Math Camp Lesson 3 (Day 1)

Calculus

UW–Madison Political Science

August 19, 2020

Overview

Calculus evaluates the behavior of functions:

- Limits
- Rate of change
- Change in the rate of change
- Area of the region they defined on

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Calculus in political science:

- Finding the fitline with the minimal distance between predicted and observed data

- Calculating the probability density in regions of continuous distributions

- Solving for the choice that maximizes a decision maker's utility

Agenda

Day 1

- Limits
- Derivatives

Day 2

- More Derivatives
- Integrals
- Applications

Limits

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The limit of a function characterizes its behavior given a certain input, or as an input value changes.

Limits: Example 1

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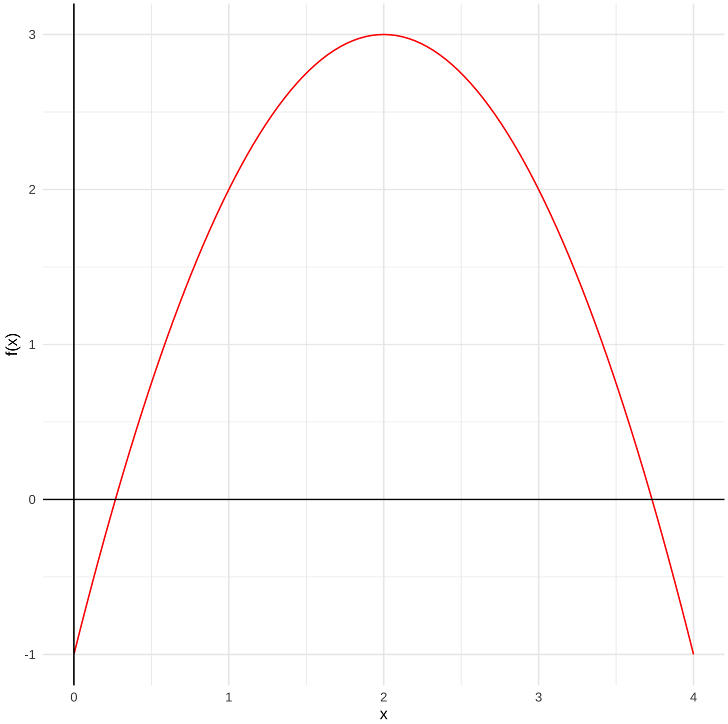
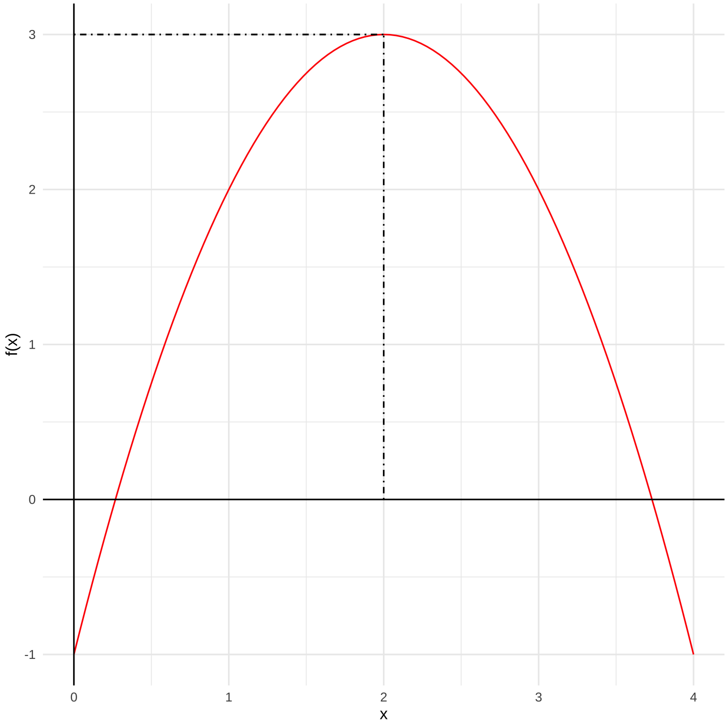


Let's consider the simple function, $f(x) = y = 3 - (x-2)^2$, plotted to the left. What is the limit of $f(x)$ as x *approaches* 2?

Limits: Example 1



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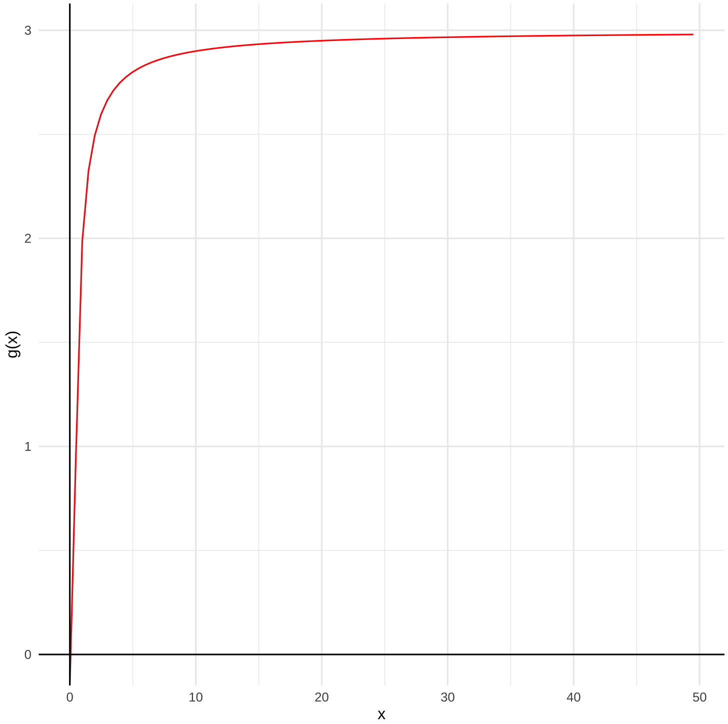
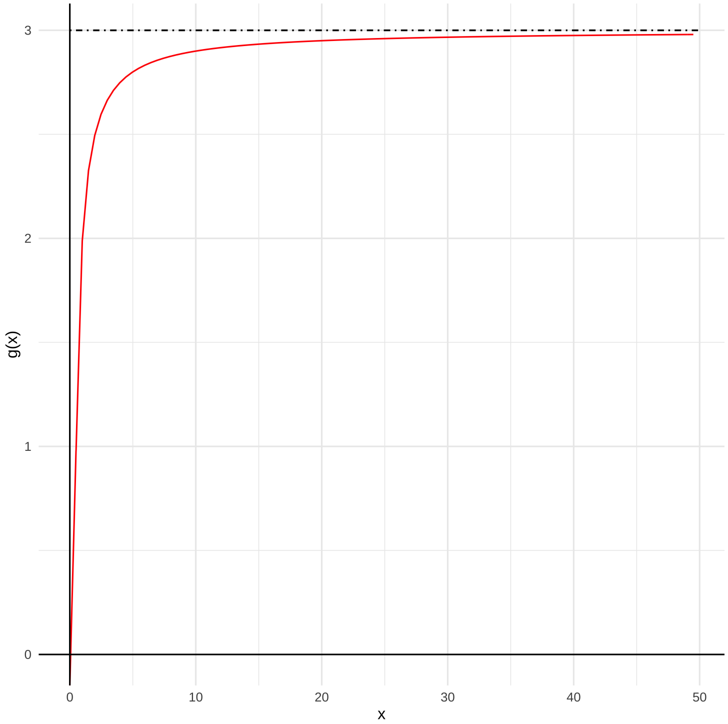
As x approaches 2, $f(x)$ or y approaches $f(2) = 3$.

Limits: Example 2



Let's consider a less simple function, $g(x) = y = 3 - \frac{1}{x}$, plotted to the left. What is the limit of $g(x)$ as x *approaches* ∞?

