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Matrices

Properties of determinants :

(i) If two rows or columns of matrix are identical, then the determinant is zero

$$\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

(ii) If two rows or columns of matrix are interchanged then the sign of determinant changes.

$$\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} \qquad -\Delta = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{vmatrix}$$

(iii) If three rows or columns of matrix are interchanged then the sign of determinant is unaltered.

$$\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

(iv) In the determinant of matrix if any column containing sum or difference of two elements then it can be split into sum or difference of two determinants.

$$\begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Delta = ad - bc$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Delta = a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Operations for solving matrix are always less than one the order of matrix.

Upper and lower triangular matrix :

If all the elements above principal diagonal are zero then its said to be lower triangular matrix and if all elements below the principal diagonal are zero it is upper triangular matrix.

$$\text{Lower triangular matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} = 18$$

$$\text{Upper triangular matrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = 24$$

Note:

If is matrix is either lower triangular or upper triangular determinant is product of principal diagonal elements.

Q. Find determinant of matrix

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

$R_1 \longrightarrow R_1 - R_2$ and $R_2 \longrightarrow R_2 - R_3$

$$\begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)$$

A,

$$|A| = |A^T|$$

Q. Find determinant of

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix}$$

$R_2 \longrightarrow R_2 - R_1$ and $R_3 \longrightarrow R_3 - R_1$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 0 \\ 0 & 0 & b \end{vmatrix} = ab$$

standard results :-

(A) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 6 \end{vmatrix} = 20$ (B) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix} = abc$

Q. Find the determinant of

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$R_1 \longrightarrow \frac{R_1}{a}$; $R_2 \longrightarrow \frac{R_2}{b}$; $R_3 \longrightarrow \frac{R_3}{b}$.

$$= abc \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$R_1 \longrightarrow R_1 + (R_2 + R_3)$

=

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & 1/b \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$C_2 \longrightarrow C_2 - C_1$ $\quad C_3 \longrightarrow C_3 - C_1$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 0 & 0 \\ 1/b & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

standard result:

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 1\times1\times1\times1\left(1+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}\right) = 5$$

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Q. Find determinant:

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$$

$$\Delta = 0(2-1) - 1(1-9) + 2(1-6)$$
$$= -2$$

or

$R_3 \longrightarrow R_3 - 3R_2$

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & -5 & -8 \end{vmatrix}$$

$$\Delta = 0 - 1(-8+10) -$$
$$= -2$$

or

$$\begin{array}{ccccc} 0 & 9 & 2 & 4 & \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \\ 12 & 0 & 1 & & \end{array}$$

$\Delta = \Sigma$ product of 1^{st} diagonal elements $- \Sigma$ product of 2^{nd} diagonal elements.

$$\Delta = 0 + 9 + 2 - 12 - 0 - 1$$

(1) $$\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \begin{matrix} 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 & 2 \end{matrix} = 1 + 8 + 8 - 4 - 4 - 4 = 4$$

(2) $$\begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{vmatrix} = \begin{matrix} 1 & 2 & 5 & 1 & 2 \\ 3 & 1 & 4 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{matrix} = 2 + 8 + 15 - 5 - 4 - 12 = 4$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{adj.\ A}{\Delta}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \qquad = 1 \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Inverse matrix exists only for non-singular matrix.

Adjoint of Higher order matrix is the transpose of the co-factor matrix.

Minor of element :

The minor of an element in square matrix is the determinant of square sub matrix in which the row and the column of particular element lies to be deleted.

$$A = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$$

$$\text{Minor of } a_{12} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = (1-9) = -8$$

$$\text{Co-factor of an element} = (-1)^{i+j} \text{ Minor of element}$$

$$= (-1)^{1+2} \times (-8)$$

$$= 8$$

i - no. of row & j - no. of column.

shortcut :-

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\Delta = 0+9+6-12-0-1 = 2$$

$$\begin{array}{c|ccccccc} 0 & 1 & & 2 & & 0 & & 1 \\ \hline 1 & 2 & -7 & 3 & 8 & 1 & -3 & 2 \\ 3 & 3 & & 1 & & 3 & & 3 \\ 0 & 1 & & 2 & & 0 & & 1 \\ 3 & 2 & & 3 & & 1 & & 2 \end{array} \rightarrow \text{Remove}$$

$\downarrow$ Remove

$$A^{-1} = \frac{+1}{2}\begin{bmatrix} -7 & 5 & -1 \\ 8 & -6 & 2 \\ -3 & 3 & -1 \end{bmatrix}$$

1) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$\Delta = 1+8+8+-4-4-4 = 5$$

$$\begin{array}{c|cccc} 1 & 2 & 2 & 1 & 2 \\ \hline 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 \end{array}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

(2) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ $\quad \Delta = 3+0-4+0+0+2 = 1$

$$\begin{array}{c|cccc} 1 & 2 & -2 & 1 & 2 \\ \hline -1 & 3 & 0 & -1 & 3 \\ 0 & -2 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 & 2 \\ -1 & 3 & 0 & -1 & 3 \end{array}$$

$$A^{-1} = \frac{-1}{1}\begin{bmatrix} 5 & 2 & +6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \Delta = -1$

$$\begin{array}{c|cccc} 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{array}$$

$$A^{-1} = -1\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of matrix :-

If all minors of order $(r+1)$ are zero's, but there is at least one non-zero minor of order 'r', if exists, is called the Rank of the matrix and is indicated by $\rho(A) = r$.

Properties of rank :-

(i) If A is null matrix or zero matrix then rank of matrix is zero.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \Rightarrow \rho(A) = 0$$

(ii) If A is a non-zero matrix then rank of A is greater than or equal to one.

$$\rho(A) \geq 1$$

(iii) If I be the unit matrix of order $n \times n$ then rank of I_n is one 'n'.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho(A) = 2$$

* (iv) If A be the matrix of order $m \times n$, then rank of matrix A is less than or equal to mininimum of m and n.

Q. Find the ranks of following Matrices :

(1) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 5 & 7 \end{bmatrix}_{2\times4}$ $\rho(A) \leq 2.$ $\rho(A) = 2$

(No sub matrix has determinant zero)

(2) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}_{2\times3}$ $\rho(A) \leq 2.$ $\rho(A) = 1$

square sub-matrix are having determinant $= 0$

$(4 - 2(2) = 0$ & $(6\times2) - (4\times3) = 0)$

(3) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3\times3}$ $\rho(A) \leq 3$

$R_3 \longrightarrow R_3 - (R_3 + R_2)$

$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$ $\rho < 3$ $\therefore$ $\rho(A) = 2$

(Non-singular square sub-matrix)

(4) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\rho(A) \leq 3$ (from order of matrix)

$\rho(A) < 3$ ($|A|_{3\times3} = 0$)

$\rho(A) = 2$ (non singular square sub matrix)

(5) $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 8 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}_{3\times4}$ $R_2 \longrightarrow R_2 + R_1$, $R_3 \longrightarrow R_3 - R_1$

$\therefore$

$\begin{bmatrix} 1 & -1 & 3 & 6 \\ 2 & 7 & 0 & 2 \\ 4 & 4 & 0 & 5 \end{bmatrix}$ $\therefore$ $\rho(A) \leq 3$

$\rho(A) = 3$

(non-zero determinant of 3X3 matrix exist)

(6) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 3 \end{bmatrix}_{3\times4}$

$\rho(A) \leq 3$.

$\rho(A) < 3$

$\rho(A) = 2$

(7) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}_{4\times4}$ $R_4 \longrightarrow R_4 - (R_1 + R_2 + R_3)$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho(A) \leq 4$

$\rho(A) < 4$

$R_2 \longrightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$ (non-zero determinant of 3×3 matrix exists)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}_{3\times3}$$

$\rho(A) \leq 3$

$\rho(A) < 3$ (3×3 matrices is zero)

$\rho(A) < 2$ (all 2×2 matrices are zero)

$\therefore \rho(A) = 1$

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}_{4\times4}$$

$R_4 \longrightarrow R_4 - (R_1 + R_3)$

$R_3 \longrightarrow R_3 - (R_1 + R_2)$

$$\therefore \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) \leq 4$

$\rho(A) < 4$ (4×4-Matrix has 0 deter)

$\rho(A) < 3$ (determinant of 3×3 is zero)

$\therefore \rho(A) = 2$

The consistency and inconsistency of system of equⁿ:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

coefficient matrix. $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

constant matrix, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

variable matrix. $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Augumented matrix. $(AB) = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$

Any system is said to be consistant system, if it has solution. Inconsistant system has no solution

(i) If $\rho(AB) = \rho(A)$ = no. of unknowns, then the system is said to be consistant and having unique solution.

(ii) If $\rho(AB) = \rho(A) <$ no. of unknowns then the system is said to be consistant and having more than one solution or infinite no. of solutions.

(iii) If $\rho(AB) \neq \rho(A)$, then system is said to be inconsistent and has no solution.

Q. Find the no. of solutions for system of equations.

$$x+y+z=3$$
$$x+2y+3z=4$$
$$x+4y+9z=6$$

Note:

(i) If Augumented matrix (AB) is rectangular matrix then last column must be excluded. (Because it will give the rank of (AB) only & we will have to calculate rank of (A) seperately.

(ii) If (AB) is square matrix then last column must be included.

(iii) Last column excluded, then represents ranks of both (A) and (AB)

(iv) If last column included gives rank of (AB) only.

$$(AB) = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{bmatrix}_{3\times4}$$

$R_2 \longrightarrow R_2 - R_1$, $R_3 \longrightarrow R_3 - R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{bmatrix}$$

$\rho(AB) = \rho(A) = 3$ (Non-zero determinant of 3×3 matrix exists)

$\therefore$ The system of equations is consistant and have unique solution.

2. $x - 2y + 3z = 2$
$2x - 3z = 3$
$x + y + z = 0$

$$(AB) = \begin{bmatrix} 1 & -2 & 3 & 2 \\ 1 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{3\times4}$$

$R_1 \longrightarrow R_1 + 2R_2$

$$= \begin{bmatrix} 3 & 0 & 5 & 2 \\ 1 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$\rho(AB) = \rho(A) = 3$

System is consistant and has unique solutions.

$3x+3y+2z=1$

$x+2y \quad =4$

$10y+3z=-2$

$2x-3y+z=-5$

$$AB = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & 1 & -5 \end{bmatrix}$$

Give preference to the equation with coefficient of 'x' as '1' to write in 1^{st} row.

$$AB = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 3R_1 \; ; \; R_4 \longrightarrow R_4 - 2R_1$

$$AB = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

To find determinant make two zeros in 3×3 matrix.

$R_2 \longrightarrow R_2 + 2R_4 , \quad R_3 \longrightarrow R_3 + 3R_4$

$$AB = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -17 & 0 & -17 \\ 0 & -11 & 0 & -11 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2 \quad \& \quad R_2 \longleftrightarrow R_4$

$$AB = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & +17 & 0 & +17 \\ 0 & -7 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(-17) \times (11)$ common out

$\rho(AB) = \rho(A) = 3 = 3$

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21$$

$$(AB) = \begin{bmatrix} 1 & 1 & -3 & -1 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 3 & 7 \end{bmatrix}_{3\times 4}.$$

$$R_2 \longrightarrow R_2 + R_1 \quad \& \quad R_3 \longrightarrow R_3 + R_1$$

$$= \begin{bmatrix} 1 & 1 & -3 & -1 \\ 3 & 0 & 0 & 3 \\ 6 & 0 & 0 & 6 \end{bmatrix} \quad \text{---- (i)}$$

$$\rho(AB) = \rho(A) = 2 < 3$$

The system is consistent and have infinite no. of solutions.

(i) $X = A^{-1} \cdot B$

(ii) Crammer's rule

$$x = \frac{\Delta_1}{\Delta}, \; y = \frac{\Delta_2}{\Delta}, \; z = \frac{\Delta_3}{\Delta}$$

(iii) Gauss-seidal

$$\begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

from —— (i)

$$x + y - 3z = -1$$
$$3x = 3 \qquad \therefore x = 1$$
$$6x = 6 \qquad x = 1$$

$$y - 3z = -2$$

if constant is zero

$$y - 3z = 0$$
$$y = 3z$$
$$\frac{y}{3} = \frac{z}{1}$$

This is called symmetrical method of solving.

$\because$ We are discussing consistency of real system of equations solutions should be real numbers.

$\therefore$ Let $z = K$ where $k \in$ Real numbers.

$$y - 3k = -2$$

$$\therefore \quad y = 3k - 2$$

$$\therefore \quad z = k.$$

$$\therefore \quad x = 1$$

For the different values of k, we get different y values. Thus, infinite number of solutions exist.

Q. Check consistency of

$$2x - y + z = 4$$
$$3x - y + z = 6$$
$$4x - y + z = 7$$
$$-x + y \pm z = 9$$

$$(AB) = \begin{bmatrix} -1 & 1 & -1 & 9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + R_1 \qquad R_3 \longrightarrow R_3 + R_4 \qquad R_4 \longrightarrow R_4 + R_1$

$$= \begin{bmatrix} -1 & 1 & -1 & 9 \\ 1 & 0 & 0 & 13 \\ 2 & 0 & 0 & 15 \\ 3 & 0 & \phi & 16 \end{bmatrix}$$

Always keep zero's either at 1^{st} row or column or last row or column.

Here, don't bring last column inside (it represents B)

$$C_1 \longleftrightarrow C_2$$

$$= \left[\begin{array}{c|ccc} 1 & -1 & -1 & 9 \\ \hline 0 & 1 & 0 & 13 \\ 0 & 2 & 0 & 15 \\ 0 & 3 & \phi & 16 \end{array}\right]$$

Here $\rho(AB) = 4$, $\rho(A) = 3$

The system of equations is inconsistent.

Q. For what values of λ and μ does the system of equations given below will be having.

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu.$$

(a) No solution
(b) Unique solution
(c) More than one number of solutions.

Case I :

$$(AB) = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$$R_3 \longrightarrow R_3 - R_2$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

Case I : If $\lambda-3=0$ & $\mu-10 \neq 0$

$$\rho(AB) = 3, \quad \rho(A) = 2$$

No-solution exists if $\lambda = 3$ & $\mu-10 \neq 0$

Case II : If $\lambda-3=0$ & $\mu-10=0$

$$\rho(AB) = \rho(A) = 2 < 3\text{-no. of unknowns}$$

$\therefore$ More than one solution exist if $\lambda=3$, $\mu=10$

Case III : If $\lambda-3 \neq 0$

$$\rho(AB) = \rho(A) = 3 = \text{no. of unknowns.}$$

unique solution exists. if $\lambda \neq 3$ & μ is any real number.

Eigen values and Eigen vectors:

Characteristic equation:

Let 'A' be the square matrix of order 'n×n', I be the unit matrix of order 'n×n', then $|A-\lambda I|=0$, is called the characteristic equation. where λ is a parameter.

The roots of characteristic equation are called the characteristic roots or Latent roots or Eigen values or proper values.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ or } [x_1\ x_2 \ldots\ x_n]^T$$

which satisfies the characteristic equation $[A-\lambda I]X=0$ is called corresponding Eigen vector of the matrix.

$$[A-\lambda I]\cdot X = 0$$

Note:

(i) The sum of the eigen values of any matrix is equal to sum of the elements of its principal diagonal.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Trace of $A = 1+5+1$

$Tr.(A) = 7 = $ (sum of eigen values)

(ii) The product of Eigen values of any matrix is equal to its determinant.

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

characteristic roots of matrix = ?

options: (a) 2.5
(b) 3.4
(c) 6.1
(d) 2.3

Sum of diagonals elements = sum of eign values
$= (5+2) = 7$

Product of characteristic roots = determinant
$= (10-4)$
$= 6$

(iii) The eigen values of symmetric matrix $(A^T = A)$ (10) are purely real.

$$S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$S^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

(iv) The eignen values of skew symmetric matrix i.e. $(A^T = -A)$ are either purely imaginary or may be zero

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A \quad \text{i.e. skew symmetric}$$

sum of eigen values $= 0$.

value of determinant $= 1$

(a) $-1, 0$ (b) $0, 1$ (c) $1, -1$ (d) $i, -i$.

Ans : (d) $i, -i$ are characteristic roots of matrix

(v) If the matrix is either lower triangular or upper triangular then principal diagonal elements ar called the eigen values.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}, \quad \lambda = 1, 3, 6$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 0 & 5 \end{bmatrix}, \quad \lambda = 1, -4, 5$$

(vi) After interchanging the rows or columns or after certain operations on matrix, if we obtain lower or upper triangular matrix, we may not necessarily get the values of eigen vector on principal diagonal.

Sunday
22^{nd} September 2013

If λ is eigen value of A then,

λ^2 is eigen value of A^2

λ^3 —— A^3

λ^4 —— A^4

$\vdots$

λ^n —— A^n

$\frac{1}{\lambda}$ —— A^{-1}

e.g. If $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ with $\lambda = 1, 5$

$$\therefore S^2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}$$

$$\therefore \lambda = 1^2, 5^2 = 1, 25$$

Q If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ Then eigen values of A^3 are.

$\therefore$ $\lambda \longrightarrow A$

$\lambda^3 \longrightarrow A^3$

(a) 9, 1. (b) 5, 4. (c) 16, -1 (d) 27, -1

(d) 27, -1 $\Rightarrow (3)^2, (-1)^3$

$\therefore \lambda^3 = 27, -1$

Q. Find the eigen values are corresponding eigen vectors of the matrix. 11

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

Characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda-1)(\lambda-6) = 0$$

$$\lambda = (1, 6)$$

Eigen vector for $\lambda = 1$.

$$[A - \lambda I] \cdot X = 0$$

$$\therefore \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{0}$$

$$4x_1 + 4x_2 = 0$$

$$x_1 + x_2 = 0$$

$$\therefore x_1 = -x_2$$

$$\frac{x_2}{x_2} = -1$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{-1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ or } [1 \ \ -1]^T$$

Also

$$\frac{x_1}{-1} = \frac{x_2}{1}$$

$$X = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ or } [-1 \ \ 1]^T$$ (usually not in options given)

For $\lambda = 6$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-x_1 + 4x_2 = 0$$

$$x_1 - 4x_2 = 0$$

$$x_1 = 4x_2$$

$$\frac{x_1}{4} = \frac{x_2}{1}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ or } [4 \quad 1]^T$$

Normolised eigen vector

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1^2 + (-1)^2}} \\ \frac{-1}{\sqrt{1^2 + (-1)^2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{4}{\sqrt{4^2 + 1^2}} \\ \frac{1}{\sqrt{4^2 + 1^2}} \end{bmatrix} = \begin{bmatrix} \frac{4}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} \end{bmatrix}$$

Q. Find the characteristic roots of matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

(a) 3, 7, 8 (b) 2, 2, 14. (c) 0, 3, 15 (d) 1, 4, 9.

sum = 18 18 18 14

$$\Delta = \begin{matrix} 8 & -6 & 2 & 8 & -6 \\ -6 & 7 & -4 & -6 & 7 \\ 2 & -4 & 3 & 2 & 4 \end{matrix}$$

$$= 168 + 48 - 48 - 28 - 128 - 108$$

$$= 0$$

(c) 0, 3, 15

(1) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ Eigen values are - (12)

(a) 1, 2, 3 (b) 0, 2, 4. (c) 1, 1, 4. (d) 2, 2, 2.

sum of eigen values $= 1+2+3 = 6$

Product of eigen values $= \Delta = 6$

$\therefore$ Eigen values are (a) 1, 2, 3

Q The eigen vector pair of matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

$D = -9-16 = -25$

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_1 \cdot \lambda_2 = -25$

(b) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_1 + \lambda_2 = 0$

$5, -5$

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 9 - 16 = 0$$

$$\lambda^2 = 25$$

$$\lambda = \pm 5$$

For $\lambda = +5$

$$\begin{bmatrix} 3\pm5 & 4 \\ 4 & -3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-2x_1 + 4x_2 = 0$$

$$\frac{x_1}{2} = \frac{x_2}{1}$$

$$X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda = -5$

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$8x_1 + 4x_2 = 0$$

$$\frac{x_1}{x_2} = \frac{1}{-2}$$

$$X_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Eigen values of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

$\therefore$ Eigen values $= \lambda = (1, -4, 7)$

For $\lambda = 1$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$2x_2 + 3x_3 = 0 \, *$$

$$-5x_2 + 2x_3 = 0 \, *$$

$$6x_3 = 0$$

If in eigen vector equation if two unknowns have two different equations, then those two unknowns have value zero.

If two unknown in the eigen vector equation are zero (for 3x3 matrix) then 3rd unknown is always non zero. (It may have all values other than zero i.e. infinity)

$\therefore$ Eigen vector for $\lambda = 1$

$$[A - \lambda I][X] = 0$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad X_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad X_4 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = -4$

$$\begin{bmatrix} 5 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$5x_1 + 2x_2 + 3x_3 = 0$$

$$2x_3 = 0$$

$$x_3 = 0$$

$$5x_1 + 2x_2 = 0$$

$$\frac{x_1}{2} = \frac{x_2}{-5}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$$

for $\lambda = 7$

$$\begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-6x_1 + 2x_2 + 3x_3 = 0$$

$$-11x_2 + 2x_3 = 0$$

$$-2x_3 = \cancel{0}\ 11x_2$$

$$\therefore \quad \frac{x_2}{-2} = \frac{x_3}{-11}$$

$$\frac{x_2}{2} = \frac{x_3}{11}$$

$$\therefore \quad x_1 = 37/6$$

$$\therefore X = \begin{bmatrix} 37/6 \\ 2 \\ 11 \end{bmatrix}$$

Q. An eigen vector of matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

(a) $[-1\ 1\ 1]^T$ (b) $[1\ 2\ 1]^T$ (c) $[1\ -1\ 2]^T$ (d) $[2\ 1\ -1]^T$

The eigen values are (1, 2, 3) (Upper triangular matrix)
But to find set of eigen vector for particular eigen value, start with least value (i.e. $\lambda = 1$)

$\therefore$ for $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore x_2 = 0$ (not in any option)

for $\lambda = 2$

$$\begin{bmatrix} 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$2x_3 = 0 \quad x_3 = 0$ (not in option

For $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 0 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 + x_2 + 0x_3 = 0$$

$$-2x_1 = -x_2$$

$$\frac{x_1}{1} = \frac{x_2}{2}$$

$\therefore$ Eigen vector $= \begin{bmatrix} 1 \\ 2 \\ \end{bmatrix}$ or $\begin{bmatrix} 1 & 2 & \end{bmatrix}^T$

Q. A real $n \times n$ matrix $A = [a_{ij}]$ is given as follow:

$$a_{ij} = i, \quad \text{if } i = j$$
$$= 0 \quad \text{if } i \neq j$$

then sum of all eigen values...

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & n \end{bmatrix}$$

Sum of Eigen values $= 1 + 2 + 3 + \cdots n$

$$= \frac{n(n+1)}{2}$$

Cayley-Hamilton theorem:

Every square matrix satisfied its own characteristic equation.

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

characteristic equation

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - \lambda(7) + 6 = 0$$

According to theorem,

$$A^2 - 7A + 6I = 0$$

I - multiplicative identity.

Proof. –

$$\text{L.H.S.} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - 7\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 35 & 28 \\ 7 & 14 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= 0$$

Application:

$$6I = 7A - A^2$$

$$6I \cdot A^{-1} = 7 \cdot A \cdot A^{-1} - A^2 \cdot A^{-1}$$

$$6A^{-1} = 7I - A.$$

$$= \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$$

If M be 3×3 matrix. & its characteristic equation $\lambda^3+1=0$ then M^{-1} is

(a) M (b) $-M^2$ (c) M^3 (d) $-M^4$

$$M^3+1=0$$
$$M^3+I=0$$
$$I=-M^3$$
$$I\cdot M^{-1}=-M^2\cdot M\cdot M^{-1}$$
$$M^{-1}=-M^2$$

For $P_{3\times3}$ if characteristic equation is $\lambda^3+\lambda^2+2\lambda+1=0$

(a) P^2+P+I (b) P^2+P+2I (c) $-(P^2+P+2I)$ (d) $-(P^2+P+I)$

$$P^3+P^2+2P+I=0$$
$$I\cdot P^{-1}=-(P^3\cdot P^{-1}+P^2\cdot P^{-1}+2P\cdot P^{-1}+I\cdot P^{-1})$$
$$P^{-1}=-(P^2+P+2I)$$

If $\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix}=0$ then line passes through.

(a) $(0,0)$ (b) $(3,4)$. (c) $(4,3)$ (d) $(4,4)$

$$\Delta = \begin{matrix} x & 2 & 4 & x & 2 \\ y & 8 & 0 & y & 8 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}$$

$$= 8x+4y-32-2y$$
$$= 8x+2y-32$$

(b) $(3,4)$ satisfies the above equation.

$A_{3\times4}$ real matrix and $AX = B$ is an inconsistant system of equations then highest possible rank of A is (15)

(a) 1 (b) 2 (c) 3 (d) 4

∵ We add column to A when we add B in matrix A

∴ If A is 3×4 ∴ $AB = 3\times5$

∴ Max. possible rank $\rho(AB) = 3$

But given that, system is inconsistant

∴ $\rho(AB) \neq \rho(A)$

In no case $\rho(A) > \rho(AB)$

∴ $\rho(A) = 2$

The value of 'a' for which the following set of equations have a non-trivial solution is

$$y + 2z = 0$$
$$2x + y + z = 0$$
$$ax + 2y + 0z = 0$$

(a) 2 (b) 4 (c) 8 (d) None.

For the system of equations $AX = 0$, $X = 0$ is always a solution and that solution is called a trivial solution.

For a non-trivial solution, determinant of coefficient matrix is always zero.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 1 \\ a & 2 & 0 \end{bmatrix}$$

$$\Delta = \begin{matrix} 0 & 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 2 & 1 \\ a & 2 & 0 & a & 2 \end{matrix}$$

$$= a + 8 - 2a$$

$$0 = 8 - a$$

$$a = 8$$

Note:

(i) $\Delta \neq 0, \quad X = 0 \quad \Rightarrow$ trivial solution

(ii) $\Delta = 0, \quad X \neq 0 \quad \Rightarrow$ non-trivial solution.

Given an orthogonal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\therefore [A \cdot A^T]^{-1} = ?$$

A matrix A is said to be an orthogonal matrix if

$$A \cdot A^T = I$$

$$\therefore [A \cdot A^T]^{-1} = I^{-1}$$

$$= I$$

The inverse of the matrix

$$A = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

We know that,

Inverse of diagonal matrix = Inverse of each element in diagonal

$$\therefore A^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Rank of diagonal matrix is equal to number of non-zero elements in the diagonal.

Q. The rank and nullity of matrix is

$$\begin{bmatrix} 3 & 0 & \phi & 2 \\ 4 & 7 & 3 & 3 \\ 1 & 7 & 2 & 1 \end{bmatrix}$$

Nullity of matrix = No. of columns − Rank.

$$= 4 - 2$$
$$= 2$$

If $|A| = 2$ $\therefore$ $|A^2| = |A|^2$

$$|A^3| = |A|^3$$
$$\vdots$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$|A^{-1}| = \frac{|\text{adj } A|}{|\,|A|\,|}$$

$$= \frac{|\text{adj } A|}{|A|^n}$$

$$\boxed{\begin{aligned} |\text{adj } A| &= |A^{-1}| \cdot |A^n| \\ &= |A|^{n-1} \end{aligned}}$$

$$|\text{adj}(\text{adj } A)| = |\text{adj } A|^{n-1}$$
$$= (|A|^{n-1})^{n-1}$$
$$= |A|^{(n-1)^2}$$

Q. If $A_{3\times3}$ is real matrix and B is adjoint of A & $|B| = 64$ then $|A| = ?$

$$|B| = |\text{adj } A| = |A|^{n-1}$$
$$64 = |A|^{3-1}$$
$$64 = |A|^2$$
$$|A| = \pm 8$$

If $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -2 & 0 \\ 4 & 2 & -1 \end{bmatrix}$ & $B = 2A^2$ then $|B| = ?$

(a) 16 (b) 32 (c) 64 (d) 128

$$B = 2A^2$$
$$|B| = |2A^2|$$
$$= 2^n |A^2|$$
$$= 2^3 (16)$$
$$= 128$$

This is not modulus but determinant

n- order of A

If $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix}$ then $a+b = ?$

$$A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix}$$

$a = \frac{0.1}{6}$ and $b = \frac{2}{6}$

$$a + b = \frac{2.1}{6}$$
$$= \frac{7}{20}$$

Let A, B, C, D be n×n matrices each with non-zero determinant & if $A \cdot B \cdot C \cdot D = I$ then $B^{-1} = ?$

(a) $D^{-1} C^{-1} A^{-1}$ (b) ADC (c) CDA (d) doesn't exist.

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$
$$(ABCD)^{-1} = I^{-1}$$
$$D^{-1} \cdot C^{-1} \cdot B^{-1} \cdot A^{-1} = I$$
$$D \cdot D^{-1} \cdot C^{-1} B^{-1} A^{-1} = D \cdot I$$
$$C \cdot C^{-1} B^{-1} \cdot A^{-1} = C \cdot D$$
$$B^{-1} \cdot A \cdot A^{-1} = C \cdot D \cdot A$$

In modern algebra (matrices)

$AB \neq BA$

For what value of a, if any, will the following system of equations i.e. 17

$$2x+3y=4$$
$$x+y+z=4$$
$$x+2y-z=a \quad \text{has a solution.}$$

$\because$ Given that,

$$\rho(AB)=\rho(A)$$

$$AB=\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & 0 & 4 \\ 1 & 2 & -1 & a \end{bmatrix}_{3\times 4}$$

$$R_3 \longrightarrow R_3 + R_1$$

$$AB=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 4 \\ 2 & 3 & 0 & a+4 \end{bmatrix}$$

Case I: If $(a+4) \neq 4$

$\rho(AB)=3$ & $\rho(A)=2$ (not true here)

Case II: If $(a+4)=4$

$$a=0$$

For the matrix $M=\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$, $M^T=M^{-1}$ then $x=?$

(a) $\frac{-4}{5}$ (b) $\frac{-3}{5}$ (c) $\frac{3}{5}$ (d) $\frac{4}{5}$

$\because$

$$M^T=\begin{bmatrix} \frac{3}{5} & \cancel{x} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \quad \text{——— (i)}$$

$$M^{-1}=\frac{1}{\frac{9}{25}-\frac{4x}{5}}\begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ x & \frac{3}{5} \end{bmatrix} \quad \text{——— (ii)}$$

For $M^T=M^{-1}$

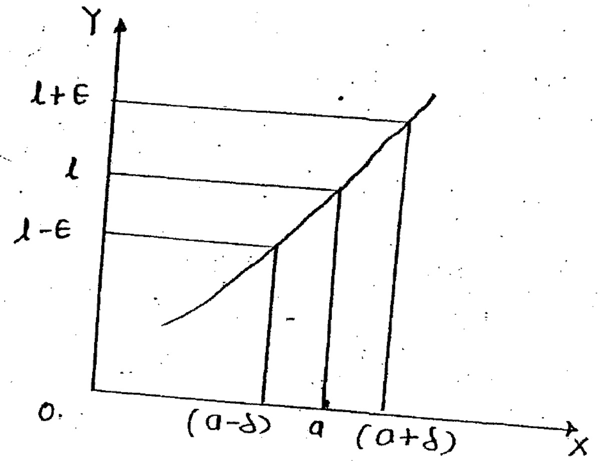
$$x=\frac{-4}{5}$$

"Differential Calculus"

A number l is said to be the limit of function $f(x)$ as $x \to a$, if for all $\epsilon > 0$ (however small) those exist a $\delta \geq 0$ such that.

$$|f(x) - l| < \epsilon \quad \forall \quad |x-a| < \delta$$

$$\therefore \quad l-\epsilon < f(x) < l+\epsilon \quad \forall \quad a-\delta < x < a+\delta$$



$$\therefore \quad \lim_{x \to a} f(x) = l$$

- Left limit :-

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h), \quad (a-\delta) < x < a$$

- Right limit :

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h), \quad (a < x < a+\delta$$

Existance of limit

The limit of a function exist if both the left and right limit are existed and are equal

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=n\cdot a^{n-1}$$

$$\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\frac{m}{n}\cdot a^{m-n}$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1 \quad =0/0$$

$$\lim_{x\to 0}\frac{e^{mx}-1}{x}=m$$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\log a$$

$$\lim_{x\to 0}\frac{a^x-b^x}{x}=\log\left(\frac{a}{b}\right)$$

$$\lim_{x\to 0}(1+x)^{1/x}=e$$

$$\lim_{x\to 0}(1+ax)^{1/x}=e^a$$

$$\lim_{x\to 0}(1+ax)^{b/x}=e^{ab}$$

$$\lim_{x\to 0}(1+ax)^{1/bx}=e^{a/b}$$

$$\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x=e$$

→ apply L hospital

$\frac{0}{0},\frac{\infty}{\infty}$

$2\sin A\cos B$
$=\sin(A+B)+\sin(A-B)$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$\lim_{x\to 0}\frac{\tan x}{x}=1$$

$$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=\frac{m^2}{n^2}$$

$$\lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2}=\frac{b^2-a^2}{2}$$

***

[$\frac{0}{0}, \frac{\infty}{0}, \frac{\infty}{\infty}, 1^\infty$ etc are the indeterminate forms]

If $\frac{0}{0}, \frac{\infty}{\infty}$ then apply the L-Hospitality rule.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

(both $f(x)$ & $g(x)$ are algebr functions)

$$=\lim_{x\to a}\frac{f''(x)}{g''(x)}=\cdots$$

Spl case

$$\lim_{x\to a}[f(x)]^{g(x)}$$

$$=e^{\lim_{x\to a} g(x)[f(x)-1]}$$

if $f(x)$ - finite

$g(x)$ - infinite.

as $x\to a$

$\cos 2x = 2\cos^2 x-1 = 1-2\sin^2 x$

$\sin 2x = 2\sin x\cos x$

$\cos(A+B)$

$\sin(A+B)$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$, if both $f(x)$ and $g(x)$ are algebric function

Case I:

If the degree of $f(x) >$ $g(x)$ degree, then result is ∞.

$$\lim_{x\to\infty} \frac{2x^3+3x+5}{3x^2+6x+7} = \infty$$

Case II:

If the degree of $f(x) <$ degree of $g(x)$, then result is zero.

$$\lim_{x\to\infty} \frac{2x^2+3x+4}{x^3+5} = 0$$

Case III:

If degrees of $f(x)$ and $g(x)$ are equal then.

