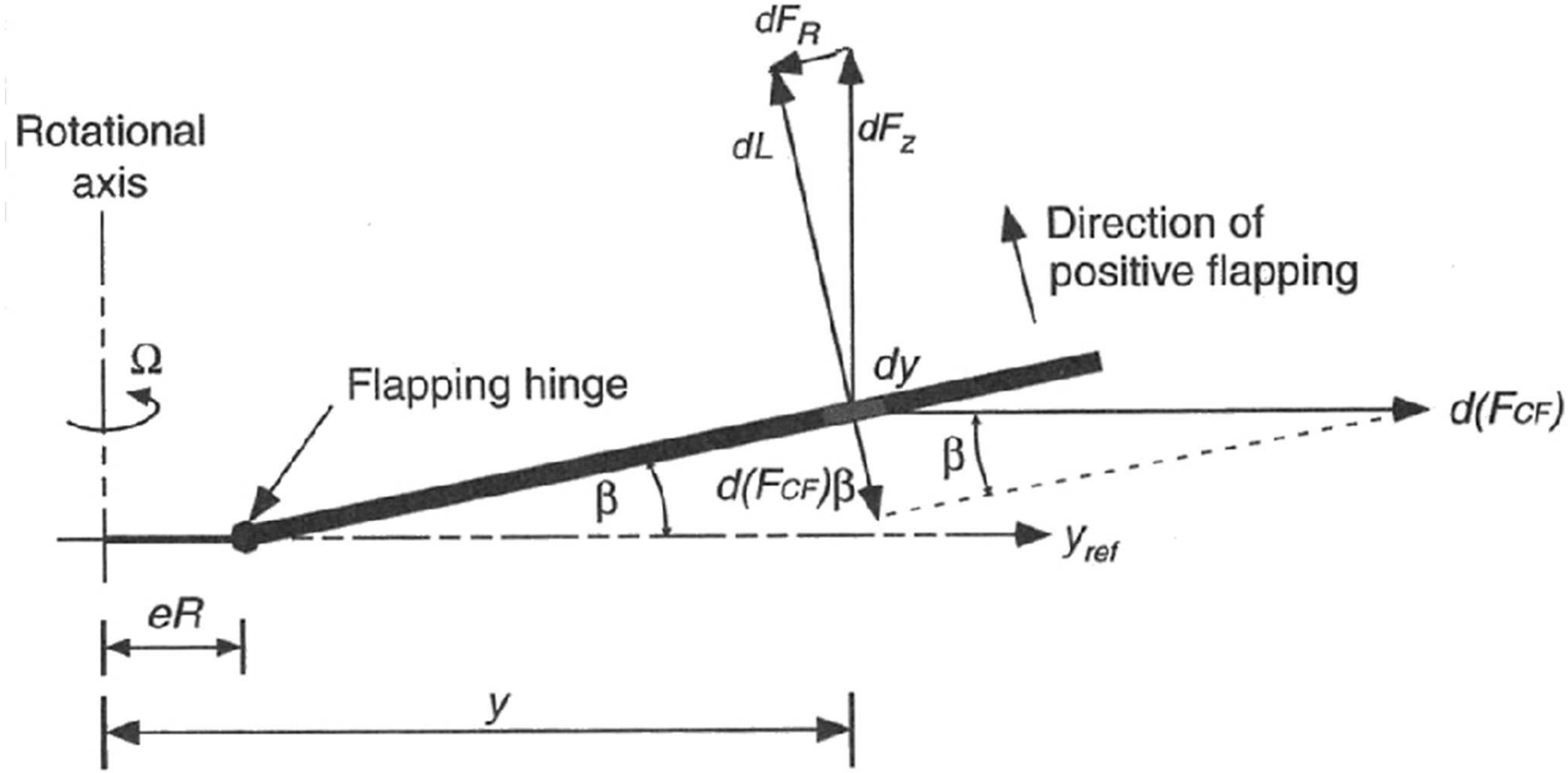
Rotating Blade motion

- As seen earlier, blades are usually hinged near the root, to alleviate high bending moments at the root.
- This allows the blades to flap up and down.
- Aerodynamic forces cause the blades to flap up.
- Centrifugal forces causes the blades to flap down.
- Inertial forces will arise, which oppose the direction of acceleration.
- In forward flight, an equilibrium position is achieved, where the net moments at the hinge due to these three types of forces (aerodynamic, centrifugal, inertial) cancel out.

