Math Camp Lesson 3 (Day 2)

Calculus

UW–Madison Political Science

August 20, 2020

Agenda

1) Second Derivatives
2) Partial Derivatives
3) Integrals
4) Optimization

Second Derivatives

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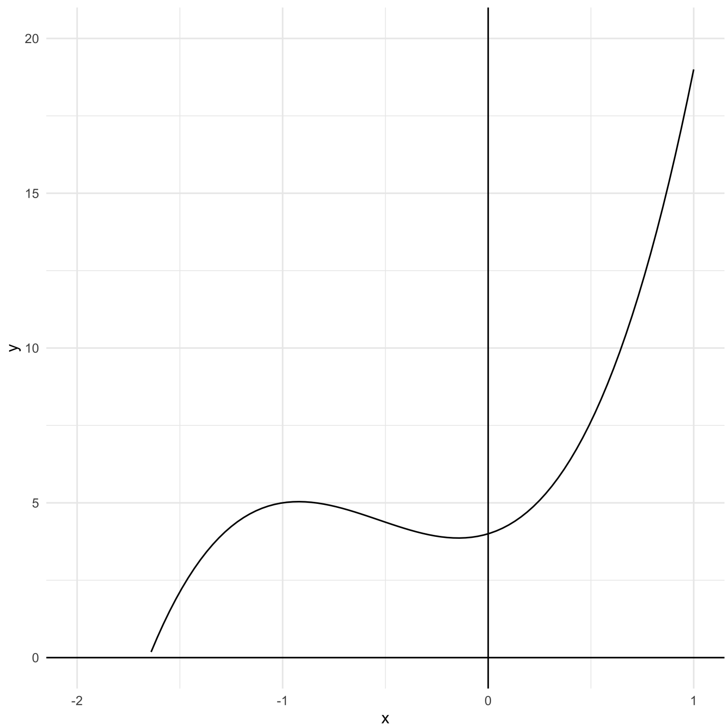
Higher order (third, fourth, etc) derivatives also exist, but are rarely relevant.

Concavity and Convexity

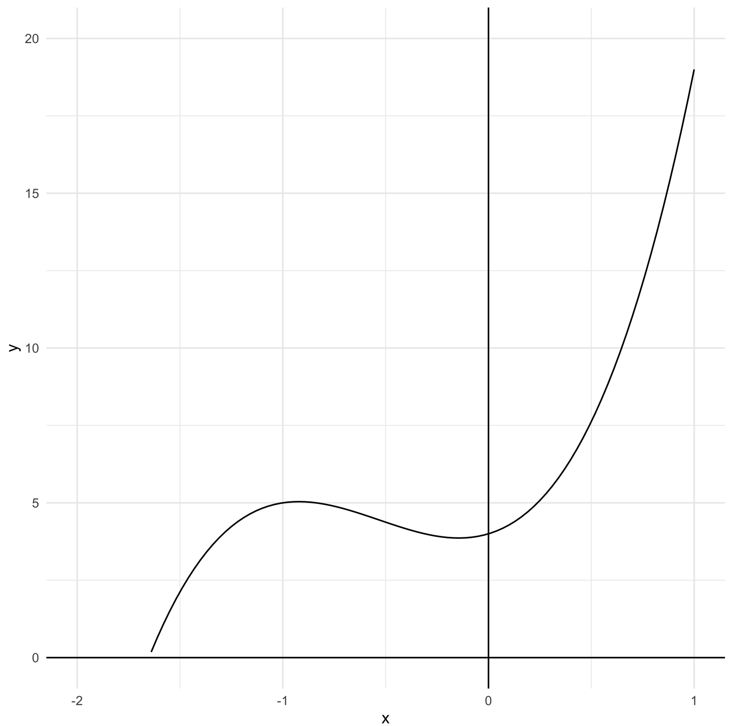


Concavity Theorem: If the function $f(x)$ is twice differentiable at $x = c$, then the graph of $f(x)$ is convex (concave upward) at $(x, f(x)) = (c, f(c))$ if $f''(c) > 0$ and concave (concave downward) if $f''(c) < 0$.

