Math Camp Lesson 3 (Day 1)

Calculus

UW–Madison Political Science

August 19, 2020

Overview

Calculus evaluates the behavior of functions:

- Limits
- Rate of change
- Change in the rate of change
- Area of the region they defined on

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Concepts from calculus underlie a wide variety of mathematics, particularly in the applied math that we use in political science

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Solving for the choice that maximizes a decision maker's utility.

Agenda

Day 1

- Limits
- Derivatives

Day 2

- More Derivatives
- Integrals
- Applications

Limits

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The limit of a function characterizes its behavior given a certain input, or as an input value changes.

Limits: Example 1

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Let's consider the simple function, $f(x) = y = 3 - (x-2)^2$, plotted to the left. What is the limit of $f(x)$ as x *approaches* 2?

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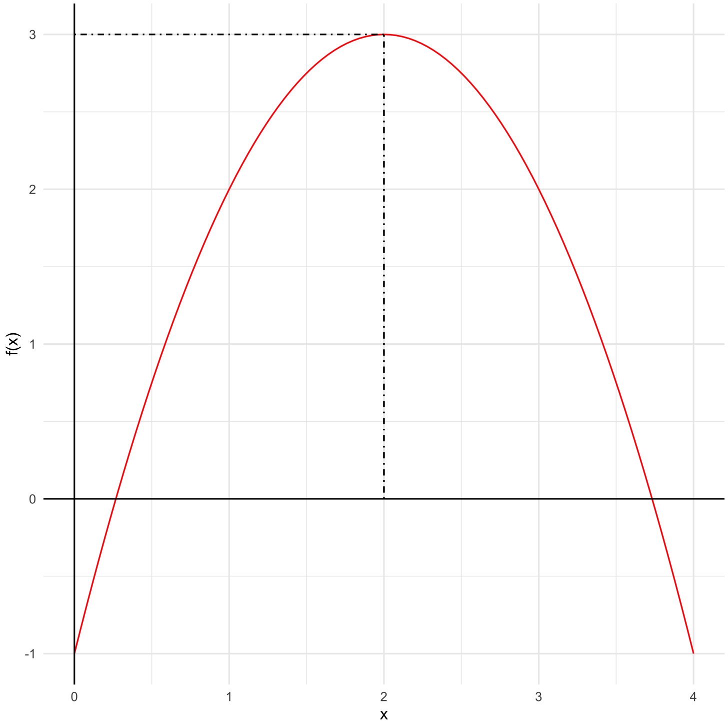

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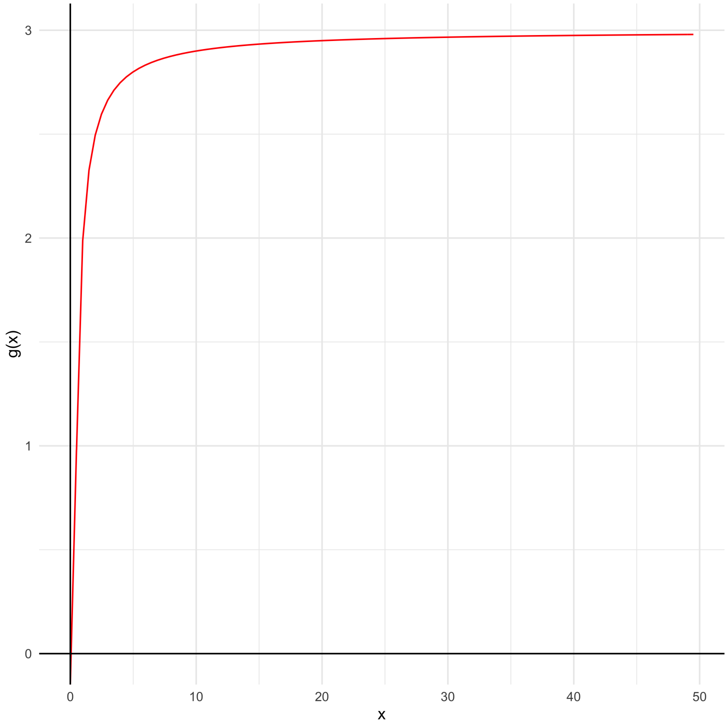
As x approaches 2, $f(x)$ or y approaches $f(2) = 3$.