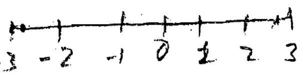
$$\lim_{x\to\infty} \frac{2x^2+3x+4}{5x^2+6x+7} = \frac{2}{5} = \frac{\text{coeff. of numerator}}{\text{coeff. of denominator}}$$

$$\begin{cases} |x| = x & \text{if } x > 0 \\ |x| = -x & \text{if } x < 0 \end{cases}$$

$[x]$ - step x - The greatest integer n of greater than x

$[2.99] = 2$

$[-2.99] = 3$



Q. 1. $\lim_{x\to a} \dfrac{1}{x-a} =$

(a) 0 (b) ∞ (c) $-\infty$ (d) does not exist

✓ $\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} \dfrac{1}{\cancel{a}-h-\cancel{a}} = \dfrac{-1}{h} = \dfrac{-1}{0} = -\infty$

✓ $\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} \dfrac{1}{\cancel{a}+h-\cancel{a}} = \dfrac{1}{h} = \infty$

lim does not exist

$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} \dfrac{1}{\cancel{a}-h-\cancel{a}} = \lim_{h\to 0} \dfrac{-h}{h} = -\infty$

$\boxed{LHS \neq R.H.S}$ doesnot exit

2. $\lim_{x\to 0} \frac{1-\cos x}{x\cdot \sin x}$

$$L = \lim_{x\to 0} \frac{1-\cos x}{x\cdot\sin x}$$

$$= \frac{1-0}{0} = \frac{0}{0}$$

Use L-hospitality rule.

$$d(\cos x) = -\sin x$$

$$d(\sin x) = \cos x$$

$$L = \lim_{x\to 0} \frac{0+\sin x}{x\cdot\cos x + \sin x\,(1)}$$

$$= \lim_{x\to 0} \frac{\cos x}{x(-\sin x) + \cos x\,(1) + \cos x}$$

$$= \frac{1}{0+1+1}$$

$$= \frac{1}{2}$$

$$\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$$

$$1+\cos x = 2\cos^2\frac{x}{2}$$

$$1-\cos x = 2\sin^2\frac{x}{2}$$

Method 2

$$L = \lim_{x\to 0} \frac{1-\cos x}{x\cdot\sin x}$$

$$= \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x\cdot 2\sin x/2\cdot\cos x/2}$$

$$= \lim_{x\to} \frac{\tan x/2}{\left(\frac{x}{2}\right)\times 2}$$

$$= \frac{1}{2}$$

Method 3

$$L = \lim_{x\to 0} \frac{1-\cos x}{x\cdot\sin x}\,\frac{(1+\cos x)}{(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x\cdot\sin x\,(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x \cdot \sin x \, (1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x \, (1+\cos x)}$$

$$= \lim_{x \to 0} \frac{(\sin x / x)}{(1+\cos x)}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

(2) $\displaystyle \lim_{x \to 0} \frac{1-\cos x}{x^2}$

$$= \frac{1-1}{0} = \frac{0}{0}$$

$$L = \lim_{x \to 0} \frac{0+\sin x}{2x} \qquad \text{(L-hospitality)}$$

$$= \frac{1}{2}$$

(3) $\displaystyle \lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$

$$L = \frac{\tan x - \sin x}{\sin^3 x}$$

If both numerator & denominator are trignometric function, don't use L-hospitality rule

$$L = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x}$$

$$= \lim_{x \to 0} \frac{1-\cos x}{\cos x \cdot \sin^2 x}$$

$$= \lim_{x \to 0} \frac{(1-\cos x)}{(1-\cos x)\,(1+\cos x) \cdot \cos x}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

(4) $\lim_{x \to 1} \dfrac{\cos\left(\frac{\pi x}{2}\right)}{1-\sqrt{x}}$

$$d(\cos ax) = -a \sin ax$$

$$d(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$L = \lim_{x \to 1} \frac{-\frac{\pi}{2} \sin \frac{\pi x}{2}}{1 - \frac{1}{2\sqrt{x}}}$$

$$= \frac{\pi (1)}{\frac{1}{\sqrt{1}}}$$

$$= \pi$$

(5) $\lim_{x \to 0} \dfrac{\log(1+x)}{x} = 0$

$$L = \lim_{x \to 0} \frac{\frac{1}{1+x} \cdot 1}{1}$$ $d(\log x) = \frac{1}{x}$

$$= \frac{1}{1}$$

$$= 1$$

(6) $\lim_{x \to 0} \dfrac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2} = \dfrac{0}{0}$

$$L = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1-x^2}}(-2x) - \frac{1}{2\sqrt{1+x^2}}(2x)}{2x}$$

$$= \lim_{x \to 0} \frac{\frac{-1}{2\sqrt{1-0}} - \frac{1}{2\sqrt{1+0}}}{1}$$

$$= \frac{-1}{2} - \frac{1}{2}$$

$$= -1$$

(7) $\lim_{x \to 0} \dfrac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x}$

$$L = \lim_{x \to 0} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}(1+x)^{-2/3} - \frac{1}{3}(1-x)^{-2/3}(-1)}{1}$$

$$= \frac{1}{3}\left[(1+0)^{-2/3} + (1)^{-2/3}\right]$$

$$= \frac{2}{3}$$

(8) $\lim_{x \to 8} \dfrac{x^{1/3}-2}{x-8}$

$$L = \lim_{x \to 8} \frac{x^{1/3} - 8^{1/3}}{(x-8)} = \frac{0}{0} \text{ form}$$

$$= \lim_{x \to 8} \frac{\frac{1}{3}x^{-2/3} - 0}{1}$$

$$= \frac{1}{3}(8)^{-2/3}$$

$$= \frac{1}{3}(2^3)^{-2/3}$$

$$= \frac{1}{3} \times \frac{1}{4}$$

$$= \frac{1}{12}$$

(9) $\lim_{x \to 0} (1+2x)^{1/3x}$

$$= \lim_{x \to 0} (1+2x)^{1/3x}$$

$$= e^{2/3}$$

(10) $\lim_{n \to \infty} \left(1-\frac{1}{n}\right)^{2n}$

$$L = \lim_{n \to \infty} \left[\left(1-\frac{1}{n}\right)^{n}\right]^2$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1$$

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{-1}$$

$$\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x = e^{-2}$$

$$\therefore\quad L = (e^{-1})^2$$

$$= e^{-2}$$

(11) $\lim_{x\to\infty}\left(\frac{x-1}{x-2}\right)^x$

$$L = \lim_{x\to\infty}\left(\frac{x-1}{x-2}\right)^x \qquad = \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{x\to\infty}\left[\frac{x\left(1-\frac{1}{x}\right)}{x\left(1-\frac{2}{x}\right)}\right]^x$$

$$= \lim_{x\to\infty}\frac{\left(1-\frac{1}{x}\right)^x}{\left(1-\frac{2}{x}\right)^x}$$

$$= \frac{e^{-1}}{e^{-2}}$$

$$= e$$

(1) $\lim_{x\to\infty}\frac{1+2+3+\dots n}{n^2}$

$$L = \lim_{n\to\infty}\frac{n(n+1)}{2n^2}$$

$$= \lim_{n\to\infty}\frac{(n+1)}{2n}$$

$$= \frac{1}{2}$$

degree of numerator and denominator $= 1$ (same)

$\therefore$ $\dfrac{\text{coeff. of num.}}{\text{coeff of denominator.}}$

(13) $\lim_{n\to\infty} \dfrac{n\,(1^3+2^3+\cdots\cdots+n^3)^2}{(1^2+2^2+\cdots\cdots+n^2)^3}$

$\because\ 1^2+2^2+3^2+\cdots\cdots = \dfrac{n(n+1)(2n+1)}{6}$

$1^3+2^3+3^3+\cdots n^3 = \dfrac{n^2(n+1)^2}{4}.$

$\therefore\ L = \lim_{n\to\infty} \dfrac{n\left[\dfrac{n^2(n+1)^2}{4}\right]^2}{\left[\dfrac{n(n+1)(2n+1)}{6}\right]^3}$

$= \lim_{n\to\infty} \dfrac{n^5(n+1)^4}{16} \times \dfrac{216}{n^3(n+1)^3(2n+1)^3}$

$= \dfrac{1\times 216}{16\times1\times1\times8}$

$= \dfrac{27}{16}$

(14) $\lim_{n\to\infty}\left[\dfrac{1}{1\cdot2}+\dfrac{1}{2\cdot3}+\dfrac{1}{3\cdot4}+\cdots\dfrac{1}{n(n+1)}\right]$

$L = \lim_{n\to\infty}\left[\Sigma\dfrac{1}{n(n+1)}\right]$

If no standardar formula is available & sum is given use

$S_n = \Sigma T_n$

$L = \lim_{n\to\infty}\Sigma\left(\dfrac{1}{n}-\dfrac{1}{n+1}\right)$ – shortcut for partial fraction, here $\dfrac{1}{n} > \dfrac{1}{n+1}$

$= \lim_{n\to\infty}\left(\dfrac{1}{1}-\cancel{\dfrac{1}{2}}+\cancel{\dfrac{1}{2}}-\cancel{\dfrac{1}{3}}+\cancel{\dfrac{1}{3}}\cdots\cdots\cancel{\dfrac{1}{n}}-\dfrac{1}{n+1}\right)$

$= 1-\dfrac{1}{n+1}$

$= 1-\dfrac{1}{\infty} = 1-0 = 1.$

15. $\lim_{x \to 0} \frac{1-\cos 2x}{1-\cos 3x}$

$$L = \lim_{x \to 0} \frac{1-\cos 2x}{1-\cos 3x}$$

$$= \frac{2^2}{3^2}$$

$$= \frac{4}{9}$$

The continuity of a function:

A function $f(x)$ is said to be continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

otherwise it is said to be a discontinuous function.

Types of discontinuous functions:

(i) Discontinuity of I-type (Jump discontinuity)

A function is said to have discontinuity of I-type if

$$\boxed{\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)}$$ (limit doesn't exist)

(ii) Discontinuity of II-type

A function is said to have discontinuity of II-type if either left limit or right limit or both does not exist.

(iii) Discontinuity of III-type (Removable discontinuity)

A function is said to have discontinuity of III-type if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$$

Note-

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(a) \to$ conti

Standard continuous functions:

(i) x^n is continuous function $\forall x$ when $n>0$ i.e. $(x, x^2, x^{2/3} \ldots)$

(ii) x^n is continuous function $\forall x$ except $x=0$, when $n<0$

(iii) $|x|$ is continuous function $\forall x$.

(iv) $\log x$ is continuous function $\forall x$ if $x>0$

(v) Every exponential function is continuous (e^x or a^x)

(vi) $\sin x$ and $\cos x$ are continuous functions for all values of x.

(vii) $\tan x$ and $\sec x$ are continuous functions $\forall x$, except $\cos x = 0$,
i.e except, $x = (2n-1)\cdot\frac{\pi}{2}$

(viii) $\cot x$ and $\operatorname{cosec} x$ are continuous functions $\forall x$ except for
$\sin x = 0$. i.e. $x = n\pi$

(ix) Every algebric function is continuous $(a_0x^n + a_1x^{n-1} + \cdots + a_n)$

(x) Every rational function $\left(\frac{f(x)}{g(x)}\right)$ is continuous except $g(x)=0$

Q. Check the continuity of the function.

$$f(x) = \frac{1}{1+2^{1/x}} \quad \text{at } x=0$$

Left limit

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \frac{1}{1+2^{1/0-h}}$$

$$2^{-1/h} = 2^{-1/0} = 2^{-\infty} = 0$$

$$2^{1/h} = 2^{1/0} \Rightarrow 2^{\infty} = \infty$$

$$L = \lim_{h\to 0} \frac{1}{1+2^{-1/h}}$$

$$= \frac{1}{1+0}$$

$$= 1$$

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$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \frac{1}{1+2^{1/h}}$$

$$= \lim \frac{1}{1+\infty}$$

$$= \frac{1}{\infty}$$

$$= 0$$

As $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$ - function is dis-continuous (1-4)

(2) $f(x) = \dfrac{x\,(e^{1/x}-1)}{(e^{1/x}+1)}$ at $x = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} \frac{(0-h)\,(e^{-1/h}-1)}{(e^{-1/h}+1)}$$

$0\left(\frac{0}{0}\right) = 0 \qquad 0\left(\frac{\infty}{\infty}\right) \neq 0$

$$e^{-1/h} = e^{-1/0} = e^{-\infty} = 0$$

$$e^{1/h} = e^{1/0} = e^{\infty} = \infty$$

$$L = \frac{-h\,(0-1)}{(0+1)} = h$$

$$= 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} \frac{(0+h)\,(e^{1/h}-1)}{(e^{1/h}+1)}$$

$$= \lim \frac{h\cdot(e^{1/h}-1)}{(e^{1/h}-1)}$$

$$= 0\cdot\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{h\to 0} \frac{h\cdot e^{1/h}\left(1-\frac{1}{e^{1/h}}\right)}{e^{1/h}\left(1+\frac{1}{e^{1/h}}\right)}$$

$$= \frac{h\left(1-\frac{1}{\infty}\right)}{\left(1+\frac{1}{\infty}\right)}.$$

$$= \frac{h\,(1-0)}{(1+0)}$$

$$= 0$$

Function is continuous.

Q. A function $f(x)$ is defined by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{2} - x & 0 < x < 1/2 \\ \frac{1}{2} & x = \frac{1}{2} \\ \frac{3}{2} - x & \frac{1}{2} < x < 1 \\ 1 & x \geq 1 \end{cases}$$

which of following is true

(a) $f(x)$ is continuous at $x=0$

✓(b) $f(x)$ is discontinuous at $x = \frac{1}{2}$

(c) $f(x)$ is continuous at $x=1$

(d) All are true.

(a) For continuity at $x=0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$x < 0$	$x > 0$	$x = 0$
0 (given)	$\frac{1}{2} - x$ (given)	
	$= \frac{1}{2} - 0 = \frac{1}{2}$	

Thus not true.

(b) At $x = \frac{1}{2}$

$x < \frac{1}{2}$	$x > \frac{1}{2}$	$x = 1/2$
$\frac{1}{2} - \frac{1}{2} = 0$	$\frac{3}{2} - x$	
	$= \frac{3}{2} - \frac{1}{2}$	
	$= \cancel{0}\,1$	

$\therefore$ Discontinuous at $x = \frac{1}{2}$

$$f(x) = -x^2 \qquad x \leq 0$$
$$= 5x-4 \qquad 0 < x \leq 1$$
$$= 4x^2-3x \qquad 1 < x \leq 2$$
$$= 3x+4 \qquad x > 2$$

At $x=0$

$x<0$	$x>0$	
$= -x^2$	$5(0)-4$	
$= 0$	$= -4$	— discontinuous

At $x=1$

$x<1$	$x>1$	$x=1$	
$5(1)-4$	$4(1)^2-3(1)$	$5(1)-4$	
$=1$	$=1$	$=1$	— continuous

At $x=2$

$x<2$	$x>2$	
$4(2)^2-3(2)$	$3(2)+4$	
$= 8-6$	$= 10$	— discontinuou
$=2$		

Monday
23rd September 2013

Differentiability :→

A function $f(x)$ is said to be differentiable at $x=a$ if

Left hand derivative $= \lim_{h \to 0} \dfrac{f(a-h)-f(a)}{-h}$

Right hand derivative $= \lim_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$

both LHD & RHD exist and are equal

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2x\sqrt{x}}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-1}{x^3}$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(e^{ax}) = a \cdot e^{ax}$$

$$\frac{d}{dx}(\cosh x) = \sinh x.$$

$$\frac{d}{dx}(a^x) = a^x \cdot \log a$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}(x^x) = x^x(1+\log x)$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(x^{1/x}) = x^{1/x} \cdot \frac{1}{x^2}(1-\log x)$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}.$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

then $\frac{dy}{dx} = \frac{2}{1+x^2}$

If $y = x^{x^{x}}$

$$\therefore \quad y = x^{y}$$

$$\log y = y \cdot \log x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\left(\frac{1}{y} - \log x\right) \frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$$

If $x^{y} = e^{x-y}$

$$\log x^{y} = \log e^{x-y}$$

$$y \log x = (x-y) \cdot \log e$$

$$y + y\log x = x$$

$$y(1 + \log x) = x$$

$$y = \frac{x}{1 + \log x}$$

$$\frac{dy}{dx} = \frac{(1+\log x) - x\left(0 + \frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{\log x}{(1+\log x)^2}$$

Find derivatives of

$x^{m} \cdot y^{n} = a^{m+n}$

$$x^{m} \cdot y^{n} = a^{m+n}$$

$$\log(x^{m} \cdot y^{n}) = \log a^{m+n}$$

$$\log x^{m} + \log y^{n} = \log a^{m+n}$$

$$m \log x + n \log y = \log a^{m+n}$$

$$m \cdot \frac{1}{x} + n \cdot \frac{1}{y} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-my}{nx}$$

$x^a \cdot y^b = (x+y)^{a+b}$

$$\log (x^a y^b) = \log (x+y)^{a+b}$$

$$a \cdot \log x + b \cdot \log y = (a+b) \cdot \log (x+y)$$

$$a \cdot \frac{1}{x} + b \cdot \frac{1}{y} \frac{dy}{dx} = (a+b) \cdot \frac{1}{(x+y)} \left(1 + \frac{dy}{dx}\right)$$

$$\frac{a}{x} + \frac{b}{y} \cdot \frac{dy}{dx} = \left(\frac{a+b}{x+y}\right) + \left(\frac{a+b}{x+y}\right) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} \left(\frac{b}{y} - \frac{a+b}{x+y}\right) = \left(\frac{a+b}{x+y}\right) - \frac{a}{x}$$

$$\frac{dy}{dx} \left(\frac{bx - by - ay - by}{y(y+x)}\right) = \frac{ax + bx - ax - ay}{x(x+y)}$$

$$\frac{dy}{dx} = \frac{bx - ay}{x(x+y)} \times \frac{y(y+x)}{bx - ay}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ then $(1-x^2)y_1 - xy =$

(a) 0 (b) 1 (c) y (d) $-y$

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

$$y^2 (1-x^2) = (\sin^{-1} x)^2$$

$$y^2(-2x) + (1-x^2) \cdot 2y \cdot y_1 = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}}$$

$$2y(-xy + (1-x^2) \cdot y_1) = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} = 2y$$

$$(1-x^2) \cdot y_1 - xy = 1$$

If $y = a\cos(\log x) + b\sin(\log x)$ then $x^2 y_2 + x y_1 =$

(a) 0 (b) 1 (c) y (d) $-y$ 26

$\therefore$ $$y_1 = -a\cdot\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

$$x y_1 = b\cos(\log x) - a\sin(\log x)$$

$$x\cdot y_2 + y_1 = \frac{-b\cdot\sin(\log x) - a\cos(\log x)}{x}$$

$$x^2 y_2 + x y_1 = -\left[(a\cos(\log x) + b\sin(\log x)\right]$$

$$= -y$$

If $y = \sqrt{\sin x\sqrt{\sin x\sqrt{\sin x\sqrt{\cdots\cdots\infty}}}}$

$$\frac{dy}{dx} = \frac{f'(x)}{2y-1}$$

$\therefore$ $$\frac{dy}{dx} = \frac{\cos x}{2y-1}$$

$y = \sqrt{\sin x + y} \Rightarrow y^2 = \sin x + y$

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$(2y-1)\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{(2y-1)}$$

If $y = \sqrt{\log x\sqrt{\log x\sqrt{\log x \cdots\infty}}}$

$$\frac{dy}{dx} = \frac{1}{x(2y-1)}$$

If $y = \sqrt{\tan x\sqrt{\tan x\sqrt{\tan x\cdots\infty}}}$

$$\frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$$

If $y = \sqrt{x\sqrt{x\sqrt{x\cdots\cdots\infty}}}$

$$\frac{dy}{dx} = \frac{1}{2y-1}$$

If $\tan^{-1}\left[\frac{(3-x)\sqrt{x}}{1-3x}\right]$ then $\frac{dy}{dx} = ?$ at $x=1$

Whenever the inside function is algebric, trignometric substitutions are required

put $x = \tan^2\theta \Rightarrow$

$$\sqrt{x} = \sqrt{\tan\theta}$$

$$\theta = \tan^{-1}(\sqrt{x})$$

$$y = \tan^{-1}\left[\frac{(3-\tan^2\theta)\cdot\tan\theta}{1-3\tan^2\theta}\right]$$

$$= \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right]$$

$$= \tan^{-1}(\tan 3\theta)$$

$$y = 3\theta$$

$$= 3\tan^{-1}(\sqrt{x})$$

$$\frac{dy}{dx} = 3\cdot\frac{1}{(1+x)}\cdot\frac{1}{2\sqrt{x}}$$

At $x=1$

$$\frac{dy}{dx} = \frac{3}{(1+1)}\cdot\frac{1}{2(1)}$$

$$= \frac{3}{4}.$$

If $y = \tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]$ then $\frac{dy}{dx} = ?$

$$y = \tan^{-1}\left[\frac{\sin(\frac{\pi}{2}-x)}{1+\cos(\frac{\pi}{2}-x)}\right]$$

$$= \tan^{-1}\left[\frac{2\sin(\frac{\pi}{4}-\frac{x}{2})\cdot\cos(\frac{\pi}{4}-\frac{x}{2})}{2\cos^2(\frac{\pi}{4}-\frac{x}{2})}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$$

$$y = \frac{\pi}{4} - \frac{x}{2}$$

$$\frac{dy}{dx} = \frac{-1}{2}$$

If $x = a\cos^3\theta$, $y = a\sin^3\theta$. find $\frac{dy}{dx} =$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(a\sin^3\theta)}{\frac{d}{d\theta}(a\cos^3\theta)}$$

$$= \frac{3a \cdot \sin^2\theta \cdot (\cos\theta)}{-3a\cos^2\theta \cdot (\sin\theta)}$$

$$= -\tan\theta$$

$x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(3\sin\theta - \sin^3\theta)}{\frac{d}{d\theta}(3\cos\theta - \cos^3\theta)}$$

$$= \frac{3\cos\theta - 3\cdot\sin^2\theta\cdot\cos\theta}{-3\sin\theta + 3\cos^2\theta\cdot\sin\theta}$$

$$= \frac{(1-\sin^2\theta)\ 3\cos\theta}{-(1-\cos^2\theta)\cdot 3\sin\theta}$$

$$= \frac{\cos^2\theta \cdot 3\cdot\cos\theta}{-\sin^2\theta \cdot 3\cdot\sin\theta}$$

$$= -\cot^3\theta$$

$x = a(\theta + \sin\theta)$, $\quad y = a(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(a(1-\cos\theta))}{\frac{d}{d\theta}(a(\theta+\sin\theta))}$$

$$= \frac{a[2\sin^2\theta/2]}{a[1+\cos\theta]}$$

$$= \frac{2\cdot\frac{1}{2}\cancel{2}\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{\cancel{2}\cos^2\theta/2}$$

$$= \frac{1}{1}\tan\theta/2$$

Mean value theorem:

Closed interval,

$$x \in [a,b] \Rightarrow a \leq x \leq b$$

Open interval,

$$x \in (a,b) \Rightarrow a < x < b$$

1. Rolle's mean value theorem:

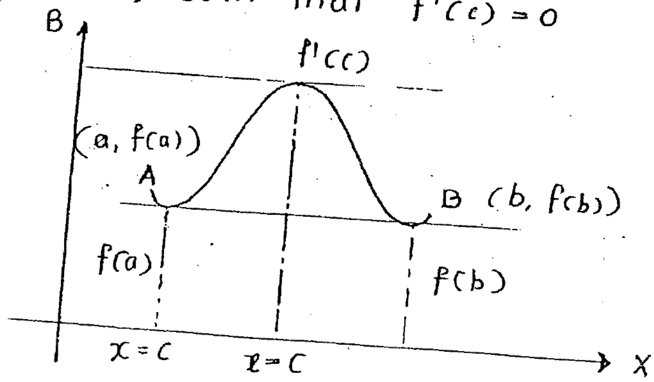
Let $f(x)$ be a function defined such that $f(x)$ is

(i) $f(x)$ is continuous $[a,b]$

(ii) $f(x)$ is differentiable (a,b)

(iii) If $f(a) = f(b)$ the $\exists$ at least (exist) one value of $c \in (a,b)$ such that $f'(c) = 0$

Belongs to



$f'(c)$ slope of tangent at $x = c$

$= \text{slope of } \overline{AB}$

$= \dfrac{y_2 - y_1}{x_2 - x_1}$ (equ^n of straight line)

$= \dfrac{f(b) - f(a)}{b - a}$

$f'(c) = 0$

Find C of Rolle's mean value of function

$f(x) = x^3 - 4x$ in $[-2, 2]$

$f'(x) = 3x^2 - 4$

$|x|$ is continuous but not differentiable at $x =$

Every differentiable function is continuous.

$f(-2) = -8 + 8 = 0$

$f(2) = 8 - 8 = 0$

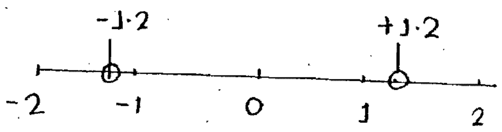
$f'(x) = 3c^2 - 4 = 0$

$3c^2 = 4$

$c^2 = \dfrac{4}{3}$

$c = \pm \dfrac{2}{\sqrt{3}} = \pm 1.2$

Given interval



— lies in the interval

$\therefore$ Answer is $c = \pm \dfrac{2}{\sqrt{3}}$

$$f(x) = \frac{\sin x}{e^x} \quad \text{in} \quad [0, \pi]$$

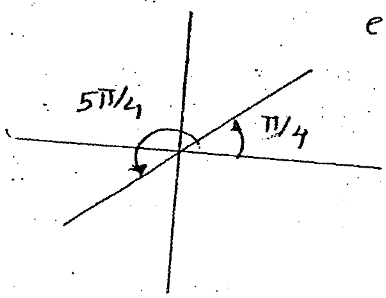
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

$$f'(x) = \frac{e^x \cdot \cos x - e^x \cdot \sin x}{e^{2x}}$$

$$f(0) = \frac{\sin 0}{e^0} = 0$$

$$f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$$

$$\frac{e^x \cdot \cos x - e^x \cdot \sin x}{e^{2x}} = 0$$



$$\cos x = \sin x.$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$\therefore \quad x = \frac{\pi}{4} \quad \because \frac{5\pi}{4}$ doesn't lie in $[0, \pi]$

$$f(x) = \frac{x^2 + ab}{x(a+b)} \quad [a, b], \quad a > 0, \quad b > 0.$$

(a) $\sqrt{ab}$ (b) $-\sqrt{ab}$ (c) $\pm\sqrt{ab}$ (d) none.

$$f(a) = \frac{a^2 + ab}{a^2 + ab} = 1$$

$$f(b) = \frac{b^2 + ba}{b^2 + ab} = 1$$

$$\therefore \quad f'(x) = \frac{1}{(a+b)} \left(\frac{x^2}{x} + \frac{ab}{x} \right)$$

$$f'(x) = \frac{1}{(a+b)} \left[1 + ab \cdot \frac{-1}{x^2} \right] = 0$$

$\because a > 0, b > 0$
roots will be positive

$$1 - \frac{ab}{x^2} = 0$$

$$x^2 = ab$$

$$x = \sqrt{ab}$$

Discuss applicability of Rolle's formula $f(x) = |x|$ in $[-2, 2]$

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$\because$ function $f(x) = |x|$ is continuous function.

$\therefore$ 0 lies in between the interval $[-2, 2]$

$\therefore$ function of $f(x) = |x|$ is not differentiable at $x = 0$

$\therefore$ Rolle's theorem is not applicable.

2. Langrange's Mean value theorem:

Let $f(x)$ be a function defined such that

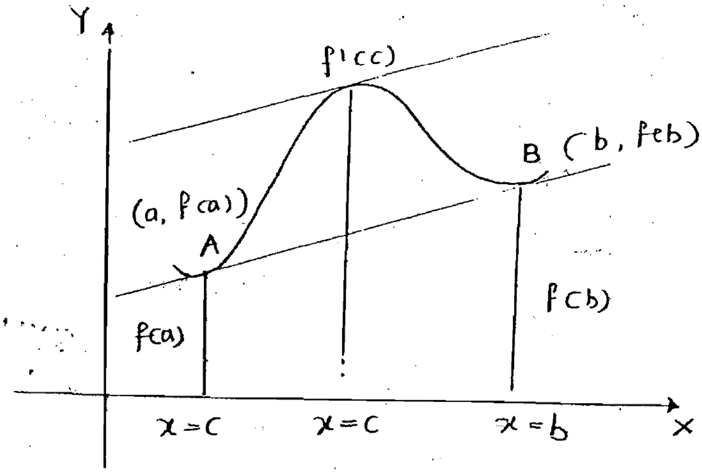
(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable in (a, b)

(iii) $f(a) \neq f(b)$

then $\exists$ at least one value $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



$f'(c) =$ slope of tgt at $x = c$

$=$ slope of AB

$= \frac{y_2 - y_1}{x_2 - x_1}$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Find 'c' of longrange's formula for the function $f(x) = \log x$ in $[1, e]$

(a) $\frac{1}{e-1}$ (b) $e-1$ (c) 1 (d) e

$f(a) = \log 1 = 0$

$f(b) = \log e = 1$

$f'(x) = \frac{1}{x}$

$$\boxed{f'(x) = \frac{1}{c} = \frac{f(b) - f(a)}{b-a}}$$

Rolles

$f(a) = f(b)$

Lagranges

$f(a) \neq f(b)$

$$= \frac{1-0}{e-1}$$

$$= \frac{1}{e-1}$$

$$\boxed{c = e-1}$$

Find 'c' $f(x) = x^3 - 6x^2 + 11x - 6$ at $[0, 4]$

$f'(x) = 3x^2 - 12x + 11$

$f(a) = -6$

$f(b) = 64 - 96 + 44 - 6$

$= 6$

$3c^2 - 12c + 11 = \frac{6 - (-6)}{4 - 0}$

$3c^2 - 12 + 11 = \frac{12}{4}$

$3c^2 - 12c + 8 = 0$

$c = \frac{+12 \pm \sqrt{144 - 96}}{6}$

$= \frac{12 \pm 4\sqrt{3}}{6}$

$= 2 \pm \frac{2}{3}\sqrt{3}$

$= 2 \pm 2/\sqrt{3}$

3. Cauchy's Mean value theorem:

Let $f(x)$ and $g(x)$ be two functions defined such that

(i) $f(x)$ and $g(x)$ are continuous in closed interval $[a,b]$

(ii) $f(x)$ and $g(x)$ are differential in (a,b)

(iii) $g'(x) \neq 0 \ \forall \ x \in (a,b)$ then

at least one value $c \in (a,b)$ such that.

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

. Find c if $f(x)$ is e^x & $g(x)$ is e^{-x} in $[a,b]$

(a) $\frac{ab}{2}$ (b) $\frac{a+b}{2}$ (c) $\sqrt{ab}$ (d) $\frac{2ab}{a+b}$

$$f'(x) = e^x$$

$$g'(x) = -e^{-x}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$-e^{2c} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$= \frac{-\left(\frac{e^a - e^b}{e^a - e^b}\right)}{e^a \cdot e^b}$$

$$e^{2c} = e^{a+b}$$

$$\therefore \quad c = \frac{a+b}{2}$$

$f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$g'(x) = \frac{-1}{2x\sqrt{x}}$$

$$\frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\cdot\sqrt{c}}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\frac{-c}{1} = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}$$

$$\frac{c}{1} = \sqrt{ab} \quad \in \quad (a, b)$$

$f(x) = \sin x, \quad g(x) = \cos x \quad$ in $\left[\overset{a}{0}, \overset{b}{\frac{\pi}{2}}\right]$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

$$\frac{\cos c}{-\sin c} = \frac{\sin\frac{\pi}{2} - \sin 0}{\cos\frac{\pi}{2} - \cos 0}$$

$$-\cot c = \frac{1-0}{0-1}$$

$$\cot c = +1$$

$$\cot c = \cot\frac{\pi}{4}$$

$$c = \frac{\pi}{4}$$

$f(x) = \frac{1}{x^2} \quad g(x) = \frac{1}{x} \quad$ in $[a, b]$

$$\frac{\frac{-2}{x^3}}{\frac{-1}{x^2}} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$$

$$\frac{2}{x} = \frac{(a^2-b^2)/a^2b^2}{(a-b)/ab}$$

$$\frac{2}{c} = \frac{(a+b)(a-b)}{ab\,(a-b)}$$

$$c = \frac{2ab}{(a+b)} \in (a, b)$$

A.M. $= \frac{a+b}{2}$

G.M. $= \sqrt{ab}$

H.M. $= \frac{2ab}{a+b}$

4. Taylor's Mean value theorem :

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Let $f(x)$ be a function defined such that

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f^{n-1}(x)$ is differentiable (a,b)

then there exist at least one value $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + h\cdot f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n!)} f^n(a+\theta h)$$

Significance:

Any non-algebric function can be represented as algebric function in series form.

$\boxed{\text{Taylor's formula}}$:

* $$f(x) = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Maclurin's formula : $a=0$

$$f(x) = f(0) + (x)\cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Q. Find expansion of $f(x) = e^x$ at $x=0$

$f(x) = e^x$ at $x=0$, $f(0) = e^0 = 1$

$f'(x) = e^x \quad = 1$

$f''(x) = e^x \quad = 1$

$f'''(x) = e^x \quad = 1$

$$e^x = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$= 1 + x(1) + \frac{(x)^2}{2} + \frac{x^3}{6}(1) + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$$

$f(x) = \sin x$ at $x=0$

$$\sin x = 0 + x \cdot \cos 0 + \frac{x^2}{2!}(-\sin 0) + \frac{x^3}{3!}(-\cos 0)$$

$$= x + 0 + \frac{x^3}{3!}(-1) + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$f(x) = \cos x$ at $x = 0$

$$\cos x = 1 + x \cdot (-\sin 0) + \frac{x^2}{2!}(-\cos 0) + \frac{x^3}{3!}(\sin 0)$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 - $$

$f(x) = \log(1+x)$

$f(x) = \log(1+x) = 0$ at $x=0$

$f'(x) = \frac{1}{1+x} = 1$

$f''(x) = \frac{-1}{(1+x)^2} = -1$

$f'''(x) = \frac{2}{(1+x)^3} = 2$

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \cdots$$

$$\boxed{= x + - \frac{x^2}{2!} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots} \quad \infty$$

$f(x) = \frac{x}{1+x}$ at $x=0$ (algebric function but not purely algebric)

$f(x) = 1 - \frac{1}{1+x} = 0 \qquad \because 1+x \overline{) x}$

$f'(x) = 0 + \frac{1}{(1+x)^2} = 1$ at $x=0$

$f''(x) = \frac{-2}{(1+x)^3} = -2$ at $x=0$

$f'''(x) = \frac{+6}{(1+x)^4} = +6$ at $x=0$

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$$\therefore \frac{x}{1+x} = 0 + x(1) + \frac{x^2}{2!}(-2) + \frac{x^3}{3!}(0) + \cdots$$

$$= x - x^2 + x^3 - x^4 + \cdots$$

or

$$x(1+x)^{-1} = x\left[1 - x + x^2 + x^3 + x^4 \cdots\right]$$

$$= x - x^2 + x^3 - x^4 + \cdots \infty$$ binomial expansion

Q. Find the coefficient of x^2 in the expansion of $f(x) = \cos^2 x$ about $x = 0$.

$$f(x) = \cos^2 x$$

$$= \cos x \cdot \cos x$$

Expansion of $f(x)$ is

$$\cos^2 x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right.$$

$$\text{coeff of } x^2 = \frac{-1}{2!} - \frac{1}{2!}$$

$$\therefore = \frac{-2}{2!}$$

$$= -1$$

or. Maclurine's formula

$$f(x) = f(0) + x \cdot f'(0) + \boxed{\frac{x^2}{2!} f''(0)} + \cdots$$

$$f(x) = \cos^2 x$$

$$f'(x) = 2\cos x\,(-\sin x)$$

$$= -2\sin x \cdot \cos x$$

$$= -\sin 2x$$

$$f''(x) = -\cos 2x \times 2 = -2 \quad \text{at } x = 0$$

$$f'''(x) = 2 \times 2 \sin 2x$$

$$\therefore \text{coeff of } x^2 = \frac{f''(0)}{2!} = \frac{-2}{2} = -1$$

$f(x) = (x-2)^4$ in the expansion of $f(x) = e^x$ about $x = 2$

(a) $\frac{1}{2!}$ (b) $\frac{1}{4!}$ (c) $\frac{e^4}{4!}$ (d) $\frac{e^2}{4!}$

$\therefore$ coefficient of $(x-2)^4 = \frac{f''''(2)}{4!}$

$$= \frac{e^2}{4!}$$

Find expansion of $f(x) = x^2$ about $x = 1$

(a) $1 - x + x^2$ (b) $1 + x^2$ (c) x^2 (d) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$

The given function is pure algebric thus there will be no expansion

$\therefore$ $f(x) = x^2$ at $x = 1$

Expansions $= x^2$ (itself)

Find the expansion of $f(x) = \sin x$ at $x = \pi$

By Taylor's formula.

$f(x) = \sin x \quad = 0 \quad$ at $x = \pi$

$f'(x) = \cos x \quad = -1 \quad$ — " —

$f''(x) = -\sin x \quad = 0 \quad$ — " —

$f'''(x) = -\cos x \quad = 1 \quad$ — " —

$\therefore$ $\sin x = 0 + (x-\pi)(-1) + \frac{(x-\pi)^2}{2!}(0) + \frac{(x-\pi)^3}{3!}(1) + \cdots$

$$= -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \frac{(x-\pi)^7}{7!} + \cdots$$

* Partial and total derivative →

Tuesday
24th September 2013

Ordinary derivative	Partial derivative
$y = f(x)$	$z = f(x, y)$
$x = g(y)$	$u = f(x, y, z)$

$$\boxed{\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

$$z = x^2 - xy + y^2$$

$$\frac{\partial z}{\partial x} = 2x - y \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \cancel{2x}^{\,0} - y$$

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = -1 \qquad \frac{\partial^2 z}{\partial x \cdot \partial y} = -1$$

$$\frac{\partial z}{\partial y} = -x + 2y$$

$$\frac{\partial z}{\partial y \cdot \partial x} = -1$$

Homogenous function:

A function $f(x, y)$ is said to be homogenous of degree of n in x and y if

$$f(kx, ky) = k^n \cdot f(x, y)$$

e.g. $f(x, y) = x^2 - xy + y^2$

$$f(kx, ky) = k^2x^2 - k^2xy + k^2y^2$$
$$= k^2(x^2 - xy + y^2)$$
$$f(kx, ky) = k^2(x, y)$$

Note:

(i) The product of two homogenous functions is again a homogenous functions.

$$f(x,y) = (x^3+y^3)(x^2-y^2)$$

$$n = 3+2 = 5$$

(ii) If function is in rational form, if both numerator and the denominator are homogenous functions then given function is also homogenous.

e.g.

$$f(x,y) = \frac{x^3+y^3}{x^2-y^2}$$

degree, $n = 3-2 = 1$

Euler's theorem:

If z is a homogenous function of degree n is x and y then

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n \cdot z$$

Note:

This formula is applicable directly if z is algebric function.

e.g. If $z = (x^2+y^2)^{1/3}$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{2}{3} z \qquad \because n = 2 \times \frac{1}{3} = \frac{2}{3}$$

If $z = \dfrac{x'y'}{x'+y'}$

$$x \cdot z_x + y \cdot z_y = 1 \cdot z \qquad n = 2-1 = 1$$

If $z = a_0 \cdot x^n + a_1 \cdot x^{n-1} y + a_2 \cdot x^{n-2} y^2 + \cdots a_n \cdot y^n$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nz$$

Note:

If z is not algebric and let $\phi(z)$ is algebric and homogerious of degree n in x and y the (34)

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{n \cdot \phi(z)}{\phi'(z)}$$

e.g. If $z = \log_e(x^2+y^2)$

$$\therefore \quad e^z = (x^2+y^2)$$

$$\therefore \quad n = 2$$

$$\therefore \quad x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial x} = \frac{2 \cdot e^z}{e^z} \qquad \phi(z) = e^z$$

$$= 2$$

1. $z = \sin^{-1}\left(\frac{x^2+y^2}{x-y}\right)$

$$\sin z = \frac{x^2+y^2}{x-y}$$

$$n = (2-1) = 1$$

$$\therefore \quad x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial x} = 1 \cdot \frac{\sin z}{\cos z}$$

$$= \tan z$$

2. $\therefore z = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\tan z = \frac{x^3+y^3}{x-y}$$

$$n = (3-1) = 2$$

$$\therefore \quad x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial x} = 2 \cdot \frac{\tan z}{\sec^2 z}$$

$$= 2 \cdot \frac{\sin z}{\cos z} \cdot \frac{\cos^2 z}{1}$$

$$= 2 \sin z$$

$$= \sin 2z$$

3. $z = x^2 \cdot \sin^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Though it contains trignometric functions for $\left(\frac{y}{x}\right)$ & $\left(\frac{x}{y}\right)$ the degree is zero $(1-1)=0$ and $(1-0)=0$.