Limits: Example 2



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As x approaches ∞, $g(x)$ approaches 3. How do we know?

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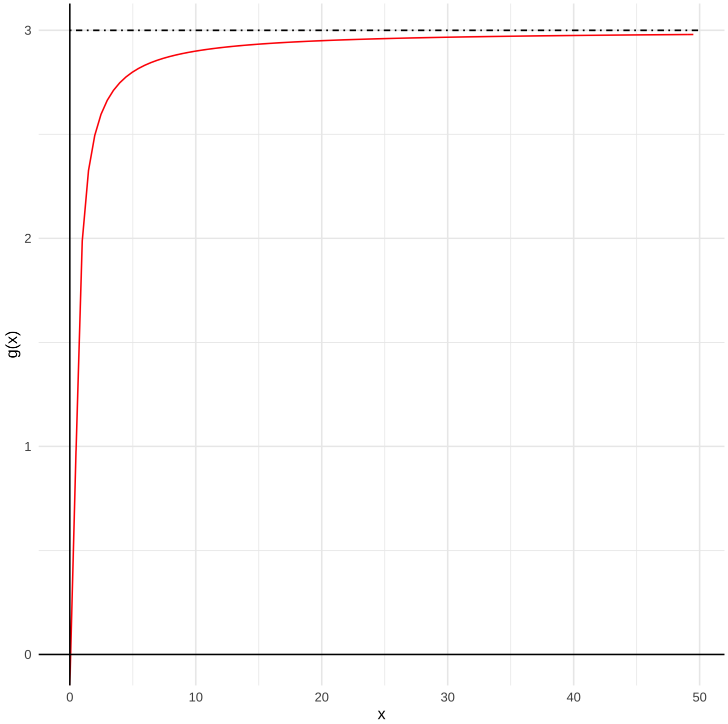

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As x approaches ∞, $g(x)$ approaches 3. Why?

As x gets larger, $\frac{1}{x}$ gets smaller and smaller.

$$\left(\frac{1}{2} > \frac{1}{20} > \frac{1}{200} \ldots\right)$$

Limits, Formally Written

Formally, limits are expressed as:

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This expression should be read as: "As x approaches c, the limit of $f(x)$ is L."

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Many times, you will see this expression written as

$\lim_{x \to c^-} f(x) = L$

or $\lim_{x \to c^+} f(x) = L$

A negative sign $(-)$ implies "As x approaches c *from the left*"

A positive sign $(+)$ implies "As x approaches c *from the right*"

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If you can, evaluate the function at the limit.

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Are these components in the numerator or denominator of a fraction?

If you can, evaluate the function at the limit.

For functions that are well-behaved, the limit as x approaches a finite point is generally the value of the function at that point (if it exists).

Finding Limits: Example 1

Let's consider $\lim_{x \to 2} x^2 - 3x + 1$

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$$\begin{aligned} \lim_{x \to 2} x^2 - 3x + 1 &= \lim_{x \to 2} x^2 - 3 \lim_{x \to 2} x + \lim_{x \to 2} 1 \\ &= 2^2 - 3(2) + 1 \\ &= -1 \end{aligned}$$

Finding Limits: Example 2

Now, let's consider $\lim_{x \to \infty} \dfrac{4x^4 + 7x^2 + 8}{3x^4}$

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$$\begin{aligned}
\lim_{x \to \infty} \frac{4x^4 + 7x^2 + 8}{3x^4} &= \lim_{x \to \infty} \frac{4x^4}{3x^4} + \lim_{x \to \infty} \frac{7x^2}{3x^4} + \lim_{x \to \infty} \frac{8}{3x^4} \\
&= \lim_{x \to \infty} \frac{4}{3} + \lim_{x \to \infty} \frac{7}{3x^2} + \lim_{x \to \infty} \frac{8}{3x^4} \\
&= \frac{4}{3} + 0 + 0 \\
&= \frac{4}{3}
\end{aligned}$$

Finding Limits: Example 2

Now, let's consider $\lim\limits_{x\to\infty} \dfrac{4x^4+7x^2+8}{3x^4}$

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&= \lim_{x\to\infty} \frac{4}{3} + \lim_{x\to\infty} \frac{7}{3x^2} + \lim_{x\to\infty} \frac{8}{3x^4} \\
&= \frac{4}{3} + 0 + 0 \\
&= \frac{4}{3}
\end{aligned}
$$

Why does $\lim_{x\to\infty} \frac{7}{3x^2} = 0$? As $x \to \infty$, $3x^2 \to \infty$, and $\frac{7}{\infty} \to 0$.

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&= \lim_{x \to 0} \frac{4}{3} + \lim_{x \to 0} \frac{7}{3x^2} + \lim_{x \to 0} \frac{8}{3x^4} \\
&= \frac{4}{3} + \frac{7}{3} \lim_{x \to 0} \frac{1}{x^2} + \frac{8}{3} \lim_{x \to 0} \frac{1}{x^4} \\
&= \frac{4}{3} + \frac{7}{3} \times \frac{1}{0} + \frac{8}{3} \times \frac{1}{0} \\
&= \dots
\end{aligned}
$$

Finding Limits: Example 3

The limit is ∞.

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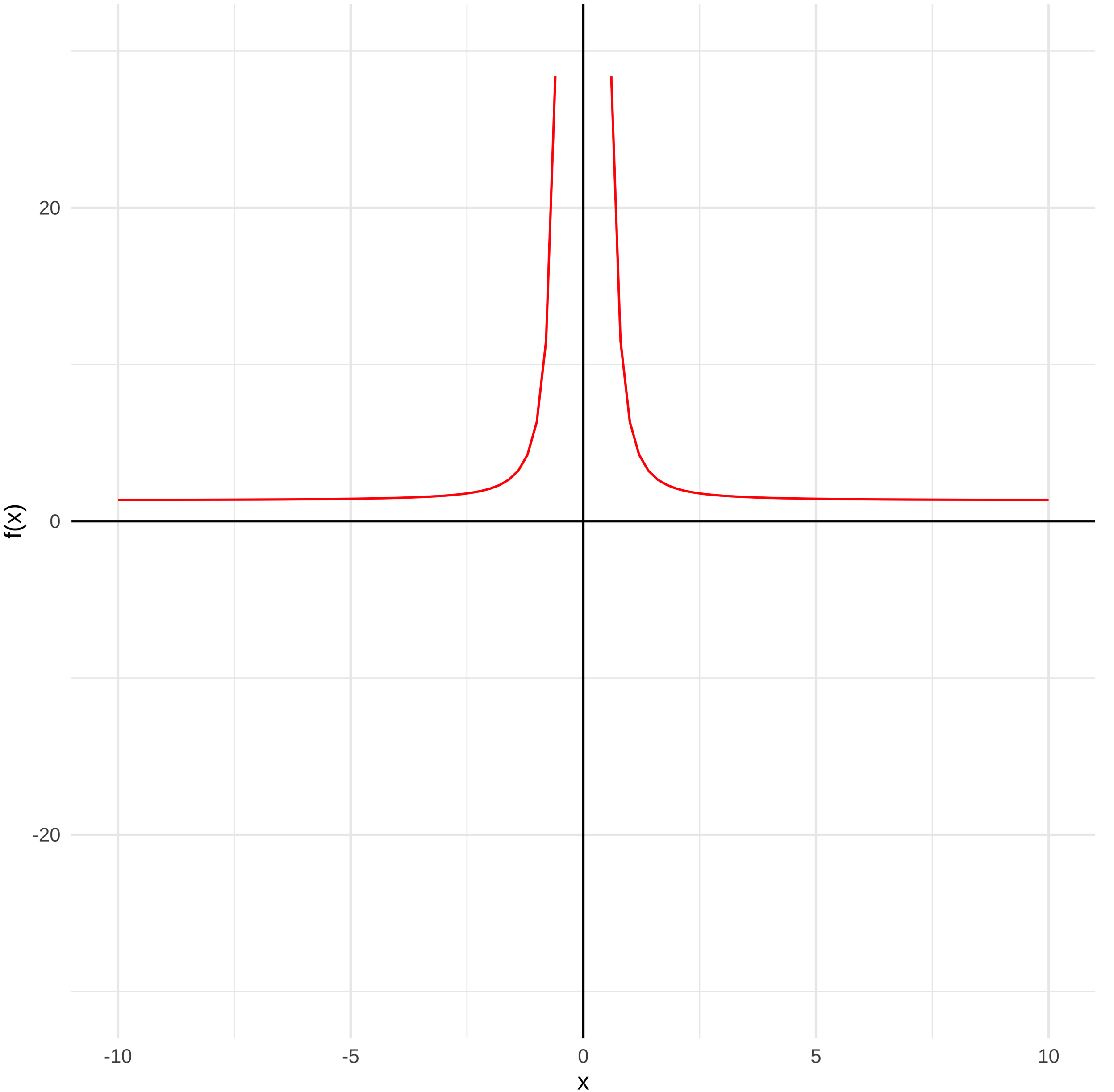
$$
\begin{aligned}
&= \lim_{x \to 0} \frac{4x^4 + 7x^2 + 8}{3x^4} \\
&= \frac{4}{3} + \frac{7}{3} \lim_{x \to 0} \frac{1}{x^2} + \frac{8}{3} \lim_{x \to 0} \frac{1}{x^4} \\
&= \infty
\end{aligned}
$$

