Math Camp Lesson 3 (Day 1)

Calculus

UW–Madison Political Science

August 19, 2020

Overview

Calculus evaluates the behavior of functions:

- Limits
- Rate of change
- Change in the rate of change
- Area of the region they defined on

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Calculus in political science:

- Finding the fitline with the minimal distance between predicted and observed data

- Calculating the probability density in regions of continuous distributions

- Solving for the choice that maximizes a decision maker's utility

Agenda

Day 1

- Limits
- Derivatives

Day 2

- More Derivatives
- Integrals
- Applications

Limits

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The limit of a function characterizes its behavior given a certain input, or as an input value changes.

Limits: Example 1

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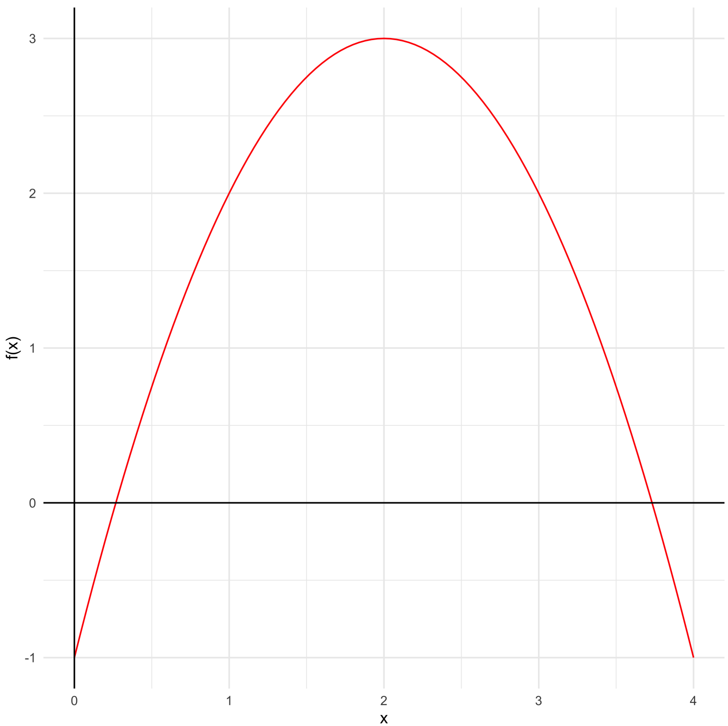
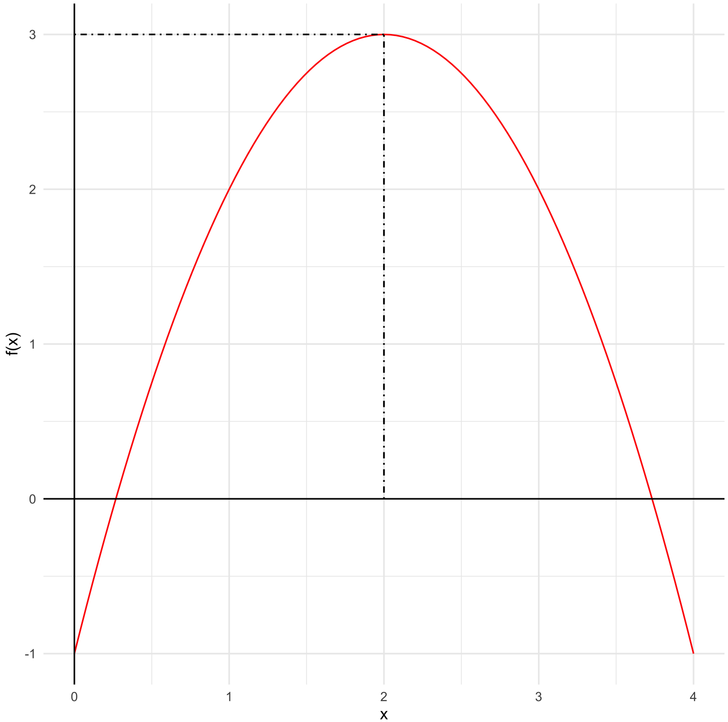


Let's consider the simple function, $f(x) = y = 3 - (x-2)^2$, plotted to the left. What is the limit of $f(x)$ as x *approaches* 2?

Limits: Example 1



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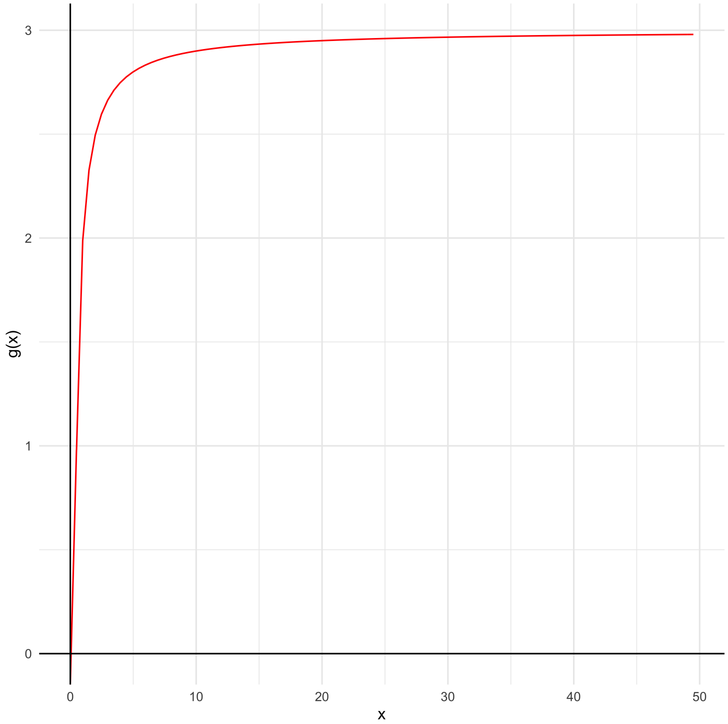
As x approaches 2, $f(x)$ or y approaches $f(2) = 3$.

Limits: Example 2



Let's consider a less simple function, $g(x) = y = 3 - \frac{1}{x}$, plotted to the left. What is the limit of $g(x)$ as x *approaches* ∞?

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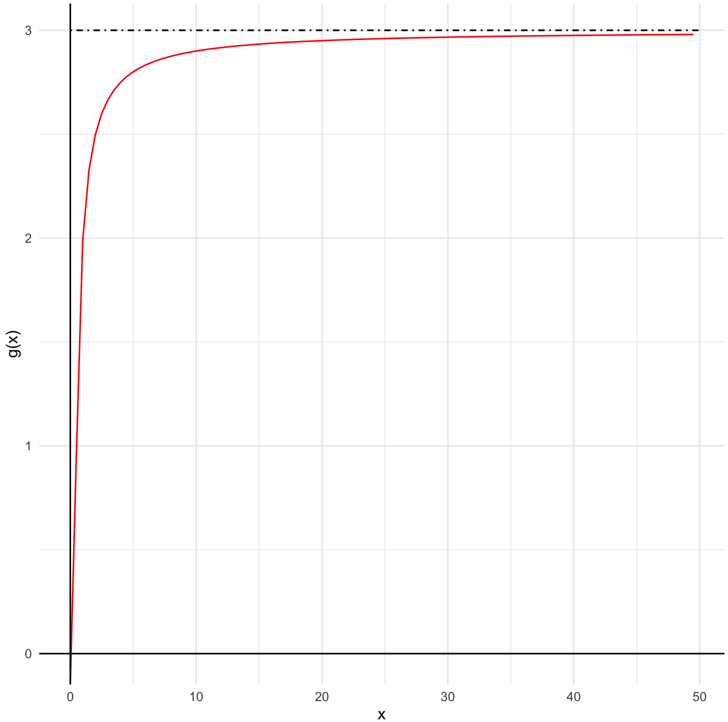

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As x approaches ∞, $g(x)$ approaches 3. How do we know?