Thus we consider there exist no inverse function.

$$\therefore \quad x\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = 2\cdot z$$

4. $z = (x^3+y^3)\cdot e^{-x/y}$

$n = 3 \qquad \because$ for $e^{-x/y}$, $n = +1-(1)$
$= 0$

Application of Euler's theorem:

If z is homogenous function of degree n in x, y then,

$$x^2\cdot\frac{\partial^2 z}{\partial x^2} + 2xy\cdot\frac{\partial^2 z}{\partial x\cdot\partial y} + y^2\cdot\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

This formula is applicable only when z is applicable

e.g. If $z = (x^2+y^2)^{1/3}$

$$x^2\cdot\frac{\partial^2 z}{\partial x^2} + 2xy\cdot\frac{\partial^2 z}{\partial x\cdot\partial y} + y^2\cdot\frac{\partial^2 z}{\partial y^2} = \frac{2}{3}\left(\frac{2}{3}-1\right)\cdot z$$

$$= \frac{2}{3}\left(\frac{-1}{3}\right)\cdot z.$$

$$= \frac{-2}{9}z$$

If $z = \frac{xy}{x+y}$

$$\therefore \quad n = (1+1)-1 = 1$$

$$\therefore \quad x^2\cdot\frac{\partial^2 z}{\partial x^2} + 2xy\cdot\frac{\partial^2 z}{\partial x\partial y} + y^2\cdot\frac{\partial^2 z}{\partial y^2} = 1(1-1)\cdot z$$

$$= 0$$

If $z = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$\therefore \quad \tan z = \frac{x^3+y^3}{x-y}$

$\therefore \quad x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = 2\cdot\frac{\tan z}{\tan^2 z}$

$= \sin 2z$

$\therefore \quad x\cdot\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}(1) + y\cdot\frac{\partial^2 z}{\partial x\cdot\partial y} = 2\cos 2z\cdot\frac{\partial z}{\partial x}$

$\therefore \quad x\cdot\frac{\partial^2 z}{\partial x^2} + y\cdot\frac{\partial^2 z}{\partial x\cdot\partial y} = (-1+2\cos 2z)\cdot\frac{\partial z}{\partial x}$

$\therefore \quad x^2\cdot\frac{\partial^2 z}{\partial x^2} + xy\cdot\frac{\partial^2 z}{\partial x\cdot\partial y} = (2\cos 2z-1)\cdot x\cdot\frac{\partial z}{\partial x}$ —— (i)

Similarly

$y^2\cdot\frac{\partial^2 z}{\partial y^2} + xy\cdot\frac{\partial^2 z}{\partial x\,\partial y} = (2\cos 2z-1)\cdot y\cdot\frac{\partial z}{\partial y}$ —— (ii)

Adding —(i) & —(ii)

$\therefore \quad x^2\frac{\partial^2 z}{\partial x^2} + 2xy\cdot\frac{\partial^2 z}{\partial x\cdot\partial y} + y^2\cdot\frac{\partial^2 z}{\partial x^2} = (2\cos 2z - 1)\cdot\sin 2z$

$= \sin 4z - \sin 2z$

* Explicit function:

Any function which we can express in the form of either $y = f(x)$ or $x = g(y)$ etc. is called an explicit functi

e.g. If

$y = ax^2+bx+c$

$\frac{dy}{dx} = 2ax+b$

$\frac{d^2y}{dx^2} = 2a$

$\frac{d^3y}{dx^3} = 0$

Implicit function :

Any function which is not explicit is, said to be an implicit function.

e.g.. $ax^2 + 2hxy + by^2 = 0$

$$\therefore a \cdot 2x + 2h\left[x \cdot \frac{dy}{dx} + y(1)\right] + 2by \cdot \frac{dy}{dx} = 0$$

$$(2h+by) \cdot \frac{dy}{dx} = -ax + hy$$

$$\boxed{\frac{dy}{dx} = \frac{-ax+hy}{(2h+by)}}$$

$$\frac{d^2y}{dx^2} = \text{------} \quad \text{(complicated \& lengthy)}$$

Partial derivatives are easier & simpler than ordinary derivatives becaus we treat y as constant while differentiating given function w.r.t. x & x as constant while differentiating given function w.r.t. y.

$$z = f(x, y)$$

Remember

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2},$$

$$s = \frac{\partial^2 z}{\partial x \cdot \partial y} \quad t = \frac{\partial^2 z}{\partial y^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-[q^2r - 2pqs + p^2t]}{q^3}$$

(for implicit function)

-** $ax^2 + 2hxy + by^2 = 1$ (particular formula)

$$* \quad \boxed{\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx+by)^3}}$$

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1. $2x^2 + 4xy + 3y^2 = 1$

$a = 2, \quad b = 3, \quad h = \frac{4}{2} = 2$

$$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx+by)^3}$$

$$= \frac{4-6}{(2x+3y)^3}$$

$$= \frac{-2}{(2x+3y)^3}$$

$2x^2 + 4xy + 3y^2 = 1$

$ax^2 + 2hxy + by^2 = 1$

$a = 2 \quad \cancel{2hxy = 4xy}$

$h = 2$

$b = 3$

2. $x^2 + xy + y^2 = 1$

$a = 1; \quad b = 1, \quad h = \frac{1}{2}$

$$\frac{d^2y}{dx^2} = \frac{1/4 - 1}{(\frac{1}{2}x + y)^3}$$

$$= \frac{-3/4}{\left(\frac{x+y}{2}\right)^3}$$

$$= \frac{-2 \times 3}{(x+y)^3}$$

$$= \frac{-6}{(x+y)^3}$$

If given function is in the form $\quad f = g = 0 \quad c = 0$

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = \text{const}$

then

$$\boxed{\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}}$$

— (standard formula)

(Applicable for all forms above)

3)

$$2x^2 + 4xy + 3y^2 + 2x + 4y + 1 = 0$$

$a = 2$, $b = 3$, $c = 1$, $h = \frac{4}{2} = 2$, $g = \frac{2}{2} = 1$, $f = \frac{4}{2} = 2$

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$$

$$= \frac{6 + 8 - 8 - 3 - 4}{(2x + 3y + 2)^3}$$

$$= \frac{-1}{(2x+3y+2)^3}$$

1. $y^2 - 5x + 4x^2 = 8$ find $\frac{d^2y}{dx^2} = ?$

$$4x^2 - 5x - y^2 - 8 = 0$$

$\therefore$ $a = 4$, $b = -1$, $h = 0$, $g = \left(\frac{-5}{2}\right)$, $f = 0$, $c = -8$.

$$\frac{d^2y}{dx^2} = \frac{32 + 0 - 0 + (-1)\frac{25}{4} - 0}{(0 - 1\cdot y + 0)^3}$$

$$= \frac{32 - \frac{25}{4}}{-y^3}$$

$$= \frac{-153}{4y^3}$$

2. $y^2 - 5x + 4x^2 = 8$ find $\frac{d^2y}{dx^2}$

$$y^2 - 5x + 4x^2 = 8 \quad \text{(explicit function)}$$

$a = 4$, $b = 1$, $g = \left(\frac{-5}{2}\right)$, $c = 0$, $h = 0$, $f = 0$

$$\frac{d^2y}{dx^2} = \frac{0 + 0 - 0 - \frac{25}{4} - 0}{(0 + y + 0)^3}$$

$$= \frac{-25}{4y^3}$$

Composite function →

Any function in function is said to be composite function.

If z is function in x and y and x, y are function in 't' then z is called a composite function in 't'.

Total derivative of composite function:

If $z = f(x, y)$, $x = g(t)$ & $y = h(t)$ then total derivative of z w.r.t. t is denoted by $\frac{dz}{dt}$ and is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

If $u = f(x, y, z)$ & (x, y, z) are function of t

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

If $z = e^{x} \cdot \sin y$. $x = \log t$, $y = t^2$

$$\frac{\partial z}{\partial x} = e^{x} \cdot \sin y. \qquad \frac{dx}{dt} = \frac{1}{t}$$

$$\frac{\partial z}{\partial y} = e^{x} \cdot \cos y. \qquad \frac{dy}{dt} = 2t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (e^{x} \sin y) \frac{1}{t} + (e^{x} \cos y) 2t$$

$$= \frac{e^{x}}{t} \cdot [\sin y + 2t^2 \cos y]$$

If $u = x^2 + y^2 + z^2$

& $x = e^{2t}$, $y = e^{2t} \cos 3t$, $z = e^{2t} \sin 3t$.

$$u = (e^{2t})^2 + (e^{2t} \cos 3t)^2 + (e^{2t} \sin 3t)^2$$

$$= e^{4t} (1 + \cos^2 3t + \sin^2 3t)$$

$$= e^{4t} (1+1)$$

$$= 2e^{4t}$$

$$\frac{du}{dt} = 8\cancel{t} \cdot e^{4t}$$

If $u = x^2 - y^2$. $x = e^t \cos t$. $y = e^t \sin t$

$$u = (e^t \cdot \cos t)^2 - (e^t \cdot \sin t)^2$$

$$u = e^{2t} [\cos^2 t - \sin^2 t]$$

$$\frac{du}{dt} = \frac{d e^{2t}}{dt} (\cos 2t)$$

$$= e^{2t} \cdot \frac{d}{dt} \cos 2t + \cos 2t \cdot \frac{d}{dt} e^{2t}$$

$$= -2 \sin 2t \cdot e^{2t} + 2 \cdot e^{2t} \cdot \cos 2t$$

$$\left(\frac{du}{dt}\right)_{t=0} = 0 + 2$$

$$= 2$$

If $u = x^3 \cdot y \cdot e^z$

$x = t$, $y = t^2$, $z = \log t$

$$u = t^3 \cdot t^2 \cdot e^{\log t}$$

$$= t^5 \cdot t$$

$$= t^6$$

$$\left(\frac{du}{dt}\right)_{t=2} = 6 \cdot t^5 = 6 \times (2)^5 = 6 \times 32 = 192$$

If $u = e^{xyz}$. $\frac{\partial^3 u}{\partial x \cdot \partial y \cdot \partial z}$.

(a) $e^{xyz}(1+x+y+z)$ (b) $e^{xyz}(x^2+y^2+z^2)$

(c) $e^{xyz}(1+3xyz+x^2y^2z^2)$ (d) None

$$u = e^{xyz}$$

$$\frac{d}{dx}(e^{ax}) = a \cdot e^{ax}$$

$$\frac{\partial u}{\partial x} = yz \cdot e^{xyz}$$

$$\frac{\partial u}{\partial x \cdot \partial y} = z\left[y \cdot xz \cdot e^{xyz} + e^{xyz} \cdot (1)\right]$$

$$\frac{\partial u}{\partial x \cdot \partial y} = xyz^2 \cdot e^{xyz} + z \cdot e^{xyz}$$

$$\frac{\partial u}{\partial x \cdot \partial y \cdot \partial z} = xy\left[z^2 \cdot xy \cdot e^{xyz} + e^{xyz} \cdot 2z\right] + \left[z \cdot e^{xyz} \cdot xy + e^{xy}\right]$$

$$= x^2y^2z^2 \cdot e^{xyz} + 2xyz \cdot e^{xyz} + z \cdot xy \cdot e^{xyz} + e^{xyz}$$

$$= e^{xyz}\left[x^2y^2z^2 + 3xyz + 1\right]$$

If $f(x) = y^x$ find $\frac{\partial^2 f}{\partial x \cdot \partial y}$ at $x = 2$, $y = 1$

$$f = y^x$$

$$\frac{\partial f}{\partial y} = x \cdot y^{x-1}$$

$$= x \cdot (1)^{x-1} \quad \text{at } y = 1$$

$$\frac{\partial f}{\partial x \cdot \partial y} = 1 .$$

If $z = f(x+ay) + \phi(x-ay)$ then which of the following is true.

(a) $\frac{\partial^2 z}{\partial x^2} = a^2 \cdot \frac{\partial^2 z}{\partial y^2}$ (b) $\frac{\partial^2 z}{\partial x^2} = -a^2 \cdot \frac{\partial^2 z}{\partial y^2}$

(b) $\frac{\partial^2 z}{\partial y^2} = a^2 \cdot \frac{\partial^2 z}{\partial x^2}$ · (b) $\frac{\partial^2 z}{\partial y^2} = -a^2 \frac{\partial^2 z}{\partial x^2}$

$$Z = f(x+ay) + \phi(x-ay)$$

$$\frac{\partial z}{\partial x} = f'(x+ay)(1) + \phi'(x-ay).(1)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay).(1) + \phi'(x-ay).(1)$$

$$\frac{\partial z}{\partial y} = f'(x+ay).a + \phi'(x-ay)(-a)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+ay).a^2 + \phi''(x-ay)\, a^2$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 . \frac{\partial^2 z}{\partial x^2}$$

If $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial^2 y} = ?$$

$$\frac{\partial^2 \theta}{\partial x \cdot \partial y} = ?$$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$

$$\frac{y}{x} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2} \quad \text{--- (i)}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{-1}{(x^2+y^2)^2} \cdot (-y) \cdot 2x$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

similarly $\frac{\partial^2\theta}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$

$$\therefore \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

from —— (i)

$$\frac{\partial\theta}{\partial x.\partial y} = \frac{(x^2+y^2).(-1) - (-y).(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

$$= \frac{-(x^2-y^2)}{(x^2+y^2)^2}$$

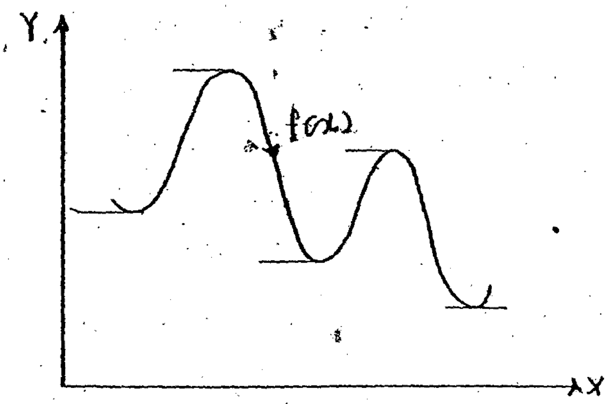
$$= \frac{-(r^2\cos^2\theta - r^2\sin^2\theta)}{(r^2\cos^2\theta + r^2\sin^2\theta)^2}$$

$$= \frac{-r^2(\cos 2\theta)}{(r^2(1)^2)}$$

$$= \frac{-r^2 \cos 2\theta}{r^4}$$

$$= \frac{-\cos 2\theta}{r^2}$$

Maxima and Minima :



Maximum value :

If a continuous function $f(x)$ increases to a certain value and then decreases, that value is called maximum value of a function.

Minimum value :

If a continuous function $f(x)$ decreases to a certain value and then increases that value is called minimum value of the function.

(i) The maxima or minima occurs alternatively.

(ii) A function can have several maximum and several minimum values.

(iii) The minimum value may be greater than maximum value.

(iv) The least minimum value is called global minimum or universal minimum and highest maximum value is called a global maximum or universal maximum.

For a function $f(x)$, for maxima or minima,

$$f'(x) = 0$$

$$x = \alpha, \beta, \gamma \ldots$$

For $x=\alpha$, if $f''(\alpha) > 0$ the $f(x)$ is minimum at α.

Minimum value $= f(\alpha)$ 40

For $x=\beta$, if $f''(\beta) < 0$, the $f(x)$ is maximum at β.

Maximum value $= f(\beta)$

For $x=\gamma$, if $f''(\gamma) = 0$. the $f(x)$ is stationary.

if $f'''(\gamma) \neq 0$. then γ is said to be inflection poi

1. $f(x) = x^x$

$$f'(x) = x^x(1+\log x)$$

$$f''(x) = x^x\left(\frac{1}{x}\right) + (1+\log x)\cdot x^x (1+\log x)$$

$$= \frac{x^x}{x} + (1+\log x)^2 \cdot x^x$$

For maxima or minima, x^x $x^x(1+\log x)$

$$f'(x) = 0$$

$$x^x(1+\log x) = 0$$

$$x^x \neq 0 \quad \text{for any value of } x$$

$\therefore$

$$(1+\log x) = 0$$

$$\log x = -1$$

$$x = e^{-1} = \frac{1}{e}$$

$2^2 = 4 > 0$

$$f''\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e - 1} + (1+\log_e \tfrac{1}{e})^2.$$

$2^{-2} = \frac{1}{4} > 0$

$$= \left(\frac{1}{e}\right)^{1/e-1} + 0 \quad > 0$$

+ve → min.

–ve → MAX

$f(x)$ is minimum at $x = \frac{1}{e}$

$$\text{Minimum value} = f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e}$$

$$= (e^{-1})^{1/e}$$

$$= e^{-1/e}$$

$f(x) = x^{1/x}$ has its maxima at $x = e$.-

Max. value $\Rightarrow e^{1/e}$

Short trick

Function	Minimum value	Maximum value	at
x^x	$e^{-1/e}$	-	$x = 1/e$
$x^{1/x}$	-	$e^{1/e}$	$x = e$

$f(x) = \frac{\log x}{x}$

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2}$$

$$= \frac{1 - \log x}{x^2}$$

$$f''(x) = \frac{x^2\left(\frac{1}{x}\right) - 2x\cdot(1 - \log x)}{x^4}$$

$$= \frac{x\,(1 - 2(1 - \log x))}{x^4}$$

$$= \frac{1 - 2(1 - \log x)}{x^3}$$

For max. or minima,

$$f'(x) = 0$$

$$\frac{1 - \log x}{x^2} = 0$$

$$\log x = 1$$

$$x = e$$

$$f''(e) = \frac{1 - 2(1 - \log e)}{e^3}$$

$$= \frac{-1}{e^3} < 0$$

Max. value $= \frac{\log e}{e} = \frac{1}{e}$

$f(x) = 3x^2 - 6x$

$$f'(x) = 6x - 6$$

$$f''(x) = 6$$

$$6x - 6 = 0$$

$$x = 1$$

$$f(1) = 3 - 6$$

$$= -3 \quad < 0$$

Maximum value $= -3$

$f(x) = x^3 + \frac{3}{x^2}$

$$f'(x) = 3x^2 - \frac{3}{x^2} = 0$$

$$3x^2 = \frac{3}{x^2}$$

$$x^4 = 1$$

$$(x^2-1)(x^2+1) = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

$$f''(x) = 6x + \frac{6}{x^3}$$

At $x = +1$

$$f''(1) = 6(1) + \frac{6}{(1)}$$

$$= 12 \quad > 0$$ Minimum at $x = 1$

At $x = -1$

$$f''(1) = 6(-1) + \frac{6}{(-1)}$$

$$= -6 - 6$$

$$= -12 < 0$$ Maximum value at $x = -1$

$f(x) = x^2 \cdot e^{-x}$

$$f'(x) = x^2 \cdot e^{-x} \cdot (-1) + e^{-x} \cdot 2x.$$

$$f''(x) = \frac{d}{dx}\left[e^{-x}(-x^2 + 2x) \right]$$

$$= 2 \cdot e^{-x} - 2 \cdot x e^{-x} - 2x \cdot e^{-x} + x^2 e^{-x}$$

$$= e^{-x}(2 - 4x + x^2)$$

$$f'(x) = 0$$

$$e^{-x}(2x - x^2) = 0$$

$$x(2 - x) = 0$$

$$x = 0, \quad \text{or} \quad x = 2$$

At $x = 0$ $\qquad f''(0) = e^{-0}(2 - 0 + 0)$

$$= 2 > 0 \qquad \text{Minimum at } x = 0$$

At $x = 2$ $\qquad f''(2) = e^{-2}(2 - 8 + 4)$

$$= \frac{-2}{e^2}$$

$f(x) = y = x^{5/2}$

$$\frac{dy}{dx} = \frac{5}{2} x^{3/2}$$

$$\frac{d^2y}{dx^2} = \frac{5}{2} \times \frac{3}{2} \cdot x^{1/2}$$

$$\frac{d^3y}{dx^3} = \frac{15}{8} \frac{1}{\sqrt{x}}$$

At $x = 0$,

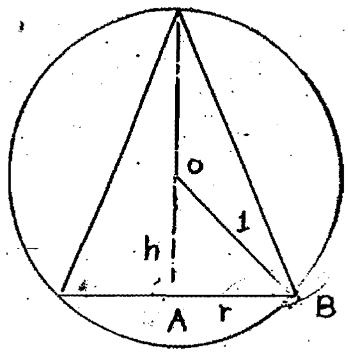
$$\frac{dy}{dx} = 0 \Rightarrow x = 0$$

At $x = 0$,

$$\frac{d^3y}{dx^3} = \frac{15}{8} \frac{1}{\sqrt{0}} = \infty \neq 0$$

inflection point at $(0,0)$

Find the height of cone of maximum valume that can be incribed in the sphere of radius 1 unit. (42)



$$OA^2 + AB^2 = OB^2$$

$$h^2 + r^2 = 1^2$$

$$r^2 = 1^2 - h^2$$

volume of cone, $V = \frac{1}{3}\pi r^2 h$

$$= \frac{\pi}{3}(1-h^2)(h+1)$$

height of cone $= h+1$

$$= \frac{\pi}{3}(1+h-h^2-h^3)$$

$$\therefore \frac{dV}{dh} = \frac{\pi}{3}(0+1-2h-3h^2)$$

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(-2-6h)$$

for maxima.

$$\frac{dV}{dh} = 0$$

$$\frac{\pi}{3}(1-2h-3h^2) = 0$$

$$3h^2+2h-1 = 0$$

$$(3h-1)\cancel{(h+1)} = 0$$

(height cannot be -ve)

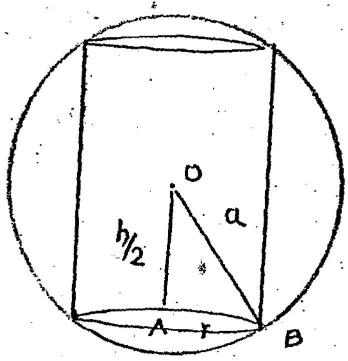
$$h = \frac{1}{3}$$

height of the cone $= 1 + h$

$$= 1 + \frac{1}{3}$$

$$= \frac{4}{3}$$

Q Find the height of cylinder of max. volume that can be incribed in the sphere of radius a units.



$$OA^2 + OB^2 = a^2$$

$$\left(\frac{h}{2}\right)^2 + r^2 = a^2$$

$$r^2 = a^2 - \frac{h^2}{4}$$

volume of cylinder,

$$= \pi r^2 . h$$

$$= \pi \left(a^2 - \frac{h^2}{4}\right). h$$

$$= \pi \left(a^2 h - \frac{h^3}{4}\right)$$

$$\frac{dv}{dh} = \pi \left(a^2 - \frac{3h^2}{4}\right)$$

For maximium volume

$$\frac{dv}{dh} = 0$$

$$a^2 - \frac{3h^2}{4} = 0$$

$$a = \frac{\sqrt{3}h}{2}$$

$$h = \frac{2a}{\sqrt{3}} \quad *$$

For $z = f(x,y)$

$p = \frac{\partial z}{\partial x}$ $\quad q = \frac{\partial z}{\partial y}$ $\quad s = \frac{\partial^2 z}{\partial x \cdot \partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$ $\quad t = \frac{\partial^2 z}{\partial y^2}$

For a maxima or minima. Let $p=0$ and $q=0$ since p and q are functions in x and y, so we have two equations in x and y. Solve these equations and get relation between x and y.
e.g. ($x=y$, $x=-y$, $x=2y$ etc.)

Put this relation either in $p=0$ or in $q=0$, then the equation transformed to either purely in x or in y. Solve the equation and get the roots

$$x = x_0, x_1, x_2 ----$$
$$y = y_0, y_1, y_2 ---$$

(x_0, y_0), (x_1, y_1), (x_2, y_2) ---- are called stationary points or critical points (points at which maxima, minima can be decided

Take point (x_0, y_0) and find values of r, s and t.

Case I → If $rt - s^2 > 0$ and $r > 0$, then function attains its minimum.

Minimum value $= f(x_0, y_0)$

Case II: If $rt - s^2 > 0$ and $r < 0$ then function attains its maximum.

Maximum value $= f(x_1, y_1)$

Case III: If $rt - s^2 < 0$ then the function has no extreme value (neither minima nor maxima)

Case IV: If $rt - s^2 = 0$ the case is doubtful and needs further investigation.

Q. The function $f(x) = x^2+y^2+6x+12$ has

(a) maximum at $(3,0)$.
(b) min at $(-3,0)$
(c) no extreme at $(-3,0)$
(d) none

$$p = \frac{\partial f}{\partial x} = 2x+6 \qquad q = \frac{\partial f}{\partial y} = 2y$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2 \qquad s = \frac{\partial^2 f}{\partial x \cdot \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial^2 y} = 2.$$

$\therefore$

$$rt - s^2 = (2\times 2) - 0 = 4 > 0$$

$$r = 2 > 2$$

$F(x)$ has minimum value at $(-3,0)$

Q. $f(x,y) = 1 - x^2 - y^2$ has

(a) ~~Minimum at $(0,0)$~~
(b) max. at $(0,0)$
(c) no extreme at $(0,0)$
(d) none.

$$p = \frac{\partial f}{\partial x} = -2x \qquad q = \frac{\partial f}{\partial y} = -2y$$

$$r = \frac{\partial^2 f}{\partial x^2} = -2 \qquad s = \frac{\partial^2 f}{\partial y \cdot \partial x} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2$$

$$rt - s^2 = (-2)(-2) - 0$$

$$= 4 > 0$$

$$r = -2 < 0$$

$f(x)$ has max. value at $(0,0)$

Q. $f(x,y) = x^3+y^3-3xy$

(a) Min at $(1,-1)$
(b) Max at $(1,-1)$
(c) Min at $(1,1)$
(d) Max. at $(1,1)$

$$p = \frac{\partial f}{\partial x} = 3x^2 - 3y \qquad q = \frac{\partial f}{\partial y} = 3y^2 - 3x$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x \qquad s = \frac{\partial f}{\partial x \cdot \partial y} = -3$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y - 6$$

$$rt - s^2 =$$

for $(1,-1)$, $\quad p \neq 0, \quad q \neq 0$

for $(1,1)$ $\quad p = 3(1)^2 - 3(1) = 0 \quad\quad q = 3(1)^2 - 3(1) = 0$

$$rt - s^2 = 6 \times 6 - 9 > 0$$

$$r = 6 > 0.$$

Function has minima at $(1,1)$

Q. $f(x,y) = x^2 + y^2 + xy + x - 4y + 5$ has

(a) Min. at $(-2,3)$ (b) Max. at $(-2,3)$

(c) Min at $(2,-3)$ (d) Max. at $(2,-3)$

$$p = \frac{\partial f}{\partial x} = 2x + y + 1 \quad\quad q = \frac{\partial f}{\partial y} = 2y + x - 4$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2 \quad\quad s = \frac{\partial f}{\partial y \cdot \partial x} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

At $(-2,3)$, $\quad p = 2(-2) + 3 + 1 = 0$

$$q = 2(3) + (-2) - 4 = 0$$

$$(rt - s^2) = 4 - 1 > 0$$

$$r > 0$$

Minimum at $(-2,3)$

Q. $f(x,y) = 4x^2 + 6y^2 + 8 - 8x - 4y$ has (value = ?)

(a) Max at $-8/3$

(b) Min at $8/3$

(c) Max at $10/3$

(d) Max $-10/3$

$$p = \frac{\partial f}{\partial x} = 8x - 8 \qquad q = \frac{\partial f}{\partial y} = 12y - 4$$

$$r = \frac{\partial f}{\partial x^2} = 8 \qquad s = \frac{\partial f}{\partial y \cdot \partial x} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 12$$

$$p = 0 \qquad q = 0$$
$$8x - 8 = 0 \qquad 12y - 4 = 0$$
$$x = 1 \qquad y = 1/3$$

At $(x = 1, y = 1/3)$

$$f(1, 1/3) = 4(1)^2 + 6(1/3)^2 + 8 - 8(1) - 4(1/3)$$
$$= \frac{10}{3}$$

$$rt - s^2 = 96 - 0 > 0$$
$$r = 8 > 0$$

Minimum value $= \frac{10}{3}$

Q. $f(x,y) = \sin x + \sin y + \sin(x+y)$ has

(a) Max $\frac{\sqrt{3}}{2}$

(b) Min $\frac{\sqrt{3}}{2}$

(c) Max. $\frac{3\sqrt{3}}{2}$

(d) Min $\frac{3\sqrt{3}}{2}$

$$p = \frac{\partial f}{\partial x} = \cos x + \cos(x+y). \qquad q = \frac{\partial f}{\partial y} = \cos y + \cos(x+y)$$

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y) \qquad s = \frac{\partial f}{\partial y \cdot \partial x} = -\sin(x+y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

$$p = 0 \qquad q = 0$$
$$\cos x + \cos(x+y) = 0 \quad \text{— (i)} \qquad \cos y + \cos(x+y) = 0 \quad \text{— (ii)}$$

solving — (i) & — (ii)

$$\cos - \cos y = 0$$
$$x = y$$

from —— (i)

$$\cos x + \cos 2x = 0$$

$$\cos x \times 2 = -\cos 2x$$

$$\cos x \times 2 = \cos(\pi - x)$$

$$2x = \pi - x$$

$$x = \frac{\pi}{3}$$

$$y = \frac{\pi}{3}$$

$$r = -\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = -\sqrt{3}$$

$$s = -\sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$t = -\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = -\sqrt{3}$$

$$rt - s^2 = 3 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= 3 - \frac{3}{4} = > 0$$

$$r = -\sqrt{3} < 0$$

Max: at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$\text{Max. value} = \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \cos\left(90 + \frac{\pi}{6}\right)$$

$$= \frac{3\sqrt{3}}{2}$$

Definite and improper integrals

$$\int_a^b f(x)\cdot dx = \int_a^c f(x)\cdot dx + \int_c^b f(x)\cdot dx.$$

$$\int_a^b f(x)\cdot dx = \int_a^b (a+b-x)\cdot dx$$

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\, dx = \frac{b-a}{2}$$

$$\int_0^a f(x)\cdot dx = \int_0^a f(a-x)\cdot dx.$$

$$\int_{-a}^{a} f(x)\cdot dx = 2\int_0^a f(x)\cdot dx, \text{ if } f(x) \text{ is even, i.e } f(-x)=f(x)$$

$$= 0 \quad \text{if } f(x) \text{ is odd i.e. } f(-x)=-f(x)$$

$$\int_0^{2a} f(x)\cdot dx = 2\int_0^a f(x)\cdot dx, \text{ if } f(2a-x) = f(x)$$

$$= 0 \quad \text{if } f(2a-x) = -f(x).$$

$$\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x)+f(\cos x)}\, dx = \int_0^{\pi/2} \frac{f(\tan x)}{f(\tan x)+f(\cot x)} = \int_0^{\pi/2} \frac{f(\sec x)}{f(\sec x)+f(\operatorname{cosec} x)}\, dx$$

$$\int_0^{\pi/2} \sin^n x\cdot dx = \int_0^{\pi/2} \cos^n x\cdot dx = \frac{(n-1)(n-3)(n-5)}{n(n-2)(n-4)} \cdots \times k.$$

$k=1$ if n is odd

$k=\pi/2$ if n is even.

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \cdot dx = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} k. \quad (91)$$

$k = 1$ if either m or n or both are odd.

$k = \frac{\pi}{2}$ if both m and n are even.

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{1}{x} \cdot dx = \log x$$

$$\int \frac{1}{x^2} \cdot dx = \frac{-1}{x}$$

$$\int \frac{1}{x^3} \cdot dx = \frac{-1}{2x^2}$$

$$\int e^{ax} \cdot dx = \frac{e^{ax}}{a}$$

$$\int \sin x \cdot dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\int \sec^2 x \cdot dx = \tan x$$

$$\int \sec x \cdot \tan x \, dx = \sec x.$$

$$\int \tan x \cdot dx = \log(\sec x)$$

$$\int \cot x \cdot dx = \log(\sin x)$$

$$\int \sinh x \cdot dx = \cosh x$$

$$\int \cosh x \cdot dx = \sinh x.$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} \, dx = \sinh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \cosh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right)$$

$$\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right)$$

$$\int \frac{f'(x)}{f(x)} \cdot dx = \log f(x)$$

$$\int \frac{f'(x)}{\sqrt{f(x)}} \, dx = 2\sqrt{f(x)}$$

$$\int \frac{f'(x)}{1 + (f(x))^2} \, dx = \tan^{-1}(f(x))$$

$$\int f(x)^n \cdot f'(x) \cdot dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$\int e^{f(x)} \cdot f'(x) \cdot dx = e^{f(x)}$$

$$\int e^x [f(x) + f'(x)] = e^x \cdot f(x).$$

$$\int f(x) \cdot g(x) \cdot dx = f(x) \int g(x) \cdot dx - \int \left[f'(x) \cdot \int g(x) \cdot dx \right] dx.$$

To chose $f(x)$, remember the world.

I L A T E

I → Inverse

L → Log

A → Algebric

T → trignometric

E → exponential

$$\int e^{ax} \cdot \sin bx \cdot dx = \frac{e^{ax}}{a^2+b^2} \left[a \sin bx \mp b \cos bx \right]$$

$$\int e^{ax} \cdot \cos bx \cdot dx = \frac{e^{ax}}{a^2+b^2} \left[a \cos bx + b \sin bx \right]$$

Examples:

shortcut 1 :

$$\int x \cdot e^x \, dx = e^x (x - 1)$$

$$\int x^2 \cdot e^x \, dx = e^x (x^2 - 2x + 2) \quad d/dx$$

$$\int x^3 \cdot e^x \, dx = e^x (x^3 - 3x^2 + 6x - 6) \quad \frac{d}{dx}$$

$$\int (x^2+x) \cdot e^x \cdot dx = e^x \left[(x^2+x) - (2x+1) + 2 \right]$$

shortcut 2:

$$\int x \cdot e^{ax} \, dx = e^{ax} \left[\frac{x}{a} - \frac{1}{a^2} \right]$$

$$\int x^2 \cdot e^{ax} \cdot dx = e^{ax} \left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right]$$

shortcut 3:

$$\int \log x \cdot dx = x (\log x - 1)$$

$$\int (\log x)^2 \cdot dx = x \left[(\log x)^2 - 2 (\log x) + 2 \right]$$

$$\int (\log x)^3 \cdot dx = x \left[(\log x)^3 - 3 (\log x)^2 + 6 (\log x) + 6 \right]$$

1. $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} \, dx$

$$I = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= \left[\sin^{-1}(x) \right]_0^{1/2}$$

$$= \sin^{-1}(1/2) - \sin(0)$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

2. $\int_{\log 2}^{\log 3} \frac{e^x}{1+e^x} \cdot dx$

While evaluating integration of rational function always take derivative of denominator.

$$I = \left[\log (1 + e^x) \right]_{\log 2}^{\log 3}$$

$$= \log (1 + e^{\log 3}) - \log (1 + e^{\log 2})$$

$$= \log 4 - \log 3 = \log \left(\frac{4}{3} \right)$$

3. $\int_0^1 \frac{1}{e^x + e^{-x}} \cdot dx$.

$$I = \int_0^1 \frac{e^x}{e^{2x}+1} \, dx$$

$$= \frac{1}{2} \cdot \int_0^1 \frac{2 \cdot e^x}{e^{2x}+1} \, dx \quad \text{(WRONG)}$$

$$= \int_0^1 \frac{e^x}{1+(e^x)^2} \, dx$$

$$= \left[\tan^{-1}(e^x)\right]_0^1$$

$$= \tan^{-1}(e^1) - \tan^{-1}(e^0)$$

$$= \tan^{-1} e - \tan^{-1}(1)$$

$$= \tan^{-1} e - \frac{\pi}{4}.$$

4. $\int_0^1 x \cdot e^x \cdot dx$

$$I = \int_0^1 x \, e^x \cdot dx$$

$$= \left[e^x (x-1)\right]_0^1$$

$$= \left[x \cdot e^x - e^x\right]_0^1$$

$$= e^1 - e^1 - (0 - e^0)$$

$$= 1$$

5. $\int_2^7 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} \cdot dx$

$$I = \int_2^7 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} \, dx.$$

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} = \frac{b-a}{2}$$

$$J = \frac{7-2}{2}$$

$$= \frac{5}{2}$$

6. $\int_0^2 |1-x| \cdot dx$

$$J = \int_0^2 |1-x| \cdot dx$$

Check the modulus, for given interval i.e. (0, 2).

If the function totally positive or totally negative then don't split; if it is partially positive and partially negative then split the function.

$$J = \int_0^1 |1-x| \cdot dx + \int_1^2 |1-x| \cdot dx$$

$\because |x| = x$ if x is +v

$|x| = -x$ if x is -v

$$= \int_0^1 (1-x) \cdot dx + \int_1^2 -(1-x) \cdot dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^1 - \left[x - \frac{x^2}{2}\right]_1^2$$

$$= \left[1 - \frac{1}{2}\right] - \left[2 - \frac{4}{2} - 1 + \frac{1}{2}\right]$$

$$= \frac{+1}{2} - \left[\frac{-2}{4}\right]$$

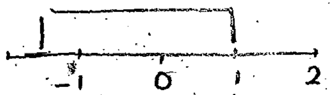
$$= 1$$

$\int_{-1.5} [x+1] \cdot dx$

step x : $[x]$ -

In step function,

If integer lies between given limits, split the function



$$[x+1] =$$

for $-1.5 < x < -1$.

Take, -1.3, $\Rightarrow$ $(-1.3+1) = (-0.3)$

(nearest smaller integer is -1)

for $-1 < x < 0$

Take -0.5 $\Rightarrow$ $(-0.5+1) = 0.5$

(nearest smaller integer to 0.5 is 0)

for $0 < x < 1$

Take, 0.5 $\Rightarrow$ $(0.5+1) = 1.5$

(nearest smaller integer to 1.5 is 1)

$$I = \int_{-1.5}^{1} -1 \cdot dx + \int_{-1}^{0} 0 \cdot dx + \int_{0}^{1} 1 \cdot dx$$

$$= -[x]_{-1.5}^{1} + 0 + [x]_0^1$$

$$= -[1 - 1.5) + (1 - 0)$$

$$= 0.5.$$

2. $\int_1^2 x \cdot [x] \cdot dx$

$$I = \int_1^2 x(1) \cdot dx.$$

$$= \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{3}{2}$$

$$\int_{-1}^{1} \frac{1}{1+x^2} \cdot dx$$

$$I = \int_{-1}^{1} \frac{1}{1+x^2} \cdot dx.$$

$$\therefore \quad f(x) = \frac{1}{1+x^2} = \frac{1}{(1+(-x)^2} = \frac{1}{1+x^2} \quad \text{i.e even.}$$

$$I = 2\int_{0}^{1} \frac{1}{1+x^2} \cdot dx$$

$$= 2\,(\tan^{-1}x)_0^1$$

$$= 2\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= 2\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{2}$$

$$\int_{-a}^{a} \sqrt{\frac{a+x}{a-x}} \cdot dx$$

$$I = \int_{-a}^{a} \sqrt{\frac{a+x}{a-x}} \cdot dx$$

$$f(x) = \sqrt{\frac{a+x}{a-x}}$$

$$f(-x) = \sqrt{\;} \; ?$$

It is difficult to state whether it is even or odd, here.

$$I = \int_{-a}^{a} \sqrt{\frac{(a+x)\cdot(a+x)}{(a-x)\cdot(a+x)}}$$

$$= \int_{-a}^{a} \frac{a+x}{\sqrt{a^2-x^2}}$$

$$= \underbrace{\int_{-a}^{a} \frac{a}{\sqrt{a^2-x^2}}}_{\text{Even}} + \underbrace{\int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}}_{\text{Odd.}}$$

$$I = 2\int_0^a \frac{a}{\sqrt{a^2-x^2}}\,dx$$

$$= 2a.\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 2a\left[\sin^{-1}\left(\frac{a}{a}\right) - \sin^{-1}\left(\frac{0}{a}\right)\right]$$

$$= 2a\left[\frac{\pi}{2} - 0\right]$$

$$= a\pi.$$

$$\int_{-a}^{a}\sqrt{\frac{a+x}{a-x}} = \int_{-a}^{a}\sqrt{\frac{a-x}{a+x}} = a\pi$$

* $$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}} = \pi$$

$$\int_{-\pi}^{\pi} x^4 \cdot \sin^5 x \cdot dx.$$

$$I = \int_{-\pi}^{\pi} x^4 \sin^5 x \cdot dx$$

$$f(x) = x^4 \cdot \sin^5 x$$

$$f(-x) = (-x)^4 \cdot \sin^5(-x)$$

$$= -x^4 \sin^5(x)$$

$$\sin(-\theta) = -\sin\theta$$

$$I = 0$$

$$\int_{-\pi}^{\pi} \cos^4 x \cdot \sin^5 x =$$

$$I = \int_{-\pi}^{\pi} \cos^4 x \cdot \sin^5 x \quad = 0$$

$$f(x) = \cos^4 x \cdot \sin^5 x$$

$$f(-x) = \cos^4(-x) \cdot \sin^5(-x)$$

$$= -\cos^4 x \cdot \sin^5 x$$

Given odd

$$\int_0^{\pi/2} \sin^5 x \cdot dx :$$

1, 2, 3, 4, 5

$$\int_0^{\pi/2} 1 \cdot \sin^5 x \cdot dx = \frac{4 \times 2}{5 \times 3 \times 1} \times 1$$ (if power is odd)

$$= \frac{8}{15}$$

odd power

$$\int_0^{\pi/2} \cos^7 x \cdot dx$$

$$\int_0^{\pi/2} \cos^7 x \cdot dx = \frac{6 \times 4 \times 2}{7 \times 5 \times 3 \times 1} \times 1$$

$$= \frac{16}{35}$$

even

$$\int_0^{\pi/2} \sin^8 x \, dx$$

1, 2, 3, 4, 5, 6, 7, 8

$$\int_0^{\pi/2} \sin^8 x \cdot dx = \frac{7 \times 5 \times 3 \times 1}{8 \times 6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$= \frac{35\pi}{256}$$

$$\int_0^1 \frac{x^6}{\sqrt{1-x^2}} \, dx$$

If $(1-x^2)$ put $x = \sin\theta$

If $(1+x^2)$ put $x = \tan\theta$ always.

put $x = \sin\theta$

$dx = \cos\theta \cdot d\theta$

If $x = 0$, $\theta = 0$

$x = 1$, $\theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{\sin^6\theta \cdot \cos\theta}{\sqrt{1-\sin^2\theta}} \cdot d\theta$$

$$= \int_0^{\pi/2} \sin^6\theta \cdot d\theta$$

$$= \frac{5\times3\times1}{6\times4\times2} \times \frac{\pi}{2}$$

$$= \frac{15\pi}{96}$$

$$= \frac{15}{32}\pi$$

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^5 x \cdot dx$$

for 4 (even) — for 5 (odd)

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^5 x \cdot dx = \frac{(3\times1)\times(4\times2)}{\underbrace{(4+5)}_{\text{sum (odd)}}\times7\times5\times3} \times 1$$

one power is odd

$9\times7\times5\times3\times1$

$$= \frac{24}{315}$$

$$\int_0^{\pi/2} \sin^6 x \cdot \cos^4 x \cdot dx$$

$$\int_0^{\pi/2} \sin^6 x \cdot \cos^4 x \cdot dx = \frac{(5\times3\times1)\times(3\times1)}{10\times8\times6\times4\times2} \times \frac{\pi}{2}$$

$$= \frac{3}{512}\,\frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^3 x \cdot \cos^5 x\, dx$$

$$\int_0^{\pi/2} \sin^3 x \cdot \cos^5 x = \frac{(2\times1)\times(4\times2)}{8\times6\times4\times2}$$

$$= \frac{1}{24}$$

$$\int_0^{\pi} \sin^3 x \cdot dx$$

$$\int_0^{2a} f(x)\cdot dx = 2\int_0^{a} f(x)\cdot dx, \quad f(2a-x) = f(x)$$

$$\therefore \int_0^{\pi} \sin^3 x \, dx = 2 \int_0^{\pi/2} \sin^3 x \cdot dx$$

$$= 2 \times \left(\frac{2}{3 \times 1} \right)$$

$$= \frac{4}{3}.$$

Improper integrals :

An integral $\int_a^b f(x) \cdot dx$ is said to be an improper integral if

i) $f(x)$ becomes infinite in interval of integration $(a, b$

ii) one or both of limits are infinite $(a = \infty$ or $b = \infty)$

convergent - finite value of integral

divergent - infinite value of integral.

e.g.

