

Measure Theory and Fourier Analysis

Summary Notes for MATH3969
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Introduction

Definition 0.1 (0.1). *Let X be a set and \mathcal{A} a collection of subsets of X . We call \mathcal{A} a σ -algebra if*

- (i) $\emptyset \in \mathcal{A}$;
- (ii) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$;
- (iii) $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ implies $\bigcup_{j \in \mathbb{N}} A_j \in \mathcal{A}$.

Definition 0.2 (0.2). *Let X be a set and \mathcal{A} a σ -algebra of subsets of X . A function $\mu: \mathcal{A} \rightarrow [0, \infty]$ is called a measure if*

- (i) $\mu(\emptyset) = 0$;
- (ii) $A_j \in \mathcal{A}, j \in \mathbb{N}$ are such that $A_k \cap A_j = \emptyset$ for $j \neq k$, then $\mu\left(\bigcup_{j \in \mathbb{N}} A_j\right) = \sum_{j=0}^{\infty} \mu(A_j)$.

1 σ -algebras

Definition 1.1 (1.1). If X is a set we call the class of all subsets of X the power set of X and denote it by $\mathcal{P}(X)$.

Definition 1.2 (1.2). Let \mathcal{A} be a collection of subsets of a set X , that is, $\mathcal{A} \subseteq \mathcal{P}(X)$. We call \mathcal{A} a σ -algebra of subsets of X if it has the following properties:

- (i) $\emptyset \in \mathcal{A}$;
- (ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;
- (iii) If $A_k \in \mathcal{A}$ for all $k \in \mathbb{N}$, then $\bigcup_{k \in \mathbb{N}} A_k \in \mathcal{A}$.

We call \mathcal{A} an algebra if instead of (iii) we only have

- (iii) If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.

Proposition 1.1 (1.5). Let \mathcal{A} be a σ -algebra of subsets of X . Then the following statements are true:

- (i) $\emptyset, X \in \mathcal{A}$;
- (ii) If $A_k \in \mathcal{A}$, $k \in \mathbb{N}$, then $\bigcap_{k \in \mathbb{N}} A_k \in \mathcal{A}$;
- (iii) If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$ and $A \cap B \in \mathcal{A}$.

Proposition 1.2 (1.6). Let I be an arbitrary index set and suppose that for every $i \in I$, \mathcal{A}_i is a σ -algebra of subsets of X . Then

$$\mathcal{A} := \bigcap_{i \in I} \mathcal{A}_i \subseteq \mathcal{P}(X)$$

is a σ -algebra of subsets of X .

Definition 1.3 (1.7). (a) Let C be a collection of subsets of X . We call $\mathcal{A}(C)$ the σ -algebra generated by C .

(b) Let X be a metric (or more generally a topological) space and C the collection of all open subsets of X . Then $\mathcal{B} := \mathcal{A}(C)$ is called the Borel σ -algebra in X . Sets in \mathcal{B} are called Borel sets.

Definition 1.4 (1.9). Let X be a set and A_k , $k \in I$, be a collection of subsets, where I is an arbitrary index set. We say that this collection consists of disjoint sets if $A_j \cap A_k = \emptyset$ whenever $j \neq k$.

Lemma 1.1 (1.10). Let $A_n \in X$ for $n \in \mathbb{N}$. Set

$$B_0 := A_0 \quad \text{and} \quad B_n := A_n \cap (A_0 \cup A_1 \cup \dots \cup A_{n-1})^c \quad \text{for } n \geq 1.$$

Then $\bigcup_{k=0}^n B_k = \bigcup_{k=0}^n A_k$ for all $n \geq 0$ and B_k , $k \in \mathbb{N}$, is disjoint.

Proposition 1.3 (1.11). Suppose that \mathcal{A} is a collection of subsets of X with the following properties.

- (i) $\emptyset \in \mathcal{A}$;

- (ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;
- (iii) If $A_k \in \mathcal{A}$, $k \in \mathbb{N}$, are disjoint, then $\bigcup_{k \in \mathbb{N}} A_k \in \mathcal{A}$;
- (iv) If $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

Then \mathcal{A} is a σ -algebra.

Proposition 1.4 (1.12). Suppose that X, Y are sets and $f: X \rightarrow Y$ a function.

- (i) If \mathcal{A} is a σ -algebra in Y , then

$$\mathcal{A}_0 := \{f^{-1}[A] : A \in \mathcal{A}\}$$

is a σ -algebra in X .

- (ii) If \mathcal{A} is a σ -algebra in X , then

$$\mathcal{A}_1 := \{A \subseteq Y : f^{-1}[A] \in \mathcal{A}\}$$

is a σ -algebra in Y .

Definition 1.5 (1.13). Let $f: X \rightarrow Y$ be a function and \mathcal{A} a σ -algebra of subsets of Y . The σ -algebra $\mathcal{A}_0 := \{f^{-1}[A] : A \in \mathcal{A}\}$ is called the σ -algebra generated by f .

2 Measures

Definition 2.1 (2.1). Let \mathcal{A} be a σ -algebra of subsets of X . A function $\mu: \mathcal{A} \rightarrow [0, \infty]$ is called a measure if

- (i) $\mu(\emptyset) = 0$;

- (ii) for every countable collection $A_k \in \mathcal{A}$, $k \in \mathbb{N}$, of disjoint sets

$$\mu\left(\bigcup_{k=0}^{\infty} A_k\right) = \sum_{k=0}^{\infty} \mu(A_k) \quad (\text{countable additivity}).$$

Moreover, if $A \in \mathcal{A}$ we call the set A measurable (with respect to μ). Finally the triple (X, \mathcal{A}, μ) is called a measure space. If $\mu(X) = 1$ we sometimes call μ a probability measure and (X, \mathcal{A}, μ) a probability space. If X is a metric space, we call μ a Borel measure if $\mathcal{B} \subseteq \mathcal{A}$, that is, if all Borel sets are μ -measurable.

Example 2.1 (2.2). Let X be a set and fix $x \in X$. For $A \subseteq X$ we set

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then $\delta_x: \mathcal{P}(X) \rightarrow [0, \infty)$ is a measure. It is called the Dirac measure concentrated at x .

Example 2.2 (2.3). Let X be a set and fix $x \in X$. For $A \subseteq X$ we set

$$\mu(A) := \begin{cases} \#A & \text{if } A \text{ has finite cardinality,} \\ \infty & \text{otherwise,} \end{cases}$$

where $\#A$ is the cardinality of A . Then $\mu: \mathcal{P}(X) \rightarrow [0, \infty]$ is a measure. It is called the counting measure on X .

Proposition 2.1 (2.5). Let (X, \mathcal{A}, μ) be a measure space, Y a set and $f: X \rightarrow Y$ a function. Define $\mathcal{A}_1 = \{A \subseteq Y : f^{-1}[A] \in \mathcal{A}\}$ and

$$\mu_f(A) := \mu(f^{-1}[A])$$

for all $A \in \mathcal{A}_1$. Then $(Y, \mathcal{A}_1, \mu_f)$ is a measure space.

Proposition 2.2 (2.6). Let (X, \mathcal{A}, μ) be a measure space.

(i) If $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

(ii) If $A, B \in \mathcal{A}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

(iii) If $A, A_k \in \mathcal{A}$, $k \in \mathbb{N}$, and $A \subseteq \bigcup_{k=0}^{\infty} A_k$, then

$$\mu(A) \leq \sum_{k=0}^{\infty} \mu(A_k).$$

The last property is referred to as the countable sub-additivity of a measure.

Proposition 2.3 (2.7). Let (X, \mathcal{A}, μ) be a measure space.