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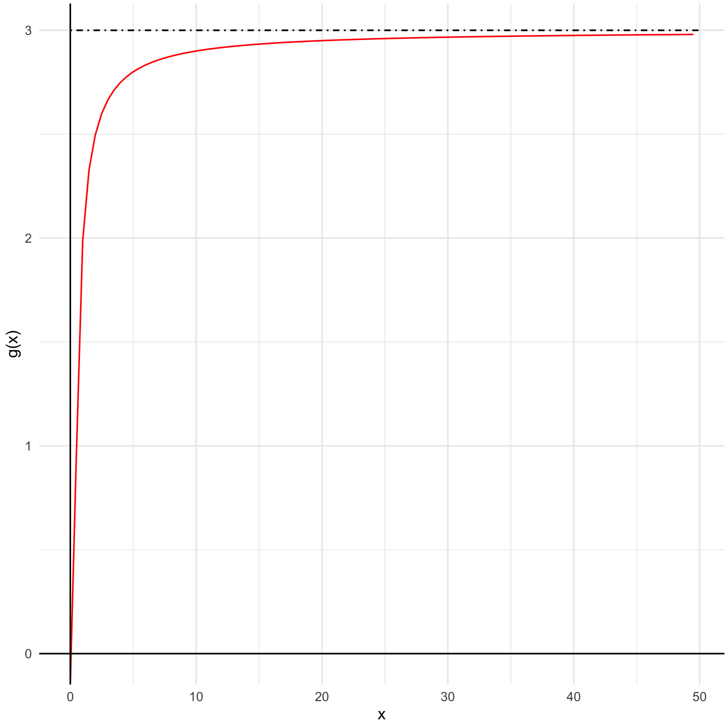

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As x gets larger, $\frac{1}{x}$ gets smaller and smaller.

$$\left(\frac{1}{2} > \frac{1}{20} > \frac{1}{200} \dots\right)$$

Limits, Formally Written

Formally, limits are expressed as:

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This expression should be read as: "As x approaches c, the limit of $f(x)$ is L."

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Many times, you will see this expression written as

$\lim_{x \to c^-} f(x) = L$

or $\lim_{x \to c^+} f(x) = L$

A negative sign $(-)$ implies "As x approaches c *from the left*"

A positive sign $(+)$ implies "As x approaches c *from the right*"

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If you can, evaluate the function at the limit.

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If you can, evaluate the function at the limit.

For functions that are well-behaved, the limit as x approaches a finite point is generally the value of the function at that point (if it exists).

Finding Limits: Example 1

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$$\begin{aligned}
\lim_{x \to 2} x^2 - 3x + 1 &= \lim_{x \to 2} x^2 - 3 \lim_{x \to 2} x + \lim_{x \to 2} 1 \\
&= 2^2 - 3(2) + 1 \\
&= -1
\end{aligned}$$

Finding Limits: Example 2

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\lim_{x\to\infty} \frac{4x^4 + 7x^2 + 8}{3x^4} &= \lim_{x\to\infty} \frac{4x^4}{3x^4} + \lim_{x\to\infty} \frac{7x^2}{3x^4} + \lim_{x\to\infty} \frac{8}{3x^4} \\
&= \lim_{x\to\infty} \frac{4}{3} + \lim_{x\to\infty} \frac{7}{3x^2} + \lim_{x\to\infty} \frac{8}{3x^4} \\
&= \frac{4}{3} + 0 + 0 \\
&= \frac{4}{3}
\end{aligned}
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&= \frac{4}{3} + 0 + 0 \\
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\end{aligned}$$

Why does $\lim_{x \to \infty} \frac{7}{3x^2} = 0$? As $x \to \infty$, $3x^2 \to \infty$, and $\frac{7}{\infty} \to 0$.

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&= \lim_{x \to 0} \frac{4}{3} + \lim_{x \to 0} \frac{7}{3x^2} + \lim_{x \to 0} \frac{8}{3x^4} \\
&= \frac{4}{3} + \frac{7}{3} \lim_{x \to 0} \frac{1}{x^2} + \frac{8}{3} \lim_{x \to 0} \frac{1}{x^4} \\
&= \frac{4}{3} + \frac{7}{3} \times \frac{1}{0} + \frac{8}{3} \times \frac{1}{0} \\
&= \ldots
\end{aligned}
$$

Finding Limits: Example 3

The limit is ∞.

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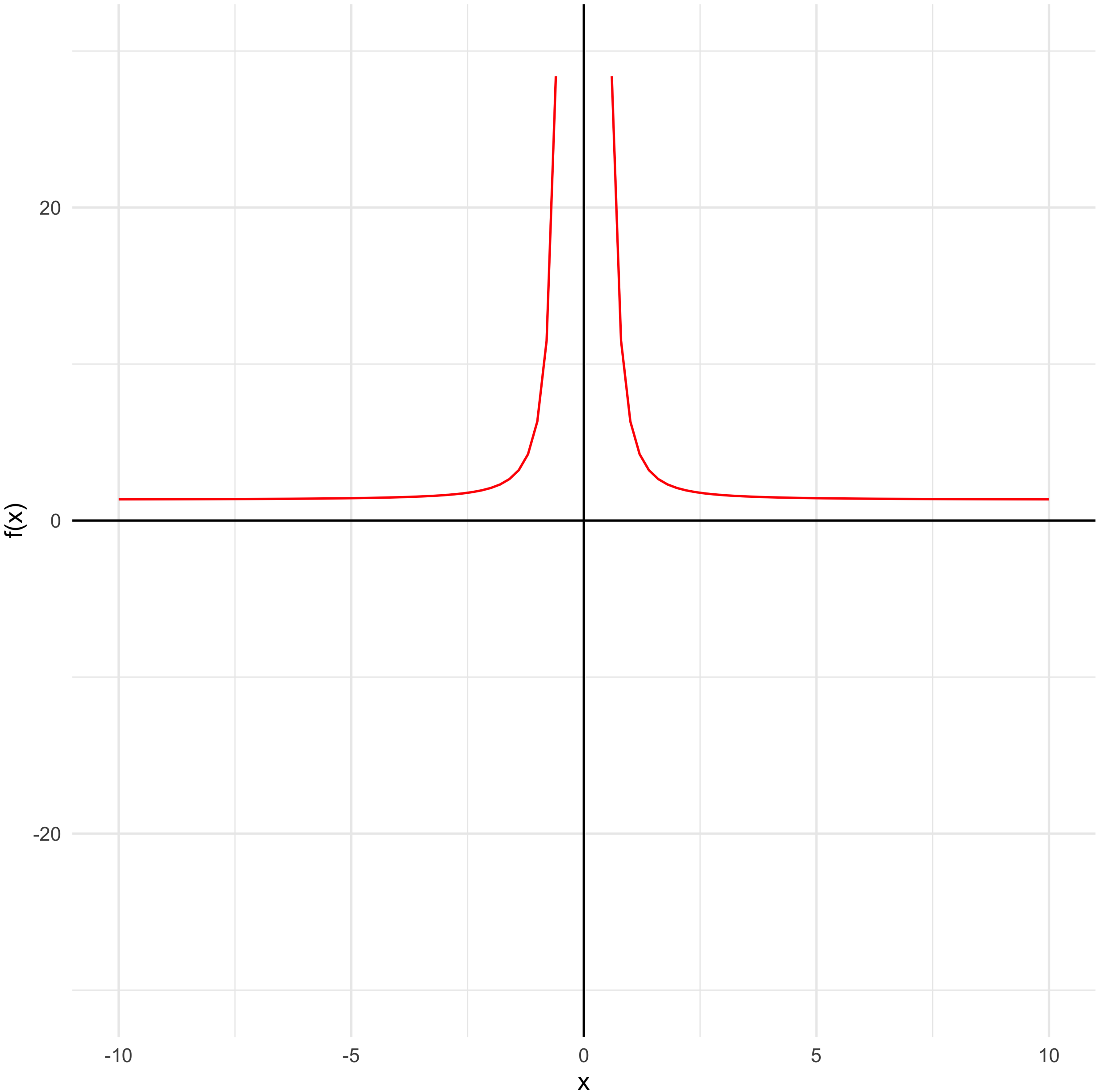
$$
\begin{aligned}
&= \lim_{x \to 0} \frac{4x^4 + 7x^2 + 8}{3x^4} \\
&= \frac{4}{3} + \frac{7}{3} \lim_{x \to 0} \frac{1}{x^2} + \frac{8}{3} \lim_{x \to 0} \frac{1}{x^4} \\
&= \infty
\end{aligned}
$$