Limits: Example 2



Let's consider a less simple function, $g(x) = y = 3 - \frac{1}{x}$, plotted to the left. What is the limit of $g(x)$ as x *approaches* ∞?

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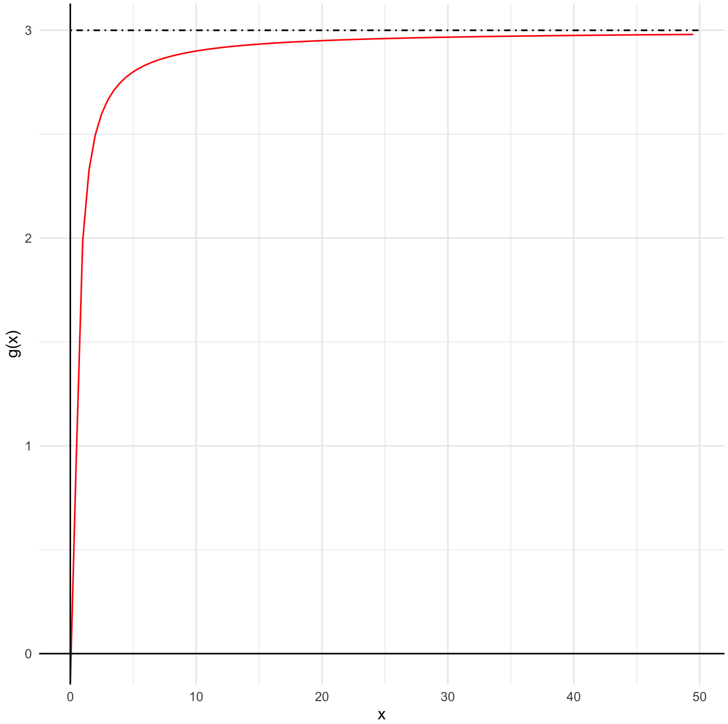

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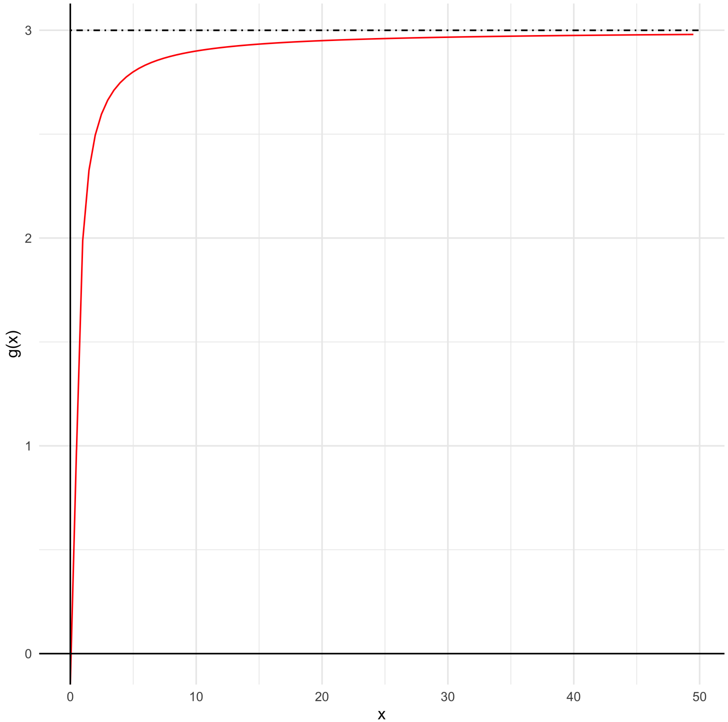

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As x gets larger, $\frac{1}{x}$ gets smaller and smaller.

$$\left(\frac{1}{2} > \frac{1}{20} > \frac{1}{200} \dots\right)$$

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Many times, you will see this expression written as $\lim_{x \to c^-} f(x) = L$ or $\lim_{x \to c^+} f(x) = L$.