(i) If $A_k \in \mathcal{A}$, $k \in \mathbb{N}$, and $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$, then

$$\lim_{k \rightarrow \infty} \mu(A_k) = \mu\left(\bigcup_{k=0}^{\infty} A_k\right).$$

(ii) If $A_k \in \mathcal{A}$, $k \in \mathbb{N}$, with $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ and $\mu(A_k) < \infty$ for some $k \in \mathbb{N}$, then

$$\lim_{k \rightarrow \infty} \mu(A_k) = \mu\left(\bigcap_{k=0}^{\infty} A_k\right).$$

Remark 2.1 (2.8). Without the assumption that $\mu(A_k) < \infty$ (and therefore $\mu(A_j) < \infty$ for all $j \geq k$) it is possible that (ii) is not true.

3 The construction of measures from outer measures

Definition 3.1 (3.1). For every subset $A \subseteq \mathbb{R}^N$ we set

$$m_N^*(A) := \inf \left\{ \sum_{k=0}^{\infty} \text{vol}(R_k) : R_k, k \in \mathbb{N}, \text{ open rectangles with } A \subseteq \bigcup_{k=0}^{\infty} R_k \right\}.$$

We call $m_N^*(A)$ the Lebesgue outer measure of the set A .

Definition 3.2 (3.3). A function $\mu^*: \mathcal{P}(X) \rightarrow [0, \infty]$ is called an outer measure if

$$(i) \quad \mu^*(\emptyset) = 0;$$

(ii) for every countable collection $A, A_k \subseteq X, k \in \mathbb{N}$ with $A \subseteq \bigcup_{k=0}^{\infty} A_k$

$$\mu^*(A) \leq \sum_{k=0}^{\infty} \mu^*(A_k) \quad (\text{countable sub-additivity}).$$

Definition 3.3 (3.5). Let $\mu^*: \mathcal{P}(X) \rightarrow [0, \infty]$ be an outer measure on the set X . We call a set $A \subseteq X$ a μ^* -measurable set if

$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c)$$

for all $S \subseteq X$.

Remark 3.1 (3.6). Since $S = (S \cap A) \cup (S \cap A^c)$ for all sets $A, S \subseteq X$ it follows that

$$\mu^*(S) \leq \mu^*(S \cap A) + \mu^*(S \cap A^c)$$

for all $A, S \subseteq X$. Hence to prove that A is μ^* -measurable we only need to show that

$$\mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \cap A^c)$$

for all $S \subseteq X$.

Proposition 3.1 (3.7). Let μ^* be an outer measure and \mathcal{A} the set of μ^* -measurable sets. Then the following assertions are true.

(i) If $\mu^*(A) = 0$, then $A \in \mathcal{A}$;

(ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;

(iii) If $A \in \mathcal{A}, B, S \subseteq X$ and $A \cap B = \emptyset$, then

$$\mu^*(S \cap (A \cup B)) = \mu^*(S \cap A) + \mu^*(S \cap B);$$

(iv) If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$ and $A \cap B \in \mathcal{A}$.

Theorem 3.1 (3.8: Carathéodory). Let $\mu^*: \mathcal{P}(X) \rightarrow [0, \infty]$ be an outer measure and let

$$\mathcal{A} := \{A \subseteq X : \mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c) \text{ for all } S \subseteq X\}.$$

Then \mathcal{A} is a σ -algebra and $\mu := \mu^*|_{\mathcal{A}}: \mathcal{A} \rightarrow [0, \infty]$ is a measure.

4 The Lebesgue measure

Definition 4.1 (4.1). *We call*

$$\mathcal{M}_N := \{A \subseteq \mathbb{R}^N : m_N^*(S) = m_N^*(S \cap A) + m_N^*(S \cap A^c) \text{ for all } S \subseteq \mathbb{R}^N\}$$

the Lebesgue σ -algebra and $m_N := m_N^|_{\mathcal{M}_N}$ the (N -dimensional) Lebesgue measure. Sets in \mathcal{M}_N are called Lebesgue measurable or simply measurable subsets of \mathbb{R}^N .*

Proposition 4.1 (4.2). *Let R be an open rectangle in \mathbb{R}^N . Then*

$$m_N^*(R) = m_N^*(\bar{R}) = \text{vol}(R),$$

where \bar{R} is the closure of R .

Proposition 4.2 (4.3). *For every $\delta > 0$ and $A \subseteq \mathbb{R}^N$ we have*

$$m_N^*(A) = \inf \left\{ \sum_{k=0}^{\infty} \text{vol}(Q_k) : Q_k \text{ open cube, } A \subseteq \bigcup_{k=0}^{\infty} Q_k, \text{diam}(Q_k) < \delta \right\}.$$

Proposition 4.3 (4.4). *Let $A, B \subseteq \mathbb{R}^N$ such that*

$$\delta := \text{dist}(A, B) := \inf\{\|x - y\| : x \in A, y \in B\} > 0.$$

Then $m_N^(A \cup B) = m_N^*(A) + m_N^*(B)$.*

Theorem 4.1 (4.5). *The Lebesgue measure is a Borel measure, that is, $\mathcal{B}_N \subseteq \mathcal{M}_N$.*

5 Regularity of the Lebesgue Measure

Proposition 5.1 (5.1). *For every $t \in \mathbb{R}^N$ and every $A \subseteq \mathbb{R}^N$ we have $m_N^*(t + A) = m_N^*(A)$.*

Proposition 5.2 (5.2). (i) *Let $A \subseteq \mathbb{R}^N$ be an arbitrary set. Then*

$$m_N^*(A) = \inf\{m_N(U) : A \subseteq U, U \text{ open}\}.$$

(ii) *Let $A \subseteq \mathbb{R}^N$ be Lebesgue measurable. Then*

$$m_N(A) = \sup\{m_N(K) : K \subseteq A, K \text{ compact}\}.$$

Definition 5.1 (5.3). *A Borel measure on a metric space X is called outer regular if for every Borel set $A \subseteq X$*

$$\mu(A) = \inf\{\mu(U) : A \subseteq U, U \text{ open}\}.$$

The measure is called inner regular if

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, K \text{ compact}\}.$$

Corollary 5.1 (5.4). *For every Lebesgue measurable set $A \subseteq \mathbb{R}^N$ the following assertions are true.*

(i) For every $\varepsilon > 0$ there exists an open set $U \subseteq \mathbb{R}^N$ such that $A \subseteq U$ and

$$m_N(U \setminus A) < \varepsilon.$$

(ii) There exist a sequence $(U_k)_{k \in \mathbb{N}}$ of open sets with $A \subseteq U_k$ for all $k \in \mathbb{N}$ and a set $S \subseteq A^c$ with $m_N(S) = 0$ such that

$$B := \bigcap_{k=1}^{\infty} U_k = A \cup S$$

(iii) There exist a sequence $(C_k)_{k \in \mathbb{N}}$ of compact sets with $C_k \subseteq A$ for all $k \in \mathbb{N}$ and a set $S \subseteq A$ with $m_N(S) = 0$ such that

$$A = \left(\bigcup_{k=1}^{\infty} C_k \right) \cup S.$$

6 Uniqueness of the Lebesgue Measure

Lemma 6.1 (6.1). Let $U \subseteq \mathbb{R}^N$ be an open set. Then there exist disjoint dyadic cubes $Q_j \in \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$ such that $U = \bigcup_{j=0}^{\infty} Q_j$.

Lemma 6.2 (6.2). Let μ be a translation invariant Borel measure on \mathbb{R}^N . If $Q_{n,k}$ is the dyadic cube given by (6.1), and $\alpha := \mu((0, 1]^N) < \infty$, then

$$\mu(Q_{n,k}) = \alpha m_N(Q_{n,k}) = \alpha 2^{-nN}$$

for all $k \in \mathbb{Z}^N$ and all $n \in \mathbb{N}$.