As x approaches 0, the function retains some positive value in the numerator while the denominator *positively* approaches 0. This means that you are dividing by a smaller and smaller fraction, which means the entire term is getting larger and approaches ∞.


20
0
-20
f(x)
-10
-5
0
5
10
x

Exercises

Find the following limits:

$\lim_{x\to 4} x^2 - 6x + 4$

$\lim_{x\to 4} \frac{x^2}{3x-2}$

$\lim_{x\to \infty} \frac{3x-4}{x+3}$

Derivatives

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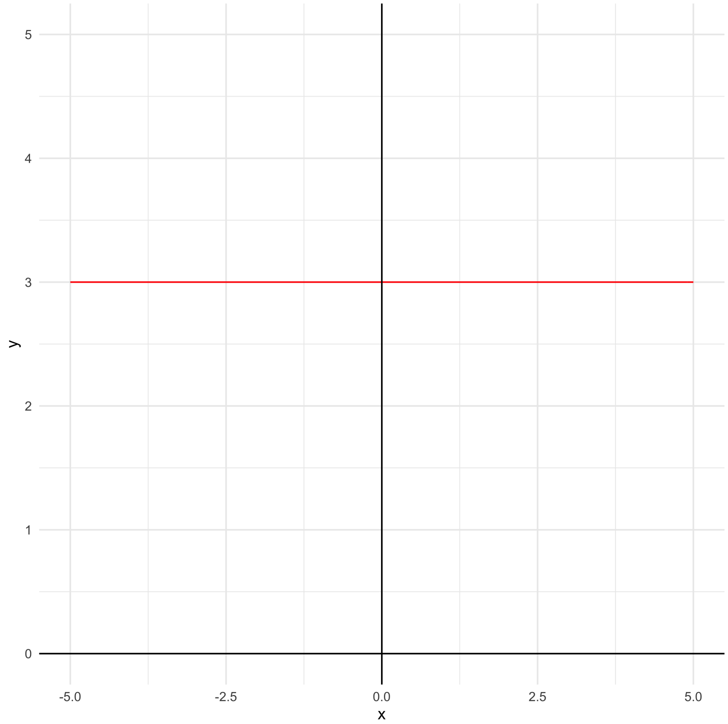
It is the instantaneous slope of the line at any given point.

The slope of a function is how much the output changes as a result of changes in the input.

Using Δ to signify 'change', this is $\frac{\Delta f(x)}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ or "rise-over-run".

Slope



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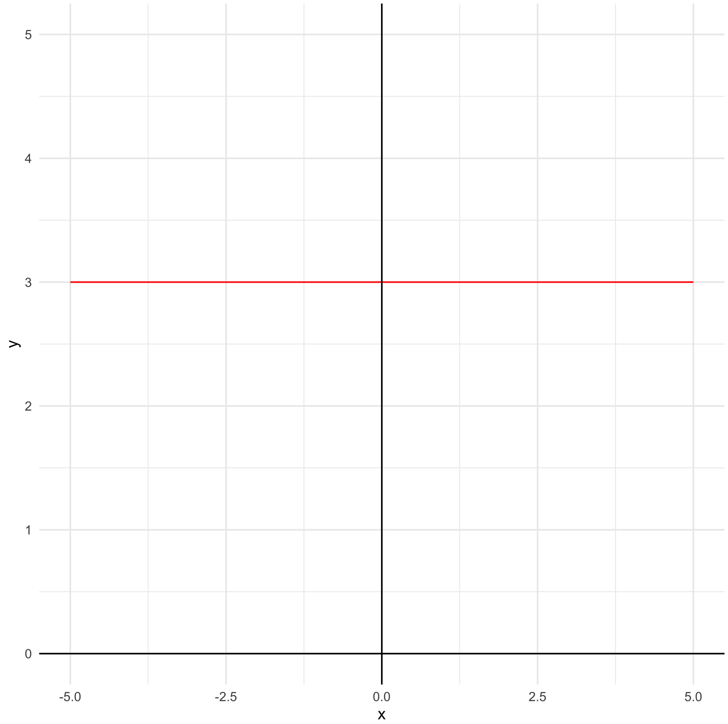
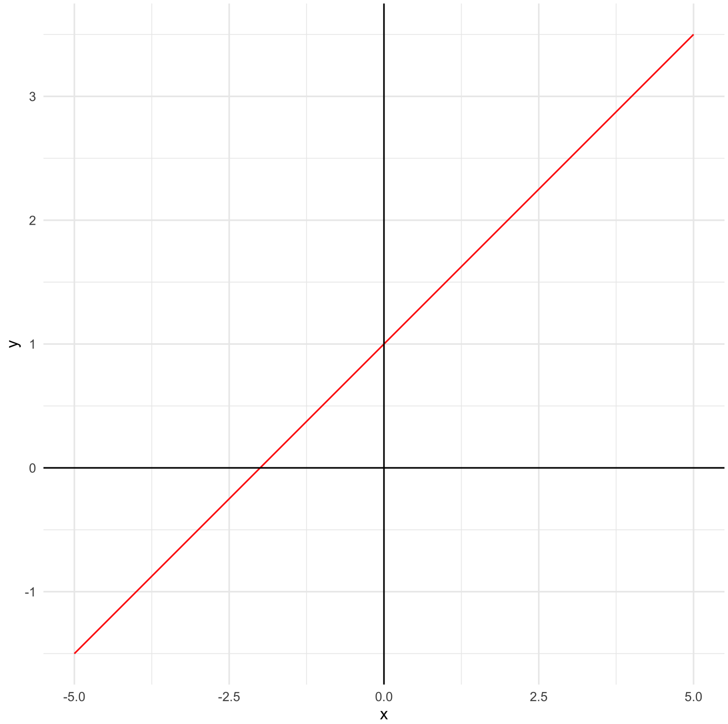


Let's consider the function, $y = 3$, plotted to the left. What is its "slope"?