As x approaches 0, the function retains some positive value in the numerator while the denominator *positively* approaches 0. This means that you are dividing by a smaller and smaller fraction, which means the entire term is getting larger and approaches ∞.



20
0
-20
f(x)
-10
-5
0
5
10
x

Exercises

Find the following limits:

$\lim_{x \to 4} x^2 - 6x + 4$

$\lim_{x \to 4} \frac{x^2}{3x-2}$

$\lim_{x \to \infty} \frac{3x-4}{x+3}$

Derivatives

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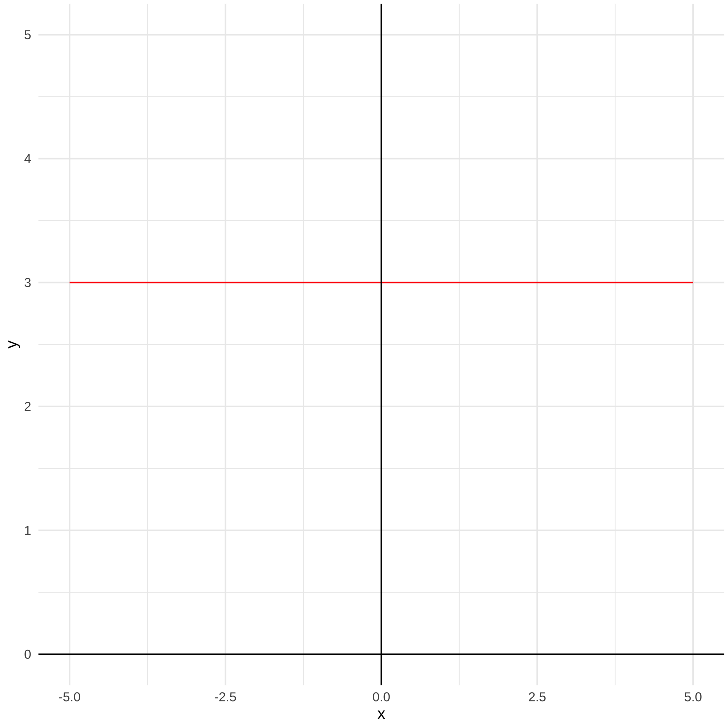
It is the instantaneous slope of the line at any given point.

The slope of a function is how much the output changes as a result of changes in the input.

Using Δ to signify ‘change’, this is $\frac{\Delta f(x)}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ or "rise-over-run".

Slope



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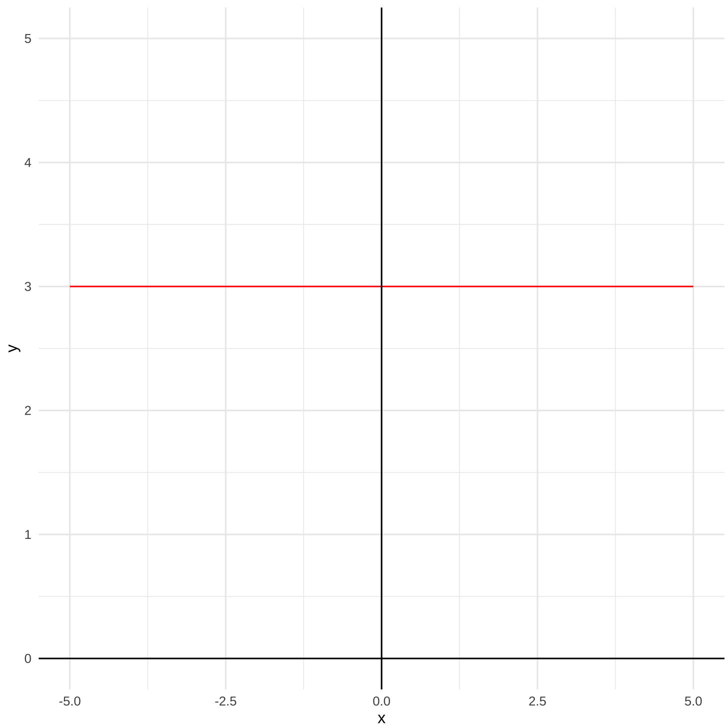
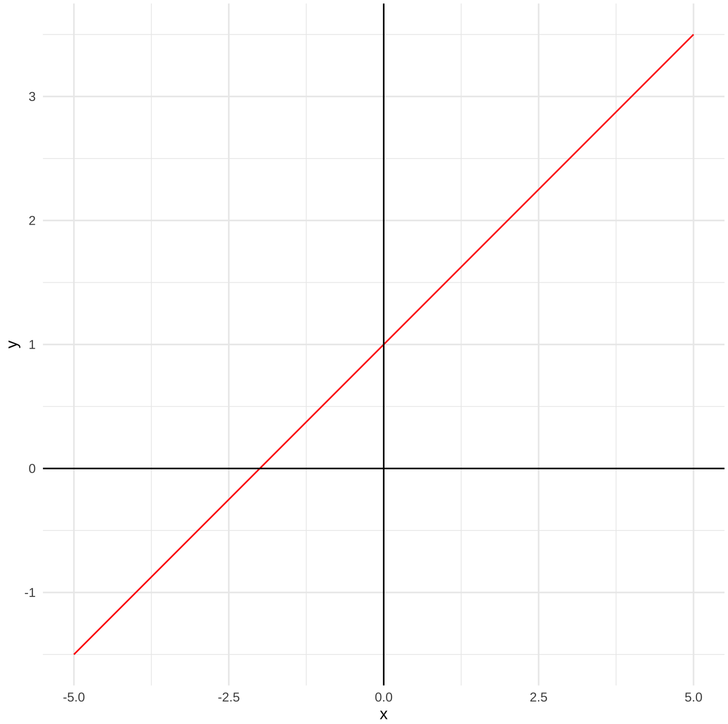


Let's consider the function, $y = 3$, plotted to the left. What is its "slope"?