Theorem 6.1 (6.3). Let μ be an outer regular Borel measure on \mathbb{R}^N . If μ is translation invariant and $\alpha := \mu((0, 1]^N) < \infty$, then

$$\mu(A) = \alpha m_N(A)$$

for all Borel sets $A \subseteq \mathbb{R}^N$.

Remark 6.1 (6.4). One can show that every Borel measure μ on \mathbb{R}^N with $\mu(K) < \infty$ for every compact set K is outer regular.

7 Lebesgue Measure and Linear Transformations

Theorem 7.1 (7.1). Let $T: \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a linear transformation. Then

$$m_N^*(T(A)) = |\det T| m_N^*(A)$$

for every $A \subseteq \mathbb{R}^N$. Moreover, if A is Lebesgue measurable, then $T(A)$ is Lebesgue measurable as well.

Remark 7.1 (7.2). To prove Theorem 7.1 it is in fact sufficient to prove that

$$m_N^*(T(A)) \leq |\det T| m_N^*(A).$$

Lemma 7.1 (7.3). Let $\alpha \neq 0$ and $A \subseteq \mathbb{R}^N$. Suppose that

$$E(x_1, \dots, x_N) := (x_1, \dots, x_{i-1}, \alpha x_i, x_{i+1}, \dots, x_N)$$

where $1 \leq i \leq N$. Then $m_N^*(E(A)) \leq |\alpha|m_N^*(A) = |\det E|m_N^*(A)$.

Lemma 7.2 (7.4). Let $A \subseteq \mathbb{R}^N$ and suppose that

$$E(x_1, \dots, x_N) := (x_1, \dots, x_{j-1}, x_j + x_i, x_{j+1}, \dots, x_N).$$

where $1 \leq i, j \leq N$ and $i \neq j$. Then $m_N^*(E(A)) \leq |\det E|m_N^*(A) = m_N^*(A)$.

Lemma 7.3 (7.5). Let $A \subseteq \mathbb{R}^N$ be an arbitrary set and T an invertible matrix. Then $m_N^*(T(A)) \leq |\det(T)|m_N^*(A)$.

Lemma 7.4 (7.6). Suppose that

$$S_r := \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

for some $0 \leq r < N$. Then $m_N^*(S_r(A)) = 0$.

8 The Lebesgue-Stieltjes measure

Proposition 8.1 (8.1). Let μ be a Borel measure on \mathbb{R} such that $\mu([a, b]) < \infty$ for every compact interval $[a, b] \subseteq \mathbb{R}$. Then the function $F: \mathbb{R} \rightarrow [0, \infty)$ given by

$$F(t) := \begin{cases} \mu((0, t]) & \text{if } t \geq 0, \\ -\mu((t, 0]) & \text{if } t < 0. \end{cases}$$

is increasing and right continuous, and $\mu((a, b]) = F(b) - F(a)$ whenever $a < b$.

Example 8.1 (8.2). (a) If $\mu = m_1$ is the Lebesgue measure, then clearly $F(x) = x$.

(b) Let $\mu := \delta_0$ be the Dirac measure on \mathbb{R} concentrated at $x = 0$. Then

$$F(t) = \begin{cases} -1 & \text{if } t < 0, \\ 0 & \text{if } t \geq 0. \end{cases}$$

Theorem 8.1 (8.3: Lebesgue-Stieltjes measure). There is a one-to-one correspondence (up to a constant) between right continuous increasing functions and regular Borel measures on \mathbb{R} :

- (i) For every regular Borel measure μ on \mathbb{R} there exists a right continuous increasing function $F: \mathbb{R} \rightarrow \mathbb{R}$ such that $\mu((a, b]) = F(b) - F(a)$ holds. Moreover, that function is unique up to an additive constant.
- (ii) For every right continuous increasing function $F: \mathbb{R} \rightarrow \mathbb{R}$ there exists a unique regular Borel measure such that $\mu((a, b]) = F(b) - F(a)$ holds.

Lemma 8.1 (8.4). For every $A \subseteq \mathbb{R}$ we have

$$\mu_F^*(A) = \inf \left\{ \sum_{k=0}^{\infty} \nu_F((a_k, b_k]): A \subseteq \bigcup_{k=0}^{\infty} (a_k, b_k) \right\}.$$

Lemma 8.2 (8.5). Let μ_F^* be the outer measure defined by (8.4). Then

$$\mu_F^*((a, b]) = F(b) - F(a).$$

Proposition 8.2 (8.6). The measure μ_F is outer regular.

9 Measurable functions

Definition 9.1 (9.1). Let (X, \mathcal{A}, μ) be a measure space and Y a topological space (often a subset of \mathbb{R} or \mathbb{C}). We call a function $f: X \rightarrow Y$ μ -measurable if $f^{-1}[U] \in \mathcal{A}$ for all open sets $U \subseteq Y$.

Definition 9.2 (9.4). Let X be a set and $A \subseteq X$ a subset. The function $1_A: X \rightarrow \mathbb{R}$ given by

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A \end{cases}$$

is called the indicator function of A .

Proposition 9.1 (9.5). Let (X, \mathcal{A}, μ) be a measure space and $A \subseteq X$. Then 1_A is a measurable function if and only if A is a measurable set.

Theorem 9.1 (9.6). Let (X, \mathcal{A}, μ) be a measure space and $f: X \rightarrow [-\infty, \infty]$ a function. Then the following assertions are equivalent:

- (i) f is measurable;
- (ii) $f^{-1}[(\alpha, \infty)]$ is measurable for all $\alpha \in \mathbb{Q}$;
- (iii) $f^{-1}[[\alpha, \infty]]$ is measurable for all $\alpha \in \mathbb{Q}$;
- (iv) $f^{-1}[[-\infty, \alpha]]$ is measurable for all $\alpha \in \mathbb{Q}$;
- (v) $f^{-1}[[-\infty, \alpha]]$ is measurable for all $\alpha \in \mathbb{Q}$.

We can also replace \mathbb{Q} by some other dense subset of \mathbb{R} in all the above statements.

Proposition 9.2 (9.7). Let (X, \mathcal{A}, μ) be a measure space and $f = (f_1, \dots, f_N): X \rightarrow \mathbb{K}^N$ a function. Then f is measurable if and only if every component function $f_k: X \rightarrow \mathbb{K}$ is measurable.

Proposition 9.3 (9.8). Let (X, \mathcal{A}, μ) be a measure space and Y, Z metric spaces. If $f: X \rightarrow Y$ is measurable and $\varphi: Y \rightarrow Z$ continuous, then $\varphi \circ f: X \rightarrow Z$ is measurable.

Theorem 9.2 (9.9). Let (X, \mathcal{A}, μ) be a measure space and $f, g: X \rightarrow [-\infty, \infty]$ measurable. Then the following functions are measurable as well:

- (i) Let

$$S := X \setminus \{x \in X : f(x) = \infty \text{ and } g(x) = -\infty, \text{ or } f(x) = -\infty \text{ and } g(x) = \infty\}.$$

Then $(f + g)1_S$ is measurable.

- (ii) fg ;
- (iii) $\frac{f}{g}$ if $g(x) \neq 0$ for all $x \in X$;
- (iv) $|f|$, $\max\{f, g\}$ and $\min\{f, g\}$.

10 Sequences of measurable functions

Proposition 10.1 (10.1). *Let (X, \mathcal{A}, μ) be a measure space and $f_n: X \rightarrow [-\infty, \infty]$ be measurable for all $n \in \mathbb{N}$. For $x \in X$ define*

$$g(x) := \inf_{n \in \mathbb{N}} f_n(x) = \inf\{f_0(x), f_1(x), f_2(x), \dots\}$$

and

$$h(x) := \sup_{n \in \mathbb{N}} f_n(x) = \sup\{f_0(x), f_1(x), f_2(x), \dots\}.$$

Then g and h are measurable.

