

Math Camp Lesson 3 (Day 1)

Calculus

UW–Madison Political Science

August 19, 2020

Overview

Calculus evaluates the behavior of functions:

- Limits
- Rate of change
- Change in the rate of change
- Area of the region they defined on

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Concepts from calculus underlie a wide variety of mathematics, particularly in the applied math that we use in political science

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Solving for the choice that maximizes a decision maker's utility.

Agenda

Day 1

- Limits
- Derivatives

Day 2

- More Derivatives
- Integrals
- Applications

Limits

Limits

The first important idea for calculus are limits.

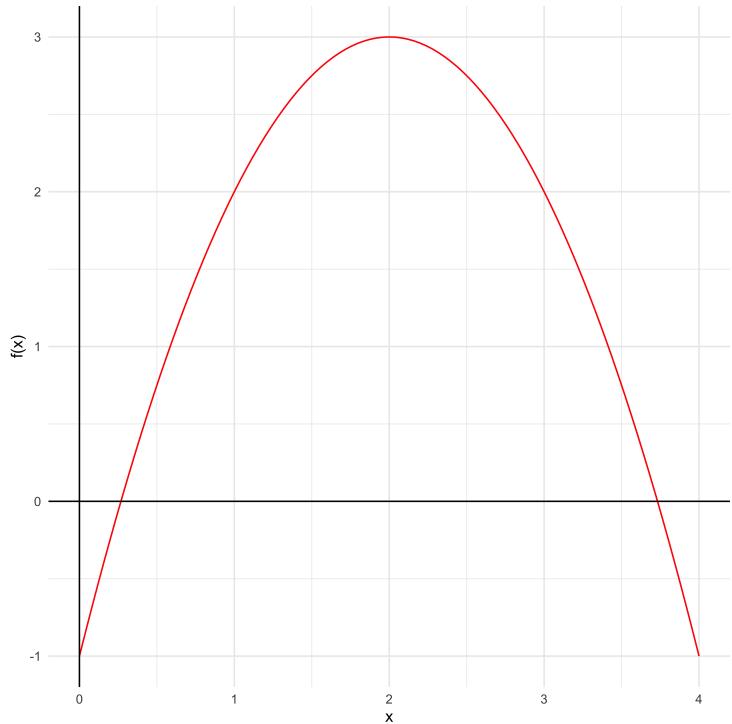
Limits

The first important idea for calculus are limits.

The limit of a function characterizes its behavior given a certain input, or as an input value changes.

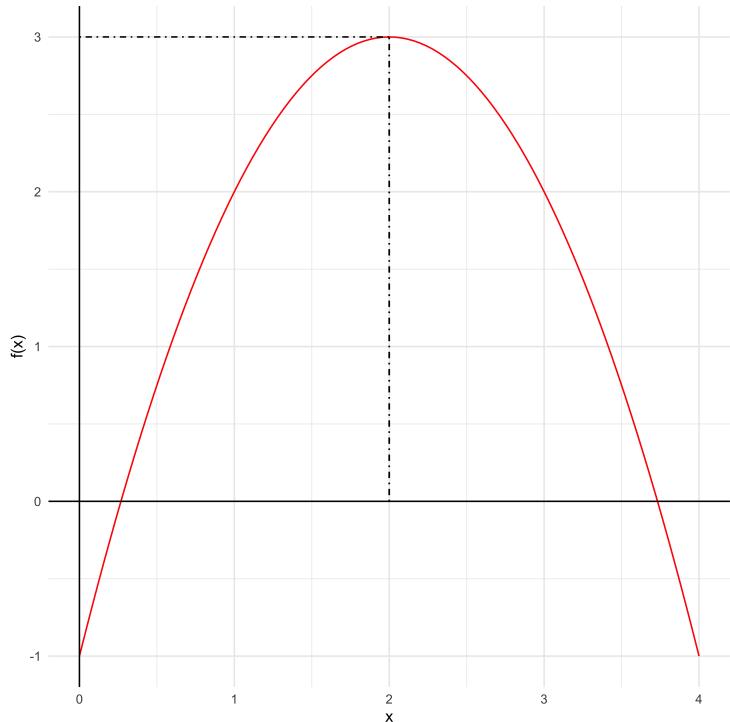
Limits: Example 1

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Let's consider the simple function,
 $f(x) = y = 3 - (x - 2)^2$, plotted to
the left. What is the limit of $f(x)$ as x
approaches 2?

