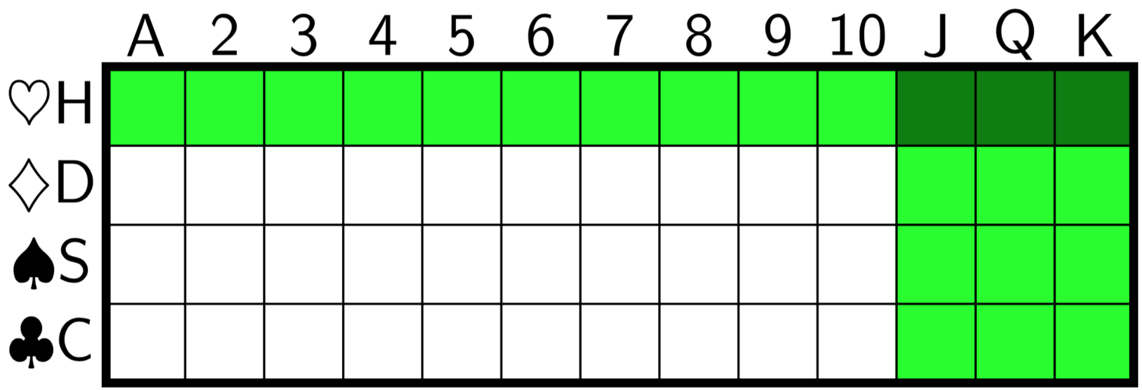
$p(H) = ?$

$p(F) = ?$

$p(H \cap F) = ?$

Probability of Unions

What is the probability that we draw a card that is *either* a Heart *or* a face card?



$p(H) = ?$

$p(F) = ?$

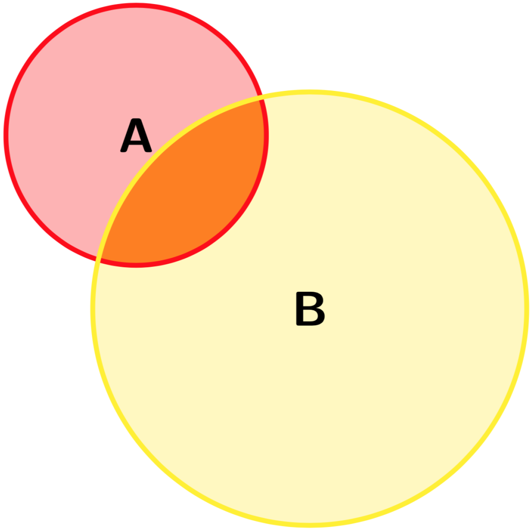
$p(H \cap F) = ?$

$p(H \cup F) = \frac{1}{4} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52}$

Probability of Intersections

The probability of $A \cap B$

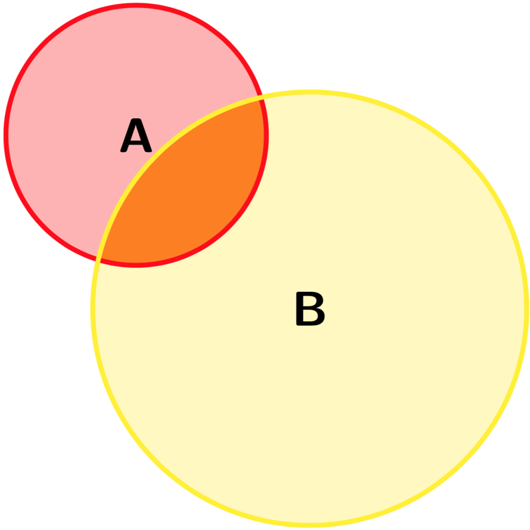
The probability that *both* A and B occur



Probability of Intersections

The probability of $A \cap B$

The probability that *both* A and B occur

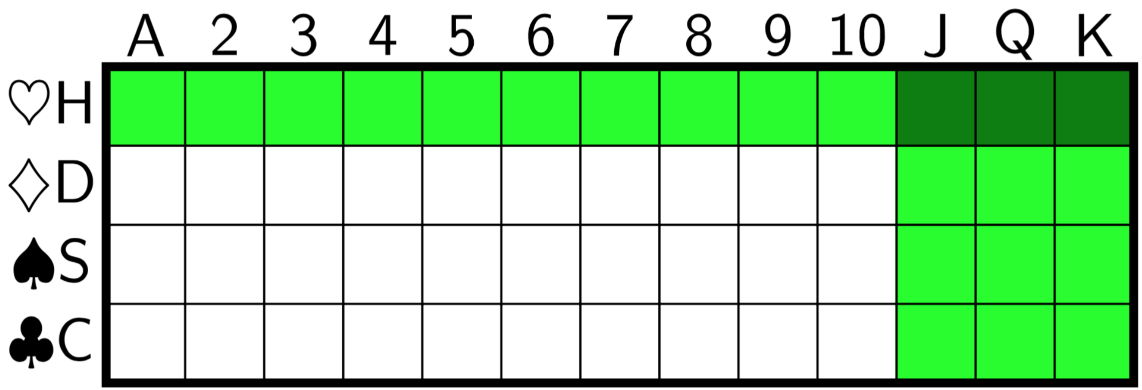


$$p(A \cap B) = p(A) + p(B) - p(A \cup B)$$

The intuition: We care only about the component that we double counted

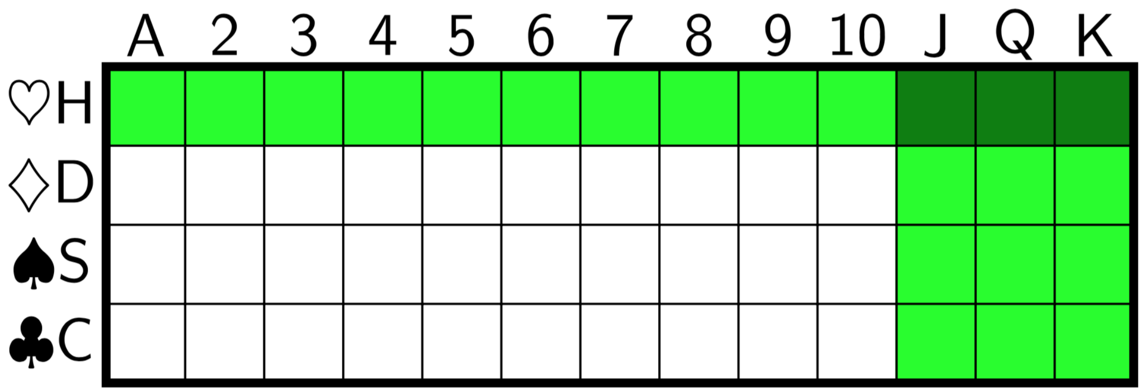
Probability of Intersections

What is the probability that we draw a card that is *both* a heart *and* a face card?



Probability of Intersections

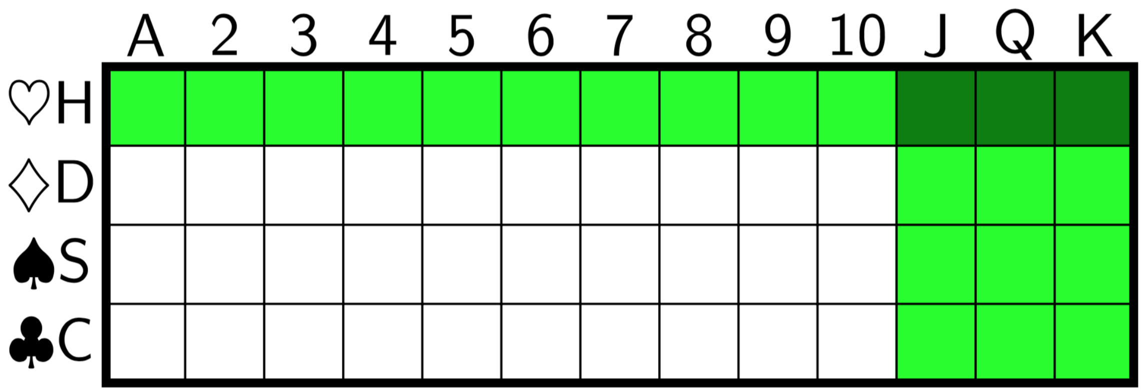
What is the probability that we draw a card that is *both* a heart *and* a face card?



$p(H) = ?$

Probability of Intersections

What is the probability that we draw a card that is *both* a heart *and* a face card?

