Derivation for 4D quantum Hall coefficient

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1 Basic Configuration

We first make assumptions that after undergoing multiple reflection, the system has reached an equilibrium where in different regions, the electromagnetic wave can always be expressed as plane waves. It is also quite simple to put that the general relations still apply for the superposition of EM waves. In the substrate region, we have incident and reflected waves

$$\vec{E}_i = \vec{E}_1 e^{i(k_y y + k_1 z - \omega t)} \tag{1}$$

$$\vec{E}_r = \vec{E}_2 e^{i(k_y y - k_1 z - \omega t)} \tag{2}$$

In the sample film, we have two waves propagating in two directions, forward and backward

$$\vec{E}_f = \vec{E}_3 e^{i(k_y y + k_3 z - \omega t)} \tag{3}$$

$$\vec{E}_b = \vec{E}_4 e^{i(k_y y - k_3 z - \omega t)} \tag{4}$$

And in the vacuum, we have the transmitted wave as

$$\vec{E}_t = \vec{E}_5 e^{i[k_y y + k_5 (z-d) - \omega t]} \tag{5}$$

All these EM waves share the same k_y according to the refraction rule ($n_1 \sin\theta_1 = n_2 \sin\theta_2 = \sin\theta_3$). Plus, k_z can be derived as (n_1, n_2 represent the refractive index of substrate and sample, respectively)

$$k_1 = n_1 k_0 \cos\theta_1,\ k_3 = n_2 k_0 \cos\theta_2,\ k_5 = k_0 \cos\theta_3 \tag{6}$$

The refractive index can also be related to the permittivity constant as

$$\varepsilon_{r1} = n_1^2,\ \varepsilon_{r2} = n_2^2 \tag{7}$$

2 Governing Equations

Starting from the Maxwell equations and setting the surface charge density as Q_i, we have

$$D_z^R - D_z^L = Q_i \tag{8}$$

$$E_x^R = E_x^L \tag{9}$$

$$E_y^R = E_y^L \tag{10}$$

$$H_x^L - H_x^R = \sigma_{xy} E_x - \sigma_{xx} E_y \tag{11}$$

$$H_y^L - H_y^R = \sigma_{xx} E_x + \sigma_{xy} E_y \tag{12}$$

$$B_z^L = B_z^R \tag{13}$$

And the relation of $\vec{E}$ and $\vec{H}$ derived from the nature of EM wave

$$\vec{H} = \frac{1}{\mu_0 \omega} \vec{k} \times \vec{E} \tag{14}$$

Plugging all these into the Maxwell equations and setting $\beta = e^{in_2 k_0 d \cos\theta_2}$ for simplicity, we have

$$n_2^2 \tan\theta_2(E_{4y} - E_{3y}) - n_1^2 \tan\theta_1(E_{2y} - E_{1y}) = \frac{Q_1}{\varepsilon_0} \tag{15}$$

$$n_2^2 \tan\theta_2(\beta E_{3y} - \beta^* E_{4y}) - \tan\theta_3 E_{5y} = \frac{Q_2}{\varepsilon_0} \tag{16}$$

$$E_{1x} + E_{2x} = E_{3x} + E_{4x} \tag{17}$$

$$E_{1y} + E_{2y} = E_{3y} + E_{4y} \tag{18}$$

$$\beta E_{3x} + \beta^* E_{4x} = E_{5x} \tag{19}$$

$$\beta E_{3y} + \beta^* E_{4y} = E_{5y} \tag{20}$$

$$-\frac{n_1}{\cos\theta_1}(E_{1y} - E_{2y}) + \frac{n_2}{\cos\theta_2}(E_{3y} - E_{4y}) = \mu_0 c[\sigma_{xy}^b(E_{1x} + E_{2x}) - \sigma_{xx}^b(E_{1y} + E_{2y})] \tag{21}$$

$$-\frac{n_2}{\cos\theta_2}(\beta E_{3y} - \beta^* E_{4y}) + \frac{E_{5y}}{\cos\theta_3} = \mu_0 c(\sigma_{xy}^t E_{5x} - \sigma_{xx}^t E_{5y}) \tag{22}$$

$$n_1 \cos\theta_1(E_{1x} - E_{2x}) - n_2 \cos\theta_2(E_{3x} - E_{4x}) = \mu_0 c[\sigma_{xx}^b(E_{1x} + E_{2x}) + \sigma_{xy}^b(E_{1y} + E_{2y})] \tag{23}$$

$$n_2 \cos\theta_2(\beta E_{3x} - \beta^* E_{4x}) - \cos\theta_3 E_{5x} = \mu_0 c(\sigma_{xx}^t E_{5x} + \sigma_{xy}^t E_{5y}) \tag{24}$$

We also have two auxiliary relations of rotation angles (here I define Faraday/Kerr rotation angles to be absolute rather than relative)

$$\tan\theta_F = \frac{1}{\cos\theta_3}\frac{E_{5y}}{E_{5x}}, \ \tan\theta_K = \frac{1}{\cos\theta_1}\frac{E_{2y}}{E_{2x}} \tag{25}$$