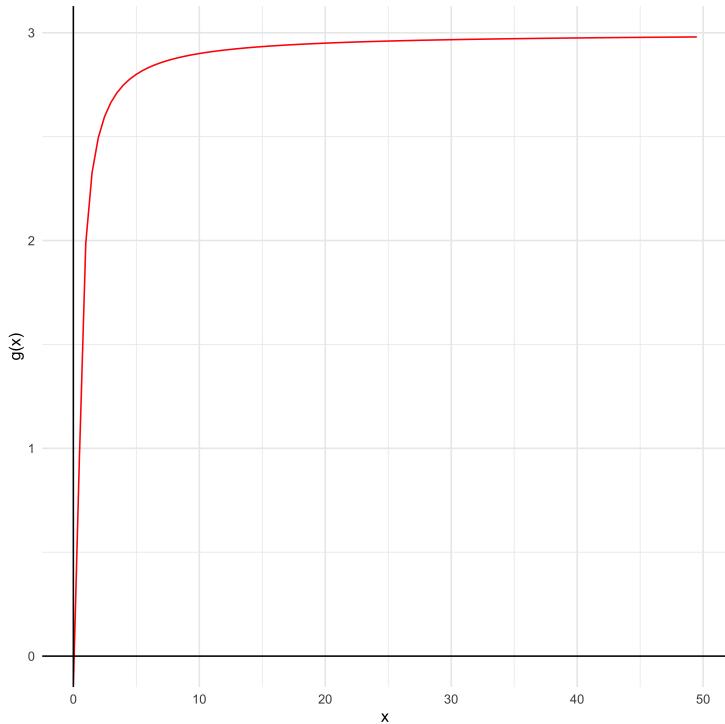
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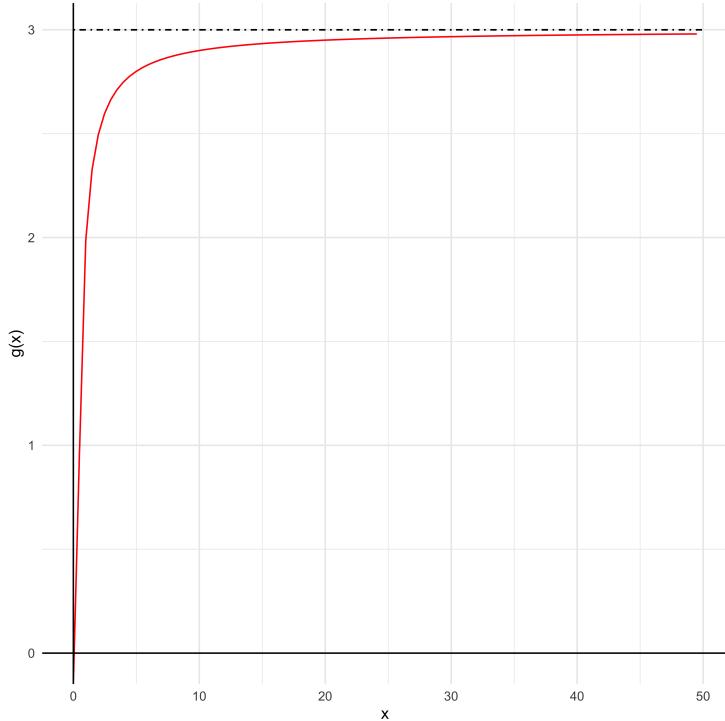
As x approaches 2, $f(x)$ or y
approaches $f(2) = 3$.

Limits: Example 2



Let's consider a less simple function,
 $g(x) = y = 3 - \frac{1}{x}$, plotted to the left.
What is the limit of $g(x)$ as x
approaches ∞ ?