A negative sign $(-)$ implies "As x approaches c *from the left*"

A positive sign $(+)$ implies "As x approaches c *from the right*"

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For functions that are well-behaved, the limit as x approaches a finite point is generally the value of the function at that point (if it exists).

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$$\begin{aligned} \lim_{x \to 2} x^2 - 3x + 1 &= \lim_{x \to 2} x^2 - 3 \lim_{x \to 2} x + \lim_{x \to 2} 1 \\ &= 2^2 - 3(2) + 1 \\ &= -1 \end{aligned}$$

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\lim_{x \to \infty} \frac{4x^4 + 7x^2 + 8}{3x^4} &= \lim_{x \to \infty} \frac{4x^4}{3x^4} + \lim_{x \to \infty} \frac{7x^2}{3x^4} + \lim_{x \to \infty} \frac{8}{3x^4} \\
&= \lim_{x \to \infty} \frac{4}{3} + \lim_{x \to \infty} \frac{7}{3x^2} + \lim_{x \to \infty} \frac{8}{3x^4} \\
&= \frac{4}{3} + 0 + 0 \\
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Why does $\lim_{x\to\infty} \frac{7}{3x^2} = 0$? As $x \to \infty$, $3x^2 \to \infty$, and $\frac{7}{\infty} \to 0$.

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&= \frac{4}{3} + \frac{7}{3} \times \frac{1}{0} + \frac{8}{3} \times \frac{1}{0} \\
&= \dots
\end{aligned}$$

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&= \infty
\end{aligned}$$

As x approaches 0, the function retains some positive value in the numerator while the denominator *positively* approaches 0. This means that you are dividing by a smaller and smaller fraction, which means the entire term is getting larger and approaches ∞.

Exercises

Find the following limits:

$\lim_{x \to 4} x^2 - 6x + 4$

$\lim_{x \to 4} \frac{x^2}{3x-2}$

$\lim_{x \to \infty} \frac{3x-4}{x+3}$

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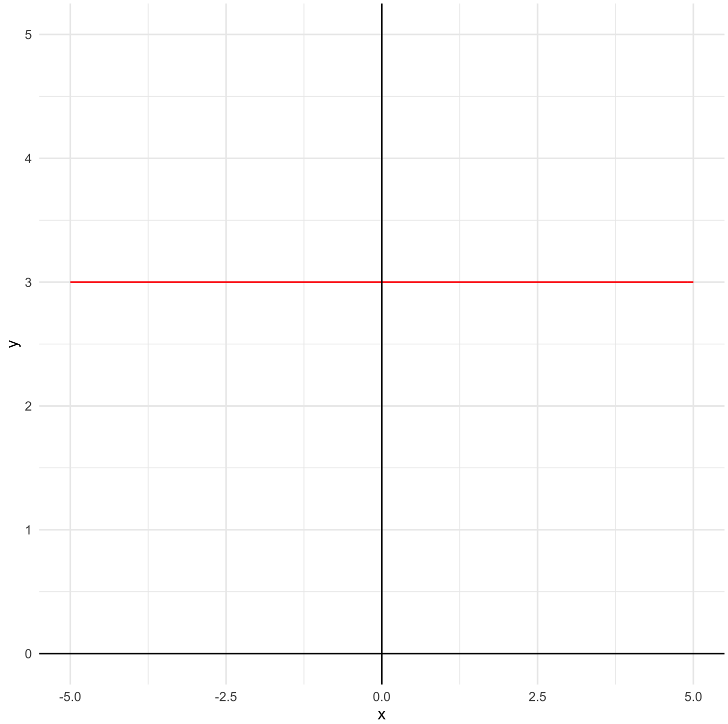
The slope of a function is how much the output changes as a result of changes in the input.

