

# Math Camp Lesson 3 (Day 1)

## Calculus

UW–Madison Political Science

August 19, 2020

# Overview

Calculus evaluates the behavior of functions:

- Limits
- Rate of change
- Change in the rate of change
- Area of the region they defined on

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Calculus in political science:

- Finding the fitline with the minimal distance between predicted and observed data
- Calculating the probability density in regions of continuous distributions
- Solving for the choice that maximizes a decision maker's utility

# Agenda

Day 1

- Limits
- Derivatives

Day 2

- More Derivatives
- Integrals
- Applications

# Limits

The first important idea for calculus are limits.

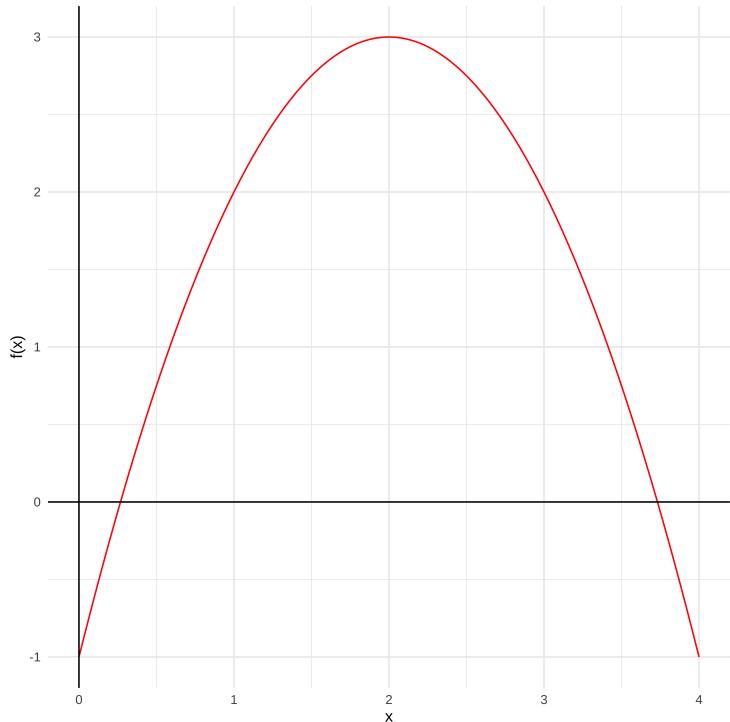
# Limits

The first important idea for calculus are limits.

The limit of a function characterizes its behavior given a certain input, or as an input value changes.

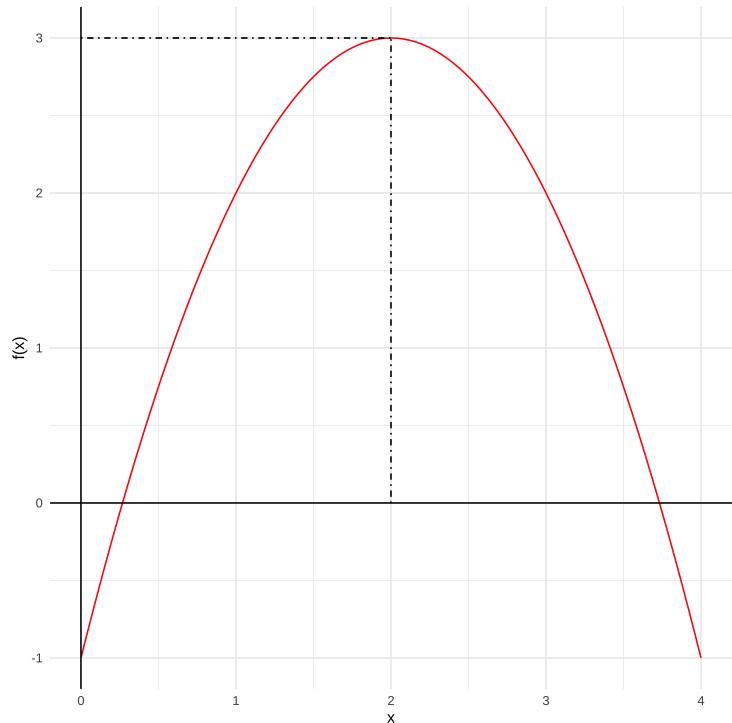
# Limits: Example 1

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Let's consider the simple function,  
 $f(x) = -x^2 + 4x$ , plotted to the  
left. What is the limit of  $f(x)$  as  
*approaches 2?*

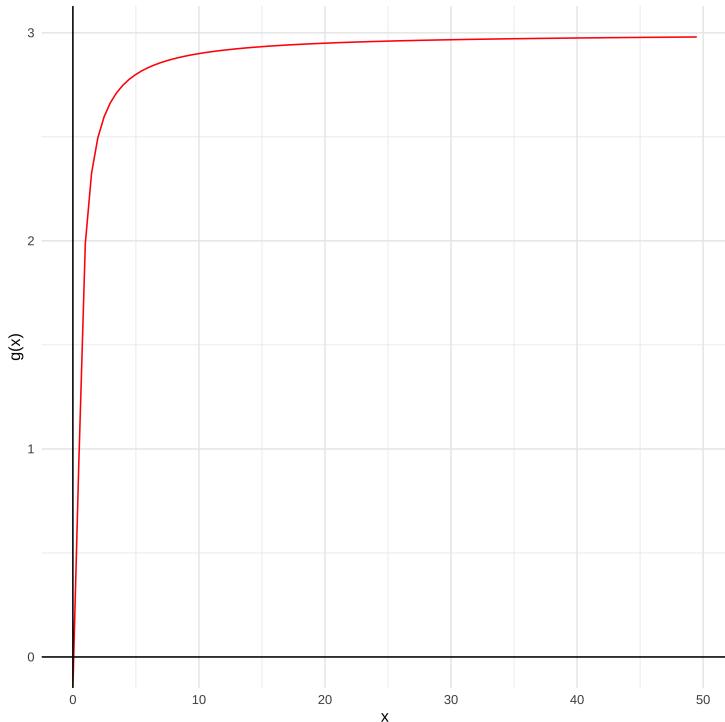
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As     approaches 2,     or     approaches

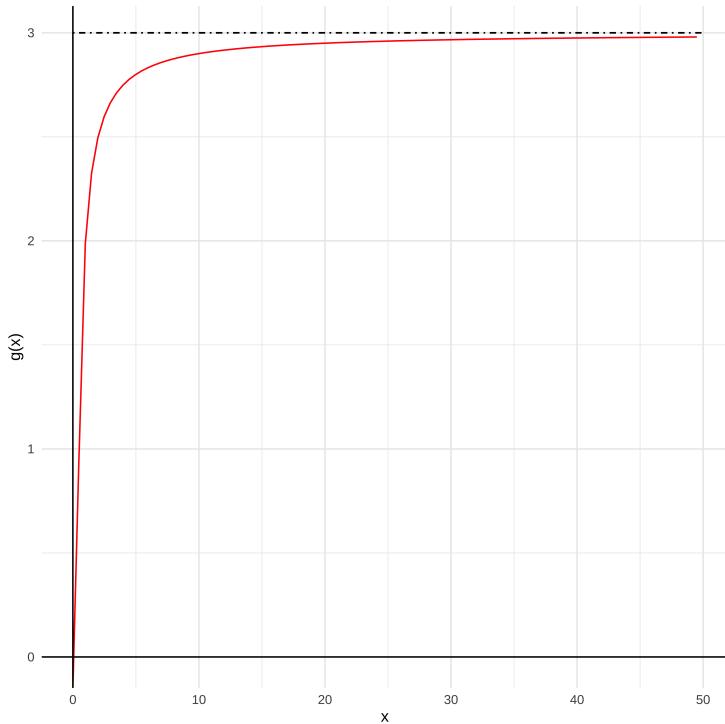
# Limits: Example 2



Let's consider a less simple function,  
—, plotted to the left.