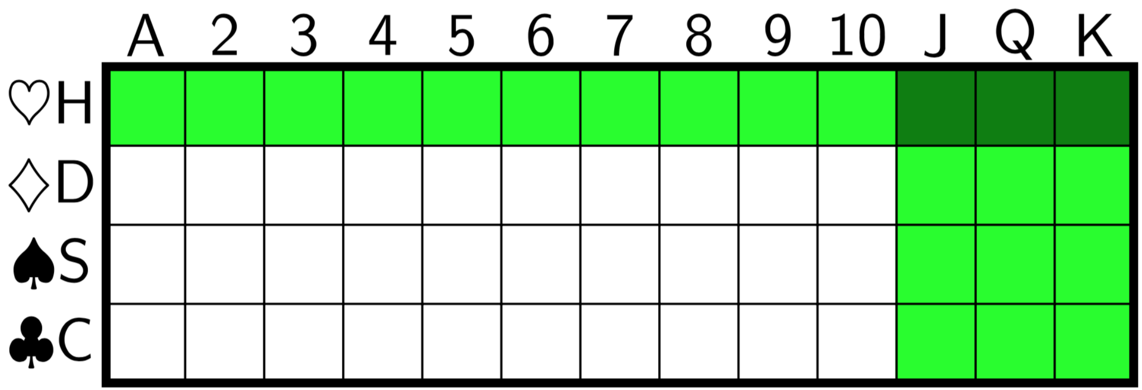


$p(H) = ?$

$p(F) = ?$

Probability of Intersections

What is the probability that we draw a card that is *both* a heart *and* a face card?



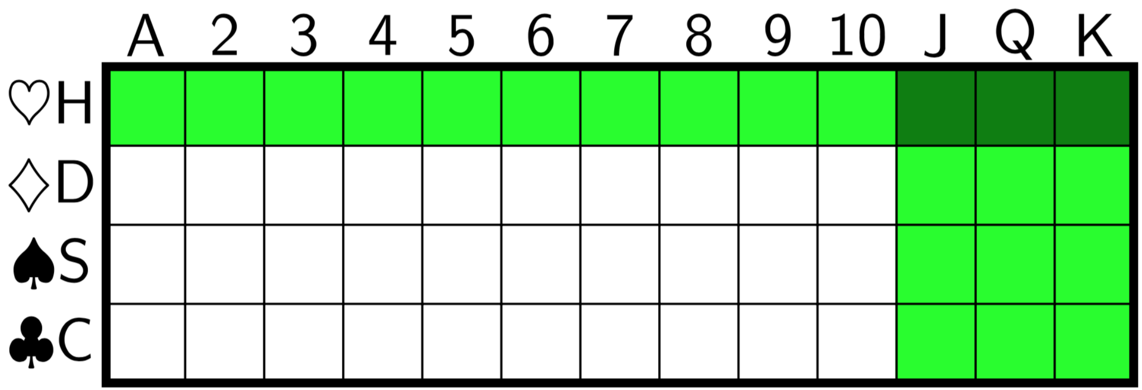
$p(H) = ?$

$p(F) = ?$

$p(H \cup F) = ?$

Probability of Intersections

What is the probability that we draw a card that is *both* a heart *and* a face card?



$p(H) = ?$

$p(F) = ?$

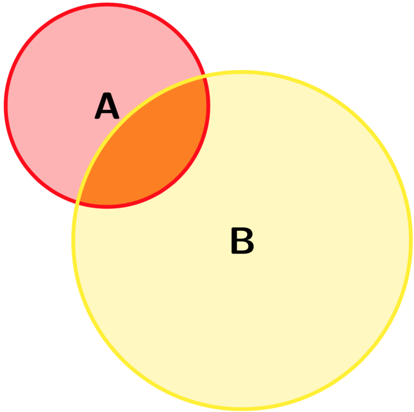
$p(H \cup F) = ?$

$p(H \cap F) = \frac{1}{4} + \frac{12}{52} - \frac{22}{52} = \frac{3}{52}$

Conditional Probability

The probability of A, given B, is expressed as $p(A \mid B)$

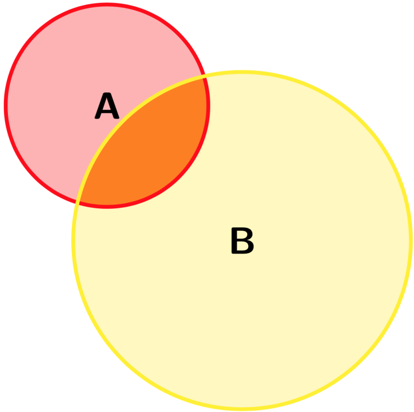
What is the probability of A, given that B also occurs?



Conditional Probability

The probability of A, given B, is expressed as $p(A \mid B)$

What is the probability of A, given that B also occurs?



$$p(A \mid B) = \frac{p(A \cap B)}{p(B)}$$

The intuition:

- If we *know* that B happened, we only care about the space within B
- the probability that both A and B happen, divided by the probability of B
- $p(\text{intersection}) \ / \ p(\text{conditioning event})$

Conditional Probability

	A	2	3	4	5	6	7	8	9	10	J	Q	K
♡H													
♢D													
♠S													
♣C													

Conditional Probability

	A	2	3	4	5	6	7	8	9	10	J	Q	K
♡H													
♢D													
♠S													
♣C													

What is the probability of drawing the Ace of Diamonds?

Conditional Probability

	A	2	3	4	5	6	7	8	9	10	J	Q	K
♡H													
♢D													
♠S													
♣C													

What is the probability of drawing the Ace of Diamonds?

What is the probability of drawing the Ace of Diamonds, *given that* we have have drawn an Ace?

Conditional Probability

	A	2	3	4	5	6	7	8	9	10	J	Q	K
♡H													
♢D													
♠S													
♣C													

What is the probability of drawing the Ace of Diamonds?

What is the probability of drawing the Ace of Diamonds, *given that* we have have drawn an Ace?

- $p(\text{Ace of Diamonds}) = \frac{1}{52}$

Conditional Probability

	A	2	3	4	5	6	7	8	9	10	J	Q	K
♡H													
♢D													
♠S													
♣C													

What is the probability of drawing the Ace of Diamonds?

What is the probability of drawing the Ace of Diamonds, *given that* we have have drawn an Ace?

- $p(\text{Ace of Diamonds}) = \frac{1}{52}$
- $p(\text{Ace}) = \frac{4}{52}$

Conditional Probability

	A	2	3	4	5	6	7	8	9	10	J	Q	K
♡H													
♢D													
♠S													
♣C													

What is the probability of drawing the Ace of Diamonds?

What is the probability of drawing the Ace of Diamonds, *given that* we have have drawn an Ace?

- $p(\text{Ace of Diamonds}) = \frac{1}{52}$
- $p(\text{Ace}) = \frac{4}{52}$
- $p(\text{Ace of Diamonds} \mid \text{Ace}) = \frac{1/52}{4/52} = \frac{1}{4}$

Conditional Probability in Research

Many questions we ask in political science are about conditional probabilities. For example:

- What is the probability that an authoritarian government increases welfare spending, given that it performs poorly in an election?
- How likely is a police officer to shoot a minority civilian, given that the officer is white/non-white?
- What is the probability that a respondent expresses positive feelings toward immigrants conditional on the randomly assigned treatment (e.g., a pro-immigrant campaign ad) she is exposed to?
- How likely is an incumbent president to be re-elected if the economy grows by 1/2/3/4 percent in the election year?

Exercises: What's the Probability?

	A	2	3	4	5	6	7	8	9	10	J	Q	K
♡H													
♢D													
♠S													
♣C													

$p(\{8, 9, 10\})$

$p(\{5, 6\} \cup \{6, 10\})$

$p(A \mid H^C)$

Exercises: What's the Probability?

We have a sample of democratic regimes and we know which of them have broken down (numbers are completely invented):

Country wealth	Breakdown	No breakdown
Poor	23	56
Wealthy	5	115

What is the probability that a randomly chosen democracy from this sample is poor and has broken down?

What is the probability that a randomly chosen democracy breaks down, given that it is wealthy?

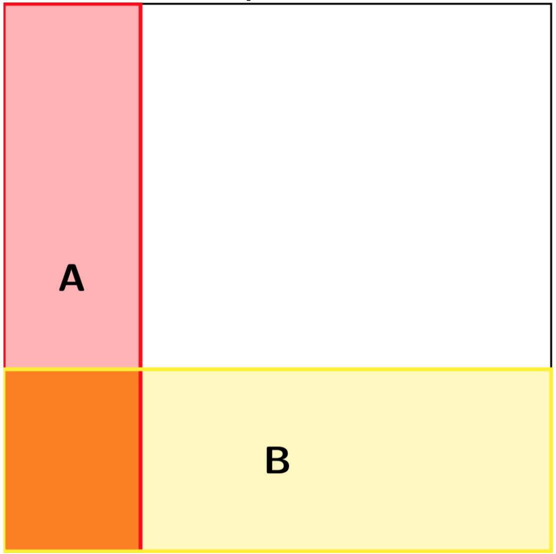
The Notion of *Independence*

Two events are **independent** if knowing the outcome of one event does not change the probability of the other

The Notion of *Independence*

Two events are **independent** if knowing the outcome of one event does not change the probability of the other