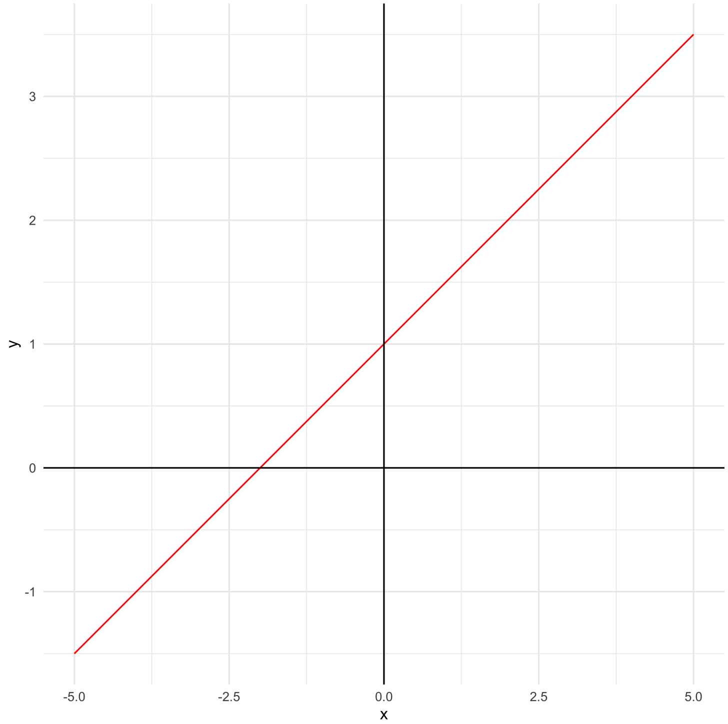
Its slope or $\frac{\Delta f(x)}{\Delta x} = 0$ because there is no "rise".

Slope



Let's consider a less simple function, $y = \frac{1}{2}x + 1$, plotted to the left. What is its "slope"?

Slope

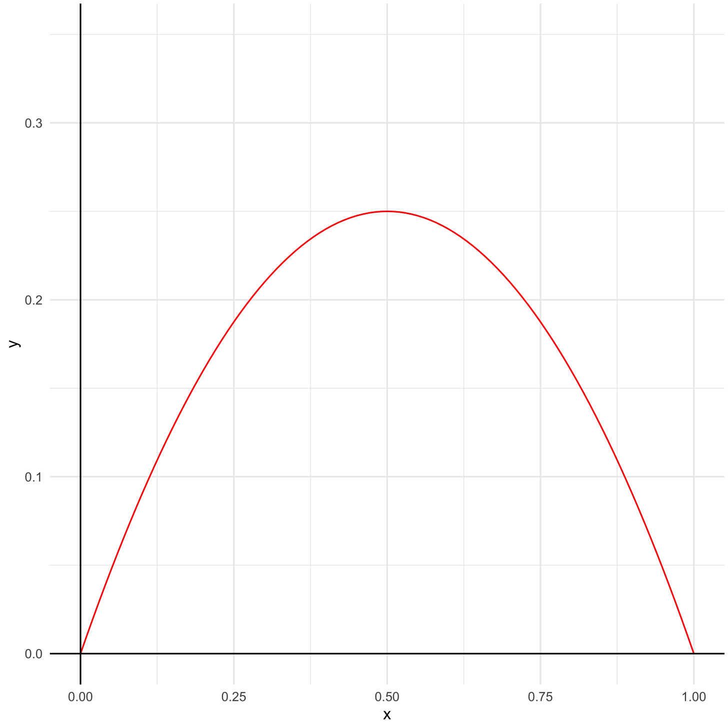


Let's consider a less simple function, $y = \frac{1}{2}x + 1$, plotted to the left. What is its "slope"?

Its slope or $\frac{\Delta f(x)}{\Delta x} = \frac{1}{2}$.

[Recall: $y = mx + b$ from Day 1]

Slope



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What is its "slope", or $\frac{\Delta f(x)}{\Delta x}$?

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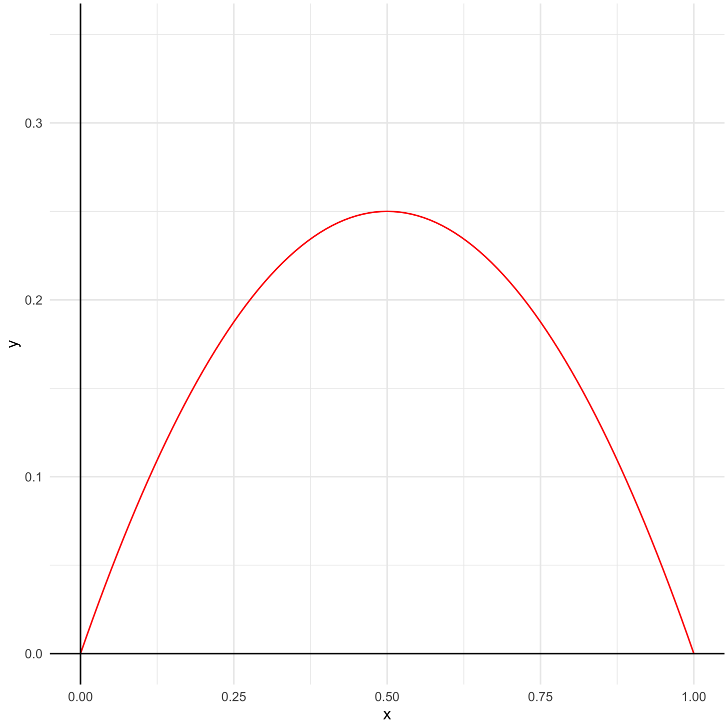

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How do we even calculate this?

Derivatives as Limits

We can approximate the slope at a certain location by picking a point nearby on the line and finding the slope of the straight line connecting these two points.

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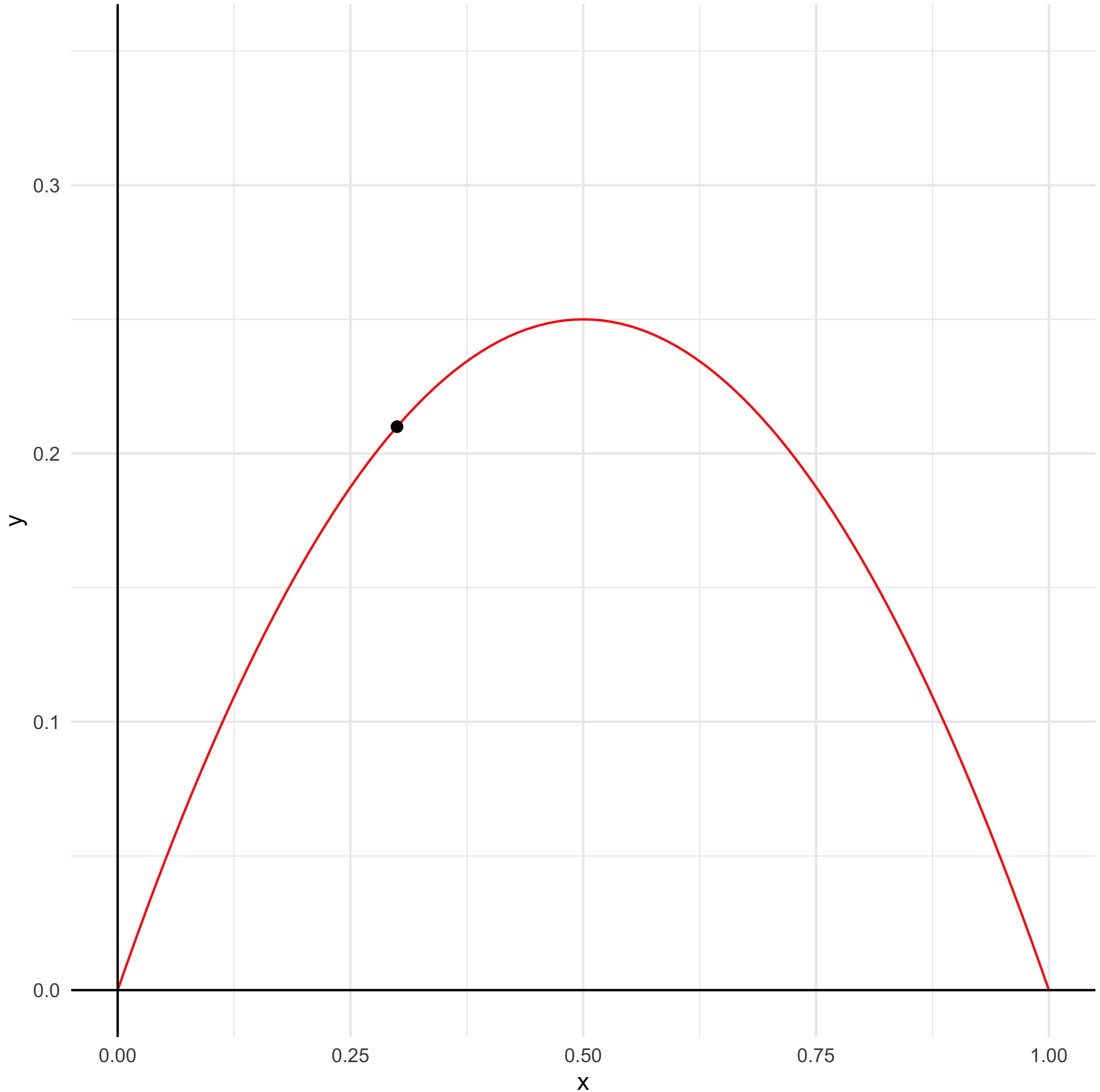
Let's consider a few examples.