What is the limit of  $g(x)$  as  $x$  approaches  $-\infty$ ?

# Limits: Example 2

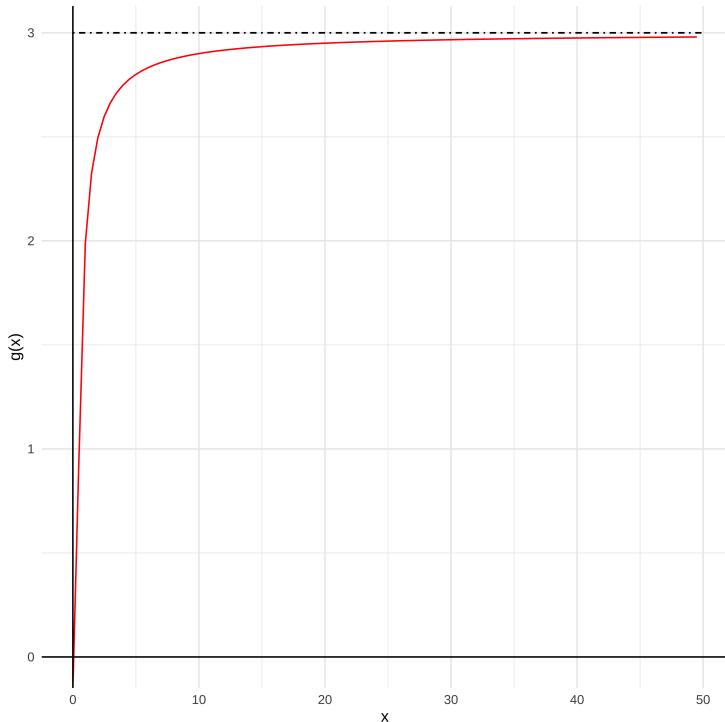


Let's consider a less simple function,  
—, plotted to the left.

What is the limit of  $\quad$  as  $\quad$  approaches  
?

As  $\quad$  approaches  $\quad$ ,  $\quad$  approaches 3.  
How do we know?

# Limits: Example 2



Let's consider a less simple function,  
 $-$ , plotted to the left.

What is the limit of  $-$  as  $x$  approaches ?

As  $x$  approaches ,  $-$  approaches 3.  
Why?

As  $x$  gets larger,  $-$  gets smaller and smaller.

— — —

# Limits, Formally Written

Formally, limits are expressed as:

This expression should be read as: "As  $x$  approaches  $a$ , the limit of  $f(x)$  is  $L$ ."

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Many times, you will see this expression written as

or

A negative sign  $\lim_{x \rightarrow a^-} f(x) = L$  implies "As  $x$  approaches  $a$  *from the left*"

A positive sign  $\lim_{x \rightarrow a^+} f(x) = L$  implies "As  $x$  approaches  $a$  *from the right*"

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If you can, evaluate the function at the limit.

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Are these components in the numerator or denominator of a fraction?

If you can, evaluate the function at the limit.

For functions that are well-behaved, the limit as  $x$  approaches a finite point is generally the value of the function at that point (if it exists).

# Finding Limits: Example 1

Let's consider

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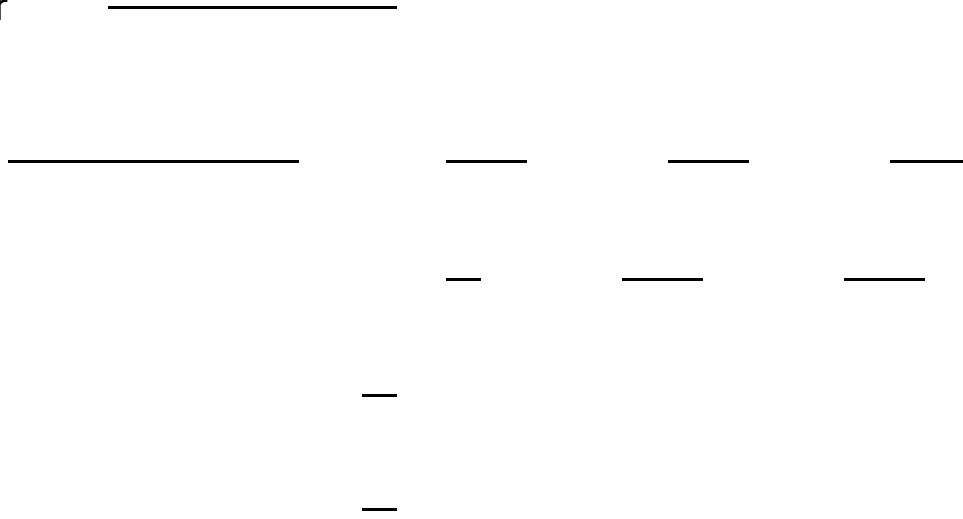
Let's consider

# Finding Limits: Example 2

Now, let's consider \_\_\_\_\_

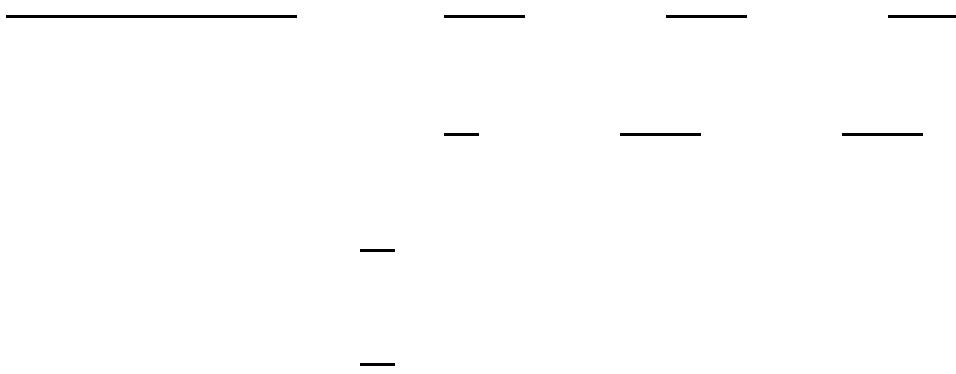
# Finding Limits: Example 2

Now, let's consider



# Finding Limits: Example 2

Now, let's consider  $\frac{\sin x}{x}$



Why does  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$  exist? As  $x \rightarrow 0$ ,  $\sin x \rightarrow 0$ , and  $\frac{\sin x}{x} \rightarrow 1$ .