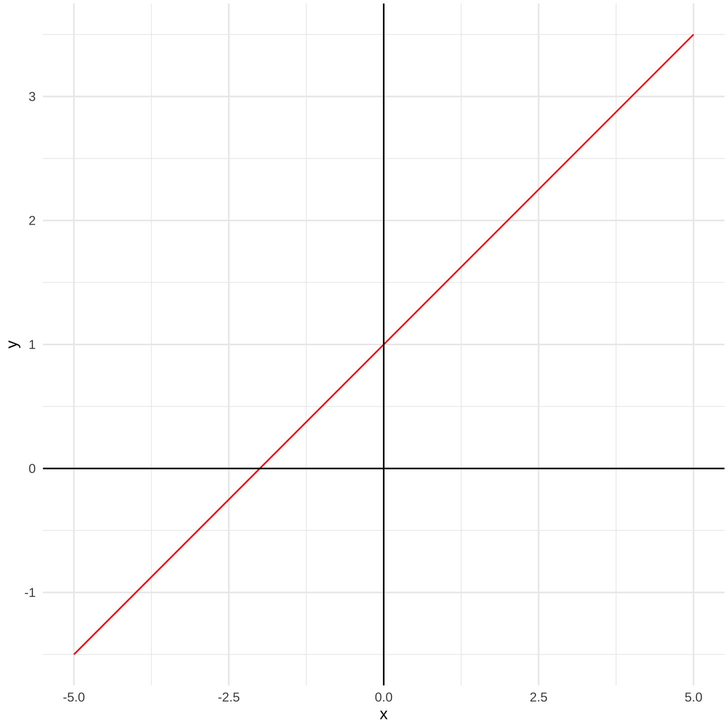
Its slope or $\frac{\Delta f(x)}{\Delta x} = 0$ because there is no "rise".

Slope



Let's consider a less simple function, $y = \frac{1}{2}x + 1$, plotted to the left. What is its "slope"?

Slope

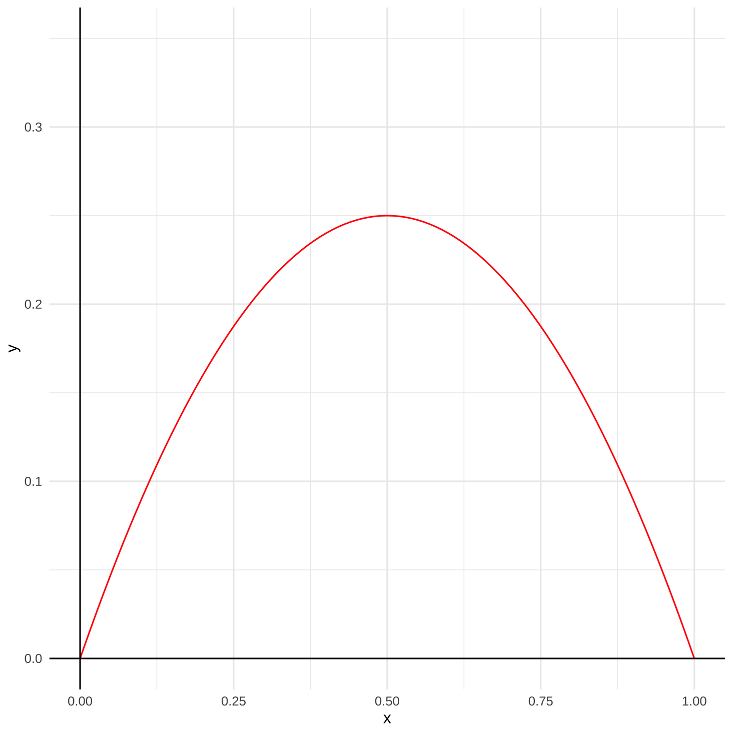


Let's consider a less simple function, $y = \frac{1}{2}x + 1$, plotted to the left. What is its "slope"?

Its slope or $\frac{\Delta f(x)}{\Delta x} = \frac{1}{2}$.

[Recall: $y = mx + b$ from Day 1]

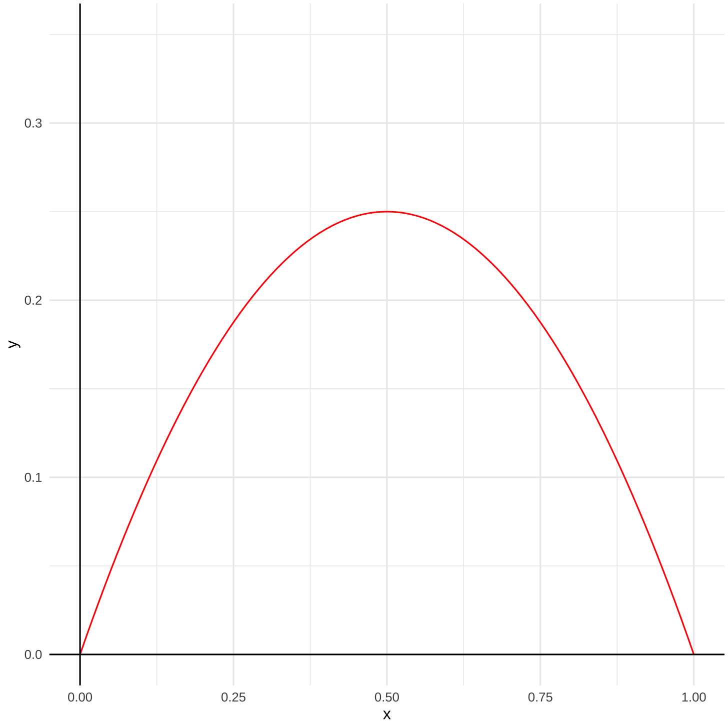
Slope



Let's consider an even more complicated function, $y = x - x^2$, plotted to the left.

What is its "slope", or $\frac{\Delta f(x)}{\Delta x}$?

Slope



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How do we even calculate this?

Derivatives as Limits

We can approximate the slope at a certain location by picking a point nearby on the line and finding the slope of the straight line connecting these two points.

