Exercise 7.1

Applying power series method, solve the following differential equations.

(1) $y' = 3y$

Solution: Given differential equation is,

$$y' = 3y \quad \text{.........(i)}$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \quad \ldots \text{(ii)}$$

be the solution of (i).

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots\ldots$$

Putting the value of y and y' in (i),

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots\ldots = 3a_0 + 3a_1x + 3a_2x^2 + 3a_3x^3 + 3a_4x^3 +$$

.... comparing coefficient of constant term, x, x^2

$a_1 = 3a_0$, $\quad 2a_2 = 3a_1 \quad 3a_3 = 3a_2 \quad 4a_4 = 3a_3 \quad$ and so on.

$$\Rightarrow a_2 = \frac{3}{2}a \qquad \Rightarrow a_3 = a_2 \qquad \Rightarrow a_4 = \frac{3}{4}a$$

$$= \frac{9}{2}a_0 \qquad = \frac{9}{2}a_0 \qquad = \frac{3}{4}\times\frac{9}{2}a_0 = \frac{27}{8}a_0$$

Putting the value of a_1, a_2, a_3 and a_4 in (ii),

$$y = a_0 + 3a_0x + \frac{9a_0}{2}x^2 + \frac{9a_0}{2}x^3 + \frac{27}{8}a_0x^4 + \ldots\ldots$$

$$= a_0\left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \ldots\ldots\right)$$

$$= a_0e^{3x}$$

(2) $y' + 2y = 0$.

Solution: Given differential equation is,

$$y' + 2y = 0 \qquad \ldots (i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad \ldots (ii)$$

be solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots\ldots\ldots$$

Putting the value of y an y' in (i) then,

$$a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots + 2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + \ldots\ldots = 0$$

$$\Rightarrow (a_1 + 2a_0) + x(2a_2 + 2a_1) + x^2(3a_3 + 2a_2) + \ldots\ldots\ldots = 0$$

Equating each coefficient to zero,

$$a_1 + 2a_0 = 0 \qquad 2a_2 + 2a_1 = 0 \qquad 3a_3 + 2a_2 = 0 \quad \text{and so on.}$$

$$\Rightarrow a_1 = -2a_0 \qquad \Rightarrow a_2 = -a_1 = 2a_0 \qquad \Rightarrow a_3 = -\frac{2}{3}a_2 = -\frac{2}{3}\times 2a_0 = -\frac{4}{3}a_0$$

Substituting the value of $a_1, a_2, a_3, \ldots$ in (ii) then,

$$y = a_0 - 2a_0x + 2a_0x^2 - \frac{4}{3}a_0x^3 + \ldots\ldots\ldots$$

$$= a_0\left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots\ldots\right)$$

$$= a_0e^{-2x}$$

(3) $y' - y = 0$.

Solution: Given differential equation is,

$$y' - y = 0 \qquad \ldots\ldots(i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad \ldots\ldots\ldots (ii)$$

be the solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots$$

Putting the value of y and y' in (i) then

$$a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots\ldots - a_0 - a_1x - a_2x^2 - a_3x^3 \ldots\ldots\ldots\ldots = 0$$

$$\Rightarrow (a_1 - a_0) + x(2a_2 - a_1) + x^2(3a_3 - a_2) + \ldots\ldots\ldots = 0.$$

Equating coefficient of like terms from both sides then,

$$a_1 - a_0 = 0, \qquad 2a_2 - a_1 = 0 \qquad 3a_3 - a_2 = 0 \text{ and so on.}$$

$$\Rightarrow a_1 = a_0 \qquad \Rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2} \qquad \Rightarrow a_3 = \frac{a_2}{3} = \frac{a_0}{6}$$

Substituting the value of $a_1, a_2, a_3, \ldots$ in (ii), we get

$$y = a_0 + a_0x + \frac{a_0}{2}x^2 + \frac{a_0}{6}x^3 \;\ldots\ldots$$

$$= a_0\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\ldots\ldots\right)$$

$$= a_0e^{x}$$

(4) $y' = 2xy$.