Independent

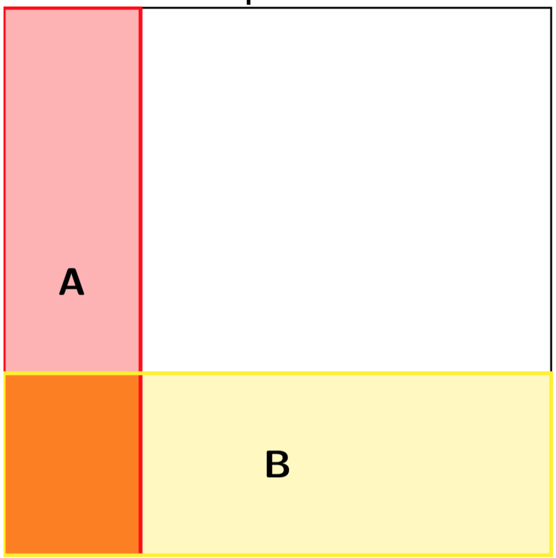


$$p(B) = p(B \mid A)$$

The Notion of *Independence*

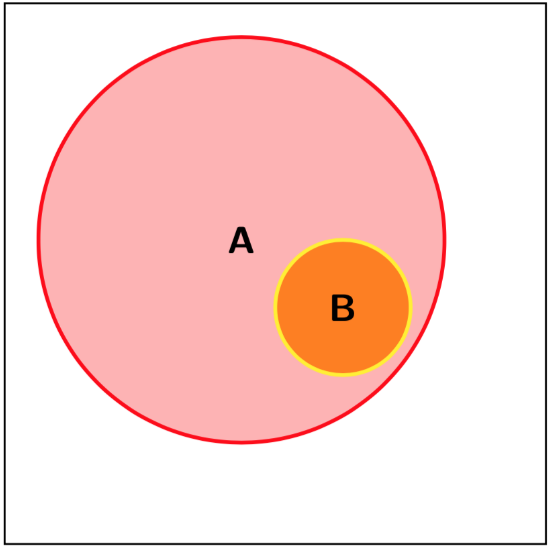
Two events are **independent** if knowing the outcome of one event does not change the probability of the other

Independent



$$p(B) = p(B \mid A)$$

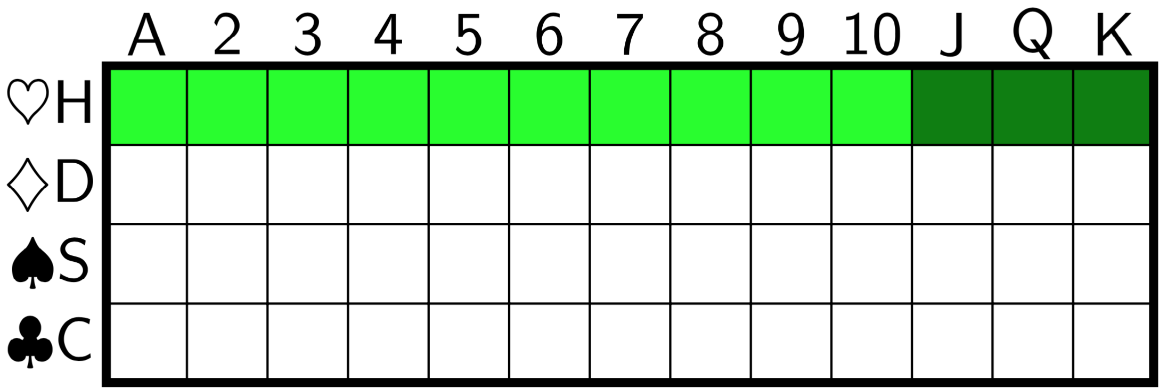
Dependent



$$p(B) \neq p(B \mid A)$$

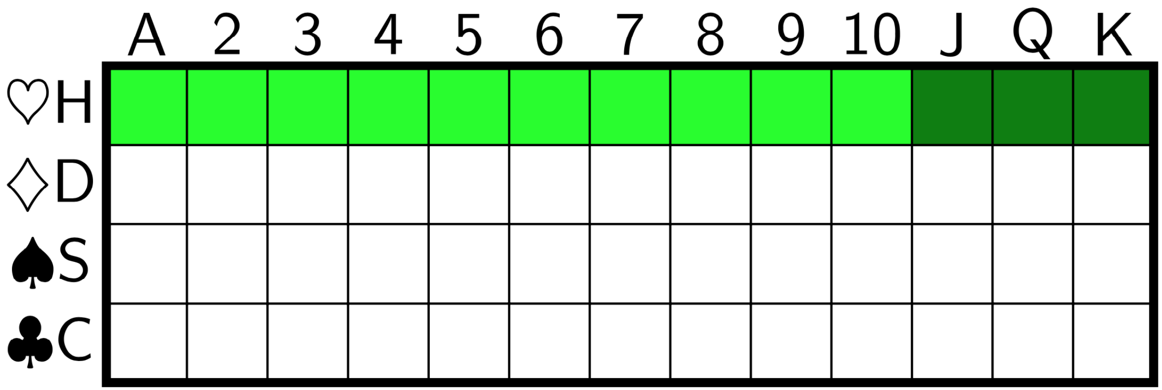
Independence of Events

Is drawing a face card independent of drawing a Hearts card?



Independence of Events

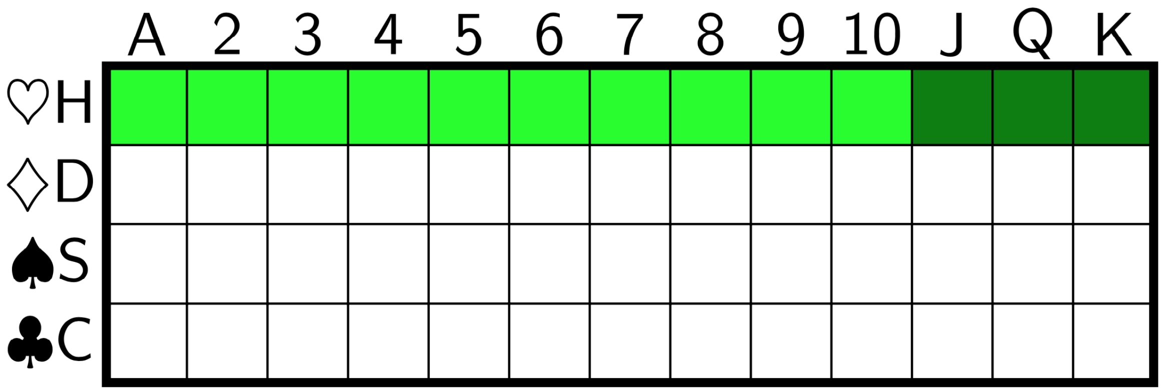
Is drawing a face card independent of drawing a Hearts card?



$p(F \mid H) = \frac{3}{13}$

Independence of Events

Is drawing a face card independent of drawing a Hearts card?

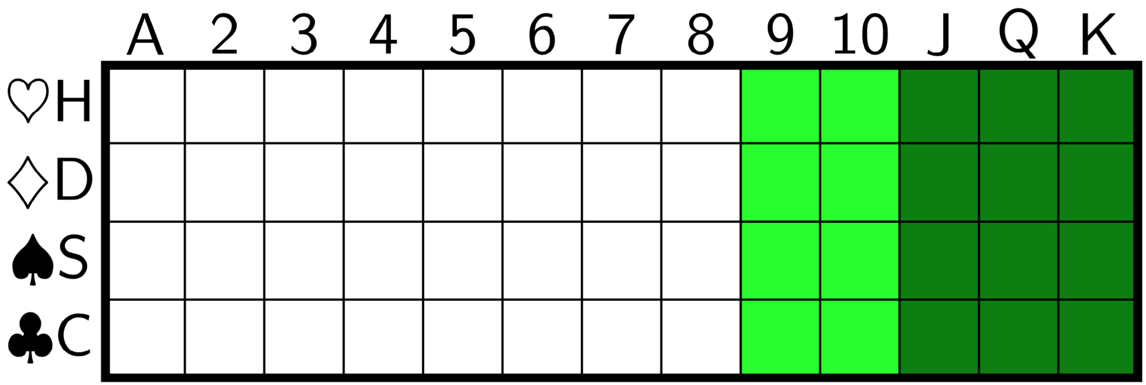


$$p(F \mid H) = \frac{3}{13}$$

$$p(F) = \frac{12}{52} = \frac{3}{13}$$

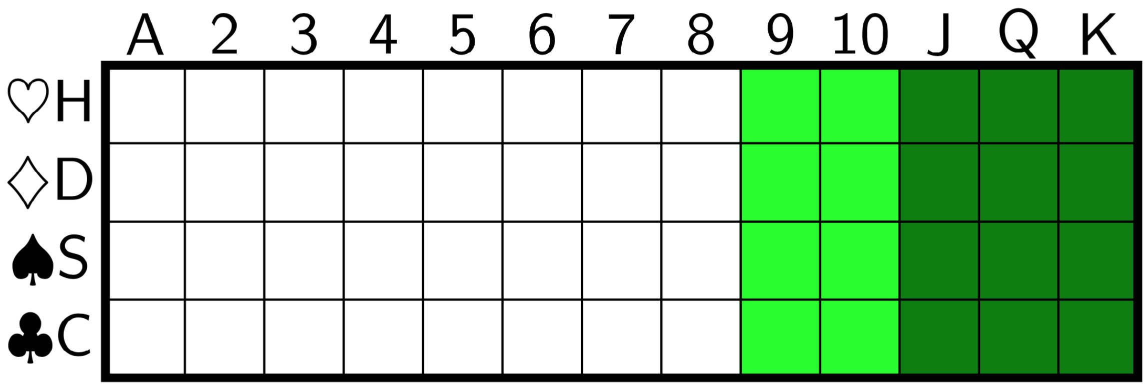
Independence of Events

What about drawing a face card independent of drawing a card greater than 8?