Equilibrium about the flapping hinge



Equilibrium about the flapping hinge

- Element of mass per unit length *m*
- Distance *y* from the rotational axis
- Performing a circular motion with speed *Ω*
- Therefore the centrifugal force is:

$$d\left(F_{CF}\right) = \left(m dy\right) a_r = \left(m dy\right) y \Omega^2$$

Equilibrium about the flapping hinge

- Neglecting, for now, the hinge offset the total centrifugal force is:

$$F_{CF} = \int_0^R m\Omega^2 y dy = \frac{m\Omega^2 R^2}{2} = \frac{M\Omega^2 R}{2}$$

- *M* is the total mass of the blade
- Since the blade has a coning angle β the Centrifugal force component acting perpendicular to the blade is:

$$d(F_{CF})\sin\beta = (mdy)y\Omega^2 \sin\beta \approx m\Omega^2 \beta y dy$$

Equilibrium about the flapping hinge

- The moment about the flapping hinge is:

$$M_{CF}=\int_0^R m\Omega^2 y^2 \beta dy=\Omega^2\beta\int_0^R y^2 m dy=\frac{m\Omega^2\beta R^3}{3}=$$

$$=\frac{M\Omega^2\beta R^2}{3}=\frac{2}{3}F_{CF}R\beta$$

- Or we could also write

$$M_{CF}=\Omega^2\beta\int_0^R y^2 m dy=I_b\Omega^2\beta$$

- I_b is the blade moment of inertia around the flapping hinge

Equilibrium about the flapping hinge

- The aerodynamic moment around the flapping hinge:

$$M_\beta = -\int_0^R Lydy$$

- In equilibrium $M_{CF}+M_\beta=0$ therefore:

$$\frac{M\Omega^2\beta R^2}{3} - \int_0^R Lydy = 0 \Rightarrow \beta_o = \frac{\int_0^R Lydy}{\left(\frac{M\Omega^2 R^2}{3}\right)}$$

Equilibrium about the flapping hinge

- Since the flapping hinge can be offset by a distance *eR(<0.15R)* we can obtain the following expressions:

$$M_{CF} = \int_{eR}^{R} m\Omega^2 y^2 \beta dy = \frac{m\Omega^2 \beta R^3 \left(1-e^3\right)}{3} =$$