1. $$\int_0^1 \frac{1}{x} \, dx = [\log x]_0^1$$

$$= \log 1 - \log 0 \qquad \log 0 \text{ - undefined}$$

$$= \log \left(\frac{1}{0}\right)$$

$$= \infty \qquad \text{- divergent}$$

2. $$\int_0^1 \sqrt{\frac{(1+x)}{(1-x)}} \, dx = \int_0^1 \sqrt{\frac{(1+x)(1+x)}{(1-x)(1+x)}} \, dx$$

$$= \int_0^1 \frac{1+x}{\sqrt{1-x^2}} \, dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx + \int_0^1 \frac{x}{\sqrt{1-x^2}} \cdot dx$$

$$= \left[\sin^{-1} x\right]_0^1 + \left[\frac{1}{2} \times 2\sqrt{1-x^2}\right]_0^1$$

$$\because \int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)}$$

$$= \left[\sin^{-1}1 - \sin^{-1}0\right] - \left[\sin^{-1}0 - \sqrt{1-0}\right]$$

$$= \frac{\pi}{2} - 0 - 0 + 1$$

$$= \frac{\pi}{2} + 1$$

$$\int_1^\infty \frac{1}{x^3} \cdot dx$$

$$I = \int_1^\infty x^{-3} \cdot dx$$

$$= \left[\frac{x^{-3+1}}{-3+1}\right]_1^\infty$$

$$= \left[\frac{x^{-2}}{-2}\right]_1^\infty$$

$$= \left[\frac{-1}{2x^2}\right]_1^\infty$$

$$= \frac{-1}{2}\left[\frac{1}{\infty} - \frac{1}{1}\right]$$

$$= \frac{-1}{2}\left[0 - 1\right]$$

$$= \frac{1}{2}$$

$$\int_{-\infty}^0 \sinh x \cdot dx$$

$$I = \left[\cosh x\right]_{-\infty}^0$$

$$= \left[\frac{e^x + e^{-x}}{2}\right]_{-\infty}^0$$

$$= \left[\frac{e^0 + e^{-0}}{2}\right] - \left[\frac{e^{-\infty} + e^{\infty}}{2}\right]$$

$= 1 - \infty$

$= -\infty$

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$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$$

$f(x) = \frac{1}{1+x^2}$ -- even

$$I = 2\int_0^{\infty} \frac{1}{1+x^2}\, dx$$

$= 2\left[\tan^{-1}(x)\right]_0^{\infty}$

$= 2\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$

$= 2\left[\frac{\pi}{2} - 0\right]$

$= \pi$

$$\int_0^{\infty} x \cdot e^{-x^2} \cdot dx$$

$f(x) = -x^2$

$f'(x) = -2x$

$$I = \int_0^{\infty} x \cdot e^{-x^2} \cdot dx$$

$= \frac{-1}{2}\int_0^{\infty} x \cdot e^{-x^2} \cdot (-2) \cdot dx$

$= -\frac{1}{2}\left(e^{-x^2}\right)_0^{\infty}$ $\quad \because \int e^{f(x)} \cdot f'(x) = e^{f(x)}$

$= \frac{-1}{2}\left[e^{-\infty} - e^{0}\right]$

$= \frac{-1}{2}\left[0 - 1\right]$

$= \frac{1}{2}$

$\int_0 e^{-x^2} \cdot dx$

put, $t = x^2$ $\Rightarrow x = \sqrt{t}$

$dt = 2x \cdot dx$

$dx = \frac{dt}{2x}$

$= \frac{dt}{2\sqrt{t}}$

$x = \infty$, $t = \infty$

$x = 0$ $t = 0$

$I = \int_0^\infty e^{-t} \cdot \frac{dt}{2\sqrt{t}}$

$= \frac{1}{2} \int_0^\infty e^{-t} \cdot t^{-1/2} \cdot dt$

$\frac{-1}{2}$ - rational no.
$\therefore$ shortcut not valid

$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} \cdot dx$ - gamma function.

$\Gamma n+1 = n \Gamma n$ if $n > 0$

$\Gamma n+1 = n!$ if $n \in N$

$\Gamma 1 = 1$

$\Gamma 1/2 = \sqrt{\pi}$

$I = \frac{1}{2} \Gamma 1/2$

$= \frac{\sqrt{\pi}}{2}$

$\because I = \int_0^\infty e^{-t} \cdot t^{1/2 - 1} \cdot dt$

$\because \Gamma 1/2 = \sqrt{\pi}$

[standard]

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\cdot dx\cdot dy$$

$$I = \int_0^\infty\int_0^\infty e^{-x^2}\cdot dx \cdot e^{-y^2}\cdot dy$$

$$= \int_0^\infty e^{-x^2}\cdot dx \cdot \int_0^\infty e^{-y^2}\cdot dy$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \frac{\pi}{4}$$

$$\int_0^\infty \frac{x}{(x^2+9)^2}\,dx$$

$$I = \int_0^\infty (x^2+9)^{-2}\cdot x\cdot dx$$

$$= \frac{1}{2}\int_0^\infty 2x\cdot(x^2+9)^{-2}\cdot dx$$

$$= \frac{1}{2}\left[\frac{(x^2+9)^{-2+1}}{-2+1}\right]_0^\infty$$

$$\because \int [f(x)]^n\cdot f'(x)\cdot dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \left[\frac{1}{2}\,\frac{(x^2+9)^{-1}}{-1}\right]_0^\infty$$

$$= \left[\frac{-1}{2(x^2+9)}\right]_0^\infty$$

$$= \left[\frac{-1}{2(\infty+9)} - \frac{1}{2}\,\frac{1}{(9)}\right] - \left[\frac{-1}{2(0+9)} - \frac{-1}{2}\right]$$

$$= -\frac{1}{2}\left(0-\frac{1}{9}\right) = \frac{1}{18}$$

vector calculus :

Gradient of scalar function :

Let $\phi(x,y,z)=c$ be any scalar function then the gradient of ϕ is denoted by grad ϕ or $\nabla\phi$ and is defined as

$$\text{grad}\ \phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

We know that,

$$\bar{r} = xi + yj + zk$$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$\therefore\ 2r\cdot\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \& \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore\ \text{grad}\ r = \nabla r = i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}$$

$$= i\left(\frac{x}{r}\right) + j\left(\frac{y}{r}\right) + k\left(\frac{z}{r}\right)$$

$$= \frac{xi + yj + zk}{r}$$

$$= \frac{\bar{r}}{r}$$

$$\therefore\ \text{grad}\ r = \nabla r = \frac{\bar{r}}{r}.$$

∇ is vector differential operator

e.g.

(i) $\nabla \log r = \frac{1}{r} \nabla r = \frac{1}{r} \cdot \frac{\bar{r}}{r} = \frac{\bar{r}}{r^2}$

(ii) $\nabla \frac{1}{r} = \frac{-1}{r^2} \nabla r = \frac{-1}{r^2} \cdot \frac{\bar{r}}{r} = \frac{-\bar{r}}{r^3}$

(iii) $\nabla r^n = n \cdot r^{n-1} \cdot \nabla r = n \cdot r^{n-1} \cdot \frac{\bar{r}}{r} = n \cdot r^{n-2} \cdot \bar{r}$

Tangent vector to a curve:

Let $\bar{r}(t)$ be the given vector curve then $\frac{d\bar{r}}{dt}$ is called the tangent vector to the given curve.

$$\bar{r} = xi + yj + zk$$

$$\frac{d\bar{r}}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} \cdot j + \frac{dz}{dt} \cdot k$$

Normol to a surface:

Let $\phi(x,y,z) = c$ be any surface, the $\nabla(\phi)$ is called normal to the surface ϕ. and $\frac{\nabla \phi}{|\nabla \phi|}$ is unit normal vector to the surface ϕ. and is denoled by N

$$N = \frac{\nabla \phi}{|\nabla \phi|}$$

Directional derivative:

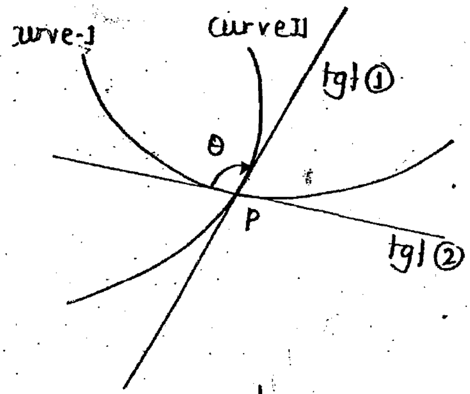
Let $\phi(x,y,z) = c$ be any surface, then directional derivative of ϕ is defined by $\nabla \phi \cdot e$ where e is the unit vectr in the direction of given vector.

If given vector is a

$$e = \frac{\bar{a}}{|a|}$$

directional derivative $= \nabla \phi \cdot e$

The max. or greatest value of directional derivative or magnitude of directional derivative of a scalar function ϕ is defined by $|\nabla \phi|$.



For tangent ①

$$a = a_1 i + a_2 j + a_3 k$$

For tangent ②

$$b = b_1 i + b_2 j + b_3 k$$

$$\therefore \cos\theta = \frac{a \cdot b}{|a||b|}$$

where,

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Note:

The two curves cut each other orthogonally $(\theta = 90^\circ)$

$$\therefore a \cdot b = 0$$

Angle between two surfaces:-

The angle between the two surfaces is the angle between the normals drawn at their point of intersection.

If f and g are any two surfaces and θ is angle between them,

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f| \cdot |\nabla g|}$$

Note:

If the two surfaces cut each other orthogonally, then

$$\nabla f \cdot \nabla g = 0$$

Q. Find the unit normal vector to the surface

$$x^3+y^3+3xyz = 3 \quad \text{at } (1, 2, -1)$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i(3x^2+3yz) + j(3y^2+3xz) + k(3xy)$$

$$(\nabla\phi)_{(1,2-1)} = -3i + 9j + 6k$$

$$N = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{3(-i+3j+2k)}{3\sqrt{1+9+4}}$$

$$= \frac{-i+3j+2k}{\sqrt{14}}$$

Q. Find unit normal vector for the surfaces.

$$x^2y+2xz = 4 \quad \text{at } (2, -2, 3)$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i(2xy+2z) + j(2x) + k(2x)$$

$$= i(-4+6) + j(4) + k(4)$$

$$= +2i + 4j + 4k$$

$$N = \frac{-2(i+2j+2k)}{\sqrt{4+16+16}}$$

$$= \frac{-i+2j+2k}{\sqrt{9}}$$

$$= \frac{-i+2j+2k}{3}$$

$x^2+y^2+z^2=1$ at $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

(a) $\frac{i+j}{2}$

(b) $\frac{j+k}{\sqrt{2}}$

(c) $\frac{i+k}{\sqrt{2}}$

(d) $\frac{i+j+k}{\sqrt{3}}$

Ans : (c) $\frac{i+k}{\sqrt{2}}$

Find the max. value of directional derivative to the surface

$$\phi = x^2 y \cdot z^3 \quad \text{at } (2, 1, -1)$$

$$\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + j \frac{\partial \phi}{\partial z}$$

$$= i(2xy \cdot z^3) + j(x^2 z^3) + k(3x^2 y \cdot z^2)$$

$$= i(-4) + j(-4) + k(12)$$

$$= -4(i + j - 3k)$$

$$|\nabla \cdot \phi| = \sqrt{16+16+144}$$

$$= \sqrt{176}$$

Find the greatest value of directional derivative to surface

$$\phi = x^2 yz \quad \text{at } (1, 4, 1)$$

$$\nabla \phi = i(2xyz) + j(x^2 z) + k(x^2 y)$$

$$= 8i + 4j + 4k$$

$$|\nabla \phi| = \sqrt{64+16+16}$$

$$= \sqrt{81}$$

$$= 9.$$

Find magnitude of gradient $u = \frac{x^2}{2} + \frac{y^2}{3}$ at $(1,3)$

$$\nabla u = \frac{2x}{2} \cdot i + \frac{2y}{3} j + 0k$$

$$= 1i + \frac{2}{3} \times 3j$$

$$= i + 2j$$

$$|\nabla u| = \sqrt{1+4}$$

$$= \sqrt{5}$$

Find directional derivative of $f = xy + yz + zx$ at $(1, 2, 0)$ in the direction of $i + 2j + 2k$.

$$\nabla f = i(y+z) + j(x+z) + k(x+y)$$

$$= i(2) + j(1) + k(3)$$

$$= 2i + j + k \times 3$$

$$e = \frac{i + 2j + 2k}{\sqrt{1+4+4}}$$

$$= \frac{i + 2j + 2k}{3}$$

Directional derivative $= \nabla f \cdot e$

$$= 2 \times \frac{1}{3} + 1 \times \frac{2}{3} + 3 \times \frac{2}{3}$$

$$= \frac{10}{3}$$

Find directional derivative of $f = 2xy + z^2$ at $(1, -1, 3)$ in direction of $i + 2j + 3k$.

$$\nabla f = i(2y) + j(2x) + k(2z)$$

$$= -2i + 2j + 6k.$$

$$e = \frac{i + 2j + 3k}{\sqrt{1+4+9}} = \frac{i + 2j + 3k}{\sqrt{14}}$$

$$\nabla f \cdot e = -2 \times \frac{1}{\sqrt{14}} + 2 \times \frac{2}{\sqrt{14}} + 6 \times \frac{3}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

Find directional derivative of $f(x^2-y^2+2z^2)$ at $P(1,2,3)$ in the direction of $\overline{PQ}$ where $Q(5,0,4)$

$$a = \overline{PQ} = Q-P$$
$$= 4i-2j+1k$$
$$e = \frac{4i-2j+k}{\sqrt{16+4+1}}$$
$$= \frac{4i-2j+k}{\sqrt{21}}$$
$$\nabla f = i(2x)+j(-2y)+k(4z)$$
$$= 2i-4j+12k$$
$$\nabla f \cdot e = \left(2\times\frac{4}{\sqrt{21}}\right)+\left(-4\times\frac{-2}{\sqrt{21}}\right)+\left(12\times\frac{1}{\sqrt{21}}\right)$$
$$= \frac{28}{\sqrt{21}}$$

Find directional derivative of $f = xy+yz+zx$ at $P(1,2,-1)$ in the direction of $\overline{PQ}$ where $Q(1,2,3)$

$$a = \overline{PQ} = Q-P$$
$$= 0i+0j+4k$$
$$e = \frac{4k}{4}$$
$$= k$$
$$\nabla \cdot f = i(y+z)+j(x+z)+k(x+y)$$
$$= i(1)+j(0)+k(3)$$
$$= i+3k$$
$$\nabla f \cdot e = 3\times 1$$
$$= 3$$

Find the directional derivative of $f = xy^2 + yz^2 + zx^2$ at $P(1,1,1)$ along the tangent to curve $x = t,\ y = t^2,\ z = t^3$

$$a = \frac{d\bar{r}}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$$

$$= i + 2t \cdot j + 3t^2 \cdot k$$

$$e = \frac{a}{|a|}$$

$$= \frac{i + 2tj + 3t^2 k}{\sqrt{1 + 4t^2 + 9t^4}}$$

Take $x = t$ (or $y = t^2$ or $z = t^3$)

$$t = 1$$

$$= \frac{i + 2j + 3k}{\sqrt{1+4+9}}$$

$$= \frac{i + 2j + 3k}{\sqrt{14}}$$

$$\nabla f = (y^2 + 2xz) i + (2xy + z^2) j + k(2yz + x^2),$$

$$\nabla f_{(1,1,1)} = 3i + 3j + 3k$$

$$\text{directional derivative} = 3\left(\frac{1}{\sqrt{14}}\right) + 3\left(\frac{2}{\sqrt{14}}\right) + 3\left(\frac{3}{14}\right)$$

$$= \frac{18}{\sqrt{14}}$$

Find the directional derivative of $f = xyz^2 + xz$ at $(1,1,1)$ in the direction of normal of $g = 3xy^2 + y - z$ at $(0,1,1)$

$$a = \nabla g$$

$$= i(3y^2) + j(6xy + 1) + k(-1)$$

$$= 3i + j - k$$

$$e = \frac{3i + j - k}{\sqrt{11}}$$

$$\nabla f = i\,(yz^2+z) + j\,(xz^2) + k\,(2xyz + x)$$
$$= 2i + j + 3k$$

Directional derivative

$$= \frac{3}{\sqrt{11}} \times 2 + \frac{1}{\sqrt{11}} \times 1 - \frac{1}{\sqrt{11}} \times 3$$
$$= \frac{4}{\sqrt{11}}$$

Thursday
26th September 2013

Q. Find the angle between $x^2+y^2+z^2 = 9$, $x^2+y^2-z = 3$ at $(2,-1,2)$

$$\Delta f = 2xi + 2yj + 2zk$$
$$= 4i - 2j + 4k$$
$$\Delta g = 2xi + 2yj \pm k$$
$$= 4i - 2j - k.$$

$$\cos\theta = \frac{\nabla f\, \nabla g}{|\nabla f||\nabla g|}$$

$$= \frac{16+4-4}{\sqrt{16+4+16}\ \sqrt{16+4+1}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Q. Find accute angle between $xy^2z - 3x + z^2$ & $3x^2 - y^2 + 2z = 1$ at $(1,-2,1)$

$$\nabla f = (y^2z - 3)i + j\,(2xyz) + k\,(xy^2 - 2z)$$
$$= i - 4j + 2k$$

$$\nabla g = (6x)i + j\,(-2y) + 2k$$
$$= 6xi + 4j + 2k$$

$$\cos\theta = \frac{|\nabla f \cdot \nabla g|}{|\nabla f| \cdot |\nabla g|}$$

$$= \frac{|16 - 16 + 4|}{\sqrt{1+16+4}\ \sqrt{36+16+4}}$$

$$\theta = \cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$$

Q. Find the constants so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at $(1,-1,2)$

$$f = ax^2 - byz - (a+2)x$$

$$\nabla f = (2ax)i - (a+2)k.$$

$$= (2ax - a + 2)i - (bz)j - (by)k$$

$$\nabla(1,-1,2) = (a-2)i - 2bj + bk$$

$$\nabla g = (8xy)i + (4x^2)j + (3z^2)k$$

$$= -8i + 4j + 12k$$

$$\nabla f \cdot \nabla g = (a-2) \times (-8) - 2b \times 4 + 12 \times b$$

$$0 = -8a + 16 - 8b + 12b$$

$$= 16 + 4b - 8a$$

$$b = 2a - 4 \qquad \text{--- (i)}$$

$\therefore$ put in 2^{nd} surface

$$a(1)^2 - b(-1)(2) = (a+2).1$$

$$a + 2b = (a+2)$$

$$b = 1$$

$$2a = b + 4$$

$$a = \frac{5}{2}$$

Divergence of a vector function:

Let $F = F_1 i + F_2 j + F_3 k$ be any differentible vector function then divergence of F is denoted by div. F or $\nabla \cdot F$ and is defined as

$$\text{div. } F \text{ or } \nabla F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\bar{r} = xi + yj + zk$$

$$\text{div. } \bar{r} = 1 + 1 + 1$$

$$\therefore \quad * \quad \text{div } \bar{r} = \nabla \cdot \bar{r} = 3$$

Solenoidal vector:

The necessary and sufficient condition for a vector function F to be solenoid is div. F = 0
i.e.

$$\nabla \cdot F = 0$$

Note:

Let ϕ is scalar and A is vector then:

$$\text{div. }(\phi \cdot A) = (\text{grad }(\phi)) A + \phi \cdot \text{div.}(A)$$

$$\nabla \cdot (\phi A) = (\nabla \phi) \cdot A + \phi \cdot (\nabla \cdot A)$$

(product of scalar and vector w.r.t divergence)

e.g.

$$\nabla^2 \frac{1}{r}$$

$$= \nabla \cdot \nabla \cdot \frac{1}{r}$$

$$= \nabla \cdot \left(\frac{-1}{r^2} \cdot \nabla r \right)$$

$$= \nabla \cdot \left(\frac{-1}{r^2} \cdot \frac{\bar{r}}{r} \right) \qquad \because \nabla r = \frac{\bar{r}}{r}$$

$$= \nabla \left(\underbrace{\frac{-1}{r^3}}_{\text{Scalar}} \cdot \underbrace{\bar{r}}_{\text{vector}} \right)$$

$$\therefore \nabla^2 \frac{1}{r} = \nabla\cdot\left(\frac{-1}{r^3}\right)\overline{r} + \left(\frac{-1}{r^3}\right)\nabla\cdot\overline{r}$$

$$= \frac{3}{r^4}\cdot\nabla r\cdot\overline{r} - \frac{1}{r^3}\times 3 \qquad \because \nabla\cdot\overline{r} = 3$$

$$= \frac{3}{r^4}\cdot\frac{\overline{r}}{r}\cdot\overline{r} - \frac{3}{r^3} \qquad \because \overline{r}\cdot\overline{r} = r^2$$

$$\because \nabla r = \frac{\overline{r}}{r}$$

$$= \frac{3r^2}{r^5} - \frac{3}{r^3}$$

$$= 0$$

Shortcut 1:

$$\nabla^2[f(r)] = f''(r) + \frac{2}{r}f'(r)$$

In above problem,

$$f(r) = \frac{1}{r}, \quad f'(r) = \frac{-1}{r^2} \quad f''(r) = \frac{2}{r^3}$$

$$\therefore \nabla^2\left(\frac{1}{r}\right) = \frac{2}{r^3} + \frac{2}{r}\left(\frac{-1}{r^2}\right)$$

$$= \frac{2}{r^3} - \frac{2}{r^3}$$

e.g. 1. $\nabla^2(\log r)$

$$= \frac{-1}{r^2} + \frac{2}{r}\left(\frac{1}{r}\right)$$

$$= \frac{-1}{r^2} + \frac{2}{r^2}$$

$$= \frac{1}{r^2}$$

$\nabla(r^n)$

$$= n(n-1)r^{n-2} + \frac{2}{r}(n\cdot r^{n-1})$$

$$= (n^2-n)\cdot r^{n-2} + \frac{2}{r}(n\cdot r^{n-1})$$

$$= n(n-1)r^{n-1} + 2\cdot n\cdot r^{n-2}$$

$$= n(n-1+2)r^{n-2}$$

$$= n(n+1)\cdot r^{n-2}$$

Q. If $\nabla\cdot(r^n\cdot\bar{r}) = 0$ then $n = ?$

$$\nabla\cdot(\phi\cdot A) = (\nabla\phi)\cdot A + \phi\cdot(\nabla\cdot A)$$

$$0 = (\nabla\cdot r^n)\cdot\bar{r} + r^n(\nabla\cdot\bar{r})$$

$$= n\cdot r^{n-1}\cdot\nabla r\cdot\bar{r} + r^n\cdot(3)$$

$$= n\cdot r^{n-1}\cdot\frac{\bar{r}}{r}\cdot\bar{r} + 3r^n$$

$$= n\cdot\frac{r^{n-1}\cdot r^2}{r} + 3r^n$$

$$= n\cdot r^2 + 3r^n$$

$$(n+3) = 0$$

$$n = -3$$

Curl of a vector function:

Let $F = F_1 i + F_2 j + F_3 k$ be any differentiable vector function then the curl of F is denoted by curl F or $\nabla\times F$ and is defined as

$$\text{curl } F = \nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If $\bar{r} = xi + yj + zk$

$$\nabla\cdot\bar{r} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - j\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$\nabla\times\bar{r} = 0$$

Note :

Let F be any non-zero vector function, then

$$\text{div.}(\text{curl } F) = 0$$

or

$$\nabla \cdot (\nabla \times F) = 0$$

Q. If $f = x^3 + y^3 + z^3 - 3xyz$ then $\text{div.}(\text{grad } f) = ?$

$$\nabla \cdot (\nabla f) = \nabla^2 . f$$

shortcut 2 :

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 6x + 6y + 6z$$

$$= 6(x+y+z)$$

Q. If $f = (x+2y)i + (y+2z)j + (x+\lambda z)k$ if this function is solenoidal then find λ.

$$\nabla \cdot F = 0$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$1 + 1 + \lambda = 0$$

$$\lambda = -2$$

Q. Find the value of λ so that vector function given is zero divergencer.

$$F = (2\lambda x^2 y + yz)i + (xy^2 - xz^2) + (2xyz - 2x^2y^2)$$

$$\nabla F = 0$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$4x \cdot \lambda \cdot y + 2xy + 2xy = 0$$

$$2xy(2\lambda + 2) = 0$$

$$\lambda = -1$$

Q. If $F = 3x^2 i + 5xy^2 j + xyz^3 k$ then $(\text{div } F)_{(1,2,3)} = ?$

$$\text{div } F = 6x + 10xy + 3xyz^2$$

$$= 6(1) + 10(2) + 3(18)$$

$$= 80$$

Irrotational vector:

The necessary and sufficient condition for a vector function F to be irrotational is curl F = zero.

$$\text{curl } F = 0$$

$$\nabla \times F = 0$$

* (i) curl represents angular velocity of the vector field
  (ii) divergence represents normal flux of the vector field.

Q. If $F = xy^2 i + 2x^2yz j \pm 3yz^2 k$ then curl (F) at (1, -1, 1) is

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right)$$

$$-j\left(\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right)$$

$$+k\left(\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial y}(xy^2)\right)$$

$$= i(-2x^2y) - j(0) + k(0 - 2xy)$$

$$= -2x^2y\, i - 2xy\, k$$

$$= 2i - 2k$$

(61)

If $F = 2xyi - x^2z\,j$ then curl (F) at (1, 1, 1) is

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & -x^2z & 0 \end{vmatrix}$$

$$= i(0 + x^2) - j(0) + k(-2xz - 2x)$$

$$\nabla \times F_{(1,1,1)} = i - 4k$$

Q. Find the constants a, b, c so that vector function given is irrotational.

$$F = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$$

$$\nabla \times F = 0$$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$i[c + 1] - j[4 - a] + k[b - 2] = 0i + 0j + 0k$$

$$c + 1 = 0$$
$$c = -1$$
$$4 - a = 0$$
$$a = 4$$
$$b - 2 = 0$$
$$b = 2$$

Q. If $F = x^2yz\,i + xyz^3\,j + 3x^2y^2\,k$ then $[\nabla \cdot (\nabla \times F)] =$ at (1, -2, 1)

$\nabla \cdot (\nabla \times F) = 0$ for any non-zero vector function.

Vector Integration :

(i) Line integral ($\int$)

(ii) Surface integral ($\iint$)

(iii) Volume integral ($\iiint$)

(i) Line integral :

Any integral which is evaluated along the curve is called Line integral.

Let F be any differentiable vector function defined along the curve C then, line integral of F is defined as

$$\int_C F \cdot d\bar{r}.$$

If C is closed curve, it is denoted by $\oint_C F \cdot d\bar{r}$

Cartesian form :

Let
$$F = F_1 i + F_2 j + F_3 k$$
$$\bar{r} = xi + yj + zk$$
$$\therefore d\bar{r} = dx \cdot i + dy \cdot j + dz \cdot k$$

$$\therefore \int_C F \cdot \bar{dr} = \int_C (F_1 \cdot dx + F_2 \cdot dy + F_3 \cdot dz).$$

Note :

If x, y, z are the functions in 't' then,

$$\int_C F \cdot \bar{dr} = \int_C \left[F_1 \cdot \frac{dx}{dt} + F_2 \cdot \frac{dy}{dt} + F_3 \cdot \frac{dz}{dt} \right] dt.$$

Q. The value of $\int_C F \cdot d\bar{r}$ where $F = (2y+3)i + xzj + (yz-x)k$ 69

where c is

(i) The curve $x = 2t^2,\ y = t,\ z = t^3$ joining $(0,0,0)$, $(2,1,$

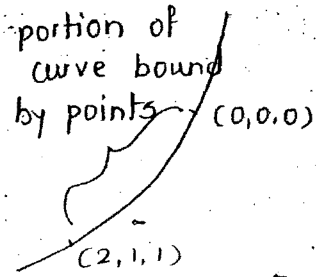
(ii) The st. line joining $(0,0,0)$ to $(0,0,-1)$ to $(0,1,1)$ to $(2,1,1)$

$$\int F \cdot d\bar{r} = \int_C \left[F_1 \cdot \frac{dx}{dt} + F_2 \cdot \frac{dy}{dt} + F_3 \cdot \frac{dz}{dt}\right] dt$$

$x = 2t^2 \qquad y = t \qquad z = t^3$

$\frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 1 \qquad \frac{dz}{dt} = 3t^2$

Take $y = t$



If $y = 0, \quad t = 0$

$y = 1, \quad t = 1$

$$\int_C F \cdot d\bar{r} = \int_0^1 \left[(2t+3) \cdot 4t + (2t^5) \cdot 1 + (t^4 - 2t^2) \cdot 3t^2\right] dt$$

$$= \int_0^1 \left[8t^3 + 12t + 2t^5 + 3t^6 - 6t^4\right] \cdot dt$$

$$= \left[8 \cdot \frac{t^4}{4} + 12 \cdot \frac{t^2}{2} + 2\frac{t^6}{6} + 3\frac{t^7}{7} - 6 \cdot \frac{t^5}{5}\right]_0^1$$

$$= \left[2 \times 1 + 6 \times 1 + \frac{1}{3} + \frac{3}{7} - 6 \times \frac{1}{5}\right]$$

$$= 8 + \frac{10}{21} - \frac{6}{5}$$

$$= \frac{288}{35}$$

$O \equiv (0,0,0)$ $A \equiv (0,0,1)$ $B \equiv (0,1,1)$, $C \equiv (2,1,1)$

① Along OA

$x = 0$ $y = 0$

$dx = 0$ $dy = 0$

$z = 0$ to 1

$$\int_{OA} (yz - x) \cdot dz = 0 \quad \because \; y = 0 \;\; \& \; x = 0$$

② Along AB

$x = 0,$ $z = 1$

$dx = 0$ $dz = 0$

$y = 0$ to 1

$$\int_{AB} x \cdot z \cdot dy = 0$$

$\therefore = 0$

③ Along BC

$y = 1$ $z = 1$

$dy = 0$ $dz = 0$

$x = 0$ to 2

$$\int_{BC} (2y + 3) \cdot dx = \int_0^2 (2(1) + 3) \cdot dx$$

$$= 5(2 - 0)$$

$$= 10$$

$$\int_C F \cdot \bar{dr} = 0 + 0 + 10$$

$$= 10$$

Q. Find $\int_c F\cdot \overline{dr}$ where $F = 3x^2 i + (2xz - y)j + 2k$ where c is the straight line joining (0, 0, 0) to (2, 1, 3) 63

(1) For OA

$O \equiv (0,0,0) \qquad A \equiv (2,1,3)$

Equ^n of st. line

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$\therefore$ Equ^n of st. line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) in the symmetric form is.

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

where t is some scalar quantity.

Equ^n of OA

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0} = t$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$x = 2t \qquad y = t \qquad z = 3t$

$\frac{dx}{dx} = 2 \qquad \frac{dy}{dt} = 1 \qquad \frac{dz}{dt} = 3$

If $y = 0 \qquad t = 0$

$y = 1 \qquad t = 1$

$$\int_{t=0}^{1} \left[3x^2 \cdot \frac{dx}{dt} + (2xy - y)\cdot\frac{dy}{dt} + 2\cdot\frac{dz}{dt}\right] \cdot dt$$

$$= \int_0^1 \left[3(2t)^2 \cdot 2 + (2\times 2t \times 3t - t)\cdot 1 + 3t\times 3\right] \cdot dt$$

$$= \int_0^1 \left[24t^2 + (12t^2 - t) + 9t\right] dt$$

$$= \int_0^1 (36t^2 + 8t)\, dt$$

$$= \left[36\left(\frac{t^3}{3}\right) + 8\left(\frac{t^2}{2}\right) \right]_0^1$$

$$= 36 \times \frac{1}{3} + 8 \times \frac{1}{2}$$

$$= 16$$

Q. Find $\int_c F \cdot d\bar{r}$ where $F = 2xyz\, i + x^2 y \cdot j + x^2 z \cdot k$ where c is straight line joining (0,0,0) to (1,1,1)

∴ Equ^n of straight line :-

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = t$$

$$\therefore \int_c F \cdot d\bar{r} = \int_0^1 2t^3 + t^3 + t^3 \cdot dt$$

$$= \int_0^1 4t^3 \cdot dt$$

$$= \frac{4}{4}\,[1]$$

$$= 1$$

Q. Find the value of $\int_c F \cdot d\bar{r}$ where $F = 3xy\, j - y^2 j$ where c is the curve $y = 2x^2$ in the xY-plane joining (0, 0) to (1,2)

$$\int_c F\cdot\bar{dr} = \int_c (3xy)\cdot dx - y^2\cdot dy$$

$\because$

$$y = 2x^2$$
$$dy = 4x\cdot dx.$$

put the limits of x i.e. 0 to 1

$$\int_c F\cdot\bar{dr} = \int_0^1 (3x\times 2x^2)\cdot dx - 4x^4\cdot 4x\cdot dx$$

$$= \int_0^1 6x^3\cdot dx - \int_0^1 16x^5\cdot dx$$

$$= \frac{6}{4}[1] - \frac{16}{6}[1].$$

$$= \frac{-7}{6}.$$

Q. Find $\int_c F\cdot\bar{dr}$ where $F = (5xy - 6x^2)i + (2y-4x)j$ where c is curve $y = x^3$ in X-Y plane joining (1,1) to (2,8)

$\because$

$$\int_c F\cdot\bar{dr} = \int_c (5xy - 6x^2)\cdot dx + (2y - 4x)\,dy$$

$$y = x^3$$
$$dy = 3x^2\cdot dx.$$

$\because$

$$\int_c F\bar{dr} = \int_1^2 (5x\cdot x^3 - 6x^2)\cdot dx + (2x^3 - 4x)\cdot 3x^2\cdot dx$$

$$= \int_1^2 5x^4 - 6x^2\cdot dx + \int_1^2 6x^5 - 12x^3\cdot dx$$

$$= \frac{5}{5}\left[2^5 - 1^5\right] - \frac{6}{3}\left[2^3 - 1^3\right] + \frac{6}{6}\left[2^6 - 1\right] - \frac{12}{4}\left[2^4\right.$$

$$= [32-1] - 2[8-1] + 1[64-1] - 3[16]$$

$= 35$

2. Find $\oint_c F\cdot d\bar{r}$ where $F = (2x-y+z)\,i + (x+y-z^2)\,j + (3x-4y+2z)k$ where c is the circle in the XY- plane having centre at origin and of radius 3 units.

$$\oint_c F\cdot d\bar{r} = \oint_c (2x-y)\,dx + (x+y)\cdot dy$$

$\because$ XY- plane, $z=0$
$dz=0$

Equation of circle,

$$x^2+y^2 = r^2$$

$$x^2+y^2 = 9$$

Parametric form of circle,

$$x = 3\cos\theta, \qquad y = 3\sin\theta$$

$$dx = -3\sin\theta\cdot d\theta \qquad dy = 3\cos\theta\cdot d\theta$$

Limits will be

$$\theta = 0 \quad \text{to} \quad \theta = 360^\circ$$

$$\therefore \oint_c F\cdot d\bar{r} = \int_0^{2\pi} [2\times 3\cos\theta - 3\sin\theta]\cdot(-3\cdot\sin\theta\cdot d\theta)$$

$$+ \int_0^{2\pi} (3\cos\theta + 3\sin\theta)\cdot 3\cos\theta\cdot d\theta$$

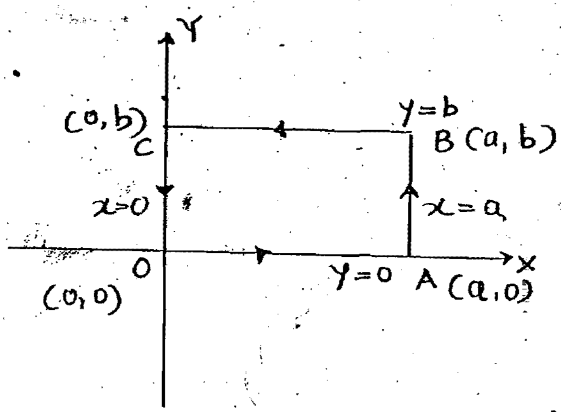
$$= \int_0^{2\pi} (-9\sin\theta\cdot\cos\theta + 9)\cdot d\theta$$

$$= \int_0^{2\pi} -9\sin\theta\cdot\cos\theta + 9\int_0^{2\pi} d\theta$$

$$= +9\left[\frac{\sin^2\theta}{2}\right]_0^{2\pi} + \left[9\theta\right]_0^{2\pi}$$

$$= 18\pi$$

Q. Find $\oint_c F\cdot\overline{dr}$ where $F = (x^2+y^2)i \pm 2xy\,j$ where c is the rectangle bounded by $x=0,\ x=a$ & $y=0,\ y=b$. 68



$$\oint_c F\cdot\overline{dr} = \oint_c (x^2+y^2)\cdot dx - (2xy)\cdot dy$$

① Along OA

$y = 0$

$dy = 0$

Limits $x=0$ to a

$$= \int_0^a (x^2+\cancel{y^2})\cdot dx = \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3}$$

② Along AB

$x = a$

$dx = 0$

Limits $y=0$ to $y=b$

$$= \int_0^b -(2a\cdot y)\cdot dy$$

$$= -2a\left[\frac{y^2}{2}\right] = -a\cdot b^2$$

③ Along BC

$y = b$

$dy = 0$

Limits $x = a$ to $x = a$.

$$= \int_a^0 (x^2+b^2)\cdot dx$$

$$= \left(\frac{x^3}{3} + b^2.x\right)_a^0$$

$$= 0 - \left(\frac{a^3}{3} + ab^2\right)$$

$$= \frac{-a^3}{3} \mp ab^2$$

④ Along CO

$$x = 0$$
$$dx = 0$$

Limits $y = b$ to $y = 0$

$$= \int_b^0 - (2 \times 0)$$

$$= 0$$

$$\oint_c F \cdot \overline{dr} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 - 0$$
$$= -2ab^2$$

(2) Surface Integrals :

Any integral which is evaluated over surface is called a surface integral.