Independence of Events

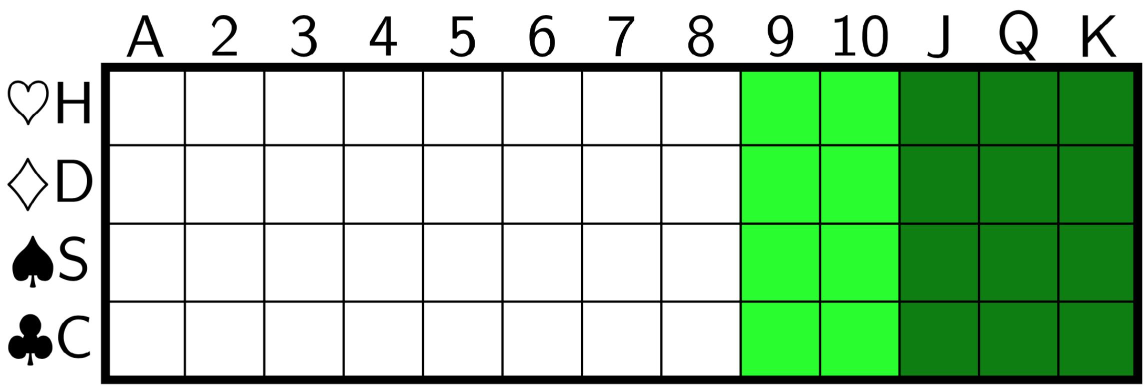
What about drawing a face card independent of drawing a card greater than 8?



$p(X = F \mid X > 8) = \frac{12}{20} = \frac{3}{5}$

Independence of Events

What about drawing a face card independent of drawing a card greater than 8?



$$p(X = F \mid X > 8) = \frac{12}{20} = \frac{3}{5}$$

$$p(F) = \frac{12}{52} = \frac{3}{13}$$

Joint Probability

What we're doing here is considering the probability of *multiple events*

Joint probability: the probability of *more than one event* occurring simultaneously

Joint Probability

What we're doing here is considering the probability of *multiple events*

Joint probability: the probability of *more than one event* occurring simultaneously

$p(A, B) \equiv p(A) \cap p(B)$

Joint Probability of Independent Events

If multiple events are independent of one another, the joint probability of all events is the *product* of the individual probabilities.

Example: we flip three coins independently of one another. What's the probability of the sequence $\{H, H, H\}$?

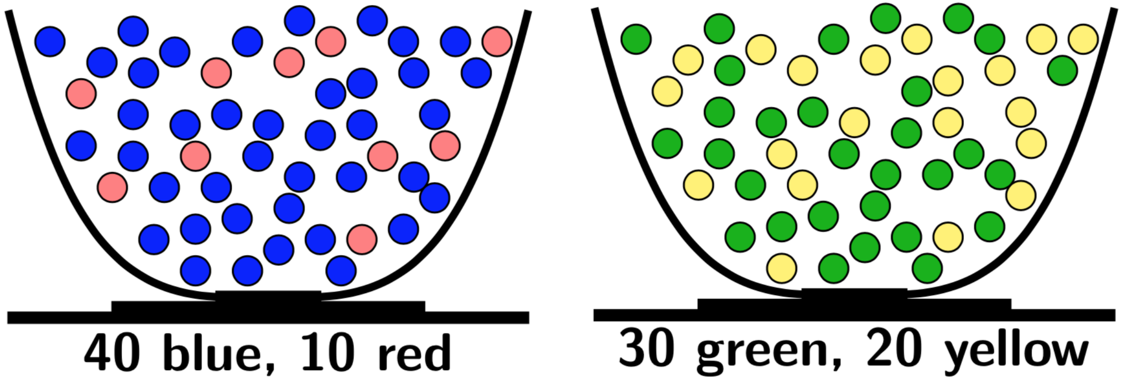
Joint Probability of Independent Events

If multiple events are independent of one another, the joint probability of all events is the *product* of the individual probabilities.

Example: we flip three coins independently of one another. What's the probability of the sequence $\{H, H, H\}$?

$$\begin{aligned} p(H) \times p(H) \times p(H) &= .5 \times .5 \times .5 \\ &= 0.125 \end{aligned}$$

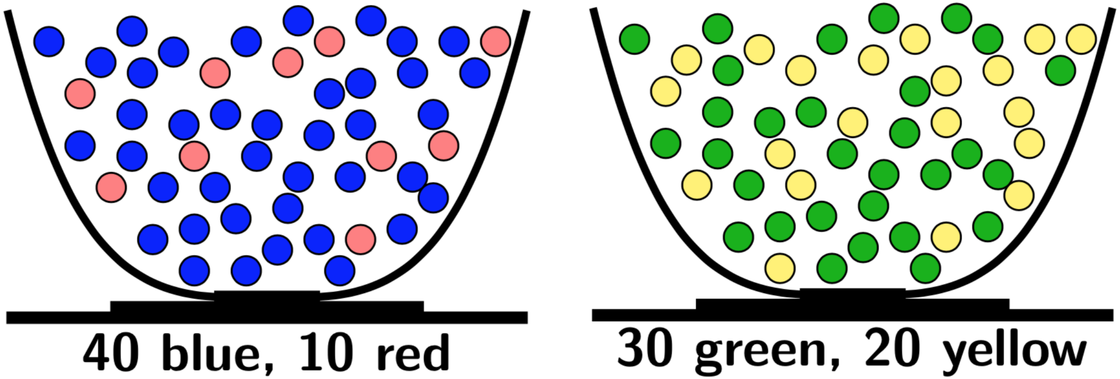
So we've got two bowls



If we draw a ball from each bowl, what is the joint probability of...

- $p(\text{blue}, \text{green}) = ?$
- $p(\text{blue}, \text{yellow}) = ?$
- $p(\text{red}, \text{green}) = ?$
- $p(\text{red}, \text{yellow}) = ?$

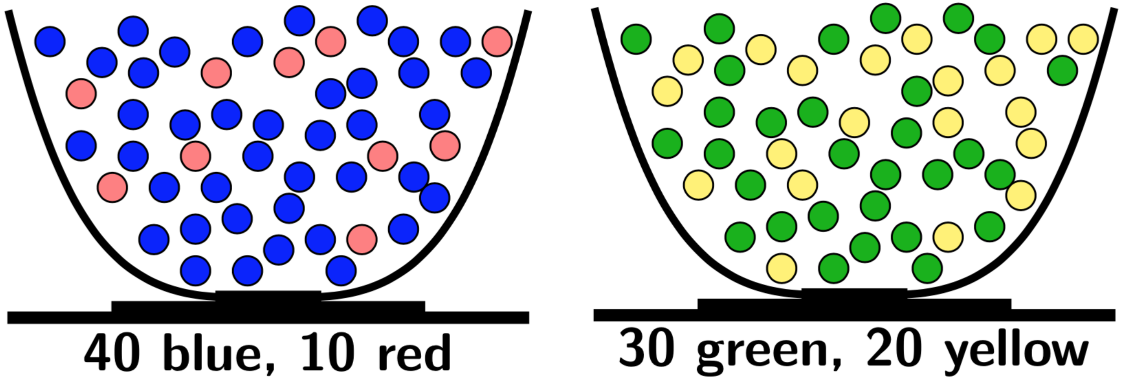
So we've got two bowls



If we draw a ball from each bowl, what is the joint probability of...

- $p(\text{blue}, \text{green}) = \left(\frac{40}{50}\right)\left(\frac{30}{50}\right) = (.8)(.6) = .48$
- $p(\text{blue}, \text{yellow}) = \left(\frac{40}{50}\right)\left(\frac{20}{50}\right) = (.8)(.4) = .32$
- $p(\text{red}, \text{green}) = \left(\frac{10}{50}\right)\left(\frac{30}{50}\right) = (.2)(.6) = .12$
- $p(\text{red}, \text{yellow}) = \left(\frac{10}{50}\right)\left(\frac{20}{50}\right) = (.2)(.4) = .08$

So we've got two bowls



If we draw a ball from each bowl, what is the joint probability of...