$$= \frac{M\Omega^2 \beta R^2 \left(1+e\right)}{3} + o\left(e^2\right)$$

- Remember that *M=m(R-eR)=mR(1-e)*

Equilibrium about the flapping hinge

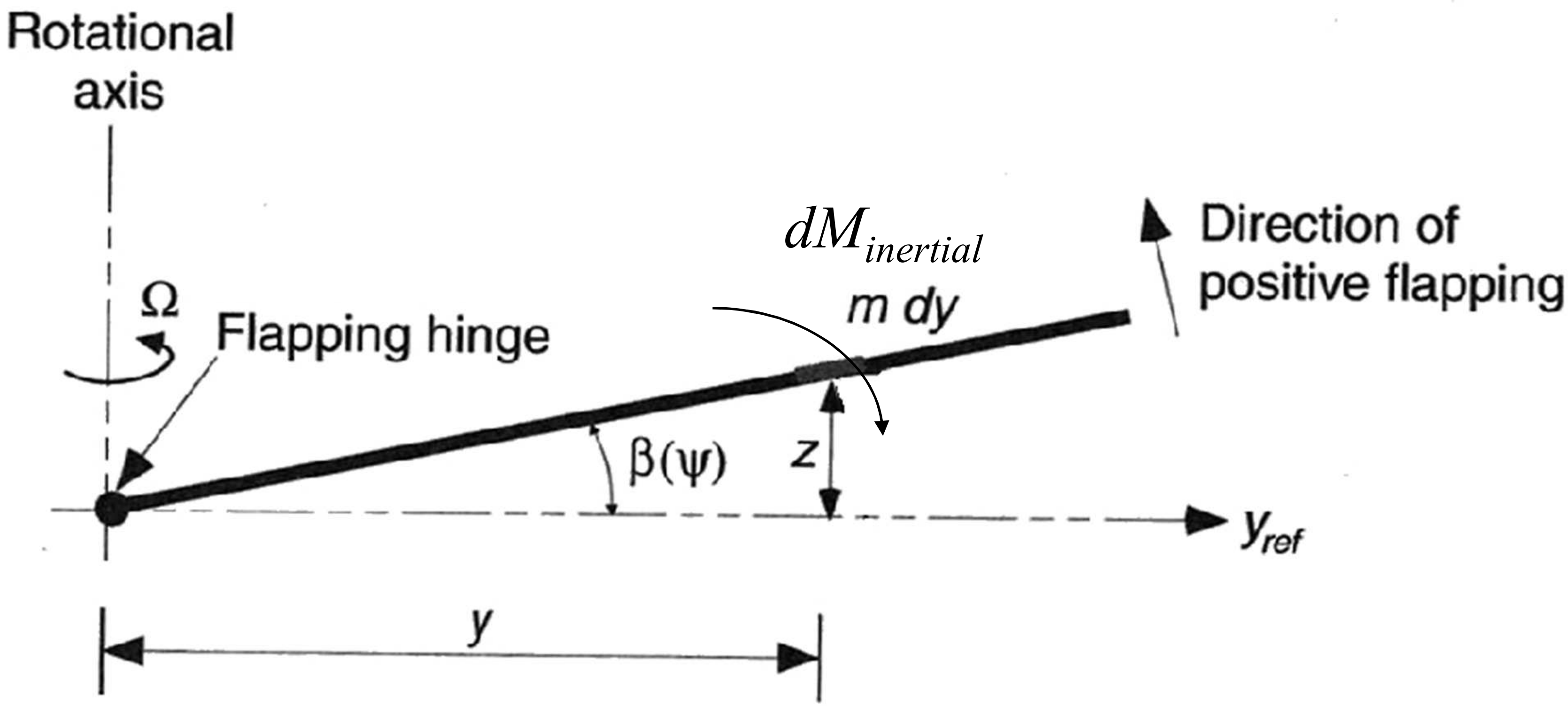
- Also the aerodynamic moment is:

$$M_\beta = -\int_{eR}^{R} Ly\,dy$$

- The equilibrium coning angle is:

$$\beta_o = \frac{\int_{eR}^{R} Ly\,dy}{\left(\frac{M\Omega^2 R^2 (1+e)}{3}\right)}$$

Motion for a Flapping Blade



Motion for a Flapping Blade

- We have already obtained the expressions for the moments:

$$dM_{CF} = m\Omega^2 y^2 \beta dy \qquad dM_{\beta} = -Lydy$$

- And the Inertial moment is given by:

$$dM_{inertial} = \left(mdy\right) y^2 \ddot{\beta}$$

- Assuming no hinge offset we have:

$$\int_0^R m\Omega^2 y^2 \beta dy + \int_0^R my^2 \ddot{\beta} dy - \int_0^R Lydy = 0$$

Motion for a Flapping Blade

- Writing:

$$\left(\int_0^R m y^2 dy\right)\left(\ddot{\beta} + \Omega^2 \beta\right) = \int_0^R L y dy$$

- And noting that the first item is I_b then:

$$\left(I_b \ddot{\beta} + I_b \Omega^2 \beta\right) = \int_0^R L y dy$$

Motion for a Flapping Blade

- Performing some mathematical modifications:

$$\psi = \Omega t$$

$$\dot{\beta} = \frac{d\beta}{dt} = \frac{d\beta}{d\psi}\frac{d\psi}{dt} = \Omega\frac{d\beta}{d\psi} = \Omega\beta^*$$

- Also

$$\ddot{\beta} = \frac{d^2\beta}{dt^2} = \Omega^2\frac{d^2\beta}{d\psi^2} = \Omega^2\beta^{**}$$

Motion for a Flapping Blade

- The flapping equation can be written as:

$$\left(\frac{d^2\beta}{d\psi^2}+\beta\right)=\frac{1}{I_b\Omega^2}\int_0^R Lydy \Rightarrow \left(\beta^{**}+\beta\right)=\frac{1}{I_b\Omega^2}\int_0^R Lydy$$

