NYQUIST PLOTS

Introduction

- Nyquist plots are the continuation of polar plots for finding the stability of the closed loop control systems by varying ω from −∞ to ∞.
- Nyquist plots are used to draw the complete frequency response of the open loop transfer function.
- The Nyquist stability criterion determines the stability of a closed-loop system from its open-loop frequency response and open-loop poles.

The Nyquist Criterion can be expressed as,

$$Z = P + N$$

where Z = number of zeros of $1 + G(s)H(s)$ on the right-half s-plane

N = net encirclements around the point $(-1+j0)$.

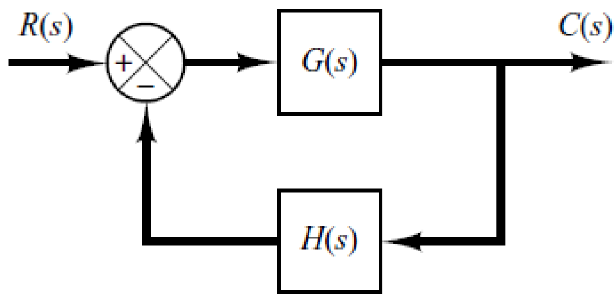
(clockwise encirclements are taken as positive and anticlockwise encirclements are negative)

P = number of poles of $G(s)H(s)$ in the right-half of s-plane

The stability of linear control systems using the Nyquist stability criterion, three possibilities can occur:

1. There is no encirclement of the (–1+j0) point. This implies that the system is stable if there are no poles of G(s)H(s) in the right-half of s- plane; otherwise, the system is unstable.
2. There are one or more counterclockwise encirclements of the (–1+j0) point. In this case the system is stable if the number of counterclockwise encirclements is the same as the number of poles of G(s)H(s) in the right-half of s- plane; otherwise, the system is unstable.
3. There are one or more clockwise encirclements of the (–1+j0) point. In this case the system is unstable.

Consider the closed-loop system shown in Fig.



The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The characteristic equation is $1 + G(s)H(s) = 0$

For stability, all roots of the characteristic equation must lie in the left-half of s- plane.

Example 1

G(s)H(s) = $\frac{(s+1)(s+2)}{s(s+3)}$

Open loop zeros: -1, -2

Open loop poles: 0, -3

The characteristic equation is 1 + G(s)H(s) = 0

$$1+\frac{(s+1)(s+2)}{s(s+3)} = 0$$

$$\frac{(s+0.38)(s+2.62)}{s(s+3)} = 0$$

Roots of the system: - 0.38, - 2.62

Closed-loop system T.F = $\frac{G(s)}{1 + G(s)H(s)} = \frac{(s+1)(s+2)}{(s+0.38)(s+2.62)}$

Closed-loop poles: - 0.38, - 2.62 [zeros of 1 + G(s)H(s)]

Example 2

G(s)H(s) = $\frac{(s+2)}{(s+1)(s-1)}$

Open loop zeros: -2

Open loop poles: 1, -1

The characteristic equation is 1 + G(s)H(s) = 0

$$1 + \frac{(s+2)}{(s+1)(s-1)} = 0$$

$$\frac{(s+0.5\pm j0.87)}{(s+1)(s-1)} = 0$$

Roots of the system: - $0.5 \pm j0.87$

Closed-loop system T.F = $\frac{G(s)}{1 + G(s)H(s)} = \frac{(s+2)}{(s+0.5\pm j0.87)}$

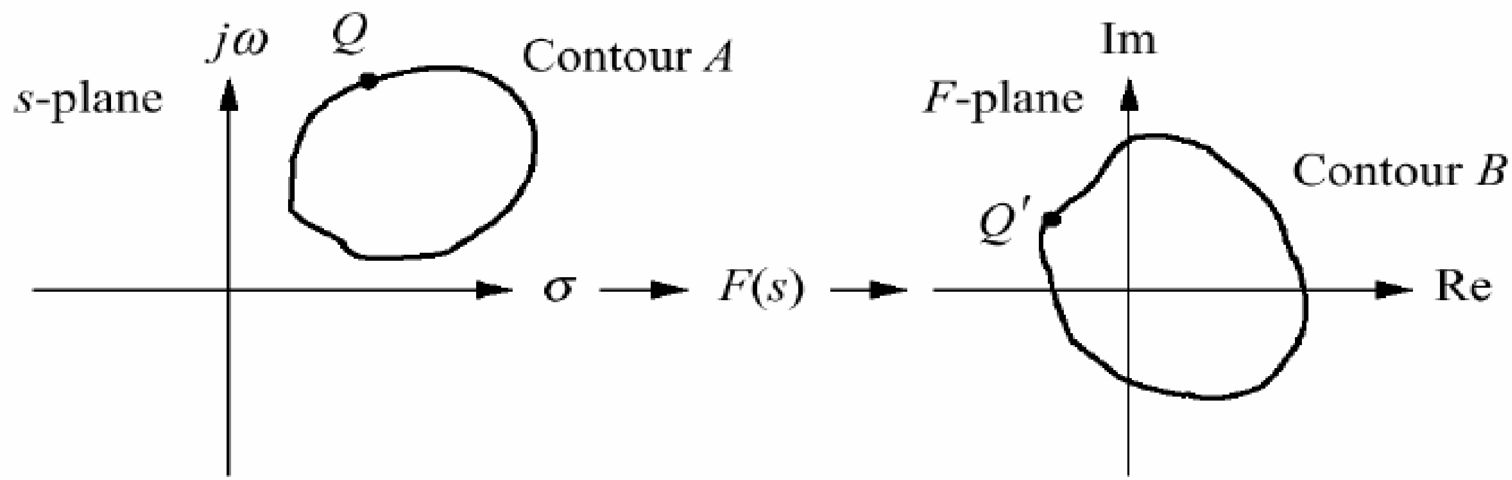
Closed-loop poles: - $0.5 \pm j0.87$ [zeros of 1 + G(s)H(s)]

- The system is stable if all the poles of the closed-loop transfer function are in the left-half of s- plane (Zeros of characteristic function). Although there may be poles and zeros of the open-loop transfer function G(s)H(s) may be in the right- half of s- plane.
- The Nyquist stability criterion relates the open-loop frequency response G(s)H(s) to the number of zeros and poles of 1+G(s)H(s) that lie in the right-half of s- plane.

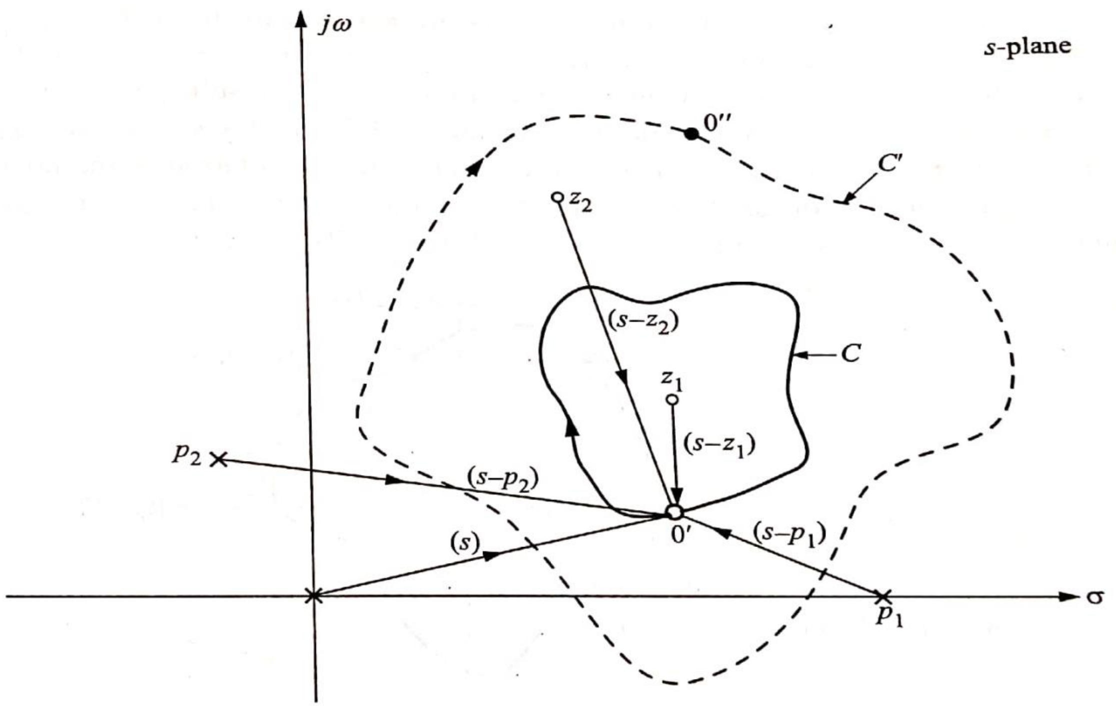
Mapping from s-plane to F -plane through a function of $F(s)$

- **For a point.** Taking a complex number in the s-plane and substituting it into a function of $F(s)$, the result is also a complex number, which is represented in a new complex-plane (called F-plane). This process is called mapping, specifically **mapping a point from s-plane to F-plane through $F(s)$.**

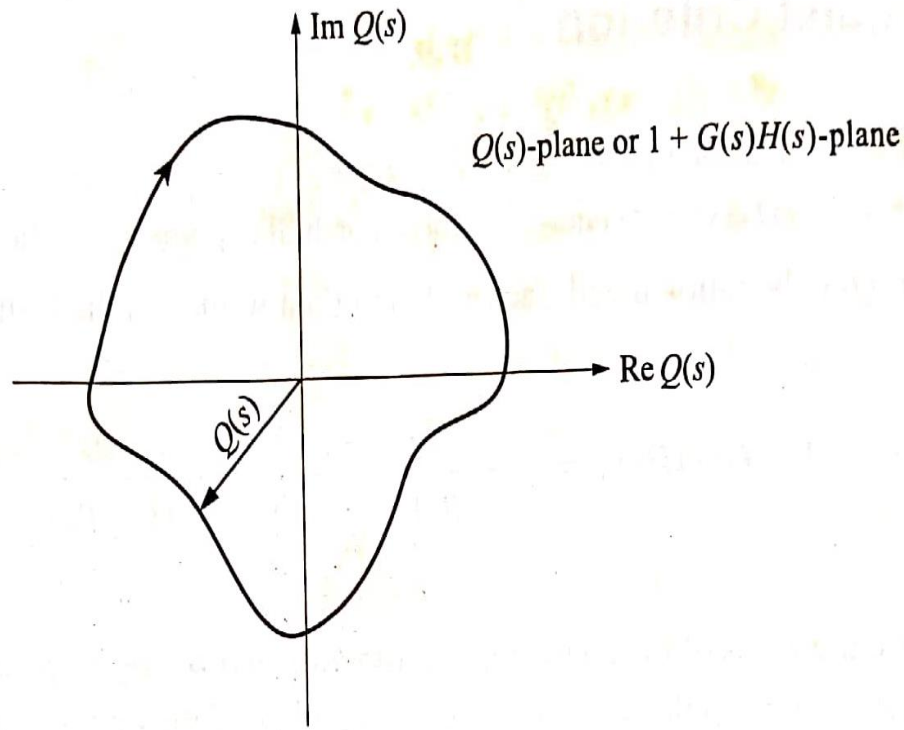
- **For a contour.** Consider the collection of points in the *s*-plane (called a contour), shown in the following figure as contour A. Using the above point mapping process through *F*(*s*), we can also get a contour in the *F*-plane, shown in the following figure contour B.



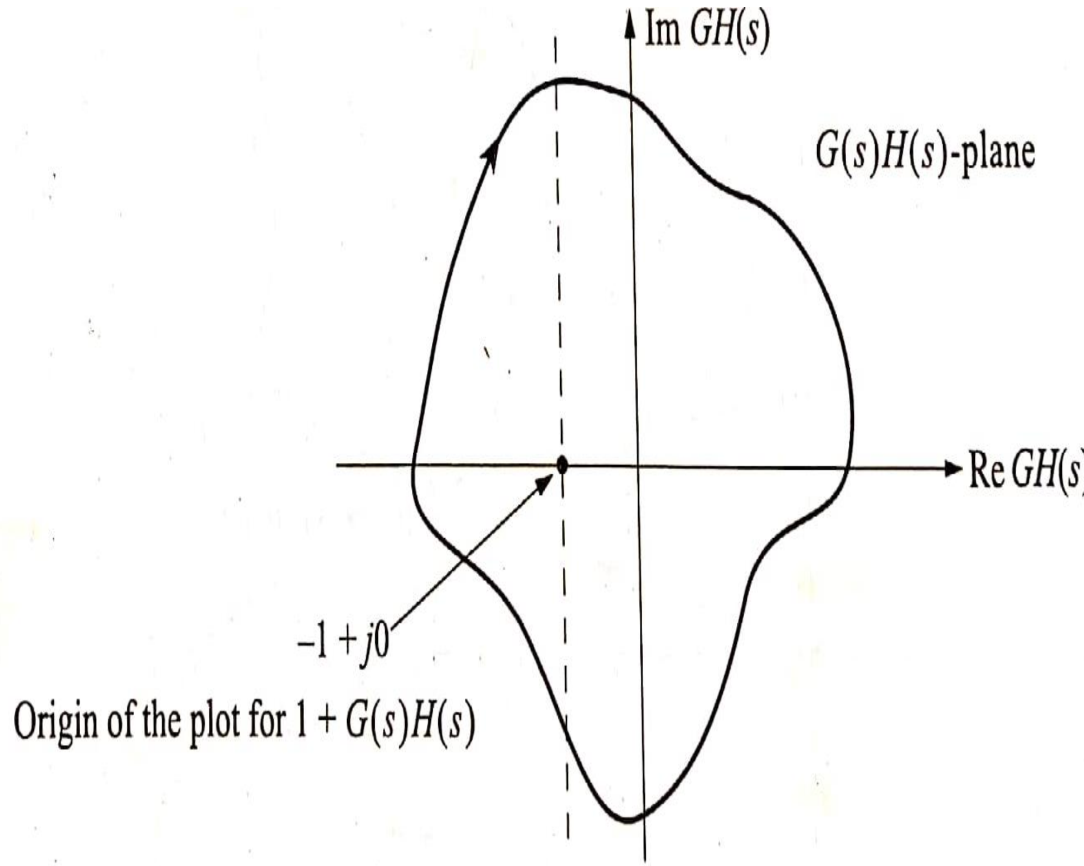
As point o’ is rotated once around the contour in the clockwise direction, the vector (s-z_1) makes one complete clock-wise revolution. Undergoes a net angle of 360° clockwise



The path traced on the 1+GH(s)-plane corresponding to one rotation of the point o’ on the s-plane also experience a net phase change of 360^{o} clockwise



- Enclosed
- Encirclement



Procedure for drawing Nyquist plot

Steps

1. Plot the poles of $G(s)H(s)$ on the s-plane. Then find P.
   P = Number of open loop poles on right-half of s-plane
2. Perform the conformal mapping or find the image of the contour 'abcda' enclosing the right-half of s-plane on the $G(s)H(s)$-plane and then determine the number of encirclements (N) of the $-1+j0$ point.
3. Determine Z:

$$Z = P + N$$

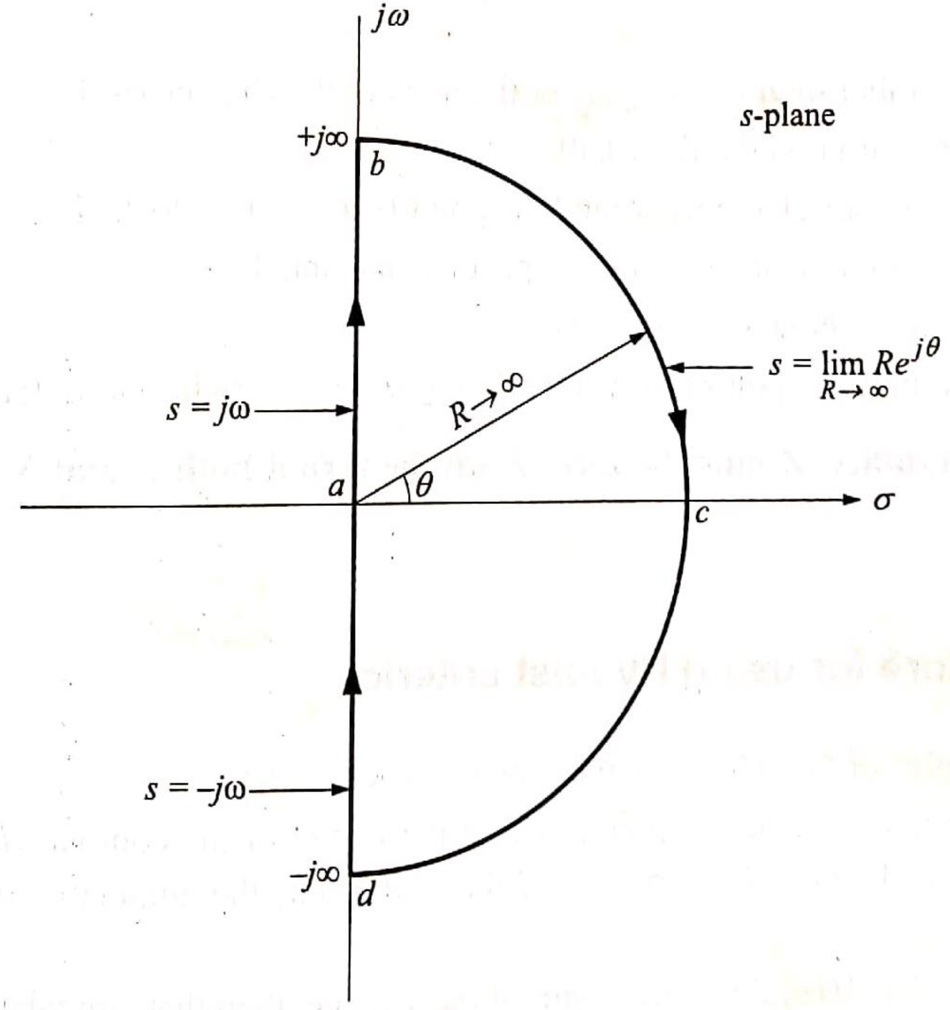
If Z is zero, the closed-loop system is stable.

The right-half of the s-plane enclosed by the semicircle

Section I : path ab

Section II : path bcd

Section III : path da



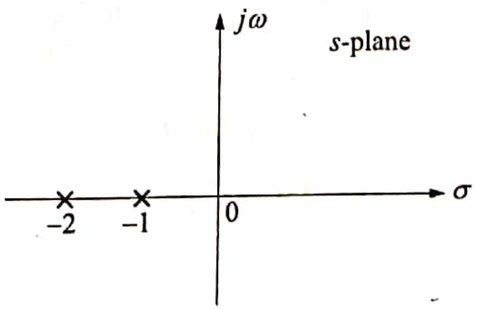
Problem

Using Nyquist stability criterion, Investigate the stability of a closed-loop system whose open-loop transfer function is given by,

$$G(s)H(s) = \frac{10}{(s+1)(s+2)}$$

Solution:

Step 1: Plot the poles of G(s)H(s) on the s-plane



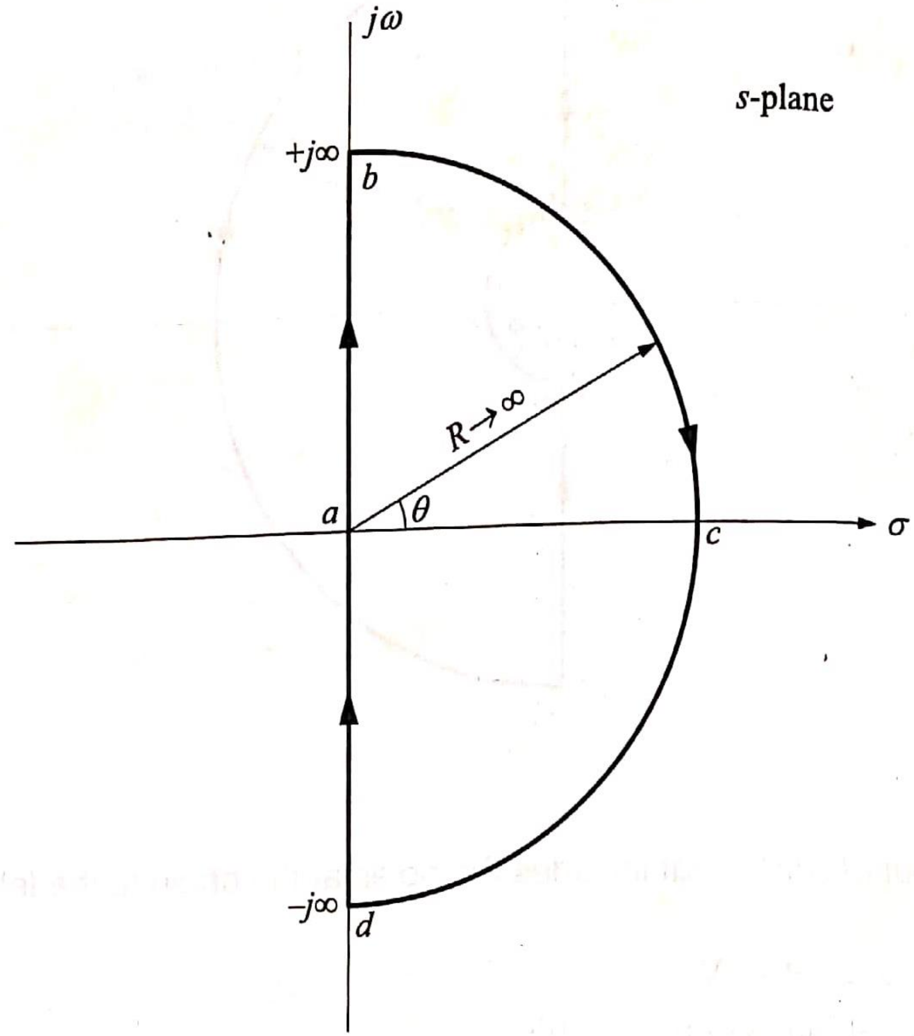
Since both poles lie on the left side of the s-plane, P = 0

Step 2: To find the image of the contour 'abcda' in G(s)H(s) plane and N

Section I: path ab

Section II: path bcd

Section III: path da



Section I : To find the image of path ab (Polar Plot):

$$G(s)H(s) = \frac{10}{(s+1)(s+2)}$$

put s = jω

$$G(j\omega)H(j\omega) = \frac{10}{(j\omega+1)(j\omega+2)}$$

$$= \frac{10}{\{\sqrt{\omega^2+1}\angle\tan -1\omega\}\{\sqrt{\omega^2+4}\angle\tan -1(\frac{\omega}{2})\}}$$