- $p(\text{blue}, \text{green}) = \left(\frac{40}{50}\right)\left(\frac{30}{50}\right) = (.8)(.6) = .48$
- $p(\text{blue}, \text{yellow}) = \left(\frac{40}{50}\right)\left(\frac{20}{50}\right) = (.8)(.4) = .32$
- $p(\text{red}, \text{green}) = \left(\frac{10}{50}\right)\left(\frac{30}{50}\right) = (.2)(.6) = .12$
- $p(\text{red}, \text{yellow}) = \left(\frac{10}{50}\right)\left(\frac{20}{50}\right) = (.2)(.4) = .08$

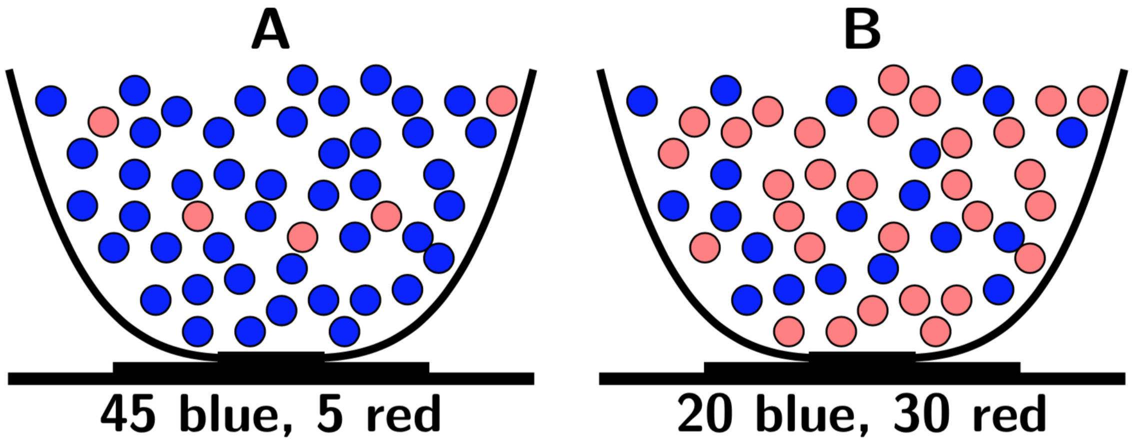
Because these are mutually exclusive and exhaustive events, probabilities sum to 1

Exercises

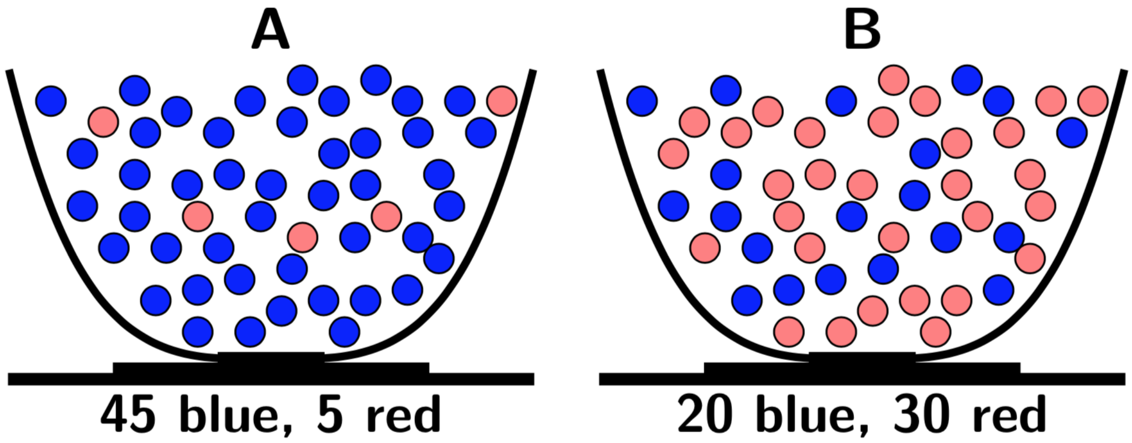
In a given district, 35% support party A, 50% support party B, and 15% support party C. The turnout in the previous local election was 20%, and supporters of all parties were equally likely to vote. What is the probability that a randomly chosen adult in the district supports party C and has voted in the last election?

In a sample of countries, 45% are democratic and 55% are authoritarian. 10% of the countries, regardless of their political regime, have experienced an oil windfall in the last 20 years. How likely is that a randomly selected country is a democracy and has experienced an oil windfall?

Imagine we flip a coin. If heads, we draw a ball from the left urn. If tails, we draw from the right.

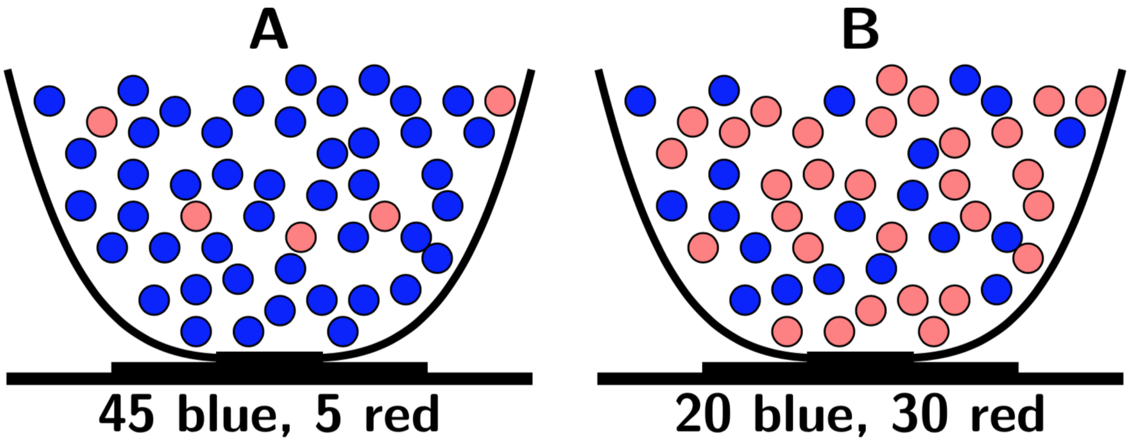


Imagine we flip a coin. If heads, we draw a ball from the left urn. If tails, we draw from the right.



This means there are two ways to choose a blue ball: $\{A, \mathrm{blue}\}$ and $\{B, \mathrm{blue}\}$

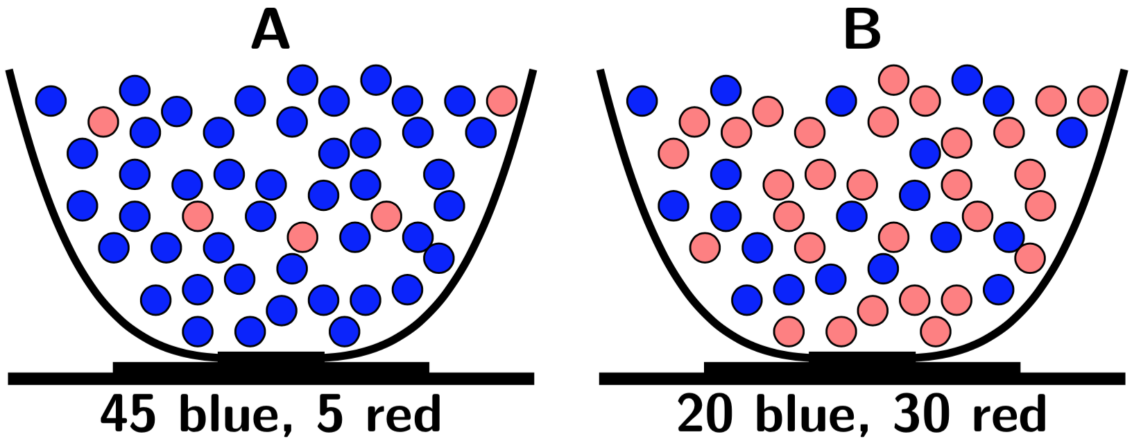
Imagine we flip a coin. If heads, we draw a ball from the left urn. If tails, we draw from the right.



This means there are two ways to choose a blue ball: $\{A, \text{blue}\}$ and $\{B, \text{blue}\}$

- $p(A, \text{blue}) = 0.5 * \frac{45}{50} = 0.45$
- $p(B, \text{blue}) = 0.5 * \frac{20}{50} = 0.20$

Imagine we flip a coin. If heads, we draw a ball from the left urn. If tails, we draw from the right.