The function to the left is $f(x) = 5x^3 + 8x^2 + 2x + 4$ and its second derivative is $f''(x) = 30x + 16$.

When is it convex? When is it concave?

Concavity and Convexity



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It is convex when $x \in \left(-\frac{16}{30}, \infty\right)$ and concave when $x \in \left(-\infty, -\frac{16}{30}\right)$.
$x = -\frac{16}{30}$ is an *inflection point.*

Exercises: Second Derivatives

Find the first and second derivative of the expressions below:

$$f(x) = 16x^3 - 3x^2 + 6$$

$$g(x) = x - x^2$$

$$h(x) = 4x^{-1} + 5x^{\frac{7}{2}}$$

Multivariate Functions & Partial Derivatives

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$$\frac{\partial[f(x,y,z)]}{\partial x} = f_x(x,y,z) = 8xy^4 + 2z^3 + 8$$
$$\frac{\partial[f(x,y,z)]}{\partial y} = f_y(x,y,z) = 16x^2y^3 + 16yz^4 + 7$$
$$\frac{\partial[f(x,y,z)]}{\partial z} = f_z(x,y,z) = 6xz^2 + 32y^2z^3 + 3$$

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Why is this useful? E.g., we are modeling election turnout as a function of age and income, and we want to know how turnout changes with respect only to changes in income.

Exercise: Partial Derivatives

Find the partial derivatives of the function below with respect to each variable

$$g(p, q) = 8p^2q + 4pq - 7pq^2 + 18$$

Partial Higher-Order Derivatives

It is possible to combine second-order (and higher) derivatives with partial derivatives. For example:

Consider $f(x, y) = 3x^3y^2$ and let's we wanted to find $\frac{\partial^2}{\partial x \partial y} f(x, y)$:

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y}(3x^3y^2) &= \frac{\partial}{\partial y}((3)3x^{3-1}y^2) \\
&= \frac{\partial}{\partial y}(9x^2y^2) \\
&= (2)9x^2y^{2-1} \\
&= 18x^2y
\end{aligned}$$

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Pay attention to the denominator to give you guidance about what operations to perform. Here, we are taking the second derivative of the entire function, but are differentiating once with respect to x and once with respect to y overall.

If instead we were given $\frac{\partial^3}{\partial x^2 \partial y}$ we would differentiate 3 times overall, twice with respect to x and once with respect to y.

Exercise: Partial Higher-Order Derivatives

Consider again $f(x, y) = 3x^3y^2$. Find:

- $\frac{\partial^3}{\partial x^2 \partial y}$
- $\frac{\partial^3}{\partial x \partial y^2}$

Integrals

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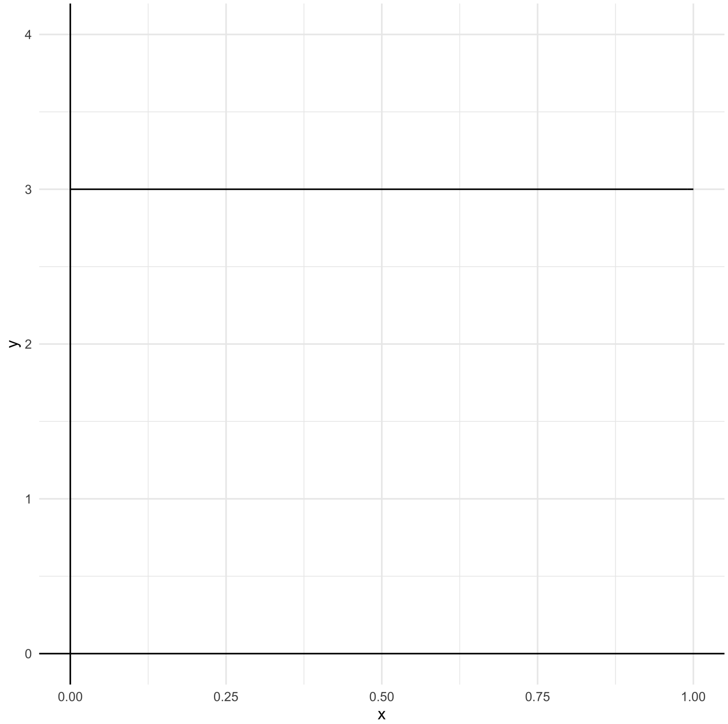
One common application of integrals in social science is to calculate the probability that a variable falls in a certain range (e.g., that a conflict will leave more than a 1,000 people dead) or takes a certain value (a politician is re-elected, an authoritarian regime breaks down).