Solution: Given differential equation is,

$$y' = 2xy \qquad \ldots\ldots\ldots (i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \qquad \ldots\ldots\ldots\ldots (ii)$$

be solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots$$

Putting the value of y and y' is equation (i), we get

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots\ldots = 2a_0x + 2a_1x^2 + 2a_2x^3 + 2a_3x^3 + \ldots\ldots$$

Equating coefficient of like terms from both sides then,

$$a_1 = 0, \qquad 2a_2 = 2a_0, \qquad 3a_3 = 2a_1, \qquad 4a_4 = 2a_2 \text{ and so on.}$$

$$\Rightarrow a_2 = a_0 \qquad \Rightarrow a_3 = \frac{2}{3}a_1 = 0 \qquad \Rightarrow a_4 = \frac{1}{2}a_2 = \frac{1}{2}a_0$$

Putting the value of $a_1, a_2, a_3, a_4, \ldots$ in (ii), we get

$$y = a_0 + 0 + a_0x^2 + 0 + \frac{a_0}{2}x^4 + \ldots$$

$$= a_0\left(1 + x^2 + \frac{1}{2}x^4 + \ldots\ldots\right)$$

$$= a_0e^{x^2}$$

(5) $y' = -2xy$ **[1999, 2001 Q. No. 5(a) OR] [2004 Fall Q. No. 5(a)]**

Solution: Given differential equation is,

$$y' = -2xy \qquad \ldots\ldots\ldots\ldots(i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \qquad \ldots\ldots\ldots (ii)$$

be solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots\ldots$$

Putting the value of y and y' in (i) then

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots = -2a_0x - 2a_1x^2 - 2a_2x^3 - 2a_3x^4 - \ldots$$

Equating coefficient of like terms from both sides then,

$a_1 = 0,$ $\quad 2a_2 = -2a_0$ $\quad 3a_3 = -2a_1$ $\quad 4a_4 = -2a_2$ and so on.

$\Rightarrow a_2 = -a_0$ $\quad \Rightarrow a_3 = \frac{-2}{3}a_1 = 0$ $\quad \Rightarrow a_4 = \frac{-1}{2}a_2 = \frac{a_0}{2}$

Putting the value of $a_1, a_2, a_3, a_4, \ldots\ldots\ldots$ in (ii) then,

$$y = a_0 + 0 - a_0x^2 + 0 - \frac{a_0}{2}x^4 \ldots\ldots\ldots$$

$$= a_0\left(1 - x^2 - \frac{x^4}{2} - \ldots\ldots\right)$$

$$= a_0e^{-x^2}$$

(6) $xy' - 3y = k$

Solution: Given differential equation is,

$$xy' - 3y = k \qquad \ldots\ldots(i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \qquad \ldots\ldots\ldots(ii)$$

be solution of (i)

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots\ldots$$

Putting the value of y_1y' in equation (i)

$$a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + \ldots\ldots\ldots - 3a_0 - 3a_1x - 3a_2x^2 - 3a_3x^3 - = k.$$

$$\Rightarrow a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + \ldots\ldots = k + 3a_0 + 3a_1x + 3a_2x^2 + 3a_3x^3 + \ldots\ldots\ldots$$

Equating coefficient of like terms from both sides then,

$(3a_0 + k) = 0,$ $\quad a_1 = 3a_1$ $\quad 3a_2 = 2a_2$ and so on.

$\Rightarrow a_0 = \frac{-k}{3}$ $\quad \Rightarrow a_1 = 0$ $\quad \Rightarrow a_2 = 0.$

Putting the value of $a_0, a_1, a_2, \ldots\ldots\ldots$ in (ii)

$$y = \frac{-k}{3}$$

(7) $y'' + 9y = 0.$ **[2000 Q. No. 5(a) OR]**

Solution: Given differential equation is,

$$y'' + 9y = 0 \qquad \ldots\ldots\ldots(i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \qquad \ldots\ldots(ii)$$

be solution of (i).

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots$$

and, $\quad y'' = 2a_2 + 6a_3x + 12a_4x^2 + \ldots\ldots\ldots$

Putting the value of y and y" in (i) then,

$$2a_2 + 6a_3x + 12x_4x^2 + \ldots\ldots + 9a_0 + 9a_1 + 9a_2x^2 + 9a_3x^3 + \ldots\ldots\ldots = 0$$

$$\Rightarrow (2a_2 + 9a_0) + x(6a_3 + 9a_1) + x^2(12a_4 + 9a_2) + \ldots\ldots\ldots = 0$$

Equating coefficient of like terms from both sides then,

$2a_2 + 9a_0 = 0$ $\quad 6a_3 + 9a_1 = 0$ $\quad 12a_4 + 9a_2 = 0$ and so on.