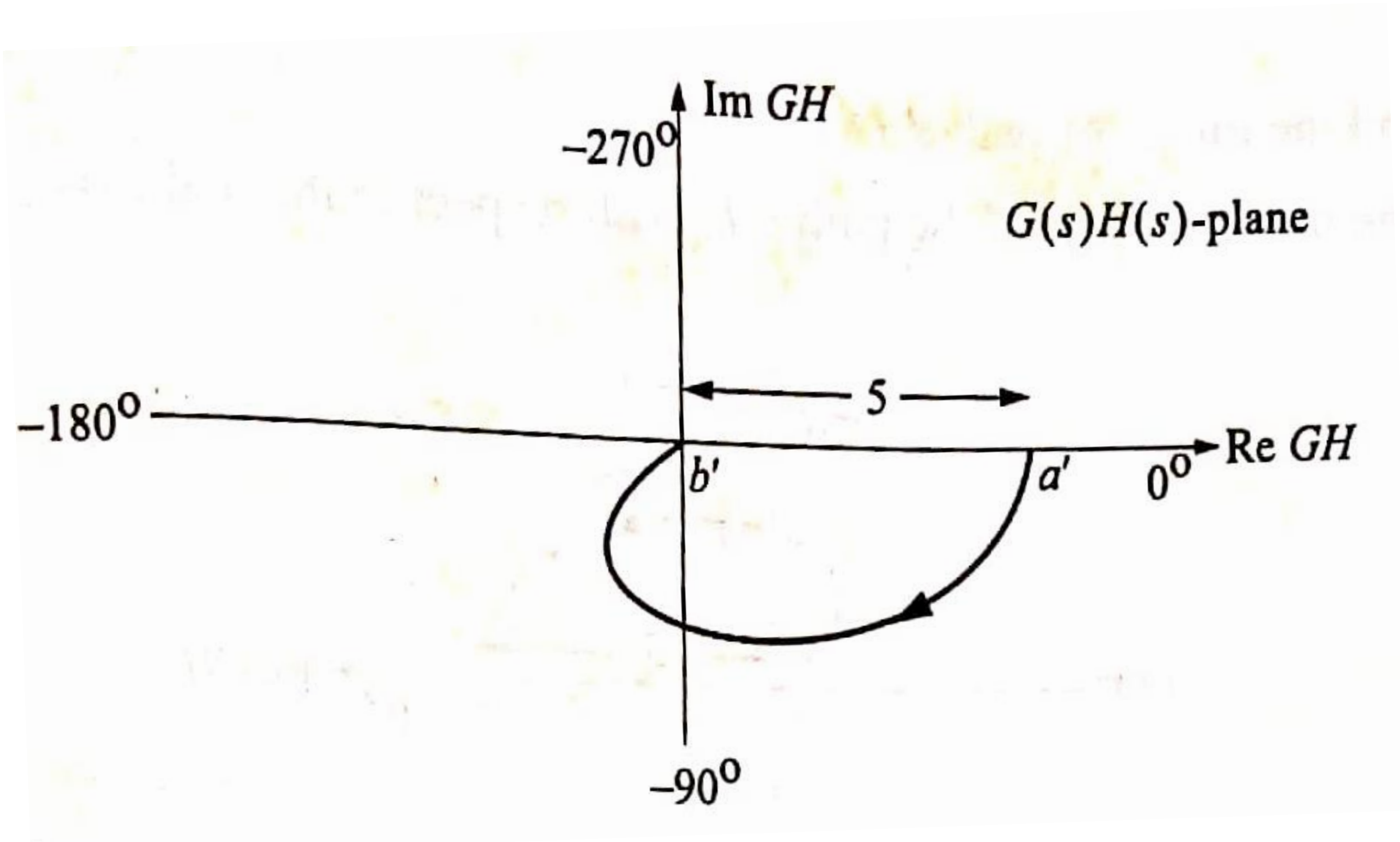
$$= \frac{10}{\{\sqrt{\omega^2+1}\}\{\sqrt{\omega^2+4}\}} - \angle\tan -1\omega \ - \ \angle\tan -1\ (\frac{\omega}{2})$$

$$M = \frac{10}{\{\sqrt{\omega^2+1}\}\{\sqrt{\omega^2+4}\}}\ ;$$

$$\emptyset = -\angle\tan -1\omega - \angle\tan -1\ (\frac{\omega}{2})$$

$\lim_{\omega \to 0} M\angle\emptyset = 5\angle 0$ (point a')

$\lim_{\omega \to \infty} M\angle\emptyset = 0\angle\text{-}180$ (point b')



Section II : To find the image of path 'bcd'

put s = $\lim_{R \to \infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from $+90 \longrightarrow 0 \longrightarrow -90$

Then, $\lim_{R \to \infty} GH(Re^{j\theta}) = \lim_{R \to \infty} \frac{10}{(Rej^{\theta}+1)(Re^{j\theta}+2)}$

$= \lim_{R \to \infty} \frac{10}{(Re^{j\theta})(Re^{j\theta})}$

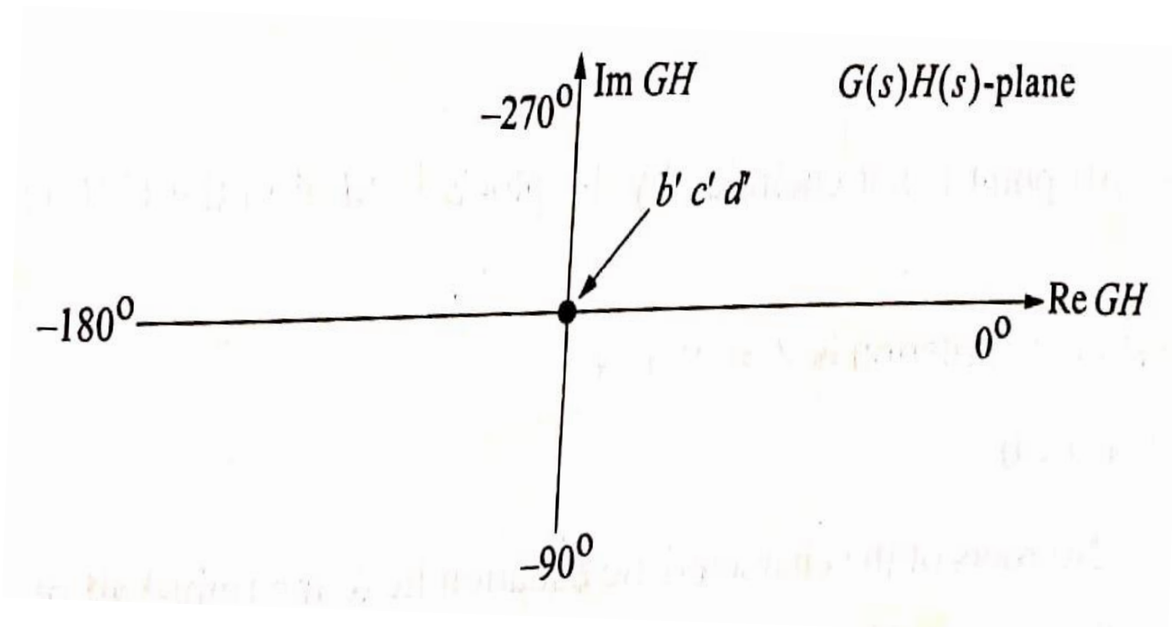
$= \lim_{R \to \infty} \frac{10}{(R^2 e^{j2\theta})}$

$= 0\angle{-2\theta}$

$= 0\angle{-180} \longrightarrow 0 \longrightarrow 180$

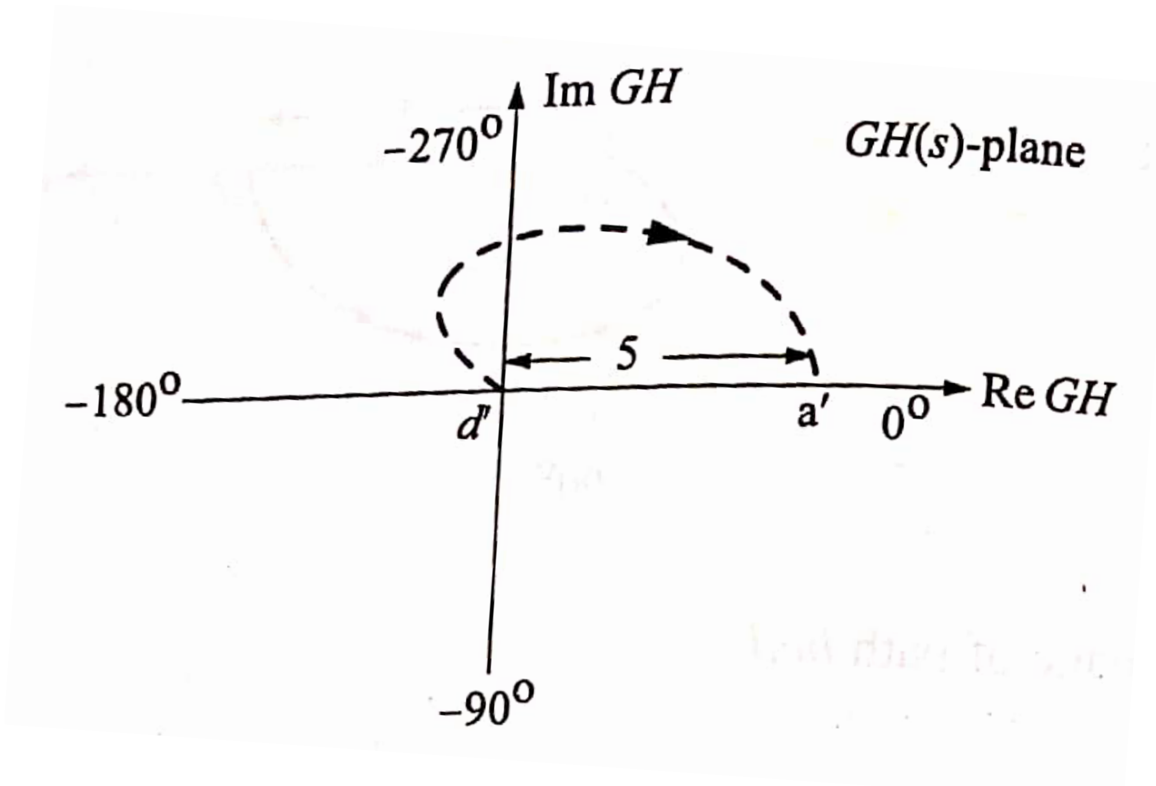
↑b' ↑c' ↑d'

Hence, the infinite semicircle ‘bcd’ on the s-plane is mapped to the origin of the G(s)H(s)-plane.

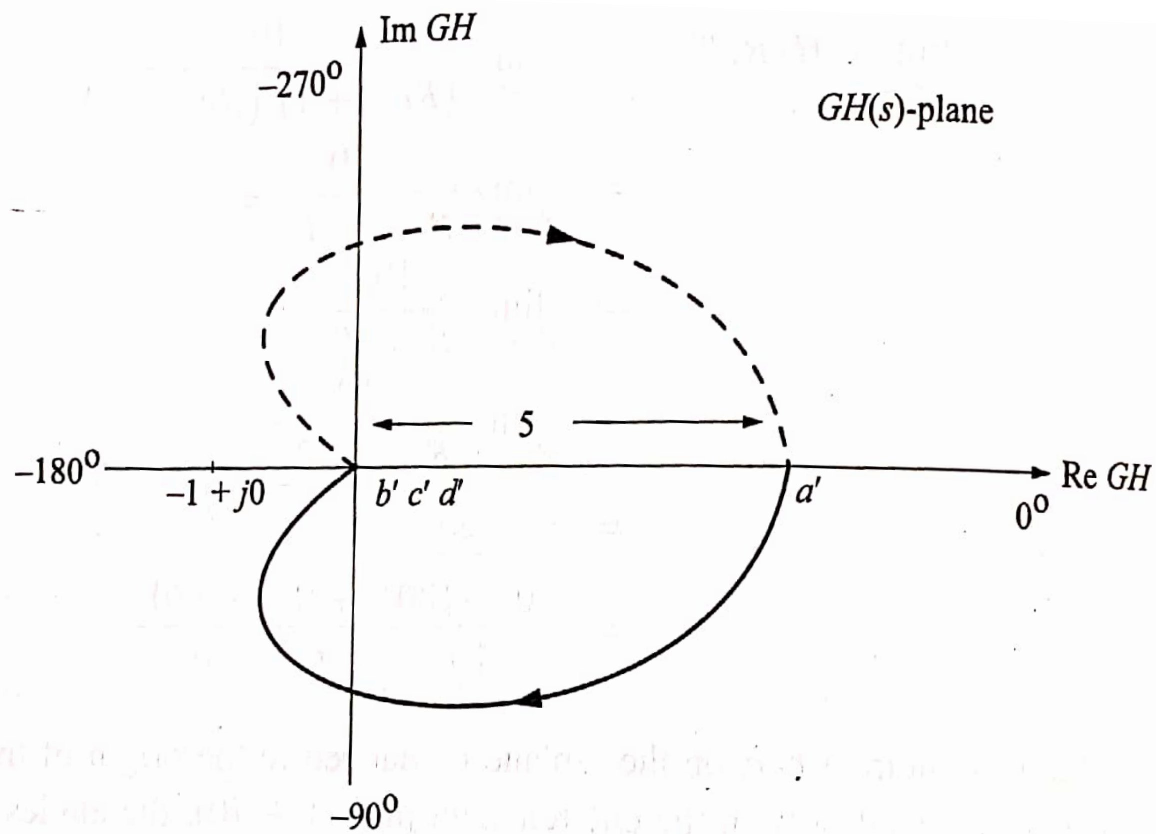


Section III: To find the image of path 'da'

Path d'a' is the mirror image of the path a'b' with respect to real axis.



The complete Nyquist plot is shown below



Since, (-1+j0) point is not encircled by the plot a’b’c’d’a’ in the GH(s) plane , N= 0

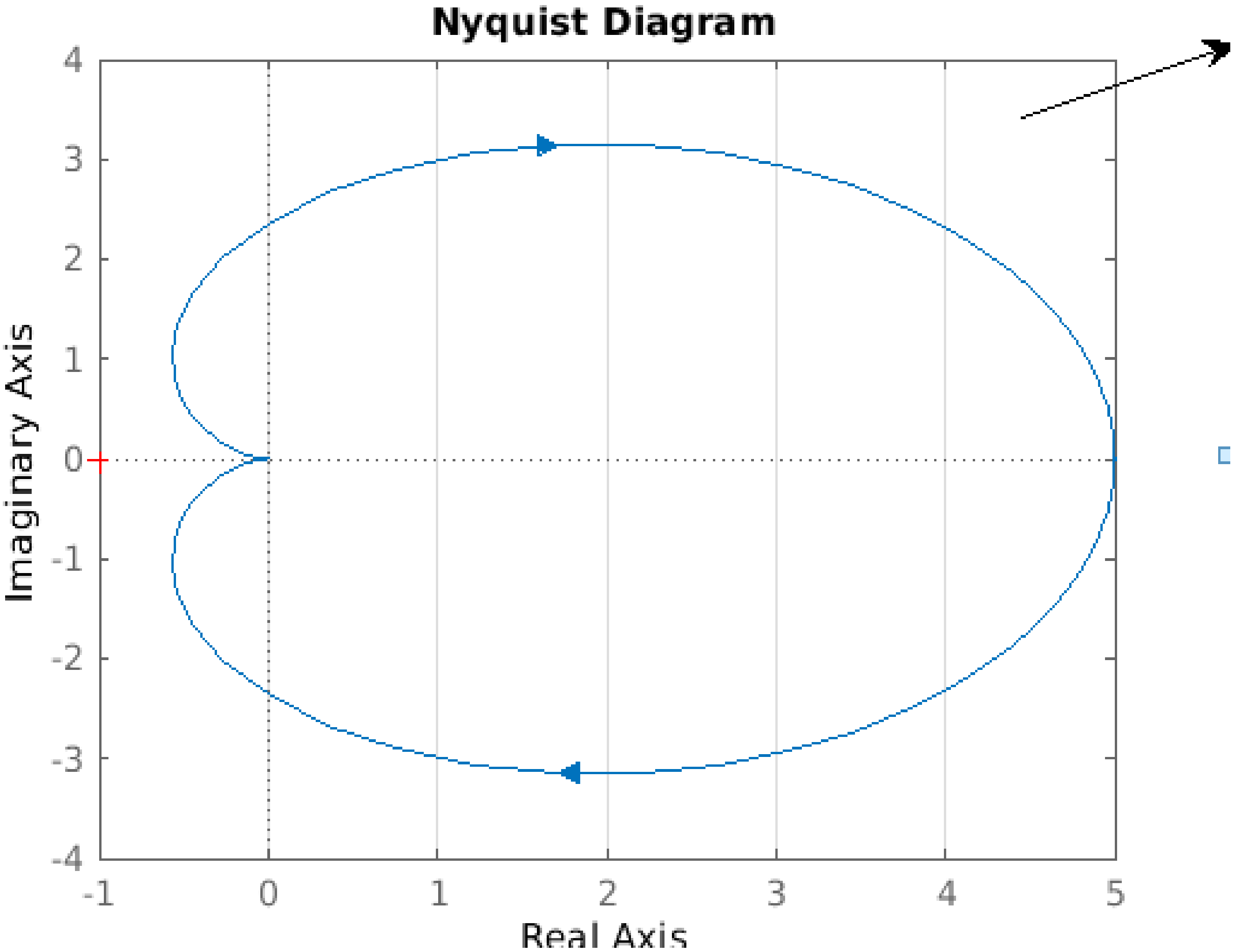
Step 3: The Nyquist stability criterion is Z = P + N

Hence, Z = 0 + 0 = 0

$\Longrightarrow$No roots of the system lie to the right-half of s-plane.

Hence the closed-loop control system is stable.

