Designing Circular and Non-Circular Gears for FDM 3d-Printing

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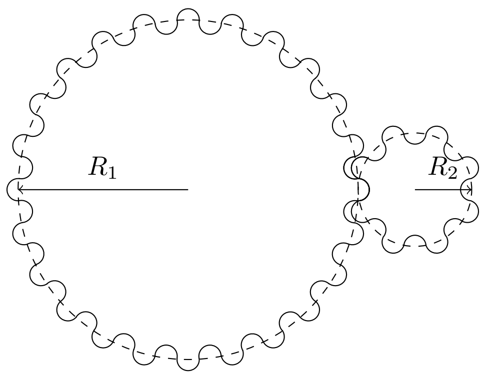

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1 Tooth Profile

Since FDM 3d printed gears are weak and therefore torque and transmission efficiency is not as important we are going to use a circular *tooth profile*. Please note that this profile is not optimal but it makes it incredibly easy to calculate and draw non circular gears. It is sufficient for most 3d printed parts. If this is not suitable for your design take a look at involute gears.

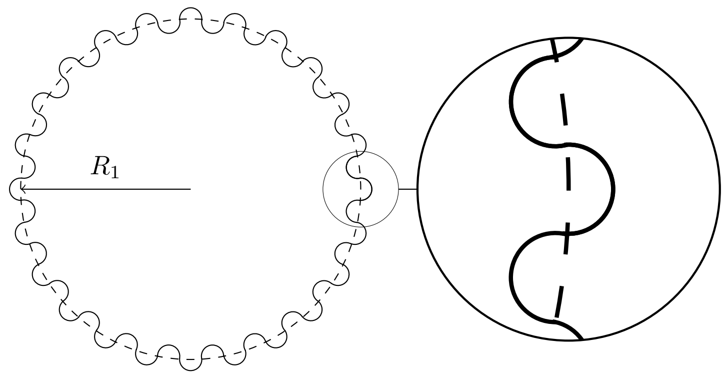


Figure 1: Gear profile construction using circles

1.1 Drawing a Gear

Start by drawing the outline of the gear. As we will see in section 3.3 this can be any convex shape. As an example we will take a look at a circular gear. Then, using a compas, mark points with distance r on the outline of your shape and draw a circle with radius r on every second mark. We will see in section 3 or in the examples how to calculate r.

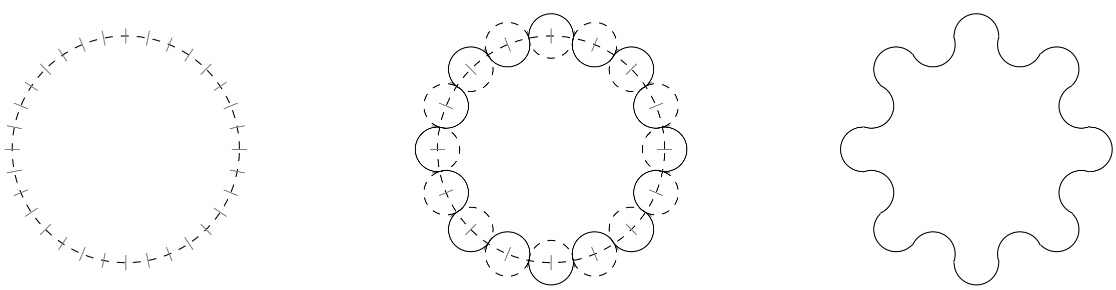


Figure 2: Drawing a circular gear

2 Notation

To analyze different gear shapes and calculate the *tooth size* r we need to grasp what a gear is mathematically. As seen in the previous chapter, a gear is constituted by the *gear shape* together with the number of teeth, the size (radius) of each tooth, a set containing the centerpoints of each tooth and the *centerpoint* of the gear itself. This leads to the following definition:

Definition 2.1
A tuple (γ, n, r, m_0, M) where $\gamma : [0, 2\pi) \to \mathbb{R}^2$ is a simple closed curve, $n \in \mathbb{N}$, $r \in \mathbb{R}$, $m_0 \in \gamma[[0, 2\pi)]$ and $M \in \mathbb{R}^2$ is called a *gear* if the enclosed area of γ is convex.