$\Rightarrow a_2 = \frac{-9}{2}a_0$ $\quad \Rightarrow a_3 = \frac{-9}{6}a_1$ $\quad \Rightarrow a_4 = \frac{-9}{12}a_2 = \frac{27}{8}a_0$

Putting the value of a_2, a_3, a_4, in (ii),

$$y = a_0 + a_1x - \frac{9}{2}a_0x^2 - \frac{3}{2}a_1x^3 + \frac{27}{8}a_0x^4 + \ldots\ldots$$

$$= a_0\left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 + \ldots\ldots\right) + a_1\left(x - \frac{3}{2}x^3 + \ldots\ldots\right)$$

$$= a_0\cos 3x + a_1\sin 3x..$$

(8) $y'' + y = 0.$ **[2006 Spring, 2008 Fall, 2011 Fall Q. No. 5(a)]**

Solution: Given differential equation is,

$$y'' + y = 0 \qquad \ldots\ldots\ldots(i)$$

Let,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \qquad \ldots\ldots(ii)$$

be solution of (i).

Differentiating (ii) w. r. t. x, then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots\ldots$$

and $\quad y'' = 2a_2 + 6a_3x + 12a_4x^2 + \ldots\ldots\ldots$

Putting the value of y and y" in eq^n. (i)

$$2a_2 + 6a_3x + 12a_4x^2 + \ldots\ldots + a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots = 0$$

$$\Rightarrow (2a_2 + a_0) + x(6a_3 + a_1) + x^2(12a_4 + a_2) + \ldots\ldots\ldots = 0$$

Equating coefficient of like terms from both sides then,

$2a_2 + a_0 = 0$ $\quad 6a_3 + a_1 = 0$ $\quad 12a_4 + a_2 = 0$ and so on.

$\Rightarrow a_2 = \frac{-a_0}{2}$ $\quad \Rightarrow a_3 = \frac{-a_1}{6}$ $\quad \Rightarrow a_4 = \frac{-a_2}{12} = -\frac{a_0}{2} \times \frac{-1}{12} = \frac{a_0}{24}$

Putting the value of $a_2, a_3, a_4, \ldots\ldots$ in (ii) then,

$$y = a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 + \frac{a_0}{24}x^4 + \ldots\ldots\ldots$$

$$= a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) + a_1\left(x - \frac{x^3}{6} + \ldots\ldots\right).$$

(9) $y' = 3x^2y.$ **[2004 Spring Q. No. 5(a) OR]**

Solution: Given differential equation is,

$$y' = 3x^2y \qquad \ldots\ldots\ldots(i)$$

Let,

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots$ ....(ii)

be solution of (i)

Differentiating (ii) w. r. t. x, then

$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots\ldots$

Putting the value of y and y' in eq^n. (i)

$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots = 3a_0x^2 + 3a_1x^3 + 3a_2x^4 + 3a_3x^5 + 3a_4x^6 + \ldots$

Equating coefficient of like terms from both sides then,

$a_1 = 0$, $2a_2 = 0$ $3a_3 = 3a_0$ $4a_4 = 3a_1$ $5a_5 = 3a_2$ and so on.

$\Rightarrow a_2 = 0.$ $\Rightarrow a_3 = a_0$ $\Rightarrow a_4 = \frac{3}{4}a_1$ $\Rightarrow a_5 = \frac{3}{5}a_2 = 0.$

Putting the value of $a_1, a_2, a_3, \ldots$ in (ii) then,

$y = a_0 + a_0x^3 + \ldots$

$= a_0(1 + x^3 + \ldots)$

$= a_0e^{x^3}$

[2009 Spring Q. No. 5(a)]

(10) $y'' + 4y = 0$.

Solution: Given differential equation is,

$y'' + 4y = 0$ ....(i)

Let,

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ .... (ii)

be solution of (i).

Differentiating (ii) w. r. t. x, then

$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$

and $y'' = 2a_2 + 6a_3x + 12a_4x^2 + \ldots$

Putting the value of y and y'' in (i) then,

$2a_2 + 6a_3x + 12a_4x^2 + \ldots + 4a_0 + 3a_1x + 4a_2x^2 + 4a_3x^3 + \ldots = 0$

$\Rightarrow (2a_2 + 4a_0) + x(6a_3 + 4a_1) + x^2(12a_4 + 4a_2) + \ldots = 0$

Equating coefficient of like terms from both sides then,

$2a_2 + 4a_0 = 0$ $6a_3 + 4a_1 = 0$ $12a_4 + 4a_2 = 0$ and so on

$\Rightarrow a_2 = -2a_0$ $\Rightarrow a_3 = \frac{-2}{3}a_1$ $\Rightarrow a_4 = \frac{-1}{3}a_2 = \frac{2}{3}a_2$

Putting the value of $a_2, a_3, a_4, \ldots$ in (ii) then,

$y = a_0 + a_1x - 2a_0a^2 - \frac{2}{3}a_1x^3 + \frac{2}{3}a_0x^4 + \ldots$

$= a_0\left(1 - 2x^2 + \frac{2}{3}x^4\right) + a_1\left(x - \frac{2}{3}x^3 + \ldots\right)$

$= a_0\cos 2x + \frac{1}{2}a_1 \sin 2x.$

[2003 Fall Q. No. 5(a)]

(11) $(1 + x)y' = y$.