# Finding Limits: Example 3

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Let's consider \_\_\_\_\_

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\_\_\_\_\_

— — —

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# Finding Limits: Example 3

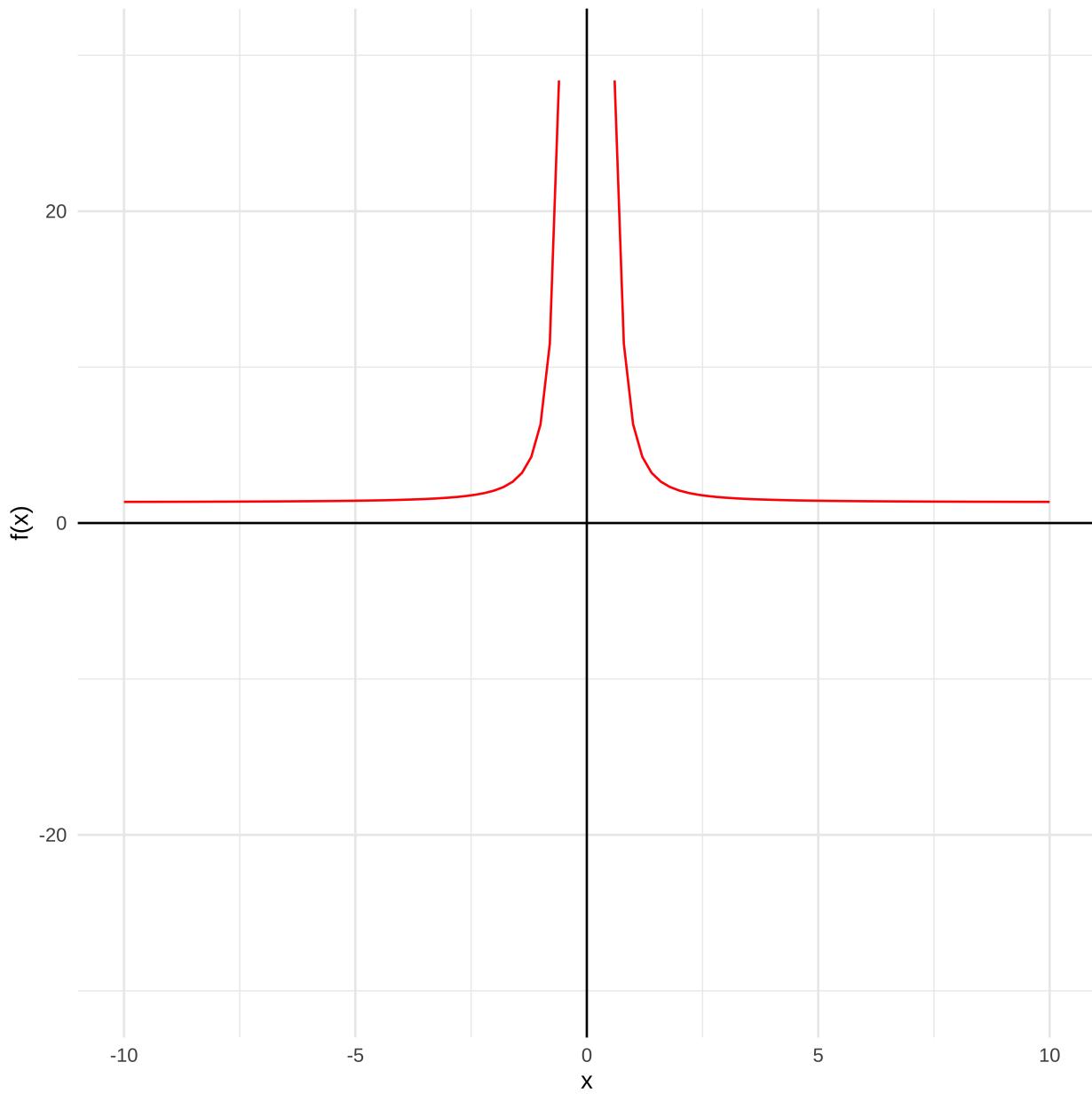
The limit is .

# Finding Limits: Example 3

The limit is .

$$\frac{\overline{\quad}}{\overline{- - - - -}}$$

As  $x$  approaches  $\infty$ , the function retains some positive value in the numerator while the denominator *positively* approaches 0. This means that you are dividing by a smaller and smaller fraction, which means the entire term is getting larger and approaches  $\infty$ .



# Exercises

Find the following limits:

—

—

# Derivatives

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The derivative of a function is its rate of change in the output as the value of its input changes.

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It is the instantaneous slope of the line at any given point.

# Derivatives

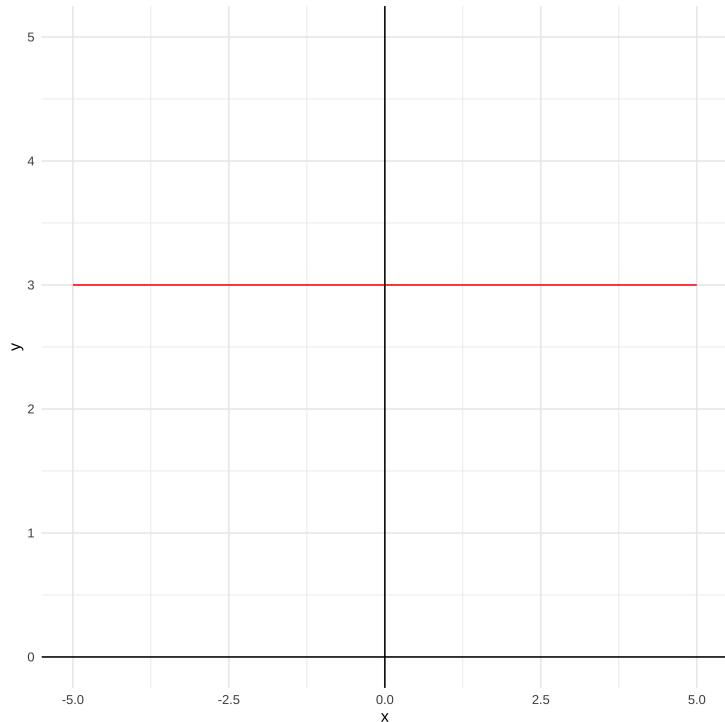
The derivative of a function is its rate of change in the output as the value of its input changes.

It is the instantaneous slope of the line at any given point.

The slope of a function is how much the output changes as a result of changes in the input.

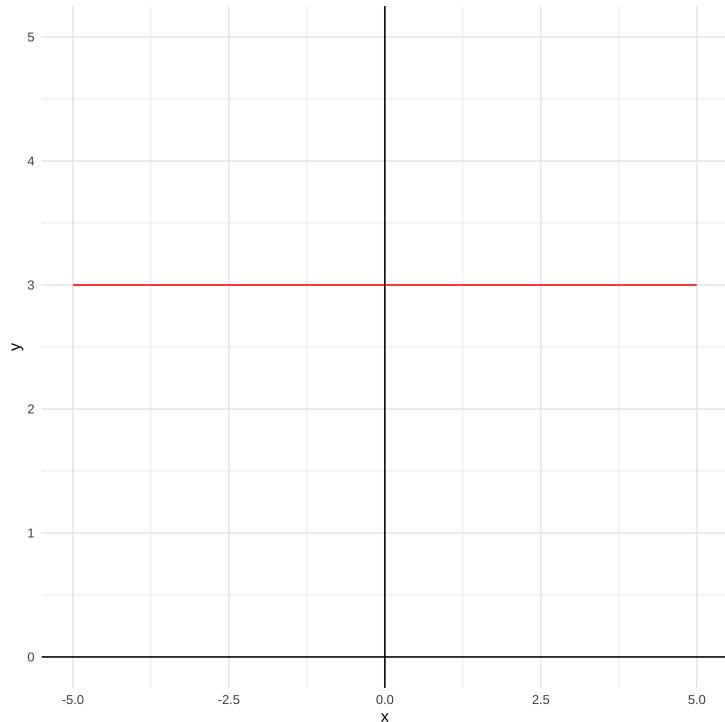
Using  $\frac{\Delta y}{\Delta x}$  to signify 'change', this is  $\frac{\text{rise}}{\text{run}}$  or "rise-over-run".

# Slope



Let's consider the function,  $\underline{\hspace{2cm}}$ , plotted to the left. What is its "slope"?