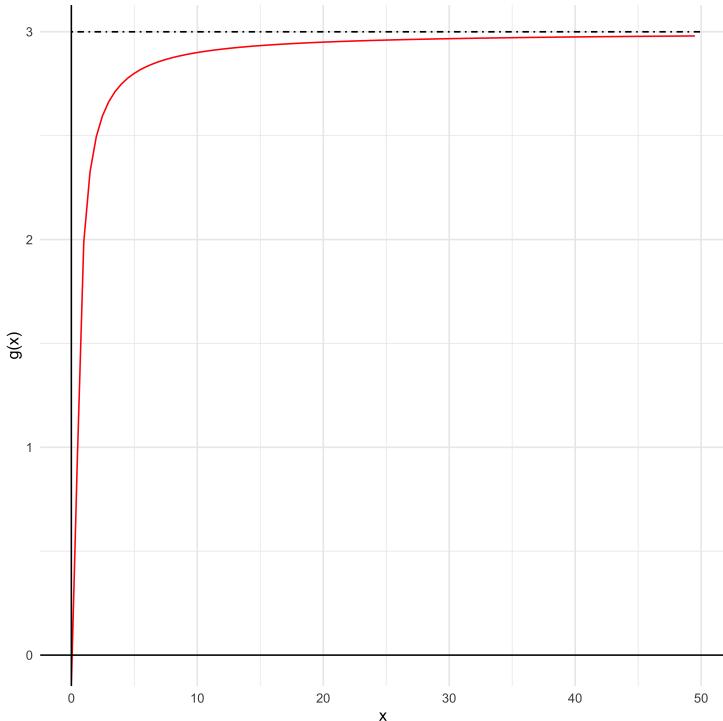
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Limits: Example 2s



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Why?

As x gets larger, $\frac{1}{x}$ gets smaller and smaller.

$$\left(\frac{1}{2} > \frac{1}{20} > \frac{1}{200} \dots \right)$$

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Many times, you will see this expression written as $\lim_{x \rightarrow c^-} f(x) = L$ or $\lim_{x \rightarrow c^+} f(x) = L$.

A negative sign ($-$) implies "As x approaches c from the left"

A positive sign ($+$) implies "As x approaches c from the right"

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For functions that are well-behaved, the limit as x approaches a finite point is generally the value of the function at that point (if it exists).

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$$\begin{aligned}\lim_{x \rightarrow 2} x^2 - 3x + 1 &= \lim_{x \rightarrow 2} x^2 - 3 \lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 1 \\&= 2^2 - 3(2) + 1 \\&= -1\end{aligned}$$

Finding Limits: Example 2

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$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{4x^4 + 7x^2 + 8}{3x^4} &= \lim_{x \rightarrow \infty} \frac{4x^4}{3x^4} + \lim_{x \rightarrow \infty} \frac{7x^2}{3x^4} + \lim_{x \rightarrow \infty} \frac{8}{3x^4} \\&= \lim_{x \rightarrow \infty} \frac{4}{3} + \lim_{x \rightarrow \infty} \frac{7}{3x^2} + \lim_{x \rightarrow \infty} \frac{8}{3x^4} \\&= \frac{4}{3} + 0 + 0 \\&= \frac{4}{3}\end{aligned}$$

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Why does $\lim_{x \rightarrow \infty} \frac{7}{3x^2} = 0$? As $x \rightarrow \infty$, $3x^2 \rightarrow \infty$, and $\frac{7}{\infty} \rightarrow 0$.

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$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{4x^4 + 7x^2 + 8}{3x^4} \\ &= \frac{4}{3} + \frac{7}{3} \lim_{x \rightarrow 0} \frac{1}{x^2} + \frac{8}{3} \lim_{x \rightarrow 0} \frac{1}{x^4} \\ &= \infty \end{aligned}$$

As x approaches 0, the function retains some positive value in the numerator while the denominator *positively* approaches 0. This means that you are dividing by a smaller and smaller fraction, which means the entire term is getting larger and approaches ∞ .