To make our deduction experimentally persuasive, we cannot express the surface charge density Q in terms of conductivity. The conductivity is the 4D quantum Hall coefficient which has to be obtained independently. For simplicity, I omitted the σ_{xx} term and of course it is negligible indeed. Thus, we can combine equation (21)~(24) to cancel σ_{xy} and obtain two more independent equations.

$$\frac{-\frac{n_1}{\cos\theta_1}(E_{1y} - E_{2y}) + \frac{n_2}{\cos\theta_2}(E_{3y} - E_{4y})}{n_1 \cos\theta_1(E_{1x} - E_{2x}) - n_2 \cos\theta_2(E_{3x} - E_{4x})} = \frac{E_{1x} + E_{2x}}{E_{1y} + E_{2y}} \tag{26}$$

$$\frac{-\frac{n_2}{\cos\theta_2}(\beta E_{3y} - \beta^* E_{4y}) + \frac{E_{5y}}{\cos\theta_3}}{n_2 \cos\theta_2(\beta E_{3x} - \beta^* E_{4x}) - \cos\theta_3 E_{5x}} = \frac{E_{5x}}{E_{5y}} \tag{27}$$

For real experiment, we cannot expect completely vanished E_{1y}, which will be evaluated by a tiny proportional coefficient in the following calculation as $E_{1y} = \lambda \cos\theta_1 E_{1x}$, which means $E_{1p} = \lambda E_{1s}$. With this in mind, from equation (26) and (27), we can derive that

$$E_{5x} = \frac{n_2(\cos^2\theta_2 + \lambda \tan\theta_F \cos\theta_1 \cos\theta_3)E_{1x} + n_2(\cos^2\theta_2 + \tan\theta_F \tan\theta_K \cos\theta_1 \cos\theta_3)E_{2x}}{n_2(\cos^2\theta_2 + \cos^2\theta_3 \tan^2\theta_F)\cos(n_2 k_0 d \cos\theta_2) - i\frac{\cos\theta_2 \cos\theta_3}{\cos^2\theta_F}\sin(n_2 k_0 d \cos\theta_2)} \tag{28}$$

$$aE_{1x}^2 = bE_{2x}^2 + cE_{1x}E_{2x} + dE_{1x}E_{5x} + eE_{2x}E_{5x} \tag{29}$$

where

$$a = (\cos^2\theta_2 + \lambda^2 \cos^2\theta_1)n_2 \cos(n_2 k_0 d \cos\theta_2) + i(1 + \lambda^2)n_1 \cos\theta_1 \cos\theta_2 \sin(n_2 k_0 d \cos\theta_2) \tag{30}$$

$$b = -n_2(\cos^2\theta_2 + \tan^2\theta_K \cos^2\theta_1)\cos(n_2 k_0 d \cos\theta_2) + i\frac{n_1 \cos\theta_1 \cos\theta_2}{\cos^2\theta_K}\sin(n_2 k_0 d \cos\theta_2) \tag{31}$$

$$c = -2n_2(\cos^2\theta_2 + \lambda \tan\theta_K \cos^2\theta_1)\cos(n_2 k_0 d \cos\theta_2) \tag{32}$$

$$d = n_2(\cos^2\theta_2 + \lambda \tan\theta_F \cos\theta_1 \cos\theta_3), \ e = n_2(\cos^2\theta_2 + \tan\theta_F \tan\theta_K \cos\theta_1 \cos\theta_3) \tag{33}$$

We use the notation that $E_{5x} = fE_{1x} + gE_{2x}$. Putting this into the equation (29), we can solve these equations numerically.

However, if we take finite σ_{xx} also into account, we can make everything more precise. For example, we set $\lambda_1 = \frac{\sigma^b_{xx}}{\sigma^b_{xy}}$ and $\lambda_2 = \frac{\sigma^t_{xx}}{\sigma^t_{xy}}$. In this way, we have

$$\frac{-\frac{n_1}{\cos\theta_1}(E_{1y} - E_{2y}) + \frac{n_2}{\cos\theta_2}(E_{3y} - E_{4y})}{n_1\cos\theta_1(E_{1x} - E_{2x}) - n_2\cos\theta_2(E_{3x} - E_{4x})} = \frac{E_{1x} + E_{2x} - \lambda_1(E_{1y} + E_{2y})}{\lambda_1(E_{1x} + E_{2x}) + E_{1y} + E_{2y}} \tag{34}$$

$$\frac{-\frac{n_2}{\cos\theta_2}(\beta E_{3y} - \beta^* E_{4y}) + \frac{E_{5y}}{\cos\theta_3}}{n_2\cos\theta_2(\beta E_{3x} - \beta^* E_{4x}) - \cos\theta_3 E_{5x}} = \frac{E_{5x} - \lambda_2 E_{5y}}{\lambda_2 E_{5x} + E_{5y}} \tag{35}$$