Areas



Let's consider the function, $y = 3$, plotted to the left. What is its area under the curve from $x = 0.25$ and $x = 0.75$?

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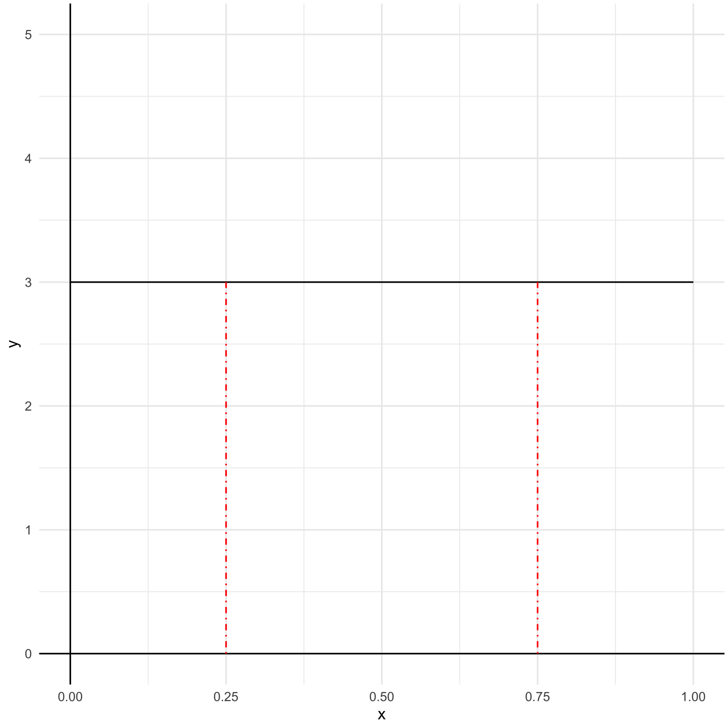

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Given that this is a rectangle, the area between the function and the x-axis is $\text{area} = (0.75 - 0.25) \times 3 = 1.5$

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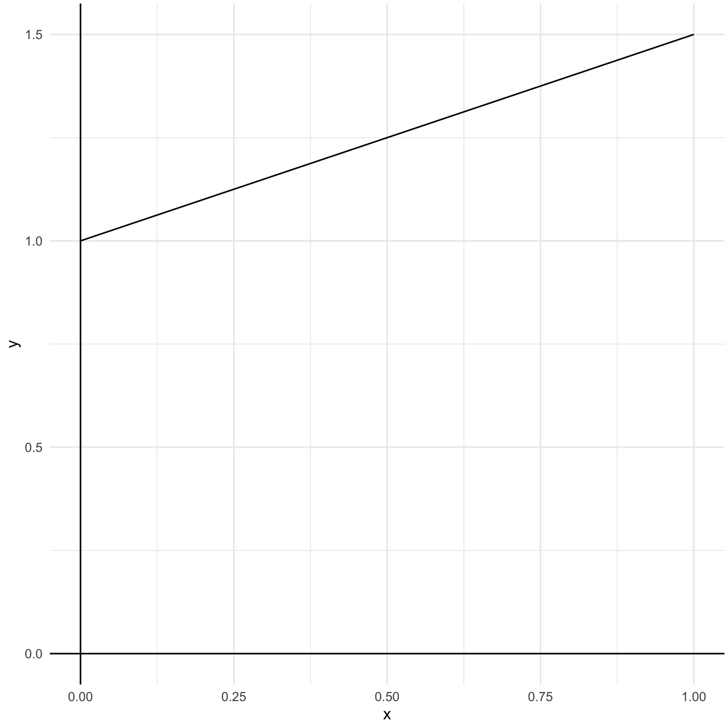