Exercises

Find the following limits:

$$\lim_{x \rightarrow 4} x^2 - 6x + 4$$

$$\lim_{x \rightarrow 4} \frac{x^2}{3x-2}$$

$$\lim_{x \rightarrow \infty} \frac{3x-4}{x+3}$$

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Derivatives

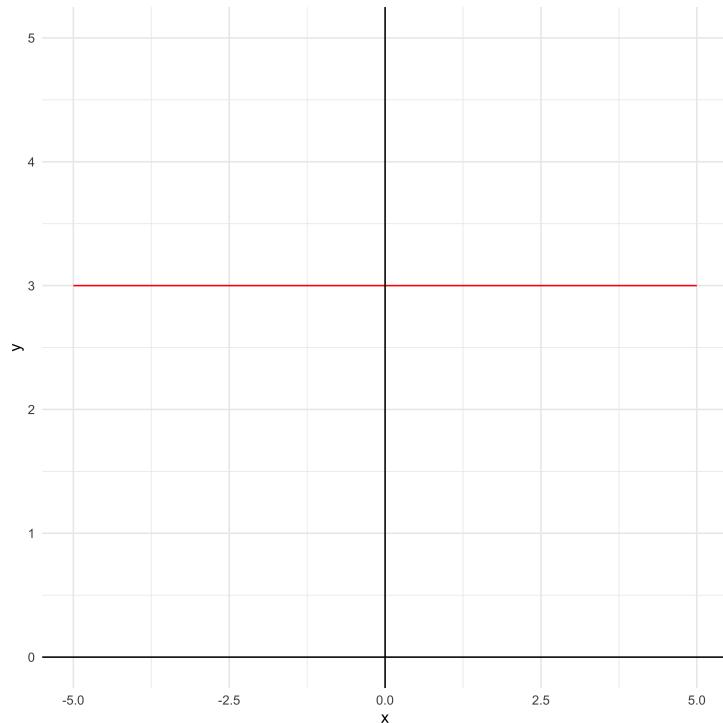
The derivative of a function is its rate of change in the output as the value of its input changes.

It is the instantaneous slope of the line at any given point.

The slope of a function is how much the output changes as a result of changes in the input.

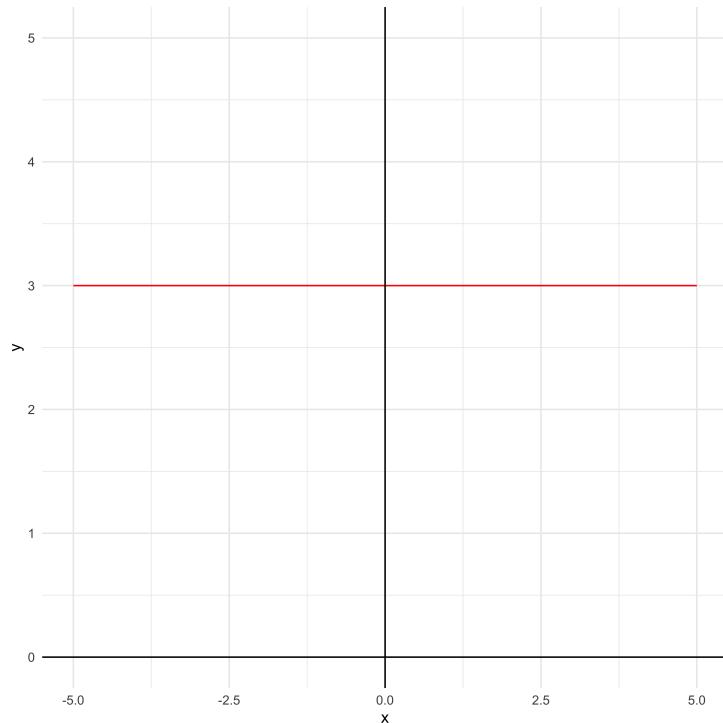
Using Δ to signify 'change', this is $\frac{\Delta f(x)}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ or "rise-over-run".

Slope



Let's consider the function, $y = 3$, plotted to the left. What is its "slope"?