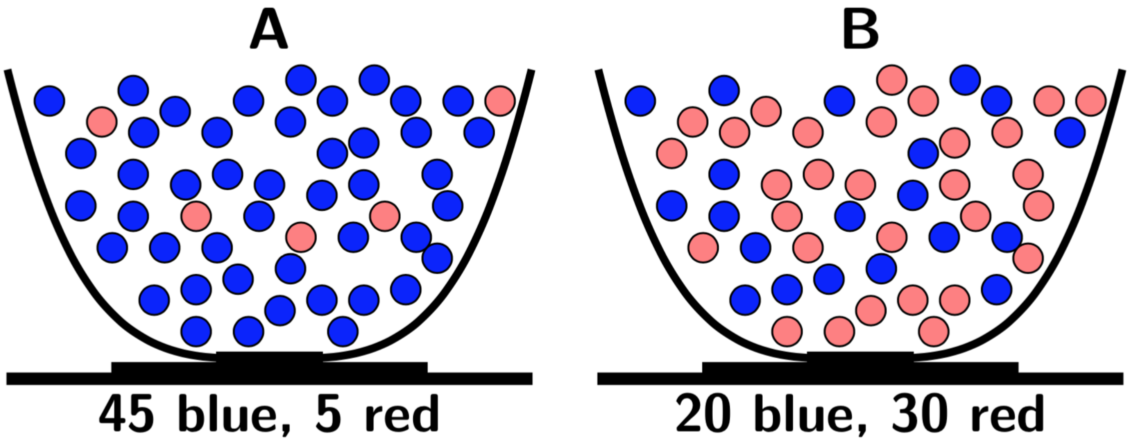


This means there are two ways to choose a blue ball: $\{A, \mathrm{blue}\}$ and $\{B, \mathrm{blue}\}$

- $p(A, \mathrm{blue}) = 0.5 * \frac{45}{50} = 0.45$
- $p(B, \mathrm{blue}) = 0.5 * \frac{20}{50} = 0.20$

Total probability of blue is the sum of the joint probabilities (a very useful principle...)

Imagine we flip a coin. If heads, we draw a ball from the left urn. If tails, we draw from the right.



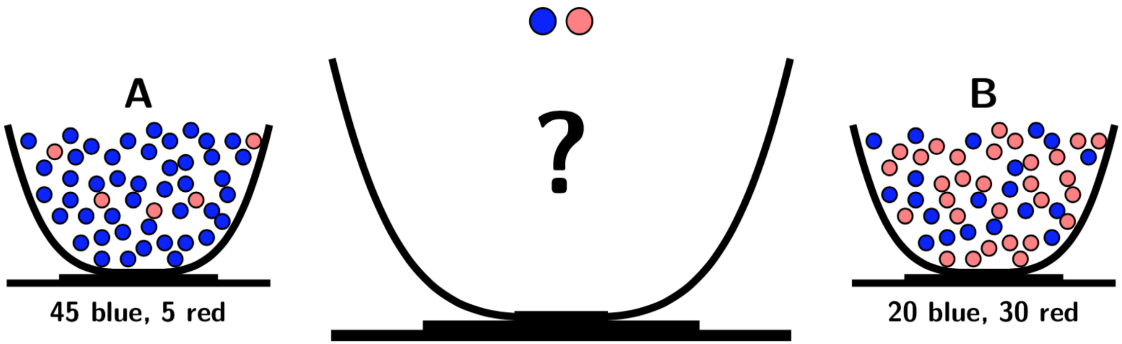
This means there are two ways to choose a blue ball: $\{A, \text{blue}\}$ and $\{B, \text{blue}\}$

- $p(A, \text{blue}) = 0.5 * \frac{45}{50} = 0.45$
- $p(B, \text{blue}) = 0.5 * \frac{20}{50} = 0.20$

Total probability of blue is the sum of the joint probabilities (a very useful principle...)

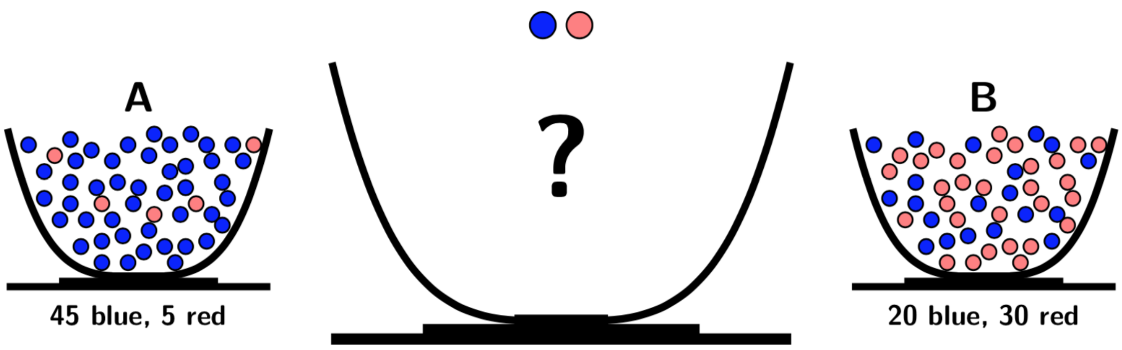
$$\begin{aligned} p(\text{blue}) &= p(\text{blue} \mid A) * p(A) + p(\text{blue} \mid B) * p(B) \\ &= p(\text{blue} \mid A) * p(A) + p(\text{blue} \mid A^C) * p(A^C) \\ &= 0.65 \end{aligned}$$

Inverse Conditional Probability



Someone flips a coin to decide whether to draw a ball from bowl A or B (each with 50% probability), but the bowl is hidden from us.

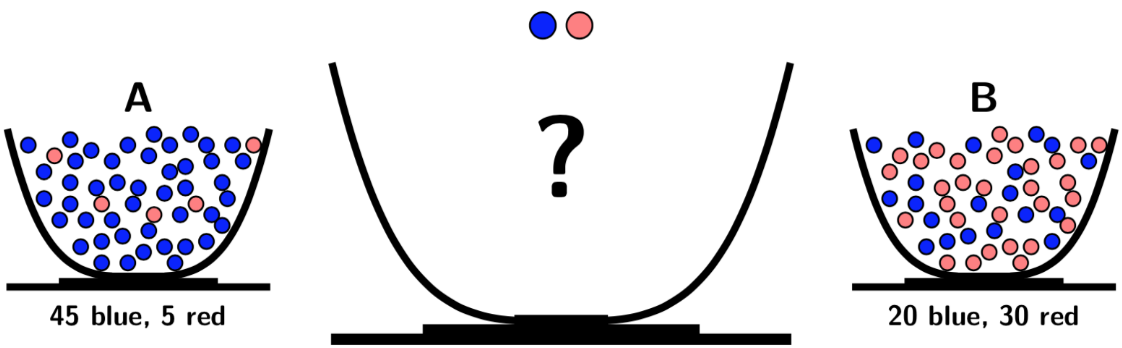
Inverse Conditional Probability



Someone flips a coin to decide whether to draw a ball from bowl A or B (each with 50% probability), but the bowl is hidden from us.

- We've drawn a *blue* ball. What's the probability that we drew from A?

Inverse Conditional Probability



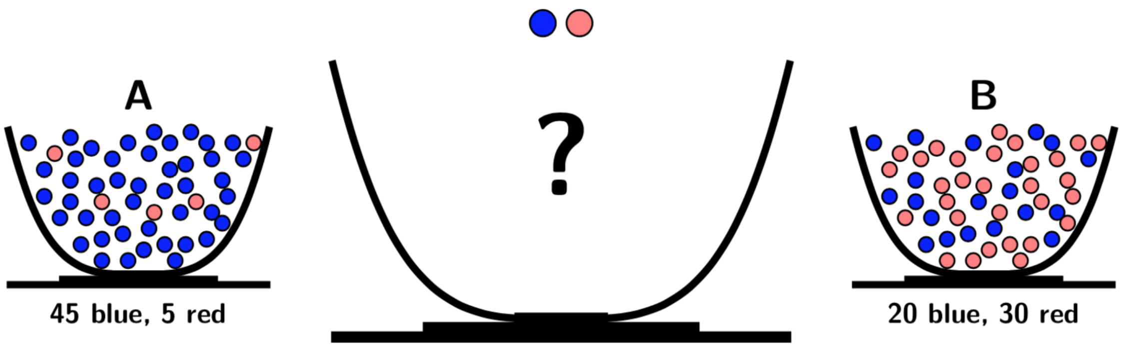
Someone flips a coin to decide whether to draw a ball from bowl A or B (each with 50% probability), but the bowl is hidden from us.

- We've drawn a *blue* ball. What's the probability that we drew from A?

"Inverse" conditional probability problem:

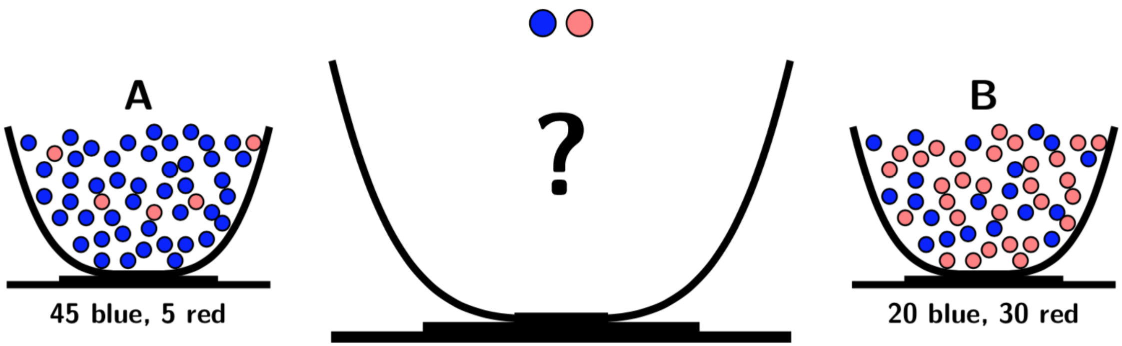
- It's easy to find $p(\text{blue} \mid A)$,
- but how can we *invert* it to find $p(A \mid \text{blue})$?