Using Δ to signify ‘change’, this is $\frac{\Delta f(x)}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ or "rise-over-run".

Slope



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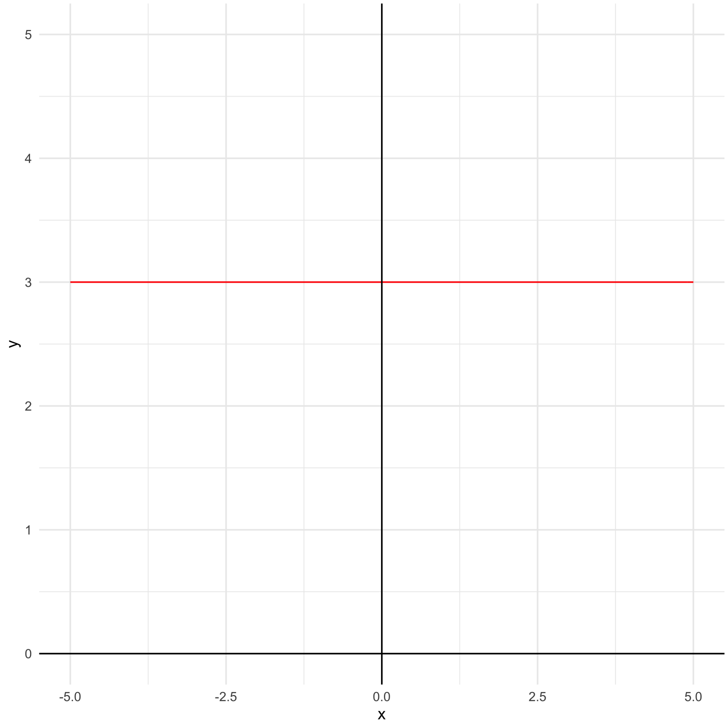

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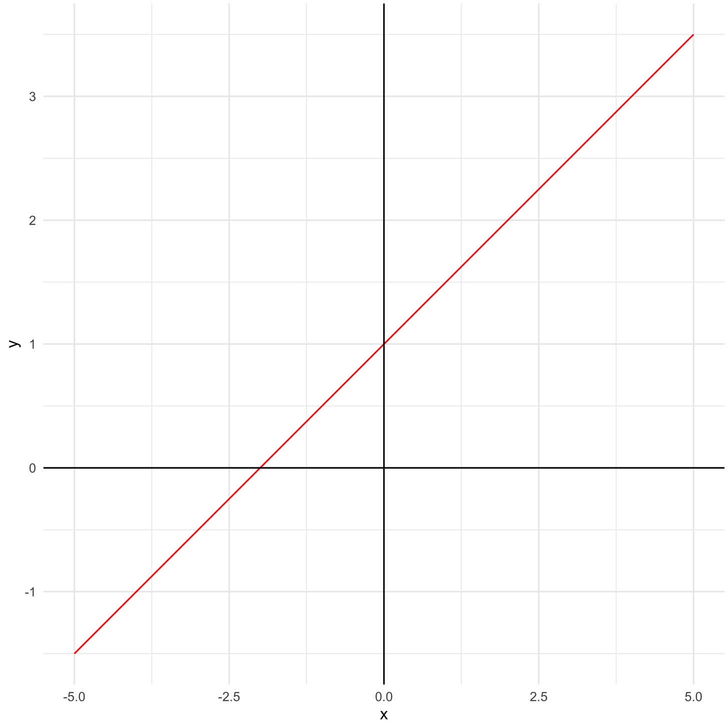
Its slope or $\frac{\Delta f(x)}{\Delta x} = 0$ because there is no "rise".