0.3
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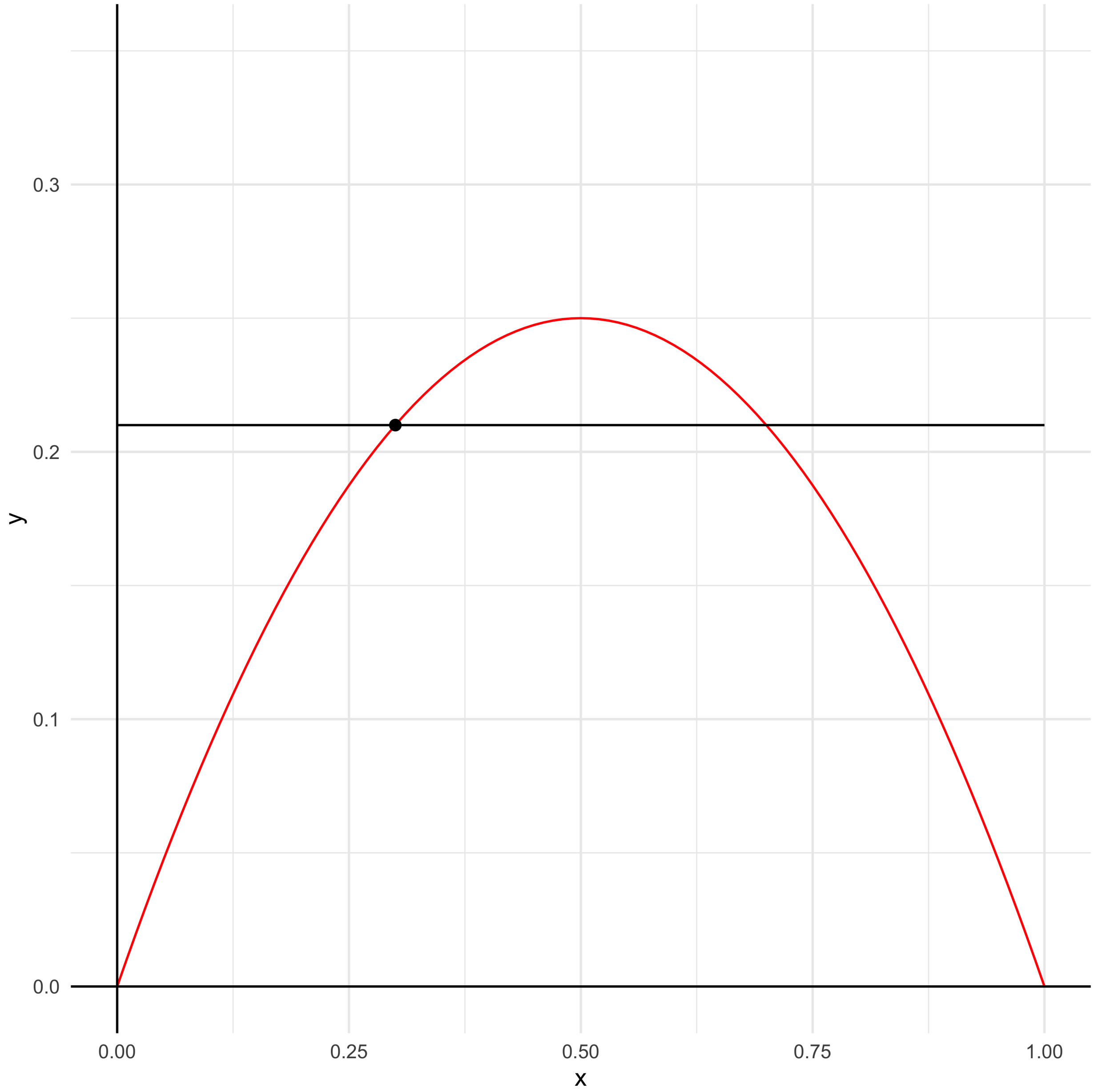
1.00
x