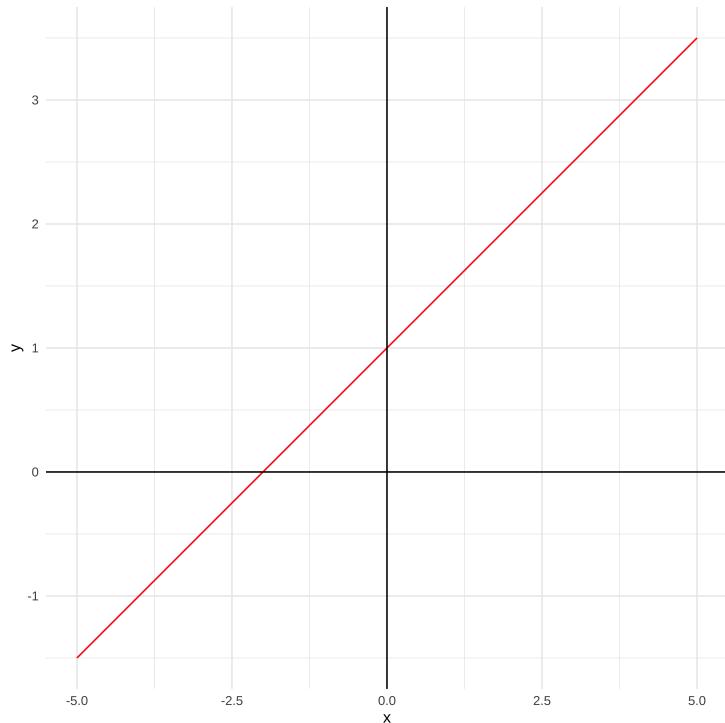
# Slope



Let's consider the function, \_\_\_\_\_, plotted to the left. What is its "slope"?

Its slope or \_\_\_\_\_ because there is no "rise".

# Slope



Let's consider a less simple function,  
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# Slope

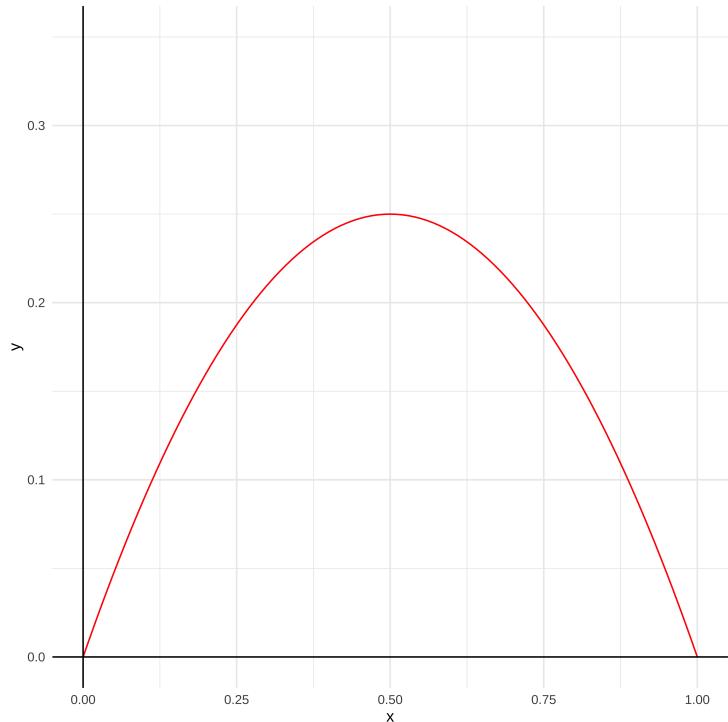


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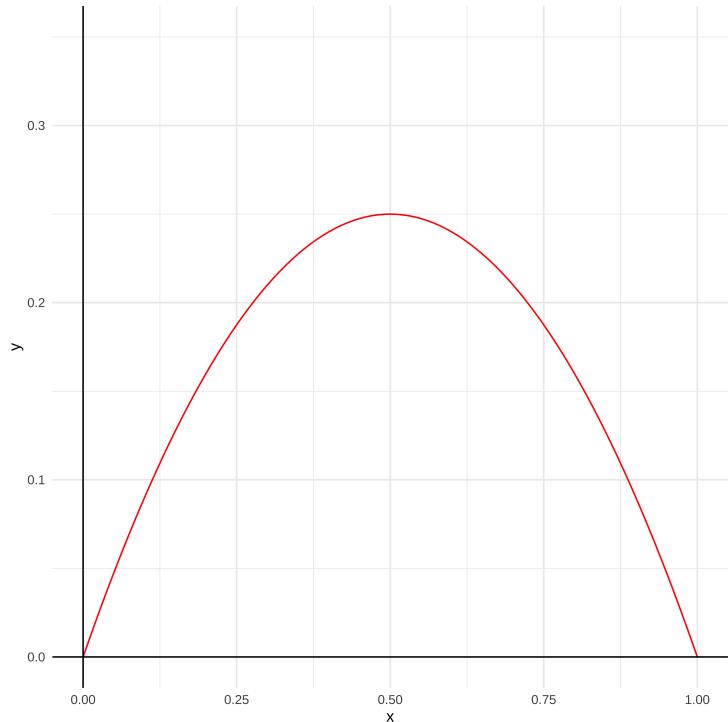
[Recall: from Day 1]

# Slope



Let's consider an even more complicated function, \_\_\_\_\_, plotted to the left.  
What is its "slope", or \_\_\_\_\_?

# Slope



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What is its "slope", or \_\_\_\_\_?

How do we even calculate this?