Slope



Let's consider a less simple function, $y = \frac{1}{2}x + 1$, plotted to the left. What is its "slope"?

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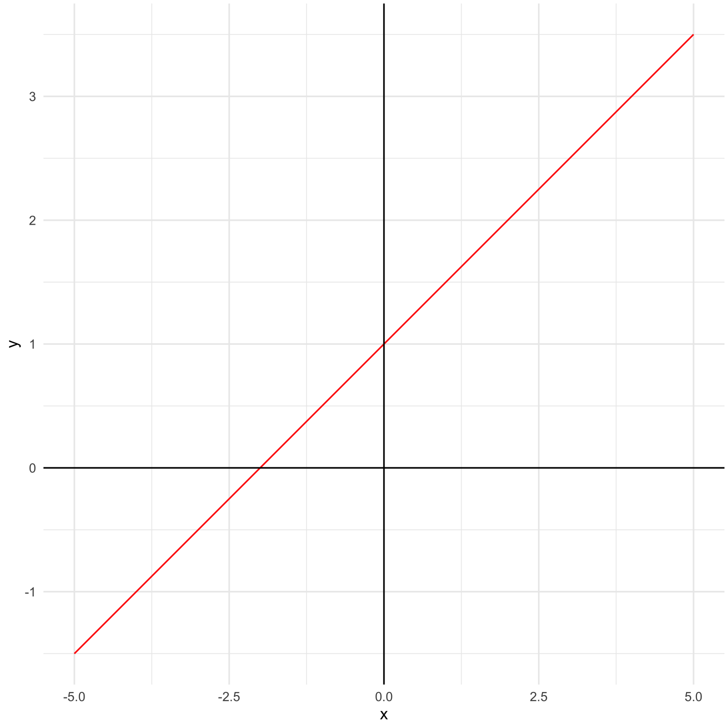
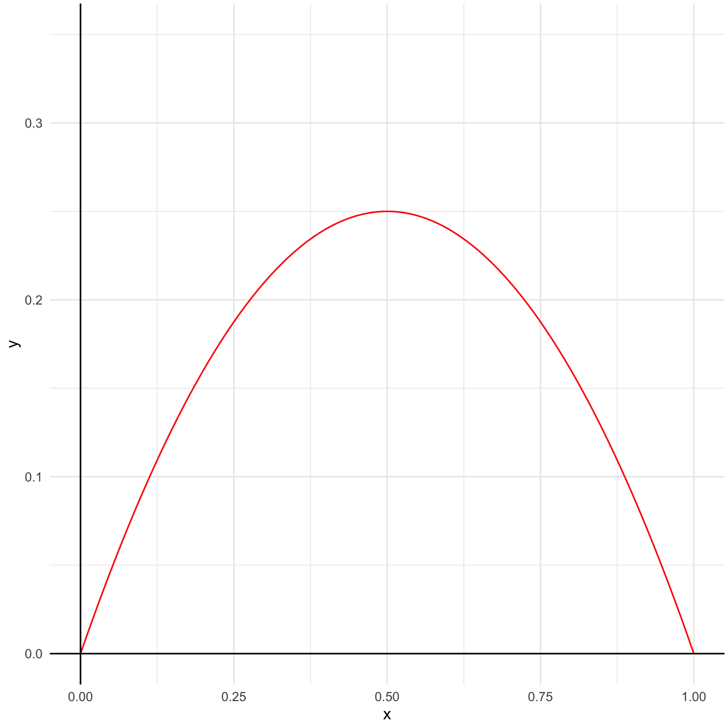


Let's consider a less simple function, $y = \frac{1}{2}x + 1$, plotted to the left. What is its "slope"?

Its slope or $\frac{\Delta f(x)}{\Delta x} = \frac{1}{2}$.