Let's consider a less simple function, $y = \frac{1}{2}x + 1$, plotted to the left. What is its area under the curve from $x = 0.25$ and $x = 0.75$?

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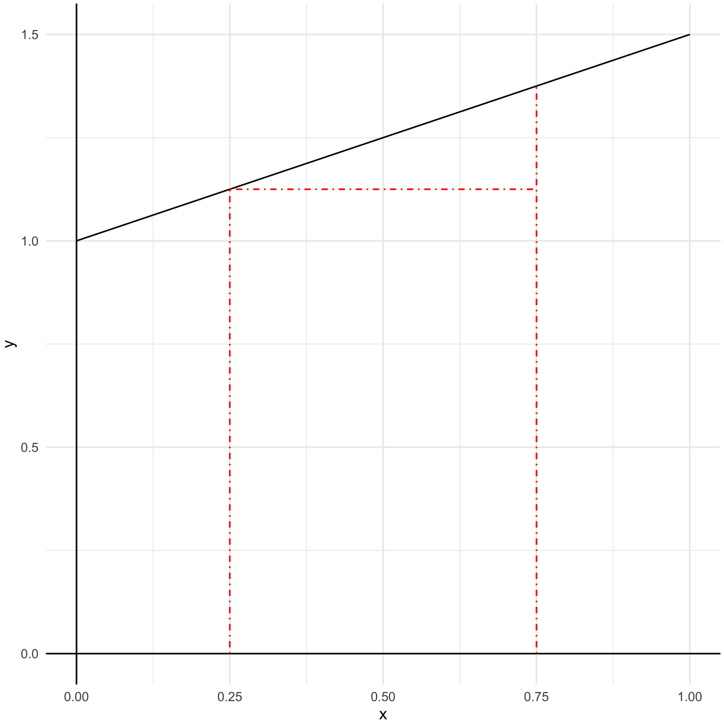

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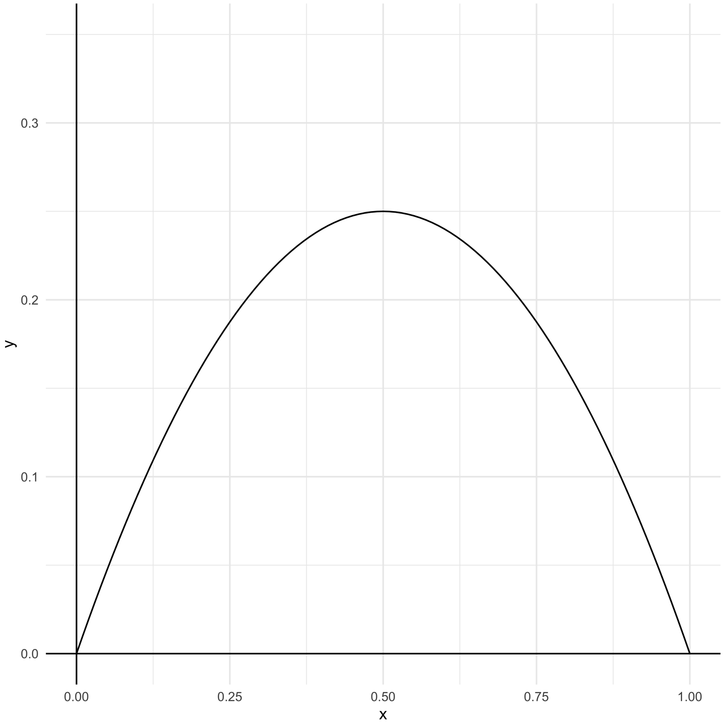
We can find the area between the function and the x-axis by taking advantage of the fact we can draw a triangle and rectangle and sum their areas. $\text{area}_{tri} = 0.0625$ and $\text{area}_{rect} = 0.5625$. Thus, the total area is 0.625.