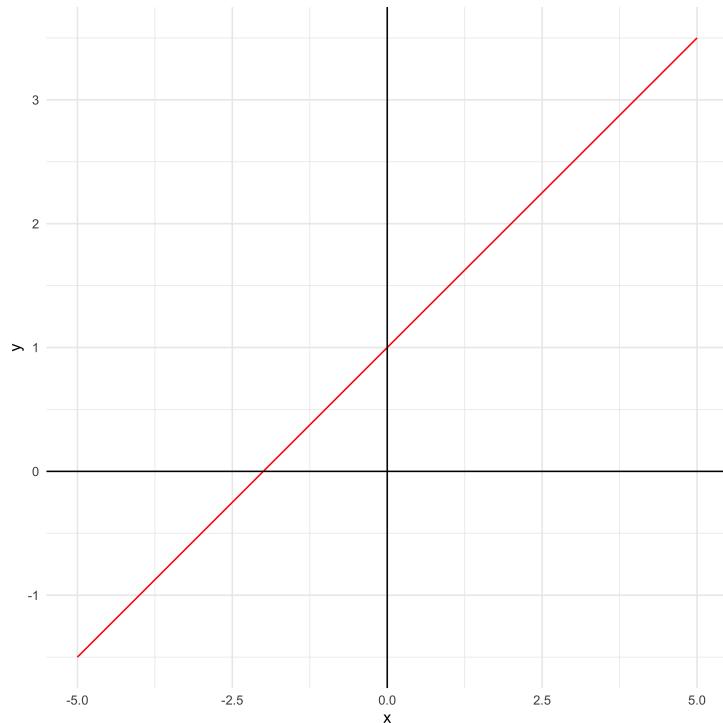
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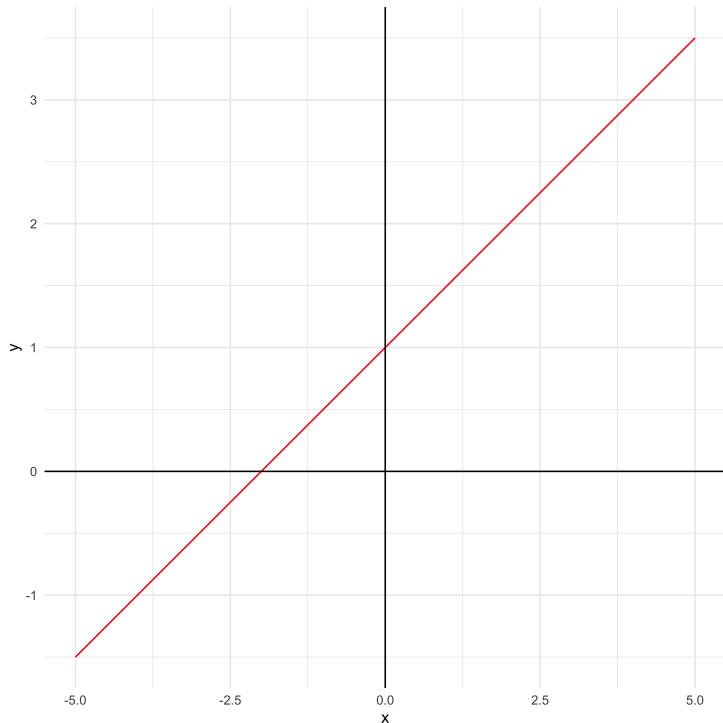
Its slope or $\frac{\Delta f(x)}{\Delta x} = 0$ because there is no "rise".

Slope



Let's consider a less simple function,
 $y = \frac{1}{2}x + 1$, plotted to the left. What is
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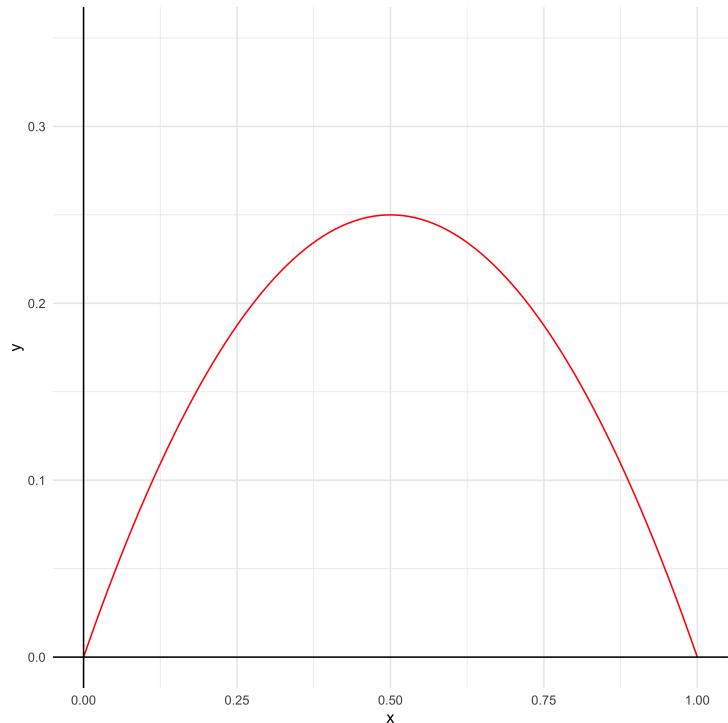


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 $y = \frac{1}{2}x + 1$, plotted to the left. What is
its "slope"?