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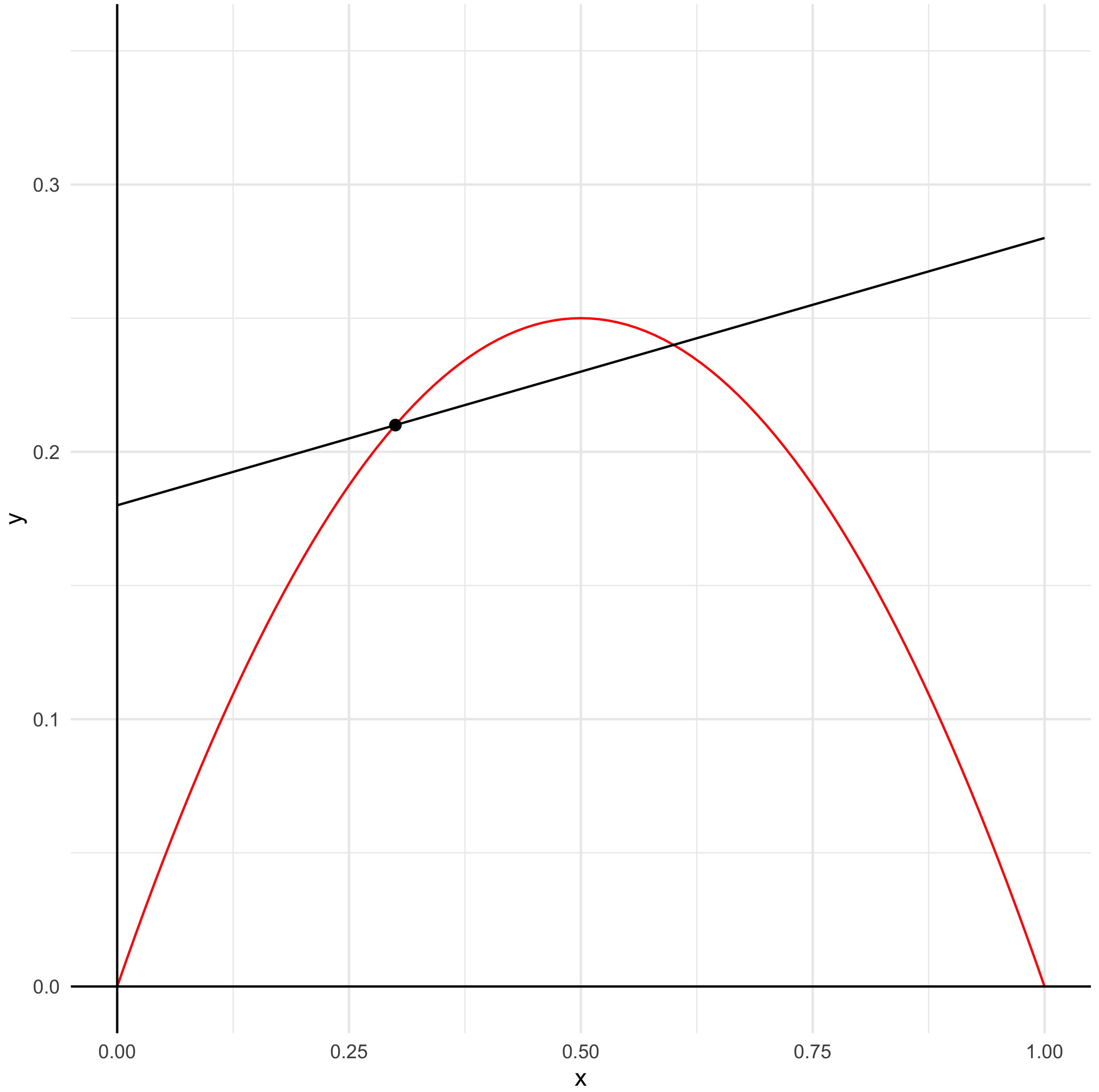
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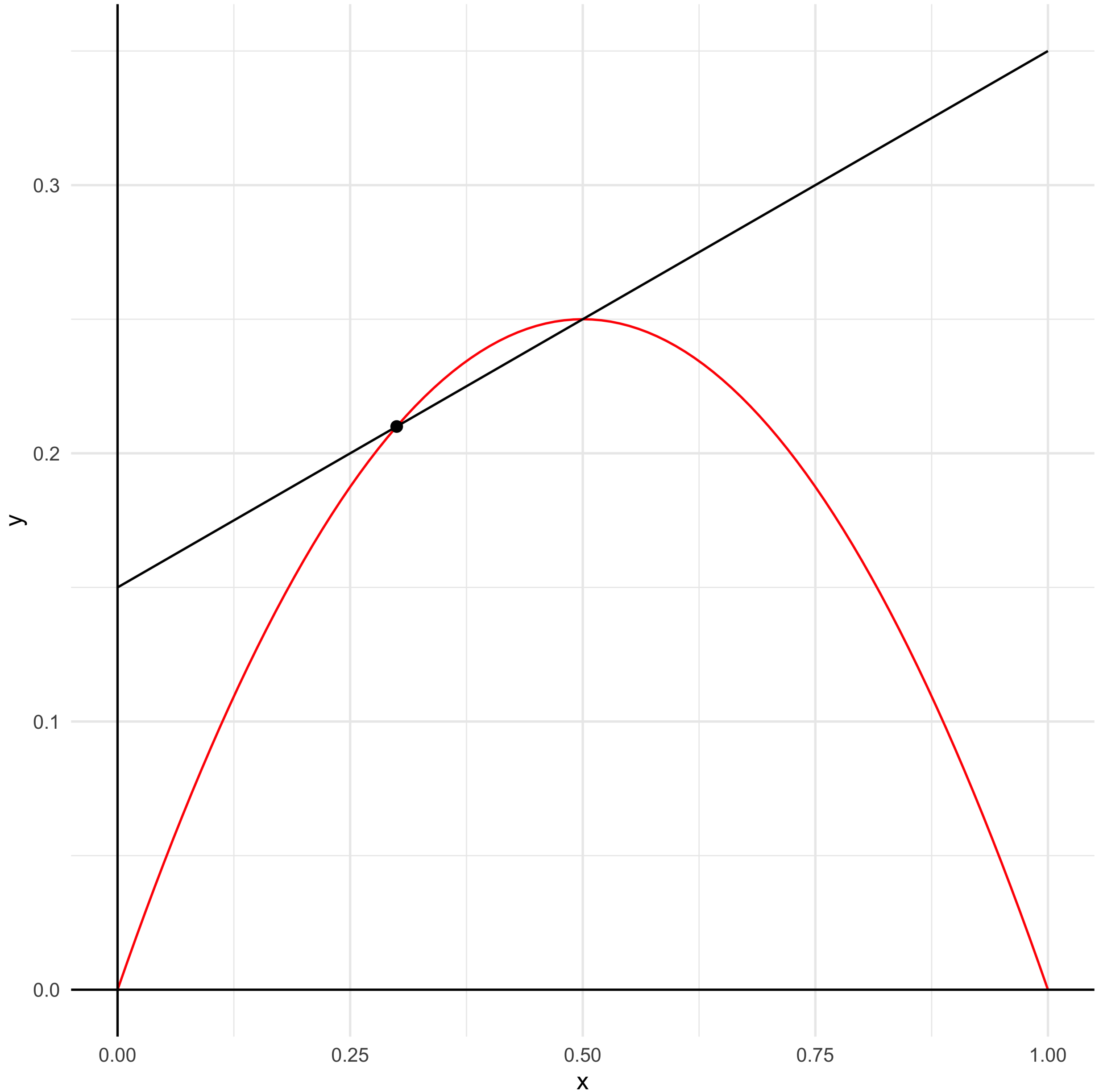
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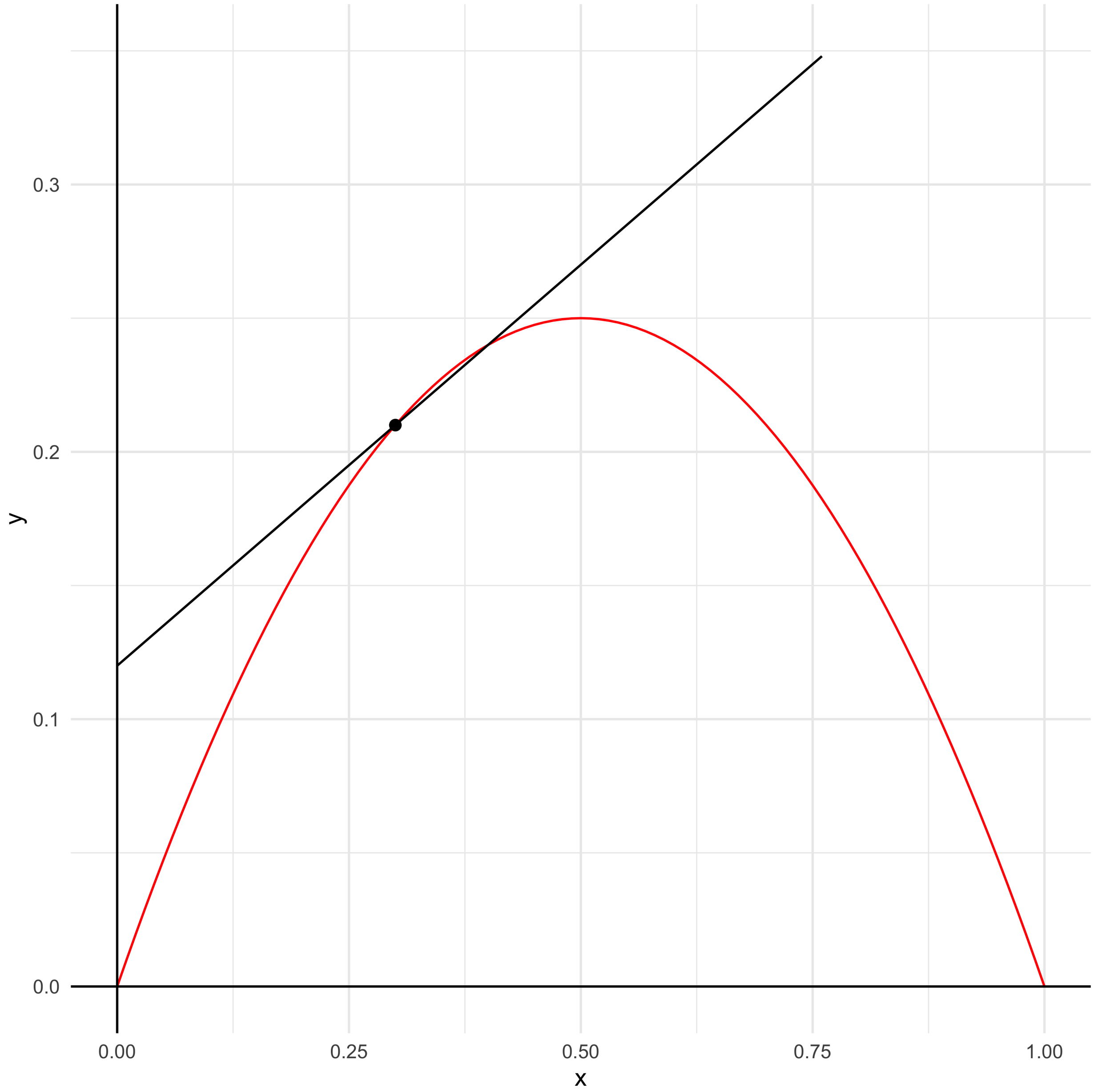