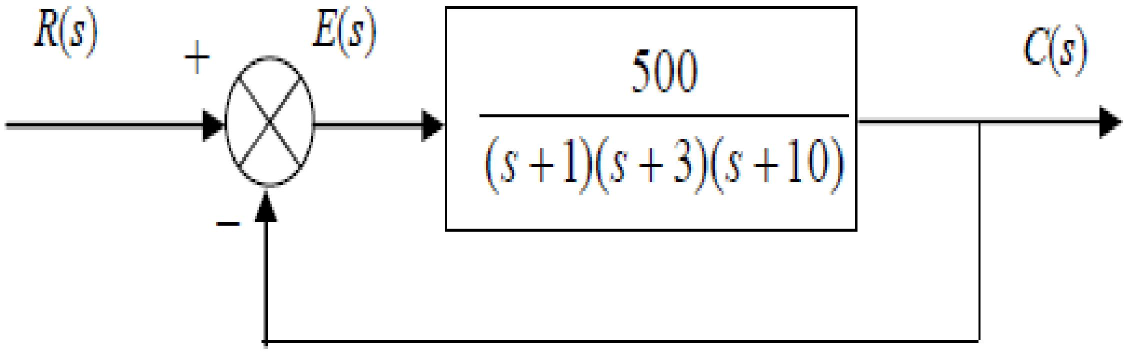
Matlab

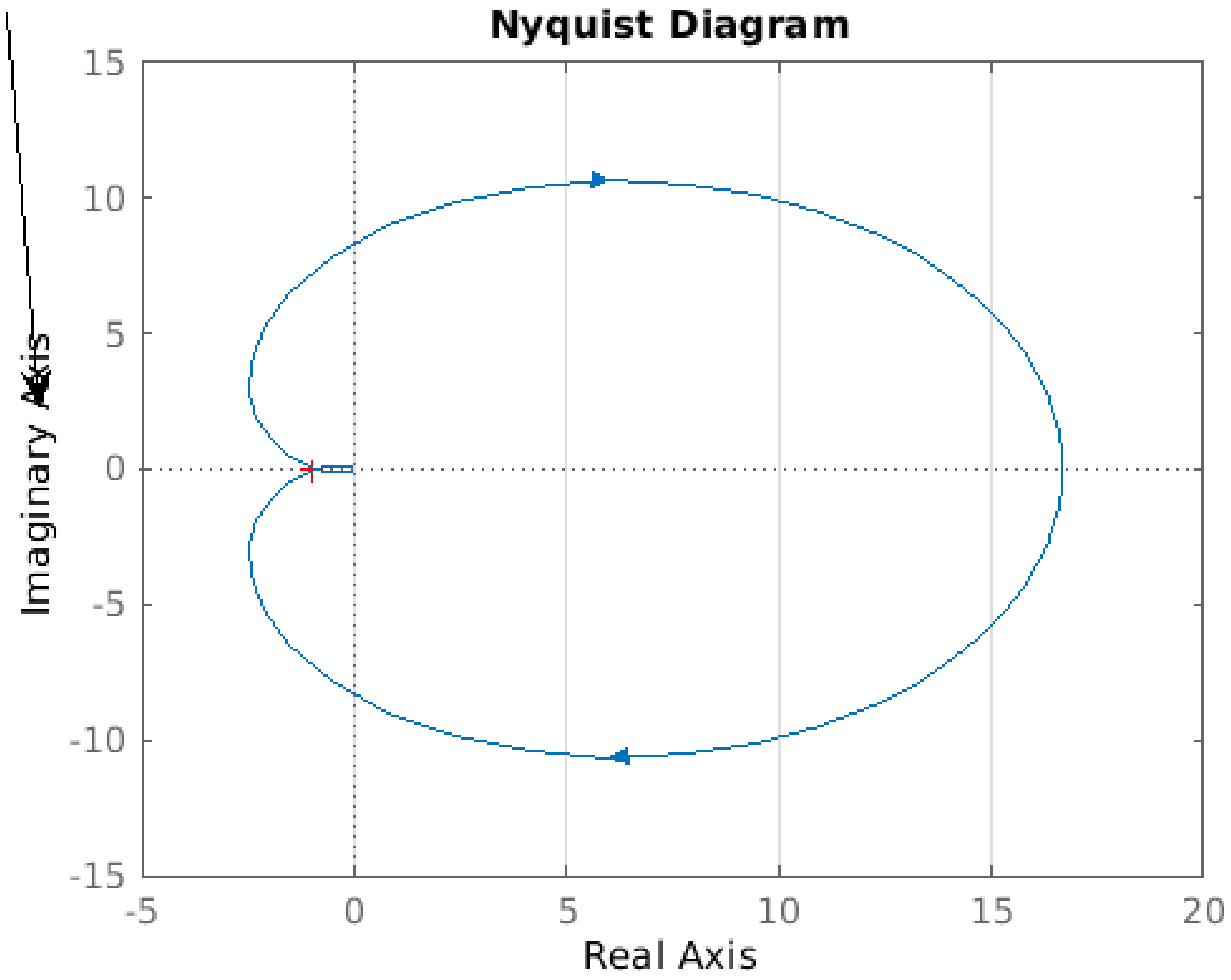


Nyquist Diagram
Imaginary Axis
Real Axis
4
3
2
1
0
-1
-2
-3
-4
-1
0
1
2
3
4
5

Problem

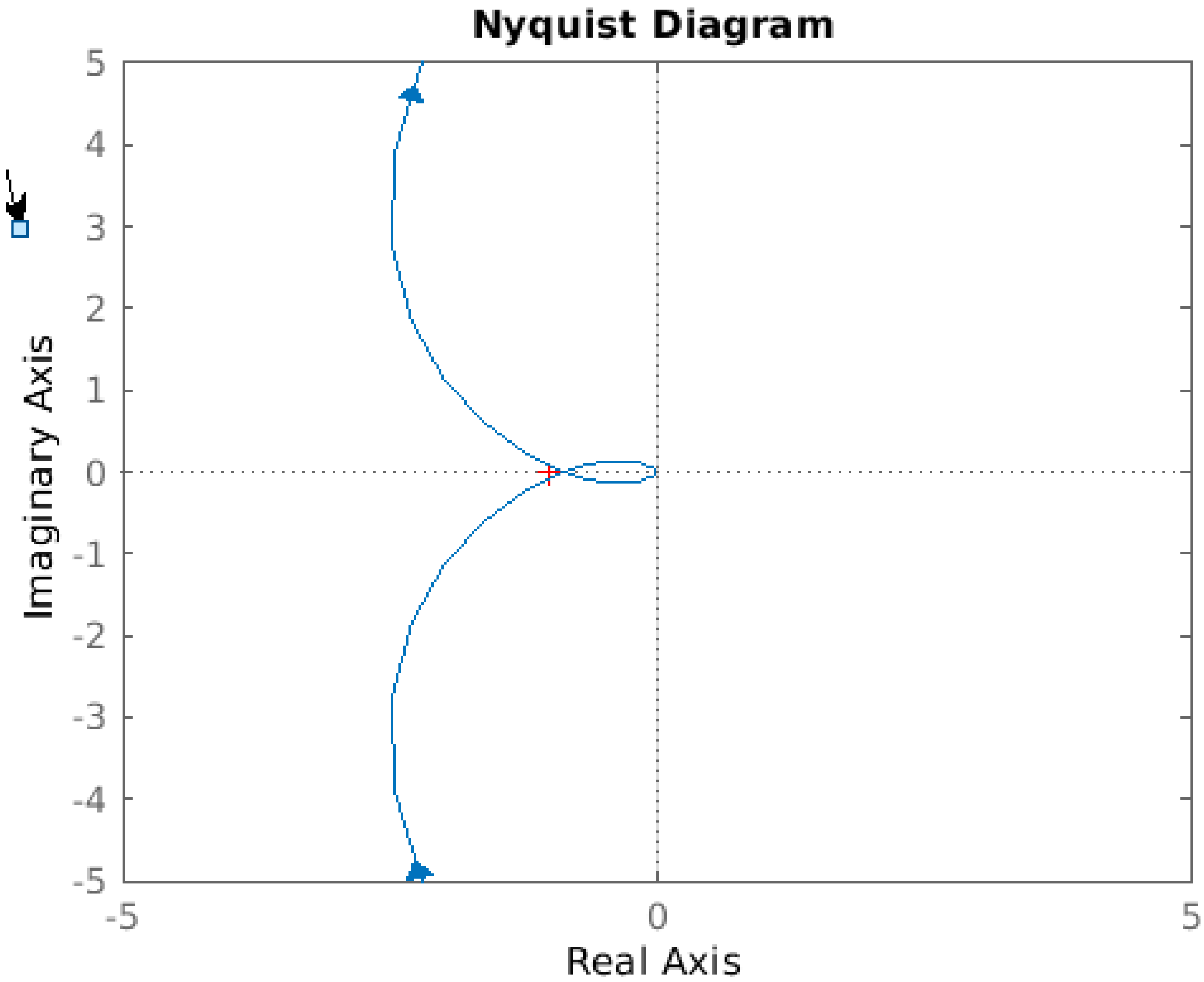
Sketch the Nyquist diagram for the system shown in the following figure, and then determine the system stability using the Nyquist criterion





Nyquist Diagram
Imaginary Axis
Real Axis
15
10
5
0
-5
-10
-15
-5
0
5
10
15
20

Matlab



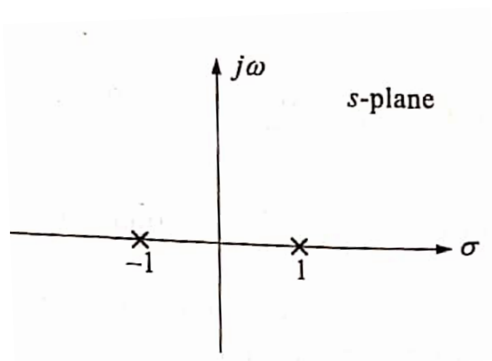
Problem

Using Nyquist stability criterion, Investigate the stability of a closed-loop system whose open-loop transfer function is given by,

$$G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$$

Solution:

Step 1: Plot the poles of G(s)H(s) on the s-plane



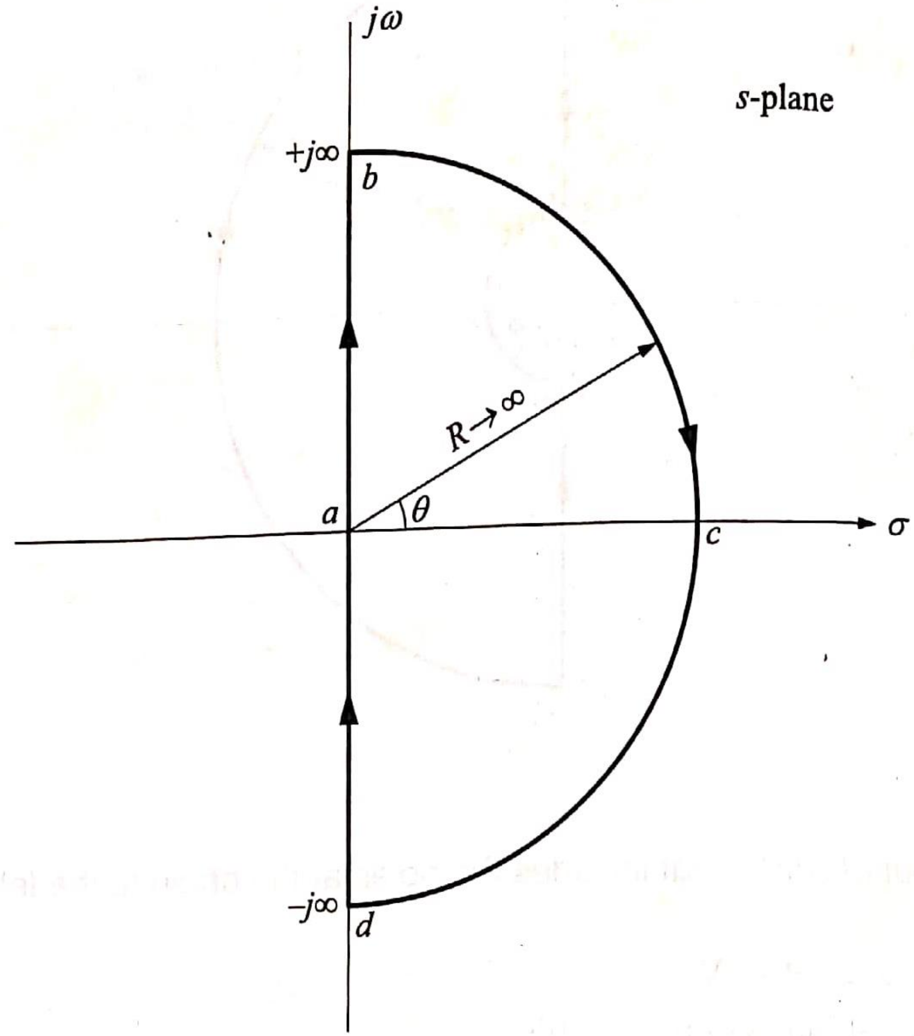
Since one pole lie on the right side of the s-plane, P = 1

Step 2: To find the image of the contour 'abcda' in G(s)H(s) plane and N

Section I: path ab

Section II: path bcd

Section III: path da



Section I : To find the image of path ab (Polar Plot):

$$G(s)H(s) = \frac{(s+2)}{(s+1)(s-1)}$$

put s = jω

$$G(j\omega)H(j\omega) = \frac{(j\omega+2)}{(j\omega+1)(j\omega-1)}$$

$$= \frac{\{\sqrt{\omega^2+4}\angle\tan^{-1}(\frac{\omega}{2})\}}{\{\sqrt{\omega^2+1}\angle\tan^{-1}\omega\}\{\sqrt{\omega^2+1}\angle\tan^{-1}(\frac{\omega}{-1})\}}$$

$$= \frac{\{\sqrt{\omega^2+4}\angle\tan^{-1}(\frac{\omega}{2})\}}{\{(\sqrt{\omega^2+1})^2\angle\tan^{-1}\omega\}\{\angle 180 - \tan^{-1}\omega\}}$$

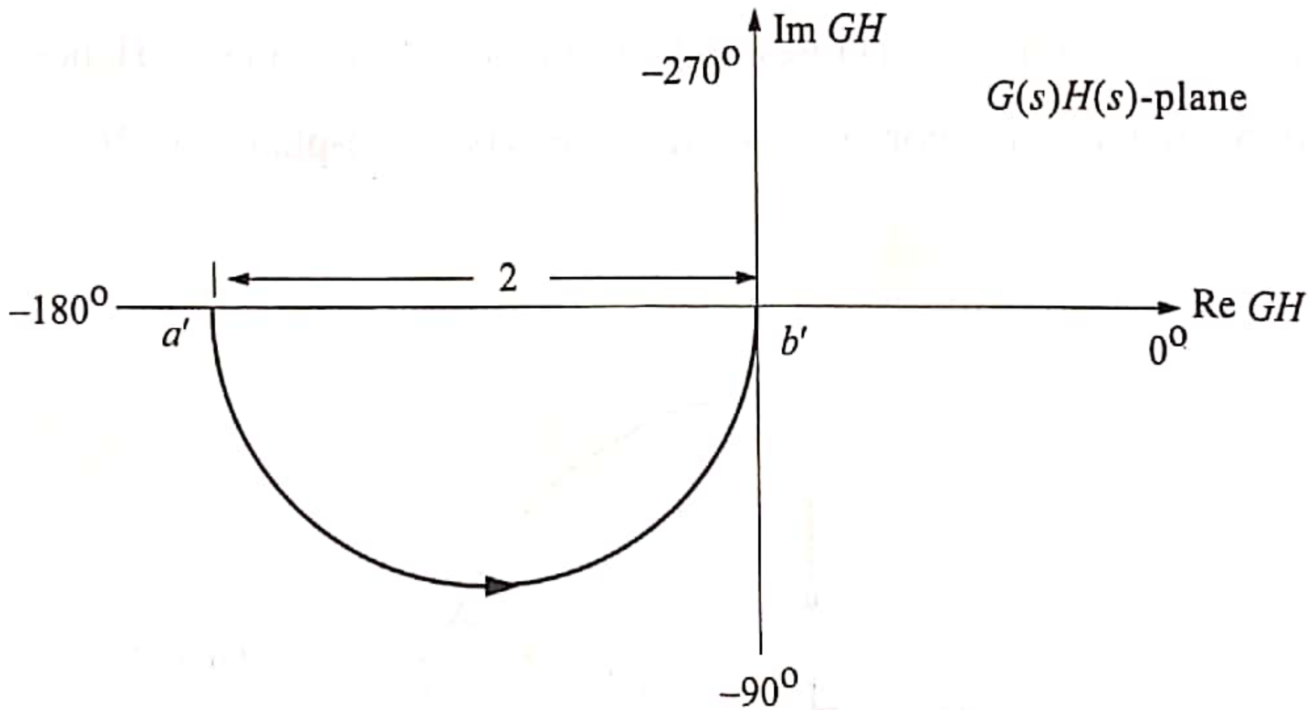
$$M = \frac{\{\sqrt{\omega^2+4}\}}{\omega^2+1};$$

$$\emptyset = \angle\tan^{-1}(\frac{\omega}{2}) - \angle\tan^{-1}\omega - 180 + \angle\tan^{-1}\omega$$

$$= \angle\tan^{-1}(\frac{\omega}{2}) - 180$$

$$\lim_{\omega \to 0} M\angle\emptyset = 2\angle{-180} \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = 0\angle{-90} \qquad \text{(point b')}$$



Section II : To find the image of path ‘bcd’

put s = $\lim_{R \to \infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 ⟶ 0 ⟶ -90

Then, $\lim_{R \to \infty} GH(Re^{j\theta}) = \lim_{R \to \infty} \frac{(Rej^{\theta}+2)}{(Rej^{\theta}+1)(Re^{j\theta}-1)}$

$= \lim_{R \to \infty} \frac{Re^{j\theta}}{(Re^{j\theta})(Re^{j\theta})}$

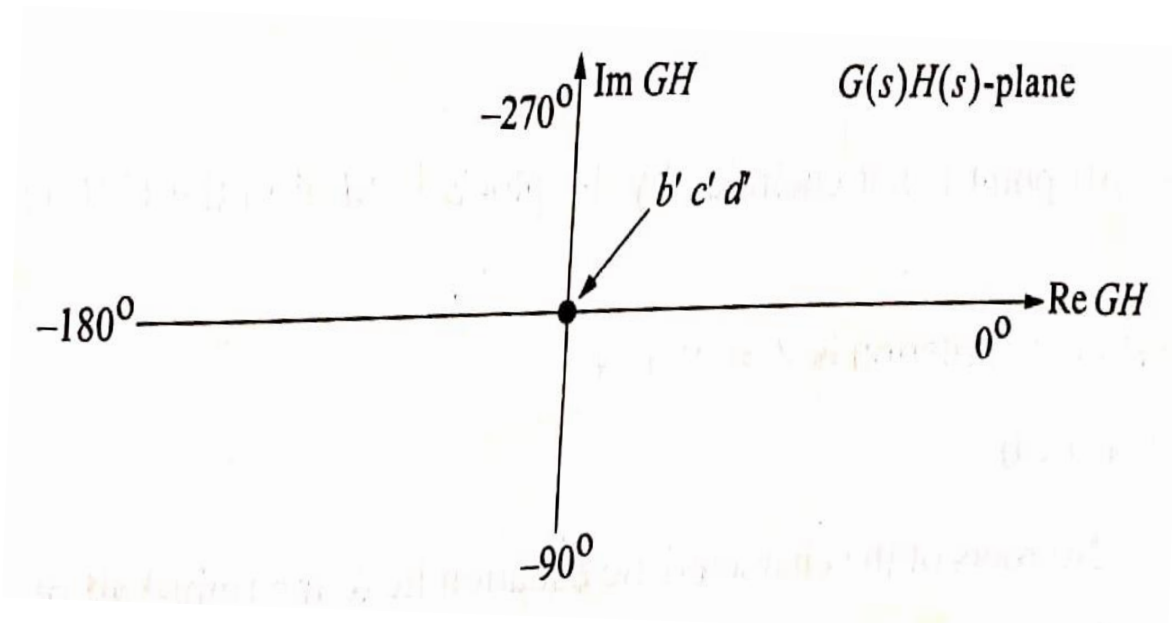
$= \lim_{R \to \infty} \frac{1}{(Rej^{\theta})}$

= 0∠-θ

= 0∠-90⟶ 0⟶ 90

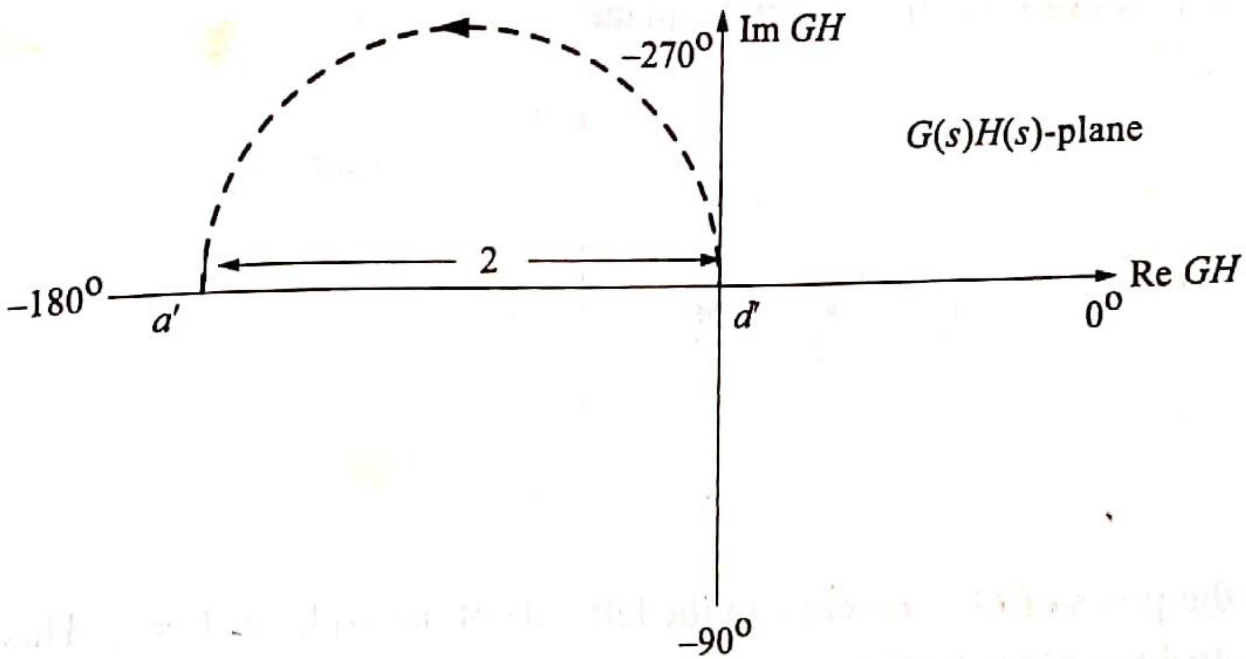
⤒b’ ⤒c’ ⤒d’

Hence, the infinite semicircle ‘bcd’ on the s-plane is mapped to the origin of the G(s)H(s)-plane.

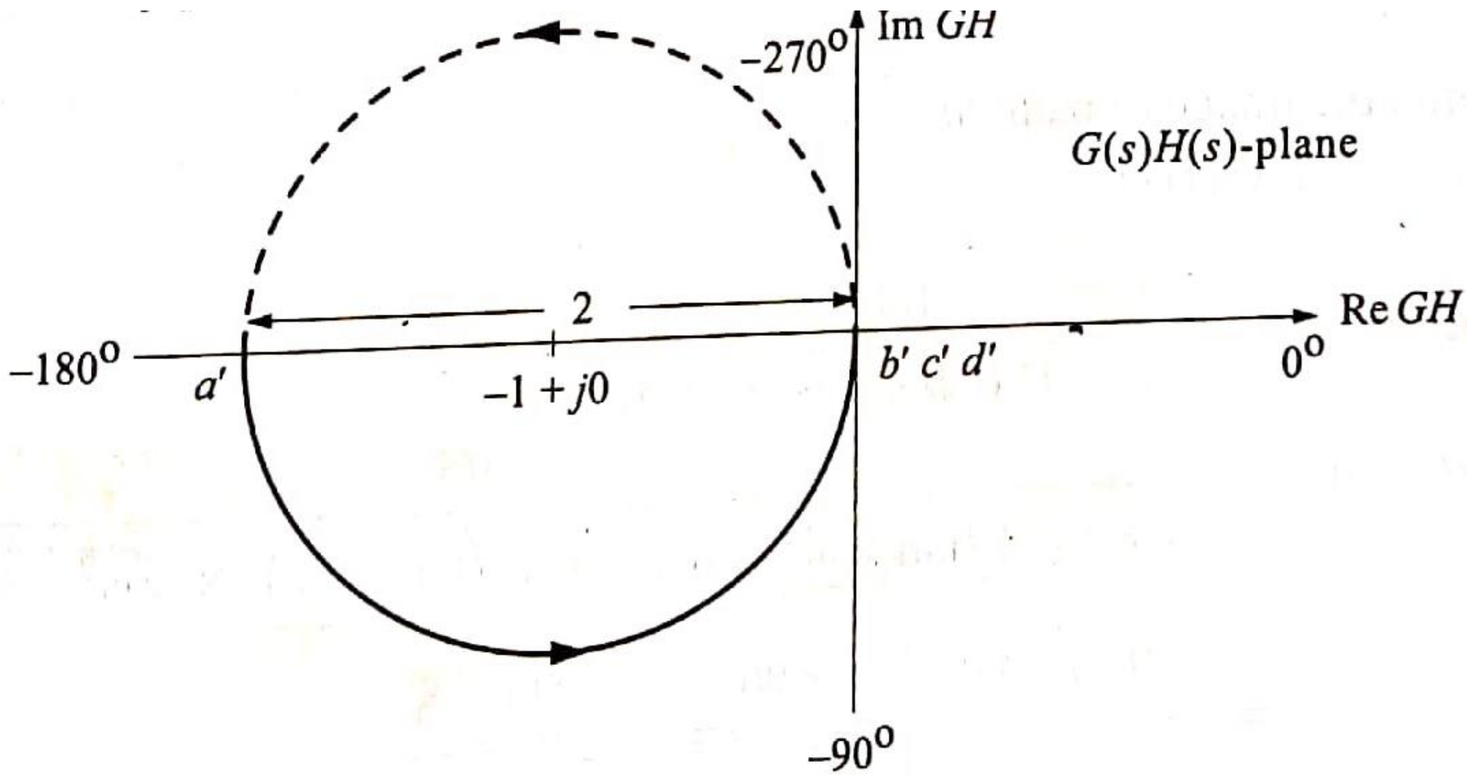


Section III: To find the image of path 'da'

Path d'a' is the mirror image of the path a'b' with respect to real axis.



