Math Camp Calculus Day 2: Solutions for Selected Exercises

Optimization

Find the local minimum and local maximum of the function below, and check mathematically which is the minimum and which is the maximum:

$x^3 - x^2 + 1$

To find local minima and maxima, find the first derivative:

$$f'(x) = (x^3 - x^2 + 1)' = 3x^2 - 2x$$

...then set it to 0 and solve for x:

$$\begin{aligned} 3x^2 - 2x &= 0 \\ x(3x-2) &= 0 \end{aligned}$$

which means $x = 0$

or

$3x - 2 = 0 \implies x = \frac{2}{3}$

So $x = 0$ and $x = \frac{2}{3}$ are local minima or maxima

Optimization (continued)

To check which is the maximum and which is the minimum, find the second derivative:

$$f''(x) = (3x^2 - 2x)' = 6x - 2$$

...and then plug in the x values where we suspect the maxima or minima to be:

$f''(0) = 6 \cdot 0 - 2 = -2 \implies$ a local maximum, because $f''x() < 0$

$f''(\frac{2}{3}) = 6 \cdot \frac{2}{3} - 2 = 2 \implies$ a local minimum, because $f''x() > 0$

Partial Derivatives

Find the partial derivatives of the function below with respect to each variable

$g(p,q) = 8p^2q + 4pq - 7pq^2 + 18$

First, the partial derivative with respect to p:

$$\frac{\partial[f(p,q)]}{\partial p} = 16pq + 4q - 7q^2$$

Now, the partial derivative with respect to q:

$$\frac{\partial[f(p,q)]}{\partial q} = 8p^2 + 4p - 14pq$$

Partial Higher-Order Derivatives

Consider again $f(x,y) = 3x^3y^2$. Find:

- $\frac{\partial^3}{\partial x^2 \partial y}$
- $\frac{\partial^3}{\partial x \partial y^2}$

$$\begin{aligned}
\frac{\partial^3}{\partial x^2 \partial y}(3x^3y^2) &= \frac{\partial^2}{\partial x^2}(2 \cdot 3x^3y^{2-1}) \\
&= \frac{\partial^2}{\partial x^2}(6x^3y) \\
&= \frac{\partial}{\partial x}(3 \cdot 6x^{3-1}y) \\
&= \frac{\partial}{\partial x}(18x^2y) \\
&= 2 \cdot 18xy \\
&= 36xy
\end{aligned}$$

Partial Higher-Order Derivatives (cont.)

$$\begin{aligned}
\frac{\partial^3}{\partial x \partial y^2}(3x^3y^2) &= \frac{\partial^2}{\partial x \partial y}(2 \cdot 3x^3y^{2-1}) \\
&= \frac{\partial^2}{\partial x \partial y}(6x^3y) \\
&= \frac{\partial}{\partial x}(6x^3y^{1-1}) \\
&= \frac{\partial}{\partial x}(6x^3) \\
&= 3 \cdot 6x^{3-1} \\
&= 18x^2
\end{aligned}$$

Integrals

Find the indefinite integral of the function below, and calculate the area under the curve between 0 and 1:

$\int(2x^3 - 3x^2 + 7x + 4)dx$

First, let's find the indefinite integral by taking an antiderivative of this function:

$$\begin{aligned}\int(2x^3 - 3x^2 + 7x + 4) &= \int 2x^3dx - \int 3x^2dx + \int 7x + \int 4 \\ &= 2\int x^3dx - 3\int x^2dx + 7\int x + \int 4 \\ &= \frac{2}{3+1}x^{3+1} - \frac{3}{2+1}x^{2+1} + \frac{7}{1+1}x^{1+1} + 4x + C \\ &= \frac{1}{2}x^4 - x^3 + \frac{7}{2}x^2 + 4x + C\end{aligned}$$

Integrals (cont.)

Now, let's substitute $x = 0$ and $x = 1$ to find the area between 0 and 1:

$$\begin{aligned}
\int_0^1 (2x^3 - 3x^2 + 7x + 4)dx &= \frac{1}{2}x^4 - x^3 + \frac{7}{2}x^2 + 4x\Big|_0^1 \\
&= (\frac{1}{2} \cdot 1^4 - 1^3 + \frac{7}{2} \cdot 1^2 + 4 \cdot 1) \\
&- (\frac{1}{2} \cdot 0^4 - 0^3 + \frac{7}{2} \cdot 0^2 + 4 \cdot 0) \\
&= (\frac{1}{2} - 1 + \frac{7}{2} + 4) - (0 - 0 + 0 + 0) \\
&= 7
\end{aligned}$$