Area



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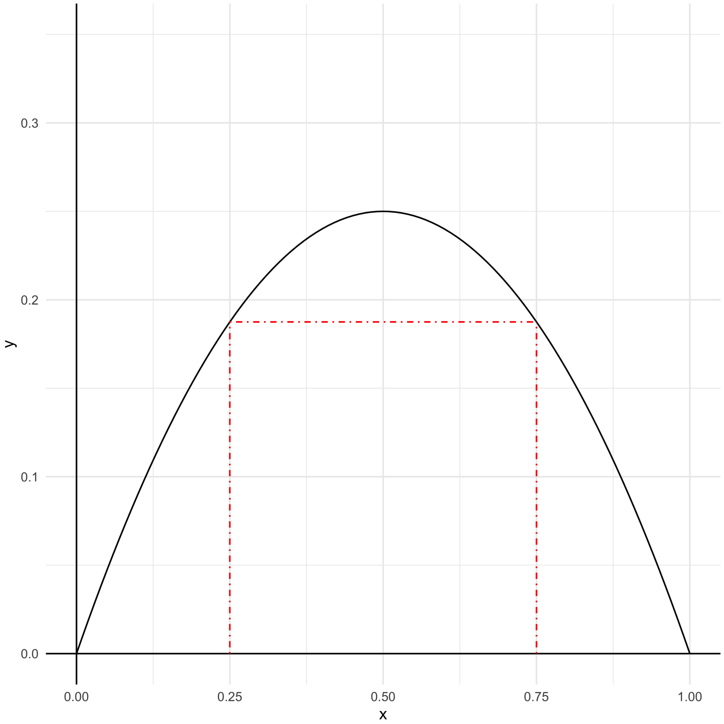

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This would require a more complex trick.

Integral as the Limit of a Sum

Imagine dividing the region into intervals and drawing a rectangle to capture the area for each interval, with height equal to the value of the function at the left (or right) of the interval, then summing the area of those rectangles.

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Let's see what happens as we add rectangles

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Approximation improves as the intervals become smaller.

Integrals

As you reduce the width of rectangles to zero, the summed areas of the rectangles converges to the area under the curve—including more and more of the area inside and less and less of the area outside.

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However, it is mathematically difficult to solve these using this approach.

Antiderivatives

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Essentially, this “unwinds” the derivative operation, or applies it backwards.

The Fundamental Theorem of Calculus

The fundamental theorem of calculus relates the derivative and the integral.

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$$

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This expression uses exactly the same antiderivative as the definite integral but there is no subtraction and there's an arbitrary constant C added (since that'd disappear when taking the derivative).

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$$\int e^x dx = e^x + C$$

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$$\begin{aligned}
\int_{.2}^{.75} x - x^2 dx &= \frac{1}{2} x^2 - \frac{1}{3} x^3 \Big|_{.25}^{.75} \\
&= \left(\frac{1}{2}(.75)^2 - \frac{1}{3}(.75)^3 \right) - \left(\frac{1}{2}(.25)^2 - \frac{1}{3}(.25)^3 \right) \\
&= \frac{11}{96}
\end{aligned}$$