[Recall: $y = mx + b$ from Day 1]

Slope



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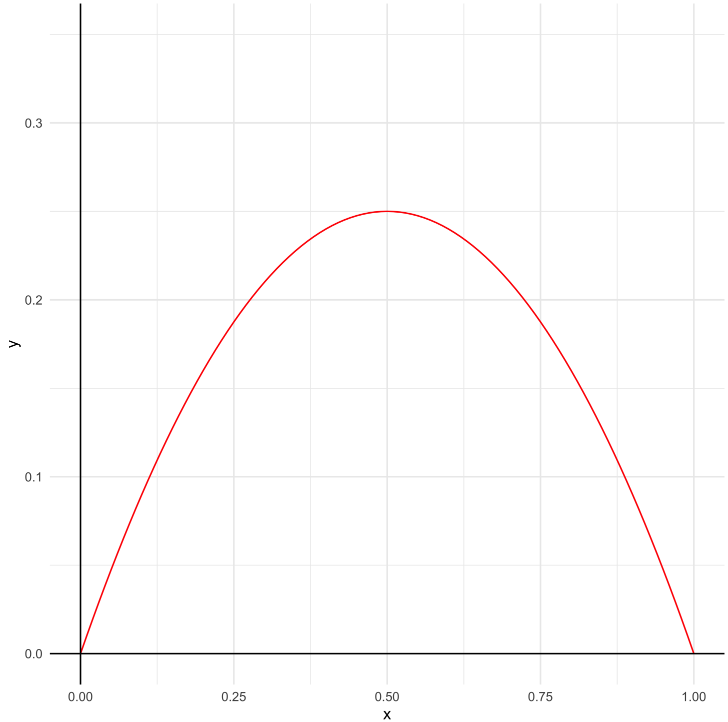

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How do we even calculate this?

Derivatives as Limits

We can approximate the slope at a certain location by picking a point nearby on the line and finding the slope of the straight line connecting these two points.

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Let's consider a few examples.

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$$= \lim_{\epsilon \to 0} \frac{\epsilon - 2\epsilon x - \epsilon^2}{\epsilon}$$

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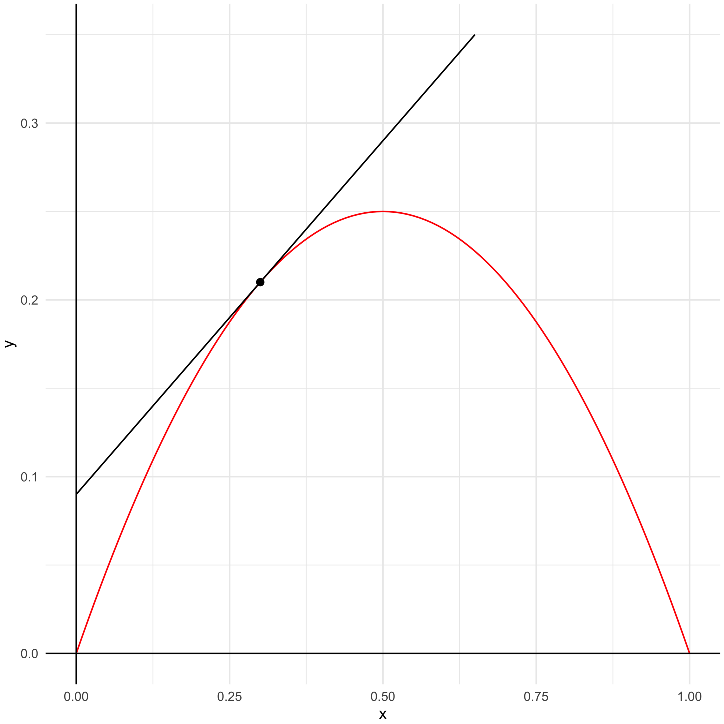
Derivative as Limits