Theorem 10.1 (10.2). *Let (X, \mathcal{A}, μ) be a measure space and let $f_n: X \rightarrow [-\infty, \infty]$ be measurable functions. Then, the functions u and v given by*

$$u(x) = \liminf_{n \rightarrow \infty} f_n(x) \quad \text{and} \quad v(x) = \limsup_{n \rightarrow \infty} f_n(x)$$

are measurable functions. Moreover, if $f_n \rightarrow f$ pointwise, then f is measurable.

11 Simple measurable functions

Definition 11.1 (11.1). *Let X, Y be sets and $f: X \rightarrow Y$ be a function. We call f a simple function if its range is a finite set.*

Proposition 11.1 (11.3). *Suppose that $f, g: X \rightarrow \mathbb{C}$ are simple measurable functions. Then the following functions are also simple measurable functions:*

- (i) $f + g$;
- (ii) αf for all $\alpha \in \mathbb{R}$ (or \mathbb{C});
- (iii) fg ;
- (iv) $\frac{f}{g}$ if $g(x) \neq 0$ for all $x \in X$.

12 Approximation by simple functions

Theorem 12.1 (12.1). *Let (X, \mathcal{A}, μ) be a measure space and $f: X \rightarrow [0, \infty]$ a measurable function. Then there exists a sequence of simple measurable functions $\varphi_n: X \rightarrow [0, \infty)$ such that*

$$0 \leq \varphi_n(x) \leq \varphi_{n+1}(x) \leq f(x)$$

for all $n \in \mathbb{N}$ and all $x \in X$. Moreover, $\varphi_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for all $x \in X$, that is, $\varphi_n \rightarrow f$ pointwise.

Remark 12.1 (12.2). *If $f: X \rightarrow [0, \infty)$ is bounded, then φ_n as constructed in the above proof converges uniformly and not just pointwise to f .*

Corollary 12.1 (12.3). *Let (X, \mathcal{A}, μ) a measure space and $f: X \rightarrow \mathbb{K}$ measurable. Then there exist simple measurable functions $\varphi_n: X \rightarrow \mathbb{K}$ such that*

$$0 \leq |\varphi_n(x)| \leq |\varphi_{n+1}(x)| \leq |f(x)|$$

for all $n \in \mathbb{N}$ and all $x \in X$. Moreover, $\varphi_n \rightarrow f$ pointwise.

13 The integration of non-negative simple functions

Definition 13.1 (14.1). Let (X, \mathcal{A}, μ) be a measure space and $\varphi = \sum_{k=0}^n \alpha_k 1_{A_k}$ be a simple measurable function. We let

$$\int_X \varphi d\mu := \sum_{k=0}^n \alpha_k \mu(A_k).$$

If $\mu(A_k) = \infty$ we set $\alpha_k \mu(A_k) = \infty$ if $\alpha_k > 0$ and $\alpha_k \mu(A_k) = 0$ if $\alpha_k = 0$.

Proposition 13.1 (14.2). Let (X, \mathcal{A}, μ) be a measure space and φ, ψ be simple measurable functions. Then the following assertions are valid.

- (i) $\int_X \varphi + \psi d\mu = \int_X \varphi d\mu + \int_X \psi d\mu$;
- (ii) If $0 \leq \varphi \leq \psi$, then $\int_X \varphi d\mu \leq \int_X \psi d\mu$;
- (iii) $\alpha \int_X \varphi d\mu = \int_X \alpha \varphi d\mu$ for all $\alpha \geq 0$;
- (iv) If $N \in \mathcal{A}$ and $\mu(N) = 0$, then $\int_X \varphi d\mu = \int_X 1_{X \setminus N} \varphi d\mu$.

14 Integration of non-negative measurable functions

Definition 14.1 (15.1). Let (X, \mathcal{A}, μ) be a measure space and $f: X \rightarrow [0, \infty]$ a measurable function. We then set

$$\int_X f d\mu = \sup \left\{ \int_X \varphi d\mu : \varphi \text{ is simple measurable and } 0 \leq \varphi \leq f \right\}.$$

Definition 14.2 (15.4). Let (X, \mathcal{A}, μ) be a measure space and (P) some property. We say that (P) holds almost everywhere on X if there exists a measurable set N such that $\mu(N) = 0$ and (P) holds for all $x \in X \setminus N$.

Lemma 14.1 (15.5: Markov's inequality). Suppose that $f: X \rightarrow [0, \infty]$ is measurable. Then for every $\alpha > 0$ we have that

$$\mu(\{x \in X : f(x) \geq \alpha\}) \leq \frac{1}{\alpha} \int_X f d\mu.$$

Proposition 14.1 (15.6). Suppose that $f: X \rightarrow [0, \infty]$ is measurable.

- (i) If $\int_X f d\mu < \infty$, then $f(x) < \infty$ almost everywhere, that is,

$$\mu(\{x \in X : f(x) = \infty\}) = 0.$$

- (ii) If $\int_X f d\mu = 0$, then $f(x) = 0$ almost everywhere, that is,

$$\mu(\{x \in X : f(x) > 0\}) = 0.$$

Remark 14.1 (15.7). From the above proposition, if the integral of a non-negative function f is zero, then we cannot conclude in general that f is zero.

15 The monotone convergence theorem

Theorem 15.1 (16.1: Monotone convergence theorem). *Let (X, \mathcal{A}, μ) be a measure space. For every $n \in \mathbb{N}$ let*

$$f_n: X \rightarrow [0, \infty]$$

be a measurable function and suppose that

$$0 \leq f_n(x) \leq f_{n+1}(x)$$

for almost every $x \in X$. Then there exists a measurable function $f: X \rightarrow [0, \infty]$ such that $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$ for almost every $x \in X$ and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

Theorem 15.2 (16.2: Fatou's Lemma). *Let (X, \mathcal{A}, μ) be a measure space. For every $n \in \mathbb{N}$ let*

$$f_n: X \rightarrow [0, \infty]$$

be measurable functions. Then

$$\int_X \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu.$$

Proposition 15.1 (16.3). *Suppose that $f, g: X \rightarrow [0, \infty]$ are measurable functions and that $\alpha \geq 0$ is a constant. Then*

$$(i) \int_X f + g d\mu = \int_X f d\mu + \int_X g d\mu;$$

$$(ii) \alpha \int_X f d\mu = \int_X \alpha f d\mu.$$

Theorem 15.3 (16.4). *For every $k \in \mathbb{N}$ let $g_k: X \rightarrow [0, \infty]$ be measurable. Then*

$$\sum_{k=0}^{\infty} \int_X g_k d\mu = \int_X \sum_{k=0}^{\infty} g_k d\mu.$$

16 Integrable functions

Definition 16.1 (17.1). *Let (X, \mathcal{A}, μ) be a measure space and $f: X \rightarrow \mathbb{K}$ measurable (with $\mathbb{K} = \mathbb{R}$ or \mathbb{C} as usual). We call f a μ -integrable function if*

$$\int_X |f| d\mu < \infty.$$

We let

$$\mathcal{L}^1(X, \mathcal{A}, \mu; \mathbb{K}) := \{f: X \rightarrow \mathbb{K} \mid f \text{ is } \mu\text{-integrable}\}$$

and call it the space of integrable functions.