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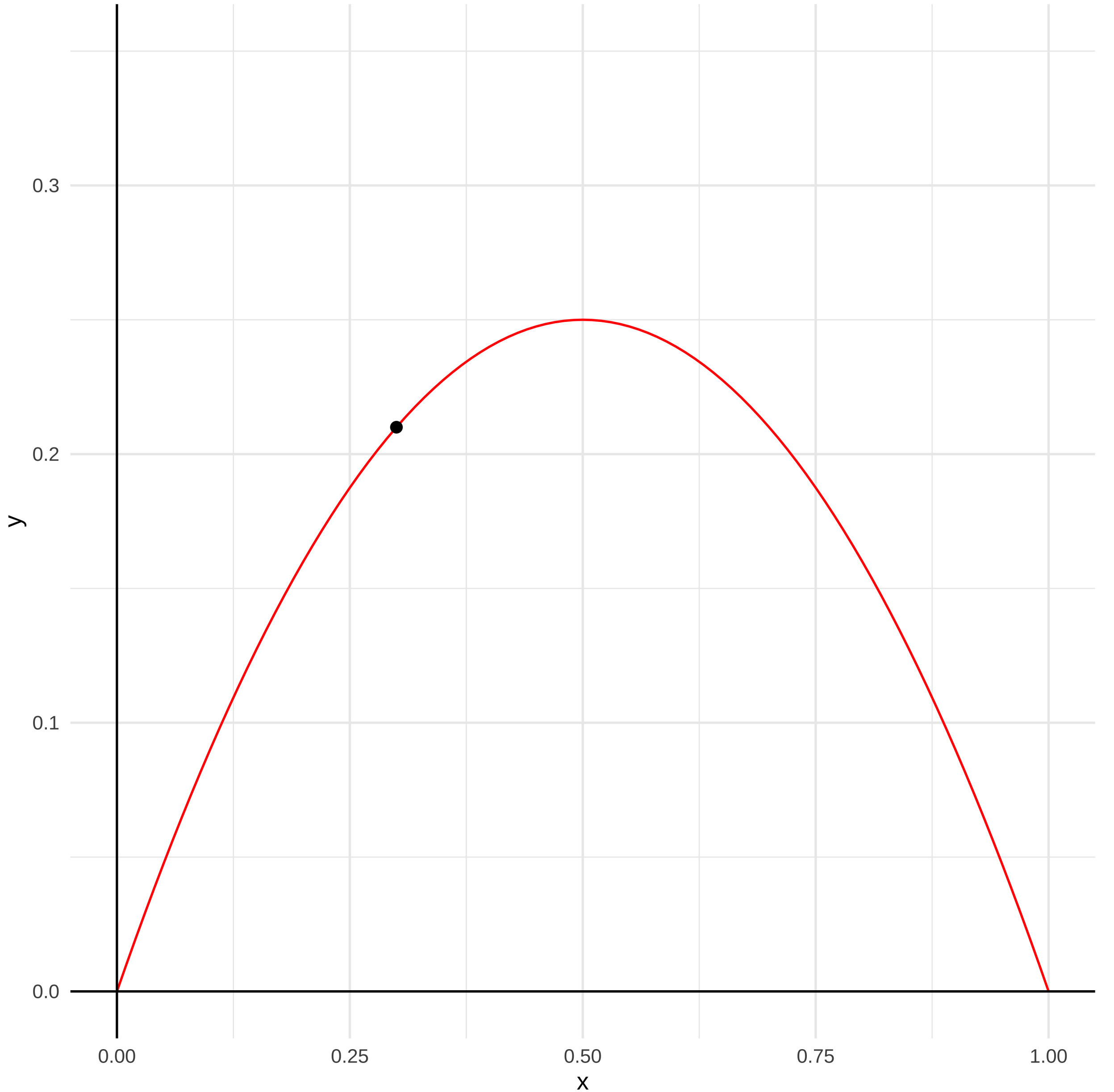
Let's consider a few examples.


0.3
0.2
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0.0
y
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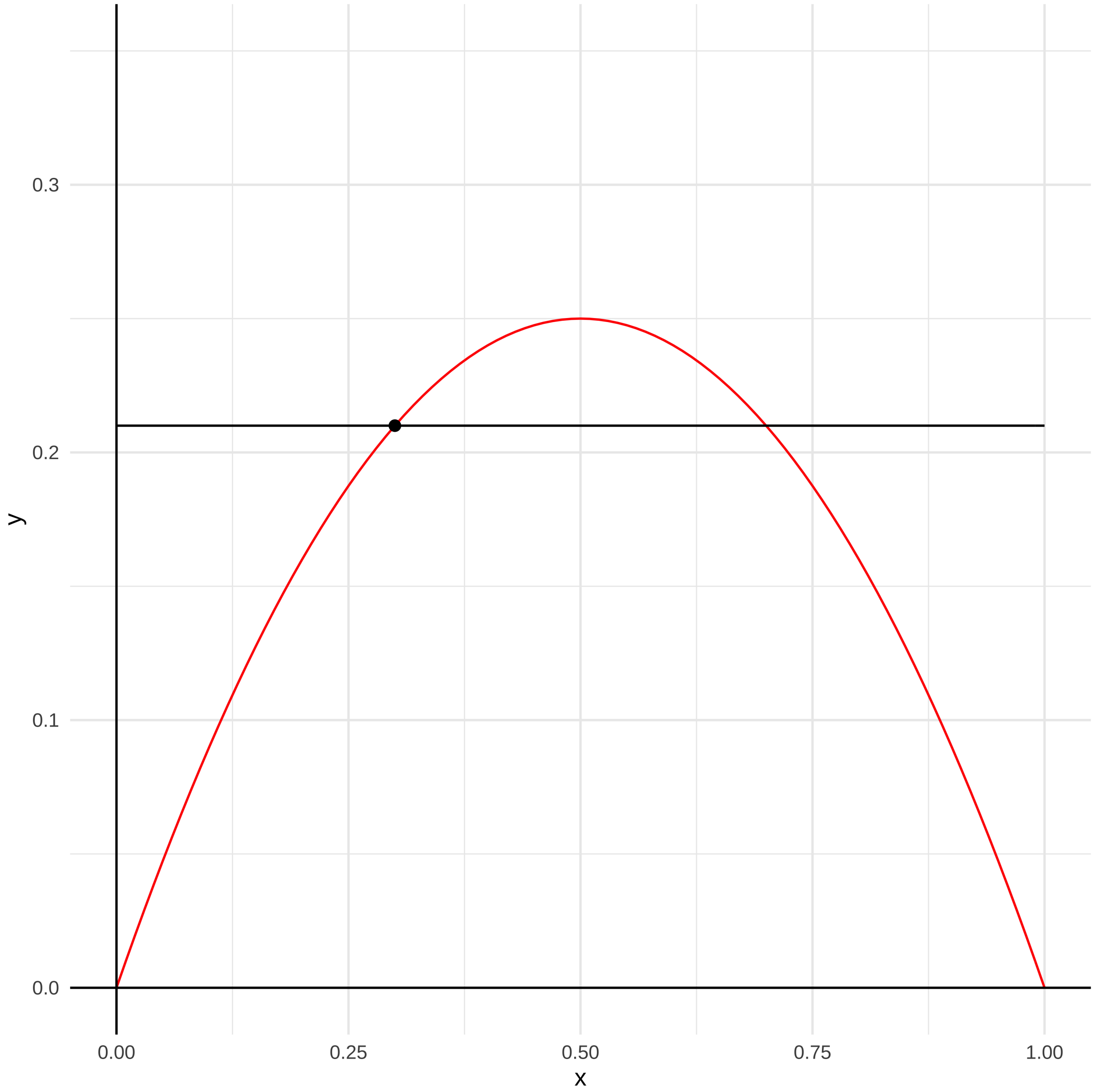
0.25
0.50
0.75
1.00
x