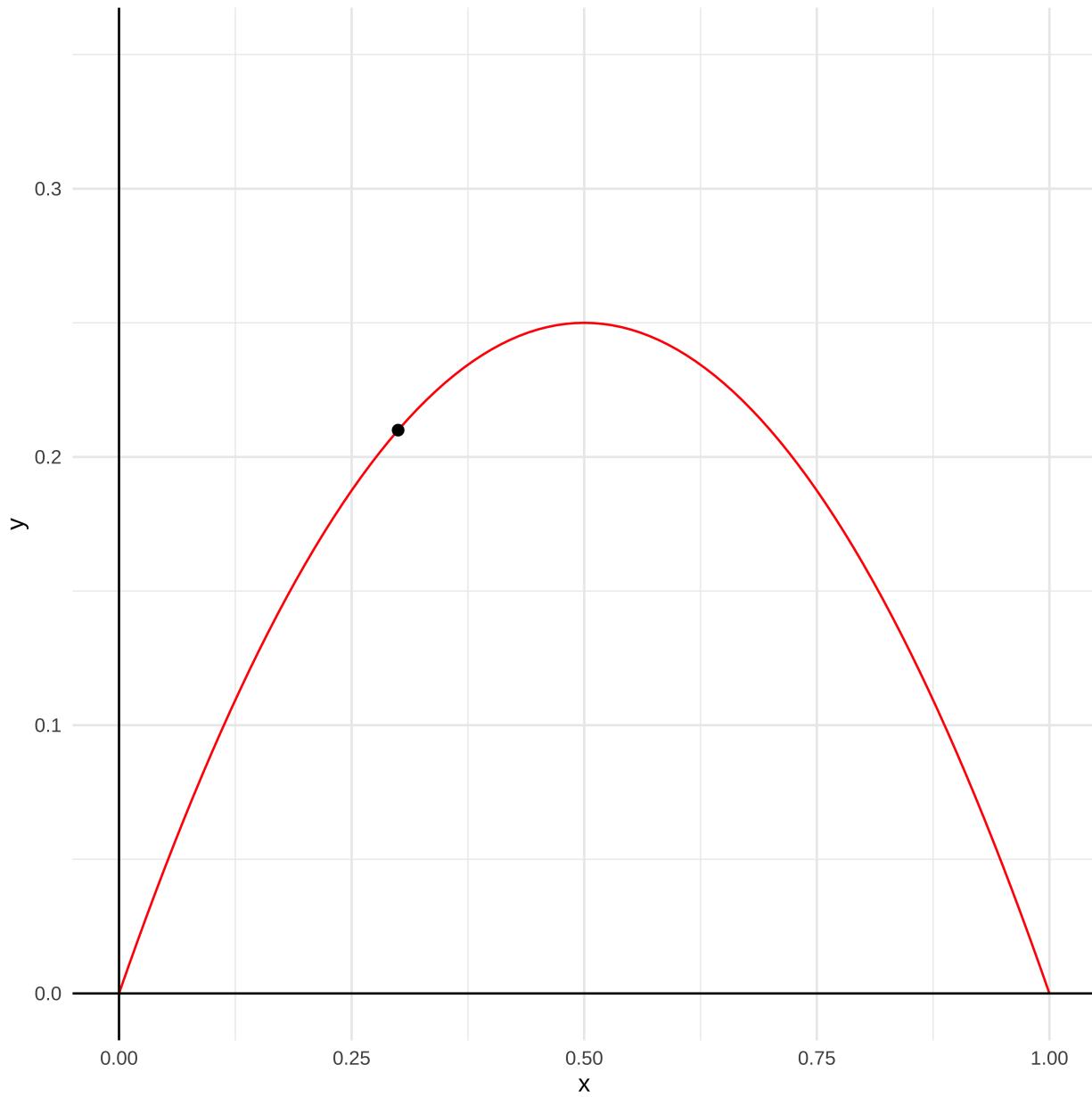
# Derivatives as Limits

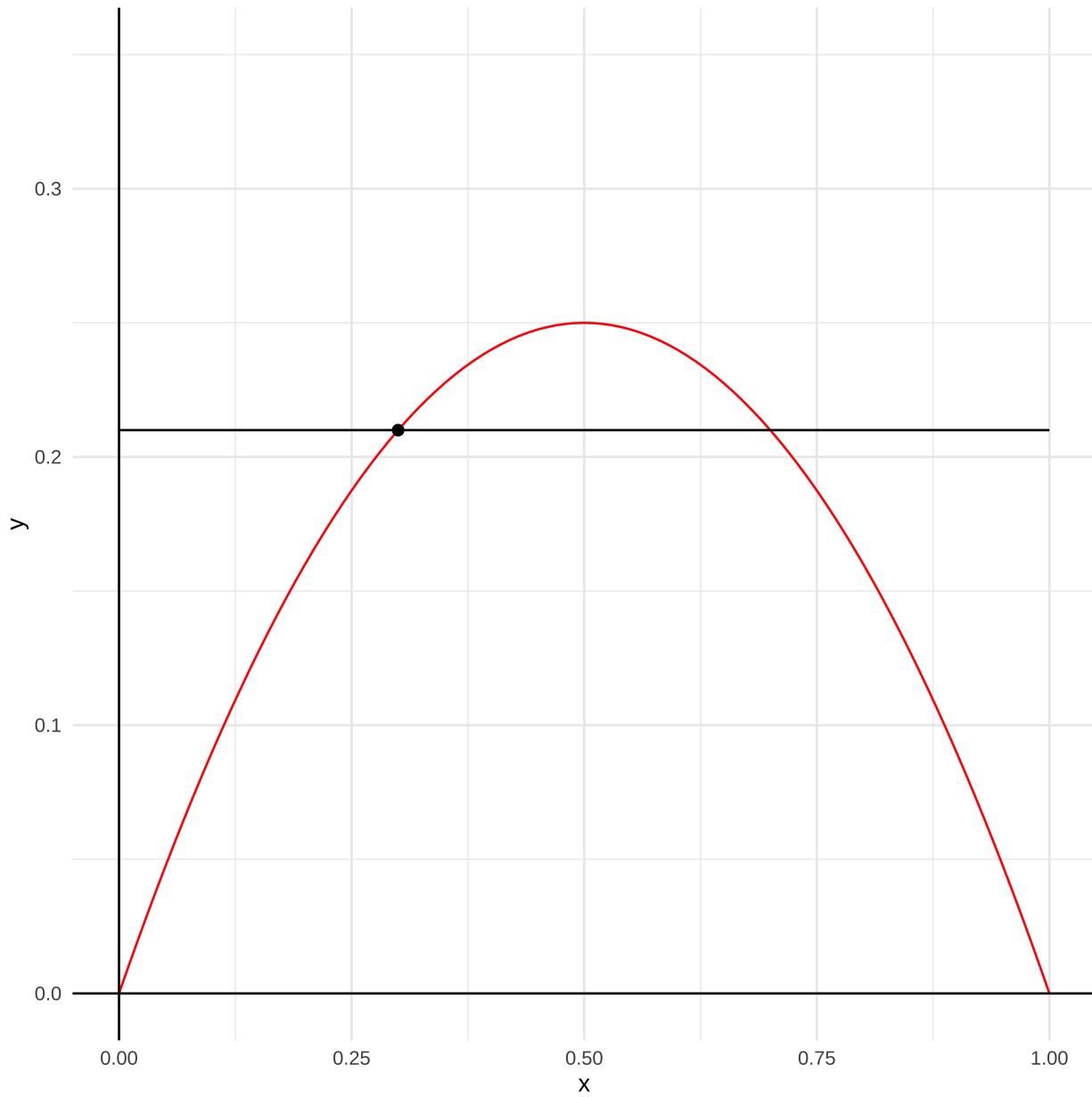
We can approximate the slope at a certain location by picking a point nearby on the line and finding the slope of the straight line connecting these two points.