Let F be any differentiable vector function defined over a surface 'S' then the surface integral is defined by $\iint_S F \cdot N \cdot ds$. where N is the outer unit normal vector to the given surface.

$$\iint_S F \cdot N \cdot ds = \iint_{R_1} F \cdot N \cdot \frac{dx \cdot dy}{|N \cdot k|}$$

R_1 - projection of S in XY-plane.

$$= \iint_{R_2} F \cdot N \cdot \frac{dy \cdot dz}{|N \cdot i|}$$

R_2 - projection of S in YZ-plane

$$= \iint_{R_3} F \cdot N \cdot \frac{dx \cdot dz}{|N \cdot j|}$$

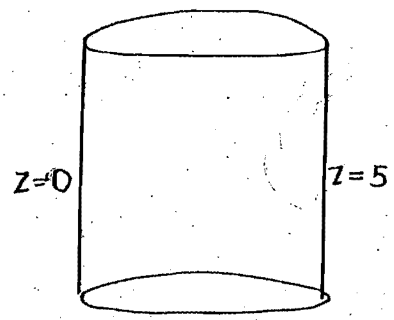
R_3 - projection of S in XZ-plane

F·N gives an idea to decide the projection.

Q. Evaluate $\iint_S F \cdot N \cdot ds$ where $F = zi + xj - 3y^2z \cdot k$ where (66)

S is the surface of the cylinder $x^2+y^2=16$, included in the 1^{st} octant bounded by $z=0$ and $z=5$.

Projection in YZ or XZ plane can be taken.



$$\therefore \quad \phi = x^2+y^2-16$$

$$\nabla\phi = 2xi + 2yj$$

$$N = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2(xi+yj)}{2\sqrt{x^2+y^2}}$$

$$= \frac{xi+yj}{\sqrt{16}}$$

$$= \frac{xi+yj}{4}$$

$$\therefore \quad F \cdot N = z\cdot\left(\frac{x}{4}\right) + x\cdot\left(\frac{y}{4}\right)$$

$$= \frac{zx+xy}{4}$$

$$= \frac{x(y+z)}{4}$$

For the limits of y, put

$$x^2+y^2 = 16 \qquad (\text{in YZ plane } x=0)$$

$$0+y^2 = 16$$

$$y = \pm 4$$

but surface is included in 1^{st} octant

$\therefore \quad y=0$ to $y=4$.

$$\iint_S F\cdot N \cdot ds = \iint_S F\cdot N \cdot \frac{dy \cdot dz}{|N \cdot i|}$$

$$= \int_{y=0}^{4} \int_{z=0}^{5} \left(\frac{x}{4}\right)(y \cdot 1 + z \cdot 1) \cdot \frac{dy \cdot dz}{(x/4)}$$

$$= \left(\frac{y^2}{2}\right)_0^4 \cdot (z)_0^5 + \left(\frac{z^2}{2}\right)_0^5 (y)_0^4$$

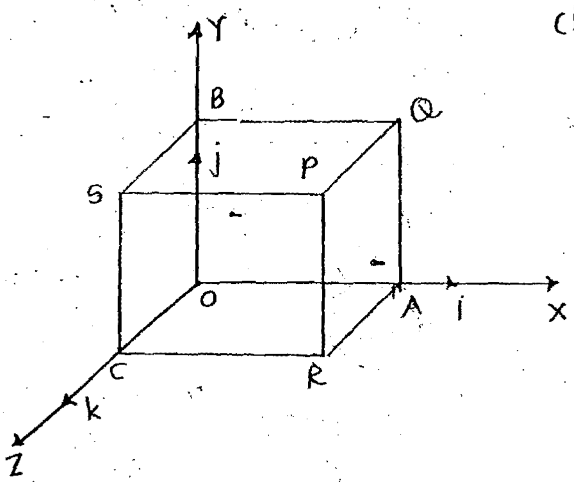
$$= \frac{16}{2} \times 5 + \frac{25}{2} \times 4^2$$

$$= 40 + 50$$

$$= 90$$

Friday
27th September 2013

Q. Find $\iint_S F \cdot N \cdot ds$ where $F = 4xzi - y^2j + yzk$ and S is cube bounded by $0 \leq x, y, z \leq 1$



(i) For PQAR

$N = i \quad x = 1$

$\therefore F \cdot N = (4xz) \cdot 1$

$= 4xz$

$= 4z \qquad (\because x = 1)$

$$\therefore = \int_0^1 \int_0^1 4z \cdot \frac{dy \cdot dz}{|N \cdot i|}$$

$$= \int_0^1 \int_0^1 4z \cdot \frac{dy \cdot dz}{|i \cdot i|}$$

$$= \int_0^1 \int_0^1 4z \cdot dy \cdot dz \qquad i \cdot i = 1$$

$$= 4\left(\frac{z^2}{2}\right)_0^1 (y)_0^1 = 4\left(\frac{1}{2}\right) \times 1 = 2$$

(ii) For OCSB,

$N = -i, \quad x = 0$

$\therefore F \cdot N = -4xz = 0 \qquad (\because x = 0)$

$$\therefore \iint F \cdot N \cdot ds = 0$$

(iii) For PQBS

$N = j \quad y = 1$ (67)

$\therefore \quad F \cdot N = -y^2 \quad = -1 \qquad (\because y = 1)$

$$\therefore \iint F \cdot N \cdot ds = \int_0^1 \int_0^1 -1 \cdot dx \cdot dz$$

$$= -\left[x\right]_0^1 \left[z\right]_0^1$$

$$= -1$$

(iv) For OARC

$N = -j \quad y = 0$

$\therefore \quad F \cdot N = 0$

$$\therefore \iint F \cdot N \cdot ds = 0$$

(v) For PRCS

$N = \bar{k} \quad z = 1$

$\therefore \quad F \cdot N = yz \quad = y \qquad (\because z = 1)$

$$\therefore \iint F \cdot N \cdot ds = \int_0^1 \int_0^1 y^2 \cdot dx \cdot dz$$

$$= \left[\frac{y^2}{2}\right]_0^1 \left[x\right]_0^1$$

$$= \frac{1}{2}$$

(vi) For QAOB

$N = -\bar{k} \quad , \quad z = 0$

$\therefore \quad F \cdot N = 0$

$$\therefore \iint F \cdot N \cdot ds = 0$$

$$\therefore \iint_S F \cdot N \cdot ds = 2 + 0 + (-1) + 0 + \left(\frac{1}{2}\right)$$

$$= \frac{3}{2}$$

(3) Volume integrals:

Any integral which is evaluated over a volume is called volume integral.

Q. Evaluate $\iiint_V \nabla \cdot F \cdot dv$ where $F = (2x^2-3z)i - 2xyj - 4xk$ and V is the reagion bounded by $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$.

$$\nabla F = \frac{\partial F_1}{\partial x} i + \frac{\partial F_2}{\partial y} j + \frac{\partial F_3}{\partial z}$$

$$= 4x - 2x + 0$$

$$= 2x$$

Limits $z=0$ to $z = 4-2x-2y$

$y=0$ to $y=2-x$

$x=0$ to $x=2$

$$\therefore \iiint_V \nabla \cdot F dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x \cdot dx \cdot dy \cdot dz$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x} x\,(4-2x-2y)\, dx \cdot dy$$

$$= 2\int_{x=0}^{2} x \int_{y=0}^{2-x} 2(2-x) - 2y \cdot dy \cdot dx$$

$$= 2\int_{x=0}^{2} x \left[2(2-x)\cdot y - 2\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= 2\int_0^2 x \left[2(2-x)(2-x) - \frac{2(2-x)^2}{2}\right] dx$$

$$= 2\int_0^2 x\,[2-x]^2\, dx$$

$$= 2\int_0^2 4x + x^3 - 4x^2 \; dx$$

$$= 2\left[\frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3}\right]_0^2$$

$$= 2\left[2\times 8 + \frac{16}{4} - \frac{4\times 8}{3}\right]$$

$$= \frac{8}{3}$$

Vector transformation:

1. Gauss - Divergence theorem $\left(\iint \rightarrow \iiint\right)$

Let s be the closed surface enclosed by a volume V. Let F be any differentiable vector function. then,

$$\iint_S F\cdot N\cdot ds = \iiint_V (\text{div}. F)\cdot dv$$ (only for closed surfaces)

or

$$\iint_S (F_1\cdot dy\cdot dz + F_2\cdot dz\cdot dx + F_3\cdot dx\cdot dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, d$$

Q. $\iint_S \bar{r}\cdot N\cdot ds$ where s is closed surface enclosed by volume V

(a) V (b) 2V (c) 3V (d) 4V

$$\iint_S \bar{F}\cdot N\cdot ds = \iiint_V \nabla . F\cdot dv$$

$$\int_S \bar{r}\cdot N\cdot ds = \int_V (\nabla\cdot\bar{r})\cdot dv$$

$$= \int_V 3\cdot dv$$

$$= 3V$$

$$\int_S 5\cdot\bar{r}\cdot N\cdot ds = 15V$$

$\int_S \bar{F}\cdot N\cdot ds$ where s is sphere $x^2+y^2+z^2=a^{-3/2}$

$$\int_S \bar{F}\cdot N\cdot ds = \int_V \bar{F}\cdot dv$$

$$= 3V$$

$$= 3\times \frac{4}{3}\pi a^3$$

$$= 4\pi a^3$$

For radius R

$$\int_S \bar{F}\cdot N\cdot ds = 4\pi R^3.$$

For $x^2+y^2+z^2=1$

$$\int_S \bar{r}\cdot N\cdot ds = 4\pi.$$

$\int_S (x\cdot dy\cdot dz + y\cdot dx\cdot dz + z\cdot dy\cdot dx)$. where s is cube of unit length

$$\nabla F = 1+1+1$$

$$= 3$$

$$\int_S F\cdot N\cdot ds = \int_V 3\cdot dv$$

$$= 3V$$

$$= 3(1) = 3$$ (unit length)

Find $\int_S (x+z)\cdot dy\cdot dz + (y+z)\cdot dz\cdot dx + (x+y)\cdot dx\cdot dy$ where s is the sphere $x^2+y^2+z^2=4$

$$\nabla\cdot F = 1+1+0$$

$$= 2$$

$$\int_S F\cdot N\cdot ds \quad \int_V 2\cdot dv$$

$$= 2V = 2\times\frac{4}{3}\pi\times 2^3$$

$$= \frac{64\pi}{3}$$

Find value of $\int_S (axi + byj + czk) \cdot N\, ds$ where S is sphere (69)

of $x^2+y^2+z^2=1$

$$\nabla F = a+b+c$$

$$\therefore \int_S F \cdot N \cdot ds = \int_V (a+b+c) \cdot dv$$

$$= (a+b+c) \cdot V$$

$$= (a+b+c) \times \frac{4}{3} \times \pi \times 1$$

$$= \frac{4}{3}\pi (a+b+c)$$

Find $\int_S (ax^2+by^2+cz^2) \cdot ds$ where S is $x^2+y^2+z^2=1$

$$\therefore N = \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}$$

$$= xi+yj+zk$$

$\therefore$ for $(axi+byj+czk) = F$.

$$\therefore F \cdot N = ax^2+by^2+cz^2$$

$$\int_S (ax^2+by^2+cz^2) \cdot ds = \int_S (axi+byj+czk) \cdot N \cdot ds.$$

$$= \frac{4}{3}\pi (a+b+c)$$

Find $\int_S (4xzi - y^2j + yzk) \cdot N \cdot ds$ where S is the cube boundeo by $0 \leqslant x, y, z \leqslant 1$.

$$\therefore \nabla F = 4z - 2y + 0$$

$$= 4z - y$$

$$\int_S F\cdot N\cdot ds = \int_V \nabla F\cdot dV$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\cdot dx\cdot dy\cdot dz$$

$$= (x)_0^1 \left[4\left(\frac{z^2}{2}\right)_0^1 (y)_0^1 - \left(\frac{y^2}{2}\right)_0^1 (z)_0^1\right]$$

$$= \frac{3}{2}$$

Find $\int_S F\cdot N\cdot ds$ where $F = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$

where s is cuboid bounded by $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$

$$\nabla F = 2x + 2y + 2z$$

$$\int_S F\cdot N\cdot ds = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} [2x\cdot 1 + 2y\cdot 1 + 2z\cdot 1]\cdot dx\cdot dy\, dz$$

$$= \left[2\cdot\frac{x^2}{2}\cdot y\cdot z + 2\cdot\frac{y^2}{2}\cdot x\cdot z + 2\cdot\frac{z^2}{2}\cdot x\cdot y\right]_{x=0,\,y=0,\,z=0}^{x=a,\,y=b,\,z=c}$$

$$= (a^2bc + b^2ac + c^2ab)$$

$$= abc\,(a+b+c)$$

A reogion of volume 10 cu. units is bounded by closed surface and N is unit normal vector to the surface.

$$\iint_S (4xi + 2yj - zk)\cdot N\cdot ds.$$

$$\int_V \nabla F dv = 10 \qquad \because \nabla F = 4+2-1$$
$$= 5$$

$$\iint_S (4xi + 2yj - zk)\cdot N\cdot ds = 5$$

2. Green's theorem in a plane $(\int \rightarrow \iint)$

Let R be closed reagion in XY plane bounded by a simple and closed curve C. Let P and Q are continuous functions of x and y possesing the first order partial derivatives then

$$\oint_C P \cdot dx + Q \cdot dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cdot dx \cdot dy$$

- Selecting limits is important
- Draw diagram to understand limits
- circle, square, rectangle - symmetrical - no need of diagram.

Find $\oint_C (x^2+y^2) \cdot dx + (-2xy) \cdot dy$ where c is rectangle bounded b $x=0$, $x=a$ and $y=0$, $y=b$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2) = 2y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(-2xy) = -2y$$

$$\oint_C (x^2+y^2) \cdot dx - 2xy \cdot dy = \int_{x=0}^{a} \int_{y=0}^{b} (-2y-2y) \cdot dx \cdot dy$$

$$= (x)_0^a \cdot \left(-4 \cdot \frac{y^2}{2}\right)_0^b$$

$$= a\left(-4 \times \frac{b^2}{2}\right)$$

$$= -2ab^2$$

Find $\oint_C (2x-y) \cdot dx + (x+y) \cdot dy$ where C is circle $x^2+y^2=9$

$$\frac{\partial P}{\partial y} = -1 \quad , \quad \frac{\partial Q}{\partial x} = 1$$

$$= \iint_R [1-(-1)] \cdot dx \cdot dy$$

$$= \int_R 2 \cdot dR$$

$$= 2 \cdot R$$

$$= 2 \times \pi (9)$$

$$= 18\pi$$

Find $\oint_C (x^2 - xy^3) \cdot dx + (y^3 - 2xy) \cdot dy$ where c is rectangle bdd by $(0,0), (2,0)\ (2,2)\ (0,2)$

$$\because \frac{\partial P}{\partial y} = -3xy^2$$

$$\frac{\partial Q}{\partial x} = -2y.$$

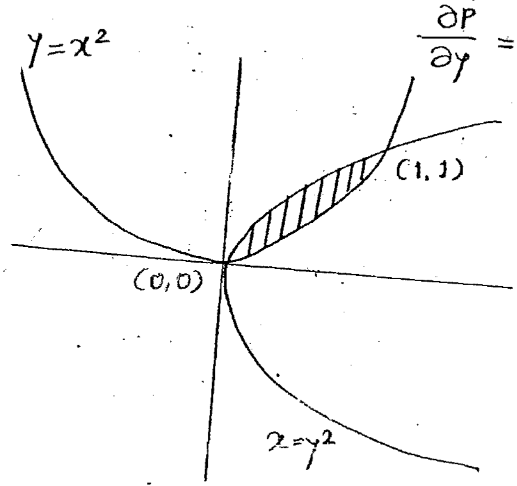
$$\therefore \oint_C P\,dx + Q\cdot dy = \int_0^2\int_0^2 (+3xy^2 - 2y.1)\cdot dx \cdot dy$$

$$= \left(-2\frac{y^2}{2}\right)_0^2 \cdot (x)_0^2 + 3\left(\frac{x^2}{2}\right)_0^2 \cdot \left(\frac{y^3}{3}\right)_0^2$$

$$= -8 + 16$$

$$= 8$$

Find $\oint_C (3x^2 - 8y^2) \cdot dx + (4y - 6xy) \cdot dy$ where c is reagion bdd by $y = \sqrt{x}$ and $y = x^2$.



$$\frac{\partial P}{\partial y} = -16y \qquad \frac{\partial Q}{\partial x} = -6y$$

Functions

limits required.

$$\because y = \sqrt{x} \Rightarrow x = y^2$$

$$\text{and} \quad y = x^2$$

$$\therefore x^2 = \sqrt{x}$$

$$x^4 = x$$

$$\therefore x^4 - x = 0$$

$$x(x^3-1)=0$$

$$x=0,1$$

$$y=0,1$$

point of intersection $(0,0)$ and $(1,1)$.

∵ If two curves bound a reagion and reagion is not symmetric about any axis, then lower curve represent lower limits and upper curve given upper limit.

∴ limits of $y = x^2$ to $\sqrt{x}$

$x = 0$ to 1

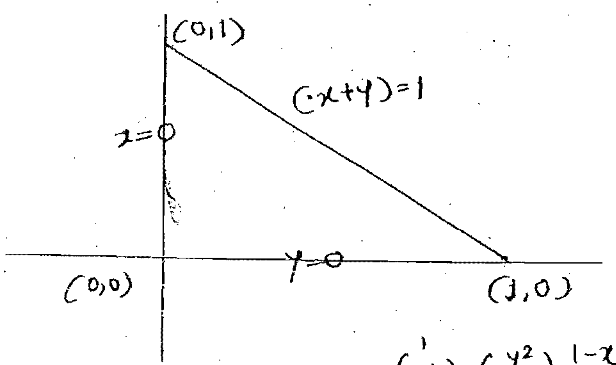
$$\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (-6y+16y)\cdot dx\cdot dy$$

$$= \int_0^1 \left(10\cdot\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}}$$

$$= 5\int_0^1 (x-x^4)\cdot dx$$

$$= \frac{3}{2}$$

Find the $\oint_c Pdx + Q\cdot dy$ of above problem for reagion bdd by $x=0,\ y=0,\ x+y=1$



Limits of y

$y=0$ to $y=(1-x)$

limits of x

$x=0$ to $x=1$

$$= \int_{x=0}^{1}\int_{0}^{1-x} (-6y+16y)\cdot dx\cdot dy$$

$$= \int_0^1 10\left(\frac{y^2}{2}\right)_0^{1-x} dx = 5\int_0^1 (1-x)^2\, dx = \frac{5}{3}$$

Find $\oint_C (y-\sin x)\cdot dx + \cos x \cdot dy$ where c is Δ- formed by

$y=0,\ x=\frac{\pi}{2},\ y=\frac{2x}{\pi}$ Ans. $-\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$

Find $\oint_C (x^2-\cosh y)\cdot dx + (y+\sin x)\cdot dy$ where c is rectangle bdd by $(0,0)$ $(\pi,0)$ $(\pi,1)$ $(0,1)$. Ans. $\pi[\cosh 1 - 1]$

5. Stoke's theorem $(\int \rightarrow \iint)$

Let 'S' be an open surface bounded by a closed and non-intersecting curve C. Let F be any differentiable vector function then.

$$\oint_C F\cdot \bar{dr} = \iint_S \text{Curl } F\cdot N\cdot ds$$

$$= \iint_S (\nabla \times F) N ds$$

Evaluate $\oint_C F\cdot \bar{dr}$, if $F = -y^3 i + x^3 j$ and S is surface of the circular disc $x^2+y^2 \leq 1$, $z=0$. (can solve by Green's theorem)

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= i(-0) - j(0-0) + k(3x^2+3y^2)$$

$$= 3(x^2+y^2)\cdot k.$$

$$= 3k \qquad \because x^2+y^2 \leq 1$$

$$\therefore \oint_C F\cdot \bar{dr} = \iint_S (\nabla \times F)\cdot N \cdot ds$$

$$= \int_S 3k\cdot k\, ds$$

$z=0 \therefore xy$ plane

$$= \int 3k^2\cdot ds$$

$$= \int_S 3 \cdot ds \qquad \because k^2 = 1$$

$$= 3 \cdot S$$

$$= 3 \times \pi (1)^2$$

$$= 3\pi$$

If $y = x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x} + \cdots \infty}}}$ $y(2) = ?$

(a) 4 or 1 (b) 1 only (c) 4 only (d) undefined.

$$y = x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots \infty}}}$$

$$y - x = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \infty}}}$$

$$y - x = \sqrt{x + (y - x)}$$

$$(y - x)^2 = x + y - x$$

$$(y-1)(y-4) = 0$$

$$y = 1 \quad \text{or} \quad y = 4$$

but value of y should be more than 2 here for $y(2)$

Find $\lim_{x \to 0} \dfrac{e^x - \left(1 + x + \frac{x^2}{2!}\right)}{x^3} =$

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (c) 1

$$L = \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(1 + x + \frac{x^2}{2!}\right)}{x^3}$$

$$= \frac{1}{6} + 0 + 0$$

$$= \frac{1}{6}$$

If $f(x) = \dfrac{2x^2-7x+3}{5x^2-12x-9}$

$\therefore \lim_{x \to 3} f(x) = ?$

(a) $-1/3$ (b) 0 (c) $\frac{2}{5}$ (d) $\frac{5}{18}$

By L-hospitality rule

$$\lim_{x \to 3} f(x) = \frac{4x-7}{10x-12}$$
$$= \frac{5}{18}$$

Find $\int_0^{\pi/3} e^{it}.\,dt$

$$I = \int_0^{\pi/3} e^{it}.\,dt$$

$\because e^{i\theta} = \cos\theta + i\sin\theta$

$$= \int_0^{\pi/3} (\cos t + i\sin t).\,dt$$
$$= \left[\sin t - i.\cos t\right]_0^{\pi/3}$$
$$= \left[\sin\frac{\pi}{3} - i\cos\frac{\pi}{3}\right] - \left[\sin 0 - i\,(1)\right]$$
$$= \frac{\sqrt{3}}{2} - \frac{i}{2} + i$$
$$= \frac{\sqrt{3}}{2} + \frac{i}{2}$$

With $y = e^{ax}$ if the sum $S = \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} + \dots \frac{d^ny}{dx^n}$ approaches to $2y$ as $n \to \infty$. Find a.

$\therefore$
$$S = 2y$$

$\therefore$
$$2y = \frac{d}{dx}(e^{ax}) + \frac{d^2}{dx^2}(e^{ax}) + \dots$$
$$2.e^{ax} = a.e^{ax} + a^2.e^{ax} + a^3.e^{ax} + \dots$$

$\therefore$
$$2 = a + a^2 + a^3 + \dots$$

$\because$ Sum of infinite Geometric progression $\Rightarrow S_\infty = \frac{a}{1-r}$

$$r = \frac{T_2}{T_1}$$

$$= \frac{a^2}{a} = a.$$

$$2 = \frac{a}{1-a}$$

$$a = \frac{2}{3}$$

If two roots of non-linear equation $f(x) = x^3 - 6x^2 + 11x - 6 = 0$ are 1 and 3 then third root is.

$\therefore$ sum of three roots

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$= \frac{-(-6)}{1}$$

$$3 + 1 + \gamma = 6$$

$$\gamma = 2$$

or

1st value always zero

	1	−6	11	−6
$x = 1$	0	1	−5	6
	1	−5	6	0
$x = 3$	0	3	−6	
	1	−2	0	0

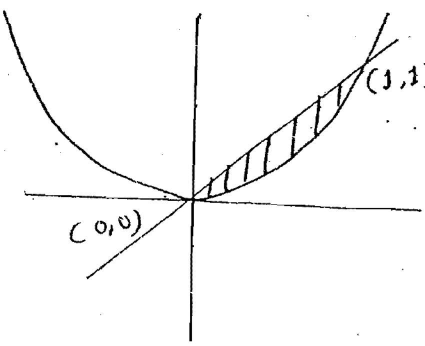
Method of the synthetic divisio

$$x - 2 = 0$$

$$x = 2$$

The area of the reagion bounded by $y = x^2$ and $y = x$

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 1



$$x^2 = x$$

$$x(x-1) = 0$$

$x = 0 \Downarrow y = 0$ (0,0)

$x = 1 \Downarrow y = 1$ (1,1)

$$\text{Area} = \int_0^1 (x - x^2)\cdot dx \quad \text{(upper curve - lower curve)}$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

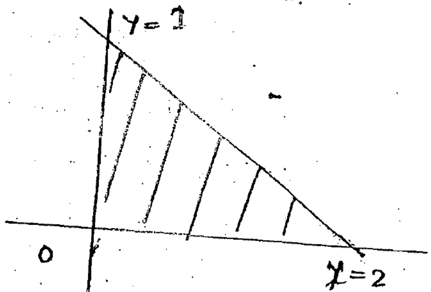
Area bounded by $y^2 = 4ax$ & $x^2 = 4ay$ $\Rightarrow \frac{16a^2}{3}$

$y^2 = 8x$ & $x^2 = 8y$ $\Rightarrow \frac{16(2)^2}{3} = \frac{64}{3}$

$y^2 = 4x$ & $x^2 = 4y$ $\Rightarrow \frac{16}{3}$ $(\because a = 1)$

$y^2 = x$ & $x^2 = y$ $\Rightarrow \frac{1}{3}$ $a = \frac{1}{4}$.

$\iint xy\cdot dx\cdot dy$ for given reagion is



Equ^n of line

$$= \frac{x}{2} + \frac{y}{1} = 1$$

$\therefore$ limits $y = 0$ to $1 - \frac{x}{2}$

$x = 0$ to 2

$$\iint xy\cdot dx\cdot dy = \int_{x=0}^{2}\int_{y=0}^{1-x/2} xy\cdot dx\cdot dy$$

$$= \int_{x=0}^{2} x\left[\frac{y^2}{2}\right]_0^{1-x/2}\cdot dx$$

$$= \int_0^2 \frac{x}{2}\left[\left(1-\frac{x}{2}\right)^2\right] dx$$

$$= \int_0^2 \frac{x}{2}\left[1 - x + \frac{x^2}{4}\right] dx$$

$$= \frac{1}{6}$$

$$\int_0^3 \int_0^x (6-x-y) \cdot dx \cdot dy$$

(a) 13.5 (b) 27.0 (c) 40.5 (d) 54.0

$$I = \int_0^3 \left(6y - xy - \frac{y^2}{2}\right)_0^x \cdot dx$$

$$= \int_0^3 \left(6x - x^2 - \frac{x^2}{2}\right) dx$$

$$= \int_0^3 \left(6x - \frac{3x^2}{2}\right) dx$$

$$6\left(\frac{x^2}{2}\right)_0^3 - \frac{3}{2}\left(\frac{x^3}{3}\right)_0^3$$

$$= 3 \times 9 - \frac{1}{2}(27)$$

$$= 27 - 12.5$$

$$= 13.5$$

A parabolic arc $y = \sqrt{x}$ $1 \leq x \leq 2$ is revolved around x-axis then the volume of solid revolution is

$$\text{volume} = \int_a^b \pi y^2 \, dx$$

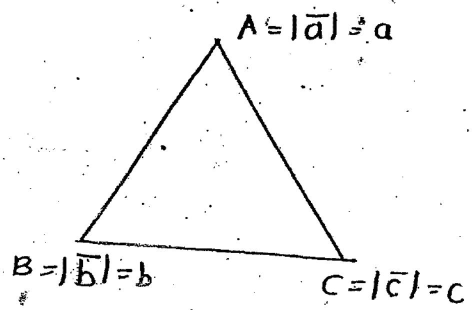
$$= \int_1^2 \pi \cdot x \cdot dx$$

$$= \pi \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{\pi}{2}(4-1)$$

$$= \frac{3\pi}{2}.$$

The area of Δ formed by tips of vector $\bar{a}, \bar{b}$ & $\bar{c}$ is



$$\text{Area} = \frac{1}{2} |AB \times AC|$$

$$= \frac{1}{2} |(B-A) \times (C-A)|$$

$$= \frac{1}{2} |(b-a) \times (c-a)|$$

$$= \frac{1}{2} |(a-b) \times (a-c)|$$

The value of α for which the vectors are co-planer.

$$a = i + 2j + k$$
$$b = 3i + k$$
$$c = 2i + \alpha j$$

vectors are co-planer

$$[a \ b \ c] = 0$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & \alpha & 0 \end{vmatrix} = 0$$

$$(0-\alpha) - 2(0-2) + 1(3\alpha - 0) = 0$$

$$2\alpha - 4 = 0$$

$$\alpha = 2$$

Differential equations:

Any equation consisting of differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots \frac{d^ny}{dx^n} \ldots$ is called Differential equation.

Degree of differential equation:

The degree of differential equ^n is, degree of higest order.

Find the degree and order of following D.F.

1. $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0$

   degree $= 1$
   order $= 2$

2. $\left(\frac{d^2y}{dx^2}\right)^3 + 5\frac{dy}{dx} + 6y = k\left(\frac{d^3y}{dx^3}\right)^2$

   degree $= 2$
   order $= 3$

3. $\left(\frac{d^2y}{dx^2}\right)^2 + \sqrt{1+\left(\frac{dy}{dx}\right)^3} = 5y$

   degre $= 4$
   order $= 2$

4. $\left(\frac{d^2y}{dx^2}\right) + 6\cos\left(\frac{dy}{dx}\right) + 7y = 6$

   degree - not defined
   order - 2.

Formation of differential equation :

A. differential equation can be formed by eliminating arbitrary constants in the solution.

$$y^2 = 4ax$$

constant (doesn't change) arbitrary constant (can change)

$$2y \cdot \frac{dy}{dx} = 4a.$$

(1). $y^2 = x \cdot 2y \cdot \frac{dy}{dx}$

$$y = 2x \cdot \frac{dy}{dx}.$$

$$2x \cdot \frac{dy}{dx} - y = 0$$

2. $x^2 + y^2 = a^2$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$x + y \cdot \frac{dy}{dx} = 0$$

5. $y^2 = 4a(x+a)$

$$2y \cdot \frac{dy}{dx} = 4a \ (1)$$

$$y \cdot \frac{dy}{dx} = 2a$$

$$y \cdot y_1 = 2a$$

$$a = \frac{y \cdot y_1}{2}$$

$$y^2 = \frac{4 \cdot y \cdot y_1}{2} \cdot \left(x + \frac{y \cdot y_1}{2}\right)$$

$$y = 2 \cdot \frac{dy}{dx} \left(x + y \cdot \frac{dy}{2dx}\right)$$

$$y \cdot \left(\frac{dy}{dx}\right)^2 - 2x \cdot \frac{dy}{dx} + y = 0$$

$y = a\cdot e^{x} + b\cdot e^{-x}$

$$\frac{dy}{dx} = a\cdot e^{x} + b\,e^{-x}(-1)$$

$$\frac{dy}{dx} = a\cdot e^{x} - b\cdot e^{-x}$$

$$\frac{dy}{dx} = -\,(a\cdot e^{x} + b\cdot e^{-x})$$

$$\therefore \frac{dy}{dx} + y = 0$$

or

$$\frac{d^2y}{dx^2} - y = 0$$

$y = e^{x}(a\cos x + b\sin x)$

$$\frac{dy}{dx} = e^{x}(-a\sin x + b\cos x) + e^{x}(a\cos x + b\sin x)$$

$$= e^{x}(a\cos x + b\cdot\sin x) + e^{x}(-a\sin x + b\cos x)$$

$$y_1 = y + e^{x}(-a\sin x + b\cos x)$$

$$y_1 - y = e^{x}(-a\sin x + b\cos x)$$

$$y_2 - y_1 = e^{x}(-a\cos x - b\sin x) + (-a\sin x + b\cos x)$$

$$y_2 - y_1 = -\,e^{x}(a\cos x + b\sin x) + (-a\sin x + b\cos x)$$

$$y_2 - y_1 = -y + (y_1 - y)$$

$$\frac{d^2y}{dx^2} - \frac{2\,dy}{dx} + 2y = 0$$

Solution of differential equation:

(1) Variable-seperable method:

1. $$\frac{dy}{dx} + \sqrt{\frac{1+y^2}{1+x^2}} = 0$$

$$\frac{dy}{dx} = \frac{-\sqrt{1-y^2}}{\sqrt{1+x^2}}$$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{-dx}{\sqrt{1+x^2}}$$

$$\int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1+x^2}}$$

$$\sinh^{-1} y = -\sinh^{-1} x + c$$

$$\sinh^{-1} x + \sinh^{-1} y = c$$

$$\frac{dy}{dx} = e^{x+y}$$

$$\frac{dy}{dx} = e^{x} \cdot e^{y}$$

$$\frac{dy}{e^{y}} = e^{x} dx$$

$$\int e^{-y} \cdot dy = \int e^{x} \cdot dx$$

$$\frac{e^{-y}}{-1} = e^{x} + c$$

$$e^{x} + e^{-y} + c = 0$$

$$e^{x} \cdot e^{y} + 1 + c \cdot e^{y} = 0$$

$$e^{x+y} + c \cdot e^{y} + 1 = 0$$

$$\frac{dy}{dx} = x^2 \cdot e^{x-y}$$

$$\frac{dy}{dx} = x^2 \cdot \frac{e^x}{e^y}$$

$$\int e^y\, dy = \int x^2 \cdot e^x \cdot dx$$

$$e^y = e^x\,(x^2 - 2x + 2) + c$$

$$\frac{dy}{dx} + y^2 = 0$$

$$\frac{dy}{dx} = -y^2$$

$$\int \frac{-1}{y^2}\, dy = \int dx$$

$$\frac{y^{-1}}{-1} = x + c$$

$$x + \frac{1}{y} + C = 0$$

$$y = \frac{1}{x+c}$$

$\frac{dy}{dx} + y^3 = 0 \quad y(1) = 1$, Find the interval of validity.

$$\frac{dy}{dx} = -y^3$$

$$\int y^{-3}\cdot dy = -\int dx$$

$$\frac{y^{-2}}{-2} = -x + c$$

$$\frac{(1)^{-2}}{-2} = -1 + c$$

$$c = \frac{1}{2} - 1 = \frac{-1}{2}$$

$$\frac{y^{-2}}{-2} = -\left(x - \frac{1}{2}\right)$$

$$\frac{1}{2y^2} = x - \frac{1}{2}$$

$$y^2 = \frac{1}{2(x - 1/2)}$$

For validity

$$\left(x - \frac{1}{2}\right) > 0 \qquad \because \text{Square of number always +ve}$$

$$x > \frac{1}{2}$$

$\therefore$ Interval of validity, $\frac{1}{2} < x < \infty$

$$e^x \cdot \tan y \cdot dx + (1+e^x) \cdot \sec^2 y \cdot dy = 0$$

$$e^x \tan y \, dx = -(1+e^x) \cdot \sec^2 y \cdot dy$$

$$\frac{e^x}{1+e^x} dx = \frac{-\sec^2 y}{\tan y} dy$$

$$\int \frac{e^x}{1+e^x} \cdot dx + \int \frac{\sec^2 y}{\tan y} \cdot dy = 0$$

$$\log(1+e^x) + \log(\tan y) = \log c$$

$$(1+e^x) \cdot \tan y = c$$

(2) Equation reducible to variable - seperable form:

If given D.E. is not directly under variable - seperable form, by making substitution again it can be transformed in variable seperable form.

$$\frac{dy}{dx} = (4x+y+1)^2$$

$$z = 4x + y + 1$$

$$\frac{dz}{dx} = 4 + \frac{dy}{dx}$$

$$\frac{dz}{dx} = 4 + z^2$$

$$\int \frac{dz}{2^2 + z^2} = \int dx$$

$$\frac{1}{2} \cdot \tan^{-1}\left(\frac{z}{2}\right) = x + c$$

$$\frac{1}{2} \tan^{-1}\left(\frac{4x+y+1}{2}\right) - x = c$$

$$\frac{dy}{dx} = \frac{a^2}{(x-y)^2}$$

$$x - y = z$$

$$1 - \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$

$$\frac{a^2}{z^2} = 1 - \frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{z^2 - a^2}{z^2}$$

$$\int \frac{z^2}{z^2 - a^2} \cdot dz = \int dx$$

$$z + a^2 \cdot \frac{1}{2a} \log\left(\frac{z-a}{z+a}\right) = x + c$$

$$x - y + a^2 \frac{1}{2a} \cdot \log\left(\frac{x-y-a}{x-y+a}\right) = x + c$$

$$y + c = \frac{a}{2} \log\left(\frac{x-y-a}{x-y+a}\right).$$

$$z^2 - a^2 \overline{\big) \; z^2 \;} \; 1$$
$$\underline{-z^2 + a}$$
$$a$$

or

$$\frac{z^2 - a^2 + a^2}{z^2 - a^2}$$

$$1 + \frac{a^2}{(z^2 - a^2)}$$

(3) Homogenous differential equation :

A D.E. is of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is said to be a homogenous differential equation if both $f(x,y)$ and $g(x,y)$ are homogenous functions of same degree.

Here substitute,

$$y = vx \quad \Rightarrow \quad v = \frac{y}{x}$$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

or

$$x = vy \quad \Rightarrow \quad v = \frac{x}{y}$$

$$\frac{dx}{dy} = v \cdot \frac{dy}{dy} + y \cdot \frac{dv}{dy}.$$

(For D.E. only algebric, trignometric & exponential function exist)

e.g.