Solution: Given differential equation is,

$(1 + x)y' = y$ ....(i)

Let.

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$ .... (ii)

be solution of (i).

Differentiating (ii) w. r. t. x, then

$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots$

Putting the value of y and y' in (i) then,

$(1 + x)(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

$\Rightarrow a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + \ldots$

$= + a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

$\Rightarrow a_1 + x(2a_2 + a_1) + x^2(3a_3 + 2a_2) + x^2(4a_4 + 3a_3) + \ldots$

$= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

Equating coefficient of like terms from both sides then,

$a_1 = a_0$ $2a_2 + a_1 = a_1$ $3a_3 + 2a_2 = a_2$ $4a_4 + 3a_3 = a_3$ and so on.

$\Rightarrow a_2 = 0$ $\Rightarrow a_3 = \frac{-a_2}{3} = 0$ $\Rightarrow 4a_4 = -2a_3$

$\Rightarrow a_4 = \frac{-1}{2}a_3 = 0.$

Putting the value of $a_1, a_2, a_3, \ldots$ in (2) then,

$y = a_0 + a_0x + 0 + 0 + 0 + \ldots$

$= a_0(1 + x).$

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

Similar Question for Practice from Final Exam:

2002 Q. No. 5(a)

Find a power series solution of the differential equation $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 2)y = 0.$

2002 Q. No. 5(a) OR; 2006 Fall; 2008 Spring; 2010 Spring Q. No. 5(a)

Solve by power series method: $y'' = 4y$.

2002 Q. No. 5(b)

Solve the initial value problem $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$; $y(0) = 10$, $y'(0) = 0$ by power series solution.

2000(OR); 2007 Fall Q. No. 5(a)

Solve $y'' = 9y$ by using power series method.

2009 Spring Q. No. 5(a)

2009 fall Solve: $y'' = 8y$ by power series method.

Legendre's Equation:

The second order differential equation of the form

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

is known as Legendre's equation.

Note: The solution of above equation is Legendre's function.

Legendre's Polynomial:

The polynomial,

$$P_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(2n-2m)!^m}{2^{2n} m!\,(n+m)!\,(n-m)!} x^{n-2m}$$

is called the Legendre's polynomial of order n.

Solution of Legendre's Equation: **[2007 Fall Q. No. 5(a) OR]**

We have Legendre's equation as

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{...... (1)}$$

Let, $$y = \sum_{m=0}^{\infty} a_m x^m \quad \text{...... (2)}$$

be the solution of (1).

Here differentiating with respect to x, we get,

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \quad \text{and} \quad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

Substituting these values in equation (1) we get,

$$(1 - x^2) \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + k \sum_{m=0}^{\infty} a_m x^m = 0$$

where $k = n(n+1)$

By writing the first expression as two separate series, we have the equation

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow 2a_1 + 3.2a_3x + 4.3a_4x^2 + \ldots\ldots + (s+2)(s+1)a_s + 2x^s + \ldots\ldots - 2.1a_2x^2 - \ldots\ldots + \ldots\ldots - 2.1a_1x - 2.2a_2x^2 - \ldots\ldots - s(s-1)a_sx^s - \ldots\ldots + ka_0 + ka_1x + ka_2x^2 + \ldots\ldots - 2sa_sx^s - \ldots\ldots = 0.$$

Comparing the coefficients of x^0, x, x^s, we get

$$2a_2 + ka_0 = 0 \Rightarrow 2a_2 + n(n+1)a_0 = 0 \quad \text{...... (3)}$$

$$6a_3 + [-2+k]a_1 = 0 \Rightarrow 6a_3 + [-2 + n(n+1)]a_1 = 0 \quad \text{...... (4)}$$

$$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + k]a_s = 0$$

$$\Rightarrow (s+2)(s+1)a_{s+2} + [-s^2 - s + n(n+1)]a_s = 0 \quad \text{...... (5)}$$

Thus, $$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \quad \text{for } s = 0, 1, 2, 3, \ldots$$

From equation (3), (4) and (5) we get,

$$a_2 = -\frac{n(n+1)}{2!} a_0; \qquad a_3 = -\frac{(n-1)(n+2)}{3!} a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4.3} a_2 = \frac{(n-2)n(n+1)(n+3)}{4!} a_0;$$

$$a_5 = -\frac{(n-3)(n+4)}{5.4} a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$$