Definition 16.2 (17.3). (i) For $f \in \mathcal{L}^1(X, \mathbb{R})$ we define

$$\int_X f d\mu := \int_X f^+ d\mu - \int_X f^- d\mu.$$

(ii) For $f \in \mathcal{L}^1(X, \mathbb{C})$ we define

$$\int_X f d\mu := \int_X \operatorname{Re} f d\mu + i \int_X \operatorname{Im} f d\mu.$$

(iii) For any μ -measurable set A and $f \in \mathcal{L}^1(X, \mathbb{K})$ we define

$$\int_A f d\mu := \int_X 1_A f d\mu.$$

Theorem 16.1 (17.4). Let $f, g \in \mathcal{L}^1(X, \mathcal{A}, \mu; \mathbb{K})$. Then the following assertions are true.

(i) for all $\alpha, \beta \in \mathbb{K}$

$$\int_X \alpha f + \beta g d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu.$$

(ii) If $A, B \in \mathcal{A}$ are disjoint, then

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu.$$

(iii) We have the "triangle inequality"

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

17 Limit Theorems

Theorem 17.1 (20.1: Dominated convergence theorem). Let $f_n: X \rightarrow \mathbb{K}$ be measurable and $f: X \rightarrow \mathbb{K}$ be such that $f_n(x) \rightarrow f(x)$ pointwise almost everywhere. Furthermore assume that there exists $g \in \mathcal{L}^1(X)$ such that

$$|f_n(x)| \leq g(x)$$

for all $n \in \mathbb{N}$ and for almost all $x \in X$. Then

(i) $f_n, f \in \mathcal{L}^1(X, \mathbb{K})$ for all $n \in \mathbb{N}$;

(ii) $\int_X |f_n - f| d\mu \rightarrow 0$ as $n \rightarrow \infty$;

(iii) $\int_X f_n d\mu \rightarrow \int_X f d\mu$ as $n \rightarrow \infty$.

Corollary 17.1 (20.2). Let $f \in \mathcal{L}^1((a, b), \mathbb{K})$. Then

$$\lim_{x \rightarrow b^-} \int_a^x f(t) dt = \int_a^b f(t) dt$$

Theorem 17.2 (20.3: Continuity of parameter integrals). *Let (X, \mathcal{A}, μ) a measure space and Y a metric space (usually a subset of \mathbb{R} or \mathbb{C}). Suppose that $f: X \times Y \rightarrow \mathbb{K}$ such that*

- $x \mapsto f(x, y)$ is μ -measurable for all $y \in Y$;
- $y \mapsto f(x, y)$ is continuous at y_0 for almost all $x \in X$;
- there exists $g \in L^1(X, \mathbb{R})$ such that

$$|f(x, y)| \leq g(x)$$

for almost all $x \in X$ and all $y \in Y$.

For $y \in Y$ define

$$F(y) := \int_X f(x, y) d\mu.$$

Then F is continuous at y_0 .

Theorem 17.3 (20.4: Differentiation of parameter integrals). *Let (X, \mathcal{A}, μ) a measure space and $J \subseteq \mathbb{R}$ an interval. Suppose that $f: X \times J \rightarrow \mathbb{K}$ is a function such that*

- $x \mapsto f(x, t)$ is μ -integrable for all $t \in J$;
- for almost all $x \in X$, $\frac{\partial f}{\partial t}(x, t)$ exists and is continuous on J ;
- there exists $g \in \mathcal{L}^1(X, \mathbb{R})$ such that

$$\left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x)$$

for almost all $x \in X$ and all $t \in J$.

For $t \in J$ define

$$F(t) := \int_X f(x, t) d\mu.$$

Then $F: J \rightarrow \mathbb{K}$ is differentiable and

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x, t) d\mu$$

for all $t \in J$.

Theorem 17.4 (20.5: Fundamental Theorem of Calculus). (i) If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then

$$\frac{d}{dt} \int_a^t f(s) ds = f(t)$$

for all $t \in [a, b]$.

(ii) If $f: [a, b] \rightarrow \mathbb{R}$ is continuous and $f: (a, b) \rightarrow \mathbb{R}$ continuously differentiable, then

$$\int_a^b f'(s) ds = f(b) - f(a).$$

18 The \mathcal{L}^p -spaces

Definition 18.1 (21.1). Let $1 \leq p < \infty$ and $f: X \rightarrow \mathbb{K}$. We call

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{1/p}$$

the L^p -norm of f . We set

$$\mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{K}) := \{f: X \rightarrow \mathbb{K} \mid f \text{ measurable and } \|f\|_p < \infty\}$$

Lemma 18.1 (21.3: Young's inequality). Let $p, q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$st \leq \frac{1}{p}s^p + \frac{1}{q}t^q$$

for all $s, t \geq 0$.

Theorem 18.1 (21.4: Hölder's inequality). Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_X |fg| d\mu \leq \|f\|_p \|g\|_q$$

for all $f \in \mathcal{L}^p(X)$ and $g \in \mathcal{L}^q(X)$.

Proposition 18.1 (21.5). Let $1 \leq p < q < \infty$. If $\mu(X) < \infty$, then $\mathcal{L}^q(X) \subseteq \mathcal{L}^p(X)$. More precisely, if $f \in \mathcal{L}^q(X)$, then

$$\|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

In general, $\mathcal{L}^p(X) \not\subseteq \mathcal{L}^q(X)$. Moreover, if $\mu(X) = \infty$, then neither $\mathcal{L}^p(X) \not\subseteq \mathcal{L}^q(X)$ nor $\mathcal{L}^q(X) \not\subseteq \mathcal{L}^p(X)$ in general.

Proposition 18.2 (21.6: Minkowski's inequality). For $1 \leq p < \infty$ we have

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

for all $f, g \in \mathcal{L}^p(X)$.

Theorem 18.2 (21.7: Properties of L^p -norm). Let $1 \leq p < \infty$. Then for $f, g \in \mathcal{L}^p(X)$ and $\alpha \in \mathbb{K}$ we have

- (i) $\|f\|_p \geq 0$ with equality if and only if $f = 0$ almost everywhere.
- (ii) $\|\alpha f\|_p = |\alpha| \|f\|_p$
- (iii) $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ (triangle or Minkowski's inequality)

Definition 18.2 (21.8). Let $f_n, f \in \mathcal{L}^p(X)$.

- (i) We say that $f_n \rightarrow f$ in $\mathcal{L}^p(X)$ if $\|f_n - f\|_p \rightarrow 0$ as $n \rightarrow \infty$.

(ii) We say that (f_n) is a Cauchy sequence in $\mathcal{L}^p(X)$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p < \varepsilon$$

for all $n, m > n_0$.

Lemma 18.2 (21.10). Suppose that (f_n) is a Cauchy sequence in $\mathcal{L}^p(X)$. If (f_{n_k}) is a convergent subsequence with $f_{n_k} \rightarrow f$ in $\mathcal{L}^p(X)$, then $f_n \rightarrow f$ in $\mathcal{L}^p(X)$.

Proposition 18.3 (21.11). Let (g_k) be a sequence in $\mathcal{L}^p(X)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|_p < \infty.$$

Then there exists a function $f \in \mathcal{L}^p(X)$ such that $f = \sum_{k=0}^{\infty} g_k$ converges pointwise almost everywhere and in $\mathcal{L}^p(X)$.

Theorem 18.3 (21.12: Completeness of \mathcal{L}^p). Let (f_n) be a Cauchy sequence in $\mathcal{L}^p(X)$. Then there exists a function $f \in \mathcal{L}^p(X)$ such that $f_n \rightarrow f$ in $\mathcal{L}^p(X)$. Moreover, (f_n) has a subsequence that converges pointwise almost everywhere.

Remark 18.1 (21.13). (a) In the above theorem we did not claim that $f_n \rightarrow f$ pointwise almost everywhere.