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad \text{or} \quad \frac{dx}{dy} = \frac{2xy}{x^2+y^2}$$

- the solution will be same for both functions but "preference is given to the function which have more no. of terms in numerator than denominator (for algebric function in both numerator and denominator)

$$\therefore \quad \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

put

$$y = vx$$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

$$\therefore \quad v + x \cdot \frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x \cdot (x^2v^2)}$$

$$x \cdot \frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$\int \frac{1}{x} \cdot dx = \log cx$$

$$\int \frac{-1}{x} \cdot dx = \log \frac{c}{x}$$

$$\int \frac{2}{x} dx = \log cx^2$$

$$\int \frac{-3}{x} dx = \log \frac{c}{x^3}$$

$$\int \frac{-2}{x} dx = \log \frac{c}{x^2}$$

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$$x \cdot \frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

$$= \frac{1-v^2}{2v}$$

$$\int \frac{-2v}{1-v^2}\, dv = \int \frac{-dx}{x}$$

$$\log(1-v^2) = \log \frac{c}{x}$$

$$1 - \frac{y^2}{x^2} = \frac{c}{x}$$

$$\frac{x^2-y^2}{x^2} = \frac{c}{x}$$

$$x^2-y^2 = c \cdot x$$

$$x^2y \cdot dx - (x^3+y^3)dy = 0$$

$$x^3y \cdot dx = (x^3+y^3) \cdot dy$$

$$\frac{dx}{dx} = \frac{(x^3+y^3)}{x^3y}$$

$$= \frac{x^3}{x^2y} + \frac{y^3}{x^2y}$$

$$\frac{dx}{dy} = \frac{x}{y} + \frac{y}{x}$$

$$v + y \cdot \frac{dv}{dy} = v + \frac{1}{v^2}$$

$$\int v^2 \cdot dv = \int \frac{dy}{y}$$

$$\frac{v^3}{3} = \log y + c$$

$$\frac{x^3}{3y^3} = \log y + c$$

$$x \cdot \frac{dy}{dx} = y + x \tan\left(\frac{y}{x}\right)$$

For angle $\frac{y}{x}$ take $\frac{dy}{dx}$

if angle $\frac{x}{y}$ take $\frac{dx}{dy}$

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

put $y = xv$

$$v + x\frac{dv}{dx} = v + \tan v$$

$$\frac{dv}{\tan\theta} = \frac{dx}{x}$$

$$\int \cot\theta \cdot dv = \int \frac{dx}{x}$$

$$\log(\sin v) = \log cx$$

$$\sin\left(\frac{y}{x}\right) = cx$$

$$(1+e^{x/y}) \cdot dx + e^{x/y}\left(1 - \frac{x}{y}\right) \cdot dy = 0$$

$$(1+e^{x/y})\,dx = e^{x/y}\left(\frac{x-y}{y}\right) \cdot dy$$

$$\frac{dx}{dy} = \frac{e^{x/y}(x/y - 1)}{(1+e^{x/y})}$$

$$v + y \cdot \frac{dv}{dy} = \frac{e^v(v-1)}{(1+e^v)} \qquad x = vy$$

$$y \cdot \frac{dv}{dy} = \frac{v \cdot e^v(v-1)}{(1+e^v)} - v$$

$$y \cdot \frac{dv}{dy} = \frac{ve^v - e^v - v - v \cdot e^v}{(1+e^v)}$$

$$y \cdot \frac{dv}{dy} = \frac{-(e^v + v)}{(1+e^v)}$$

$$\int \frac{1+e^v}{v+e^v}\,dv = \int \frac{-dy}{y}$$

$$\log(v+e^{v}) = \log(c/y)$$

$$\frac{x}{y} + e^{x/y} = \frac{c}{y}$$

$$x + y\cdot e^{x/y} = c$$

(4) Non-homogenous equation.

The D.E. is of form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is said to be a non-homogenous differential equation.

because of constants $\therefore$ equ^n is not homogenc

Case I: If $a_1b_2 - a_2b_1 \neq 0$ then,

substitute $x = X+h$, $y = Y+k$.

where h and k are constants to be determined.

Case II: If $a_1b_2 - a_2b_1 = 0$ then, some part of numerator and denominator which is equal, is considered as substitution.

Find the substitution that transforms the given D.E. from non-homogenous to homogenous form.

$$\frac{dy}{dx} = \frac{3x+y-5}{2x+2y-2}$$

$$6-2 \neq 0$$

Case I :

h	k	1	h
3	1	-5	3
2	2	-2	2

$$\frac{h}{-2+10} = \frac{k}{-10+6} = \frac{1}{6-2}$$

$$\frac{h}{8} = \frac{k}{-4} = \frac{1}{4}$$

$$h = \frac{8}{4} = 2$$

$$k = \frac{-4}{4} = -1$$

$$x = X + 2$$

$$y = Y - 1$$

$$\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$$

h	k	1	h
1	2	-3	1
2	1	-3	2

$$\frac{h}{-6+3} = \frac{k}{-6+3} = \frac{1}{1-4}$$

$$\frac{h}{-3} = \frac{k}{-3} = \frac{1}{-5}$$

$$h = k = 1$$

$$x = X + 1$$

$$y = Y + 1$$

$$\frac{dy}{dx} = \frac{x + 2y + 1}{2x + 4y + 1}$$

$$4 - 2(2) = 0$$

$$\frac{dy}{dx} = \frac{(x + 2y) + 1}{2(x + 2y) + 1}$$

substitute $(x + 2y) = z$.

$$\frac{dy}{dx} = \frac{x+2y+3}{2x+y+5}$$

$$1-4 \neq 0$$

h	k	1	h
1	2	3	1
2	1	5	2

$$\frac{h}{10-3} = \frac{k}{6-5} = \frac{1}{1-4}$$

$$\frac{h}{7} = \frac{k}{1} = \frac{1}{-3}$$

$$h = \frac{-7}{3} \qquad k = \frac{-1}{3}$$

$$x = X \mp \frac{7}{3}$$

$$y = Y - \frac{1}{3}$$

Exact differential equation:

A differential equation is of the form $M \cdot dx + N \cdot dy = 0$ is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

General solution is

$$\int_0^x M \cdot dx + \int (\text{terms of } N \text{ not containing } x) \cdot dy = c$$

$(x^2+y^2) \cdot dx + 2xy \cdot dy = 0$

$$\frac{\partial M}{\partial y} = 2y. \qquad \frac{\partial N}{\partial x} = 2y. \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ solution is

$$\int_0^x (x^2+y^2) \cdot dx + \int 0 \cdot dy = c$$

$$\frac{x^3}{3} + xy^2 = c$$

$$2xy \cdot dx + (x^2+y^2) \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$$\int_0^x 2xy \cdot dx + \int y^2 \cdot dy = c$$

$$x^2 y + \frac{y^3}{3} = c$$

shortcut :

$$(ax + hy + g)\,dx + (hx + by + f)\,dy = 0$$

$$\frac{\partial M}{\partial y} = h \qquad \frac{\partial M}{\partial x} = h$$

$\therefore$ solution is

$$\frac{ax^2}{2} + hy(x) + gx + \frac{by^2}{2} + fy = c$$

$$ax^2 + 2hyx + 2gx + by^2 + 2fy = 2c$$

$$e^x \cdot \tan y \cdot dx + (1+e^x) \cdot \sec^2 y \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = e^x(\sec^2 y). \qquad \frac{\partial N}{\partial x} = \sec^2 y\,(e^x).$$

$$\int^x e^x \cdot \tan y \, dx + \int \sec^2 y \cdot dy = c$$

$$\tan y \cdot e^x + \tan y = c$$

$$(e^x + 1)\tan y = c$$

$$(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right) \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = e^{x/y}\left(\frac{-x}{y^2}\right). \qquad \frac{\partial N}{\partial x} = e^{x/y}\left(-\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right) e^{x/y} \frac{1}{y}$$

$$= e^{x/y}\left(\frac{-x}{y^2}\right)$$

$$\int (1 + e^{x/y}) \cdot dx +$$

$$= x + y \cdot e^{x/y} = c$$

Non-exact differential equation:

The D.E is of the form $M\,dx + N\cdot dy = 0$ is said to be non-exact if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Method 1:

To form suitable integrating factors:

(i) $x\,dy + y\cdot dx = d(xy)$

(ii) $\frac{x\cdot dy - y\cdot dx}{x^2} = d\left(\frac{y}{x}\right)$

(iii) $\frac{x\cdot dy - y\cdot dx}{xy} = d(\log y/x)$

(iv) $\frac{x\cdot dy - y\cdot dx}{x^2+y^2} = d(\tan^{-1} y/x)$

(v) $\frac{y\cdot dx - x\cdot dy}{y^2} = d(x/y)$

(vi) $\frac{y\cdot dx - x\cdot dy}{xy} = d\left(\log \frac{x}{y}\right)$

(vii) $\frac{y\cdot dx - x\cdot dy}{x^2+y^2} = d(\tan^{-1}(x/y))$

(viii) $\frac{y\cdot e^x\cdot dx - e^x\cdot dy}{y^2} = d(e^x/y)$

(ix) $\frac{2xy\cdot dx - x^2\cdot dy}{y^2} = d(x^2/y)$

(x) $\frac{x\cdot dx + y\cdot dy}{x^2+y^2} = d\left(\frac{1}{2}\log(x^2+y^2)\right)$

Q. $y(1+xy)\,dx + x(1-xy)\cdot dy = 0$

$$(y + xy^2)\cdot dx + (x - x^2y)\cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 1 + 2xy \qquad \frac{\partial N}{\partial x} = 1 - 2xy \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\therefore\ y\cdot dx + xy^2\cdot dx + x\cdot dy - x^2y\cdot dy$$

$$\frac{d\,xy}{(xy)^2} + \frac{xy\,(y\cdot dx - x\cdot dy)}{(xy)^2} = 0$$

(by divinding terms by certain Factor make it integrable)

$$\frac{1}{(xy)^2}\cdot d(xy) + \boxed{\frac{y\cdot dx - x\cdot dy}{xy}} = 0$$

standard.

$$\int \frac{1}{(xy)^2}\cdot d(xy) + \int d\left(\log\frac{x}{y}\right) = 0$$

$$\frac{-1}{xy} + \log\frac{x}{y} = c$$

$$y\cdot dx - x\cdot dy + (1+x^2)\cdot dx + x^2\sin y\cdot dy = 0$$

$$y\,dx - x\cdot dy + (1+x^2)\,dx + x^2\sin y\cdot dy = 0$$

$$-\frac{(x\cdot dy - y\cdot dx)}{x^2} + \frac{(1+x^2)\cdot dx}{x^2} + \frac{x^2\sin y\cdot dy}{x^2} = 0$$

perfect integral factor

$$\boxed{-\frac{(x\cdot dy - y\cdot dx)}{x^2}} + \left(\frac{1}{x^2} + 1\right)\cdot dx + \sin y\cdot dy = 0$$

$$-\int d\left(\frac{y}{x}\right) + \int\left(\frac{1}{x^2}+1\right)\cdot dx + \int \sin y\cdot dy = 0$$

$$\frac{-y}{x} - \frac{1}{x} + x - \cos y = c$$

$$y\,(axy + e^x).dx - e^x.dy = 0$$

Perfect integration factor (83)

$$\frac{ay.xy.dx}{y^2} + \boxed{\frac{e^x.dx - e^x.dy}{y^2}} = 0$$

$$\int ax.dx + \int d\left(\frac{e^x}{y}\right) = 0$$

$$\frac{ax^2}{2} + \frac{e^x}{y} = C$$

$$(y^2.e^x + 2xy).dx - x^2.dy = 0$$

$$\frac{y^2.e^x dx}{y^2} + \frac{2xy.dx - x^2.dy}{y^2} = 0$$

$$\int e^x.dx + \int d\left(\frac{x^2}{y}\right) = 0$$

$$e^x + \frac{x^2}{y} = C$$

$$\cancel{y\,(2x^2} \; 2x^2y^2.dx + y.e^x.dx - e^x.dy - y^3.dy = 0$$

$$\frac{2x^2y^2.dx}{y^2} + \frac{y.e^x.dx - e^x.dy}{y^2} - \frac{y^3.dy}{y^2} = 0$$

$$2\int x^2.dx + \int d\left(e^x/y\right) - \int y.dy = 0$$

$$\frac{2x^3}{5} + \frac{e^x}{y} - \frac{y^2}{2} = 0$$

Method 2

If $M\,dx + N\,dy = 0$ is non-exact but homogenous and if $Mx + Ny \neq 0$ then $\dfrac{1}{Mx+Ny}$ is Integrating factor.

$$(x^2+y^2)\,dx - 2xy\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial M}{\partial x} = -2y$$

$$Mx + Ny = (x^2+y^2)\cdot x + (-2xy)\cdot y$$

$$= x^3 + xy^2 - 2xy^2 \neq 0$$

then

$$\text{Integrating factor} = \frac{1}{Mx+Ny}$$

$$= \frac{1}{x^3 - xy^2} = \frac{1}{x(x^2-y^2)}$$

Any non-exact equation multiplied by its suitable integrating factor, then it becomes exact DE.

$$\int^{x} \frac{x^2+y^2}{x\cdot(x^2-y^2)}\,dx - \frac{2xy}{x(x^2-y^2)}\,dy = 0$$

Minor fraction d/dx Major fraction d/dx

$$\int \left(\frac{2x}{(x^2-y^2)} - \frac{1}{x} \right) dx - 0 = \int 0$$

$$\log \frac{(x^2-y^2)}{x} = \log c$$

$$x^2 - y^2 = cx$$

$x^2y\cdot dx - (x^3+y^3)\cdot dy = 0$

$$\text{Integrating factor} = \frac{1}{Mx+Ny}$$

$$= \frac{1}{(x^2y)x \pm (x^3+y^3)\cdot y}$$

$$= \frac{1}{x^3y - x^3y - y^4}$$

$$= \frac{1}{-y^4}$$

In Integrating factor, importance is not given to sign and constant

$$I.F. = \frac{1}{y^4}$$

$(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\cdot dy = 0$

$$\text{Integrating factor} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$= \frac{1}{x^2y^2}$$

$xy\cdot dx - (x^2+2y^2)\cdot dy = 0$

$$\text{Integrating factor} = \frac{1}{x^2y - x^2y - 2y^3}$$

$$= \frac{1}{-2y^3}$$

$$I.F. = \frac{1}{y^3}$$

Method 3.

If $M\,dx + N\,dy = 0$ is non-exact but is in the form $y \cdot f(x,y)\,dx + x \cdot g(x,y)\,dy = 0$ and $Mx - Ny \neq 0$, then $\dfrac{1}{Mx - Ny}$ is Integrating factor.

$$f(x,y) = x^3 - 5y^2 + 6x + 7y + 8$$
$$f(xy) = x^3y^3 - 5x^2y^2 + xy + 7$$

$$y(1+xy)\,dx + x(1-xy)\,dy = 0$$

$$Mx - Ny = xy + x^2y^2 - xy + x^2y^2$$
$$= 2x^2y^2 \neq 0$$

$$\therefore \quad \frac{1}{Mx-Ny} = \frac{1}{x^2y^2} \quad \text{is an integrating factor.}$$

solution is

$$\int^{x} \frac{y(1+xy)}{x^2y^2}\,dx + \int \frac{x(1-xy)}{x^2y^2}\,dy = 0$$

terms in x removed.

$$\therefore \int \frac{y}{x^2y^2} + \frac{xy^2}{x^2y^2}\,dx + \int -\frac{1}{y}\,dy = 0$$

$$\therefore \frac{-1}{xy} + \log\left(\frac{x}{y}\right) = c$$

$$y(x^2y^2 + xy + 1)\,dx + x(x^2y^2 - xy + 1)\,dy = 0$$

$$\text{Integrating factor} = \frac{1}{x^2y^2}$$

$$y(x^3y^3 + x^2y^2 + xy + 1)\,dx + x(x^3y^3 - x^2y^2 - xy + 1)\,dy = 0$$

$y(x^2y^2+2)\,dx + x(2-x^2y^2)\,dy = 0$

$\therefore$ Integrating factor $= \dfrac{1}{x^3y^3}$

$y(xy\sin xy + \cos xy)\,dx + x(xy\cdot\sin xy - \cos xy)\,dy = 0$

Integrating factor $= \dfrac{1}{xy\cos xy}$

(i) In both the terms, cancel terms with same degree and same sign.

(ii) In first bracket, take the remaining terms and increase its degree by 1.

(iii) Reciprocal of above obtained term is Integrating factor.

Method 4 :

If $M\cdot dx + N\cdot dy = 0$ is non-exact i.e. $\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$ and if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is purely function of x then integrating factor is $e^{\int f(x)\cdot dx}$

Method 5 :

If $M\cdot dx + N\cdot dy = 0$ is non exact and $\frac{1}{N}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is purely function of y then integrating factor is $e^{\int g(y)\cdot dy}$

Function	Integrating factor
$1/x$	x
$-1/x$	$1/x$
$2/x$	x^2
$-2/x$	$1/x^2$
$3/x$	x^3
$-3/x$	$1/x^3$

$(x^2+y^2+1)\cdot dx - 2xy\cdot dx = 0$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = -2y$$

$$\left.\begin{aligned} \therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} &= 2y + 2y = 4y \\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= -2y - 2y = -4y \end{aligned}\right\} \text{single term}$$

$\because$ M has 3 terms while N has only <u>one</u> term

$\therefore$

$$\text{I.F.} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$f(x) = \frac{1}{-2xy}\left(-4y\right) = \frac{-2}{x}$$

$\therefore$ Integrating factor $= 1/x^2$

$(y+xy^3)\cdot dx + 2(x^2y^2+x+y^4)\cdot dy = 0$

$$\therefore \frac{\partial M}{\partial y} = 1 + 3xy^2 \qquad \frac{\partial N}{\partial x} = 2(2xy^2+1)$$

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + 3xy^2 - 4xy^2 - 2$$

$$= 1 - xy^2$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 4xy^2 + 2 - 1 - 3xy^2$$

$$= xy^2 + 1$$

$$\therefore f(x) = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{(1+xy^2)}{y+xy^3}$$

$$= \frac{1}{y}\left(\frac{xy^2+1}{xy^2+1}\right)$$

$$f(x) = \frac{1}{y}$$

$$\text{I.F} = y$$

$$\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \frac{1}{4}\left(x + xy^2\right) \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 1 + y^2 \qquad \frac{\partial N}{\partial x} = \frac{1}{4} + \frac{y^2}{4}$$

$\therefore$ difference of $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$ will have two terms

$$\therefore \quad f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$= \frac{1}{\left(\frac{x+xy^2}{4}\right)}\left(1 + y^2 - \frac{1}{4} - \frac{y^2}{4}\right)$$

$$= \frac{4}{x(1+y^2)}\left(\frac{3}{4} y^2 + \frac{3}{4}\right)$$

$$= \frac{3(y^2+1)}{x(1+y^2)}$$

$$= \frac{3(1+y^2)}{x(1+y^2)} = \frac{3}{x}$$

$\therefore$ I.F. $= x^3$

$$(y^4 + 2y) \cdot dx + (xy^3 + 2y^4 - 4x) \, dy = 0$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2 \qquad \frac{\partial N}{\partial x} = y^3 - 4$$

$$f(x) = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{1}{y^4+2y}\left(y^3 - 4 - 4y^3 - 2\right)$$

$$= \frac{-3y^3 - 6}{y^4 + 2y}$$

$$= \frac{-3(y^3+2)}{y(y^3+2)}$$

$$= \frac{-3}{y} \qquad \therefore \text{ I.F.} = 1/y^3$$

$$(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\cdot dy=0$$

$$\frac{\partial M}{\partial y}=12x^2y^3+2x$$

$$\frac{\partial N}{\partial x}=6x^2y^3-2x$$

∵ difference has + sign divisible by M (i.e. having + sign)

$$f(x)=\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$$

$$=\frac{1}{(3x^2y^4+2xy)}\left(6x^2y^3-2x-12x^2y^3-2x\right)$$

$$=\frac{-6x^2y^3-4x}{(3x^2y^4+2xy)}$$

$$=\frac{-2(3x^2y^3+2x)}{y(3x^2y^3+2x)}$$

$$=\frac{-2}{y}$$

$$\therefore\ \text{I.F.}=1/y^2$$

Linear equation:

A differential equation is of the form $\frac{dy}{dx}+Py=Q$ where P and Q are function of x, is said to be function of x.

Then integrating factor $=e^{\int P\cdot dx}$

General solution is

$$y\cdot e^{\int P\cdot dx}=\int (Q\cdot e^{\int P\cdot dx})\cdot dx+c.$$

For $\frac{dx}{dy}+Px=Q$.

G.S. $$x\cdot e^{\int P\cdot dy}=\int (Q\cdot e^{\int P\cdot dy})\cdot dy+c.$$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{\log x}{x}$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right) \cdot y = \frac{\log x}{x}$$

$$y(x) = \int \frac{\log x}{x} \cdot (x) \cdot dx + c$$

$$xy = x \cdot \log x - x + c$$

$$(1-x^2) \cdot \frac{dy}{dx} - 2xy = x^2$$

$$\frac{dy}{dx} - x^2 \cdot \frac{dy}{dx} - 2xy = x^2$$

$$\frac{dy}{dx} - \left(\frac{2x}{1-x^2}\right) \cdot y = \frac{x^2}{1-x^2}$$

$$\text{I.F.} = e^{\int P \cdot dx} = e^{\int \frac{-2x}{1-x^2} \cdot dx}$$

$$= e^{\log(1-x^2)}$$

$$= 1-x^2$$

General solution is

$$y(1-x^2) = \int \frac{x^2}{(1-x^2)} \cdot (1-x^2) \cdot dx + c$$

$$= \frac{x^3}{3} + c$$

$$\cancel{x} \log x \frac{dy}{dx} + 2xy = e^{-xy} \qquad y(0) = 1$$

$$\frac{dy}{dx} + (2x) \cdot y = e^{-xy}$$

$$\text{I.F.} = e^{\int 2x \cdot dx}$$

$$= e^{x^2}$$

solution: $\quad y \cdot (e^{x^2}) = \int e^{-x^2} \cdot e^{x^2} \cdot dx + c$

$$y \cdot e^{x^2} = x + c$$

$$1 \cdot e^0 = 0 + c \qquad y(0) = 1$$

$$c = 1$$

$$y \cdot e^{x^2} = x + 1$$

$$y = \frac{x+1}{e^{x^2}}$$

$$y = e^{x^{-2}} \cdot (x+1)$$

- $x \cdot \log x \cdot \frac{dy}{dx} + y = 2\log x$

$$\frac{dy}{dx} + \left(\frac{1}{x \cdot \log x}\right) \cdot y = \frac{2\log x}{x}$$

$$\text{I.F.} = e^{\int \frac{1}{x \log x} \cdot dx}$$

$$= e^{\int \frac{\frac{1}{x}}{\log x} \cdot dx}$$

$$= e^{\log(\log x)}$$

$$= \log x$$

solution:

$$y(\log x) = \int \frac{2\log x}{x} \cdot dx + c$$

$$= 2\int \log x \cdot \frac{1}{x} \cdot dx + c$$

$$= 2\frac{(\log x)^2}{2} + c$$

$$y = \log x + \frac{c}{\log x}$$

$$(x+2y^3)\cdot\frac{dy}{dx} = y$$

$$\frac{dx}{dy} = \left(\frac{x+2y^3}{y}\right) \qquad \because \text{ linear in } x$$

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \left(\frac{1}{y}\right)\cdot x = 2y^2$$

$$\therefore \quad I.F. = \frac{1}{y}$$

General studies,

$$x\left(\frac{1}{y}\right) = \int 2y^2\cdot\frac{1}{y}\cdot dy + c$$

$$\frac{x}{y} = y^2 + c$$

$$x = y^3 + cy.$$

$$(1+y^2)\cdot dx + (x - e^{\tan^{-1}y})\cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 2y. \qquad \frac{\partial N}{\partial x} = 1 - 0$$

Method I -

Method II - Not homogenous

Method III - No. xy term

Method IV & V } - Not divisible by M & N also

$$\therefore \quad \frac{dx}{dy} = \frac{e^{\tan^{-1}y}}{1+y^2} - \frac{x}{1-y^2}$$

$$\frac{1}{1+y^2}\cdot x + \frac{dx}{dy} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

which is linear.

$$\therefore \quad I.F = e^{\int \frac{1}{1+y^2}\cdot dy}$$

$$= e^{\tan^{-1}y}$$

solution,

$$x \cdot e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \cdot e^{\tan^{-1}y} \cdot dy + c$$

$$= \frac{1}{2} \int e^{2\tan^{-1}y} \cdot \frac{2}{1+y^2} \cdot dy + \bar{c}$$

$$x \cdot e^{\tan^{-1}y} = \frac{1}{2} \cdot e^{2\tan^{-1}y} + c.$$

Bernoullis equation:

A D.E. of the form $\frac{dy}{dx} + Py = Q \cdot y^n$, where P & Q are functions of x, is said to be Bernoullis equation.

$$\frac{dy}{dx} + P \cdot y = Q \cdot y^n$$

$$\frac{1}{y^n} \frac{dy}{dx} + \frac{P \cdot y}{y^n} = \frac{Q \cdot y^n}{y^n}$$

$$y^{-n} \frac{dy}{dx} + P \cdot y^{1-n} = Q$$

$$z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n) \cdot y^{1-n-1} \cdot \frac{dy}{dx}$$

$$y^{-n} \frac{dy}{dx} = \frac{1}{(1-n)} \cdot \frac{dz}{dx}$$

$$P \cdot z - Q = \frac{1}{(1-n)} \cdot \frac{dz}{dx}$$

$$\frac{1}{(1-n)} \frac{dz}{dx} + Pz = Q$$

$$\frac{dz}{dx} + P(1-n) \cdot z = Q \cdot (1-n)$$ which is linear.

$$I.F. = e^{\int P(1-n) \cdot dx}$$

General solution is

$$y^{1-n} \cdot e^{\int P(1-n) \cdot dx} = \int (Q \cdot (1-n) \cdot e^{\int P(1-n) \cdot dx}) \cdot dx + c.$$

If.

$$\frac{dx}{dy} + Px = Q \cdot x^n.$$

$$\text{Integrating factor} = e^{\int P(1-n) \cdot dy}.$$

General solution.

$$x^{1-n} \cdot e^{\int P(1-n) \cdot dy} = \int [Q(1-n) \cdot e^{\int P(1-n) \cdot dy}] \cdot dy + c.$$

$$\frac{dy}{dx} + \frac{y}{x} = x^2 \cdot y^3.$$

$$\text{I.F.} = e^{\int P(1-n) \cdot dx}$$

$$= e^{\int \frac{1}{x}(1-3) \cdot dx}$$

$$= e^{-\int 2/x \cdot dx}$$

$$= \frac{1}{x^2}$$

$$\frac{dy}{dx} \cdot (xy + x^2y^3) = 1$$

$$\frac{dy}{dx} = xy + x^2 \cdot y^3$$

$$\frac{dx}{dy} - yx = y^3 \cdot x^2$$

$$\text{I.F.} = e^{\int -y(1-2) \cdot dy}$$

$$= e^{\int y \cdot dy}$$

$$= e^{y^2/2}$$

Orthogonal trajectory :

The family of curves which cuts every member of given family orthogonally is called Orthogonal trajectory of given family.

$$f(x, y, c) = 0 \quad \text{—— (i)}$$

To remove c, differentiate

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \quad \text{—— (ii)}$$

To get orthogonal trajectory, replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$

$$g\left(x, y, \frac{-dx}{dy}\right) = 0 \quad \text{—— (iii)}$$

The solution of equ^n —— (iii) is called orthogonal trajectory of given family.

(i) If equation —— (ii) & —— (iii) are identical then the curve is said to be self orthogonal. (perpendicular to itself)

Q. Find the orthogonal trajectory of family of lines passing through origins. is

(a) family of lines
(b) family of circles having centre at origin
(c) family of parabolas
(d) family of ellipses.

equ^n of lines passing through origin

$$y = mx + c \quad \text{—— (i)}$$

$$\frac{dy}{dx} = m$$

$$y = x \cdot \frac{dy}{dx} \quad \text{—— (ii)}$$

Replace $\frac{dy}{dx}$ by $\left(\frac{-dx}{dy}\right)$

$$y = -x \cdot \frac{dx}{dy} \quad \text{--- (iii)}$$

$$\int y \cdot dy = \int -x \cdot dx$$

$$\frac{y^2}{2} + \frac{x^2}{2} = c$$

$$x^2 + y^2 = 2c$$

i.e.

$$x^2 + y^2 = r^2$$

— equation of circle with centre origin

Find trojectory of family of parabolas $y^2 = 4ax$

$$y^2 = 4ax \quad \text{--- (i)}$$

$$2y \cdot \frac{dy}{dx} = 4a$$

$$\therefore \quad y^2 = 2y \cdot \frac{dy}{dx} \cdot x.$$

$$y = 2x \cdot \frac{dy}{dx} \quad \text{--- (ii)}$$

replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$

$$\therefore \quad y = 2x\left(-\frac{dx}{dy}\right)$$

$$\therefore \quad \int y \cdot dy = \int -2x \cdot dx$$

$$\therefore \quad \frac{y^2}{2} + x^2 = c$$

$$\frac{x^2}{c} + \frac{y^2}{2c} = -1$$

- equation of ellipse

Find trajectory of hyperbola $x^2-y^2=a^2$ - (rectangular hyperbola)

$$x^2-y^2=a^2 \quad \text{---(i)}$$

$$2x-2y.\frac{dy}{dx}=0$$

$$x-y.\frac{dy}{dx}=0 \quad \text{---(ii)}$$

replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$

$$x+y.\frac{dx}{dy}=0$$

$$x.dy+y.dx=0$$

$$\int d(xy)=\int 0$$

$$xy=c$$ equ^n of asymptole

Find trajectory for parabola $y^2=4a(x+a)$

$$y^2=4a(x+a) \quad \text{---(i)}$$

$$2y.\frac{dy}{dx}=4a$$

$$a=\frac{y.y_1}{2}$$

$$\therefore y^2=\frac{4y.y_1}{2}.(x+\frac{yy_1}{2})$$

$$y=2xy_1+y.y_1^2 \quad \text{---(ii)}$$

replace y_1 by $\frac{-1}{y_1}$

$$y=2x.\frac{-1}{y_1}+y.\frac{1}{y_1^2}$$

$$y=\frac{-2xy_1+y}{y_1^2}$$

$$y.y_1^2=-2xy_1+y$$

$$y=2xy_1+y.y_1^2 \quad \text{---(iii)}$$

$\because$ —(ii) & —(iii) are same

Given curve is self orthogonal.

$$\therefore \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad - \text{ self orthogonal.}$$

* first order D.E. has only one arbitrary constant.

Orthogonal trajectory of $x^2+y^2+2gx-5=0$ is
$x^2+y^2-2fy+5=0$

Monday
30^{th} September 201

Differential equation of 1^{st} order but not 1^{st} degree :

A D.E is of the form

$$a_0\left(\frac{dy}{dx}\right)^n + a_1\left(\frac{dy}{dx}\right)^{n-1} + a_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots\cdots a_n = 0$$

where $a_0, a_1, a_2 \cdots a_n$ are functions of x & y and constants, is said to be D.E of 1^{st} order but not first degree.

Let $\frac{dy}{dx} = P$

$$\therefore \quad a_0 \cdot P^n + a_1 P^{n-1} + a_2 \cdot P^{n-2} + \cdots\cdots a_n = 0$$

Let $f(x, y, P) = 0$

Thus, equation containing three variables x, y & P
$\therefore$ It may contain three types of solutions

- solvable for P
- solvable for x
- solvable for y

If solvable for P, then it must be factorised,

$$[P - f_1(x,y)]\,[P-f_2(x,y)]\,[P-f_3(x,y)] - [P-f_n(x,y)]$$

$$P - f_1(x,y) = 0$$

$$\frac{dy}{dx} = f_1(x,y)$$

$p^2 - 5p + 6 = 0$ where $p = \frac{dy}{dx}$

$(p-2)(p+3) = 0$

(If factorisation is not possible - it is not solvable for p)

$p - 2 = 0$

$\frac{dy}{dx} - 2 = 0$

$\int dy = 2\int dx$

$y - 2x - c_1 = 0$

$\therefore$ solⁿ is $(y - 2x - c_1)(y - 3x - c_2) = 0$

1. $x^2p^2 + xyp - 6y^2 = 0$

$(x^2p^2 + 3xyp - 2xyp - 6y^2) = 0$

$xp(xp + 3y) - 2y(xp + 3y) = 0$

$(xp + 3y)(xp - 2y) = 0$

If $x \cdot \frac{dx}{dy}$ or $y \cdot \frac{dy}{dx}$ is there go for variable seperable.

If $x \cdot \frac{dy}{dx}$ $\therefore$ $y \cdot \frac{dx}{dy}$ go for linear.

$xp + 3y = 0$

$x \cdot \frac{dy}{dx} + 3y = 0$

$\frac{dy}{dx} + \left(\frac{3}{x}\right)y = 0$

$y \cdot e^{\int}$

$x \cdot \frac{dy}{dx} - 2y = 0$

$\frac{dy}{dx} - \frac{2}{x}y = 0$

$y\left(\frac{1}{x^2}\right) - c_2 = 0$

$(x^3y - c_1)(y - c_2x^2) = 0$

2. $xyp^2 + xyp - 6y^2 = 0$

$xyp^2 - x^2p - y^2p + xy = 0$

$xp(yp - x) - y(yp - x) = 0$

$(yp - x)(xp - y) = 0$

$(yp - x) = 0$

$$y \cdot \frac{dy}{dx} - x = 0$$

$y\frac{dy}{dx}$ - go for variable seperable

$$\int y \cdot dy - \int x \cdot dx = \int 0$$

$$\frac{y^2}{2} - \frac{x^2}{2} = c_1$$

$$y^2 - x^2 - 2c_1 = 0$$

$$(xp - y) = 0$$

$$x \cdot \frac{dy}{dx} - y = 0$$

$$\frac{dy}{dx} - \left(\frac{1}{x}\right) \cdot y = 0$$

$$y\left(\frac{1}{x}\right) - c_2 = 0$$

$$(y^2 - x^2 - 2c_1)\left(\frac{y}{x} - c_2\right) = 0$$

solvable for y	solvable for x.
(i) degree of y must be 1	(i) degree for x must be 1
(ii) Seperate y from x and p $y = f(x,p)$	(ii) Seperate x from y and p $x = f(y, p)$
(iii) Differentiating on both sides w.r.t. x	(iii) Differentiating on both sid w.r.t. y
(iv) Replace $\frac{dy}{dx}$ by p.	(iv) Replace $\frac{dx}{dy}$ by $\left(\frac{1}{p}\right)$
(v) Neglect the terms or concel the terms which does not contain $\frac{dp}{dx}$	(v) Neglect the terms which does not contain $\frac{dp}{dy}$

1. $y + px = p^2x^4$

$$y + px = p^2x^4$$ where $p = \frac{dy}{dx}$

(there are only three terms, all having coefficient 1. thus factorisation not possible - Not solvable for p)

$$y = p^2x^4 - px$$

$$\frac{dy}{dx} = p^2(4x^3) + x^4\left(2p\cdot\frac{dp}{dx}\right) - p(1) - x\cdot\frac{dp}{dx}$$

$$p - 4p^2x^3 + p = x^4\left(2p\cdot\frac{dp}{dx}\right) - x\cdot\frac{dp}{dx}$$

$$2p(1 - 2px^3) = -x(1 - 2px^3)\cdot\frac{dp}{dx}$$

$$2p = -x\cdot\frac{dp}{dx}$$

by variable - seperable

$$\int \frac{-2\,dx}{x} = \int \frac{dp}{p}$$

$$\log(c/x^2) = \log p$$

$$p = \frac{c}{x^2}$$ — (sol^n should be y in terms of x)

$$y + x\cdot\left(\frac{c}{x^2}\right) = \left(\frac{c}{x^2}\right)^2 x^4$$

$$y + \frac{c}{x} = c^2$$ — (solution)

or

$$\frac{dy}{dx} = \frac{c}{x^2}$$

$$\int dy = \int \frac{c}{x^2}\,dx$$

$$y = \frac{-c}{x} + c'$$

$$y + \frac{c}{x} = c'$$ — (solution)

2. $y = 2px + y^2p^3$

$$2px = y - y^2p^3$$

$$2x = \frac{y}{p} - y^2p^2$$

$$2x = \left(\frac{1}{p}\right)y - y^2p^2$$

$$2\cdot\frac{dx}{dy} = \left(\frac{1}{p}\right) + y\left(\frac{-1}{p^2}\right)\frac{dp}{dx} - y^2\left(2p\,\frac{dp}{dx}\right) - p^2\cdot 2y$$

If equ^n is in the form $y = px + \phi(p)$

Replace p by c

solution is

$y = cx + \phi(c)$

(general s.

$$\therefore\ 2p - \frac{1}{p} - p^2\cdot 2y = -\frac{1}{p^2}\cdot y\cdot\frac{dp}{dy} - y^2\left(2p\,\frac{dp}{dy}\right)$$

$$-p\left(2 + \frac{1}{p^2} + 2py\right) = -yp\cdot\frac{dp}{dy}\left(\frac{1}{p^2} + 2y\cdot p\right)$$

$$p = yp\cdot\frac{dp}{dy}$$

$$\int\frac{dy}{y} = \int dp$$

$$\log y = p + c$$

Differential equation with constant coefficients:

A D.E is of the form.

$$a_0 \cdot \frac{d^ny}{dx^n} + a_1 \frac{dy^{n-1}}{dx^{n-1}} + a_2 \cdot \frac{d^{n-2}y}{dx^{n-2}} + \cdots a_n y = Q$$

where $a_0, a_1 \cdots a_n$ are constants and Q be function of x, is said to be D.E. with constant coefficients.

$$\frac{d}{dx} = D$$

$$\therefore a_0 \cdot D^n y + a_1 D^{n-1} \cdot y^{n-1} + a_2 \cdot D^{n-2} y^{n-2} + \cdots a_n \cdot D = Q.$$

$$f(D)y = Q$$

This equ^n may contain two types of sol^n. One is called complimentory function other is called particular integral.

Note:

(i) If $Q = 0$, then complimentory function is called the general solution.

(ii) If $Q \neq 0$, then only particular integral existed.

$$G.S. = C.F. + P.I.$$

1. To find the complementory function.

(i) Replace D by m, $f(m) = 0$ is called auxillary equ^n.

$$a_0 \cdot m^n + a_1 \cdot m^{n-1} + a_2 \cdot m^{n-2} + \cdots + a_n = 0$$

Case I: The roots are real and unequal.

$$\therefore y_c = c_1 \cdot e^{\alpha x} + c_2 \cdot e^{\beta x} + c_3 \cdot e^{\gamma x} + \cdots$$

where α, β, γ are roots of equ^n.

e.g.

$$\frac{d^2y}{dx^2} - 5\cdot\frac{dy}{dx} + 6y = 0$$

$$m^2 - 5m + 6y = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

$$\therefore \quad y = C_1 \cdot e^{2x} + C_2 \cdot e^{3x}$$

Case II : The roots are real and equal.

$$m = \alpha, \alpha, \beta$$

$$\therefore \quad y_c = (C_1 + C_2 x)\cdot e^{\alpha x} + C_3 \cdot e^{\beta x}$$

If $m = \alpha, \alpha, \alpha, \beta$

$$y_c = (C_1 + C_2 x + C_3 x^2)\cdot e^{\alpha x} + C_4 \cdot e^{\beta x}$$

If $m = \alpha, \alpha, \beta, \beta$

$$y_c = (C_1 + C_2 x)\, e^{\alpha x} + (C_3 + C_4 x)\, e^{\beta x}$$

e.g.

$$\frac{d^2y}{dx^2} - 4\cdot\frac{dy}{dx} + 4y = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2.$$

$\therefore$ soln. $\quad y = (C_1 + C_2 \cdot x)\cdot e^{2x}$

Case III : If the roots are imaginary.