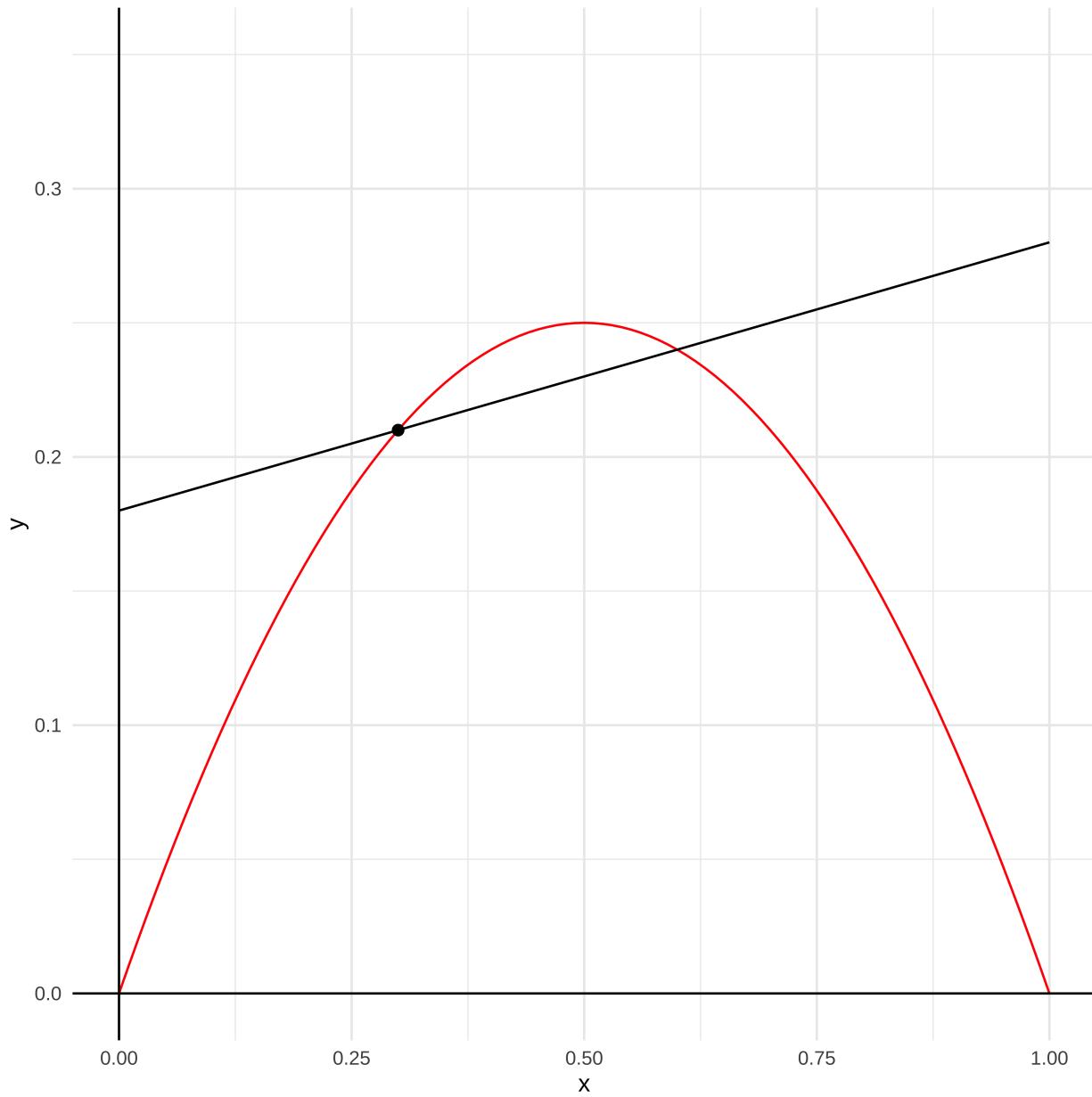
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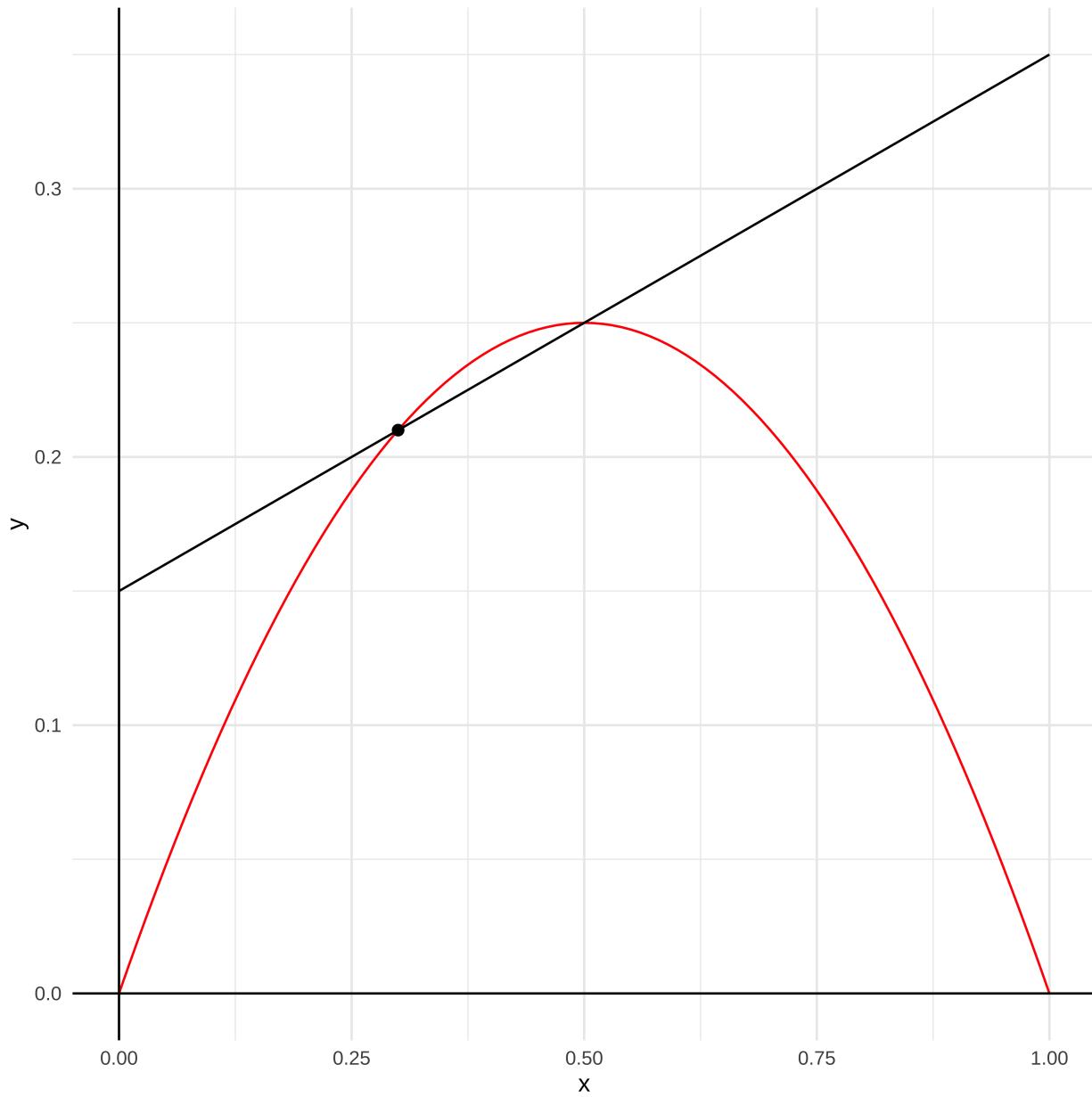
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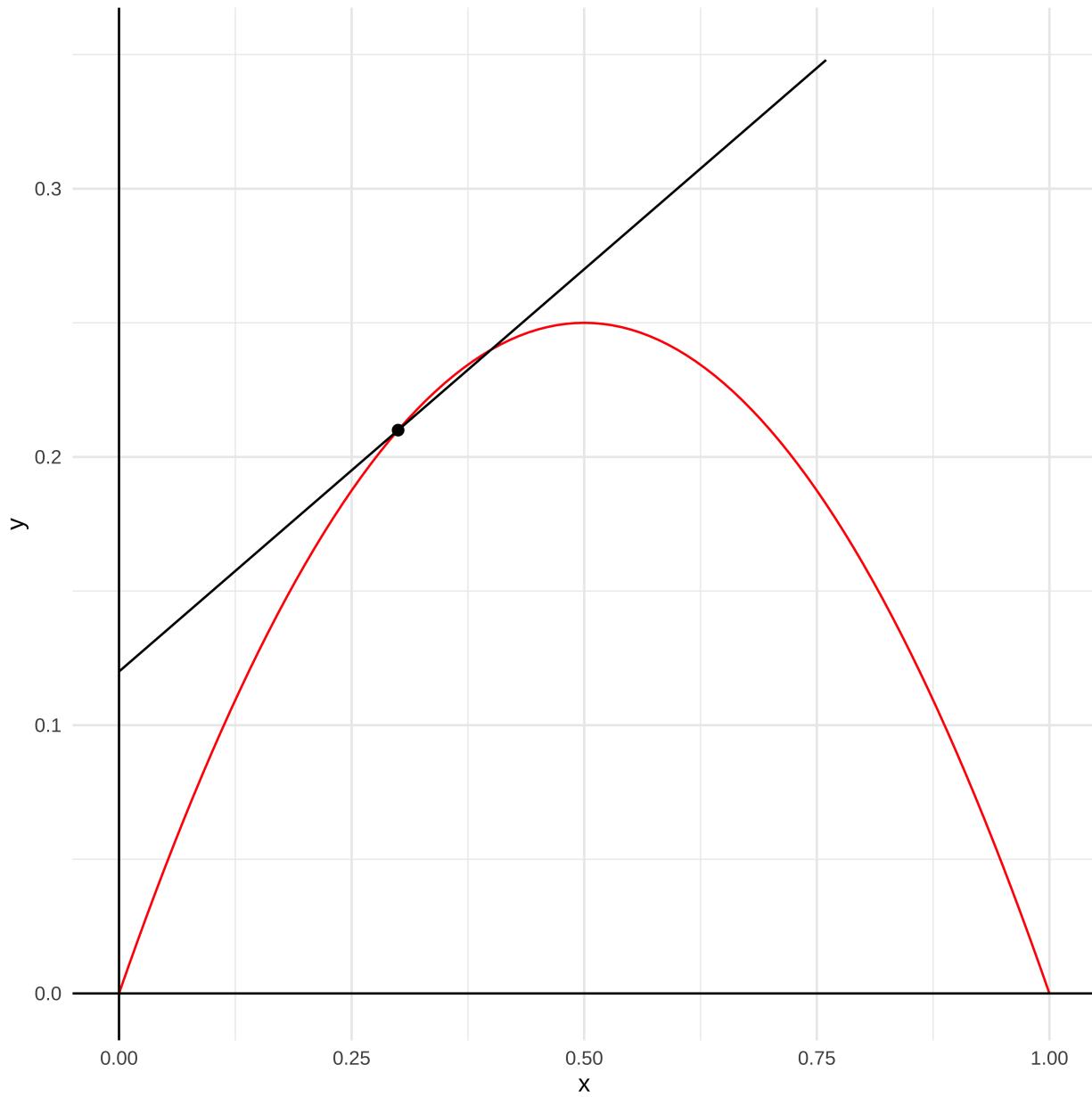
Let's consider a few examples.