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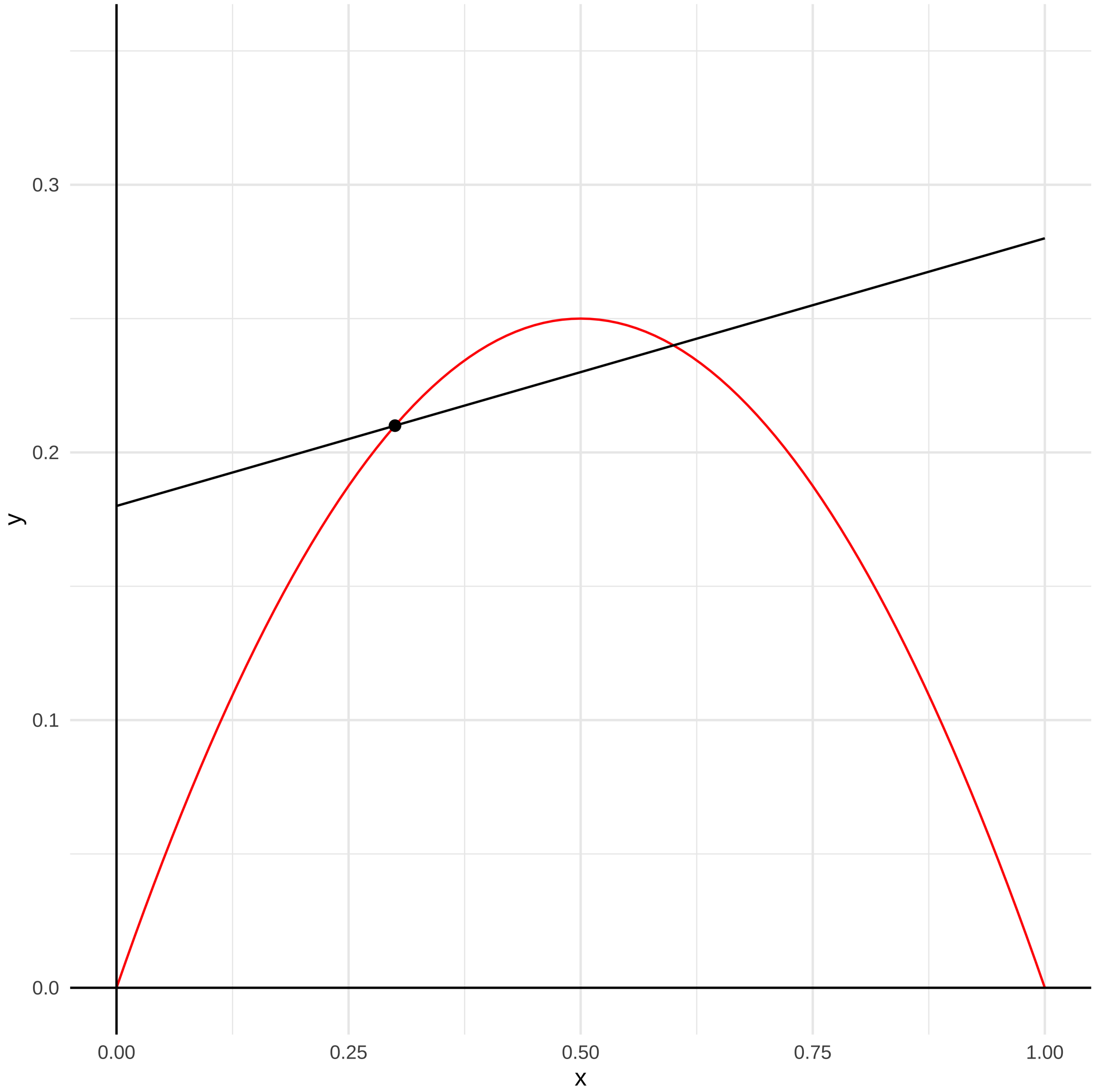
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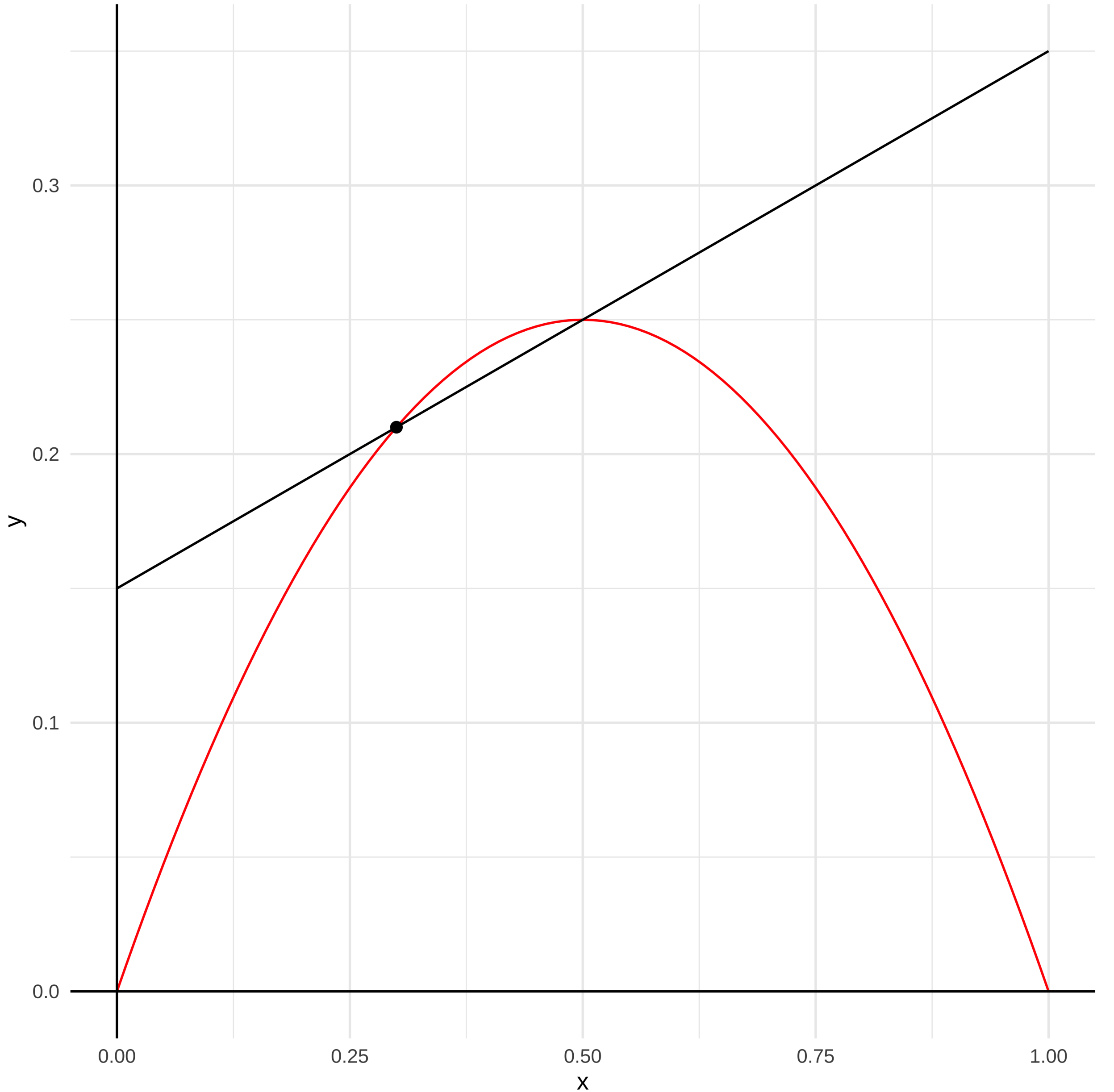
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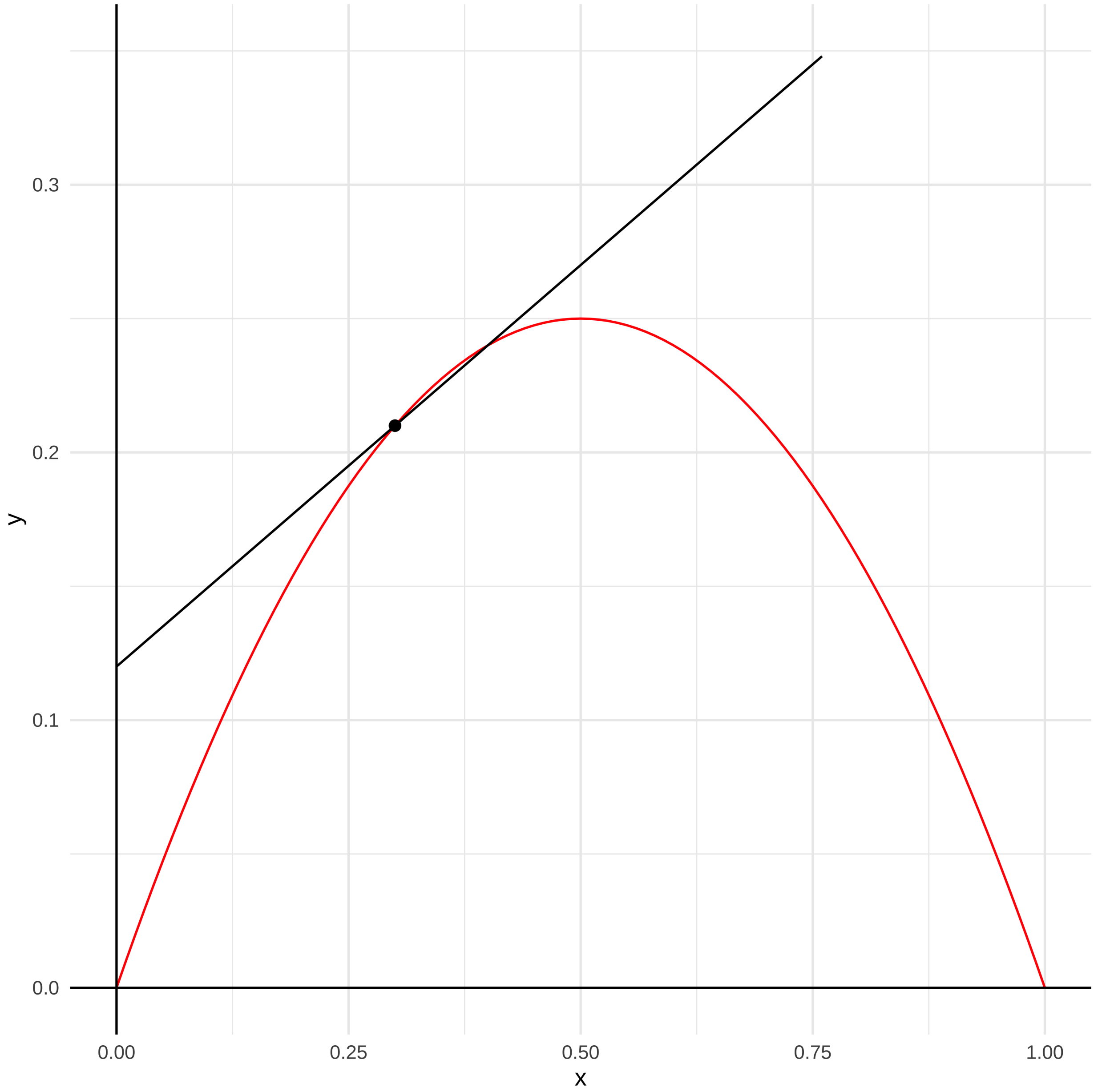