At this point it is not clear if a gear is well-defined as Definition 2.1 demands only a single tooth-centerpoint m_0 and not a set containing every tooth-centerpoint. The fact that $(m_i)_{i \in \mathbb{Z}_{2n}}$ can be represented by m_0 is one of the findings of Theorem 3.3.
If γ is a circle we will sometimes replace it with its radius $R \in \mathbb{R}$. Furthermore M is defined as point with distance $d(M, \gamma(\varphi)) = ||\gamma(\varphi)||_2, \forall \varphi \in [0, 2\pi)$. Since M is defined by γ it is not required. However we will only omit M if it is 0.

Definition 2.2
Two gears A and B *fit together* if

$$r_A = r_B.$$

Definition 2.3
Two gears A and B are of the *same size* or shape, if it is possible to map the image of γ_A to the image of γ_B without changing its size:

$$\exists \iota : d(\iota(x), \iota(y)) = d(x, y) \wedge \iota(\gamma_A[[0, 2\pi)]) = \gamma_B[[0, 2\pi)]. \tag{2.1}$$

We call two gears A and B *similar* and write $A \sim B$ if they are of the same size and

$$n_A = n_B$$
$$r_A = r_B$$
$$\iota(m_{A,0}) = m_{B,0}$$

where ι is an isometric function that satisfies (2.1).
Furthermore $A = B$ if $A \sim B$ and $\iota = \text{id}$.

Proof

For this definition to make sense $\sim$ needs to be reflexive, symmetric and transitive.

"$A \sim A$":

Let ι be the identity function.

"$A \sim B \implies B \sim A$":

As ι is an isometric isomorphism it is bijective and therefore its inverse ι^{-1} exists. Using ι^{-1}, $B \sim A$.

"$A \sim B \land B \sim C \implies A \sim C$":

The composition of two isometric isomorphisms is an isometric isomorphism and therefore $A \sim C$. $\square$

3 Calculating Gear and teeth Size

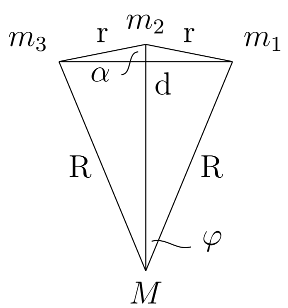
3.1 Circular Gears

We have already seen how to draw a circular gear $K = (R, n, r, m_0, M)$ in section 1.1. However r has to be chosen in a way that it is possible to find a set of $2n$ points $(m_i)_{i \in \mathbb{Z}_{2n}} \in \gamma[[0, 2\pi)] : d(m_i, m_{i+1}) = r$.

Theorem 3.1
Given a circle γ with radius R and $n \in \mathbb{N}$, it is possible to find a sequence of $2n$ points $(m_i)_{i \in \mathbb{Z}_{2n}} \in \gamma[[0, 2\pi)] : d(m_i, m_{i+1}) = r$. Let $\varphi = \frac{Pi}{n}$ then

$$R = \frac{r \cdot \cos\left(\frac{\varphi}{4}\right)}{\sin(\frac{\varphi}{2})}. \tag{3.1}$$

Proof
Let K be a circle with radius R and center $(0, 0)$ and $n = 2k$, $k \in \mathbb{N}$ we can, using polar coordinates, construct a sequence $(m_i)_{i \in \mathbb{Z}_{2n}}$, $m_i = \left(R, i \cdot \frac{\pi}{n}\text{rad}\right)$ of $2n$ points on K. They are evenly spaced and $r := d(m_0, m_1)$. We can now construct a sequence $(k_j)_{j \in \mathbb{Z}_n}$of n circles with radius r and center $m_j : j \equiv 0 \pmod 2$. Therefore $\{x \in K : \exists i, j \in \mathbb{Z}_{2n} : d(x, m_{2i}) \leq r \wedge d(x, m_{2j}) \leq r\} = \{m_i : i \equiv 1 \pmod 2, i \in \mathbb{Z}_{2n}\}$ and $|\{m_i : i \equiv 1 \pmod 2, i \in \mathbb{Z}_{2n}\}| = n$.