The complete Nyquist plot is shown below



Since, (-1+j0) point is encircled in anticlockwise direction by the plot a’b’c’d’a’ in the GH(s) plane, N = -1

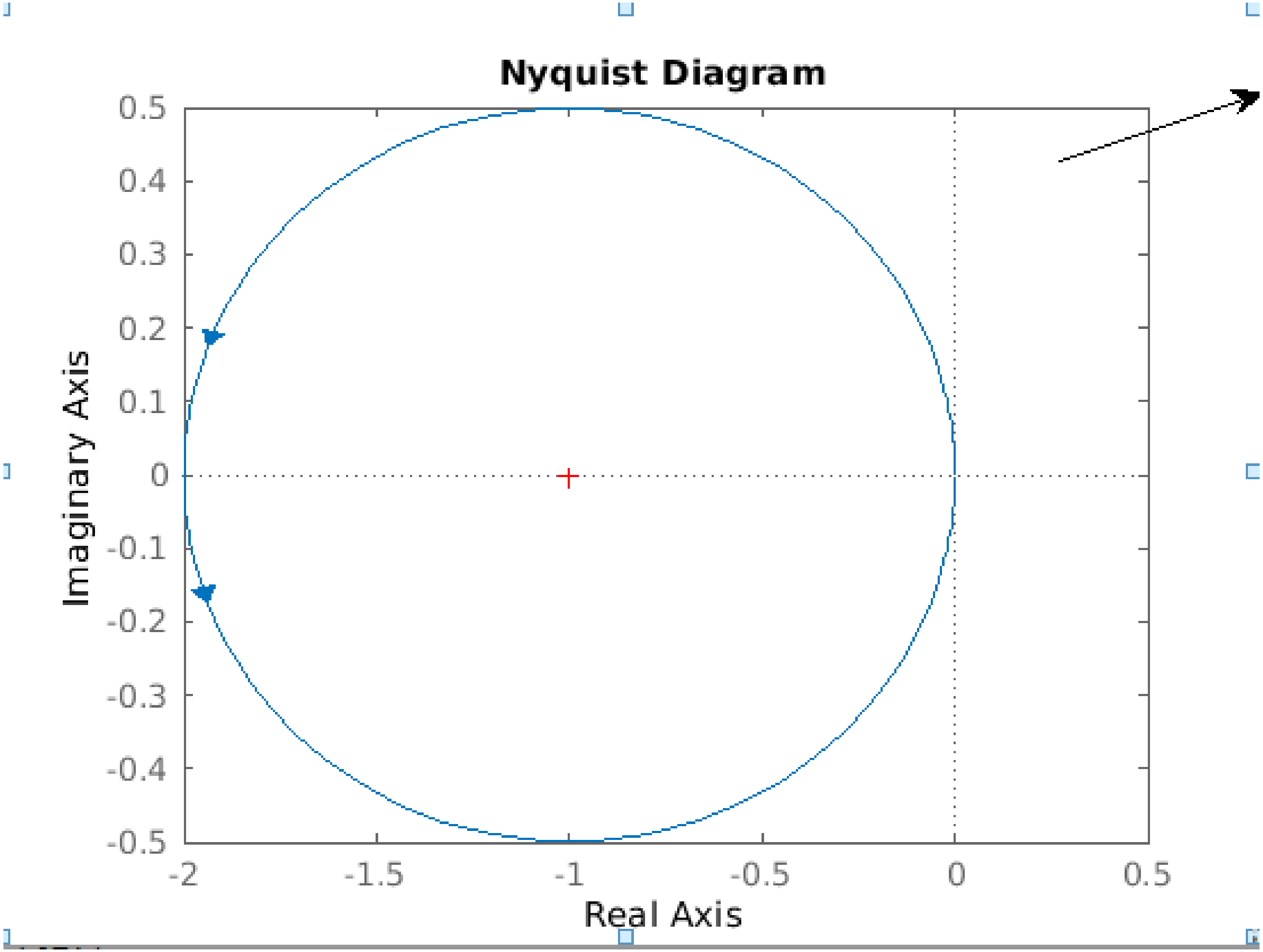
Step 3:

The Nyquist stability criterion is Z = P + N

Hence, Z = 1 - 1 = 0

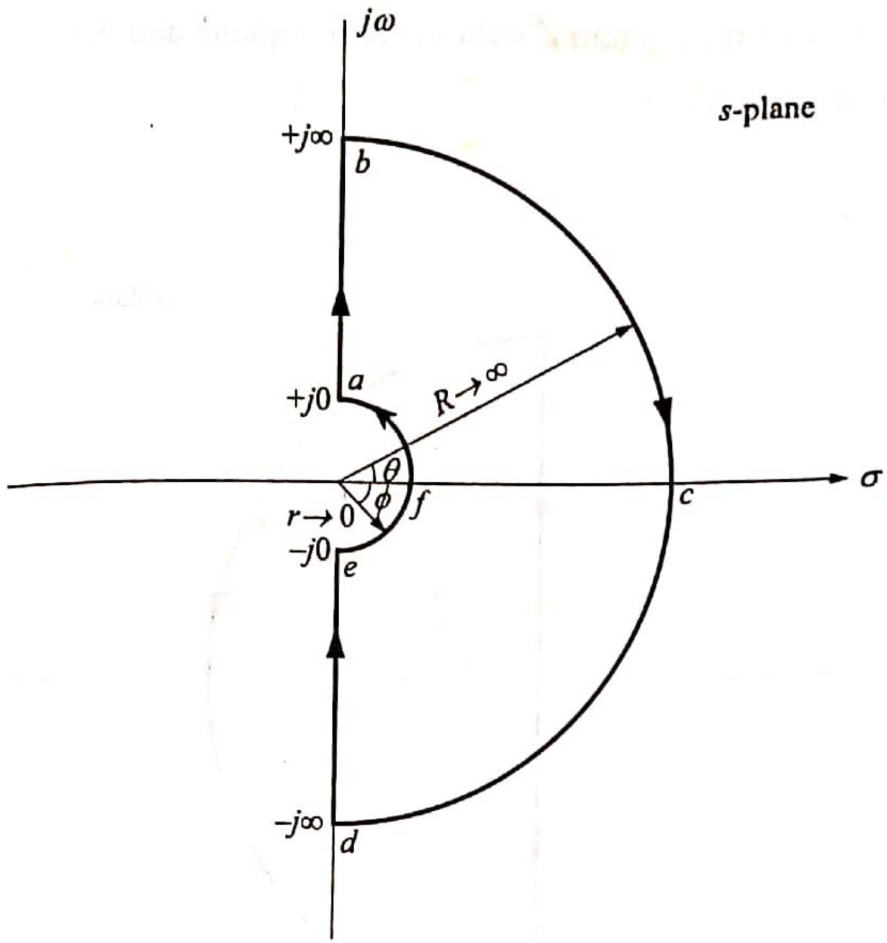
$\Longrightarrow$No roots of the system lie to the right-half of s-plane. Hence the closed-loop control system is stable.

Using Matlab

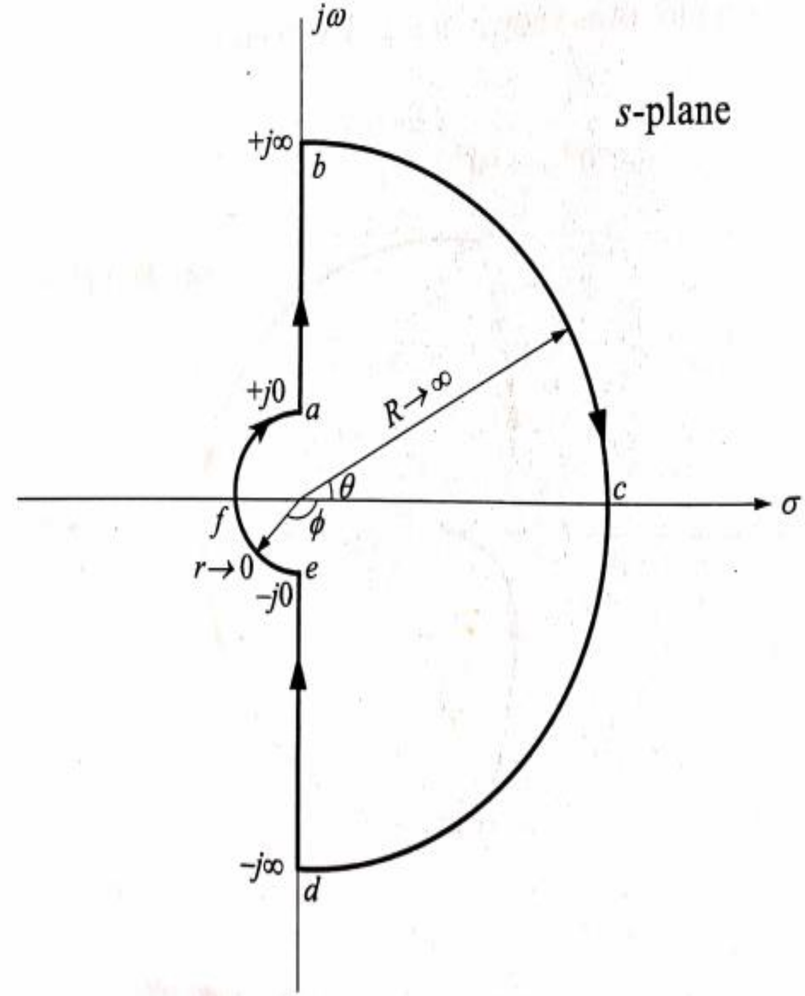


Special Case

If the poles of G(s)H(s) lie at the origin of the s- plane, then they are taken to the left-side of the s- plane (right-side of the s- plane) by drawing an indent 'efa' of radius, $r \longrightarrow 0$ as shown. Then find the number of encirclements made by the image of the contour 'abcdefa' about (- 1+j0) point on the GH(s)- plane.



jω
s-plane
+j∞
b
a
+j0
R→∞
θ
φ
f
c
σ
r→0
−j0
e
−j∞
d

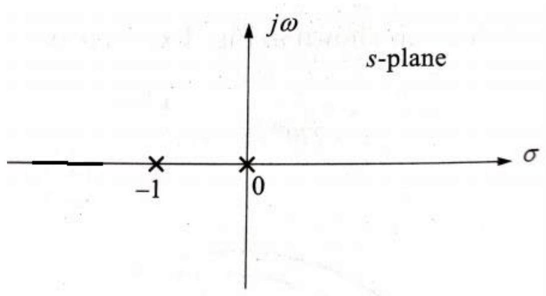


jω
s-plane
+j∞
b
+j0
a
R→∞
θ
c
σ
f
φ
r→0
e
−j0
−j∞
d

Problem

A negative feedback control system is characterized by an open-loop transfer function, G(s)H(s) = $\frac{5}{s(s+1)}$.

Investigate the closed-loop stability of the system using Nyquist stability criterion.



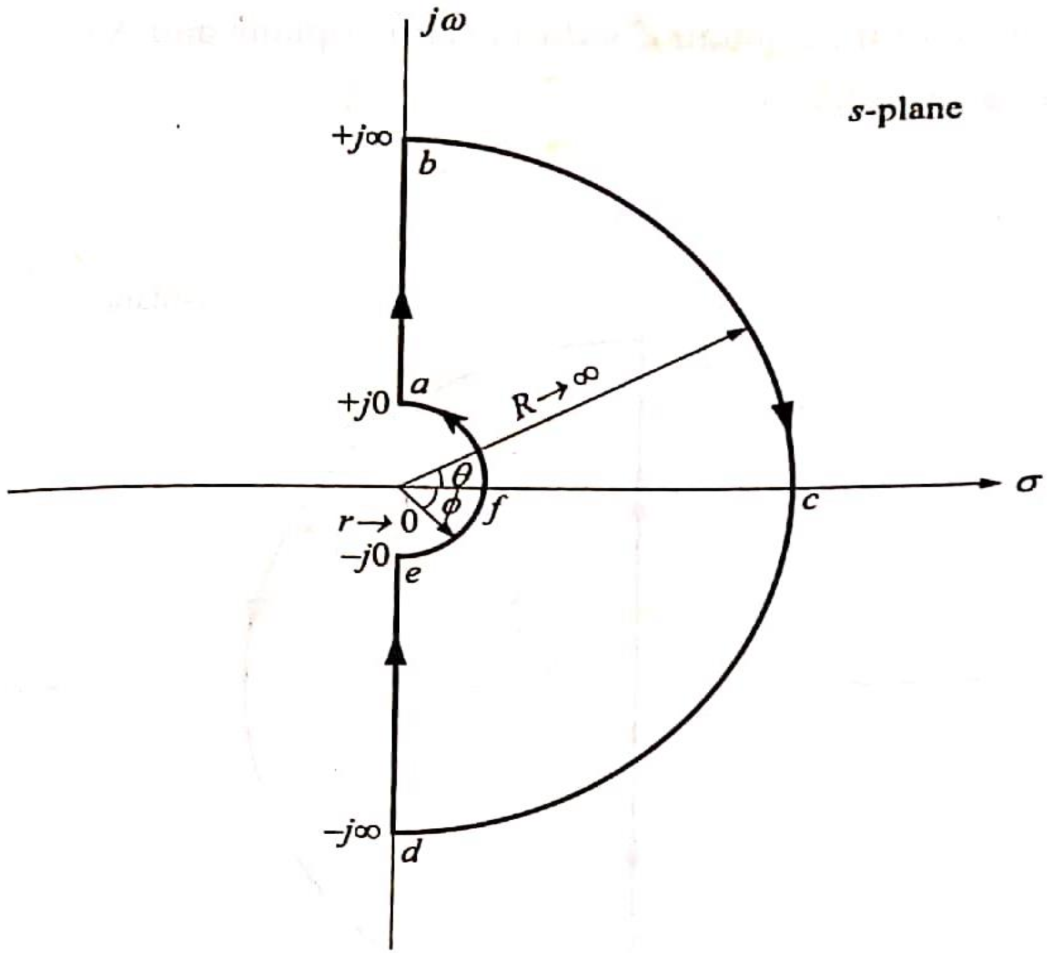
Solution

Step 1: Plot the poles of G(s)H(s) on the s-plane.

The pole at the origin is taken to the left-side of the s-plane by drawing an indent of zero radius around this pole.

Since the pole at the origin is taken to the left-side of the s-plane, P=0.

The contour ‘abcdefa’ that includes at the origin to the left side of the s-plane



Step 2: To find N:

Section I : To find the image of path ab.

$$G(s)H(s) = \frac{5}{s(s+1)}$$

Put s = jω

$$G(j\omega)H(j\omega) = \frac{5}{j\omega(j\omega+1)}$$

$$= \frac{5}{\omega\angle 90\ \sqrt{(\omega^2+1)}\angle \tan^{-1}\omega}$$

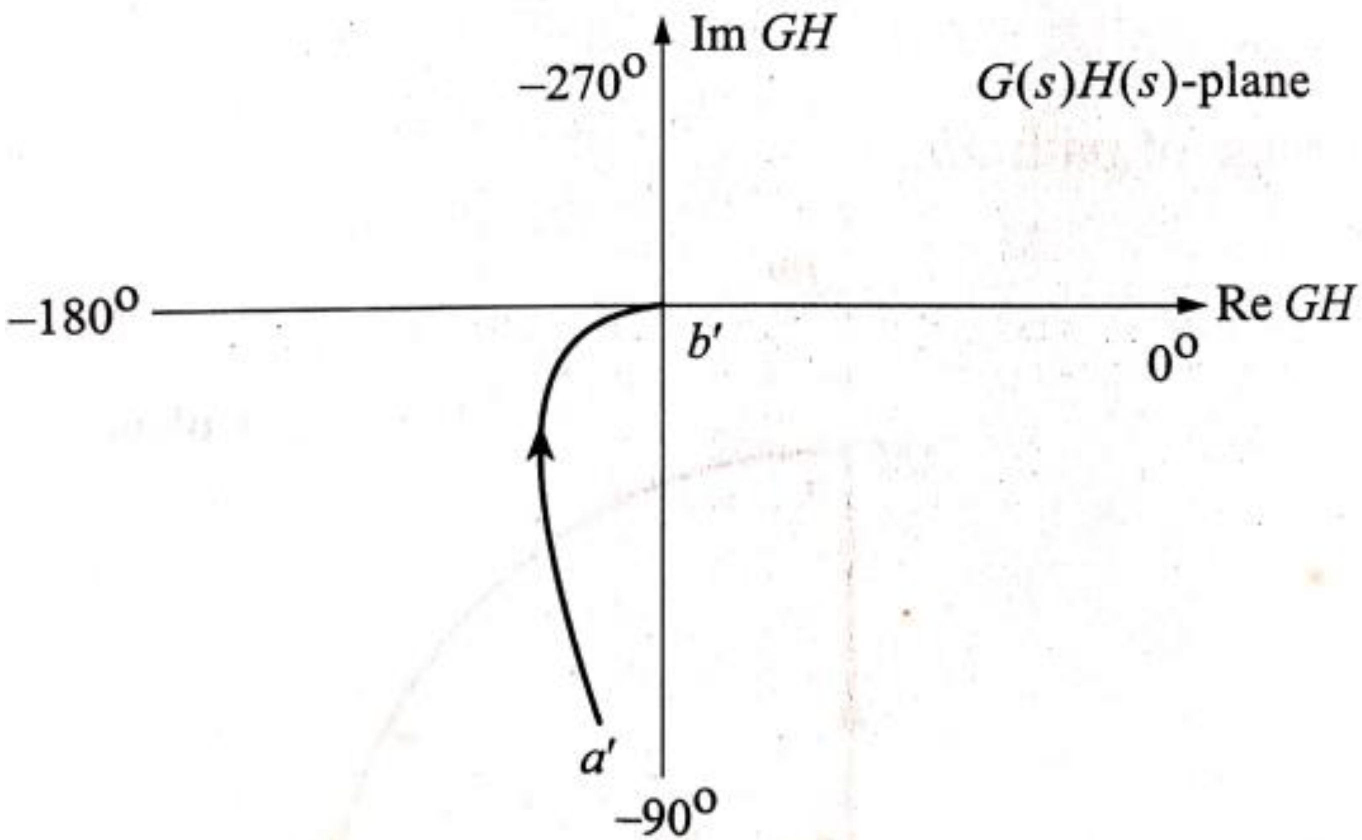
$$= \frac{5}{\omega\sqrt{(\omega^2+1)}\angle 90+\tan^{-1}\omega}$$

$$M = \frac{5}{\omega\sqrt{(\omega^2+1)}}$$

$$\emptyset = -90 - \tan^{-1}\omega$$

$$\lim_{\omega \to 0} M\angle\emptyset = \infty\angle{-90} \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = 0\angle{-180} \qquad \text{(point b')}$$



Section II : To find the image of path 'bcd'

put s = $\lim_{R \to \infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 ⟶ 0 ⟶ -90

Then, $\lim_{R \to \infty} GH(Re^{j\theta}) = \lim_{R \to \infty} \frac{5}{Re^{j\theta}(Re^{j\theta}+1)}$

$$= \lim_{R \to \infty} \frac{5}{(Re^{j\theta})(Re^{j\theta})}$$

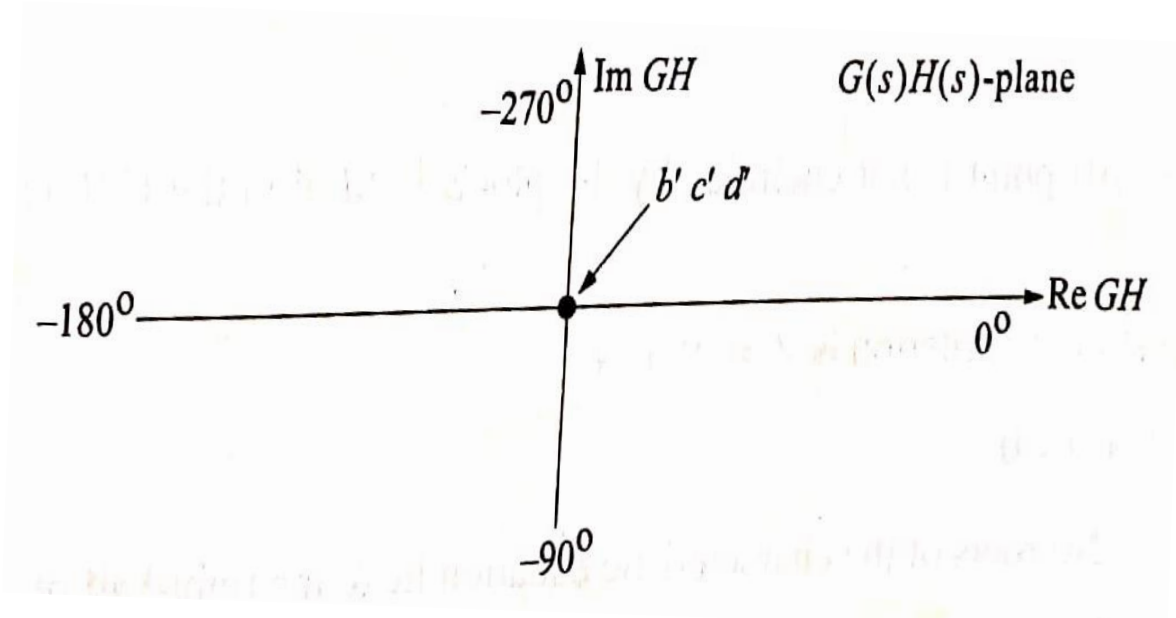
$$= \lim_{R \to \infty} \frac{5}{(R^2 e^{j2\theta})}$$

= 0∠-2θ

= 0∠-180⟶ 0⟶ 180

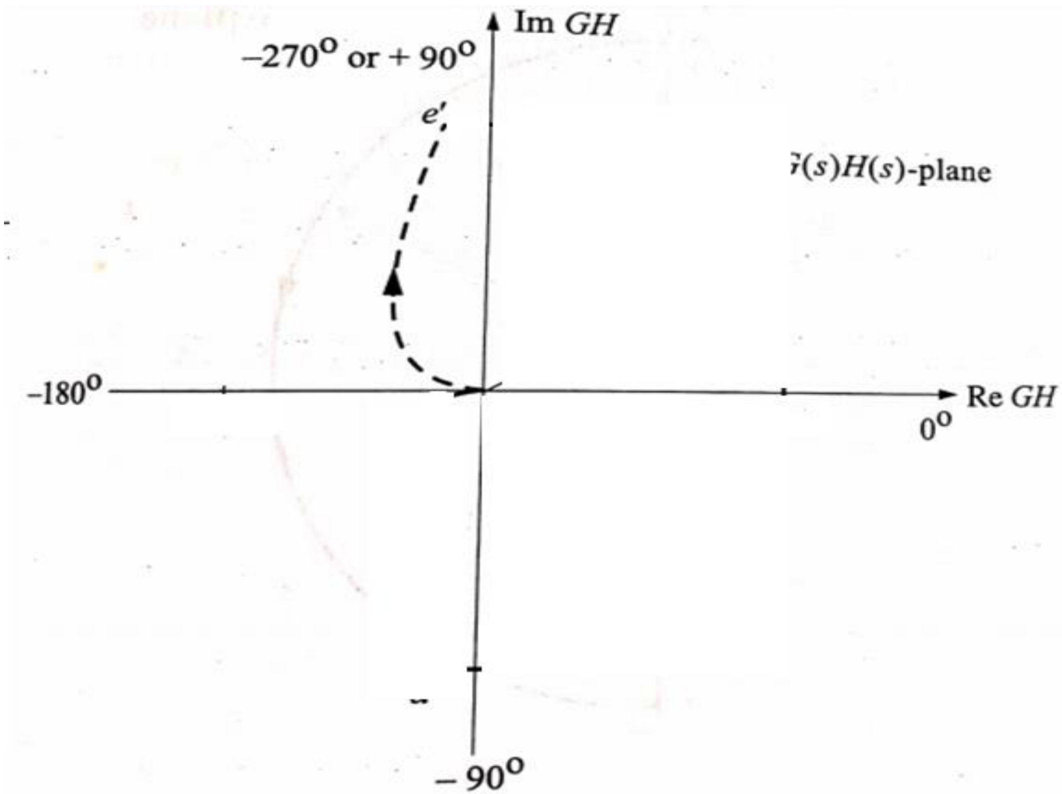
↑b' ↑c' ↑d'

Hence, the infinite semicircle ‘bcd’ on the s-plane is mapped to the origin of the G(s)H(s)-plane.