Substituting the coefficients in equation (2), we get

$$y = a_0 + a_1x + \frac{(-n)(n+1)}{2!} a_0x^2 + \frac{(-)(n-1)(n+2)}{3!} a_1x^3 + \frac{(n-2)n(n+1)(n+3)}{4!} a_0x^4 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1x^5 + \ldots\ldots$$

$$\Rightarrow y = a_0\left(1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \ldots\ldots\right)$$

$$\Rightarrow y = a_0y_1 + a_1y_1 \quad \text{...... (6)}$$

where

$$y_1 = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \ldots\ldots$$

And, $$y_2 = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots\ldots$$

These y_1 and y_2 be power series, which are convergent for $|x| < 1$. Thus $y = a_0y_1 + a_1y_2$ is the Legendre solution of the given Legendre's equation (1).

Definition of Bessel's Function of First Kind:

The Bessel's function of first kind of order n is denoted by $J_n(x)$ and is defined as,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

Bessel's Equation:

A differential equation of second order of the form

$$x^2y'' + xy' + (x^2 - \gamma^2)y = 0 \quad \ldots\ldots (1)$$

where γ is real and non-negative number; is said to be Bessel equation.

Bessel Function of first kind of order n:

The function of the form,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

is called Bessel function of first kind of order n.

Solution of Bessel Equation:

Consider a Bessel's equation,

$$x^2y'' + xy' + (x^2 - \gamma^2)y = 0 \quad \ldots\ldots (1)$$

where γ is real and non-negative number.

Let
$$y = \sum_{m=0}^{\infty} a_m x^{m+r} \quad \ldots\ldots (2)$$

with ($a_0 \neq 0$), be a solution of (1). Then,

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \gamma^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

Equating the coefficient of x^{s+r} to zero, we get

$$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \gamma^2 a_s = 0 \quad \ldots\ldots (3)$$

For s=0, we get,

$$r(r-1)a_0 + ra_0 - \gamma^2 a_0 = 0 \Rightarrow (r^2 - r + r - \gamma^2) = 0$$
$$\Rightarrow (r^2 - \gamma^2) = 0$$
$$\Rightarrow (r-\gamma)(r+\gamma) = 0 \Rightarrow r = \gamma, -\gamma$$

Let the roots of r is, $r_1 = \gamma$ and $r_2 = -\gamma$.

For $r = \gamma$, we have $(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \gamma^2 a_s = 0$

$$\Rightarrow (s^2+2sr+r^2-s-r+s+r-\gamma^2)a_s + a_{s-2} = 0$$
$$\Rightarrow (s^2+2sr+r^2-\gamma^2)a_s + a_{s-2} = 0$$
$$\Rightarrow [(s+r)^2-\gamma^2]a_s + a_{s-2} = 0$$
$$\Rightarrow (s+r-\gamma)(s+r+\gamma)a_s + a_{s-2} = 0$$

If $r = \gamma$ then $s(s+2\gamma)a_s + a_{s-2} = 0 \quad \ldots\ldots (4)$

Since, $a_1 = 0$ and $\gamma \geq 0$; it gives $a_3 = 0$, $a_5 = 0$, ... successively.

So to evaluate the coefficient of even numbers s=2m. Put s = 2m in equation (4) we get,

$$(2m+2\gamma)2ma_{2m} + a_{2m-2} = 0$$

$$\Rightarrow a_{2m} = \frac{1}{2^2 m(\gamma+m)} a_{2m-2}; \quad \text{for } m = 1, 2, 3, \ldots\ldots$$

Thus we get,

$$a_2 = \frac{-a_0}{2^2(\gamma+1)} \quad \text{and} \quad a_4 = \frac{(-a_2)}{2^2\, 2(\gamma+2)}$$

Therefore, $a_4 = \dfrac{a_0}{2^4\, 2!(\gamma+1)(\gamma+2)}$

So in general,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(\gamma+1)(\gamma+2)\ldots\ldots(\gamma+m)}, \quad \text{for } m = 1, 2, \ldots\ldots$$

Put $\gamma = n$, then,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\ldots\ldots(n+m)}$$

Here a_0 is still arbitrary. Let us choose $a_0 = \dfrac{1}{2^n n!}$, because $n!(n+1)\ldots\ldots(n+m) = (n+m)!$.

Then, $a_{2m} = \dfrac{(-1)^n}{2^{2m+n} m!(n+m)!}$ for m = 1, 2, 3, ...