Its slope or $\frac{\Delta f(x)}{\Delta x} = \frac{1}{2}$.

[Recall: $y = mx + b$ from Day 1]

Slope



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What is its "slope", or $\frac{\Delta f(x)}{\Delta x}$?

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How do we even calculate this?

Derivatives as Limits

We can approximate the slope at a certain location by picking a point nearby on the line and finding the slope of the straight line connecting these two points.

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Let's consider a few examples.

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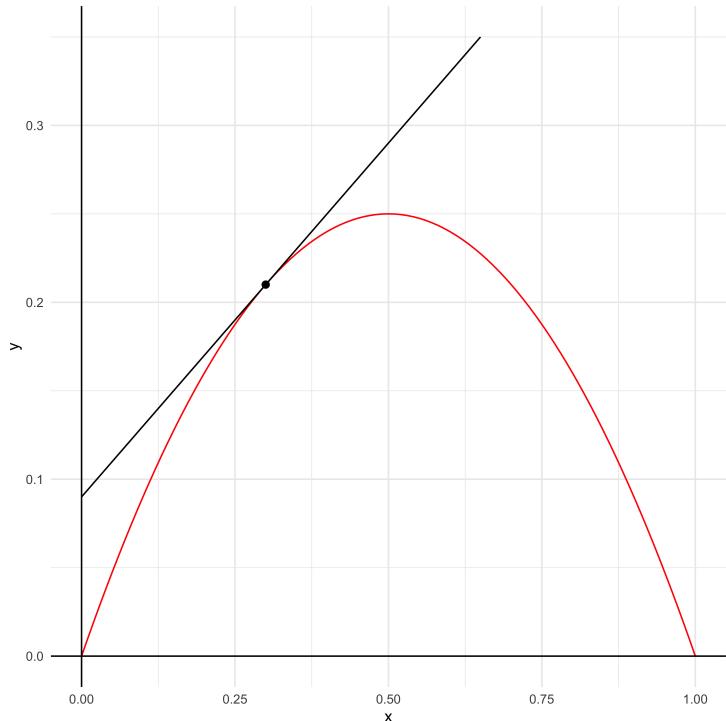
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Derivative as Limits



Using this formula, the slope of the curve at $x = .3$ (the point from the previous examples) is exactly:

$$\begin{aligned}\text{slope} &= 1 - 2(.3) \\ &= 0.4\end{aligned}$$

Or, if we want to find the point at which the slope is 0 (rate of change is 0):

$$\begin{aligned}0 &= 1 - 2x \\ 2x &= 1 \\ x &= 0.5\end{aligned}$$

Derivatives

Now, we can formally state that the derivative is equivalent to:

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Using this approach, we can find:

- A general equation for the slope at any point
- The exact value of the slope at a given point
- The point that has a given slope

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We'll be using $f'(x)$ and $f''(x)$ but you are free to choose whatever makes sense to you!

Cautionary Notes on Derivatives

A few assumptions in using this approach to find the slope:

- The function is continuous (no gaps or jumps)
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For nearly all political science applications, these are fine assumptions. But it is important to state them explicitly and be aware that they're there.

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A special case of the power rule is that the *derivative of a constant is zero*.

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$$\begin{aligned}h'(x) &= \left(\frac{1}{2}\right) 7x^{\frac{1}{2}-1} \\&= \frac{7}{2}x^{-\frac{1}{2}}\end{aligned}$$

Exercises

Find the following derivatives, and calculate the instantaneous slope of the curves at the point $x = 2$:

$$f(x) = \frac{1}{4}x^4$$

$$g(x) = \frac{2}{x^3}$$
 [Hint: How else can we express fractions?]

$$h(x) = 4x^{\frac{5}{2}}$$

$$j(x) = \sqrt[3]{x}$$
 [Hint: How else can we express roots?]