Section III: To find the image of path 'de'

Path d'e' is the mirror image of the path a'b' with respect to real axis.



Section IV : To find the image of path efa

$$\text{put } s = \lim_{r \to 0} re^{j\emptyset} \text{ in } G(s)H(s)$$

Here , $\emptyset$ changes from $-90 \longrightarrow 0 \longrightarrow +90$

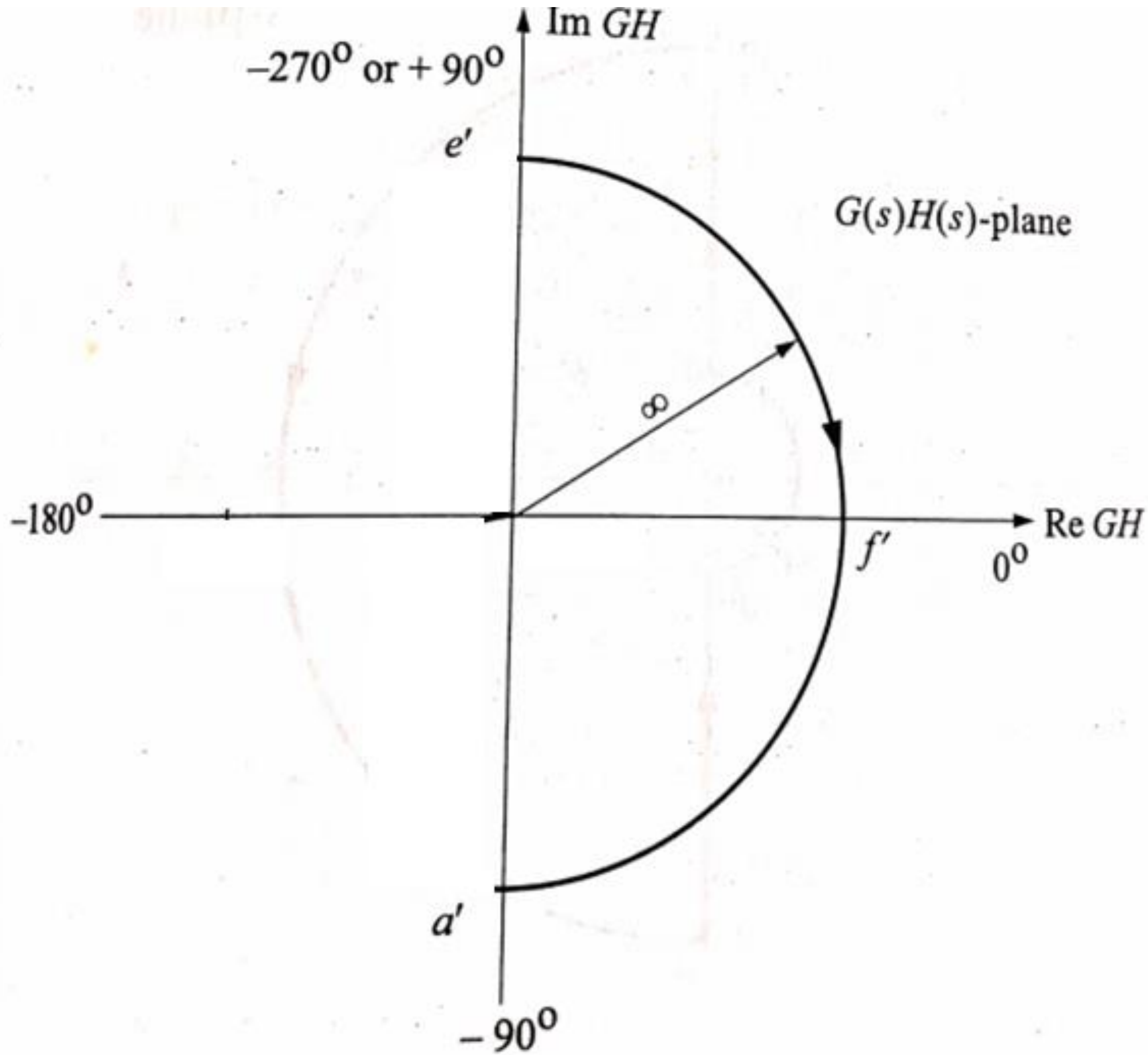
Then, $\lim_{r \to 0} GH(re^{j\emptyset}) = \lim_{r \to 0} \frac{5}{re^{j\emptyset}(re^{j\emptyset}+1)}$

$$= \lim_{r \to 0} \frac{5}{(re^{j\emptyset})}$$

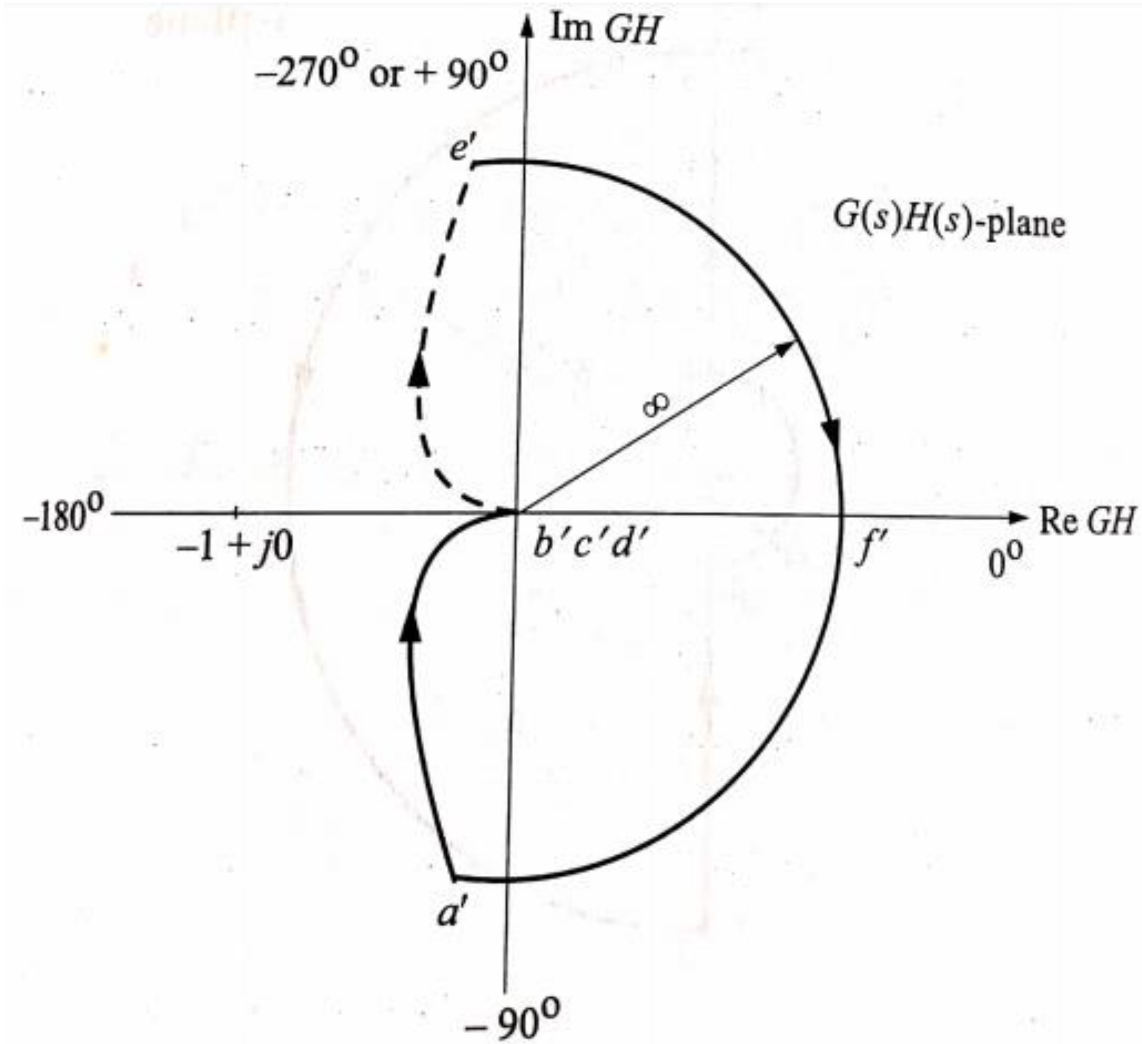
$$= \infty\angle{-\emptyset}$$

$$= \infty\angle 90 \longrightarrow 0 \longrightarrow -90$$

$$\uparrow e' \qquad \uparrow f' \qquad \uparrow a'$$


Im GH
-270^{o} or $+90^{o}$
e'
$G(s)H(s)$-plane
∞
-180^{o}
Re GH
f'
0^{o}
a'
-90^{o}

The complete Nyquist plot is shown

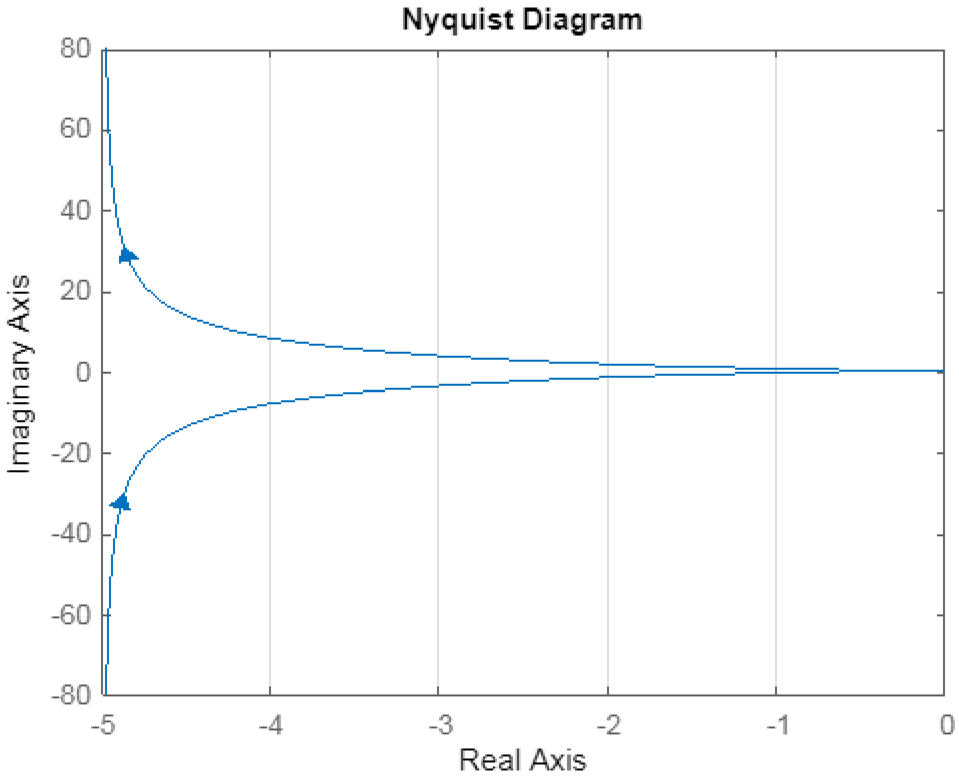


Since -1+j0 point is not encircled by the plot ,
N = 0

Step 3 : Z = P + N = 0 + 0 = 0

Hence the closed loop system is stable.

Matlab

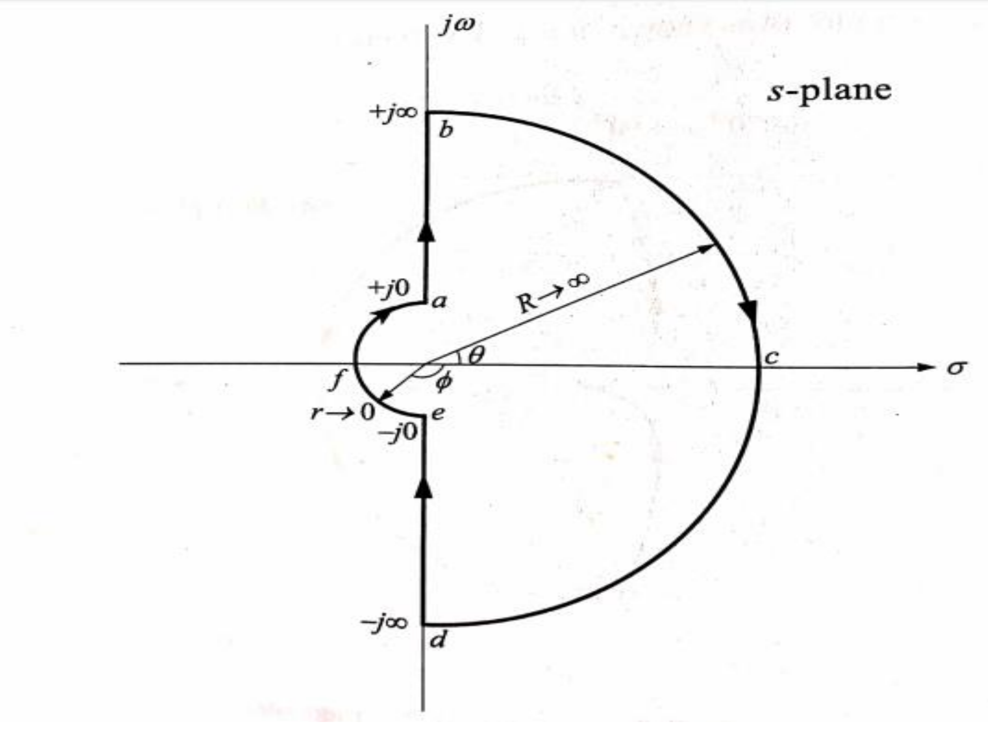


Nyquist Diagram
80
60
40
20
0
-20
-40
-60
-80
Imaginary Axis
-5
-4
-3
-2
-1
0
Real Axis

Let us re work the problem by including the pole at origin to the right-half of the s-plane and the contour ‘abcdefa’

Step 1: P = 1

The mapping of sections ‘ab’, ’bcd’, and ‘de’ on to the G(s)H(s)-plane remains same



Section IV :

Mapping of section ‘efa’:

put s = $\lim\limits_{r \longrightarrow 0} re^{j\emptyset}$ in G(s)H(s)

Here , Ø changes from -90 ⟶ -180 ⟶ -270

Then, $\lim\limits_{r \longrightarrow 0} GH(re^{j\emptyset})$ = $\lim\limits_{r \longrightarrow 0} \frac{5}{re^{j\emptyset}(rej^{\emptyset}+1)}$

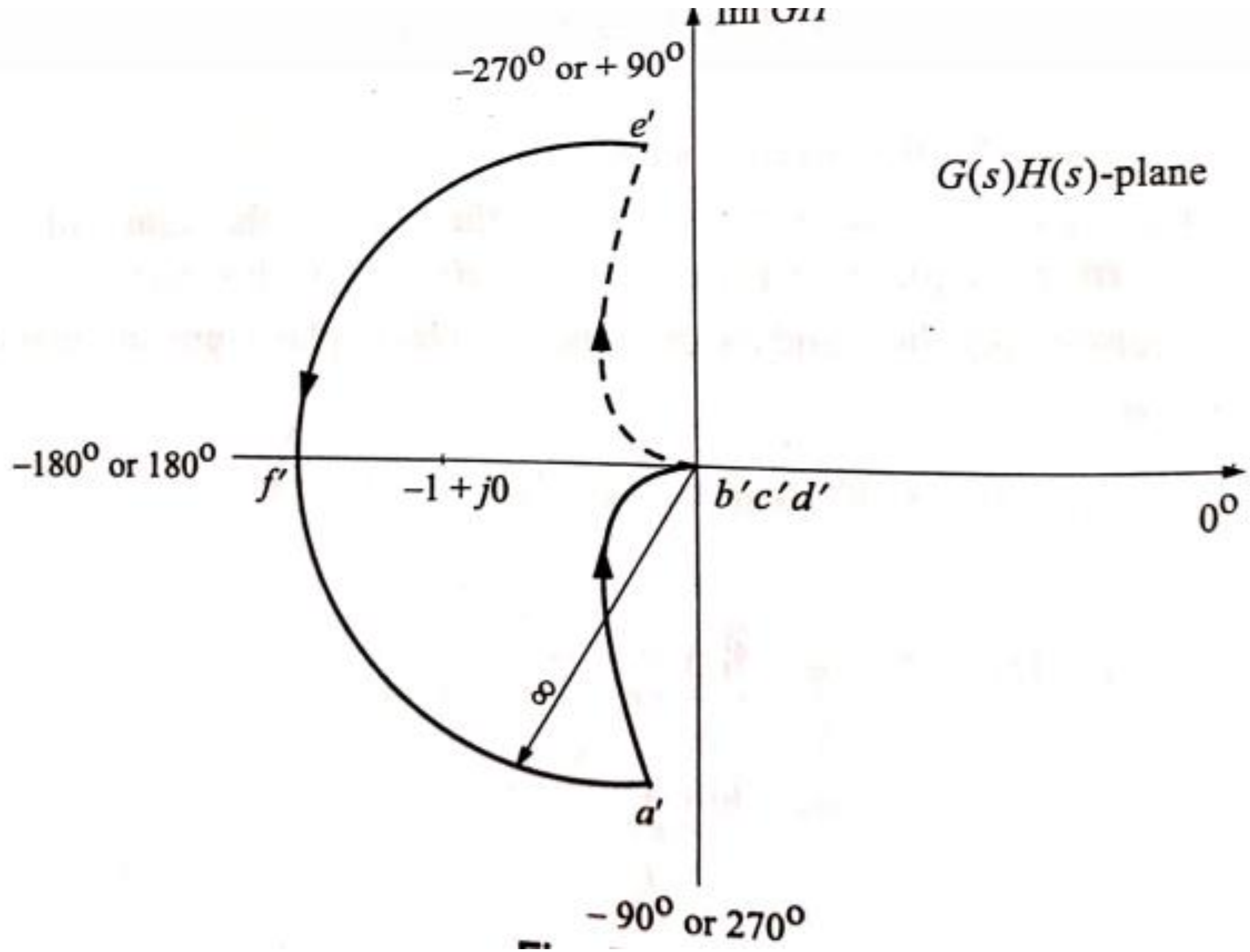
$= \lim\limits_{r \longrightarrow 0} \frac{5}{(rej^{\emptyset})}$

$= \infty\angle\text{-}\emptyset$

$= \infty\angle 90 \longrightarrow 180 \longrightarrow 270$

$\uparrow$e’ $\uparrow$f’ $\uparrow$a’

The complete Nyquist plot is shown



Since -1+j0 point is encircled by the plot in anticlockwise direction, $N = -1$

Step 3 : $Z = 1 - 1 = 0$

Hence the closed loop system is stable.

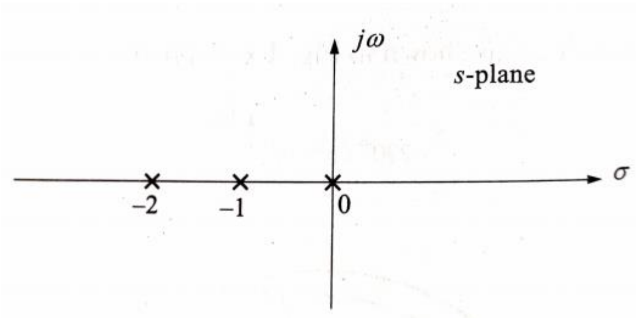
Problem

Sketch the Nyquist plot for G(s)H(s) = $\frac{k}{s(s+1)(s+2)}$

Find the range of k for closed-loop stability

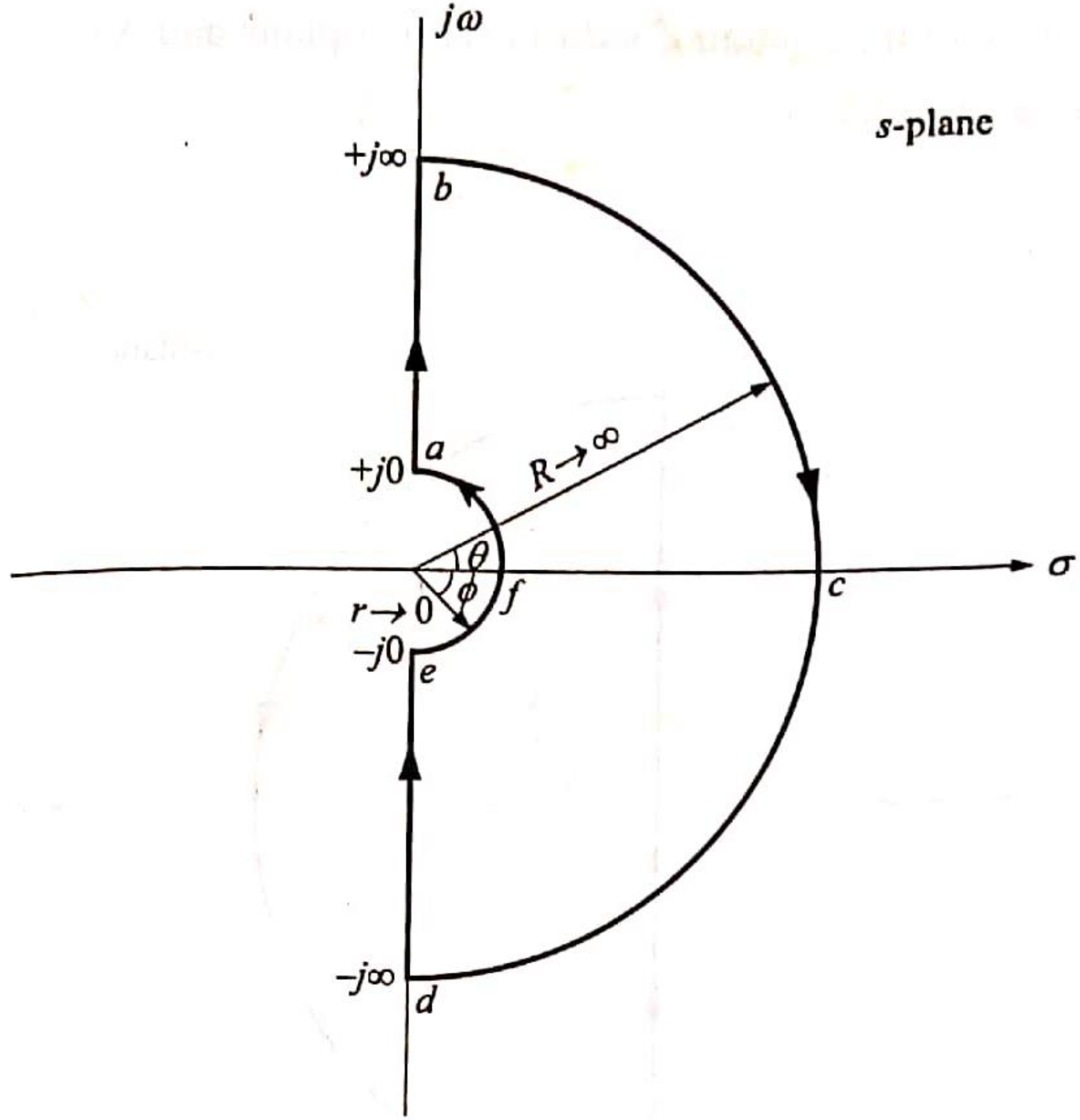
Solution

Step 1: Plot the poles of GH(s) on the s-plane



The pole at the origin is taken to the left-side of the s-plane by drawing an indent of zero radius around this pole.