If $m = \alpha \pm i\beta$

$$y = [C_1 \cos\beta x + C_2 \sin\beta x]\, e^{\alpha x}$$

If $m = \alpha \pm i\beta,\ \alpha \pm i\beta$

$$y = [(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x]\, e^{\alpha x}.$$

e.g.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

$$m^2 - 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= (1 \pm i)$$

$$y = (c_1 \cos x + c_2 \sin x) \cdot e^x$$

2. To find the roots of the equation.

(i) Every third degree equ^n must contain at least one real root.

(ii) If sum of all the coefficients is zero then $(m=1)$ satisfies the equation or $(m-1)$ is a factor.

(iii) If sum of coefficients of even power equal to sum of coefficients of odd power then $m=-1$ satisfies the equation or $(m+1)$ is factor.

e.g.

$$(D^3 - 3D + 2)y = 0$$

$$m^3 - 3m + 2 = 0$$

sum of coefficients $= 1 - 3 + 2 = 0$.

$m = 1$ is root

By Hobner's synthetic division

	m^3	m^2	m	con
	1	0	-3	2
$m=1$	0	1	1	-2
	1	1	-2	0

	1	1	-2	0
$m=1$	0	1	2	
	1	2	0	

$$m+2=0$$
$$m=-2$$

sol^n is
$$y=(C_1+C_2x)\cdot e^{x}+C_3\cdot e^{-2x}$$

1. $(D^3-4D^2+5D-2)y=0$

$$m^3-4m^2+5m-2=0$$

sum of coeff. $=1-4+5-2=0$

$m=1$ is root of equ^n

	1	-4	5	-2
$m=1$	0	1	-3	2
	1	-3	2	0
$m=1$	0	1	-2	
	1	-2	0	

$$m-2=0$$
$$m=2$$

sol^n is
$$y=(C_1+C_2x)\cdot e^{x}+C_3\cdot e^{2x}$$

2. $(D^4-2D^3+2D-1)y=0$

$$m^4-2m^3+2m-1=0$$

$m=1$ is root of eq^n

	1	-2	0	2	-1
$m=1$	0	1	-1	-1	1
	1	-1	-1	1	0
$m=1$	0	1	0	-1	
	1	0	-1	0	
$m=1$	0	1	1		
	1	1	0		

$m+1=0$
$m=1$

sol^n is

$$y = (c_1 + c_2 x + c_3 x^2) \cdot e^{x} + c_4 \cdot e^{-x}$$

3. $(D^3 + 6D^2 + 11D + 6)y = 0$

$$m^3 + 6m^2 + 11m + 6 = 0$$

sum. of coeff. of odd powers $= 1 + 11 = 12$

even powers $= 6 + 6 = 12$

$\therefore$ $m = -1$ root of equation.

	1	6	11	6
$m=-1$	0	-1	-5	-6
	1	5	6	0

$\therefore$ $m^2 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

$m = -2, -3$

$\therefore$ $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}.$

4. $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$

$$(m^4 - m^3 - 9m^2 - 11m - 4) = 0.$$

sum of coeff of odd powers $= -1 - 11 = -12$

even powers $= 1 - 9 - 4 = -12$

$\therefore$ $m = -1$ root of equ^n

	1	-1	-9	-11	-4
$m=-1$	0	-1	2	7	4
	1	-2	-7	-4	0
$m=-1$	0	-1	3	4	
	1	-3	-4	0	
$m=-1$	0	-1	4		
	1	-4	0		

odd $= 1 - 7 = -6$
even $= -2 - 4 = -6$

odd $= 1 - 4 = -3$
even $= -3$

$$m-4=0$$
$$m=4.$$
$$\therefore \quad y=(c_1+c_2x+c_3x)\cdot e^{-x}+c_4e^{4x}.$$

Q.

$$m^4+2m^2+1=0$$
$$(m^2+1)^2=0$$
$$m^2+1=0$$
$$m^2=-1$$
$$m=\pm i,\ \pm i$$

$$y=(C_1+C_2x)\cdot\cos x+(C_3+C_4x)\cdot\sin x.$$

3. To find particular integral :

$$f(D)y = Q.$$
$$y_P = P.I. = \frac{1}{f(D)}\,Q \qquad (\cancel{I}\,\cancel{L}\,ATE)$$

Q may be A,T

$$\frac{1}{D-\alpha}\,Q = e^{\alpha x}\int Q\cdot e^{-\alpha x}\,dx.$$

e.g..

$$\frac{1}{D^2-3D+2}\,e^{3x}$$
$$=\frac{1}{(D-2)(D-1)}\cdot e^{3x}$$
$$=\frac{1}{D-1}\cdot e^{2x}\int e^{3x}\cdot e^{-2x}\,dx$$
$$=\frac{1}{D-1}\cdot e^{2x}\int e^{x}\cdot dx$$
$$=\frac{1}{D-1}\cdot e^{2x}\cdot e^{x}$$

$$= e^{x}\int e^{3x}\cdot e^{-x}\cdot dx$$

$$= e^{x}\int e^{2x}\cdot dx$$

$$= \frac{e^{x}\cdot e^{2x}}{2}$$

$$= \frac{e^{3x}}{2}$$

is solution of equation (particular integral)

Case I: To find P.I. of form $\frac{1}{f(D)}\cdot e^{x}$

$$\frac{1}{f(D)}\, e^{ax} = \frac{1}{f(a)}\cdot e^{ax}. \quad \text{if } f(a) \neq 0$$

e.g.

$$\frac{1}{D^2-3D+2}\, e^{3x} = \frac{1}{9-3(3)+2}\, e^{3x}$$

$$= \frac{e^{3x}}{2}$$

Case II :

If

$$\frac{1}{f(D)}\, e^{ax} = \frac{x}{f'(D)}\cdot e^{ax}. \quad \text{if } f(a) = 0$$

$$= \frac{x^2}{f''(D)}\cdot e^{ax} \quad \text{if } f'(a) = 0$$

$$= \frac{x^3}{f'''(D)}\cdot e^{ax} \quad \text{if } f''(a) = 0$$

1. $(D^2-7D+6)\cdot y = e^{6x}$

$$y = \frac{1}{D^2-7D+6}\cdot e^{6x}$$

$$= \frac{x}{2D-7}\cdot e^{6x}$$

$$= \frac{x}{2(6)-7}\cdot e^{6x}$$

$$= \frac{x}{5}\cdot e^{6x}$$

2. $(D^2+4D+4)y = e^{2x}+e^{-2x}$

$$y = \frac{1}{D^2+4D+4}\cdot e^{2x}+e^{-2x}$$

$$= \frac{1}{D^2+4D+4}\cdot e^{2x} + \frac{1}{D^2+4D+4}\, e^{-2x}$$

$$= \frac{1}{2^2+4(2)+4}\, e^{2x} + \frac{1}{4+4(-2)+4}\cdot e^{-2x}$$

$$= \frac{1}{16}\cdot e^{2x} + \frac{x}{2D+4}\cdot e^{-2x}$$

$$= \frac{1}{16}\cdot e^{2x} + \frac{x^2}{2(1)}\cdot e^{-2x}$$

$$= \frac{1}{16}\cdot e^{2x} + \frac{x^2\cdot e^{-2x}}{2}$$

3. $(D^2-3D+2)y = \cos hx$

$$= \frac{1}{2}\left[\frac{1}{D^2-3D+2}\cdot e^{x} + \frac{1}{D^2-3D+2}\cdot e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{x}{2D-D}\cdot e^{x} + \frac{1}{1+3+2}\cdot e^{-x}\right]$$

$$= \frac{1}{2}\left[-x\cdot e^{x} + \frac{1}{6}\cdot e^{-x}\right]$$

$$= -\frac{x}{2}\, e^{x} + \frac{1}{12}\, e^{-x}$$

4. $(D^4 - D^3 - 9D^2 - 11D - 4) \cdot y = e^{-x}$

$$= \frac{1}{D^4 - D^3 - 9D^2 - 11D - 4} e^{-x}$$

$$= \frac{x}{4D^3 - 3D^2 - 18D - 11} e^{-x}$$

$$= \frac{x^2}{12D^2 - 6D - 18} e^{-x}$$

$$= \frac{x^3}{24D - 6} \cdot e^{-x}$$

$$= \frac{x^3 \cdot e^{-x}}{24(-1) - 6}$$

$$= \frac{-x^3}{30} \cdot e^{-x}$$

Case III: To find P.I. of form $\frac{1}{f(D)} \cdot \sin ax$, $\frac{1}{f(D)} \cdot \cos ax$,

replace D^2 by $-a^2$ if $\phi(-a^2) \neq 0$.

e.g. $\frac{1}{D^2 - 2D + 5} \cdot \sin 2x$

$$= \frac{1}{(-a)^2 + 2D + 5} \sin 2x$$

$$= \frac{1}{-4 + 2D + 5} \sin 2x$$

$$= \frac{1}{1 - 2D} \sin 2x$$

$$= \frac{(1 + 2D)}{(1 - 2D)(1 + 2D)} \cdot \sin 2x$$

$$= \frac{(1 + 2D)}{1 - 4D^2} \cdot \sin 2x$$

$$= \frac{(1 + 2D)}{1 - 4(-4)} \cdot \sin 2x$$

$$= \frac{1+2D}{17} \cdot \sin 2x$$

$$= \frac{\sin 2x \cdot 2\, D\, (\sin 2x)}{17}$$

$$= \frac{\sin 2x \cdot 2\times 2 \cdot \cos 2x}{17}$$

$$= \frac{\sin 2x - 4 \cdot \cos 2x}{17}$$

Case IV: To find P.I. of form $\frac{1}{D^2+a^2} \cdot \sin ax$, $\frac{1}{D^2+a^2} \cdot \cos$

$$\frac{1}{D^2+a^2} \sin ax = \frac{x}{2D} \sin ax$$

$$= \frac{x}{2}\left[\frac{-\cos ax}{a}\right]$$ $\because$ D = derivati $\frac{1}{D}$ = integrati

$$= \frac{-x}{2a} \cdot \cos ax$$

$$\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \cdot \sin ax.$$

e.g

$$\frac{1}{D^2+4} \cdot \cos 2x = \frac{x}{4} \cdot \sin 2x$$

$$\frac{1}{D^2+9} \cdot \cos 3x = \frac{x}{6} \cdot \sin 3x.$$

$$\frac{1}{D^2+16} \cdot \sin 4x = \frac{-x}{8} \cdot \cos 4x$$

* $$\frac{1}{D^2+1} \cdot \sin 2x = \frac{1}{-4+1} \sin 2x = \frac{-1}{3} \cdot \sin 2x$$

1. $(D^2-D-2)y = \cos 2x$

$$P.I. = \frac{1}{D^2-D-2} \cos 2x$$

$$= \frac{1}{-4-D-2} \cdot \cos 2x$$

$$= \frac{1}{-6-D} \cdot \cos 2x$$

$$= \frac{-6+D}{(-6-D)(-6+D)} \cos 2x$$

$$= \frac{-6+D}{36-D^2} \cos 2x$$

$$= \frac{-6+D}{36+4} \cdot \cos 2x$$

$$= \frac{-6\cos 2x - 2\sin 2x}{40}$$

2. $(D^2-4)y = \sin^2 x$

$$P.I. = \frac{1}{D^2-4} \sin^2 x$$

$$= \frac{1}{D^2-4} \cdot \left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4} \underbrace{(1)}_{=e^{0x}} - \frac{1}{D^2-4} \cdot \cos 2x\right].$$

$$= \frac{1}{2}\left[\frac{1}{0-4} - \frac{1}{-4-4} \cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{-1}{4} + \frac{1}{8} \cos 2x\right]$$

$$= \frac{-1}{8} + \frac{1}{16} \cdot \cos 2x$$

3. $(D^2+1)y = \sin 2x \cdot \cos x.$

$$P.I. = \frac{1}{D^2+1} \sin 2x \cdot \cos x$$

$$= \frac{1}{2} \frac{1}{D^2+1} 2\sin 2x \cdot \cos x$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1} \cdot \sin 3x + \frac{1}{D^2+1} \cdot \sin x\right]$$

$2\sin A \cdot \cos B = \sin(A+B) \cdot \sin(A-B).$

$$= \frac{1}{2}\left[\frac{1}{-9+1} \cdot \sin 3x + \frac{x}{2(1)} \cdot \cos x\right]$$

$$= \frac{-1}{16} \cdot \sin 3x - \frac{x}{4} \cdot \cos x.$$

4. $(D^2-2D+5) \cdot y = \cos 3x.$

$$P.I. = \frac{1}{D^2-2D+5} \cdot \cos 3x$$

$$= \frac{1}{-9-2D+5} \cos 3x$$

$$= \frac{1}{-4-2D} \cos 3x$$

$$= \frac{-4+2D}{16-4D^2} \cos 3x$$

$$= \frac{-4+2D}{16+36} \cos 3x$$

$$= \frac{1}{52}(-4\cos 3x + 6\sin 3x)$$

Case V: To find P.I. of the form $\frac{1}{f(D)} \cdot x^m$.

$$\frac{1}{f(D)} \cdot x^m = [f(D)]^{-1} \cdot x^m$$

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$$

$\cos^4\theta$

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$$

$$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots$$

1. $(D^2-4)y = x^2$

$$P.I = \frac{1}{D^2-4} \cdot x^2$$

In $f(D)$, if algebric function, any const. other than 1 is taken outside.

$$= \frac{1}{-4\left(1-\frac{D^2}{4}\right)} \cdot x^2$$

$$= \frac{-1}{4}\left[1-\frac{D^2}{4}\right]^{-1} \cdot x^2$$

$$= \frac{-1}{4}\left[1+\frac{D^2}{4}+\cancel{\frac{D^4}{16}}\right] \cdot x^2$$

$$= \frac{-1}{4}\left(x^2+\frac{2}{4}\right)$$

$$= \frac{-1}{4}\left(x^2+\frac{1}{2}\right)$$

2. $(D^3+8)y = x^4+2x+1$

$$P.I = \frac{1}{D^3+8}(x^4+2x+1)$$

$$= \frac{1}{8\left(1+\frac{D^3}{8}\right)}(x^4+2x+1)$$

$$= \frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1}(x^4+2x+1)$$

$$= \frac{1}{8}\left[1-\frac{D^3}{8}\right]\cdot(x^4+2x+1)$$

$$= \frac{1}{8}\left[x^4+2x+1-\frac{1}{8}(24x+0+0)\right]$$

$$= \frac{1}{8}(x^4+2x+1-3)$$

$$= \frac{x^4+2x-2}{8}$$

3. $(D^2+2D+1)\,y = x^2+2x$

$$P.I. = \frac{1}{(1+D)^2}\cdot x^2+2x$$

$$= (1+D)^{-2}\cdot(x^2+2x)$$

$$= (1-2D+3D^2)\cdot(x^2+2x)$$

$$= \left[x^2+2x-2(2x+2)+3(2+0)\right]$$

$$= \left[x^2+2x-4x-4+6\right]$$

$$= (x^2-2x+2)$$

Case VI To find PI of form $\frac{1}{f(D)}\cdot e^{ax}\cdot V$ where V is a function of x.

$$\frac{1}{f(D)}\,e^{ax}\cdot V = e^{ax}\cdot\frac{1}{f(D+a)}\cdot V$$

V-can be triagnometri or algebric. but NOT exponential

e.g.

1. $(D^2-2D+1)\,y = x^2\cdot e^x$

$$P.I = e^x\,\frac{1}{(D-1)^2}\cdot x^2$$

$$= e^x\,\frac{1}{(D+1-1)^2}\,x^2$$

$$= e^{x}\frac{1}{D^2}\cdot x^2 \qquad \frac{1}{D^2} = \iint$$

$$= e^{x}\cdot\frac{x^4}{12}$$

2. $(D^2-2D+5)y = e^{x}\sin x$

$$P.I. = \frac{1}{(D^2-2D+5)}\cdot e^{x}\sin x$$

$$= e^{x}\frac{1}{(D+1)^2-2(D+1)+5}\cdot\sin x$$

$$= e^{x}\frac{1}{D^2+2D+1-2D-1+5}\sin x$$

$$= e^{x}\frac{1}{D^2+4}\cdot\sin x$$

$$= e^{x}\frac{1}{-1+4}\cdot\sin x$$

$$= e^{x}\cdot\frac{\sin x}{3}$$

3. $(D^2-4D+4)y = e^{2x}\cos 3x$

$$P.I. = \frac{1}{D^2-4D+4}\cdot e^{2x}\cdot\cos 3x$$

$$= e^{2x}\frac{1}{(D-2)^2}\cos 3x$$

$$= e^{2x}\frac{1}{(D+2-2)^2}\cos 3x$$

$$= e^{2x}\frac{1}{D^2}\cos 3x$$

$$= e^{2x}\cdot\frac{-\cos 3x}{9}$$

Case VII : To find P.I. of the form $\frac{1}{f(D)} \cdot xV$ where, V is function of x.

$$\frac{1}{f(D)} \cdot xV = x \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} \cdot V$$

V - will be only trignom

1. $(D^2+4)y = x \cdot \cos x$

$$P.I. = \frac{1}{(D^2+4)} x \cdot \cos x$$

$$= x \cdot \frac{1}{D^2+4} \cdot \cos x - \frac{2D}{(D^2+4)^2} \cdot \cos x$$

$$= x \cdot \frac{1}{-1+4} \cdot \cos x - \frac{-2 \cdot \sin x}{(-1+4)^2}$$

$$= \frac{x}{3} \cdot \cos x + \frac{2}{9} \cdot \sin x$$

2. $(D^2-2D+1)y = x e^x \sin x$

$$\therefore P.I. = \frac{1}{(D^2-2D+1)} x \cdot e^x \sin x$$

$$= e^x \left[x \frac{1}{(D-1)^2} \cdot \sin x - \frac{2D-2}{(D^2-2D+1)^2} \cdot \sin x \right]$$

$$= e^x \left[x \cdot \frac{1}{D^2-2D+1} \sin x - \frac{2\cos x - 2\sin x}{(D^2-2D+1)^2} \right]$$

or $$= e^x \left[\frac{1}{(D+1-1)^2} x \cdot \sin x \right]$$

$$= e^x \left[\frac{1}{D^2} \cdot x \sin x \right]$$

$$= e^x \left[x \frac{1}{D^2} \cdot \sin x - \frac{2D}{D^4} \cdot \sin x \right]$$

$$= e^x \left[x \frac{1}{-1} \sin x - \frac{2\cos x}{(-1)^2} \right]$$

$$= e^x \left[-x \sin x - 2\cos x \right]$$

Differential equation with variable coefficient

A D.E. is of the form

$$a_0 x^n \frac{d^n y}{dy^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots a_n y = Q$$

where, $a_0, a_1, \dots a_n$ are constants & Q be the function of x, is said to be homogenerious D.E. with variables constant or Cauchy- Euler form of D.E.

(Denominator function is independant variable)

$$x = e^z$$

$$z = \log x$$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{d}{dz} = \frac{d}{dx} \frac{dx}{dz} = x \frac{d}{dx}$$

$$x \frac{d}{dx} = \frac{d}{dz} = \theta$$

$$D = \frac{d}{dx}$$

$$\theta = \frac{d}{dz}$$

Proof :

$$\frac{d}{dz}\left(x \frac{d}{dx} \right) = \frac{d^2}{dz^2}$$

$$\frac{d}{dx} \cdot x \cdot \frac{d}{dx} \cdot \frac{dx}{dz} = \theta^2$$

$$x \left[x \frac{d^2}{dx^2} + \frac{d}{dx}(1) \right] = \theta^2$$

$$x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} = \theta^2$$

$$x^2 \cdot \frac{d^2}{dx^2} + \theta = \theta^2$$

$$\frac{x^2 d^2}{dx^2} = \theta^2 - \theta = \theta(\theta-1)$$

similarly $\quad \frac{x^3 d^3}{dx^3} = \theta(\theta-1)(\theta-2)$

$$= \theta^3 - 3\theta^2 + 2\theta$$

$$xD = \theta$$

$$x^2D^2 = \theta^2 - \theta$$

$$x^3D^3 = \theta^3 - 3\theta^2 + 2\theta$$

1. $(x^2D^2 - xD - 3)y = 0$

$$(\theta^2 - \theta - \theta - 3)y = 0$$

$$(\theta^2 - 2\theta - 3)y = 0$$

$$m^2 - 2m - 3 = 0$$

$$(m-3)(m+1) = 0$$

$$m = 3, -1$$

sol^n is $\quad y = c_1 \cdot e^{3z} + c_2 \cdot e^{-z}$

$$= \cancel{c_1 \cdot x^3 + c_2}$$

$$= c_1 \cdot (e^z)^3 + c_2 \cdot (e^z)^{-1}$$

$$y = c_1 \cdot x^3 + c_2 \cdot x^{-1}$$

2. $(x^3D^3 + 3x^2D^2 - 2xD + 2)y = 0$

$$\theta^3 - 3\theta^2 + 2\theta + 3(\theta^2 - \theta) - 2(\theta) + 2 = 0$$

$$\theta^3 - 3\theta^2 + 3\theta^2 - 3\theta + 2 = 0$$

$$\theta^3 - 3\theta + 2 = 0$$

$$m^3 - 3m + 2 = 0$$

sum $= 1 - 3 + 2 = 0$

$\therefore m = 1$ is root

	1	0	−3	2
m=1	0	1	1	−2
	1	1	−2	0
m=1	0	1	2	
	1	2	0	

$$m+2=0$$
$$m=-2$$

solution is:

$$y = (c_1+c_2 z)e^{z} + c_3 \cdot e^{-2z}$$
$$= (c_1 + c_2 \log x)\cdot x + c_3 \cdot x^{-2}$$

3 . $(x^2D^2+xD-4)y=0$ $\therefore y(0)=0$

$$(\theta^2-\theta+\theta-4)y=0$$
$$(\theta^2-4)y=0$$
$$m^2-4=0$$
$$m^2=+4$$
$$m=\pm 2$$
$$y = c_1 e^{2z} + c_2 \cdot e^{-2z}$$
$$= c_1 \cdot x^{2} + c_2 \cdot x^{-2}$$
$$y = c_1 \cdot x^2 + \frac{c_2}{x^2}$$

At $y(0)=0$

$0 = 0 + c_2$ $\therefore c_2 = 0$

At $y(1)=1$

$1 = c_1 + c_2$ $\therefore c_1 = 1$

$$y = x^2$$

4. $x^2y'' + xy' - y = 0$

$$(\theta^2 - \theta + \theta - 1)y = 0$$

$$(\theta^2 - 1)y = 0$$

$$m^2 = 1$$

$$m = \pm 1$$

$$y = c_1 \cdot e^{z} + c_2 \cdot e^{-z}$$

$$= c_1 x + c_2 \cdot \frac{1}{x}$$

5. $x^2 \cdot \dfrac{d^3y}{dx^3} + 2x \cdot \dfrac{d^2y}{dx^2} - 2 \cdot \dfrac{dy}{dx} = 0$

$$(x^2 D^3 + 2x \cdot D^2 - 2D)y = 0$$

$$(x^3 D^3 + 2x^2 D^2 - 2xD)y = 0$$

$$(\theta^3 - 3\theta^2 + 2\theta + 2(\theta^2 - \theta) - 2\theta)y = 0$$

$$(\theta^3 - 3\theta^2 + 2\theta + 2\theta^2 - 2\theta - 2\theta)y = 0$$

$$(\theta^3 - \theta^2 - 2\theta)y = 0$$

$$m^3 - m^2 - 2m = 0$$

$$m(m^2 - m - 2) = 0$$

$$m(m-2)(m+1) = 0$$

$$m = 0, 2, -1$$

$$y = c_1 \cdot e^{0z} + c_2 \cdot e^{2z} + c_3 e^{-z}$$

$$= c_1 + c_2 \cdot x^2 + c_3 \cdot \frac{1}{x}$$

6. $(4D^2x^2 + 12xD + 3)y = 0$

$$(4(\theta^2 - \theta) + 12\theta + 3)y = 0$$

$$(4\theta^2 - 4\theta + 12\theta + 3)y = 0$$

$$(4\theta^2 + 8\theta + 3)y = 0$$

$$4m^2+8m+3=0$$

$$(2m+1)(2m+3)=0$$

$$m=-\frac{1}{2},\ \frac{-3}{2}$$

$$\therefore y = C_1 x^{-1/2} + C_2 \cdot x^{-3/2}$$

Laplace's Transformation:

Let $f(t)$ be function in t defined for all positive values of 't'. then Laplace transform of $f(t)$ is denoted by $L[f(t)]$ and is defined as

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\cdot dt = F(s)$$

then

$f(t) = L^{-1}[F(s)]$ is called inverse Laplace's transform of $F(s)$.

$$L(t^n) = \int_0^\infty e^{-st}\, t^n \cdot dt$$

Let

$$x = st \Rightarrow t = \frac{x}{s}$$

$$dt = \frac{dx}{s}$$

if $t=0$, $x=0$

$t=\infty$ $x=\infty$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \cdot \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \cdot x^n \cdot dx.$$

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n+1} \cdot dx$$

$\Gamma n+1 = n\,\Gamma n$ if $n > 0$

$\Gamma n+1 = n!$ if $n \in N$

$\Gamma 1 = 1$ $\Gamma 1/2 = \sqrt{\pi}$

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}} \qquad \text{if } n \in N$$

$$= \frac{n\,\Gamma(n)}{s^{n+1}} \qquad \text{if } n > 0$$

$$L(1) = \frac{1}{s} \qquad \text{if } n = 0$$

$$L(t) = \frac{1}{s^2} \qquad , \text{ if } n = 1$$

$$L(t^2) = \frac{2}{s^3}$$

$$L(e^{at}) = \frac{1}{s-a} \qquad L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(e^{-at}) = \frac{1}{s+a} \qquad L(\sinh at) = \frac{a}{s^2-a^2}$$

$$L(\sin at) = \frac{a}{s^2+a^2} \qquad L(\cosh at) = \frac{s}{s^2-a^2}$$

1. $L(t^2+6t+8)$

$$= L(t^2) + 6\,L(t) + 8\,L(1)$$

$$= \frac{2}{s^3} + \frac{6}{s^2} + \frac{8}{s}$$

2. $L(\sqrt{t})$

$$= L(t^{1/2})$$

$$= \frac{\Gamma(1/2+1)}{s^{1/2+1}}$$

$$= \frac{\frac{1}{2}\,\Gamma(1/2)}{s^{3/2}} \qquad \tfrac{1}{2} \text{ - positive number.}$$

$$= \frac{\sqrt{\pi}}{2 \cdot s^{3/2}}$$

3. $L\left(\frac{1}{\sqrt{t}}\right)$

$$= L\left(t^{-1/2}\right)$$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$$

$$= \frac{\Gamma\frac{1}{2}}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

4. $L\left(e^{2t}+3\cdot e^{-2t}\right)$

$$\Rightarrow L\cdot\left(e^{2t}\right) + 3\ L\left(e^{-2t}\right)$$

$$= \frac{1}{s-2} + 3\ \frac{1}{s+2}$$

5. $L\left(\sin^2 t\right)$

$$= L\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2}\left[L(1) - L(\cos 2t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$

$$= \frac{2}{s(s^2+4)}$$

6. $L\left(\cos^3 t\right)$

$$= L\left[\frac{\cos 3t - 3\cos t}{4}\right]$$

$\cos 3A = 4\cos^3 A - 3\cos A$

$$= \frac{1}{4}\ L(\cos 3t) - 3\ (\cos t)$$

$$= \frac{1}{4}\left[\frac{s}{s^2+9} + \frac{3\times 1}{s^2+1}\right]$$

7. $L(\sin 2t \cdot \cos t)$

$$= \frac{1}{2} L(2\sin 2t \cos t)$$

$$= \frac{1}{2} L(\sin 3t + \sin t)$$

$$= \frac{1}{2}\left[\frac{3}{9+s^2} + \frac{1}{1+s^2}\right]$$

$f(t) = 0 \quad 0 < t < 2$

$\quad = 3 \quad t \geq 2$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 e^{-st} \cdot 0 \cdot dt + \int_2^\infty e^{-st} \cdot 3 \cdot dt$$

$$= 0 + 3\left(\frac{e^{-st}}{-s}\right)_2^\infty$$

$$= \frac{-3}{s}(e^{-\infty} - e^{-2s})$$

$$= \frac{-3}{5}(0 - e^{-2s})$$

$$= \frac{3}{s} e^{-2s}$$

Shifting theorem:

If $L[f(t)] = \int_0^\infty e^{-st} \cdot f(t) \cdot dt = F(s)$

then,

$$L[e^{at} f(t)] = F(s-a)$$

L.H.S.

$$= \int_0^\infty e^{-st} \cdot e^{at} f(t) \cdot dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot f(t) \cdot dt$$

$$= F(s-a).$$

$\therefore \quad L(t^n) = \dfrac{n!}{s^{n+1}}$ if $n \in N$

$L(e^{at} \cdot t^n) = \dfrac{n!}{(s-a)^{n+1}}$

$L(e^{-at} \cdot t^n) = \dfrac{n!}{(s+a)^{n+1}}$

$L(e^{at} \cdot \sin bt) = \dfrac{b}{(s-a)+b^2}$ $\qquad L(e^{-at} \sin bt) = \dfrac{b}{(s+a)+b^2}$

$L(e^{at} \cdot \cos bt) = \dfrac{s-a}{(s-a)+b^2}$ $\qquad L(e^{-at} \cos bt) = \dfrac{s+a}{(s+a)+b^2}$

$L(e^{at} \sinh bt) = \dfrac{b}{(s-a)^2-b^2}$ $\qquad L(e^{+at} \cosh bt) = \dfrac{s-a}{(s-a)^2-b^2}$

e.g. $L(t \cdot \sin at) \Rightarrow$

We know $e^{ait} = \cos at + \underbrace{i \sin at}_{\text{imaginary part}}$

$=$ Imaginary part of $L(t \cdot e^{ait})$ $\qquad L(t) = \dfrac{1}{s^2}$

$=$ Imaginary part of $\left[\dfrac{1}{(s-ai)^2} \times \dfrac{(s+ai)^2}{(s+ai)^2}\right]$

$=$ I. P. of $\left[\dfrac{s^2-a^2+2ias}{(s^2+a^2)^2}\right]$

* $L(t \cdot \sin at) = \dfrac{2as}{(s^2+a^2)^2}$

* $L(t \cdot \cos at) =$ Real part of $\left[\dfrac{s^2-a^2+2ias}{(s^2+a^2)^2}\right]$

$= \dfrac{s^2-a^2}{(s^2+a^2)^2}$

Inverse Laplace Transform :

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L\left(\frac{t^n}{n!}\right) = \frac{1}{s^{n+1}}$$

$$L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$$

$$L^{-1}\left(\frac{1}{s}\right) = 1$$

$$L^{-1}\left(\frac{1}{s^2}\right) = t$$

$$L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}$$

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\cdot \sin at$$

$$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\cdot \sinh at$$

$$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

Find: $L^{-1}\left(\frac{1}{2s-5}\right)$

$$= \frac{1}{2} L^{-1}\left(\frac{1}{s-5/2}\right)$$

$$= \frac{1}{2}\cdot e^{5/2}$$

$$L^{-1}\left(\frac{1}{s(s+1)}\right)$$

$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$= L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$= 1 - e^{-t}$$

$$L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$= e^{-st}\cdot L^{-1}\left(\frac{1}{s^2}\right) \quad \text{shifting theorem}$$

$$= e^{-st}\cdot t$$

$L\left(\frac{}{(s+a)^2}\right)$

$$= L^{-1}\left(\frac{s+a-a}{(s+a)^2}\right)$$

when denominator is perfect square, arrange numerator in terms of denominator

$$= e^{-at}\cdot L^{-1}\left(\frac{s}{s^2} - \frac{a}{s^2}\right)$$

- shifting thm.

$$= e^{-at}\cdot L^{-1}\left(\frac{1}{s} - \frac{a}{s^2}\right)$$

$$= e^{-at}(1-at)$$

$L^{-1}\left(\frac{s+23}{s^2+13-4s}\right)$

not perfect square - arrange as perfect square) (don't go for partial factor)

$$= L^{-1}\left(\frac{s+23}{s^2-2\cdot s\cdot 2+4+9}\right)$$

$$= L^{-1}\left(\frac{s-2+25}{(s-2)^2+3^2}\right)$$

$$= e^{2t}\, L^{-1}\left(\frac{s}{s^2+3^2} + 25\,\frac{1}{s^2+3^2}\right)$$

$$= e^{2t}\left(\cos 3t + \frac{25}{3}\sin 3t\right)$$

$L^{-1}\left(\frac{s-2}{s^2+2s+2}\right)$

$$= L^{-1}\left(\frac{s-2}{s^2+2s+1+1}\right)$$

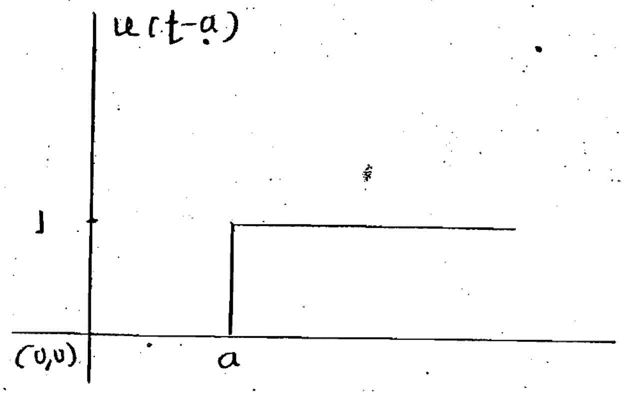
$$= L^{-1}\left(\frac{s+1-3}{(s+1)^2+1}\right)$$

$$= e^{-t}\cdot L^{-1}\left(\frac{s\cancel{+1}-3}{s^2+1}\right)$$

$$= e^{-t}\cdot L^{-1}\left(\frac{s}{s^2+1} - \frac{3}{s^2+1}\right)$$

$$= e^{-t}(\cos t - 3\sin t)$$

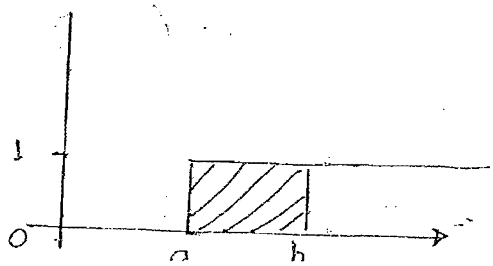
Unit step function:



$$u(t-a) = 0 \qquad 0 < t < a$$
$$= 1 \qquad t \geq a.$$

$$L[u(t-a)] = \int_0^\infty e^{-st} \cdot u(t-a) \cdot dt$$
$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} \cdot 1 \cdot dt$$
$$= 0 + \left(\frac{e^{-st}}{-s}\right)_a^\infty$$
$$= \frac{e^{-\infty} - e^{-as}}{-s}$$
$$= \frac{0 - e^{-as}}{-s}$$
$$= \frac{e^{-as}}{s}$$

Find Laplace's transform of function (shaded)



$$L(f(t)) = \frac{1}{s}e^{-as} - \frac{1}{s}e^{-bs}$$
$$= \frac{1}{s}(e^{-as} - e^{-bs})$$

Periodic function:

A function is $f(t)$ said to be periodic function with a period T

Upper finite bound of the function is always period of a function ($T=4$)

$$L[f(t)] = \frac{1}{1-e^{-4s}}\left[\int_0^2 e^{-st}(3)\cdot dt + \int_2^4 e^{-st}(0)\cdot dt\right]$$

$$= \frac{3}{1-e^{-4s}}\left[\frac{e^{-st}}{-s}\right]_0^2 + 0$$

$$= \frac{3}{1-e^{-4s}}\left[\frac{e^{0}-e^{-2s}}{s}\right]$$ (lower limit put 1^{st})

$$= \frac{3(1-e^{-2s})}{s(1-e^{-2s})(1+e^{-2s})}$$

$$= \frac{3}{s(1+e^{-2s})}$$

$$f(t) = 1 \qquad 0 < t < b$$
$$\quad = -1 \qquad b < t < 2b$$

$$f(t+2b) = f(t)$$

$$L[f(t)] = \frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}(1)\,dt + \int_b^{2b} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\left[\frac{e^{-st}}{-s}\right]_0^b + \left[\frac{e^{-st}}{s}\right]_b^{2b}\right]$$

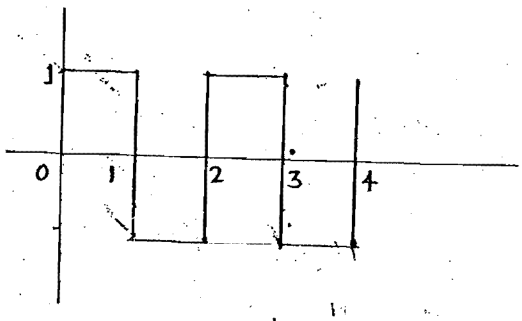
$$= \frac{1}{1-e^{-2bs}}\left[\frac{1-e^{-bs}+e^{-2bs}-e^{-bs}}{s}\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{1-2e^{-bs}+e^{-2bs}}{s}\right]$$

$$= \frac{(1-e^{-bs})^2}{(1-e^{-bs})(1+e^{-bs})s}$$

$$= \frac{(1-e^{-bs})}{s(1+e^{-bs})} \quad * \text{(shortcut)}$$

Q. A function f(t) is defined in the graph find Laplace's transform of the function.



$$f(t) = 1 \qquad 0 < t < 1$$
$$\quad = -1 \qquad 1 < t < 2$$

$$\therefore L[f(t)] = \frac{1-e^{-s}}{s(1+e^{-1})}$$

If $L[f(t)] = F(s) = \dfrac{5s^2+23s+6}{s(s^2+2s+2)}$

then.