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For example, let's consider $f(x) = 4x^2$ and $g(x) = 5x^3$. What is $(f(x) + g(x))'$?

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For example, let's consider $f(x) = 4x^2$ and $g(x) = 5x^3$. What is $(f(x) + g(x))'$?

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$$\begin{aligned}f'(x) - g'(x) &= (x)' - (x^2)' \\&= (1)x^{1-1} - (2)x^{2-1} \\&= 1 - 2x\end{aligned}$$

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$$\begin{aligned} h'(1) &= 25x^4 - 30x^2 + 12x \\ h'(1) &= 25(1)^4 - 30(1)^2 + 12(1) \\ h'(1) &= 7 \end{aligned}$$

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This is referred to as **the quotient rule**.

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$$\begin{aligned}\left(\frac{f(x)}{g(x)}\right)' &= \left(\frac{x^2 + 1}{x^3 - 4x}\right)' \\ &= \frac{(x^2 + 1)'(x^3 - 4x) - (x^2 + 1)(x^3 - 4x)'}{(x^3 - 4x)^2} \\ &= \frac{(2x)(x^3 - 4x) - (x^2 + 1)(3x^2 - 4)}{(x^3 - 4x)^2} \\ &= \frac{2x^4 - 8x^2 - (3x^4 - 4x^2 + 3x^2 - 4)}{(x^3 - 4x)^2} \\ &= \frac{-x^4 - 7x^2 + 4}{(x^3 - 4x)^2}\end{aligned}$$

Exercises

Find the derivatives of the following expressions:

$$(3x^2 - 4x + 2)(x^3 - x^2 + x - 1)$$

$$\frac{4x+1}{3x^2-2}$$

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This looks messy, but is actually fairly straightforward and extremely useful as a way to find derivatives of complex functions by treating them as nested chains of functions.

Derivatives of Nested Functions

Let $h(x) = 6(3x^2 + 2)^4$. Observe that this can be thought of as two nested functions, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $h(x) = f(g(x))$.

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What is $h'(x)$?

$$\begin{aligned} h(x)' &= (f(g(x)))' = (6(3x^2 + 2)^4)' \\ &= (4)6(3x^2 + 2)^{4-1}(3x^2 + 2)' \\ &= 24(3x^2 + 2)^3(6x) \\ &= 144x(3x^2 + 2)^3 \end{aligned}$$

Derivatives of Nested Functions

Let $k(x) = 3(6x^4)^2 + 2$. Observe that this can be thought of the same two functions nested in the reverse order, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $k(x) = g(f(x))$.

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What is $k'(x)$?

$$\begin{aligned}k(x)' &= (g(f(x)))' = (3(6x^4)^2 + 2)' \\&= (3(6x^4)^2)' + (2)' \\&= (2)3(6x^4)^{2-1}(6x^4)' + 0 \\&= (2)3(6x^4)^{2-1}(24x^{4-1}) \\&= (2)3(6x^4)(24x^3) \\&= 864x^7\end{aligned}$$

Exercises

Express the functions below as the nested result of two simpler functions, and use the chain rule to find the derivative:

$$(3x - 1)^4$$

$$2(x^4 + x^3) + 7$$

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$$\begin{aligned} (\log_e(x))' &= (\ln(x))' = \frac{1}{\ln(e)x} \\ &= \frac{1}{x} \end{aligned}$$

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$$\begin{aligned} g'(x) &= (\ln(3x^2 + 4))' = \frac{1}{3x^2 + 4} \times (3x^2 + 4)' \\ &= \frac{6x}{3x^2 + 4} \end{aligned}$$

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Consider the limit: $\lim_{x \rightarrow 2} \frac{x^2+x-6}{x^2-4}$.

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If both the numerator and the denominator are $0, \infty, -\infty$, we cannot evaluate the limit.

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If both the numerator and the denominator are $0, \infty, -\infty$, we cannot evaluate the limit.

However, **L'Hôpital's Rule** says:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

Then,

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} &= \lim_{x \rightarrow 2} \frac{(x^2 + x - 6)'}{(x^2 - 4)'} \\ &= \lim_{x \rightarrow 2} \frac{2x + 1}{2x} \\ &= 1.25\end{aligned}$$

End Day 3