Since the pole at the origin is taken to the left-side of the s-plane, P=0



$j\omega$
s-plane
$+j\infty$
b
a
$+j0$
$R \to \infty$
θ
ϕ
σ
f
c
$r \to 0$
$-j0$
e
$-j\infty$
d

Step 2: To find N:

Section I : To find the image of path ab.

$$G(s)H(s) = \frac{k}{s(s+1)(s+2)}$$

Put s = jω

$$G(j\omega)H(j\omega) = \frac{k}{j\omega(j\omega+1)(j\omega+2)}$$

$$= \frac{k}{\{\omega\angle 90\}\{\sqrt{(\omega^2+1)}\angle \tan^{-1}\omega\}\{\sqrt{(\omega^2+4)}\angle \tan^{-1}(\frac{\omega}{2})\}}$$

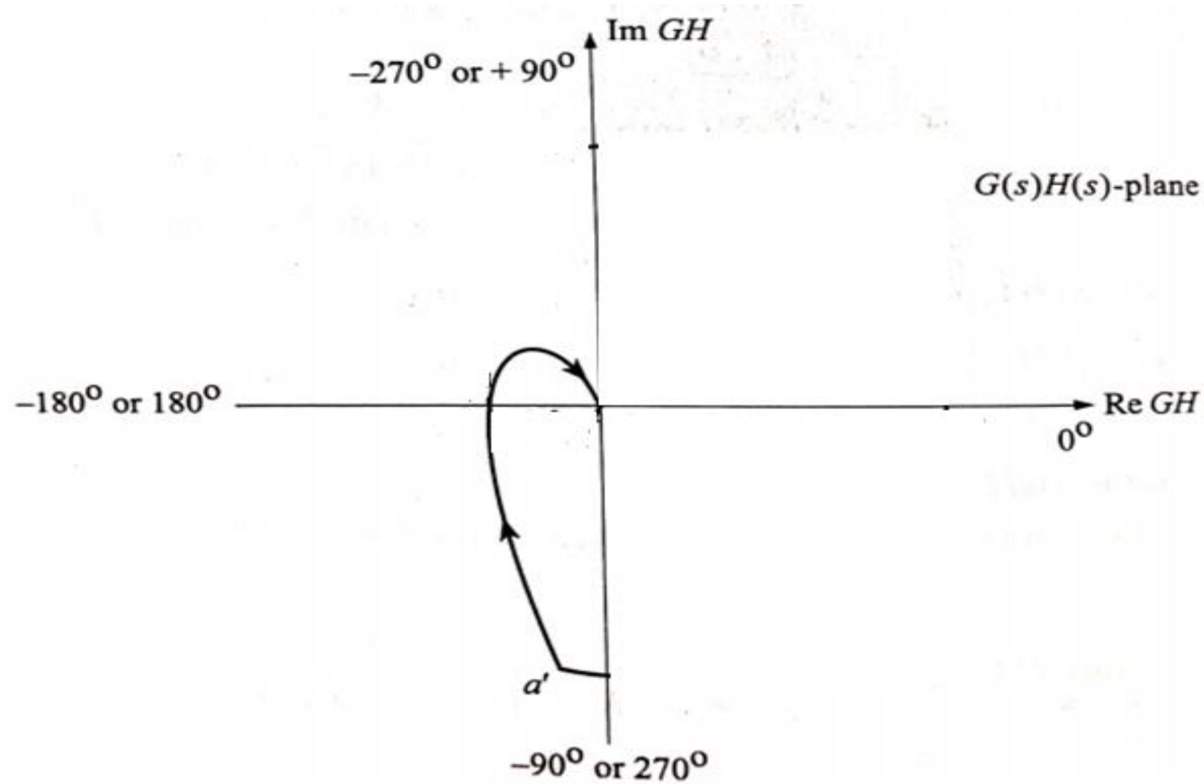
$$= \frac{k}{\omega\sqrt{(\omega^2+1)}\sqrt{(\omega^2+4)}\angle 90+\tan^{-1}\omega+\tan^{-1}(\frac{\omega}{2})}$$

$$M = \frac{k}{\omega\sqrt{(\omega^2+1)}\sqrt{(\omega^2+4)}}$$

$$\emptyset = -90 - \tan^{-1}\omega - \tan^{-1}(\frac{\omega}{2})$$

$$\lim_{\omega \to 0} M\angle\emptyset = \infty\angle\text{-}90 \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = 0\angle\text{-}270 \qquad \text{(point b')}$$



Section II : To find the image of path 'bcd'

put s = $\lim_{R \longrightarrow \infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 $\longrightarrow$ 0 $\longrightarrow$ -90

Then, $\lim_{R \longrightarrow \infty} GH(Re^{j\theta}) = \lim_{R \longrightarrow \infty} \dfrac{k}{Re^{j\theta}(Re^{j\theta}+1)(Rej^{\theta}+2)}$

$$= \lim_{R \longrightarrow \infty} \frac{k}{(Re^{j\theta})(Re^{j\theta})(Re^{j\theta})}$$

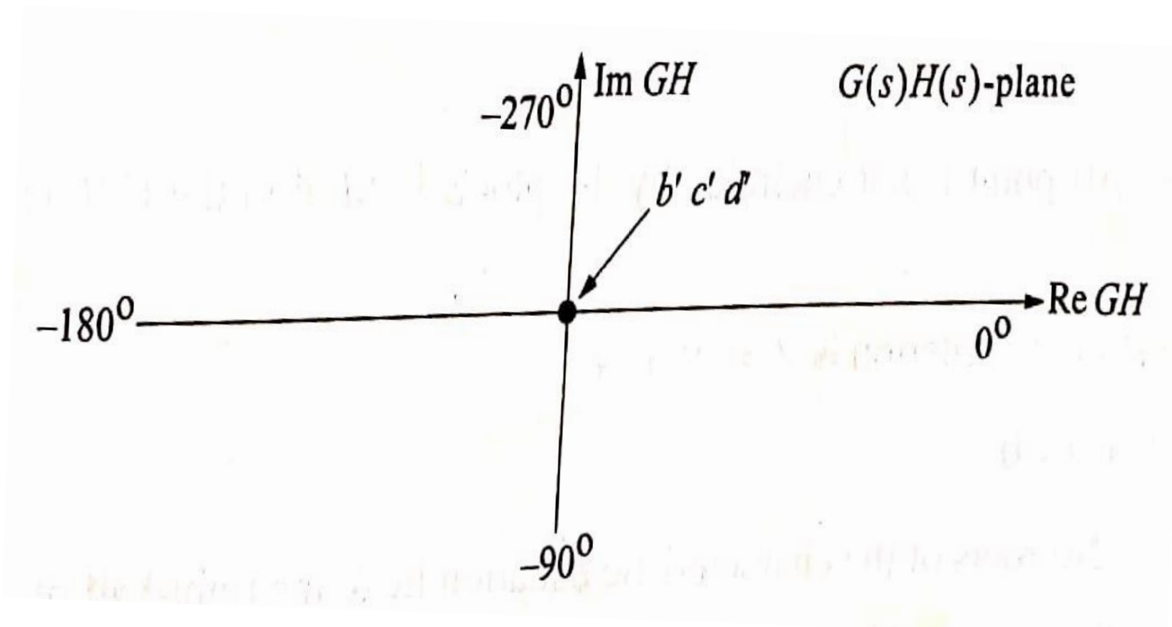
$$= \lim_{R \longrightarrow \infty} \frac{k}{(R^3 e^{j3\theta})}$$

= 0∠-3θ

= 0∠-270$\longrightarrow$ 0$\longrightarrow$ 270

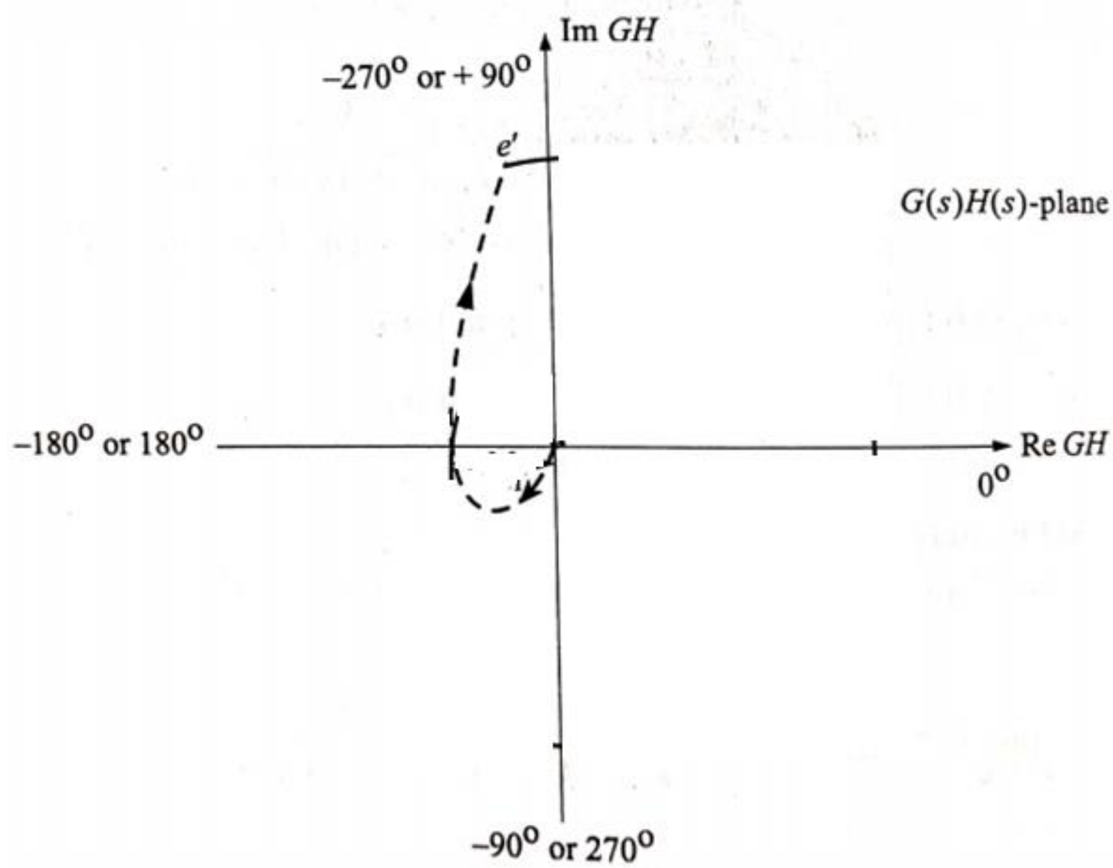
↑b'　　↑c'　↑d'

Hence, the infinite semicircle ‘bcd’ on the s-plane is mapped to the origin of the G(s)H(s)-plane.



Section III: To find the image of path 'de'

Path d'e' is the mirror image of the path a'b' with respect to real axis.



Section IV : To find the image of path efa

put s = $\lim_{r \longrightarrow 0} re^{j\emptyset}$ in G(s)H(s)

Here , Ø changes from -90 $\longrightarrow 0 \longrightarrow +90$

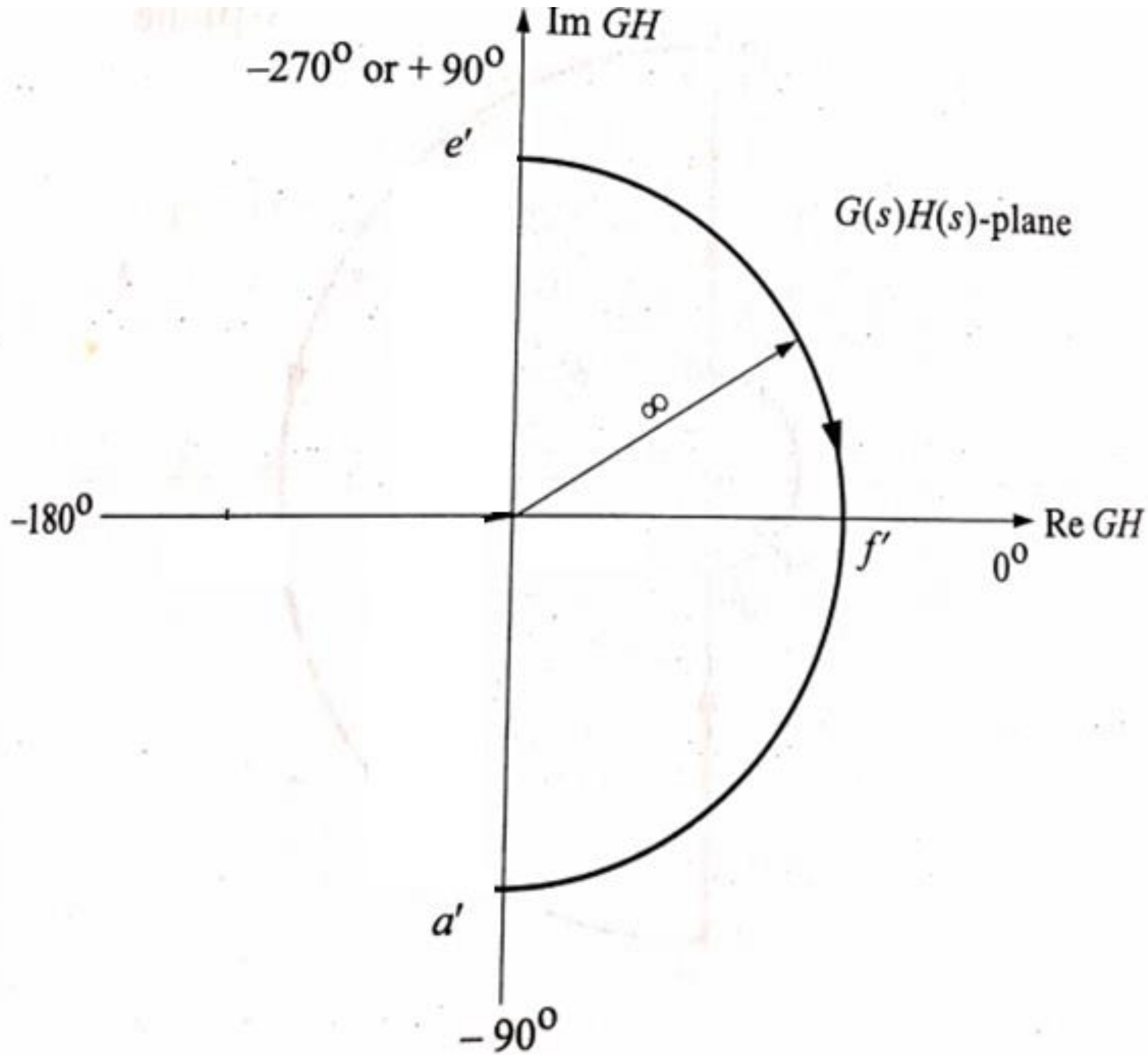
Then, $\lim_{r \longrightarrow 0} GH(re^{j\emptyset})$ = $\lim_{r \longrightarrow 0} \frac{k}{re^{j\emptyset}(re^{j\emptyset}+1)(rej^{\emptyset}+2)}$

$= \lim_{r \longrightarrow 0} \frac{k}{(re^{j\emptyset})}$

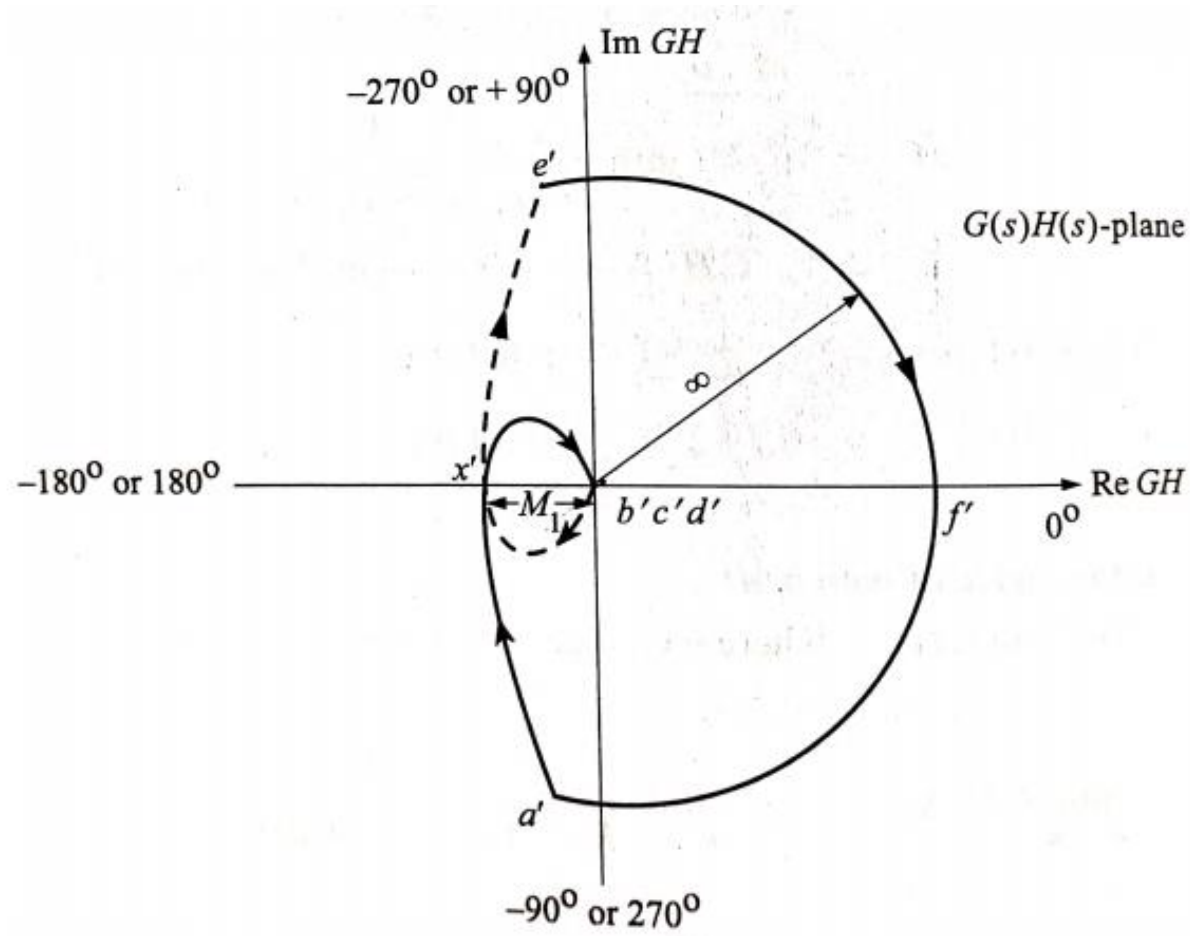
$= \infty \angle -\emptyset$

$= \infty \angle 90 \longrightarrow 0 \longrightarrow -90$

$\uparrow$e' $\uparrow$f' $\uparrow$a'


Im GH
−270° or + 90°
e′
G(s)H(s)-plane
∞
−180°
Re GH
f′
0°
a′
− 90°

The complete Nyquist plot is shown



To find M_1 :

At point x’, phase = -180

$$\Longrightarrow -90 - \tan^{-1}\omega - \tan^{-1}\left(\frac{\omega}{2}\right) = -180$$

$$\tan^{-1}\frac{3\omega}{2-\omega^2} = 0$$

$$2 - \omega^2 = 0$$

$$\omega = \sqrt{2}\ \text{rad/sec}$$

$$M_1 = \left|GH(j\omega)\right|_{\omega=\sqrt{2}}$$

$$= \frac{k}{\omega\sqrt{(\omega^2+1)}\sqrt{(\omega^2+4)}} = \frac{k}{6}$$

Since P is zero , N must be zero for Z to be zero.

N will be zero if and only if -1+j0 is not encircled by the Nyquist plot

For N to be zero, $M_1 < 1$

Hence, $\frac{k}{6} < 1$

$\Longrightarrow k < 6$

Since k is always positive, for closed-loop stability : $0 < k < 6$

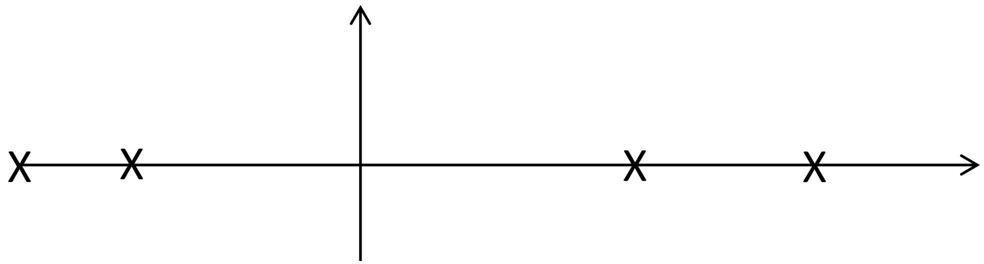
Problem

The open loop transfer function of a negative unity feed back system is given by $\frac{k(s+3)(s+5)}{(s-2)(s-4)}$.