- Knowing that (from BET)

$$L=\tfrac{1}{2}\rho U_T^2 c C_{l_\alpha}\left(\theta-\frac{\dot{\beta}y}{U_T}-\frac{v_i}{U_T}\right)$$

Motion for a Flapping Blade

- The aerodynamic moment is:

$$\int_0^R Lydy = \int_0^R \tfrac{1}{2}\rho U_T^2 c C_{l_\alpha}\left(\theta - \frac{\dot{\beta}y}{U_T} - \frac{v_i}{U_T}\right) ydy =$$

$$= \tfrac{1}{2}\rho\Omega^2 c C_{l_\alpha}\int_0^R \left(\theta - \frac{\dot{\beta}}{\Omega} - \frac{v_i}{\Omega y}\right) y^3 dy =$$

$$= \tfrac{1}{8}\rho\Omega^2 c C_{l_\alpha} R^4\left[\left(\theta - \frac{\dot{\beta}}{\Omega} - \frac{4\lambda_i}{3}\right)\right]$$

Motion for a Flapping Blade

- The flapping equation is therefore:

$$\left(\beta^{**}+\beta\right)=\frac{1}{I_b\Omega^2}\left(\tfrac{1}{8}\rho\Omega^2 cC_{l_\alpha}R^4\left[\left(\theta-\frac{\dot{\beta}}{\Omega}-\frac{4\lambda_i}{3}\right)\right]\right)\Rightarrow$$

$$\Rightarrow\left(\beta^{**}+\beta\right)=\frac{\rho cC_{l_\alpha}R^4}{I_b}\frac{1}{8}\left(\theta-\beta^{*}-\frac{4\lambda_i}{3}\right)$$

- Defining the Lock number as: $\gamma=\dfrac{\rho cC_{l_\alpha}R^4}{I_b}$

Motion for a Flapping Blade

- The final form of the flapping equation is:

$$\beta^{**} + \frac{\gamma}{8}\beta^{*} + \beta = \frac{\gamma}{8}\left(\theta - \frac{4\lambda_i}{3}\right)$$

- If we had left the aerodynamic moment unintegrated:

$$\beta^{**} + \beta = \gamma\,\overline{M}_{\beta} \quad \text{with} \quad \overline{M}_{\beta} = \frac{1}{\rho c C_{l_\alpha} R^4 \Omega^2}\int_0^R Lydy$$

Motion for a Flapping Blade

- Comparing the equation obtained :

$$\beta^{**} + \frac{\gamma}{8}\beta^{*} + \beta = \frac{\gamma}{8}\left(\theta - \frac{4\lambda_i}{3}\right)$$

- With a spring-mass-damper system:

$$m\ddot{x} + c\dot{x} + kx = F$$

- We can conclude that the undamped natural frequency of the blade is

$$\varpi_n = \left(\sqrt{k/m}\right) \quad = 1/rev \quad \text{or} \quad \Omega rad/s$$

Motion for a Flapping Blade

- For the study of the flapping equation let's first consider the case of the rotor in vacuum (no aerodynamic forces)

$$\beta^{**} + \beta = 0 \quad \text{with the solution} \quad \beta = \beta_{1c}\cos\psi + \beta_{1s}\sin\psi$$

- The rotor acts like a gyroscope
- With the introduction of the aerodynamic forces the rotor will precess to a new orientation until the aerodynamic damping causes equilibrium to be obtained again

Motion for a Flapping Blade

- Let's now assume that we have uniform inflow (in forward flight) and a linearly twisted blade. Using BET:

$$\overline{M}_\beta = \frac{1}{\rho c C_{l_\alpha} R^4 \Omega^2} \int_0^R y dF_z = \frac{1}{2} \int_0^1 r \left[\left(\frac{U_T}{\Omega R} \right)^2 \theta + \left(\frac{U_P}{\Omega R} \right) \left(\frac{U_T}{\Omega R} \right) \right] dr$$