Using this formula, the slope of the curve at $x = .3$ (the point from the previous examples) is exactly:

$$\begin{aligned} \text{slope} &= 1 - 2(.3) \\ &= 0.4 \end{aligned}$$

Or, if we want to find the point at which the slope is 0 (rate of change is 0):

$$\begin{aligned} 0 &= 1 - 2x \\ 2x &= 1 \\ x &= 0.5 \end{aligned}$$

Derivatives

Now, we can formally state that the derivative is equivalent to:

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Using this approach, we can find:

- A general equation for the slope at any point
- The exact value of the slope at a given point
- The point that has a given slope

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We'll be using $f'(x)$ and $f''(x)$ but you are free to choose whatever makes sense to you!

Cautionary Notes on Derivatives

A few assumptions in using this approach to find the slope:

- The function is continuous (no gaps or jumps)
- The derivative exists (the limit of the slope is the same from the left as it is from the right), or *no sharp corners*.

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For nearly all political science applications, these are fine assumptions. But it is important to state them explicitly and be aware that they're there.

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A special case of the power rule is that the *derivative of a constant is zero*.

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Let $h(x) = 7x^{\frac{1}{2}}$, then:

$$\begin{aligned} h'(x) &= \left(\frac{1}{2}\right) 7x^{\frac{1}{2}-1} \\ &= \frac{7}{2}x^{-\frac{1}{2}} \end{aligned}$$

Exercises

Find the following derivatives, and calculate the instantaneous slope of the curves at the point $x = 2$:

$f(x) = \frac{1}{4}x^4$

$g(x) = \frac{2}{x^3}$ [Hint: How else can we express fractions?]

$h(x) = 4x^{\frac{5}{2}}$

$j(x) = \sqrt[3]{x}$ [Hint: How else can we express roots?]

Derivative of a Sum (or Difference)

The derivative of a sum (difference) is the sum (difference) of the derivatives.

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$$\begin{aligned} f'(x) + g'(x) &= (4x^2)' + (5x^3)' \\ &= (2)4x^{2-1} + (3)5x^{3-1} \\ &= 8x + 15x^2 \end{aligned}$$

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Let $f(x) = x$ and $g(x) = x^2$. What is $(f(x) - g(x))'$?

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Let $f(x)=x$ and $g(x)=x^2$. What is $(f(x)-g(x))'$?

$$\begin{aligned} f'(x)-g'(x) &= (x)' - (x^2)' \\ &= (1)x^{1-1} - (2)x^{2-1} \\ &= 1 - 2x \end{aligned}$$

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$$\begin{aligned} h'(x) &= (5x^5 - 10x^3 + 6x^2 - 3)' \\ &= (5x^5)' - (10x^3)' + (6x^2)' - (3)' \\ &= 5 \times 5x^{5-1} - 3 \times 10x^{3-1} + 2 \times 6x^{2-1} - 0 \\ &= 25x^4 - 30x^2 + 12x \end{aligned}$$

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What is the rate of change when x is equal to one?

$$\begin{aligned} h'(1) &= 25x^4 - 30x^2 + 12x \\ h'(1) &= 25(1)^4 - 30(1)^2 + 12(1) \\ h'(1) &= 7 \end{aligned}$$

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This is **the product rule**.

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&= (2x)(x^3 - 4x) + (x^2 + 1)(3x^2 - 4) \\
&= 2x^4 - 8x^2 + 3x^4 - 4x^2 + 3x^2 - 4 \\
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This is easy to check by multiplying out the polynomial:
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This is easy to check by multiplying out the polynomial: $(x^2+1)(x^3-4x) = x^5 - 3x^3 - 4x$, and then taking its derivative:

$$(x^5 - 3x^3 - 4x)' = 5x^4 - 9x^2 - 4$$

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This is referred to as **the quotient rule**.