Find $p(A \mid \text{blue})$



How do we approach any conditional probability problem?

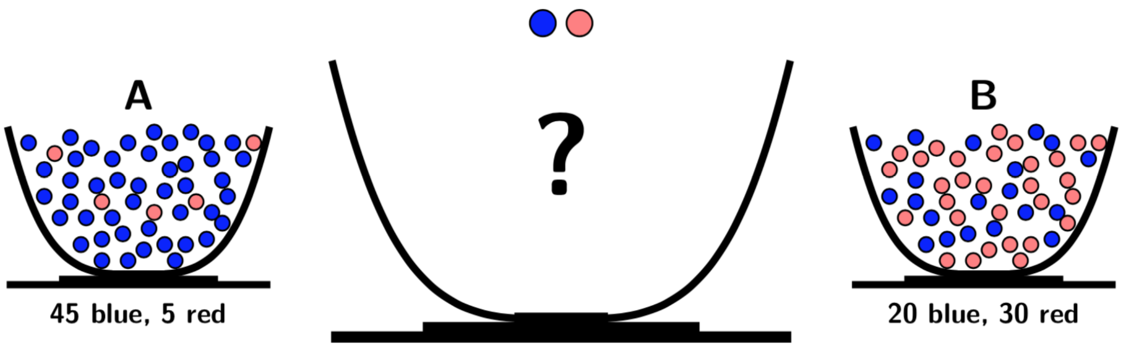
Find $p(A \mid \text{blue})$



How do we approach any conditional probability problem?

$$p(y \mid x) = \frac{p(y \cap x)}{p(x)}$$

Find $p(A \mid \text{blue})$

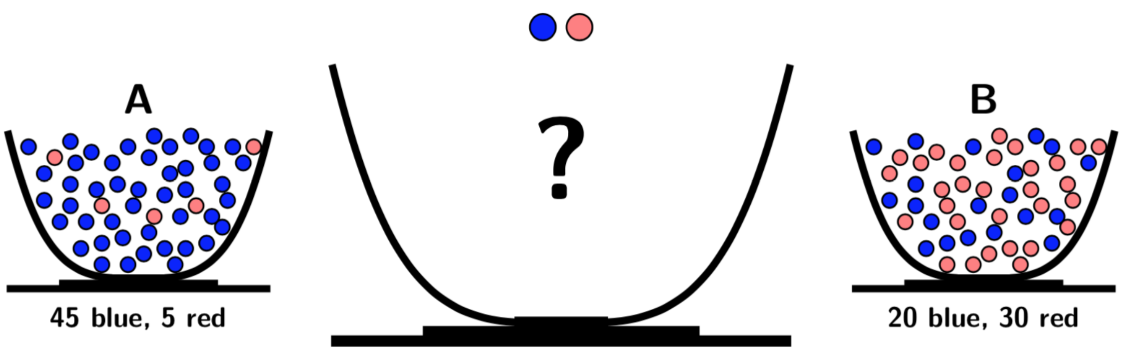


How do we approach any conditional probability problem?

$$p(y \mid x) = \frac{p(y \cap x)}{p(x)}$$

So what do we need for $p(A \mid \text{blue})$?

Find $p(A \mid \text{blue})$



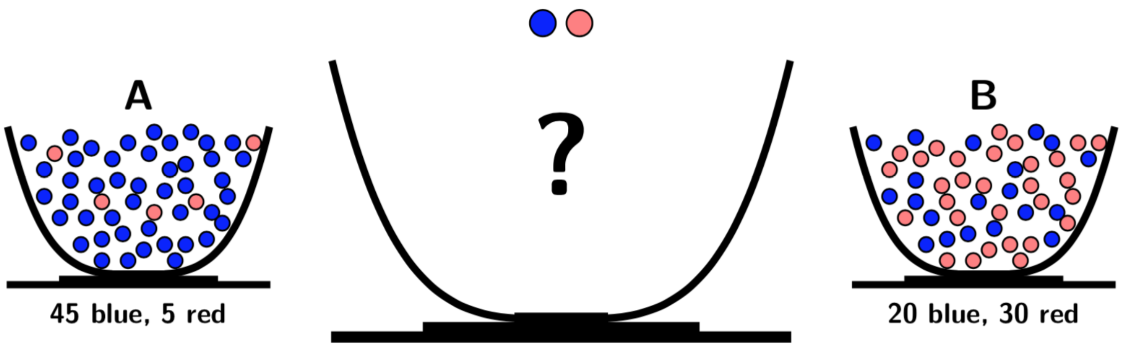
How do we approach any conditional probability problem?

$$p(y \mid x) = \frac{p(y \cap x)}{p(x)}$$

So what do we need for $p(A \mid \text{blue})$?

- $p(A \cap \text{blue})$

Find $p(A \mid \text{blue})$



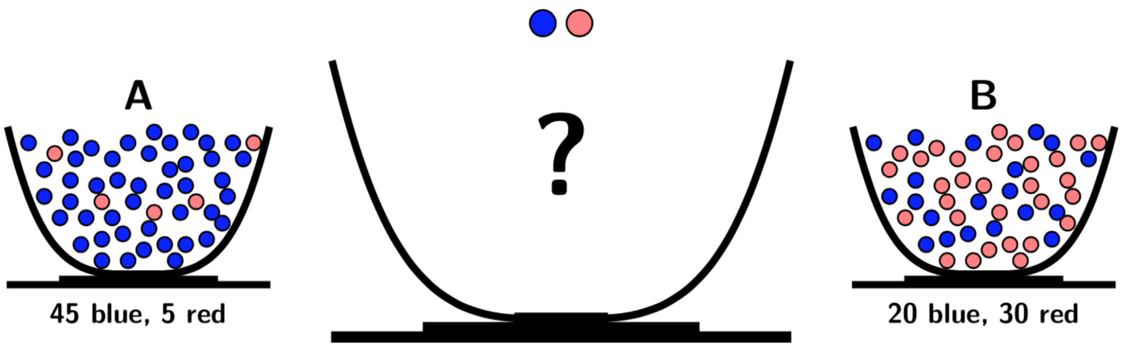
How do we approach any conditional probability problem?

$$p(y \mid x) = \frac{p(y \cap x)}{p(x)}$$

So what do we need for $p(A \mid \text{blue})$?

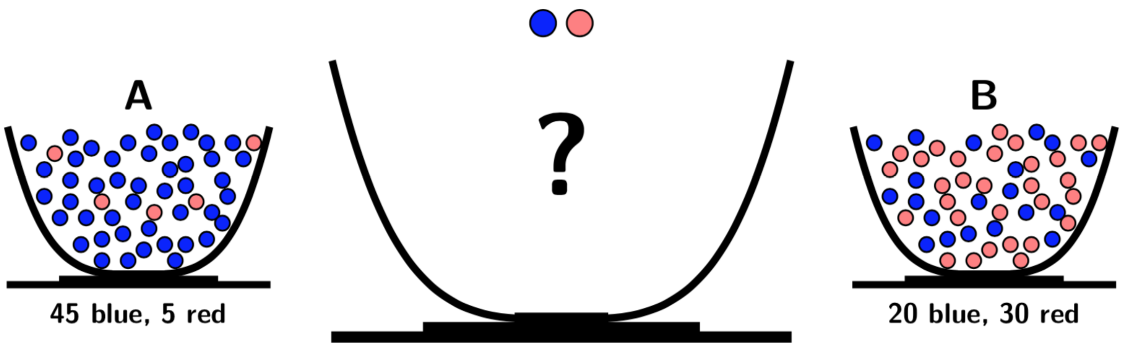
- $p(A \cap \text{blue})$
- $p(\text{blue})$

Find $p(A \mid \text{blue})$



$p(A \cap \text{blue})$?

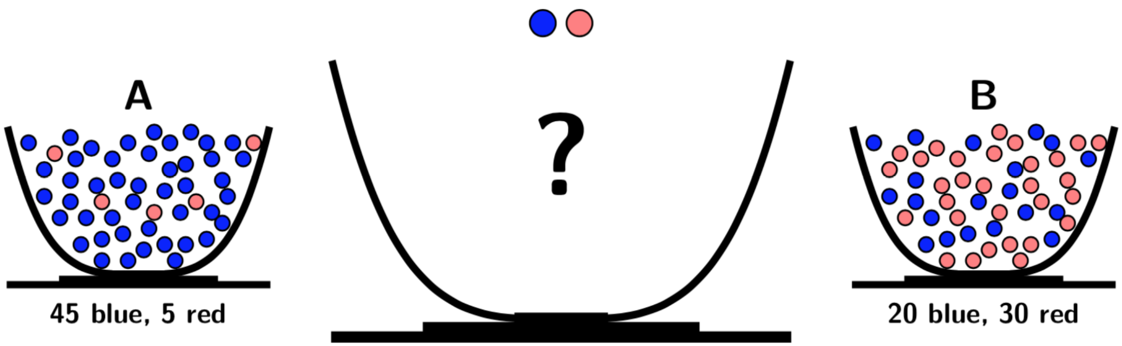
Find $p(A \mid \text{blue})$



$p(A \cap \text{blue})$?