- Substituting U_T and U_P for the expressions obtained with BET and solving the integral

$$\overline{M}_\beta = \theta_0 \left(\tfrac{1}{8} + \tfrac{\mu}{3} \sin\psi + \tfrac{\mu^2}{4} \sin^2\psi \right) + \theta_{tw} \left(\tfrac{1}{10} + \tfrac{\mu}{4} \sin\psi + \tfrac{\mu^2}{6} \sin^2\psi \right) -$$

$$- \lambda \left(\tfrac{1}{6} + \tfrac{\mu}{4} \sin\psi \right) + \beta^* \left(\tfrac{1}{8} + \tfrac{\mu}{6} \sin\psi \right) - \beta\mu\cos\psi \left(\tfrac{1}{6} + \tfrac{\mu}{4} \sin\psi \right)$$

Motion for a Flapping Blade

- In forward flight $\mu \neq 0$ and the flapping equation does not have a analytical solution
- The damping term (associated with β^*) is of aerodynamic origin.

$$\frac{\gamma}{8}\left(1+\frac{4}{3}\mu\sin\psi\right)$$

- For hover and knowing for example that $\gamma=8$ we get a damping which is 50% of the critical value. Therefore the flapping motion is well damped and stable

Motion for a Flapping Blade

- To solve the equation we can:
  - Prescribe the values for:
    - Collective pitch θ_0
    - Lateral cyclic θ_{1c}
    - Longitudinal cyclic θ_{1s}
    - Inflow λ_i
  - Integrate numerically
  - However it does not give any insight as how the blade flapping response is affected by the various parameters

Motion for a Flapping Blade

- Alternatively we can:
  - Find a periodic solution
    - Steady state periodic solution in the form of a Fourier series
    - It is not valid for transient situations such as manoeuvres

- Let's then assume a first harmonic solution :

$$\beta(\psi) = \beta_0 + \beta_{1c}\cos\psi + \beta_{1s}\sin\psi$$

Motion for a Flapping Blade

- Harmonically matching constant and periodic terms on both sides of the derived flapping equation:

$$
\begin{cases}
\beta_0 = \gamma\left[\frac{\theta_0}{8}\left(1+\mu^2\right)+\frac{\theta_{tw}}{10}\left(1+\frac{5}{6}\mu^2\right)+\frac{\mu}{6}\theta_{1s}-\frac{\lambda}{6}\right] \\
\beta_{1s}-\theta_{1c} = \left(-\frac{4}{3}\mu\beta_0\right)\Big/\left(1+\frac{1}{2}\mu^2\right) \\
\beta_{1c}+\theta_{1s} = \left(-\frac{8}{3}\mu\right)\left[\theta_0-\frac{3}{4}\lambda+\frac{3}{4}\mu\theta_{1s}+\frac{3}{4}\theta_{tw}\right]\Big/\left(1-\frac{1}{2}\mu^2\right)
\end{cases}
$$

Motion for a Flapping Blade

- In hover flight $\mu=0$: $\begin{cases} \beta_{1s} - \theta_{1c} = 0 \\ \beta_{1c} + \theta_{1s} = 0 \end{cases}$

- That is if the cyclic pitch motion is assumed as:

$$\theta = \theta_0 + \theta_{1c} \cos\psi + \theta_{1s} \sin\psi$$

- The flapping response is:

$$\beta(\psi) = \beta_0 + \theta_{1c} \cos\left(\psi - \tfrac{\pi}{2}\right) + \theta_{1s} \sin\left(\psi - \tfrac{\pi}{2}\right)$$

- The flapping response lags the blade pitch inputs by 90º

Motion for a Flapping Blade

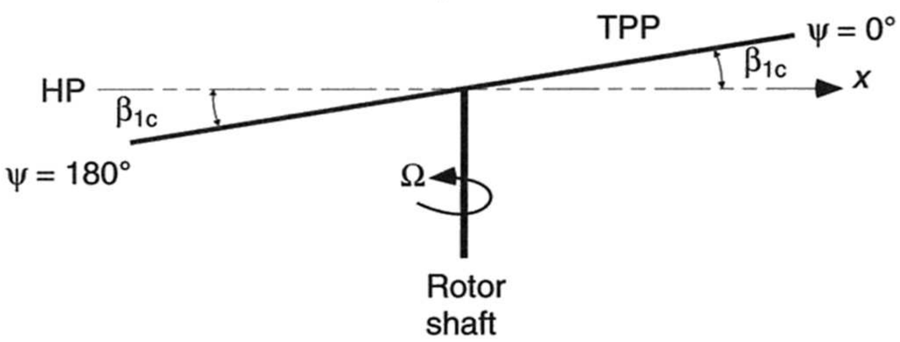
- We have seen that the flapping motion is of the type:

$$\beta(\psi) = \beta_0 + \beta_{1c}\cos\psi + \beta_{1s}\sin\psi$$

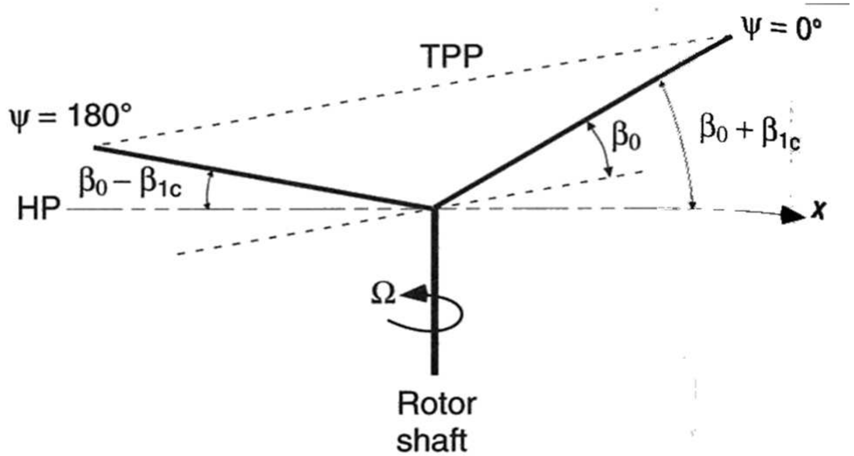
- The term β_0 is the average or mean part of the flapping motion that is independent of the blade azimuth position.

Motion for a Flapping Blade

- The term β_{1c} is the amplitude of the pure cosine flapping motion. This represents the longitudinal tilt of the rotor tip path plane:



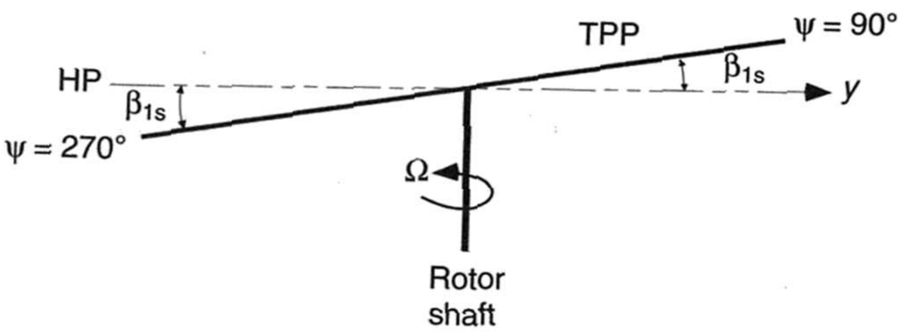
Pure longitudinal tilt (no coning)



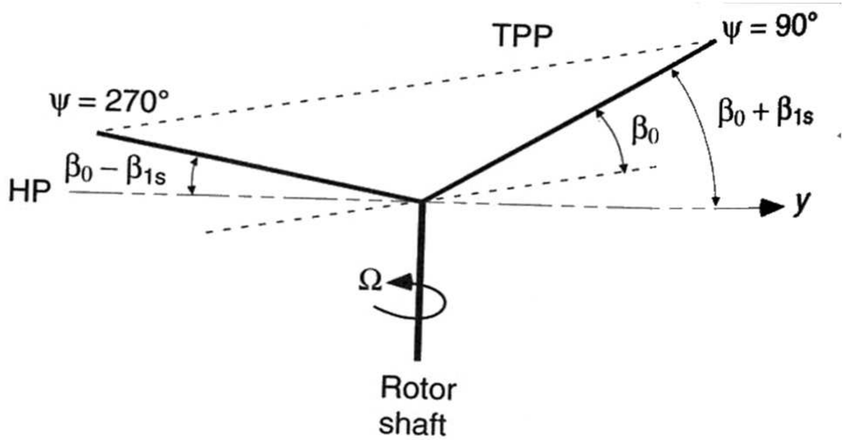
Longitudinal tilt (with coning)