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Now consider the area of the curve specifically between 2 and 5:

$$\begin{aligned}\int_2^5 9x^2 + 10x + 4dx &= 3x^3 + 5x^2 + 4x|_2^5 \\ &= (3(5)^3 + 5(5)^2 + 4(5)) - (3(2)^3 + 5(2)^2 + 4(2)) \\ &= 468\end{aligned}$$

Exercise: Straightforward Integrals

Find the indefinite integral of the function below, and calculate the area under the curve between 0 and 1:

$$\int(2x^3 - 3x^2 + 7x + 4)dx$$

Advanced Integrals

There are techniques for computing the integrals of more complicated functions:

- integration by parts ("reverse product rule")
- integration by substitution ("reverse chain rule")

These are beyond the scope of what we have time to cover here and, for the most part, beyond the scope of what you will need to do by hand in political science.

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- We have specified a function that shows the utility a president gets out of some policy in terms of their chances of reelection. Policy is an input, probability of reelection is an output.
  - We want to see when this function is maximized---that is, what values of policy would give the maximum chance of reelection.
- Or we have a function that determines the bias in our statistical estimates of said probability of reelection. Model parameters are inputs, bias (how much the estimate deviates from the truth) is an output.
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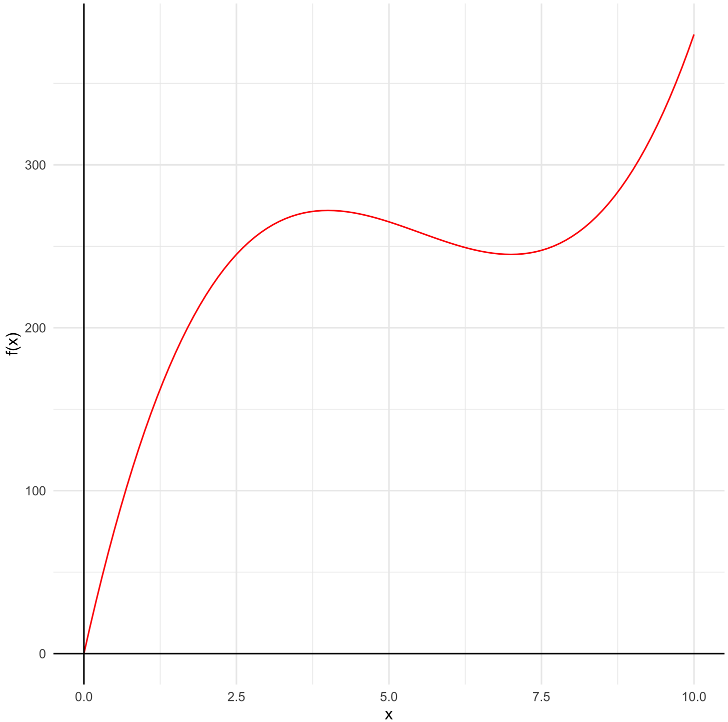
How do we do this? Derivatives can help!

Optimization: An Example



Suppose we want to find all the maxima and minima for the function $f(x) = 2x^3 - 33x^2 + 168x$ when $x \in [0, 10]$.