Let $\alpha := \sphericalangle m_3 m_2 m_1$, $\varphi := \sphericalangle m_1 M m_3$ and $d := \overline{m_3 m_1}$. Using the inscribed angle theorem(Weisstein 2020), we know that $\alpha = \frac{2\pi - \varphi}{2}$. By drawing a line from M to m we get two right triangles. $\overline{Mm_2}$ also bisects φ, α and d. Thus

$$\frac{d}{2} = r \cdot \sin\left(\frac{\alpha}{2}\right) \tag{3.2}$$

and

$$R = \frac{d}{2\sin\left(\frac{\varphi}{2}\right)}. \tag{3.3}$$

By substituting in α in eq. (3.2) and using $\cos(x) = \sin(\frac{\pi}{2} - x)$, we see that

$$R = \frac{r \cdot \cos\left(\frac{\varphi}{4}\right)}{\sin(\frac{\varphi}{2})}. \tag{3.4}$$

□

3.2 Rectangular Gears

Theorem 3.2
Let K be a rectangle with side lengths of a and b. It is possible to find a sequence of circles $(k_i)_{i\in I}$ with the same properties as in Theorem 3.1 if $4r \mid a$ and $4r \mid b$. Furthermore all four corners of K are in $\{m_i : i \in I\}$, where m_i is the center of k_i.

Proof It is easy to see that Theorem 3.2 holds true. □

3.3 Other Non Circular Gears

Theorem 3.3
Let γ be a simple, closed curve where the enclosed area is convex. γ can be used as a gear outline. Furthermore the sequence of circlecenters $\{m_i : i \in I\}$ can be calculated using:

Algorithm 1: Calculating $\{m_i : i \in I\}$

```
Result: Write here the result
I = 2 times number of teeth;
while While condition do
    instructions;
    if condition then
        instructions1;
        instructions2;
    else
        instructions3;
    end
end
```

3.4 Examples

Example 3.4.1 Circular gears with given gear ratio and centerdistance

Lets say we want a 3:1 gear reduction and the center points of each gear should be 30mm apart. We can now start to build a system of equations:

$$
\begin{aligned}
R_1 + R_2 &= 30 \\
n_1 &= 3n_2
\end{aligned}
$$

Where R_1 and R_2 are the radii of the two gears and n_1 and n_2 denote the number of teeth on each. Using eq. (3.1) we also know that:

$$
\begin{aligned}
\varphi_1 &= \frac{2Pi}{n_1} \\
R_1 &= \frac{r * \cos\left(\frac{\varphi_1}{4}\right)}{\sin(\frac{\varphi_1}{2})} \\
\varphi_2 &= \frac{2Pi}{n_2} \\
R_2 &= \frac{r * \cos\left(\frac{\varphi_2}{4}\right)}{\sin(\frac{\varphi_2}{2})}
\end{aligned}
$$

If we specify the number of teeth on either one of them lets say

$$n_2 = 16.$$

we can solve the system of equations using wolframalpha, maple or another computerprogram to find a unique solution. In this example we get:

$$
\begin{aligned}
r &= 1.471832758 \\
R_1 &= 22.49196236 \\
R_2 &= 7.508037642 \\
n_1 &= 48 \\
n_2 &= 16 \\
\varphi_1 &= 0.1308996939 \\
\varphi_2 &= 0.3926990818
\end{aligned}
$$

If r is too small/large change the number of teeth you specified.