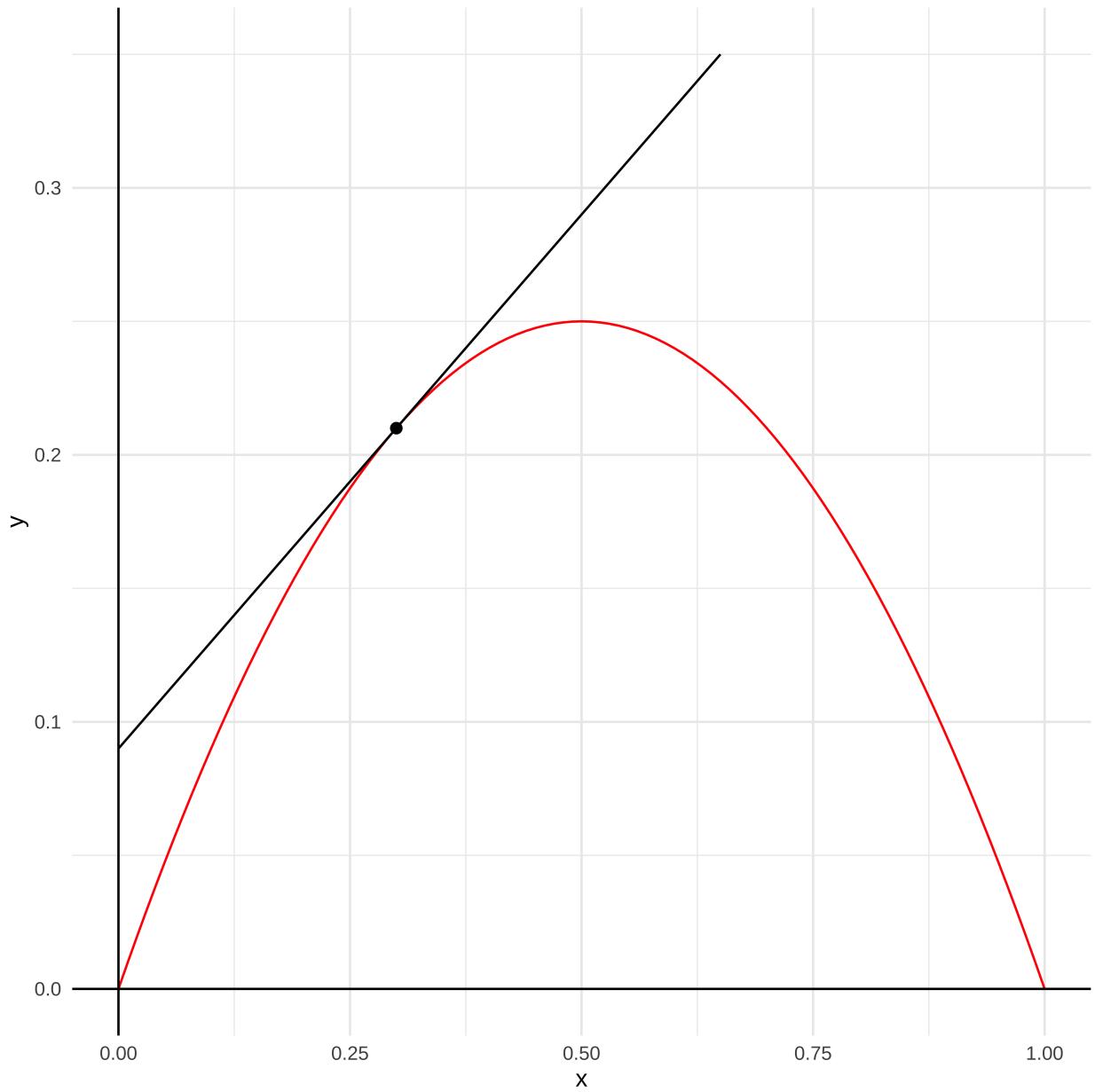












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So, as you reduce the interval size to  $\Delta x$ , this line converges in the limit to the line that lies tangent to the curve at that point. Recall that  $\Delta x \rightarrow 0$  and consider a very small interval

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Recall that  $\frac{f(x+h) - f(x)}{h}$  and consider a very small interval

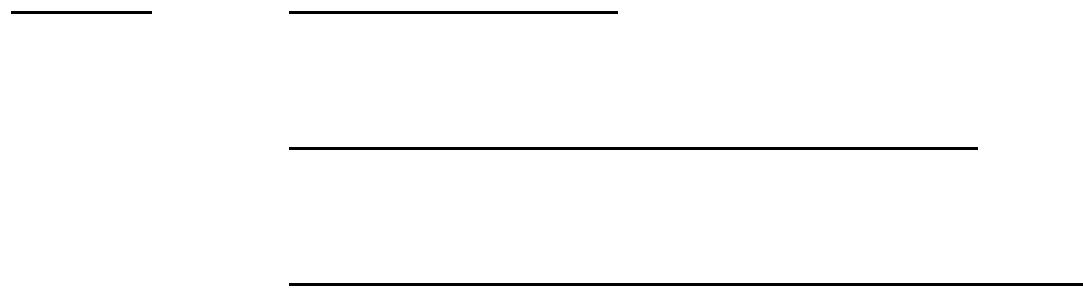


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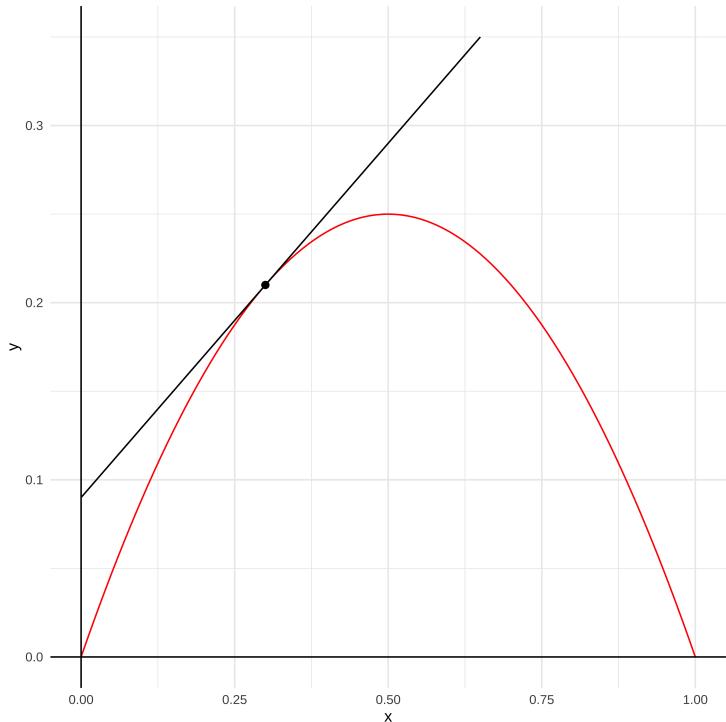


# Derivatives as Limits

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# Derivative as Limits



Using this formula, the slope of the curve at (the point from the previous examples) is exactly:

Or, if we want to find the point at which the slope is (rate of change is ):

# Derivatives

Now, we can formally state that the derivative is equivalent to:

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Using this approach, we can find:

- A general equation for the slope at any point
- The exact value of the slope at a given point
- The point that has a given slope

# Derivative notation

Different notation systems for expressing derivatives:

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Different notation systems for expressing derivatives:

First derivative:  $\frac{dy}{dx}$ ,  $y'$ ,  $\frac{\Delta y}{\Delta x}$ , or

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Second derivative:  $\frac{d^2y}{dx^2}$ ,  $y''$ ,  $\frac{d}{dx}\frac{dy}{dx}$ , or

We'll be using  $\frac{dy}{dx}$  and  $y'$  but you are free to choose whatever makes sense to you!