(b) If we know that $f_n \rightarrow g$ pointwise almost everywhere and $f_n \rightarrow f$ in $\mathcal{L}^p(X)$, then we can conclude that $f = g$ almost everywhere.

(c) If we happen to know that $f_n \rightarrow f$ in $\mathcal{L}^p(X)$ and the increments $\|f_k - f_{k-1}\|_p$ converge "fast enough", then $f_n \rightarrow f$ pointwise almost everywhere.

19 The L^p -spaces

Definition 19.1 (22.1). For $1 \leq p < \infty$ we define

$$L^p(X, \mathcal{A}, \mu, \mathbb{K}) := \{f : f \in \mathcal{L}^p(X, \mathcal{A}, \mu, \mathbb{K})\}$$

We set

$$\|f\|_p := \|f\|_p.$$

Similarly we set

$$f + g := f + g \quad \text{and} \quad \alpha f := \alpha f$$

for all $f, g \in L^p(X)$ and $\alpha \in \mathbb{K}$.

Theorem 19.1 (22.4). For $1 \leq p < \infty$, the space $L^p(X)$ is a complete normed space with respect to the norm $\|\cdot\|_p$.

Remark 19.1 (22.5). (a) A complete normed vector space is called a Banach space, and the Lebesgue spaces $L^p(X)$ is one of the most important example of such spaces.

(b) The case $p = 2$ plays an important role since then the L^2 -norm is induced by the inner product

$$(f | g) = \int_X f \bar{g} d\mu$$

20 Basic density theorems

Theorem 20.1 (23.1: Density of simple functions in L^p). *Let (X, \mathcal{A}, μ) be a measure space and $1 \leq p < \infty$. Then the simple functions are dense in $L^p(X)$. That is, for every $f \in L^p(X)$ and every $\varepsilon > 0$, there exists a simple function φ such that*

$$\|f - \varphi\|_p < \varepsilon.$$

Theorem 20.2 (23.2: Density of continuous functions in $L^p(\mathbb{R}^N)$). *Let $1 \leq p < \infty$. Then the space $C_c(\mathbb{R}^N)$ of continuous functions with compact support is dense in $L^p(\mathbb{R}^N)$.*

Theorem 20.3 (23.3: Density of step functions in $L^p(\mathbb{R}^N)$). *Let $1 \leq p < \infty$. Then the step functions (finite linear combinations of characteristic functions of rectangles) are dense in $L^p(\mathbb{R}^N)$.*

21 The space of bounded measurable functions

Definition 21.1 (24.1). *Let (X, \mathcal{A}, μ) be a measure space. We denote by $\mathcal{B}(X)$ the space of all bounded measurable functions $f : X \rightarrow \mathbb{K}$, equipped with the supremum norm*

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}.$$

Proposition 21.1 (24.2). *The space $\mathcal{B}(X)$ is a Banach space with respect to the supremum norm $\|\cdot\|_\infty$.*

Definition 21.2 (24.3). *For $f \in \mathcal{L}^\infty(X)$, we define the essential supremum norm by*

$$\|f\|_\infty := \inf\{M \geq 0 : |f(x)| \leq M \text{ for almost every } x \in X\}.$$

The space $L^\infty(X)$ consists of equivalence classes of essentially bounded measurable functions.

Theorem 21.1 (24.4: Properties of L^∞). *The space $L^\infty(X)$ is a Banach space with respect to the essential supremum norm $\|\cdot\|_\infty$.*

Proposition 21.2 (24.5: Hölder's inequality for $p = \infty$). *If $f \in L^1(X)$ and $g \in L^\infty(X)$, then $fg \in L^1(X)$ and*

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$

22 Fubini's Theorem

Definition 22.1 (25.1). *Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be measure spaces. The product σ -algebra $\mathcal{A} \otimes \mathcal{B}$ is the σ -algebra generated by all sets of the form $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

Theorem 22.1 (25.2: Existence of product measure). *There exists a unique measure $\mu \otimes \nu$ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ such that*

$$(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Theorem 22.2 (25.3: Fubini's Theorem). Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be σ -finite measure spaces, and let $f : X \times Y \rightarrow \mathbb{K}$ be $\mu \otimes \nu$ -integrable. Then:

(i) For almost every $x \in X$, the function $y \mapsto f(x, y)$ is ν -integrable.

(ii) For almost every $y \in Y$, the function $x \mapsto f(x, y)$ is μ -integrable.

(iii) The functions

$$x \mapsto \int_Y f(x, y) d\nu(y) \quad \text{and} \quad y \mapsto \int_X f(x, y) d\mu(x)$$

are integrable on X and Y respectively.

(iv)

$$\int_{X \times Y} f d(\mu \otimes \nu) = \int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) d\mu(x) \right) d\nu(y).$$

Theorem 22.3 (25.4: Tonelli's Theorem). Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be σ -finite measure spaces, and let $f : X \times Y \rightarrow [0, \infty]$ be measurable. Then:

(i) For almost every $x \in X$, the function $y \mapsto f(x, y)$ is measurable.

(ii) For almost every $y \in Y$, the function $x \mapsto f(x, y)$ is measurable.

(iii) The functions

$$x \mapsto \int_Y f(x, y) d\nu(y) \quad \text{and} \quad y \mapsto \int_X f(x, y) d\mu(x)$$

are measurable.

(iv)

$$\int_{X \times Y} f d(\mu \otimes \nu) = \int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x, y) d\mu(x) \right) d\nu(y).$$

23 Translation of a function

Definition 23.1 (26.1). For $h \in \mathbb{R}^N$ and a function $f : \mathbb{R}^N \rightarrow \mathbb{K}$, we define the translation of f by h as

$$(\tau_h f)(x) := f(x - h).$$

Proposition 23.1 (26.2: Properties of translation). Let $f, g \in L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, and $h, k \in \mathbb{R}^N$. Then:

(i) $\tau_h(\tau_k f) = \tau_{h+k} f$

(ii) $\|\tau_h f\|_p = \|f\|_p$

(iii) $\lim_{h \rightarrow 0} \|\tau_h f - f\|_p = 0$ for $1 \leq p < \infty$

Definition 23.2 (26.3). A linear operator $T : L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ is called translation invariant if

$$T(\tau_h f) = \tau_h(Tf)$$

for all $f \in L^p(\mathbb{R}^N)$ and $h \in \mathbb{R}^N$.

24 Convex functions and Jensen's inequality

Definition 24.1 (27.1). A function $\varphi : (a, b) \rightarrow \mathbb{R}$ is called convex if for all $x, y \in (a, b)$ and $\lambda \in [0, 1]$,

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y).$$

Proposition 24.1 (27.2). If φ is twice differentiable on (a, b) and $\varphi''(x) \geq 0$ for all $x \in (a, b)$, then φ is convex.