Example 3.4.2 Internal ring gear

A internal ring gear K_r with outer diameter $D_r = 5$ should be calculated such that a smaller gear K can be used to archive a $5:1$ gear reduction. The two shafts

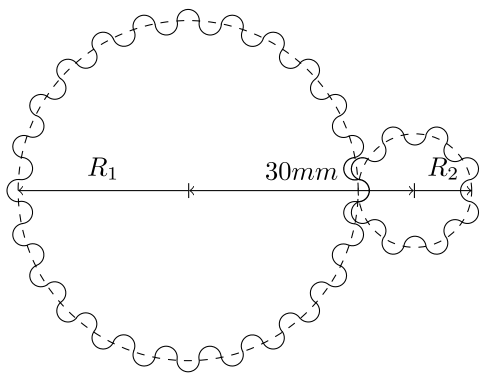


Figure 3: Resulting Gears

should be $1cm$ apart.
As the two shafts are $15mm$ apart we know that

$$R_r - R = 15.$$

Furthermore

$$n_r = 5 \cdot n$$

to achieve the required $5:1$ ratio. Using eq. (3.1) we also know that:

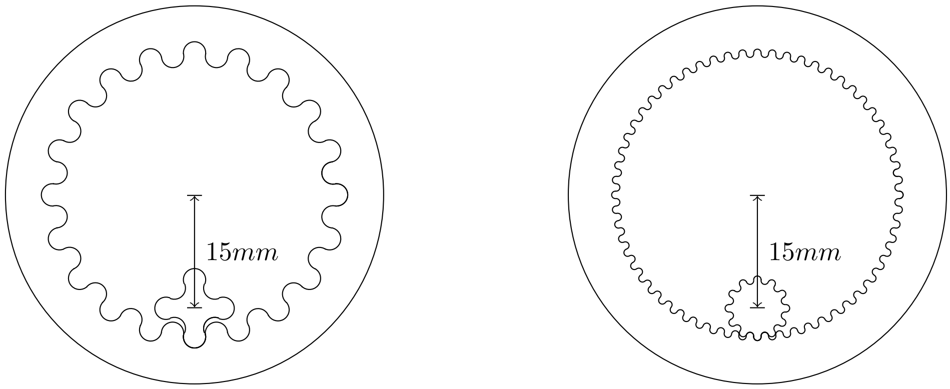
$$\begin{aligned}
\varphi_r &= \frac{2Pi}{n_r} \\
R_r &= \frac{r * \cos\left(\frac{\varphi_r}{4}\right)}{\sin(\frac{\varphi_r}{2})} \\
\varphi &= \frac{2Pi}{n} \\
R &= \frac{r * \cos\left(\frac{\varphi}{4}\right)}{\sin(\frac{\varphi}{2})}
\end{aligned}$$

If we specify the number of teeth on either one of them lets say

$$n = 8,$$

we can solve the system of equations using wolframalpha, maple or another computerprogram to find a unique solution. In this example we get:

$$
\begin{aligned}
r &= 1.474527091\\
R_r &= 18.77908826\\
R &= 3.779088256\\
n_r &= 40\\
n &= 8\\
\varphi_r &= 0.1570796327\\
\varphi &= 0.7853981635
\end{aligned}
$$



(a) $n = 8$ (b) $n = 24$

Figure 4: Resulting internal ring gears for different n

Example 3.4.3 Planetary gearbox (epicyclic gear train)
A planetary gearbox with not rotating ring gear and a gear ratio of 10:1 should be calculated. Let S be the sun gear, P the planet gear and R the outer ring gear and C the carrier.
We know that for planetary gear systems

$$n_R = 2 \cdot n_P + n_S \implies \frac{n_S}{n_P}\omega_S + (2 + \frac{n_S}{n_P})\omega_R - 2(1 + \frac{n_S}{n_P})\omega_C = 0$$

where ω denotes the angular momentum. As the the outer ring gear R is not rotating $\omega_R = 0$. We can deduce that

$$
\begin{aligned}
\frac{n_S}{n_P}\omega_S &= 2(1 + \frac{n_S}{n_P})\omega_C\\
\frac{\omega_S}{\omega_C} &= \frac{2(1 + \frac{n_S}{n_P})}{\frac{n_S}{n_P}}.
\end{aligned}
$$

As S will be used as the input and C as the output we know that

$$\frac{\omega_S}{\omega_C} = \frac{10}{1}$$

and because the gears need to touch each other the radii have to satisfy

$$R_R = 2 \cdot R_P + R_S.$$

We will use eq. (3.1) to get one more equation per gear.[1] After specifying the radius of the outer ring gear $R_R = 30$ and $n_S = 24$ we can solve the system of equation to get a unique solution. Since we want 4 planet gears, the sun gear and the outer ring gear have to be symmetrical in both the x and y direction. This is guaranteed if $4 \mid n_S$.