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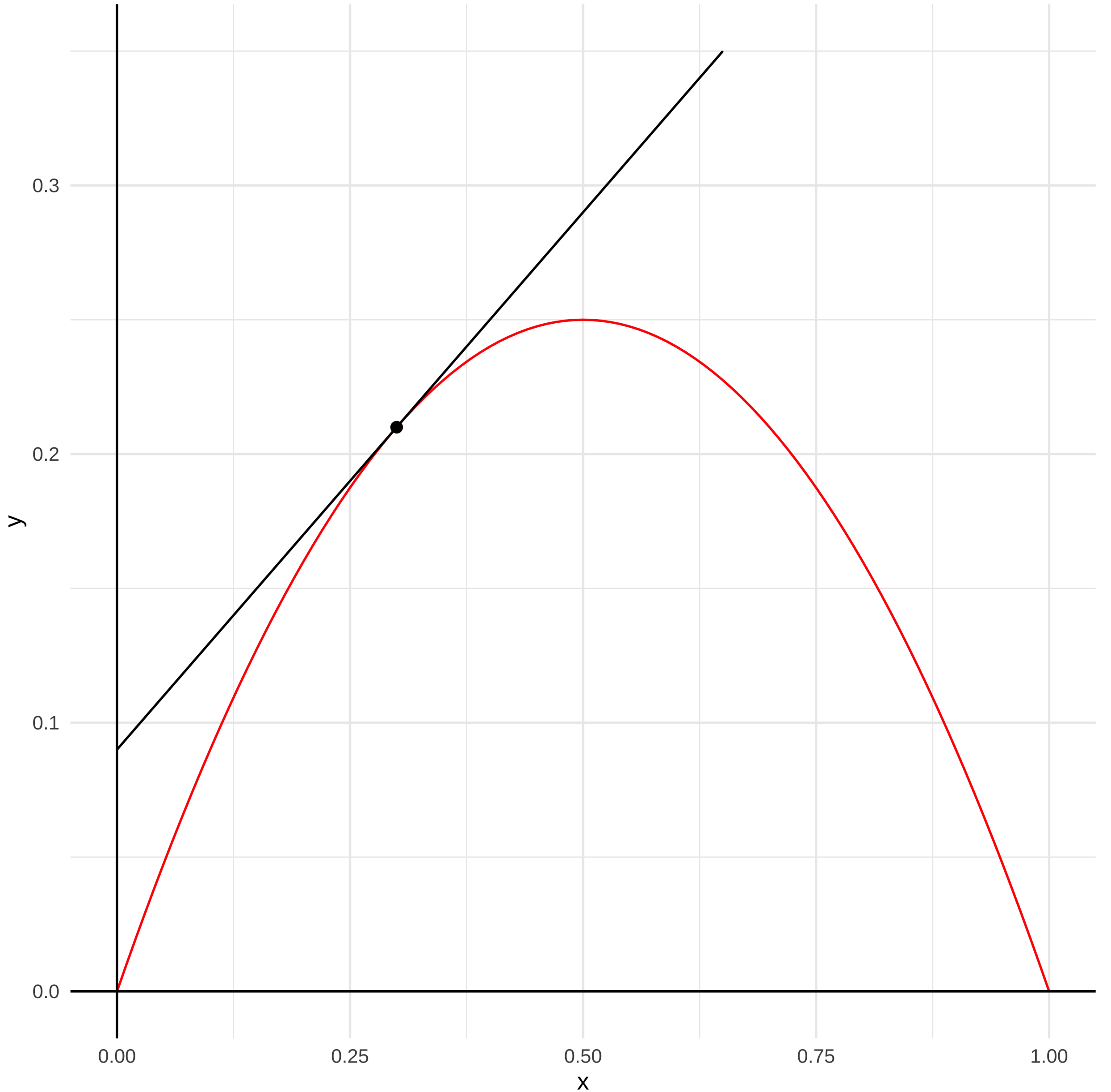

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Derivatives as Limits

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So, as you reduce the interval size to 0, this line converges in the limit to the line that lies tangent to the curve at that point. Recall that $f(x) = x - x^2$ and consider a very small interval $\epsilon \ldots$

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\end{aligned}$$

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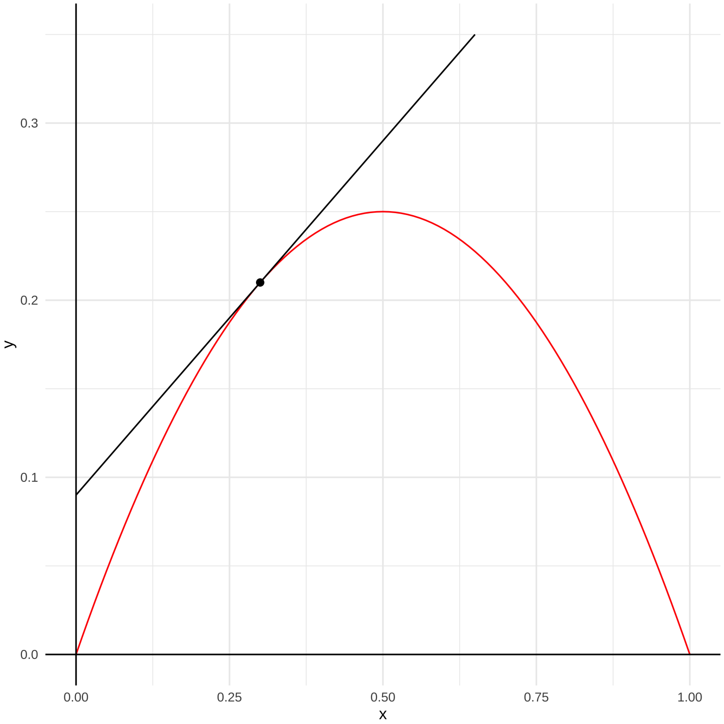
&= 1 - 2x
\end{aligned}$$