Substituting these values of coefficients in equation (2) we get,

$$y = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

Let y is denoted by $J_n(x)$. That is,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

This is the solution of Bessel's equation (1).

Some Remarks on Bessel's Function of First Kind:

1. **Show that $J_{-n}(x) = (-1)^n J_n(x)$.**

Solution: We have,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

Put n = -n we get,

$$J_{-n}(x) = x^{-n}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m-n}\, m!(-n+m)!}$$

$$= \sum_{m=n}^{\infty}\frac{(-1)^m x^{2m-n}}{2^{2m-n}\, m!(m-n)!} = \sum_{s=0}^{\infty}\frac{(-1)^{n+s}\, x^{2s+n}}{2^{2s+n}(n+s)!s!} \quad \text{when } s = m-n$$

$$= (-1)^n \sum_{s=0}^{\infty}\frac{(-1)^s x^{2s+n}}{2^{2s+n}\,(n+s)!s!} = (-1)^n J_n(x)$$

Thus, $J_{-n}(x) = (-1)^n J_n(x)$.

2. **Show that $\frac{d}{dx}[x^\gamma J_\gamma(x)] = x^\gamma J_{\gamma-1}(x)$.**

[2004(Spring)–Short; 2004 Spring Q. No. 5(b)]

Solution: We have,

$$x^\gamma J_\gamma(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+\gamma}\, m!\Gamma(\gamma+m+1)}$$

Differentiating with respect to x, we get

$$\frac{d}{dx}[x^\gamma J_\gamma(x)] = \sum_{m=0}^{\infty}\frac{(-1)^m 2(m+\gamma)x^{2m+2\gamma-1}}{2^{2m+\gamma}\, m!\Gamma(\gamma+m+1)}$$

$$= x^\gamma x^{\gamma-1}\sum_{m=0}^{\infty}\frac{(-1)^m\, x^{2m}}{2^{2m+\gamma-1}\, m!\Gamma(\gamma+m)} = x^\gamma J_{\gamma-1}(x).$$

$$\Rightarrow \frac{d}{dx}[x^\gamma J_\gamma(x)] = x^\gamma J_{\gamma-1}(x).$$

3. **Show that $\frac{d}{dx}[x^{-\gamma} J_\gamma(x)] = -x^{-\gamma} J_{\gamma+1}(x)$**

Solution: We have,

$$x^{-\gamma} J_\gamma(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+\gamma}\, m!\Gamma(\gamma+m+1)}$$

Differentiating with respect to x, we get,

$$\frac{d}{dx}[x^{-\gamma} J_\gamma(x)] = \sum_{m=0}^{\infty}\frac{(-1)^m\, 2m x^{2m-1}}{2^{2m+\gamma}\, m!\Gamma(\gamma+m+1)}$$

$$= \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m+\gamma-1}\,(m-1)!\Gamma(\gamma+m+1)}$$

$$= \sum_{m=1}^{\infty}\frac{(-1)^m x^{2(m-1)+1}}{2^{2(m-1)+\gamma+1}\,(m-1)!\Gamma(\gamma+m-1+2)}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^{s+1} x^{2s+1}}{2^{2s+\gamma+1} s!\Gamma(\gamma+s+2)} \quad \text{by putting } s = m-1$$

$$= -x^{-\gamma}\sum_{m=0}^{\infty}\frac{(-1)^s x^{2s+1+\gamma}}{2^{2s+\gamma+1} s!\Gamma(\gamma+s+1+1)} = -x^{-\gamma} J_{\gamma+1}(x).$$

Thus, $\frac{d}{dx}[x^{-\gamma}J_\gamma(x)] = -x^{-\gamma} J_{\gamma+1}(x)$.

4. **Show that $\gamma x^{\gamma-1} J_\gamma(x) + x^\gamma J'_\gamma(x) = x^\gamma J_{\gamma-1}(x)$**

Solution: We have,

$$\frac{d}{dx}[x^\gamma J_\gamma(x)] = x^\gamma J_{\gamma-1}(x) \qquad \text{[By 2]}$$

$$\Rightarrow x^\gamma J'_\gamma(x) + \gamma x^{\gamma-1} J_{\gamma-1}(x) = x^\gamma J_{\gamma-1}(x)$$

5. **Show that $J_{\gamma-1}(x) + J_{\gamma+1}(x) = \frac{2\gamma}{x} J'_\gamma(x)$**

Solution: We have,

$$\frac{d}{dx}[x^\gamma J_\gamma(x)] = x^\gamma J_{\gamma-1}(x) \qquad \ldots\ldots (1)$$

and

$$\frac{d}{dx}[x^{-\gamma} J_\gamma(x)] = -x^{-\gamma} J_{\gamma+1}(x) \qquad \ldots\ldots (2)$$