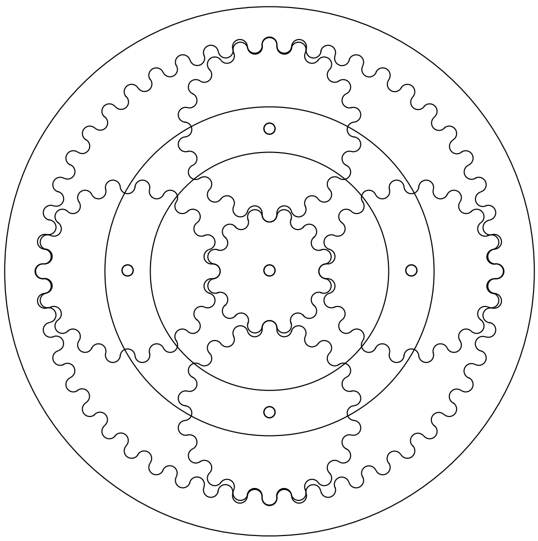


Figure 5: Resulting planetary gear system

Example 3.4.4 Rectangular gear with given tooth size r

Example 3.4.5 Rectangular and circular gears with given gear ratio and centerdistance

[1] Take a look at Example 3.4.1 to see this step in more detail.

4 Rack and Pinion

In order to translate linear motion to rotation and vice versa a rack and pinion can be used. As seen in section 3 it is possible to calculate and draw a gearprofile.

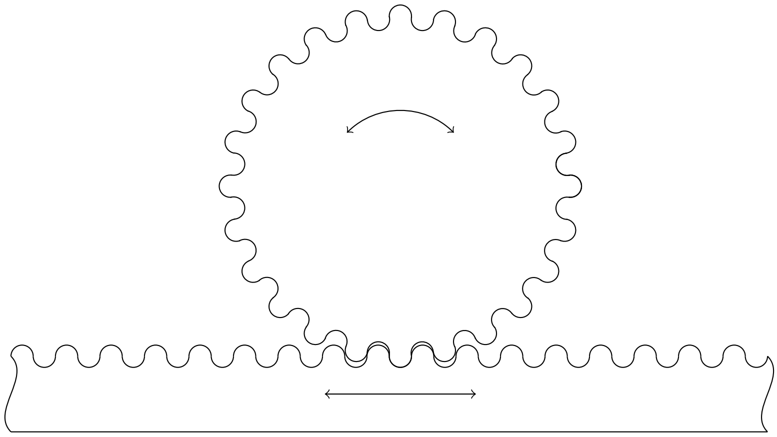


Figure 6: Rack and Pinion

4.1 Constant rotational Speed

Theorem 4.1
Let K be a circular gear with not moving center M and R a corresponding rack. There exists a homomorphism Ψ such that $\Psi(s) = \varphi$, where s is the distance R is moved and φ the angle K is rotated around M. If R is moved at a constant speed, the rotational speed of K will be constant.

Proof Assuming there is no play in this system K can be viewed as a circle with radius r and the rack R as a tangent line. The length of an arc can be calculated using

$$l = r \cdot \varphi$$

and therefore

$$\varphi = \frac{l}{r}.$$

Since there is no play, the length l of the arc will be equal to the distance s, R was moved and

$$\Psi(s) = \frac{s}{r}.$$

It is easy to see that $\Psi(\lambda \cdot s) = \lambda \cdot \Psi(s)$. As Ψ is a composition of continuous functions it is continuous.