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\left(\frac{f(x)}{g(x)}\right)' &= \left(\frac{x^2+1}{x^3-4x}\right)' \\
&= \frac{(x^2+1)'(x^3-4x) - (x^2+1)(x^3-4x)'}{(x^3-4x)^2} \\
&= \frac{(2x)(x^3-4x) - (x^2+1)(3x^2-4)}{(x^3-4x)^2} \\
&= \frac{2x^4 - 8x^2 - (3x^4 - 4x^2 + 3x^2 - 4)}{(x^3-4x)^2} \\
&= \frac{-x^4 - 7x^2 + 4}{(x^3-4x)^2}
\end{aligned}$$

Exercises

Find the derivatives of the following expressions:

$$(3x^2 - 4x + 2)(x^3 - x^2 + x - 1)$$

$$\frac{4x+1}{3x^2-2}$$

Derivatives of Nested Functions

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This looks messy, but is actually fairly straightforward and extremely useful as a way to find derivatives of complex functions by treating them as nested chains of functions.

Derivatives of Nested Functions

Let $h(x) = 6(3x^2 + 2)^4$. Observe that this can be thought of as two nested functions, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $h(x) = f(g(x))$.

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$$\begin{aligned} h(x)' = (f(g(x)))' &= (6(3x^2 + 2)^4)' \\ &= (4)6(3x^2 + 2)^{4-1}(3x^2 + 2)' \\ &= 24(3x^2 + 2)^3(6x) \\ &= 144x(3x^2 + 2)^3 \end{aligned}$$

Derivatives of Nested Functions

Let $k(x) = 3(6x^4)^2 + 2$. Observe that this can be thought of the same two functions nested in the reverse order, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $k(x) = g(f(x))$.

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Derivatives of Nested Functions

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What is $k'(x)$?

$$\begin{aligned} k(x)' = (g(f(x)))' &= (3(6x^4)^2 + 2)' \\ &= (3(6x^4)^2)' + (2)' \\ &= (2)3(6x^4)^{2-1}(6x^4)' + 0 \\ &= (2)3(6x^4)^{2-1}(24x^{4-1}) \\ &= (2)3(6x^4)(24x^3) \\ &= 864x^7 \end{aligned}$$

Exercises

Express the functions below as the nested result of two simpler functions, and use the chain rule to find the derivative:

$$(3x - 1)^4$$

$$2(x^4 + x^3) + 7$$

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$$\begin{aligned}(log_e(x))' = (ln(x))' &= \frac{1}{ln(e)x} \\ &= \frac{1}{x}\end{aligned}$$

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Let $g(x) = ln(3x^2 + 4)$. What is $g'(x)$? (using the Chain Rule):

$$\begin{aligned} g'(x) = (ln(3x^2 + 4))' &= \frac{1}{3x^2 + 4} \times (3x^2 + 4)' \\ &= \frac{6x}{3x^2 + 4} \end{aligned}$$

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$$\begin{aligned}(e^x)' &= ln(e)e^x \\ &= 1 \times e^x \\ &= e^x\end{aligned}$$

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$$\begin{aligned} g'(x) = (2^{3x})' &= ln(2) \times 2^{3x} \times (3x)' \\ &= 3ln(2) \times 2^{3x} \end{aligned}$$

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Consider the limit: $\lim_{x \to 2} \frac{x^2+x-6}{x^2-4}$.

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If both the numerator and the denominator are 0, ∞, $-\infty$, we cannot evaluate the limit.

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If both the numerator and the denominator are 0, ∞, $-\infty$, we cannot evaluate the limit.

However, L'Hôpital's Rule says:

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Then,

$$\begin{aligned} \lim_{x\to 2} \frac{x^2+x-6}{x^2-4} &= \lim_{x\to 2} \frac{(x^2+x-6)'}{(x^2-4)'} \\ &= \lim_{x\to 2} \frac{2x+1}{2x} & = 1.25 \end{aligned}$$

End Day 3