From (1), $\gamma x^{\gamma-1} J_\gamma(x) + x^\gamma J'_\gamma(x) = x^\gamma J_{\gamma-1}(x)$

$$\Rightarrow \frac{\gamma}{x} J_\gamma(x) + J'_\gamma(x) = J_{\gamma-1}(x) \qquad \ldots\ldots (3)$$

From equation (2),

$$-\gamma x^{-\gamma-1} J_\gamma(x) + x^{-\gamma} J'_\gamma(x) = -x^{-\gamma} J_{\gamma+1}(x)$$

$$\Rightarrow \frac{-\gamma}{x} J_\gamma(x) + J'_\gamma(x) = -J_{\gamma+1}(x) \qquad \ldots\ldots (4)$$

Subtracting (4) from (3) we get,

$$\frac{2\gamma}{x} J_\gamma(x) = J_{\gamma-1}(x) + J_{\gamma+1}(x).$$

6. Show that $J_{\gamma-1}(x) - J_{\gamma+1}(x) = 2J'_{\gamma}(x)$ [2003 Fall Q. No. 5(a) OR]

Solution: We have,

$$\frac{d}{dx}[x^{\gamma}J_{\gamma}(x)] = x^{\gamma}J_{\gamma-1}(x)$$

$$\Rightarrow \frac{\gamma}{x}J_{\gamma}(x) + J'_{\gamma}(x) = J_{\gamma-1}(x) \quad \text{...... (1)}$$

And $$\frac{d}{dx}[x^{-\gamma}J_{\gamma}(x)] = -x^{-\gamma}J_{\gamma+1}(x)$$

Also, $$\frac{-\gamma}{x}J_{\gamma}(x) + J'_{\gamma}(x) = -J_{\gamma+1}(x) \quad \text{...... (2)}$$

Adding (1) and (2) we get,

$$2J'_{\gamma}(x) = J_{\gamma-1}(x) - J_{\gamma+1}(x).$$

7. Show that $\int x^{\gamma}J_{\gamma-1}(x)\,dx = x^{\gamma}J_{\gamma}(x) + c$

Solution: We have,

$$\frac{d}{dx}[x^{\gamma}J_{\gamma}(x)] = x^{\gamma}J_{\gamma-1}(x)$$

Integrating with respects to x, we get,

$$\int x^{\gamma}J_{\gamma-1}(x)\,dx = x^{\gamma}J_{\gamma}(x) + c.$$

8. Show that $\int x^{-\gamma}J_{\gamma+1}(x)\,dx = -x^{-\gamma}J_{\gamma}(x) + c$

Solution: We have,

$$\frac{d}{dx}[x^{-\gamma}J_{\gamma}(x)] = -x^{-\gamma}J_{\gamma+1}(x)$$

Integrating with respect to x, we get

$$x^{-\gamma}J_{\gamma}(x) + c = -\int x^{-\gamma}J_{\gamma+1}(x)\,dx$$

$$\Rightarrow \int x^{-\gamma}J_{\gamma+1}(x)\,dx = -x^{-\gamma}J_{\gamma}(x) + c$$

9. Show that $\int J_{\gamma+1}(x)dx = \int J_{\gamma-1}(x)dx - 2J_{\gamma}(x)$

Solution: We have,

$$\int J_{\gamma-1}(x) - J_{\gamma+1}(x) = 2J'_{\gamma}(x)$$

Integrating both side with respects to x

$$\int J_{\gamma+1}(x)\,dx - \int J_{\gamma-1}(x)dx = 2J'_{\gamma}(x)$$

$$\Rightarrow \int J_{\gamma+1}(x)dx = \int J_{\gamma-1}(x)dx - 2J_{\gamma}(x).$$

10. Show that $xJ_r'(x) = rJ_r(x) - xJ_{r+1}(x)$

Solution: Since we have,

$$\frac{d}{dx}(x^{-r}J_r(x)) = -x^{-r}J_{r+1}$$

$$\Rightarrow x^{-r}J_r'(x) - r\,x^{-r-1}J_r(x) = -x^{-r}J_{r+1}$$

$$\Rightarrow x^{-r}[J_r'(x) - rx^{-1}J_r(x)] = -x^{-r}J_{r+1}$$

$$\Rightarrow J_r'(x) - r\,x^{-1}(x) = -J_{r+1}$$

$$\Rightarrow xJ_r'(x) = rJ_r(x) - xJ_{r+1}(x).$$

Exercise 7.2

(1) Show that $J_0'(x) = -J_1(x)$.