- $(0.5)(0.9) = 0.45$, or $p(\text{blue} \mid A)p(A)$

Find $p(A \mid \text{blue})$



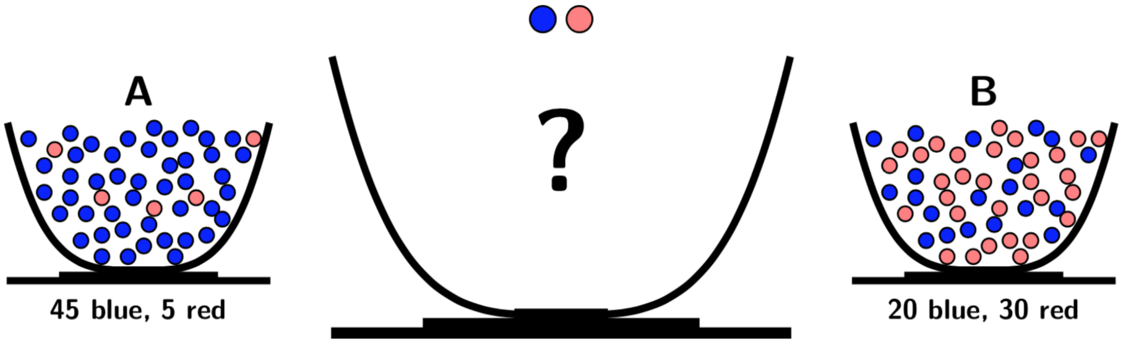
$p(A \cap \text{blue})$?

- $(0.5)(0.9) = 0.45$, or $p(\text{blue} \mid A)p(A)$

$p(\text{blue})$?

- $p(A \cap \text{blue}) + p(B \cap \text{blue})$

Find $p(A \mid \text{blue})$



$p(A \cap \text{blue})$?

- $(0.5)(0.9) = 0.45$, or $p(\text{blue} \mid A)p(A)$

$p(\text{blue})$?

- $p(A \cap \text{blue}) + p(B \cap \text{blue})$
- $(0.5)(0.9) + (0.5)(0.4) = 0.45 + 0.20 = 0.65$

Find $p(A \mid \text{blue})$

$$p(A \mid \text{blue}) = \frac{p(A \cap \text{blue})}{p(\text{blue})}$$

$$p(A \mid \text{blue}) = \frac{p(\text{blue} \mid A)p(A)}{p(\text{blue})}$$

$$p(A \mid \text{blue}) = \frac{0.45}{0.65} \qquad \approx 0.69$$

Find $p(A \mid \text{blue})$

$$p(A \mid \text{blue}) = \frac{p(A \cap \text{blue})}{p(\text{blue})}$$

$$p(A \mid \text{blue}) = \frac{p(\text{blue} \mid A)p(A)}{p(\text{blue})}$$

$$p(A \mid \text{blue}) = \frac{0.45}{0.65} \qquad \approx 0.69$$

This is inverse conditional probability: how we find $p(A \mid \text{blue})$ by starting with $p(\text{blue} \mid A)$.

Find $p(A \mid \text{blue})$

$$p(A \mid \text{blue}) = \frac{p(A \cap \text{blue})}{p(\text{blue})}$$

$$p(A \mid \text{blue}) = \frac{p(\text{blue} \mid A)p(A)}{p(\text{blue})}$$

$$p(A \mid \text{blue}) = \frac{0.45}{0.65} \qquad \approx 0.69$$

This is inverse conditional probability: how we find $p(A \mid \text{blue})$ by starting with $p(\text{blue} \mid A)$.

This is also an example of a formulation of Bayes' Theorem, which more generally is stated as:

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)}$$

Bayes' Theorem

Bayes' Theorem describes how to solve the equation by beginning with its inverse

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)}$$

Or, even more generally,

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y \mid x)p(x) + p(y \mid x^c)p(x^c)}$$

Bayes' Theorem has many applications:

- calculating risks of food allergies or rare diseases
- finding the sources of mechanical errors
- machine learning and prediction; e.g., how likely is a candidate to win an election
- formal modeling (how actors derive their optimal responses)

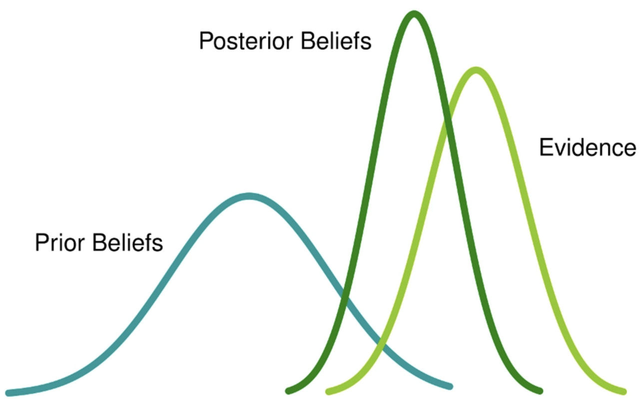
It is the basis for Bayesian inference and Bayesian statistics

Bayesian Inference

Instead of just numbers (as in previous examples), Bayesian statistics work with distributions

- We have *prior beliefs*, say, about the probability of a "heads" on a coin, and these beliefs are uncertain
- Flip the coin several times, calculate the likelihood of *evidence* given the prior
- Calculate revised (posterior) beliefs:

$$\text{Posterior} = \frac{p(\text{evidence} \mid \text{prior}) \times \text{prior}}{p(\text{evidence})}$$



Plot from Analytics Vidhya

Two Ways to Think About Statistics

"Frequentism"

- Over a large number of repeated trials, probability is the fraction of trials in which an event occurs
- Statistical properties come from **repeated sampling assumptions**
- There exists a **fixed** true parameter, which we estimate
- We can calculate the probability that our data were created by different assumed parameter values
- Low probability of data can be used to reject parameter values
- Focus is on the probability of the **data**, assuming a fixed parameter

Two Ways to Think About Statistics

"Frequentism"

- Over a large number of repeated trials, probability is the fraction of trials in which an event occurs
- Statistical properties come from **repeated sampling assumptions**
- There exists a **fixed** true parameter, which we estimate
- We can calculate the probability that our data were created by different assumed parameter values
- Low probability of data can be used to reject parameter values
- Focus is on the probability of the **data**, assuming a fixed parameter

"Bayesianism"

- Probability is our belief about how likely an event is, given our most reasonable information
- Statistical properties come from the **posterior distribution**
- Parameters are "random" (not fixed), only approximated with a distribution, given the model and data
- We have prior notions about plausible parameter values
- We update our prior based on the likelihood of data at different prior values to form posterior beliefs
- Focus is on the probability of the **parameter**, updating a prior with data

Looking Ahead

Methods Courses

If you want to understand statistical work in political science:

- 812, 813, MLE

Formal theory courses:

- 835 (intro to game theory)
- Formal models of international politics (837)

Advanced methods courses include

- Machine learning, Time series, Bayesian analysis, Experimental methods, Multilevel modeling, Agent-based modeling

Courses outside the department:

- Ag econ: applied regression, choice models
- Sociology: causal inference, survey methods
- Statistics: networks, machine learning

Methods Pathways

812 is required, 813 and MLE highly recommended

First field: "I want to study *how to study* politics" (develop new statistical estimators, etc.). You still need a substantive interest

Second field: "I want to teach and research about/use advanced methods." Most grad students in the department who take advanced methods training take this path

Minor: 3 courses. Depends on your dissertation focus. See reqs, talk to your advisors

Advice for Methods Courses

Take as many as you feasibly can, and don't delay MLE

Even if you a qualitative researcher, the *epistemological* lessons of large-N analysis are valuable

Pick something you like and get good at it

- Time series, Bayes, formal models, text as data, causal inference, experiments

Do replication projects (in MLE and beyond)

- Dataverse is a great resource where scholars post data and code for their papers

Advice for Methods in the *Discipline*

Invest in math skills in the beginning (math department has intro to calculus and other classes; self-learning also works)

If you aim for advanced knowledge of methods, classes are necessary but not sufficient:

- Google is your best friend
- Use Cross Validated for questions on stats and Stack Overflow for learning R
- Books and online courses help
- Carefully read empirical work with applications of new methods

Take the open science and the "replication crisis" seriously

Learn to organize your workflow. Some recommended resources:

- The Plain Person's Guide to Plain Text Social Science
- R for Data Science
- Intro to LaTeX
- Git/GitHub for R users

If you might leave academia for data science, consider machine learning and Python