Theorem 24.1 (27.3: Jensen's inequality). Let (X, \mathcal{A}, μ) be a probability space (i.e., $\mu(X) = 1$), and let $f \in L^1(X, \mathbb{R})$ be such that $f(x) \in (a, b)$ for almost every $x \in X$. If $\varphi : (a, b) \rightarrow \mathbb{R}$ is convex, then

$$\varphi\left(\int_X f d\mu\right) \leq \int_X (\varphi \circ f) d\mu.$$

Corollary 24.1 (27.4). For $f \in L^p(X)$, $1 \leq p < \infty$, we have

$$\left| \int_X f d\mu \right|^p \leq \int_X |f|^p d\mu.$$

25 Convolution

Definition 25.1 (28.1). For $f, g \in L^1(\mathbb{R}^N)$, the convolution of f and g is defined by

$$(f * g)(x) := \int_{\mathbb{R}^N} f(x - y)g(y) dy.$$

Theorem 25.1 (28.2: Properties of convolution). Let $f, g, h \in L^1(\mathbb{R}^N)$. Then:

- (i) $f * g \in L^1(\mathbb{R}^N)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$
- (ii) $f * g = g * f$ (commutativity)
- (iii) $(f * g) * h = f * (g * h)$ (associativity)
- (iv) $f * (g + h) = f * g + f * h$ (distributivity)

Theorem 25.2 (28.3: Young's inequality for convolution). Let $1 \leq p, q, r \leq \infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

If $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, then $f * g \in L^r(\mathbb{R}^N)$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proposition 25.1 (28.4: Smoothing property of convolution). If $f \in L^1(\mathbb{R}^N)$ and $g \in C^k(\mathbb{R}^N)$ with bounded derivatives, then $f * g \in C^k(\mathbb{R}^N)$ and

$$\partial^\alpha(f * g) = f * (\partial^\alpha g)$$

for all multi-indices α with $|\alpha| \leq k$.

26 Approximate identities

Definition 26.1 (29.1). A family $\{\varphi_\varepsilon\}_{\varepsilon>0}$ of functions in $L^1(\mathbb{R}^N)$ is called an approximate identity if:

- (i) $\int_{\mathbb{R}^N} \varphi_\varepsilon(x) dx = 1$ for all $\varepsilon > 0$
- (ii) $\sup_{\varepsilon>0} \|\varphi_\varepsilon\|_1 < \infty$
- (iii) For every $\delta > 0$, $\lim_{\varepsilon \rightarrow 0} \int_{|x|>\delta} |\varphi_\varepsilon(x)| dx = 0$

Theorem 26.1 (29.2: Approximation by convolution). Let $\{\varphi_\varepsilon\}$ be an approximate identity and $f \in L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$. Then

$$\lim_{\varepsilon \rightarrow 0} \|f * \varphi_\varepsilon - f\|_p = 0.$$

If f is bounded and uniformly continuous, then $f * \varphi_\varepsilon \rightarrow f$ uniformly as $\varepsilon \rightarrow 0$.

Example 26.1 (29.3). The Poisson kernel on \mathbb{R} :

$$P_\varepsilon(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}$$

is an approximate identity.

Example 26.2 (29.4). The Gauss-Weierstrass kernel:

$$G_\varepsilon(x) = (4\pi\varepsilon)^{-N/2} e^{-|x|^2/(4\varepsilon)}$$

is an approximate identity.

27 Approximation theorems

Theorem 27.1 (30.1: Meyers-Serrin Theorem). Let $\Omega \subseteq \mathbb{R}^N$ be open and $1 \leq p < \infty$. Then $C^\infty(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$.

Theorem 27.2 (30.2: Density of smooth functions with compact support). Let $\Omega \subseteq \mathbb{R}^N$ be open and $1 \leq p < \infty$. Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.

Theorem 27.3 (30.3: Urysohn's Lemma). Let X be a normal topological space, and let A, B be disjoint closed subsets of X . Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

Theorem 27.4 (30.4: Partition of unity). Let X be a locally compact Hausdorff space, and let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X . Then there exists a family $\{\varphi_\alpha\}_{\alpha \in I}$ of continuous functions $\varphi_\alpha : X \rightarrow [0, 1]$ such that:

- (i) Each φ_α has compact support contained in U_α
- (ii) The family $\{\text{supp}(\varphi_\alpha)\}$ is locally finite
- (iii) $\sum_{\alpha \in I} \varphi_\alpha(x) = 1$ for all $x \in X$

28 Definition and basic properties of the Fourier transform

Definition 28.1 (31.1). For $f \in L^1(\mathbb{R}^N)$, the Fourier transform of f is defined by

$$\hat{f}(\xi) = \mathcal{F}f(\xi) := \int_{\mathbb{R}^N} f(x)e^{-2\pi ix \cdot \xi} dx.$$

Proposition 28.1 (31.2: Basic properties of Fourier transform). Let $f, g \in L^1(\mathbb{R}^N)$.

- (i) $\|\hat{f}\|_\infty \leq \|f\|_1$
- (ii) \hat{f} is uniformly continuous on \mathbb{R}^N
- (iii) $\widehat{f * g} = \hat{f} \cdot \hat{g}$
- (iv) $\widehat{\tau_h f}(\xi) = e^{-2\pi i h \cdot \xi} \hat{f}(\xi)$
- (v) $e^{2\pi i h \cdot x} \widehat{f(x)}(\xi) = \hat{f}(\xi - h)$

Theorem 28.1 (31.3: Riemann-Lebesgue Lemma). If $f \in L^1(\mathbb{R}^N)$, then $\hat{f} \in C_0(\mathbb{R}^N)$, i.e., \hat{f} is continuous and $\lim_{|\xi| \rightarrow \infty} \hat{f}(\xi) = 0$.

Proposition 28.2 (31.4: Fourier transform of derivatives). If $f \in C^1(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and $\partial_j f \in L^1(\mathbb{R}^N)$, then

$$\widehat{\partial_j f}(\xi) = 2\pi i \xi_j \hat{f}(\xi).$$

Proposition 28.3 (31.5: Derivative of Fourier transform). If $f \in L^1(\mathbb{R}^N)$ and $x_j f(x) \in L^1(\mathbb{R}^N)$, then \hat{f} is differentiable with respect to ξ_j and

$$\partial_{\xi_j} \hat{f}(\xi) = -2\pi i x_j \widehat{f(x)}(\xi).$$

29 Fundamental properties of the Fourier transform

Theorem 29.1 (32.1: Fourier inversion theorem). Let $f \in L^1(\mathbb{R}^N)$ and suppose $\hat{f} \in L^1(\mathbb{R}^N)$. Then for almost every $x \in \mathbb{R}^N$,

$$f(x) = \int_{\mathbb{R}^N} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

If f is continuous, the equality holds everywhere.

Definition 29.1 (32.2). The inverse Fourier transform is defined by

$$\check{f}(x) = \mathcal{F}^{-1}f(x) := \int_{\mathbb{R}^N} f(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

Theorem 29.2 (32.3: Plancherel's theorem). The Fourier transform extends uniquely to a unitary operator on $L^2(\mathbb{R}^N)$. That is, for $f \in L^2(\mathbb{R}^N)$,

$$\|\hat{f}\|_2 = \|f\|_2.$$

Moreover, the Fourier transform is a bijection on $L^2(\mathbb{R}^N)$ with inverse \mathcal{F}^{-1} .

Theorem 29.3 (32.4: Parseval's identity). *For $f, g \in L^2(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} f(x) \overline{g(x)} dx = \int_{\mathbb{R}^N} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi.$$

Proposition 29.1 (32.5: Fourier transform of Gaussian). *The Fourier transform of the Gaussian function $f(x) = e^{-\pi|x|^2}$ is*

$$\hat{f}(\xi) = e^{-\pi|\xi|^2}.$$

Theorem 29.4 (32.6: Convolution theorem). *For $f, g \in L^1(\mathbb{R}^N)$,*

$$\widehat{f * g} = \hat{f} \cdot \hat{g}.$$

For $f, g \in L^2(\mathbb{R}^N)$,

$$\widehat{f \cdot g} = \hat{f} * \hat{g}.$$

30 The Fourier transform on $L^2(\mathbb{R}^N)$

Definition 30.1 (33.1). *For $f \in L^2(\mathbb{R}^N)$, the Fourier transform is defined by*

$$\hat{f}(\xi) = \lim_{R \rightarrow \infty} \int_{|x| \leq R} f(x) e^{-2\pi i x \cdot \xi} dx,$$

where the limit is in the L^2 -sense.