Proof: We have,

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+n}}{2^{2m+n}\,m!\,(n+m)!}$$

For, n = 1,

$$J_1(x) \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}\,m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} \quad \text{........(i)}$$

For n = 0, $$J_0(x) \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}\,m!\,m!} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{64+36} \quad \text{.........(ii)}$$

Differentiating w. r. t. x, then

$$J_0'(x) = 0 - \frac{2x}{4} + \frac{4x^3}{64} - \frac{6x^5}{64+36} + \text{........}$$

$$\Rightarrow J_0'(x) = -\left(\frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384}\text{.......}\right)$$

$$\Rightarrow J_0'(x) = -J_1(x) \quad \text{(using (i))}$$

Alternative method:

Since we have,

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

Set n = 0 then,

$$xJ_0'(x) = 0 - x\,J_1(x)$$

$$\Rightarrow J_0'(x) = -J_1(x)$$

2. Show that, $J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$

Solution: Since we have,

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

Set n = 2 then,

$$J_1(x) - J_3(x) = 2J_2'(x)$$

$$\Rightarrow \quad J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$$

3. Repeated question to 1

4. Show that $J_1'(x) = J_0(x) - x^{-1}J_1(x)$

Solution: Since we have,

$$nJ_n(x) + xJ_n'(x) = xJ_{n-1}(x)$$

Set n = 1, then

$$J_1(x) + xJ_1'(x) = xJ_0(x)$$

$$\Rightarrow \quad x^{-1}J_1(x) + J_1'(x) = J_0(x)$$

$$\Rightarrow \quad J_1'(x) = J_0(x) - x^{-1}J_1(x)$$

5. Evaluate

(i) $\int J_3(x)\,dx$ **(ii)** $\int x^3J_2(x)\,dx$ **(iii)** $\int J_5(x)\,dx$

Solution:

(i) Since we have,

$$\int x^{-n}J_{n+1}(x)\,dx = -x^{-n}J_n(x) + C \quad \text{.......(i)}$$

and $$\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2J_n(x) \quad \text{.......(ii)}$$

Set n = 0 in (i) then,

$$\int J_1(x)\,dx = -J_0(x) + C \quad \text{........(iii)}$$

And set n = 2 in (ii) then,

$$\int J_3(x)\,dx = \int J_1(x)\,dx - 2J_2(x)$$

$$= J_0(x) + C - 2J_2(x) \quad [\because \text{using (iii)}]$$

$$= -2J_2(x) - J_0(x) + C$$

(ii) Since we have,

$$\int x^nJ_{n-1}(x)\,dx = x^nJ_n(x) + C$$

Set n = 3 then,

$$\int x^3J_2(x)\,dx = x^3J_3(x) + C$$

(iii) Set, n = 4 in (ii) then [2002 – Short]

$$\int J_5(x)\,dx = \int J_3(x)\,dx - 2J_4(x)$$

$$= -2J_2(x) - J_0(x) + C - 2J_4(x) \quad [\because \text{using Q. 1}]$$

$$= -2J_4(x) - 2J_2(x) - J_0(x) + C$$

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

1999 Q. No. 5(a)

Write down the Legendre's equation and its general solution. Also, define the Legendre's polynomial of order 2.

Solution: See the Legendre's equation.

Second Part: See the solution of Legendre's equation.

Third Part: Since we have the Legendre's polynomial of order n is,

$$P_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(2n - 2m)!^m}{2^{2n}\, m!\,(n + m)!\,(n - m)!}\, x^{n-2m}$$

Set n = 2, then,

$$P_2(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(4 - 2m)!^m}{2^4\, m!\,(2 + m)!\,(2 - m)!}\, x^{2-2m}$$

2000 Q. No. 5(a)

Write down the Leendre's and Bessel equation and then also write down the general solution of the Legendre' equation and Bessel function of first kind $J_p(x)$.

Solution: See the Legendre's equation and Bessel's equation.

Second Part: See the solution of Legendre's equation.

Third Part: See the solution of Bessel's equation.

2001 Q. No. 5(a)

Write down the Legendre's equation and its general solution. Also define the Legendre's polynomial of order n and then find Legendre's polynomial of order 2.

Solution: See Solution of 1999.

2002 Q. No. 5(a)

Define Bessel function of the first kind. Show that: $\frac{d}{dx}[x^v J_v(x)] = x^v J_{v-1}(x)$.

Solution: See the definition of Bessel's function.

See the result 2.

2004 Spring; 2009 Spring: 2010 Spring (OR) Q. No. 5(a)

What is Legendre's equation? Find its solution.

Solution: See definition of Legendre's equation.