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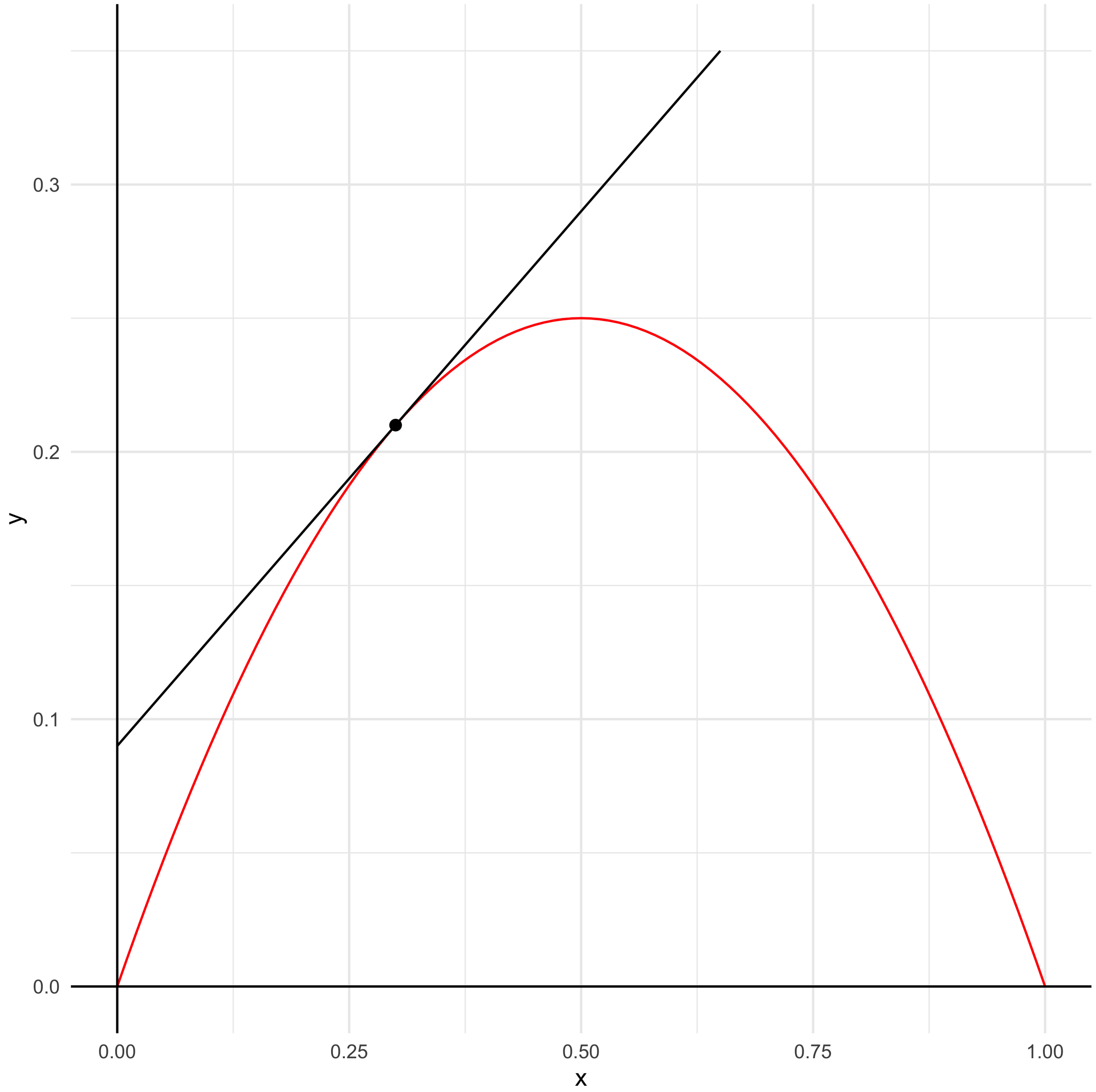

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So, as you reduce the interval size to 0, this line converges in the limit to the line that lies tangent to the curve at that point.

Recall that $f(x) = x - x^2$ and consider a very small interval $\epsilon \ldots$

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\end{aligned}$$

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\end{aligned}
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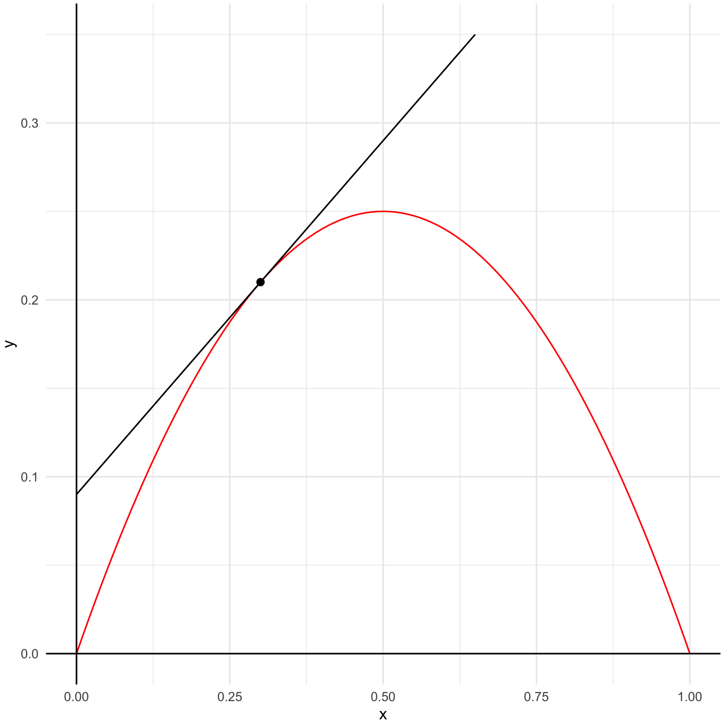

Using this formula, the slope of the curve at $x = .3$ (the point from the previous examples) is exactly:

$$\begin{aligned} \text{slope} &= 1 - 2(.3) \\ &= 0.4 \end{aligned}$$

Or, if we want to find the point at which the slope is 0 (rate of change is 0):

$$\begin{aligned} 0 &= 1 - 2x \\ 2x &= 1 \\ x &= 0.5 \end{aligned}$$

Derivatives

Now, we can formally state that the derivative is equivalent to:

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$$\frac{d[f(x)]}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{(x+\epsilon) - x}$$

Using this approach, we can find:

- A general equation for the slope at any point
- The exact value of the slope at a given point
- The point that has a given slope

Derivative notation

Different notation systems for expressing derivatives:

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First derivative: $\frac{d[f(x)]}{dx}$, $\frac{d}{dx}[f(x)]$, $f'(x)$, or $f^1(x)$

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We'll be using $f'(x)$ and $f''(x)$ but you are free to choose whatever makes sense to you!