# Cautionary Notes on Derivatives

A few assumptions in using this approach to find the slope:

- The function is continuous (no gaps or jumps)
- The derivative exists (the limit of the slope is the same from the left as it is from the right), or *no sharp corners*.

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For nearly all political science applications, these are fine assumptions. But it is important to state them explicitly and be aware that they're there.

# Straightforward Derivatives

Fortunately (for humans), it is not necessary to take limits with ' $\Delta x$ 's to find derivatives every time. A few rules generate the derivatives of most functions.

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The simplest example: consider the line  $y = x^2$ . What's its derivative?

A special case of the power rule is that the *derivative of a constant is zero*.

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Let  $y = f(x)$ , then:

Let  $y = u^v$ , then:

Let  $y = \frac{u}{v}$ , then:

# Straightforward Derivatives

Let  $y = f(x)$ , then:

Let  $y = g(x)$ , then:

Let  $y = h(x)$ , then:

$$\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx}$$

# Exercises

Find the following derivatives, and calculate the instantaneous slope of the curves at the point :

—

- [Hint: How else can we express fractions?]

—

- [Hint: How else can we express roots?]

# Derivative of a Sum (or Difference)

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What is the rate of change when is equal to one?

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This is [the product rule](#).

# Derivative of a Product

As an example consider  $f(x) = x^2$  and  $g(x) = \sin(x)$ .

What is  $f'(x)g(x) + f(x)g'(x)$ ?

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# Derivative of a Product

As an example consider  $x^2$  and  $e^x$ .

What is  $(x^2)(e^x)$ ?

This is easy to check by multiplying out the polynomial:  
 $x^2 e^x$ , and then taking its derivative:

# Derivative of a Product

As an example consider  $x^2$  and  $e^x$ .

What is  $(x^2 e^x)'$ ?

This is easy to check by multiplying out the polynomial:  
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The derivative of a quotient of two functions (let's say  $f(x)$  and  $g(x)$ ) is:

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The derivative of a quotient of two functions (let's say  $\frac{u}{v}$  and  $\frac{w}{x}$ ) is:

$$\frac{u'v - uv'}{v^2}$$

# Derivative of a Quotient

The derivative of a quotient of two functions (let's say  $\frac{f}{g}$  and  $\frac{h}{j}$ ) is:

$$\frac{\underline{\hspace{2cm}} - \underline{\hspace{2cm}}}{\underline{\hspace{2cm}}^2}$$

or the bottom *times* the derivative of the top *minus* the top *times* the derivative of the bottom, all *divided* by the bottom *squared*.

# Derivative of a Quotient

The derivative of a quotient of two functions (let's say  $\frac{f}{g}$  and  $\frac{h}{j}$ ) is:

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or the bottom *times* the derivative of the top *minus* the top *times* the derivative of the bottom, all *divided* by the bottom *squared*.

This is referred to as **the quotient rule**.

# Derivative of a Quotient

Let  $f(x)$  and  $g(x)$  be differentiable functions. What is  $\left(\frac{f}{g}\right)'$ ?

# Derivative of a Quotient

Let  $\underline{\hspace{2cm}}$  and  $\underline{\hspace{2cm}}$ . What is  $\left( \underline{\hspace{1cm}} \right)$ ?

$\underline{\hspace{5cm}}$

$\underline{\hspace{5cm}}$

$\underline{\hspace{5cm}}$

$\underline{\hspace{5cm}}$

# Exercises

Find the derivatives of the following expressions:

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# Derivatives of Nested Functions

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The derivative of one function nested inside another is:

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This looks messy, but is actually fairly straightforward and extremely useful as a way to find derivatives of complex functions by treating them as nested chains of functions.

# Derivatives of Nested Functions

Let  $y = f(g(x))$ . Observe that this can be thought of as two nested functions, such that  $y = f(u)$  and  $u = g(x)$ , and

What is  $\frac{dy}{dx}$ ?

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# Derivatives of Nested Functions

Let  $f(x) = g(h(x))$ . Observe that this can be thought of the same two functions nested in the reverse order, such that  $h$  is the inner function and  $f$  is the outer function, and

What is  $f'(x)$ ?

# Derivatives of Nested Functions

Let  $f(x) = g(h(x))$ . Observe that this can be thought of the same two functions nested in the reverse order, such that  $h(x) = u$  and  $f(u) = g(u)$ , and

What is  $\frac{dy}{dx}$ ?

# Exercises

Express the functions below as the nested result of two simpler functions, and use the chain rule to find the derivative:

# Derivatives of Logarithms

The derivative for any logarithm base  $b$  is

# Derivatives of Logarithms

The derivative for any logarithm base  $a$  is

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It is important to note that a very special case of this is the derivative of a natural logarithm (or log base  $e$ ), which is

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Let  $y = \ln(x)$ . What is  $y'$ ?

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# Derivatives of Logarithms

Let  $y = \ln(x)$ . What is  $y'$ ?

\_\_\_\_\_

Let  $y = \ln(u)$ . What is  $y'$ ?

(using the chain rule):

\_\_\_\_\_

\_\_\_\_\_

# Derivatives of Exponentials

The derivative for any exponential base  $b$  is

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# Derivatives of Exponentials

Let  $y = a^x$ . What is  $\frac{dy}{dx}$ ?

# Derivatives of Exponentials

Let  $y = e^{kt}$ . What is  $\frac{dy}{dx}$ ?

# Derivatives of Exponentials

Let  $y = a^x$ . What is  $\frac{dy}{dx}$ ?

Let  $y = e^{kx}$ . What is  $\frac{dy}{dx}$ ?

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Let  $y = a^x$ . What is  $y'$ ?

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# Derivatives of Exponentials

Let  $y = a^x$ . What is  $y'$ ?

Let  $y = e^{ax}$ . What is  $y'$ ?

Let  $y = b^{cx}$ . What is  $y'$ ?

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Let  $y = a^x$ . What is  $y'$ ?

Let  $y = e^{ax}$ . What is  $y'$ ?

Let  $y = b^{cx}$ . What is  $y'$ ?

# L'Hôpital's Rule

Sometimes when we calculate limits, it helps to take the derivative first.

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Consider the limit:

$$\frac{\text{_____}}{\text{_____}}$$

If both the numerator and the denominator are  $\infty$ ,  $-\infty$ , or  $0$ , we cannot evaluate the limit.

# L'Hôpital's Rule

Sometimes when we calculate limits, it helps to take the derivative first.

Consider the limit:  $\frac{\text{_____}}{\text{_____}}$

$\frac{\text{_____}}{\text{_____}}$

If both the numerator and the denominator are  $0$ ,  $\infty$ , or  $\text{_____}$ , we cannot evaluate the limit.

However, L'Hôpital's Rule says:  $\frac{\text{_____}}{\text{_____}}$

Then,

$\frac{\text{_____}}{\text{_____}}$

$\frac{\text{_____}}{\text{_____}}$

$\frac{\text{_____}}{\text{_____}}$

# End Day 3