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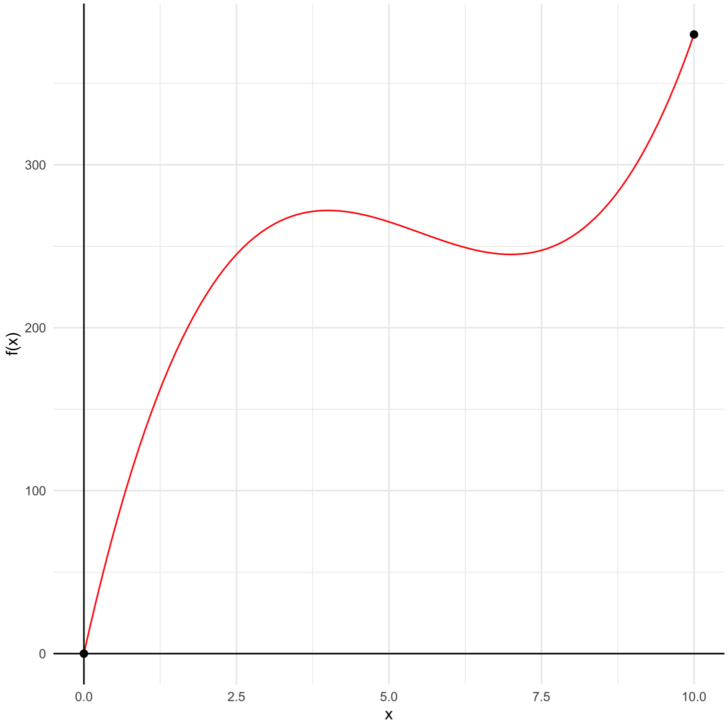
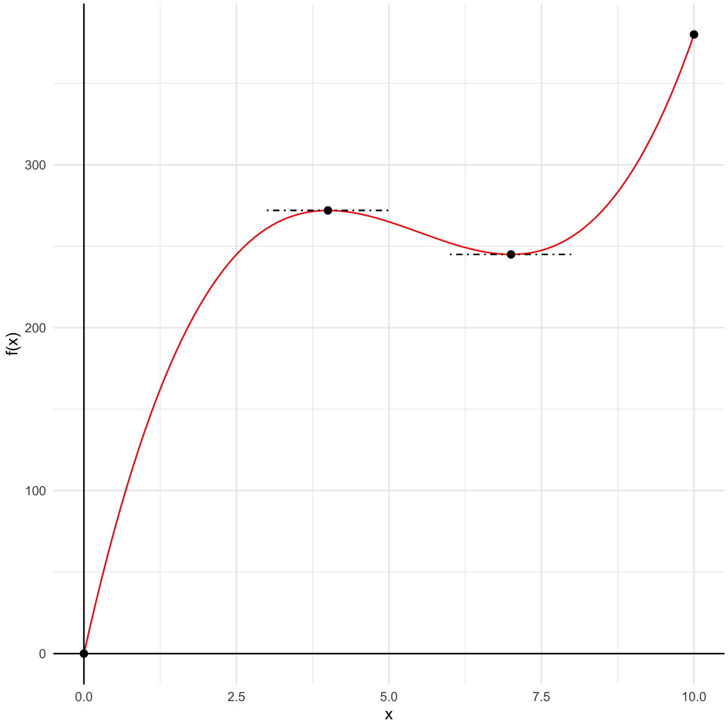


Suppose we want to find all the maxima and minima for the function $f(x) = 2x^3 - 33x^2 + 168x$ when $x \in [0, 10]$.

The *absolute maximum* occurs at $x = 10$, and the *absolute minimum* occurs at $x = 0$, or the endpoints. But there also appear to be a local maximum and a local minimum in between them.

Optimization: An Example



Suppose we want to find all the maxima and minima for the function $f(x) = 2x^3 - 33x^2 + 168x$ when $x \in [0, 10]$.

The *absolute maximum* occurs at $x = 10$, and the *absolute minimum* occurs at $x = 0$, or the endpoints. But there also appear to be a local maximum and a local minimum in between them.

To determine the precise location of these local maxima and minima, note that at these points, the slope of the line is flat. This means the derivative, which captures the slope of the tangent line, is 0.

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$$\begin{aligned} x &= \frac{11 \pm \sqrt{(-11)^2 - 4(1)(28)}}{2(1)} \\ x &= \frac{11 \pm 3}{2} \\ x &= 4 \text{ or } 7 \end{aligned}$$

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when we evaluate the second derivatives at the local minimum and maximum, the results are $f''(4) = -18$ and $f''(7) = 18$.

An Exercise

Find the local minimum and local maximum of the function below, and check mathematically which is the minimum and which is the maximum:

$$x^3 - x^2 + 1$$

End Day 4