$\text{Find the range of k for closed} - \text{loop stability}$

Solution

Step 1 : Plot the poles of GH(s) on the s-plane



Since there are 2 poles lies in the right-side of the s-plane, P=2

Step 2: To find N:

Section I : To find the image of path ab:

$$G(s)H(s) = \frac{k(s+3)(s+5)}{(s-2)(s-4)}$$

Put s = jω

$$G(j\omega)H(j\omega) = \frac{k(j\omega+3)(j\omega+5)}{(j\omega-2)(j\omega-4)}$$

$$= \frac{k\,\{\sqrt{(\omega^2+9)}\angle\tan^{-1}(\frac{\omega}{3})\}\,\{\sqrt{(\omega^2+25)}\angle\tan^{-1}(\frac{\omega}{5})\}}{\{\sqrt{(\omega^2+4)}\angle\tan^{-1}(\frac{\omega}{-2})\}\,\{\sqrt{(\omega^2+16)}\angle\tan^{-1}(\frac{\omega}{-4})\}}$$

$$= \frac{k\,\{\sqrt{(\omega^2+9)}\sqrt{(\omega^2+25)}\}\,\{\angle\tan^{-1}\left(\frac{\omega}{3}\right)+\angle\tan^{-1}(\frac{\omega}{5})\}}{\{\sqrt{(\omega^2+4)}\sqrt{(\omega^2+16)}\}\quad\{\angle\tan^{-1}\left(\frac{\omega}{-2}\right)+\angle\tan^{-1}(\frac{\omega}{-4})\}}$$

$$M = \frac{k\,\{\sqrt{(\omega^2+9)}\,\sqrt{(\omega^2+25)}\,\}}{\{\sqrt{(\omega^2+4)}\sqrt{(\omega^2+16)}\,\}}$$

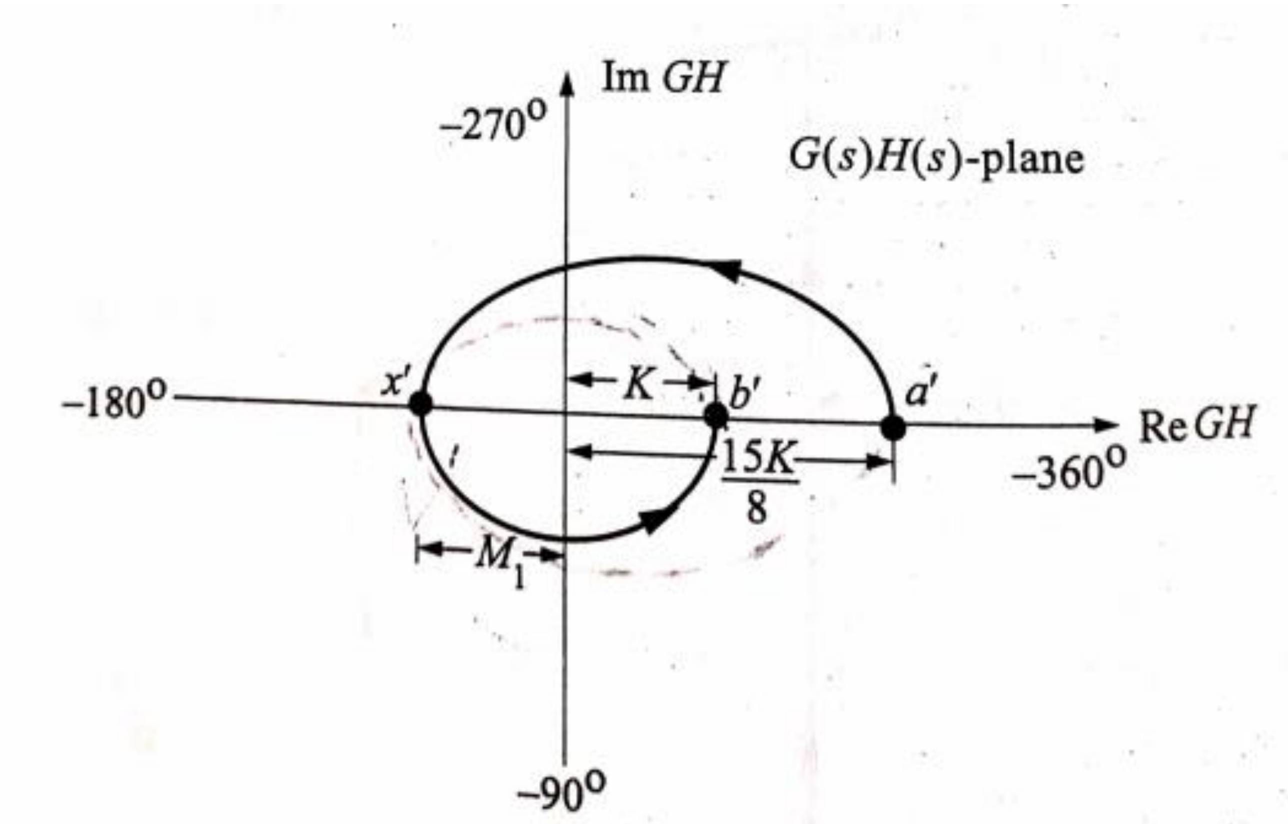
$$\varnothing = \angle \tan^{-1}\left(\frac{\omega}{3}\right) + \angle \tan^{-1}(\frac{\omega}{5})\} - \{\angle \tan^{-1}\left(\frac{\omega}{-2}\right) + \angle \tan^{-1}(\frac{\omega}{-4})\}$$

$$= \tan^{-1}\left(\frac{\omega}{3}\right) + \tan^{-1}(\frac{\omega}{5})\} - \{180 - \tan^{-1}\left(\frac{\omega}{2}\right) + 180 - \tan^{-1}(\frac{\omega}{4})\}$$

$$\varnothing = -360 + \tan^{-1}\left(\frac{\omega}{3}\right) + \tan^{-1}(\frac{\omega}{5}) + \tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}(\frac{\omega}{4})$$

$$\lim_{\omega \to 0} M\angle\emptyset = \frac{15k}{8}\angle\text{-}360 \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = k\,\angle 0 \qquad \text{(point b')}$$



Section II : To find the image of path 'bcd'

$$G(s)H(s) = \frac{k(j\omega+3)(j\omega+5)}{(j\omega-2)(j\omega-4)}$$

put s = $\lim\limits_{R \longrightarrow \infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 ⟶ 0 ⟶ -90

Then, $\lim\limits_{R \longrightarrow \infty} GH(\, Re^{j\theta}) = \lim\limits_{R \longrightarrow \infty} \frac{k(Re^{j\theta}+3)(Rej^{\theta}+5)}{(Re^{j\theta}-2)(Rej^{\theta}-4)}$

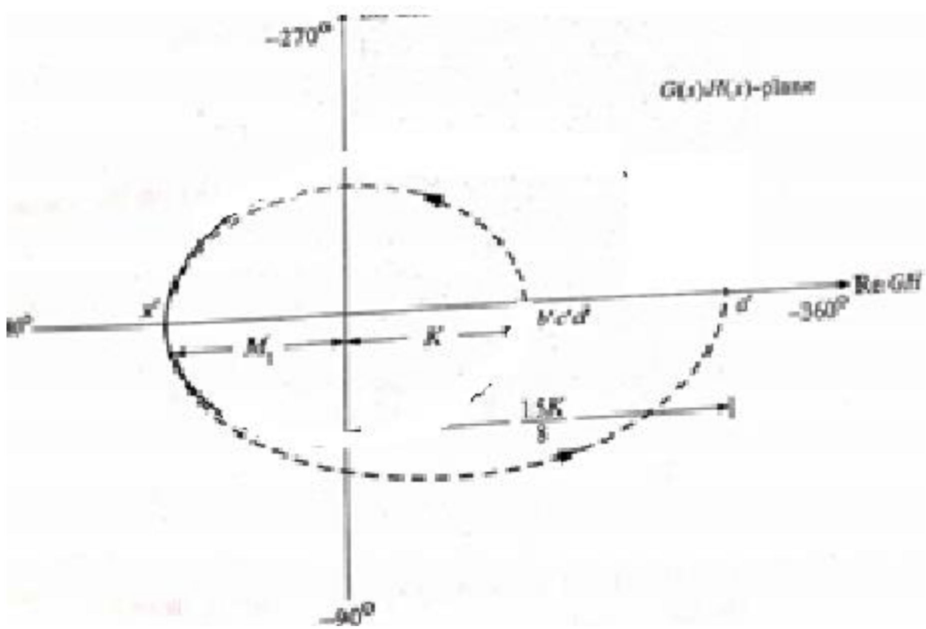
$$= \lim_{R \longrightarrow \infty} \frac{k(Rej^{\theta})(Rej^{\theta})}{(Re^{j\theta})(Re^{j\theta})}$$

$$= \lim_{R \longrightarrow \infty} k$$

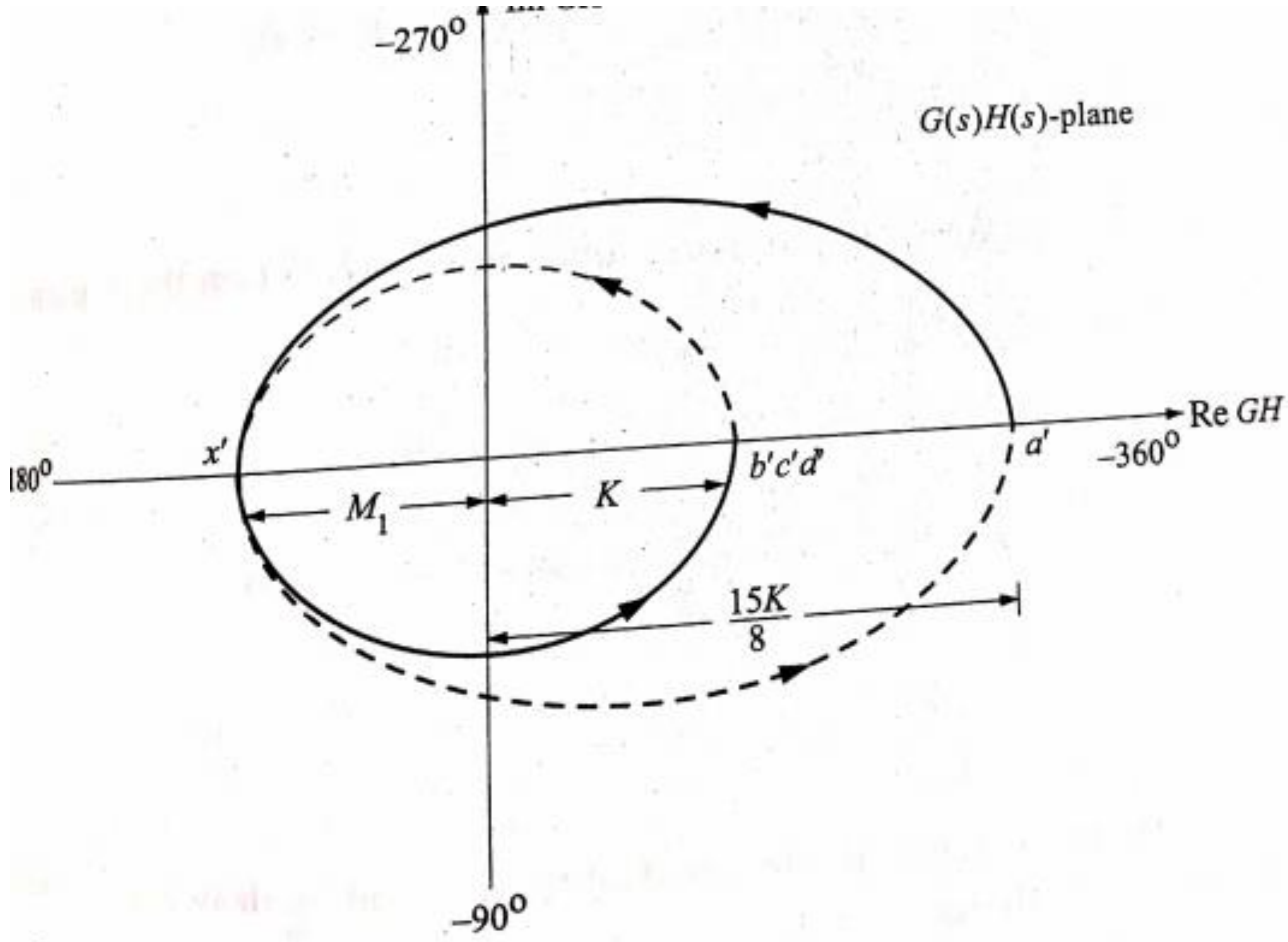
$$= k\angle 0$$

Section III: To find the image of path 'de'

Path d'e' is the mirror image of the path a'b' with respect to real axis.



The Complete Nyquist plot is shown



To find M_1 :

At point x', phase = -180

$\Longrightarrow$ - 360 + $\tan^{-1}\left(\frac{\omega}{3}\right) + \tan^{-1}(\frac{\omega}{5}) + \tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}\left(\frac{\omega}{4}\right)$ = -180

By solving, $\omega = \sqrt{}11$ rad/sec

$$M_1 = \left|GH(j\omega)\right|_{\omega = \sqrt{}11}$$

$$= \frac{k\,\{\sqrt{(\omega^2+9)}\,\sqrt{(\omega^2+25)}\,\}}{\{\sqrt{(\omega^2+4)}\sqrt{(\omega^2+16)}\,\}}$$

= 1.33k

Since P is 2 , N must be -2 for Z to be zero.

N will be -2 if and only if -1+j0 is encircled twice in the anticlockwise direction by the Nyquist plot

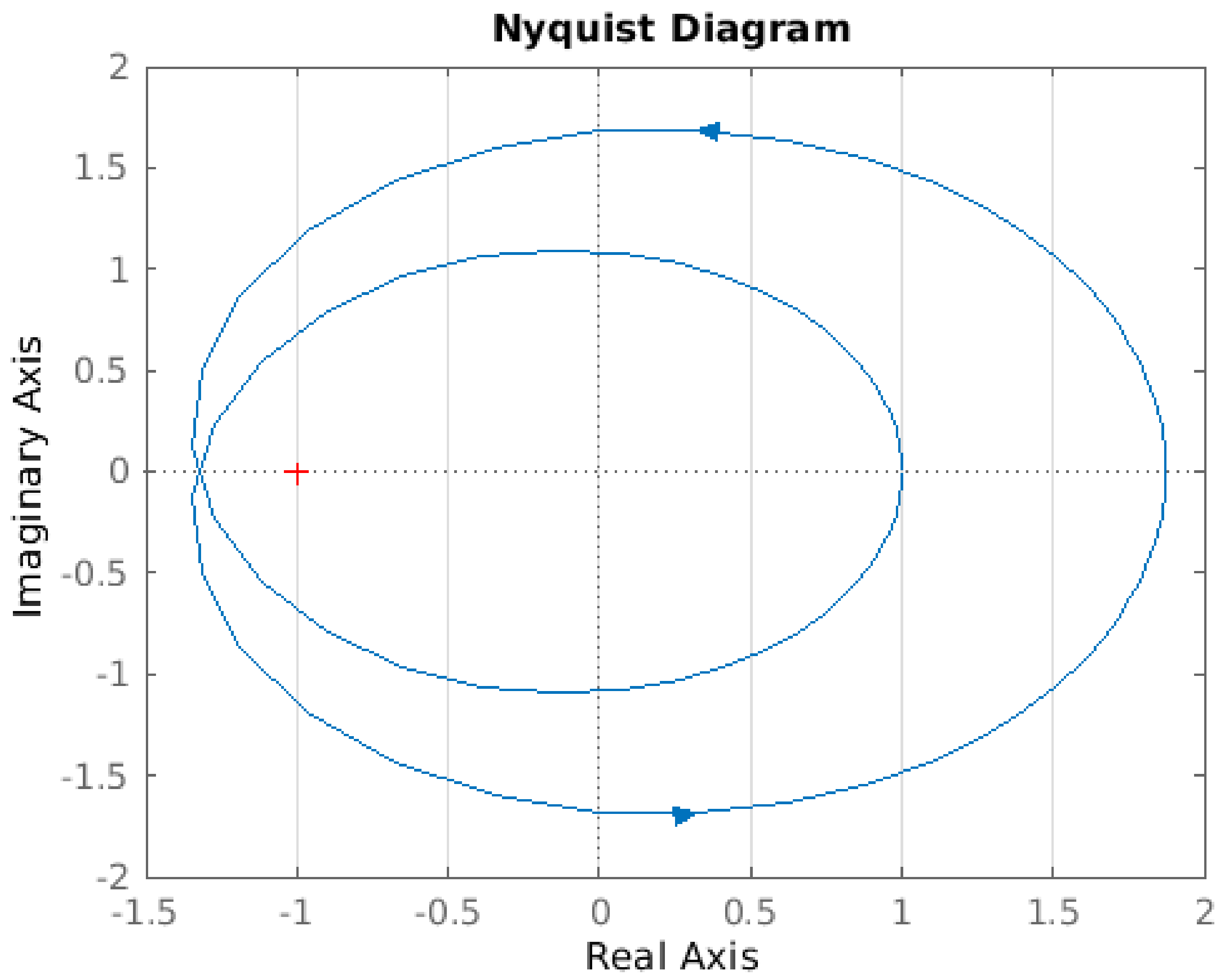
For this to happen, $M_1 > 1$

Hence, $1.33k > 1$

$$\Longrightarrow k > 0.75$$

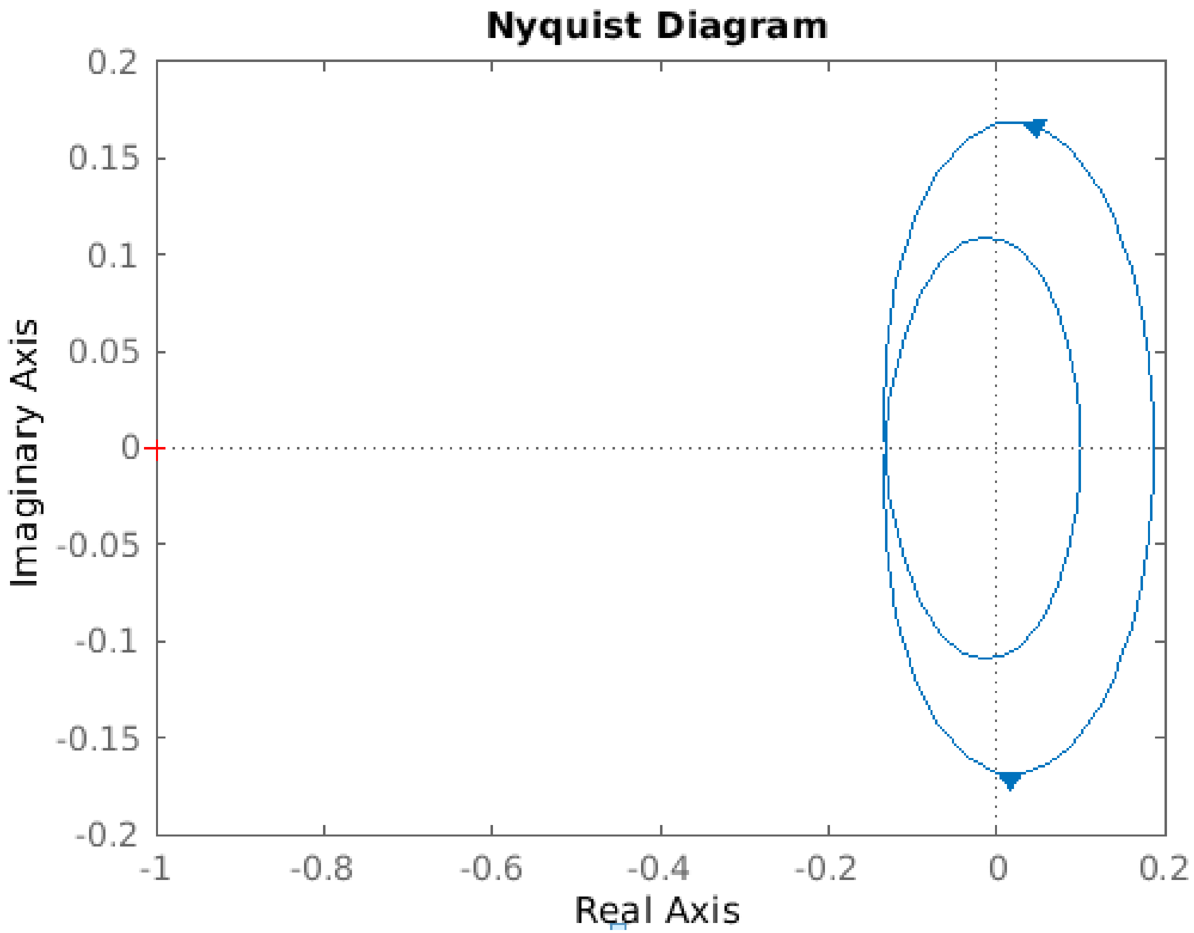
Since k is always positive, for closed-loop stability : $0.75 < k < \infty$

Matlab (k=1)



Nyquist Diagram
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
Real Axis
Imaginary Axis

K=0.1

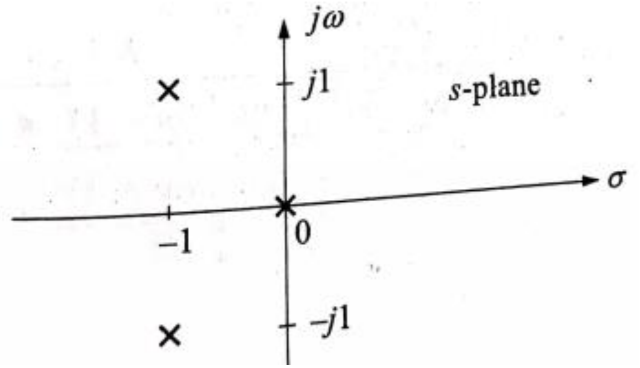

Nyquist Diagram
Imaginary Axis
Real Axis
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2

Problem

The open loop transfer function of a negative unity feed back system is given by $\frac{k}{s(s^2+2s+2)}$.
$\text{Find the range of k for closed} - \text{loop stability}$

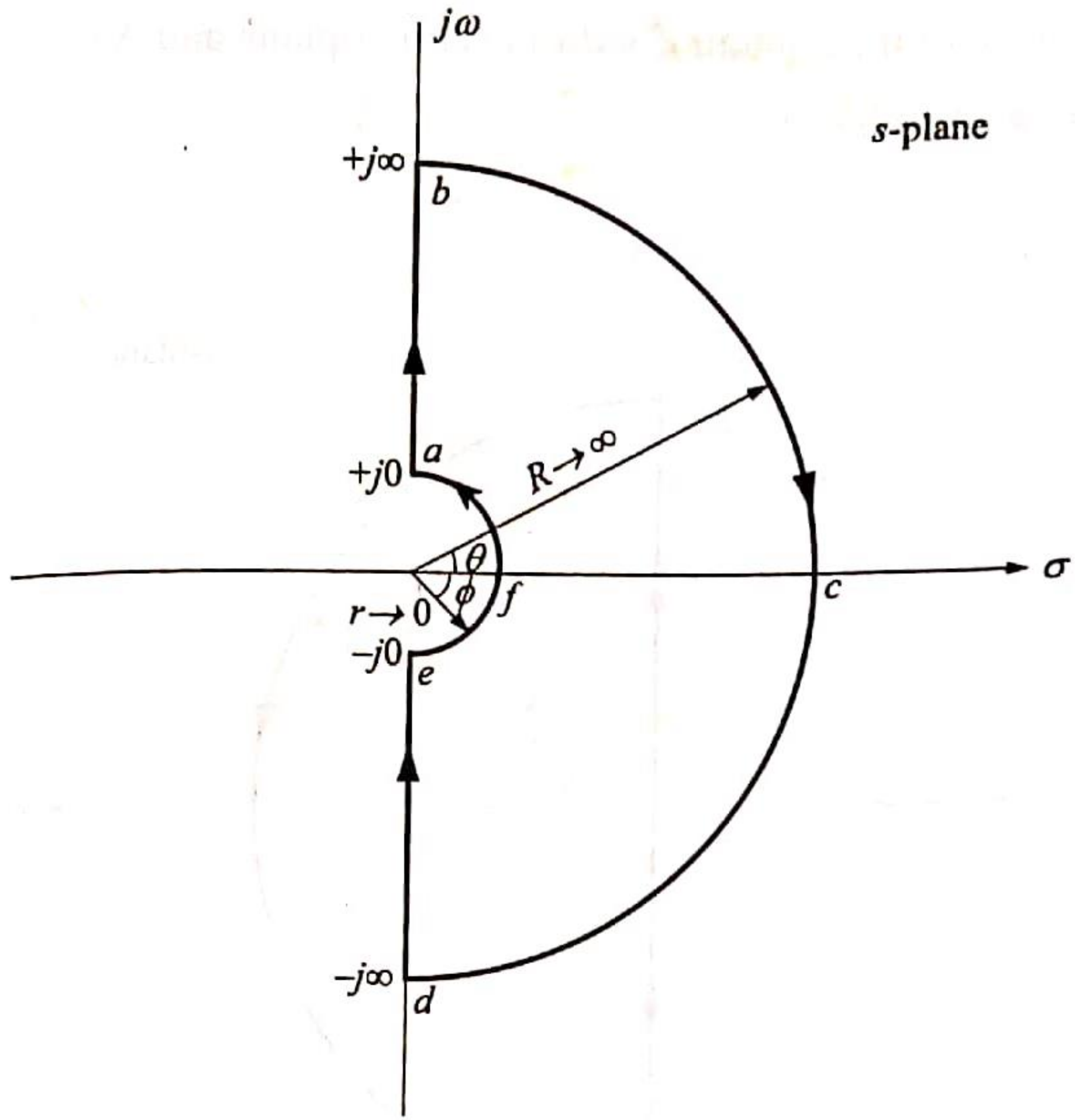
Solution

Step 1 : Plot the poles of GH(s) on the s-plane



The pole at the origin is taken to the left-side of the s-plane by drawing an indent of zero radius around this pole.