Theorem 30.1 (33.2). *The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is a unitary operator, i.e., it is bijective and preserves the inner product:*

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \quad \text{for all } f, g \in L^2(\mathbb{R}^N).$$

Proposition 30.1 (33.3). *The Fourier transform maps the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ onto itself.*

Definition 30.2 (33.4). *The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ consists of all infinitely differentiable functions $f : \mathbb{R}^N \rightarrow \mathbb{C}$ such that for all multi-indices α, β ,*

$$\sup_{x \in \mathbb{R}^N} |x^\alpha \partial^\beta f(x)| < \infty.$$

Theorem 30.2 (33.5). *The Fourier transform is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ onto itself.*

31 The Riesz representation theorem

Definition 31.1 (34.1). *Let X be a topological space. A linear functional $\Lambda : C_c(X) \rightarrow \mathbb{C}$ is called positive if $\Lambda(f) \geq 0$ whenever $f \geq 0$.*

Theorem 31.1 (34.2: Riesz-Markov-Kakutani representation theorem). *Let X be a locally compact Hausdorff space, and let $\Lambda : C_c(X) \rightarrow \mathbb{C}$ be a positive linear functional. Then there exists a unique Radon measure μ on X such that*

$$\Lambda(f) = \int_X f d\mu \quad \text{for all } f \in C_c(X).$$

Definition 31.2 (34.3). A Radon measure on a locally compact Hausdorff space X is a Borel measure that is:

- (i) Finite on compact sets
- (ii) Outer regular on all Borel sets
- (iii) Inner regular on open sets

Theorem 31.2 (34.4: Riesz representation theorem for $C_0(X)$). Let X be a locally compact Hausdorff space, and let $\Lambda : C_0(X) \rightarrow \mathbb{C}$ be a continuous linear functional. Then there exists a unique complex Radon measure μ on X such that

$$\Lambda(f) = \int_X f d\mu \quad \text{for all } f \in C_0(X),$$

and $\|\Lambda\| = |\mu|(X)$.

32 The Radon-Nikodym theorem

Definition 32.1 (35.1). Let μ and ν be measures on (X, \mathcal{A}) . We say that ν is absolutely continuous with respect to μ (written $\nu \ll \mu$) if $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \in \mathcal{A}$.

Definition 32.2 (35.2). A measure ν is σ -finite if X can be written as a countable union of sets with finite ν -measure.

Theorem 32.1 (35.3: Radon-Nikodym theorem). Let μ and ν be σ -finite measures on (X, \mathcal{A}) with $\nu \ll \mu$. Then there exists a measurable function $f : X \rightarrow [0, \infty)$ such that

$$\nu(A) = \int_A f d\mu \quad \text{for all } A \in \mathcal{A}.$$

The function f is unique up to equality μ -almost everywhere, and is called the Radon-Nikodym derivative, denoted $f = \frac{d\nu}{d\mu}$.

Theorem 32.2 (35.4: Lebesgue decomposition theorem). Let μ and ν be σ -finite measures on (X, \mathcal{A}) . Then there exist unique measures ν_a and ν_s such that:

- (i) $\nu = \nu_a + \nu_s$
- (ii) $\nu_a \ll \mu$
- (iii) $\nu_s \perp \mu$ (i.e., there exists $E \in \mathcal{A}$ with $\mu(E) = 0$ and $\nu_s(X \setminus E) = 0$)

Proposition 32.1 (35.5: Chain rule for Radon-Nikodym derivatives). If $\lambda \ll \nu \ll \mu$ and ν and μ are σ -finite, then

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \frac{d\nu}{d\mu} \quad \mu\text{-a.e.}$$

33 Basic concepts of probability theory

Definition 33.1 (36.1). A probability space is a measure space (Ω, \mathcal{F}, P) with $P(\Omega) = 1$. The set Ω is called the sample space, \mathcal{F} is called the σ -algebra of events, and P is called the probability measure.

Definition 33.2 (36.2). A random variable is a measurable function $X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition 33.3 (36.3). The distribution of a random variable X is the probability measure μ_X on \mathbb{R} defined by

$$\mu_X(B) = P(X \in B) = P(\{\omega \in \Omega : X(\omega) \in B\}).$$

Definition 33.4 (36.4). The expectation of a random variable X is defined by

$$\mathbb{E}[X] = \int_{\Omega} X dP,$$

provided the integral exists.

Definition 33.5 (36.5). The variance of a random variable X is defined by

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

Theorem 33.1 (36.6: Law of the unconscious statistician). If X is a random variable and $g : \mathbb{R} \rightarrow \mathbb{R}$ is measurable, then

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) d\mu_X(x).$$

34 The distribution of a random variable

Definition 34.1 (37.1). The cumulative distribution function (CDF) of a random variable X is defined by

$$F_X(x) = P(X \leq x) = \mu_X((-\infty, x]).$$

Proposition 34.1 (37.2). The cumulative distribution function F_X has the following properties:

- (i) F_X is non-decreasing
- (ii) $\lim_{x \rightarrow -\infty} F_X(x) = 0$ and $\lim_{x \rightarrow \infty} F_X(x) = 1$
- (iii) F_X is right-continuous

Definition 34.2 (37.3). A random variable X is called discrete if there exists a countable set $S \subseteq \mathbb{R}$ such that $P(X \in S) = 1$.

Definition 34.3 (37.4). A random variable X is called continuous if its distribution μ_X is absolutely continuous with respect to Lebesgue measure. In this case, the Radon-Nikodym derivative $f_X = \frac{d\mu_X}{dx}$ is called the probability density function (PDF) of X .

Definition 34.4 (37.5). *The characteristic function of a random variable X is defined by*

$$\phi_X(t) = \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} e^{itx} d\mu_X(x).$$

Theorem 34.1 (37.6: Inversion theorem for characteristic functions). *If $\int_{\mathbb{R}} |\phi_X(t)| dt < \infty$, then X has a continuous bounded density given by*

$$f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi_X(t) dt.$$

Theorem 34.2 (37.7: Uniqueness theorem for characteristic functions). *If two random variables have the same characteristic function, then they have the same distribution.*

35 Conditional expectation

Definition 35.1 (38.1). *Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{G} \subseteq \mathcal{F}$ be a sub- σ -algebra. The conditional expectation of an integrable random variable X given \mathcal{G} is a \mathcal{G} -measurable random variable $\mathbb{E}[X|\mathcal{G}]$ such that for all $G \in \mathcal{G}$,*

$$\int_G \mathbb{E}[X|\mathcal{G}] dP = \int_G X dP.$$

Theorem 35.1 (38.2: Existence and uniqueness of conditional expectation). *For any integrable random variable X and any sub- σ -algebra \mathcal{G} , the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ exists and is unique almost surely.*

Proposition 35.1 (38.3: Properties of conditional expectation). *Let X, Y be integrable random variables and \mathcal{G}, \mathcal{H} be sub- σ -algebras.*

- (i) *Linearity: $\mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$*
- (ii) *Tower property: If $\mathcal{H} \subseteq \mathcal{G}$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$*
- (iii) *If X is \mathcal{G} -measurable, then $\mathbb{E}[X|\mathcal{G}] = X$*
- (iv) *If X is independent of \mathcal{G} , then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$*
- (v) *Monotonicity: If $X \leq Y$, then $\mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$*

Theorem 35.2 (38.4: Conditional Jensen's inequality). *If φ is convex and $X, \varphi(X)$ are integrable, then*

$$\varphi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\varphi(X)|\mathcal{G}].$$

Definition 35.2 (38.5). *The conditional probability of an event $A \in \mathcal{F}$ given \mathcal{G} is defined by*

$$P(A|\mathcal{G}) = \mathbb{E}[1_A|\mathcal{G}].$$