Cautionary Notes on Derivatives

A few assumptions in using this approach to find the slope:

- The function is continuous (no gaps or jumps)
- The derivative exists (the limit of the slope is the same from the left as it is from the right), or *no sharp corners*.

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For nearly all political science applications, these are fine assumptions. But it is important to state them explicitly and be aware that they're there.

Straightforward Derivatives

Fortunately (for humans), it is not necessary to take limits with ϵ's to find derivatives every time. A few rules generate the derivatives of most functions.

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If $f(x) = cx^n$, then its derivative is $f'(x) = ncx^{n-1}$.

The simplest example: consider the line $f(x) = 3x$. What's its derivative?

Straightforward Derivatives

Fortunately (for humans), it is not necessary to take limits with ϵ's to find derivatives every time. A few rules generate the derivatives of most functions.

The first rule is the *power rule*.

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f'(x) &= (3x)' \\
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A special case of the power rule is that the *derivative of a constant is zero*.

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$$\begin{aligned} h'(x) &= \left(\frac{1}{2}\right)7x^{\frac{1}{2}-1} \\ &= \frac{7}{2}x^{-\frac{1}{2}} \end{aligned}$$

Exercises

Find the following derivatives, and calculate the instantaneous slope of the curves at the point $x = 2$:

$f(x) = \frac{1}{4}x^4$

$g(x) = \frac{2}{x^3}$ [Hint: How else can we express fractions?]

$h(x) = 4x^{\frac{5}{2}}$

$j(x) = \sqrt[3]{x}$ [Hint: How else can we express roots?]

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$$\begin{aligned} f'(x) + g'(x) &= (4x^2)' + (5x^3)' \\ &= (2)4x^{2-1} + (3)5x^{3-1} \\ &= 8x + 15x^2 \end{aligned}$$

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Let $f(x)=x$ and $g(x)=x^2$. What is $(f(x)-g(x))'$?

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Let $f(x) = x$ and $g(x) = x^2$. What is $(f(x) - g(x))'$?

$$\begin{aligned} f'(x) - g'(x) &= (x)' - (x^2)' \\ &= (1)x^{1-1} - (2)x^{2-1} \\ &= 1 - 2x \end{aligned}$$

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$$\begin{aligned} h'(x) &= (5x^5 - 10x^3 + 6x^2 - 3)' \\ &= (5x^5)' - (10x^3)' + (6x^2)' - (3)' \\ &= 5 \times 5x^{5-1} - 3 \times 10x^{3-1} + 2 \times 6x^{2-1} - 0 \\ &= 25x^4 - 30x^2 + 12x \end{aligned}$$

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&= 25x^4 - 30x^2 + 12x
\end{aligned}
$$

What is the rate of change when x is equal to one?

$$
\begin{aligned}
h'(1) &= 25x^4 - 30x^2 + 12x \\
h'(1) &= 25(1)^4 - 30(1)^2 + 12(1) \\
h'(1) &= 7
\end{aligned}
$$

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This is **the product rule**.

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&= (2x)(x^3-4x) + (x^2+1)(3x^2-4) \\
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This is easy to check by multiplying out the polynomial:
$(x^2+1)(x^3-4x) = x^5 - 3x^3 - 4x$, and then taking its derivative:

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or the bottom *times* the derivative of the top *minus* the top *times* the derivative of the bottom, all *divided* by the bottom *squared*.

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This is referred to as **the quotient rule**.

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\left(\frac{f(x)}{g(x)}\right)' &= \left(\frac{x^2+1}{x^3-4x}\right)' \\
&= \frac{(x^2+1)'(x^3-4x) - (x^2+1)(x^3-4x)'}{(x^3-4x)^2} \\
&= \frac{(2x)(x^3-4x) - (x^2+1)(3x^2-4)}{(x^3-4x)^2} \\
&= \frac{2x^4 - 8x^2 - (3x^4 - 4x^2 + 3x^2 - 4)}{(x^3-4x)^2} \\
&= \frac{-x^4 - 7x^2 + 4}{(x^3-4x)^2}
\end{aligned}$$

Exercises

Find the derivatives of the following expressions:

$$(3x^2 - 4x + 2)(x^3 - x^2 + x - 1)$$

$$\frac{4x+1}{3x^2-2}$$

Derivatives of Nested Functions

Let's say $h(x) = f(g(x))$.

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This looks messy, but is actually fairly straightforward and extremely useful as a way to find derivatives of complex functions by treating them as nested chains of functions.

Derivatives of Nested Functions

Let $h(x) = 6(3x^2 + 2)^4$. Observe that this can be thought of as two nested functions, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $h(x) = f(g(x))$.

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What is $h'(x)$?

$$\begin{aligned}
h(x)' = (f(g(x)))' &= (6(3x^2 + 2)^4)' \\
&= (4)6(3x^2 + 2)^{4-1}(3x^2 + 2)' \\
&= 24(3x^2 + 2)^3(6x) \\
&= 144x(3x^2 + 2)^3
\end{aligned}$$

Derivatives of Nested Functions

Let $k(x) = 3(6x^4)^2 + 2$. Observe that this can be thought of the same two functions nested in the reverse order, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $k(x) = g(f(x))$.

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Derivatives of Nested Functions

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What is $k'(x)$?

$$\begin{aligned} k(x)' = (g(f(x)))' &= (3(6x^4)^2 + 2)' \\ &= (3(6x^4)^2)' + (2)' \\ &= (2)3(6x^4)^{2-1}(6x^4)' + 0 \\ &= (2)3(6x^4)^{2-1}(24x^{4-1}) \\ &= (2)3(6x^4)(24x^3) \\ &= 864x^7 \end{aligned}$$

Exercises

Express the functions below as the nested result of two simpler functions, and use the chain rule to find the derivative:

$(3x - 1)^4$

$2(x^4 + x^3) + 7$

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$$\begin{aligned}(log_e(x))' = (ln(x))' &= \frac{1}{ln(e)x} \\ &= \frac{1}{x}\end{aligned}$$

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Let $g(x) = ln(3x^2 + 4)$. What is $g'(x)$?

(using the chain rule):

$$\begin{aligned} g'(x) = (ln(3x^2 + 4))' &= \frac{1}{3x^2 + 4} \times (3x^2 + 4)' \\ &= \frac{6x}{3x^2 + 4} \end{aligned}$$

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$$\begin{aligned}(e^x)' &= ln(e)e^x \\ &= 1 \times e^x \\ &= e^x\end{aligned}$$

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$$\begin{aligned} g'(x) = (2^{3x})' &= ln(2) \times 2^{3x} \times (3x)' \\ &= 3ln(2) \times 2^{3x} \end{aligned}$$

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Consider the limit: $\lim_{x \to 2} \frac{x^2+x-6}{x^2-4}$

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If both the numerator and the denominator are 0, ∞, or $-\infty$, we cannot evaluate the limit.

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If both the numerator and the denominator are 0, ∞, or $-\infty$, we cannot evaluate the limit.

However, L'Hôpital's Rule says: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Then,

$$\begin{aligned}
\lim_{x \to 2} \frac{x^2+x-6}{x^2-4} &= \lim_{x \to 2} \frac{(x^2+x-6)'}{(x^2-4)'} \\
&= \lim_{x \to 2} \frac{2x+1}{2x} \\
&= 1.25
\end{aligned}$$

End Day 3