Since the pole at the origin is taken to the left-side of the s-plane, P=0



$j\omega$
s-plane
$+j\infty$
b
$R \to \infty$
a
$+j0$
θ
ϕ
σ
f
c
$r \to 0$
$-j0$
e
$-j\infty$
d

Step 2: To find N:

Section I : To find the image of path ab.

$$G(s)H(s) = \frac{k}{s(s^2+2s+2)}$$

Put s = jω

$$G(j\omega)H(j\omega) = \frac{k}{j\omega[(j\omega)^2+2j\omega+2]}$$

$$= \frac{k}{j\omega[-\omega^2+2j\omega+2]}$$

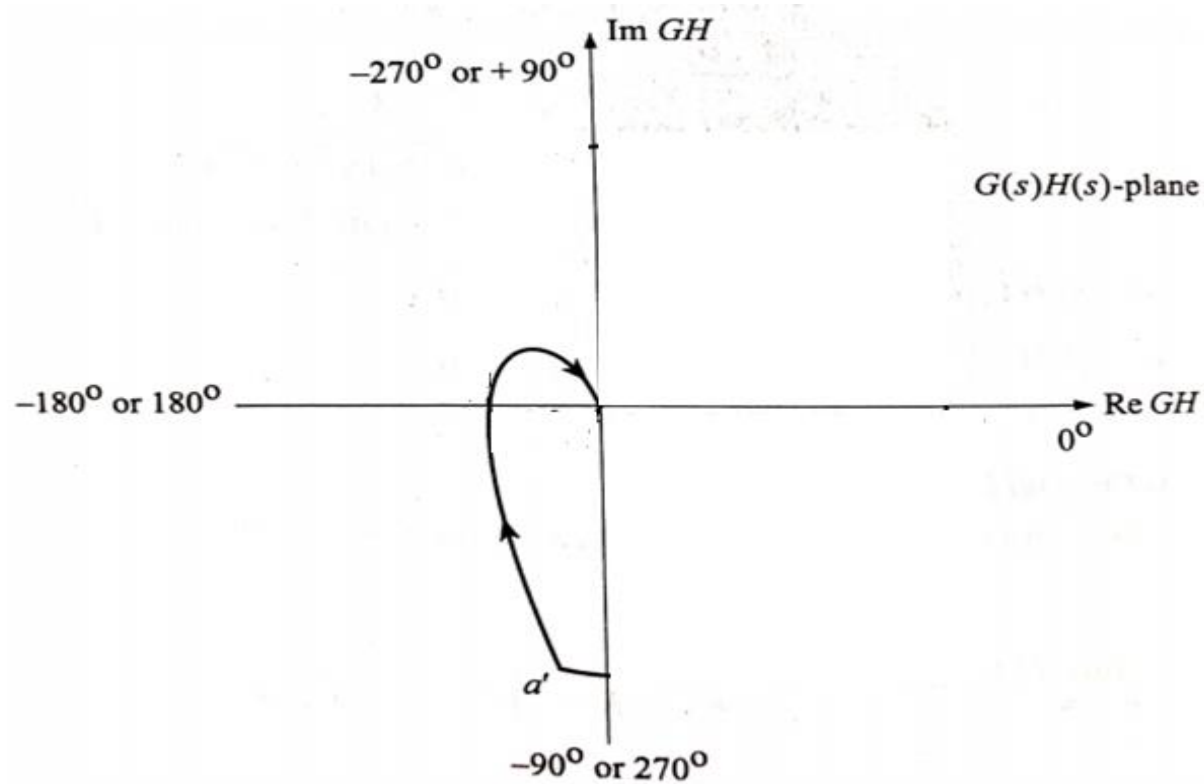
$$= \frac{k}{\omega\angle 90 \sqrt{(2\omega)^2+(2-\omega^2)^2}\angle \tan^{-1}(\frac{2\omega}{2-\omega^2})}$$

$$M = \frac{k}{\omega\sqrt{(2\omega)^2+(2-\omega^2)^2}}$$

$$\emptyset = -90 - \tan^{-1}(\frac{2\omega}{2-\omega^2})$$

$$\lim_{\omega \to 0} M\angle\emptyset = \infty\angle{-90} \qquad \text{(point a')}$$

$$\lim_{\omega \to \infty} M\angle\emptyset = 0\angle{-270} \qquad \text{(point b')}$$



Section II : To find the image of path 'bcd'

put s = $\lim_{R \longrightarrow \infty} Re^{j\theta}$ in G(s)H(s)

Here , θ changes from +90 ⟶ 0 ⟶ -90

Then, $\lim_{R \longrightarrow \infty} GH(Re^{j\theta})$ $= \lim_{R \longrightarrow \infty} \frac{k}{Re^{j\theta}[(Re^{j\theta})^2+2Re^{j\theta}+2]}$

$= \lim_{R \longrightarrow \infty} \frac{k}{(Re^{j\theta})[(R^2e^{j2\theta})+2Re^{j\theta}+2]}$

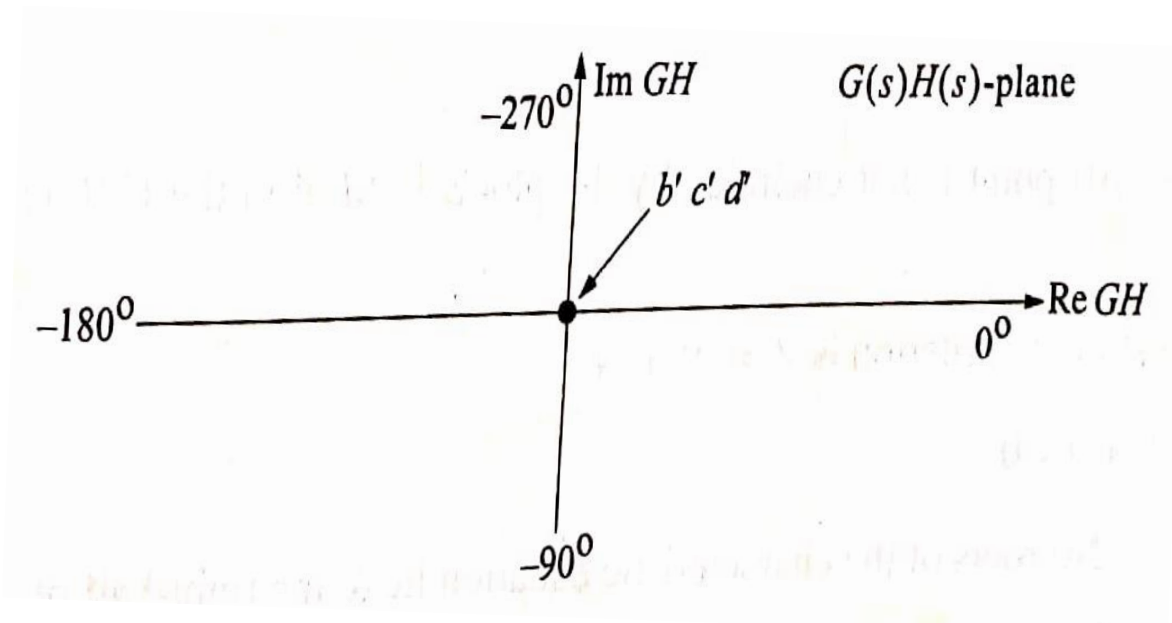
$= \lim_{R \longrightarrow \infty} \frac{5}{(R^3e^{j3\theta})}$

$= 0\angle -3\theta$

$= 0\angle -270 \longrightarrow 0 \longrightarrow 270$

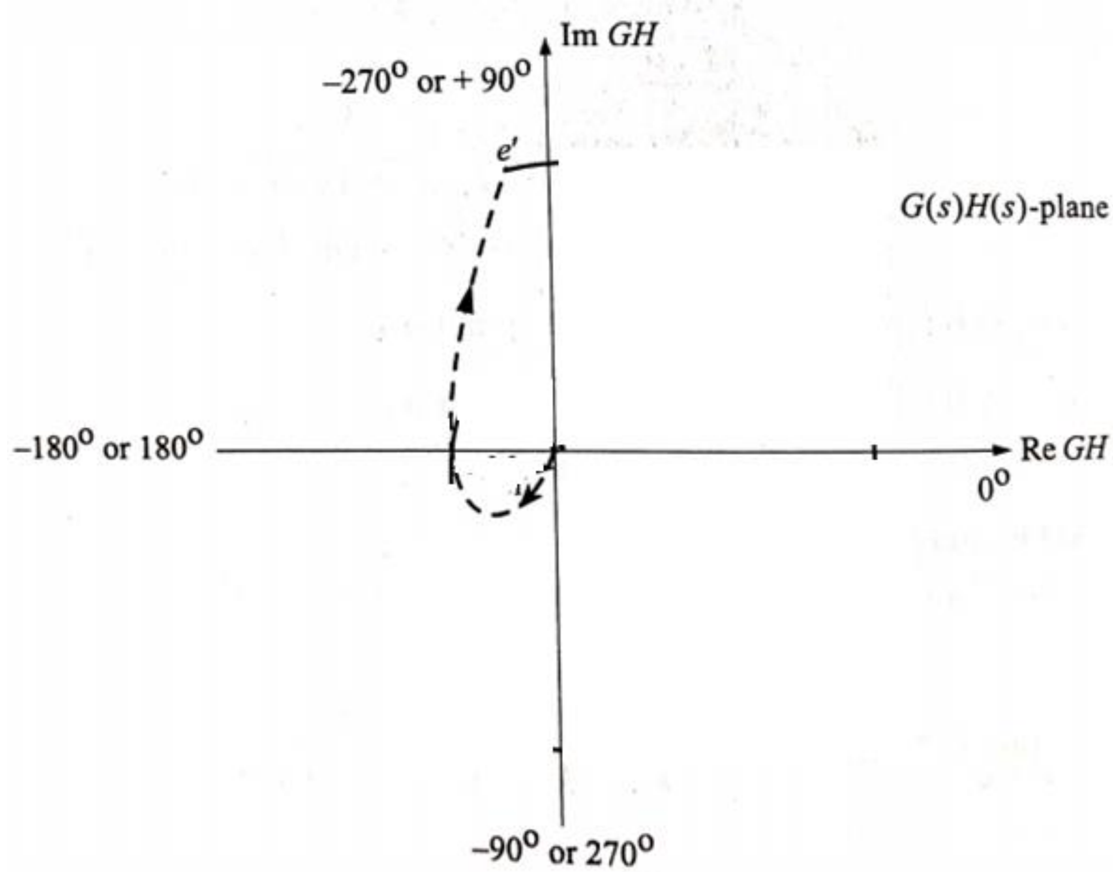
↑b' ↑c' ↑d'

Hence, the infinite semicircle ‘bcd’ on the s-plane is mapped to the origin of the G(s)H(s)-plane.



Section III: To find the image of path 'de'

Path d'e' is the mirror image of the path a'b' with respect to real axis.



Section IV : To find the image of path efa

put s = $\lim_{r \longrightarrow 0} re^{j\emptyset}$ in G(s)H(s)

Here , Ø changes from -90 $\longrightarrow$ 0 $\longrightarrow$ +90

Then, $\lim_{r \longrightarrow 0} GH(re^{j\emptyset}) = \lim_{r \longrightarrow 0} \frac{k}{re^{j\emptyset}[(re^{j\emptyset})^2 + 2re^{j\emptyset} + 2]}$

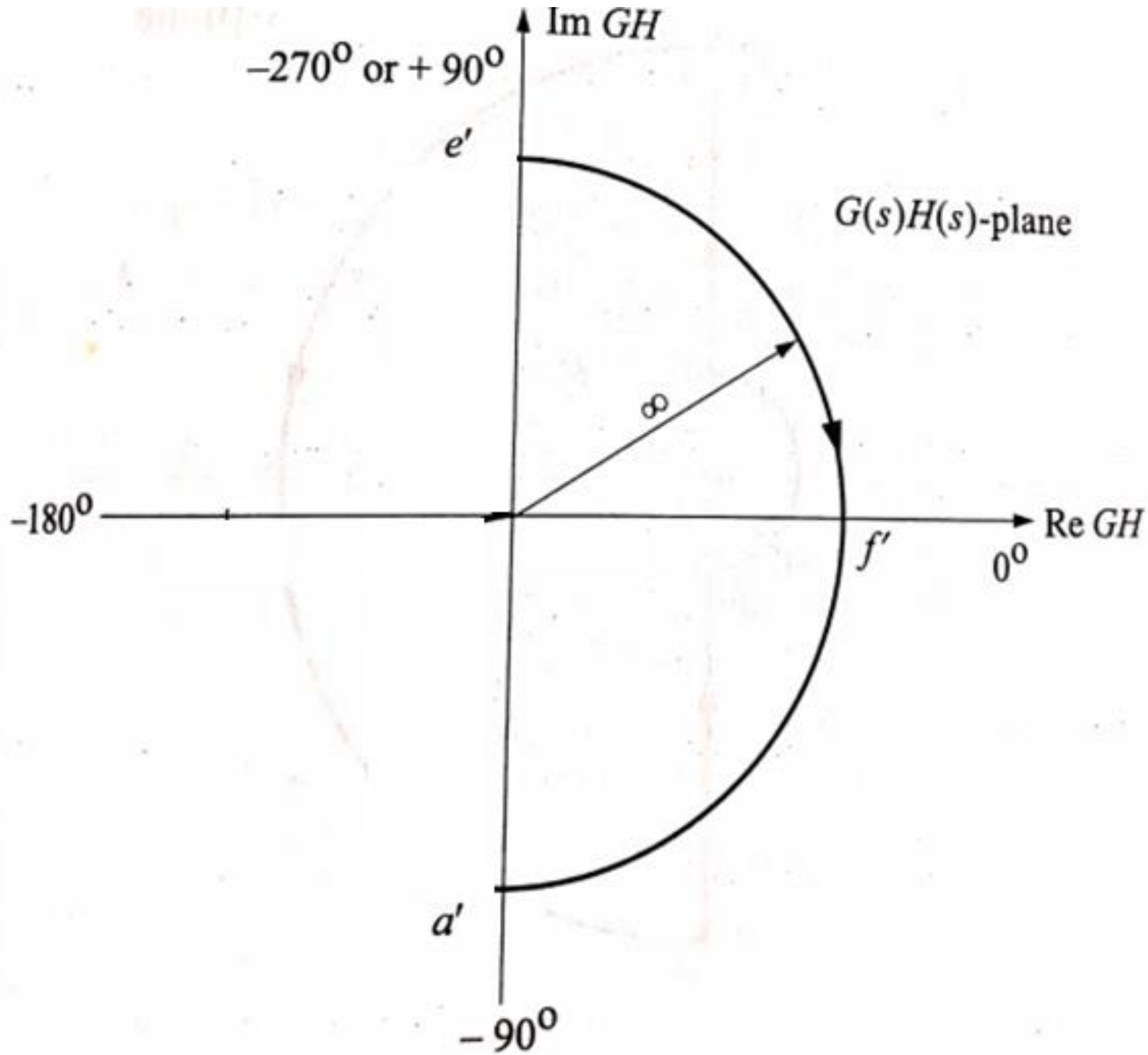
$$= \lim_{r \longrightarrow 0} \frac{k}{(re^{j\emptyset})[(r^2 e^{j2\emptyset}) + 2re^{j\emptyset} + 2]}$$

$$= \lim_{r \longrightarrow 0} \frac{k}{(re^{j\emptyset})}$$

$$= \infty\angle -\emptyset$$

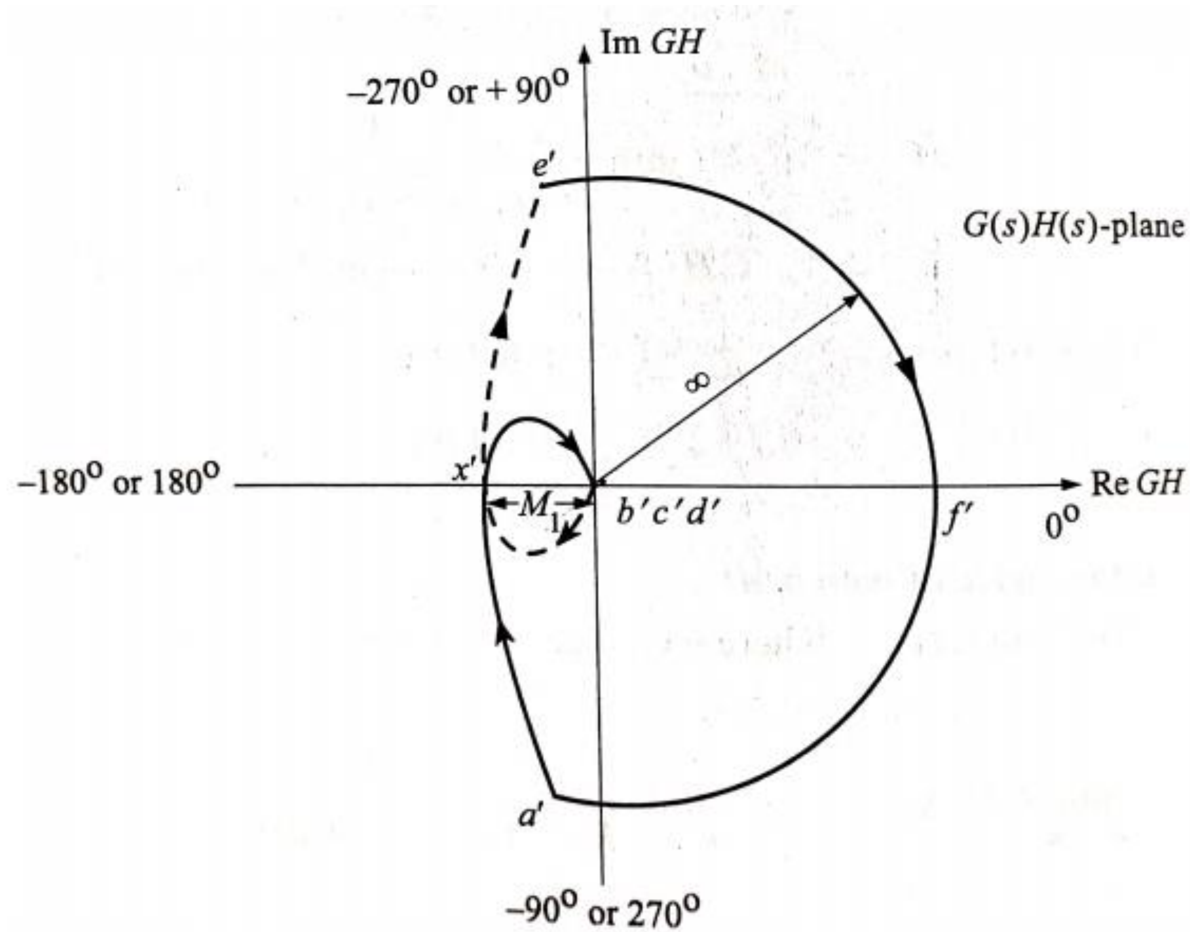
$$= \infty\angle 90 \longrightarrow 0 \longrightarrow -90$$

↑e' ↑f' ↑a'



Im GH
-270^{o} or $+90^{o}$
e'
$G(s)H(s)$-plane
∞
-180^{o}
Re GH
f'
0^{o}
a'
-90^{o}

The complete Nyquist plot is shown



To find M_1 :

At point x', phase = -180

$$\Longrightarrow -90 - \tan^{-1}\left(\frac{2\omega}{2-\omega^2}\right) = -180$$

$$\tan^{-1}\frac{2\omega}{2-\omega^2} = -90$$

$$2 - \omega^2 = 0$$

$$\omega = \sqrt{2} \text{ rad/sec}$$

$$M_1 = |GH(j\omega)|_{\omega = \sqrt{2}}$$

$$= \frac{k}{\omega\sqrt{(2\omega)^2+(2-\omega)^2}} = \frac{k}{4}$$

Since P is zero , N must be zero for Z to be zero.

N will be zero if and only if -1+j0 is not encircled by the Nyquist plot

For N to be zero, $M_1 < 1$

Hence, $\frac{k}{4} < 1$

$\Longrightarrow k < 4$

Since k is always positive, for closed-loop stability : $0 < k < 4$

References

1. Control Engineering by Nagrath & Gopal, New Age International Publishers
2. Engineering control systems - Norman S. Nise, John WILEY & sons , fifth Edition
3. Modern control Engineering-Ogata, Prentice Hall
4. Automatic Control Systems- B.C Kuo, John Wiley and Sons