If R is moving with constant speed $v(t) = \frac{s(t)}{t} = s(1)$, $v' = 0$. The rotational speed of K, $\omega(t) = \frac{\varphi}{t}$ can be calculated using Ψ.

$$\omega(t) = \frac{\Psi(s(t))}{t} = \frac{t \cdot \Psi(s(1))}{t} = \Psi(s(1)) = \Psi(v(t))$$

As v is constant so is $\Psi(v)$ and thus ω is constant.

□

4.2 Non-constant rotational Speed

If a non-circular gear is used as a pinion and a rack is made in a way that the center of the gear does not move the resulting rotational speed will be non constant. This behavior can be used to calculate a special gear shape such that the rotational speed follows a given curve. Leading to the following theorem:

Theorem 4.2
Given a periodic, continuous function f. If $\gamma''(\varphi) \leq K''(\varphi)$ where

$$\gamma(\varphi) = \begin{pmatrix} \frac{1}{f(\varphi)} \cos(\varphi) \\ \frac{1}{f(\varphi)} \sin(\varphi) \end{pmatrix}, \tag{4.1}$$

γ will be simple, closed and the enclosed area will be convex. γ can therefore be used as a gear outline. Furthermore, if a fitting rack moves with constant speed c, the rotational speed of this gear will be $cf(\varphi)$.

Proof

□

4.3 Non-constant shaft position

If it is necessary that the shaft rotates and changes position a control cam can be used (fig. 7). By restricting the shaft position in the x direction it will follow the racks shape if they stay in contact. If no rotation is required a control cam in the required shape should be used on its own.

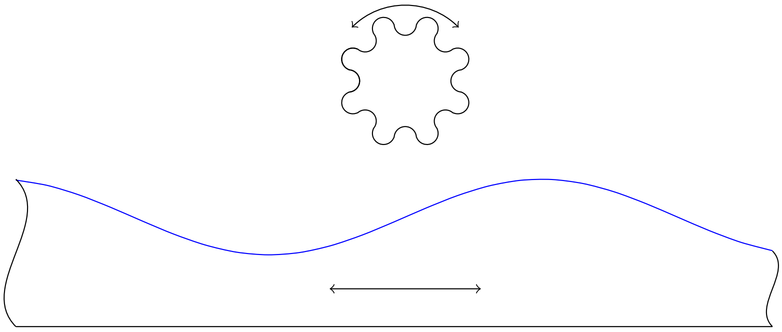


Figure 7: Rack and Pinion

Theorem 4.3
Let K be a circular gear with radius r and center m where m_x is constant and $f \in C^2([0, 2\pi))$ with $|f''(\varphi)| < |K''(\varphi)|$. f can be used as an outline for a rack R and m_y will follow the curve $f + c, c \in \mathbb{R}$.

Proof Using Theorem 3.3 it is possible to calculate a gear profile for K and R. Since K is circular and always touching R, the distance between m and R is r. It is easy to see that $|f''(\varphi)| < |K''(\varphi)|$ implies that K is touching R in no more than one point. Thus $m_y = f(\varphi) + r + c$ where c depends on the total thickness of R. A proper proof is omitted since it is easy to see that section 4.3 holds true. $\square$

4.4 Examples

Example 4.4.1 Shaft following a Sinus(t) curve
We want a gear such that the shaft is following a $\sin(t)$ curve. We know that $\sin(t)$ is a 2π periodic function with $\sin(t) > 0, t \in (0, \pi)$ and $\sin(t) < 0, t \in (\pi, 2\pi)$. As seen in **??** this will result in a non convex gear shape. To use the method described in section 3.3 we need a new function $g(t) := sin(t) + |min(sin[0, 2\pi])|$. Thus $g(t) \geq 0, t \in (0, 2\pi)$ resulting in a convex gear shape. However the shaft will now follow $g(t)$ and not $\sin(t)$. Therefore we need to offset both the rack and pinion by $|min(sin[0, 2\pi])|$.
We can now start by choosing a minimum radius $R := 5$ for the gear.

Example 4.4.2 cos(t)

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References

Weisstein, Eric W. (2020). *Inscribed Angle.* URL: `https://mathworld.wolfram.com/InscribedAngle.html` (visited on 05/05/2020) (cit. on p. 6).

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