Derivative as Limits



Using this formula, the slope of the curve at $x = .3$ (the point from the previous examples) is exactly:

$$\begin{aligned} \text{slope} &= 1 - 2(.3) \\ &= 0.4 \end{aligned}$$

Or, if we want to find the point at which the slope is 0 (rate of change is 0):

$$\begin{aligned} 0 &= 1 - 2x \\ 2x &= 1 \\ x &= 0.5 \end{aligned}$$

Derivatives

Now, we can formally state that the derivative is equivalent to:

$$\frac{d[f(x)]}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{(x+\epsilon) - x}$$

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$$\frac{d[f(x)]}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{(x+\epsilon) - x}$$

Using this approach, we can find:

- A general equation for the slope at any point
- The exact value of the slope at a given point
- The point that has a given slope

Derivative notation

Different notation systems for expressing derivatives:

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First derivative: $\frac{d[f(x)]}{dx}$, $\frac{d}{dx}[f(x)]$, $f'(x)$, or $f^1(x)$

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We'll be using $f'(x)$ and $f''(x)$ but you are free to choose whatever makes sense to you!

Cautionary Notes on Derivatives

A few assumptions in using this approach to find the slope:

- The function is continuous (no gaps or jumps)
- The derivative exists (the limit of the slope is the same from the left as it is from the right), or *no sharp corners*.

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For nearly all political science applications, these are fine assumptions. But it is important to state them explicitly and be aware that they're there.

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Fortunately (for humans), it is not necessary to take limits with ϵ's to find derivatives every time. A few rules generate the derivatives of most functions.

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The simplest example: consider the line $f(x) = 3x$. What's its derivative?

Straightforward Derivatives

Fortunately (for humans), it is not necessary to take limits with ϵ's to find derivatives every time. A few rules generate the derivatives of most functions.

The first rule is the *power rule*.

If $f(x) = cx^n$, then its derivative is $f'(x) = ncx^{n-1}$.

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A special case of the power rule is that the *derivative of a constant is zero*.

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$$\begin{aligned} h'(x) &= \left(\frac{1}{2}\right) 7x^{\frac{1}{2}-1} \\ &= \frac{7}{2}x^{-\frac{1}{2}} \end{aligned}$$

Exercises

Find the following derivatives, and calculate the instantaneous slope of the curves at the point $x = 2$:

$f(x) = \frac{1}{4}x^4$

$g(x) = \frac{2}{x^3}$ [Hint: How else can we express fractions?]

$h(x) = 4x^{\frac{5}{2}}$

$j(x) = \sqrt[3]{x}$ [Hint: How else can we express roots?]

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$$\begin{aligned} f'(x) + g'(x) &= (4x^2)' + (5x^3)' \\ &= (2)4x^{2-1} + (3)5x^{3-1} \\ &= 8x + 15x^2 \end{aligned}$$

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Let $f(x)=x$ and $g(x)=x^2$. What is $(f(x)-g(x))'$?

$$\begin{aligned} f'(x)-g'(x) &= (x)' - (x^2)' \\ &= (1)x^{1-1} - (2)x^{2-1} \\ &= 1 - 2x \end{aligned}$$

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&= (5x^5)' - (10x^3)' + (6x^2)' - (3)' \\
&= 5 \times 5x^{5-1} - 3 \times 10x^{3-1} + 2 \times 6x^{2-1} - 0 \\
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What is the rate of change when x is equal to one?

$$\begin{aligned}
h'(1) &= 25x^4 - 30x^2 + 12x \\
h'(1) &= 25(1)^4 - 30(1)^2 + 12(1) \\
h'(1) &= 7
\end{aligned}$$

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This is **the product rule**.

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&= (2x)(x^3 - 4x) + (x^2 + 1)(3x^2 - 4) \\
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This is referred to as **the quotient rule**.

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\left(\frac{f(x)}{g(x)}\right)' &= \left(\frac{x^2+1}{x^3-4x}\right)' \\
&= \frac{(x^2+1)'(x^3-4x) - (x^2+1)(x^3-4x)'}{(x^3-4x)^2} \\
&= \frac{(2x)(x^3-4x) - (x^2+1)(3x^2-4)}{(x^3-4x)^2} \\
&= \frac{2x^4 - 8x^2 - (3x^4 - 4x^2 + 3x^2 - 4)}{(x^3-4x)^2} \\
&= \frac{-x^4 - 7x^2 + 4}{(x^3-4x)^2}
\end{aligned}$$

Exercises

Find the derivatives of the following expressions:

$$(3x^2 - 4x + 2)(x^3 - x^2 + x - 1)$$

$$\frac{4x+1}{3x^2-2}$$

Derivatives of Nested Functions

Let's say $h(x) = f(g(x))$.

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This looks messy, but is actually fairly straightforward and extremely useful as a way to find derivatives of complex functions by treating them as nested chains of functions.

Derivatives of Nested Functions

Let $h(x) = 6(3x^2 + 2)^4$. Observe that this can be thought of as two nested functions, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $h(x) = f(g(x))$.

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$$\begin{aligned} h(x)' = (f(g(x)))' &= (6(3x^2 + 2)^4)' \\ &= (4)6(3x^2 + 2)^{4-1}(3x^2 + 2)' \\ &= 24(3x^2 + 2)^3(6x) \\ &= 144x(3x^2 + 2)^3 \end{aligned}$$

Derivatives of Nested Functions

Let $k(x) = 3(6x^4)^2 + 2$. Observe that this can be thought of the same two functions nested in the reverse order, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $k(x) = g(f(x))$.

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Derivatives of Nested Functions

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$$
\begin{aligned}
k(x)' = (g(f(x)))' &= (3(6x^4)^2 + 2)' \\
&= (3(6x^4)^2)' + (2)' \\
&= (2)3(6x^4)^{2-1}(6x^4)' + 0 \\
&= (2)3(6x^4)^{2-1}(24x^{4-1}) \\
&= (2)3(6x^4)(24x^3) \\
&= 864x^7
\end{aligned}
$$

Exercises

Express the functions below as the nested result of two simpler functions, and use the chain rule to find the derivative:

$(3x - 1)^4$

$2(x^4 + x^3) + 7$

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$$\begin{aligned}(log_e(x))' = (ln(x))' &= \frac{1}{ln(e)x} \\ &= \frac{1}{x}\end{aligned}$$

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Let $g(x) = ln(3x^2 + 4)$. What is $g'(x)$?

(using the chain rule):

$$\begin{aligned} g'(x) = (ln(3x^2 + 4))' &= \frac{1}{3x^2 + 4} \times (3x^2 + 4)' \\ &= \frac{6x}{3x^2 + 4} \end{aligned}$$

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$$\begin{aligned}(e^x)' &= ln(e)e^x \\ &= 1 \times e^x \\ &= e^x\end{aligned}$$

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Consider the limit: $\lim_{x \to 2} \frac{x^2+x-6}{x^2-4}$

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If both the numerator and the denominator are 0, ∞, or $-\infty$, we cannot evaluate the limit.

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If both the numerator and the denominator are 0, ∞, or $-\infty$, we cannot evaluate the limit.

However, L'Hôpital's Rule says: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Then,

$$\begin{aligned} \lim_{x \to 2} \frac{x^2+x-6}{x^2-4} &= \lim_{x \to 2} \frac{(x^2+x-6)'}{(x^2-4)'} \\ &= \lim_{x \to 2} \frac{2x+1}{2x} \\ &= 1.25 \end{aligned}$$

End Day 3