Motion for a Flapping Blade

- The term β_{1s} is the amplitude of the pure sine flapping motion. This represents the lateral tilt of the rotor tip path plane:



Pure lateral tilt (no coning)



Lateral tilt (with coning)

Motion for a Flapping Blade

- The analysis of the blade flapping with an hinge offset is similar to the one just performed.
- The differences are:
  - Inertial force is $m(y\text{-}eR)\ddot{\beta}\,dy$ acting at a distance $(y\text{-}eR)$ from the hinge
  - Centrifugal force $my\Omega^2 dy$ acting at a distance $(y\text{-}eR)\beta$ from the hinge
  - Aerodynamic lift forces Ldy acting at a distance $(y\text{-}eR)$ from the hinge

Motion for a Flapping Blade

- The moments equation about the flapping hinge:

$$\int_{eR}^{R} m\Omega^2 y(y-eR)\beta dy + \int_{eR}^{R} m(y-eR)^2 \ddot{\beta} dy - \int_{eR}^{R} L(y-eR)dy = 0$$

- In this case the blade moment of inertia about the flapping hinge is:

$$I_b = \int_{eR}^{R} m(y-eR)^2 dy$$

Motion for a Flapping Blade

- The equation of the flapping blade is:

$$I_b\left\{\ddot{\beta}+\Omega^2\left(1+\frac{eR\int_{eR}^{R} m(y-eR)dy}{I_b}\right)\beta\right\}=\int_{eR}^{R} L(y-eR)dy$$

- or

$$I_b\left\{\beta^{**}+v_\beta^2\beta\right\}=\frac{1}{\Omega^2}\int_{eR}^{R} L(y-eR)dy$$

Motion for a Flapping Blade

- In the last expression

$$\nu_\beta^2 = 1 + \frac{eR\int_{eR}^{R} m(y - eR)\,dy}{I_b}$$

- With the analogy of the mass-spring-damper system, the undamped frequency of the rotor is:

$$\nu_\beta = \varpi_n = \sqrt{1 + \frac{3e}{2(1-e)}} \approx \sqrt{1 + \frac{3e}{2}}$$

- Since the values of *e* are small the undamped natural frequency is only slightly higher than *1/rev*

Motion for a Flapping Blade

- This also means that the phase lag between the forcing and the rotor flapping response must be less than 90º. In this case the flapping equation is:

$$\overset{**}{\beta} + \nu_\beta^2 \beta = \gamma\, \overline{M}_\beta$$

- Therefore in hover the flapping response to cyclic pitch input is given by:

$$\begin{cases} \beta_{1c}\left(\nu_\beta^2 - 1\right) + \beta_{1s}\dfrac{\gamma}{8} = \dfrac{\gamma}{8}\theta_{1c} \\ \beta_{1s}\left(\nu_\beta^2 - 1\right) - \beta_{1c}\dfrac{\gamma}{8} = \dfrac{\gamma}{8}\theta_{1s} \end{cases}$$

Motion for a Flapping Blade

- Which gives for the longitudinal flapping angle

$$\beta_{1c} = \frac{-\theta_{1s} + \left(\nu_\beta^2 - 1\right)\frac{8}{\gamma}\theta_{1c}}{1 + \left[\left(\nu_\beta^2 - 1\right)\frac{8}{\gamma}\right]^2}$$

Motion for a Flapping Blade

- And gives for the lateral flapping angle

$$\beta_{1s} = \frac{\theta_{1c} + \left(\nu_\beta^2 - 1\right)\frac{8}{\gamma}\theta_{1s}}{1 + \left[\left(\nu_\beta^2 - 1\right)\frac{8}{\gamma}\right]^2}$$

Motion for a Flapping Blade

- Finally the forcing frequency 1/rev is less than the natural flapping frequency and it can be shown that the phase lag will be less than 90º as given by:

$$\phi = \tan^{-1}\left(\frac{\gamma\left(1-\frac{8e}{3}\right)}{8\left(\nu_\beta^2-1\right)}\right) \approx \tan^{-1}\left(\frac{\frac{\gamma}{8}}{\left(\nu_\beta^2-1\right)}\right)$$