$\lim_{t\to\infty} f(t) = ?$

(a) 3 (b) 5 (c) $\frac{17}{2}$ (d) ∞

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\cdot F(s)$$

$$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\left[\frac{5s^2+23s+6}{s(s^2+2s+2)}\right]$$

$$= \frac{5(0)+23(0)+6}{0+2(0)+2}$$

$$= 3$$

If $L(f(t)) = \dfrac{3s+1}{s^3+4s^2+(k-3)s}$ and $\lim_{t\to\infty} f(t) = 1$

then $k = ?$

$$\lim_{s\to 0} s\cdot f(s) = 1$$

$$= \lim_{s\to 0} s\left[\frac{3s+1}{s(s^2+4s+k-3)}\right]$$

$$1 = \frac{1}{0+0+k-3}$$

$$k-3 = 1$$

$$k = 4.$$

If $L[f(t)] = \dfrac{1}{s^2(s+1)}$ then $f(t) = ?$

(a) $t-1+e^{-t}$

(b) $= t+1+e^{-t}$

(c) $-1+e^{-t}$

(d) $2t+e^{t}$

for ⓐ $\because \quad L(t) = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$

$$= \frac{(s+1) - (s+1)\cdot s + s^2}{s^2(s+1)}$$

$$= \frac{1}{s^2(s+1)}$$

Complex Variables:

$$Z = x + iy$$

$$\bar{Z} = x - iy$$

$$\therefore \quad Z \cdot \bar{Z} = |z|^2 = x^2 + y^2$$

$\therefore$ $Z = x + iy = r(\cos\theta + i\sin\theta)$ is called modulus- amplitud form or polar form

$$\therefore \quad \text{mod. } Z = r = \sqrt{x^2+y^2}$$

$$\text{amp. } Z =$$

$$= \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \quad e^{i\theta} = \cos\theta + i\sin\theta$$

$$\therefore \quad (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n.\theta$$

$$1 = \cos\theta + i\sin\theta = e^{i\theta}$$

$$\therefore \quad i = \cos \pi/2 + i\sin \pi/2 = e^{i\pi/2}$$

$$(1+i) = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$= \sqrt{2}\cdot e^{i\pi/4}$$

$$\therefore \quad i^i = (e^{i\pi/2})^i = e^{i^2 \cdot \pi/2} = e^{-\pi/2}$$

$\because \quad i = \sqrt{-1}$

$i^2 = -1$

1. $\dfrac{i+3}{i+1}$

$$= \frac{(i+3)(i-1)}{(i+1)(i-1)}$$

$$= \frac{i^2-i+3i-3}{i^2-1}$$

$$= \frac{-1+2i-3}{-2}$$

$$= \frac{-4+2i}{-2}$$

$$= \frac{-2(2-i)}{-2}$$

$$= (2-i)$$

2. $\dfrac{1+2i}{i-2}$

$$= \frac{(1+2i)(i+2)}{(i-2)(i+2)}$$

$$= \frac{i+2+2i^2+4i}{i^2-4}$$

$$= \frac{-2+5i+2}{-5}$$

$$= -i$$

5. $\dfrac{-5+10i}{3+4i}$

$$= \frac{(-5+10i)(3-4i)}{9-16i^2}$$

$$= \frac{-15+20i+30i-40i^2}{9-16i^2}$$

$$= \frac{25+50i}{25}$$

$$= 1+2i$$

4. $\left|\dfrac{3+4i}{1-2i}\right|$

$$= \frac{|3+4i|}{|1-2i|}$$

$$= \frac{\sqrt{9+16}}{\sqrt{1+4}}$$

$$= \frac{5}{\sqrt{5}}$$

$$= \sqrt{5}$$

5. $(1+i)^8$

$$(1+i)^8 = [(1+i)^2]^4$$

$$= [1+2i+i^2]^4$$

$$= [1+2i-1]^4$$

$$= 16i^4$$

$$= 16$$

$\because \quad i^4 = (-1)(-1) = 1$

6. The polar form of $2+2i$ is

(a) $\frac{\pi}{4}e^{i\sqrt{2}}$ (b) $\frac{\pi}{4}e^{i2\sqrt{2}}$ (c) $\sqrt{2}\cdot e^{i\pi/4}$ (d) $2\sqrt{2}\cdot e^{i}$

$$\because \quad r = \sqrt{4+4}$$

$$= 2\sqrt{2}$$

polar form $= 2\sqrt{2}\cdot e^{i\pi/4}$.

Analytic function:

A single valued function which is defined and differential at each point of domain D is said to be an analytical function, in the domain.

Note:

The necessary and sufficient condition for a function $f(z) = u(x,y) + i \cdot v(x,y)$ to be analytic is, it should satisfy Caauchy-Reemann's (C-R) equation as.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

Harmonic function:

Any function of x and y satisfying the Laplace's equation is said to be harmonic function.

$$u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= 2-2$$

$$\nabla^2 u = 0$$

Note:

The real and imaginary parts of an analytic function satisfies the Laplace's equation.

$$\nabla^2 u = 0$$

$$\nabla^2 v = 0$$

Construction of analytic function:

If $u = f(x,y)$

shortcut* $f'(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ci$

e.g. $u = x^2+y^2$

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{z^2}{4} - \frac{z^2}{4i^2}$$

$$= \frac{z^2}{4} + \frac{z^2}{4}$$

$$= \frac{z^2}{2}$$

$$\therefore f(z) = 2\left(\frac{z^2}{2}\right) - 0 + ci$$

$$= z^2 + ci$$

1. $u = x^3 - 3xy^2 + 3x + 1$

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \left(\frac{z}{2}\right)^3 - 3\left(\frac{z}{2}\right)\left(\frac{z^2}{4i^2}\right) + 3\left(\frac{z}{2}\right) + 1$$

$$= \frac{z^3}{8} + \frac{3z^3}{8} + \frac{3z}{2} + 1$$

$$= \frac{4z^3}{8} + \frac{3z}{2} + 1$$

$$= \frac{z^3}{2} + \frac{3z}{2} + 1$$

$$\therefore f(z) = 2\left(\frac{z^3}{2} + \frac{3z}{2} + 1\right) - 1 + ci$$

$$= z^3 + 3z + 2 - 1 + ci$$

$$= z^3 + 3z + 1 + ci.$$

To find v (imaginary part)

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy \qquad \text{Total derivative}$$

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy.$$

$$dv = \frac{-\partial u}{\partial y}\cdot dx + \frac{\partial u}{\partial x}\cdot dy \quad \text{which is exact}$$

$u = x^2 - y^2$

$$\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{-\partial u}{\partial y} = \frac{-\partial v}{\partial x}.$$

$$\frac{\partial u}{\partial x} = 2x. \qquad \frac{\partial u}{\partial y} = -2y$$

$$\int dv = \int +2y\cdot dx + \int \cancel{2x\cdot dy}$$

C terms not containing x

$$V = 2xy$$

2. $u = e^x \cdot \cos y$

$$\frac{\partial u}{\partial x} = e^x\cdot\cos y. \qquad \frac{\partial u}{\partial y} = e^x(-\sin y)$$

$$v = \int e^x\cdot \sin y \; dx + \int \cancel{e^x\cdot\cos y}$$

$$v = e^x\cdot \sin y$$

$$f(z) = u + iv$$

$$= e^x\cdot\cos y + (e^x \sin y)i$$

$$= e^x(\cos y + i\sin y)$$

$$= e^x\cdot e^{iy}$$

$$= e^{x+iy} = e^z$$

3. $V = xy$ find u

$$\frac{\partial u}{\partial x} = y. \qquad \frac{\partial u}{\partial y} = x$$

$$V = \int x\cdot dx$$

$$du = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy$$

$$= \frac{\partial v}{\partial y}\cdot dx - \frac{\partial v}{\partial x}\cdot dy$$

$$\int du = \int x \cdot dx - \int y \cdot dy$$

$$= \frac{x^2}{2} - \frac{y^2}{2}$$

$$= \frac{1}{2}(x^2 - y^2)$$

Singular points:

The singular points are those at which the given function $f(z)$ is not analytic. These points are also called singularities or poles.

Q. Find the singular points of function. $f(z) = \dfrac{1-2z}{z(z-1)(z-2)}$

singular points $= z = (0, 1, 2.$

all the poles above are of order 1. (simple poles)

$$f(z) = \frac{z^2}{(z-1)(z+1)(z-i)(z+i)} = \frac{z^2}{z^4-1} = \frac{z^2}{(z^2-1)(z^2+1)}$$

poles $= \pm 1, \pm i$ of order 1

$$f(z) = \frac{z^2}{z^2 - 3z + 2}$$

$$z^2 - 3z + 2 = 0$$

$$(z-1)(z-2) = 0$$

$z = 1, 2,$ are poles

$$f(z) = \frac{z^2}{(z-1)^2 (z-2)^3}$$

poles, $z = 1$, of order 2

$z = 2$ of order 3

Cauchy's theorem :

If $f(z)$ is an analytic function and its derivative $f'(z)$ is continuous at all points inside and on a simple and closed curve c, then.

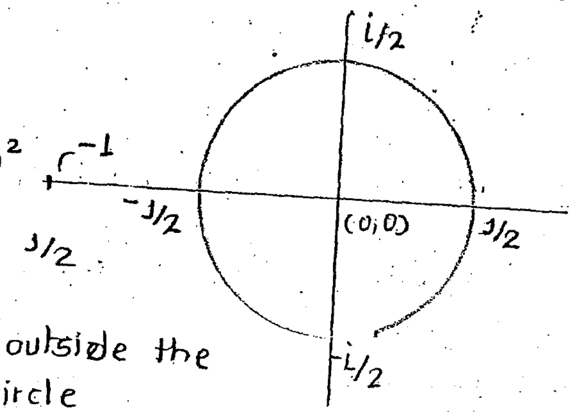
$$\int_c f(z) \cdot dz = 0$$ (pole is outside the reagion)

Q. The value of $\int_c \frac{3z^2+7z+1}{z+1} dz$ where c is circle, $|z| = 1/2$

$|z| = 1/2$

$(x^2+y^2)^2 = (1/2)^2$, $r=1$

centre $(0,0)$ & radius $1/2$



pole, $z = -1$ i.e. outside the circle

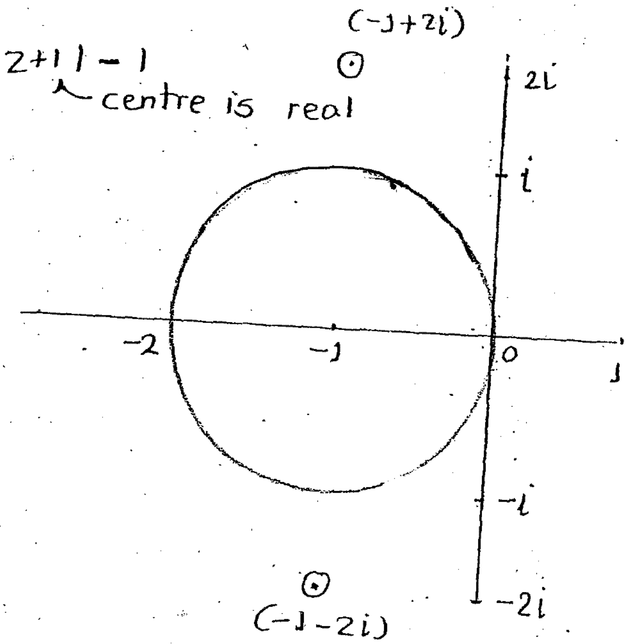
$$\int_c \frac{3z^2+7z+1}{z+1} dz = 0$$

Q. $\int_c \frac{z+4}{z^2+2z+5} dz$ where c is $|z+1| = 1$ ← centre is real

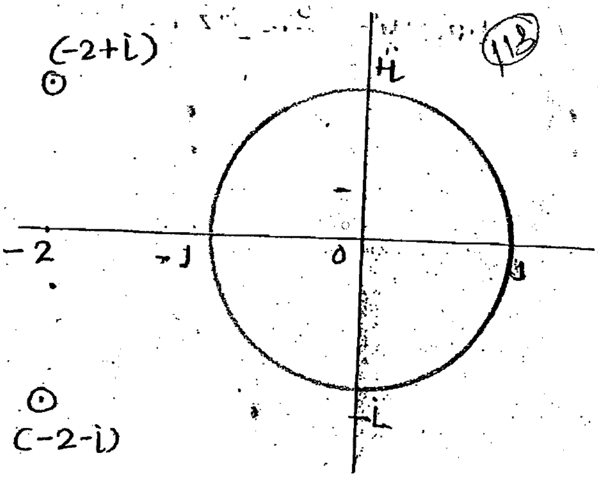
$z^2+2z+5 = 0$

$$z = \frac{-2 \pm \sqrt{4-20}}{2}$$

$= -1 \pm 2i$



3. $\oint \dfrac{-3z+4}{z^2+4z+5}\,dz = ?\quad |z|=1$



$$z^2+4z+5=0$$

$$z=\frac{-4\pm\sqrt{16-20}}{2}$$

$$=\frac{-4\pm 2i}{2}$$

poles $= -2\pm i$

$\therefore$ poles are outside circle.

$$\oint \frac{-3z+4}{z^2+4z+5}\,dz = 0$$

2014 GATE

Cauchy's integral formula:

If $f(z)$ is an analytic function inside and on a simple and closed curve 'C', then, if 'a' is any point inside c then,

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\oint_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i\, f'(a)$$

$$\oint_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!}\, f''(a)$$

$$\vdots$$

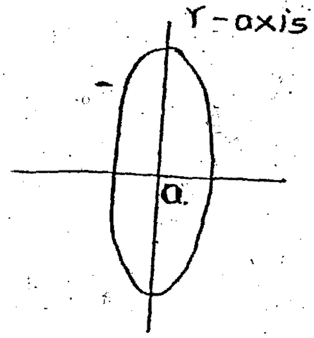
$$\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i\, f^{n-1}(a)}{(n-1)!}$$

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!}\, f^{n}(a)$$

Q. Find $\oint_C \frac{\cos z}{z}\, dz$ where c is ellipse $9x^2+4y^2=1$.

$$\frac{x^2}{(1/9)} + \frac{y^2}{(1/4)} = 1$$

More. →

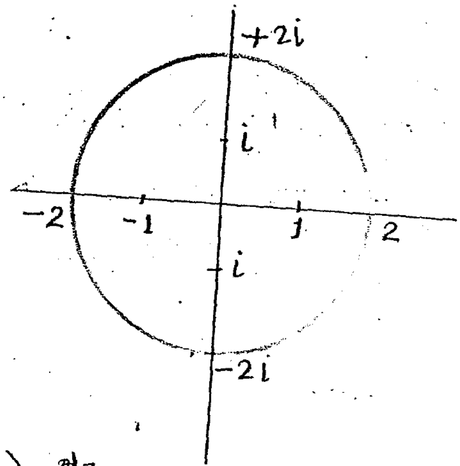


$$\oint_C \frac{\cos z}{z}\, dz = 2\pi i \left[\cos z\right]_{z=0}$$

$$= 2\pi i \cdot \cos 0$$

$$= 2\pi i$$

Q The value of $\oint_C \frac{1}{z^2-1}$ where c is circle of $x^2+y^2=4$



$$z^2-1=0$$

$$(z-1)(z+1)=0$$

poles, $z = 1, -1$

both are inside the circle.

Method 1 (partial fraction)

$$\oint_C \frac{1}{z^2-1} = \oint_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right)\frac{dz}{2}$$

$$= \frac{1}{2}\oint_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right) dz$$

$$= \frac{1}{2}\left[2\pi i(1) - 2\pi i(1)\right]$$

$$= 0$$

Method 2.

$(z-1)$ take as pole
$(z+1)$ - as function

& $(z+1)$ - as function
$(z-1)$ - as pole.

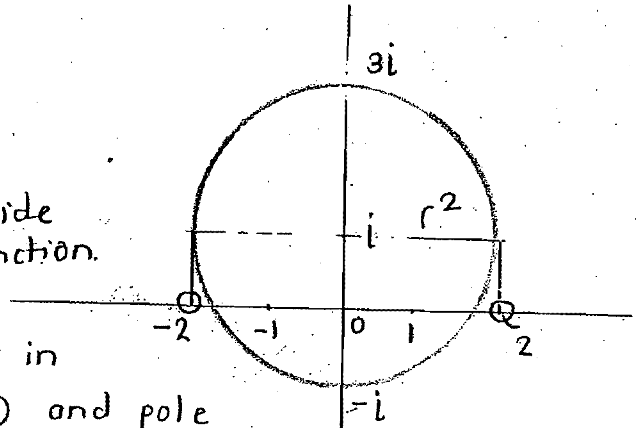
$$\oint_C \frac{1}{z^2-1} = \int_C \frac{\frac{1}{z+1}}{(z-1)}\, dz + \int \frac{\frac{1}{(z-1)}}{(z+1)}\, dz$$

$$= 2\pi i \left(\frac{1}{z+1}\right)_{z=1} + 2\pi i \left(\frac{1}{z-1}\right)_{z=-1}$$

$$= 2\pi i \left(\frac{1}{2}\right) + 2\pi i \left(\frac{-1}{2}\right)$$

$$= 0$$

Q. Find $\oint_c \frac{z-1}{(z+1)^2 (z-2)} \, dz$ where c is circle $|z-i| = 2$



$(z+1)^2 (z-2) = 0$

$z = -1$ and $z = 2$

inside the function — outside function.

* pole inside function, put in denominator (as pole) and pole outside function put in numerator (as function)

$$\oint_c \frac{\left(\frac{z-1}{z-2}\right)}{(z+1)^2} \, dz$$

$$f(z) = \frac{z-1}{z-2} = 1 + \frac{1}{z-2}$$

$$f'(z) = 0 - \frac{1}{(z-2)^2}$$

$$\therefore \oint_c \frac{\left(\frac{z-1}{z-2}\right)}{(z+1)^2} \, dz = 2\pi i \left(\frac{-1}{(z-2)^2}\right)_{z=-1}$$

$$= \frac{-2\pi i}{(-1-2)^2}$$

$$= \frac{-2\pi i}{9}$$

Q. For $|z-i| = 2$ find $\oint_c \frac{z^2-4}{z^2+4}\, dz$

$$z^2+4 = 0$$

$$(z+2i)(z-2i) = 0$$

$$z = 2i, \quad -2i$$

inside $\quad$ outside

$$\oint_c \frac{\left(\frac{z^2-4}{z+2i}\right)}{z-2i} = 2\pi i \left(\frac{z^2-4}{z+2i}\right)_{z=2i}$$

$$= 2\pi i \left(\frac{-4-4}{2i+2i}\right)$$

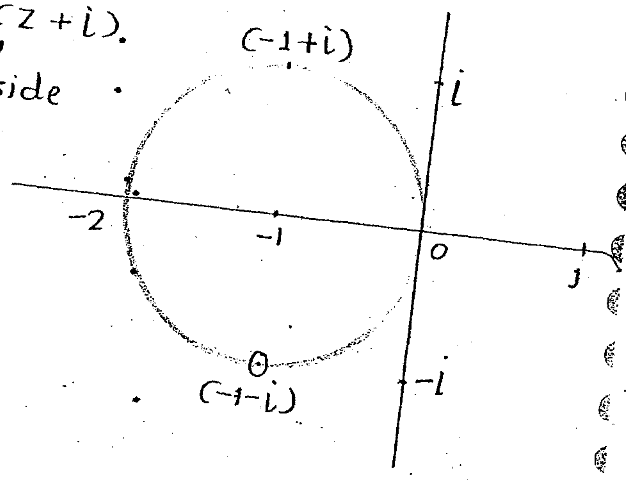
$$= \frac{-8\pi}{2}$$

$$= -4\pi$$

Q. Find $\oint_c \frac{z^2}{z^4-1}\, dz$ where c is the circle $|z+1| = 1$

$$z^4-1 = (z-1)(z+1)(z-i)(z+i).$$

outside $\quad$ inside $\quad$ outside



$$\oint_c \frac{z^2}{z^4-1} = \oint_c \frac{\frac{z^2}{(z-1)(z^2-i^2)}}{(z+1)}$$

$$= \oint_c \frac{\frac{z^2}{(z-1)(z^2+1)}}{(z+1)}$$

$$= 2\pi i \left[\frac{z^2}{(z-1)(z^2+1)}\right]_{z=-1}$$

$$= 2\pi i \left(\frac{1}{-2\times 2}\right)$$

$$= -\frac{\pi i}{2}$$

Residue at $z=a$ where 'a' is the pole of order 'n'. (not asked)

$$\lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \cdot f(z) \right]$$

Q. Find residue of function at respective poles

$$f(z) = \frac{1-2z}{z(z-1)(z-2)}$$

poles of function $= 0, 1, 2$ (of first order, $n=1$)

At $z=0$

$$\text{Residue} = \lim_{z \to 0} \frac{z \cdot (1-2z)}{z(z-1)(z-2)}$$

$$\frac{(1-0)}{(-1)(-2)} = \frac{1}{2}$$

At $z=1$

$$= \lim_{z \to 1} \frac{(z-1)\cdot(1-2z)}{z(z-1)(z-2)}$$

$$= \frac{-1}{-1} = 1$$

At $z=2$

$$= \lim_{z \to 2} \frac{(z-2)\cdot(1-2z)}{z(z-1)(z-2)}$$

$$=$$

Q. Find residue

$$f(z) = \frac{1}{z(z+2)^3}$$

At $z=-2$, $n=3$

$$\text{Residue} = \lim_{z \to -2} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z+2)^3 \cdot \frac{1}{z(z+2)^3} \right]$$

$$= \lim_{z \to -2} \frac{1}{2!} \left(\frac{2}{z^3} \right) = \frac{1}{(-2)^3} = \frac{-1}{8}$$

Numerical methods :

Solution of non-linear equations :

Let $f(x)=0$ be any non-linear equation. To find the approximate root of $f(x)=0$, chose any two values of $x = a, b$ particularly adjacent values in such a way that there functional values must have opposite signs. Then we must say that root lies between A and b. This approximate root can be found by

(i) Method of bisection

(ii) Regula - Falsi method

(iii) Newton - Raphson method

(i) Method of bisection

$$x_1 = \frac{a+b}{2}$$

if $f(x_1) < 0$

$f(a) \cdot f(x_1) \cdot f(b)$

$$x_2 = \frac{x_1+b}{2}$$

$f(x_2) > 0$

$$x_3 = \frac{x_1+x_2}{2}$$

(must go upto 13^{th} to 14^{th} approximation)

The rate of convergence in bisection method is very-very slow. Comparatively bisection method, the Regula Falsi method has little fast rate of convergence.

(ii) Regula-falsi method :

$$x_1 = \frac{a\,f(b) - b\cdot f(a)}{f(b) - f(a)}$$

(must go upto 7^{th} to 8^{th} approximation)

(i) Newton-Raphson method:

It has very fast rate of convergence. The convergen. of Newton-Raphson method is quadratic convergence or the 2nd order convergence.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\vdots$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

1. $f(x) = x^3+2x-5=0 \qquad x_0=1 \quad$ then $x_1 = ?$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{(1)^3+2(1)-5}{3(1)^2+2}$$

$$= 1 - \frac{-2}{5}$$

$$= 1.4.$$

2. $f(x) = x^3-2x-5=0 \quad , x_1 = ?$

$$\left.\begin{aligned} f(0) &= -5 \quad < 0 \\ f(1) &= 1-2-5 \\ &= -6 \quad < 0 \\ f(2) &= 8-4-5 \\ &= -1 \quad < 0 \\ f(3) &= 27-6-5 \\ &= 16 \quad > 0 \end{aligned}\right\} \therefore x_0 = 2 \ \text{- nearer to 0}$$

$$x_1 = 2 - \frac{(2)^3-2(2)-5}{3(2)^2-2}$$

$$x_1 = 2.1$$

3. Starting from $x_0 = 1$. $f(x) = x^3+3x-7=0$, find x_1

$$x_1 = 1 - \frac{3(1)^3+3(1)-7}{3(1)+3}$$

$$= 1 - \frac{-3}{6}$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2} \qquad 1.5$$

4. $f'(x) = x^2-2x-1$, $x_0 = 2$, $x_1 = ?$

$$x_1 = 2 - \frac{(2)^2-2(2)-1}{2(2)-2}$$

$$= 2 - \frac{-1}{2}$$

$$= \frac{5}{2} = 2.5$$

$$x_2 = 2.5 - \frac{(2.5)^2-2(2.5)-1}{2(2.5)-2}$$

$$= 2.5 - \frac{6.25-5-1}{5-2}$$

$$= 2.5 - \frac{0.25}{3}$$

$$= \frac{7.25}{3} = 2.417$$

5. $f(x) = e^x - 1 = 0$, $f_0 = -1$

$$x_1 = -1 - \frac{e^{-1}-1}{e^{-1}}$$

$$= -1 - \frac{(1-e)\cdot e}{e}$$

$$= -1 \mp 1 + e$$

$$= e - 2$$

$$= 2.71 - 2$$

$$= 0.71$$

Q. The Newton - Raphson iteration $x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$ can be used to compute.

(a) square of R (b) square root of R

(c) $\log R$ (d) R^{-1}

value of $x_{n+1} = x_n = \alpha$ we stop iterations.

$$\alpha = \frac{1}{2}\left(\alpha + \frac{R}{\alpha}\right)$$

$$2\alpha - \alpha = \frac{R}{\alpha}$$

$$\alpha^2 = R$$

$\alpha = \sqrt{R}$. used to find square root of R.

The value $x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$ converges to ..

ans $= \sqrt{5}$

Q. Recursion relation.

$$f(x) = x - e^{-x},$$

It can be solved by Newton - Raphson method

(a) $x_{n+1} = e^{-x_n}$

(b) $x_{n+1} = x_n - e^{-x_n}$

✓ (c) $x_{n+1} = \dfrac{(1+x_n) \cdot e^{-x_n}}{1+e^{-x_n}}$

(d) $x_{n+1} = \dfrac{x^2 - e^{-x_n}(1+x_n)-1}{x_n - e^{-x_n}}$

Ans. $f'(x) = 1 + e^{-x}$

Consider the series $x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n+1}$ & $x_0 = 0.5$ is obtained by Newton's - Raphson method, then series converges to.

(a) 1.4 (b) 1.414 (c) 1.5 (d) 1.6.

$$x_{n+1} = x_n = \alpha$$

$$\alpha = \frac{\alpha}{2} + \frac{9}{8\alpha+1}$$

$$= \frac{8\alpha^2+18}{16\alpha+2}$$

$$16\alpha^2 + 2\alpha = 8\alpha^2 + 18$$

$$8\alpha^2 = 18 - 2\alpha$$

$$\alpha^2 = \frac{9}{4}$$

$$\alpha = \sqrt{\frac{9}{4}} = 1.5$$

The solution of variables x_1 and x_2 for the equations

$$u = 10\, x_2 \sin x_1 - 0.8 = 0$$

$$v = 10x_2^2 - 10\, x_2 \cdot \cos x_1 - 0.6 = 0$$ is to be solved by Newton-Raphson method assuming initial values of $x_1 = 0.0$ and $x_2 = 1.0$ the $J(x_1, x_2)$ is

$$J(x_1, x_2) = \frac{\partial(u,v)}{\partial(x_1, x_2)} = \begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial u}{\partial x_1} = 10\, x_2 \cdot \cos x_1 \qquad \frac{\partial u}{\partial x_2} = 10 \cdot \sin x_1$$

$$\frac{\partial v}{\partial x_1} = 10\, x_2 \cdot \sin x_1 \qquad \frac{\partial v}{\partial x_2} = 20\, x_2 - 10 \cos x_1 .$$

At $x_1 = 0$, $x_2 = 1$.

$$J(x_1, x_2) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

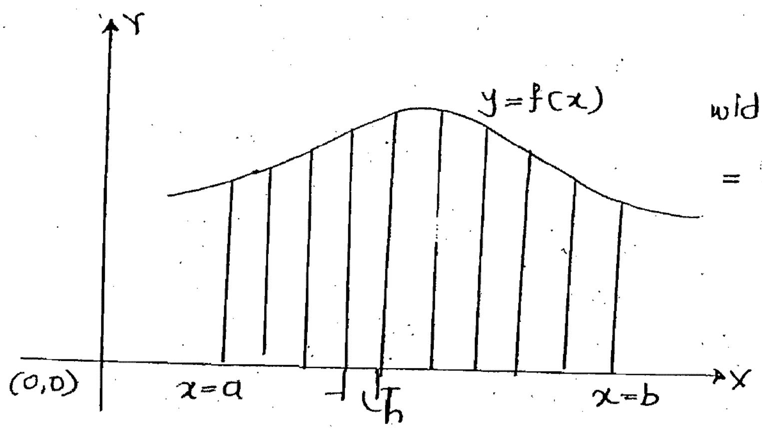
Q. The Newton Raphson method is used to solve the equation. $f(x) = (x-1)^2 + x - 3 = 0$, the method will fail in very first iteration if the initial guess is ?

$$f(x) = x^2 - 2x + 1 + x - 3$$
$$= x^2 - x - 2$$
$$f'(x) = 2x - 1$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - x_0 - 2}{2x_0 - 1}$$

Method will fail if denominator becomes zero

$$2x_0 - 1 = 0$$
$$2x_0 = 1$$
$$x_0 = 1/2 = 0.5$$

Numerical integration :



width of each interval $= h = \frac{b-a}{n}$

Trapezoidal rule :

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots y_{n-1})\right]$$

Simpson's $\frac{1}{3}$rd rule :

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 \cdots) + 2(y_2 + y_4 + \cdots)\right]$$

The accuracy of trapezoidal rule or the order of integration of trapezoidal rule is $o(h^2)$ and accuracy of the simpson's rule (1/3) is $o(h^4)$.

Q. The value of integral $\int_0^1 \frac{1}{1+x^2}\, dx$ by trapezoidal and the simpson's rule by dividing interval into 4 equal parts. & hence find value of π.

$$y = \frac{1}{1+x^2} \qquad h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x	0	0.25	0.50	0.75	1.0
y	1	0.9412	0.8	0.6400	0.5

Trapezoidal rule,

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{0.25}{2}\left[(1+0.5)+2\cdot(0.9412+0.8+0.64)\right]$$

$$(\tan^{-1}x)_0^1 = 0.7828.$$

$$\tan^{-1}1 - \cancel{\tan^{-1}0} = 0.7828$$

$$\frac{\pi}{4} = 0.7828$$

$$\pi = 3.1312$$

Actual $\pi = 3.1415$
(difference in 2nd decimal
i.e. $o(h^2)$)

Simpson's rule :

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{0.25}{3}\left[(1+0.5)+4(0.9412+0.64)+2(0.8)\right]$$

$$\frac{\pi}{4} = 0.7854$$

$$\pi = 3.1416$$

Actual $\pi = 3.1415$
(difference in 4th decimal
i.e $o(h^4)$)

Q. The value of $\int_0^6 \frac{1}{1+x^2} \cdot dx$ by trapezoidal rule by dividing the interval into 6 equal parts is? (119)

(a) 1·4132 (b) 1·3756 (c) 1·2326 (d) 1·0987

$$\int_0^6 \frac{1}{1+x^2} = \left(\tan^{-1}x\right)_0^6$$

$$= \tan^{-1}6 - \tan^{-1}0$$

$$= 1\cdot4056^c$$

(if there is no comparison between th options, then only integrate)

Q. The value of $\int_0^{\pi/4} \frac{\sin x}{\cos^3 x} \cdot dx$ by Simpson's rule. (n=4) is

(a) 0·4344 (b) 0·3787 (c) 0·5044 (d) 0·5529

$$\int_0^{\pi/4} \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x} dx$$

$$= \int_0^{\pi/4} \tan x \cdot \sec^2 x \cdot dx$$

$$= \left(\frac{\tan^2 x}{2}\right)_0^{\pi/4}$$

$$= \frac{1}{2}\left(1\right. \qquad = 0\cdot5$$

Q. $\int_1^3 \log_e x \, dx$ by Simpson's rule, (n=2)

(a) 0.5 (b) 0·8 (c) 1.00 (d) 1·29

$$\int_1^3 \log_e x \cdot dx = \left[x \cdot \log x - x\right]_1^3$$

$$= (3\log 3 - 3) - (0)$$

$$= 3\cdot1098 - 3 + 1$$

$$= 1\cdot294.$$

$\int_1^3 \frac{1}{x} \cdot dx = ?$ (n=2)

(a) 1·000 (b) 1·098 (c) 1·111 (d) 1·120

x	1	2	3
y	1	0.5	0.33

$$\int_1^3 \frac{1}{x} \cdot dx = \frac{1}{3}\left[(1+0.33)+2(0.5)\right]$$

$$= \frac{3.333}{3} = 1.111$$

Solutions of ordinary differential equations:

$$\frac{dy}{dx} = f(x,y), \quad x = x_0, \quad y = y_0, \quad y \cdot (x_1) = ?$$

It can be solved by,

(i) Taylor series method
(ii) Picard's method
(iii) Runge-kutta method.

Q. Match the following:

E- Newton-Raphson	I- sol^n of non-linear equ^n
F- Runge-Kutta	ii) sol^n of linear algebric equ^n (simultaneous equ^n)
G- Simpson's rule.	(iii) sol^n of ordinary diff equ^n
H - Gauss eliminator	(iv) Numerical integration.

E-I
-III
IV
II

Probability:

If an experiment is conducted under essentially given conditions upto n times, let m cases are favourable to an event E then the propability of E is defined by $P(E)$ and is defined as

$$P(E) = \frac{\text{no. of favourable cases to the event } (m)}{\text{Total no. of events } (n)}$$

$$P(E) = \frac{m}{n}$$

$$\therefore \quad P(\bar{E}) = \frac{n-m}{n}$$

$$= \frac{n}{n} - \frac{m}{n}$$

$$= 1 - P(E)$$

$$P(E) + P(\bar{E}) = 1.$$

Axioms of probability.

(i) Positivity, $0 \leqslant P(E) \leqslant 1$

(ii) Certainity, $P(S) = 1$

(iii) Union, $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E \cap E_2)$

Note:

If E_1, E_2 are mutually exclusive or digjoint

$$E_1 \cap E_2 = \phi$$

$$\therefore \quad P(\phi) = 0$$

Sample space:

The set of all possible outcomes in an experiment is said to be the sample space.

e.g. If two coins are tossed

$$S = \{HH, HT, TH, TT\}$$

If dice is thrown.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Conditional probability :

Let S be the sample space and E_1, E_2 are any two events of S. then the conditional probability of E_1 after the occurance of E_2 is denoted by $P\left(\frac{E_1}{E_2}\right)$ and is defined as

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)} \quad , \quad P(E_2) \neq 0$$

similarly

$$P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)} \quad , \quad P(E_1) \neq 0$$

e.g. If two coins are tossed

$$S = \{HH, HT, TH, TT\}$$

If getting Head on any of coin is success then Random variable, $X = \{0, 1, 2\}$ - subset of sample space

Mean of random variable :

If $\sum x_i \cdot P(x = x_i)$ exist, it is called mean of the random variable and is denoted by μ. (Expectation)

Variance of a or $V(\sigma^2) = \sum x_i^2 \cdot (P(X = x_i)) - \mu^2$

Standard deviation :

It is the square root of variance. and is denoted by σ.

Binomial distribution:

If, n - no. of trials.
p - probability of success
q - probability of failure.

$$p + q = 1$$

$$P(X=k) = {}^{n}C_{k} \cdot q^{n-k} \cdot p^{k}$$

If $p = q$.

$$P(X=k) = {}^{n}C_{k} \cdot p^{n}$$

Mean ~~variance~~ $= n \cdot p$

variance $= n \cdot p \cdot q$

standard deviation $= \sqrt{n \cdot p \cdot q}$

Poisson's distribution:

$$P(X=k) = \frac{\lambda^{k}}{k!} e^{-\lambda}$$ where λ is parameter.

Mean $= \lambda$

Variance $= \lambda$

S.D. $= \sqrt{\lambda}$

Q What is chance of leap year having 53 sundays.

leap years $= 366$ days.

$52 \times 7 = 364$

$$P(E) = \frac{2}{7}$$

Q. For non-leap year having 53 Mondays

365 days

$$P(E) = \frac{1}{7}$$

Q A cwa is drawn from deck of plane cards & gambler bets that it is, a spade or ace. What are odds against his winning this bet (odds - ratio).

$$\text{No. of favourable cards} = \underset{\text{spades}}{13} + (\underset{\text{Ace}}{4} - \overset{\text{spade ace}}{1}) = 16.$$

$$\text{prob. of winning} = \frac{16}{52}$$

$$\text{losing} = 1 - \frac{16}{52} = \frac{36}{52}$$

$$\text{odds against winning} = \frac{36}{52} : \frac{16}{52}$$

$$= \frac{9}{4}.$$

Q. A card is drawn from (two cards) deck of plane cards at random. What is probability of both cards being king if

(i) First card is not replaced

(ii) Replacement is allowed.

$$\text{probability of drawing kings} = \frac{4}{52}$$

(i) without replacement

$$= \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

(ii) with replacement

$$= \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}.$$

Q. A husband and wife appear in an interview for two vacancies in same post. The probability of husband selection is $\frac{1}{7}$ & that of wife is $\frac{1}{5}$. What is probability that bol

(i) both of them are selected

(ii) only one of them is selected

(iii) none of them selected

$p(A) = \frac{1}{7}$ $p(B) = \frac{1}{5}$

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(i) $p(A \cap B) = p(A) \cdot p(B) = \frac{1}{7} \times \frac{1}{5}$

p(both getting selected) $= \frac{1}{35}$

(ii) p(only one get selected) $= p(A \cap \bar{B}) + p(\bar{A} \cap B)$

$= \frac{1}{7}(1 - \frac{1}{5}) + \frac{1}{5}(1 - \frac{1}{7})$

$= \frac{4}{35} + \frac{6}{35}$

$= \frac{10}{35}$

(iii) p(none of them selected) $= p(\bar{A} \cap \bar{B})$ (dont do 1-p)

$= (1 - \frac{1}{5})(1 - \frac{1}{7})$ cause it

$= \frac{4}{5} \times \frac{6}{7}$ includes indivi

$= \frac{24}{35}$ Selection

A bag contain 3 red, 6 black, 7 white balls. What is the probability that 2 balls drawn are white & black.

$$= \frac{^6C_1 \times {^7C_1}}{^{16}C_2}$$

$$= \frac{6 \times 7}{\frac{16 \times 15}{2 \times 1}}$$

$$= \frac{7}{20}$$

There are 5 duplicate & 10 original items in automobile shop. 3 items are brought at random by customers. Find p(no item is duplicate)

$$\frac{^{10}C_3}{^{15}C_3} = \frac{10 \times 9 \times 8}{15 \times 14 \times 13} = \frac{24}{91}$$

Q. In a cetrain college 25% of students failed in mathematics and 15% failed in chemistry & 10% failed in both. If a student is selected at random, if he failed in chemistry then what is probability that he failed in maths. (ii) Vice-versa. (iii) Neither maths nor chemistry.

$P(M) = 0.25$

$P(C) = 0.15$

$P(M \cap C) = 0.10$

Conditional probability.

(i) $P(M/C) = \dfrac{P(M \cap C)}{P(C)} = \dfrac{0.10}{0.15} = \dfrac{2}{3}$

(ii) $P(C/M) = \dfrac{P(M \cap C)}{P(M)} = \dfrac{0.10}{0.25} = \dfrac{2}{5}$

(iii) $P(\bar{M} \cup \bar{C}) = 1 - P(M \cup C)$

$= 1 - [P(M) + P(C) - P(M \cap C)]$

$= 1 - [0.25 + 0.15 - 0.10]$

$= 0.7$

Q. A problem in structure is given to students A, B & C. whose chances of solving are $\frac{1}{2}$, $\frac{1}{3}$ & $\frac{1}{4}$ respectively. What is the probability that problem will be solved.

P(problem will be solved)

$= 1 - P(\text{no one solves})$

$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

$= 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$

$= 1 - [(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4})]$

$= 1 - (\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4})$

$= \frac{3}{4}$.

Q. A single die is thrown twice. find the probability that there sum is neither 8 nor 9.

$$n(S) = 36.$$

for sum = 8 → $E_1 = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

$$P(E_1) = \frac{5}{36}$$

for sum = 9 → $E_2 = \{(3,6), (4,5), (5,4), (6,3)\}$

$$P(E_2) = \frac{4}{36}.$$

These are mutually exclusive events.

$$\therefore \quad P(\bar{E_1} \cup \bar{E_2}) = 1 - P(E_1 \cup E_2)$$

$$= 1 - [P(E_1) + P(E_2)]$$

$$= 1 - \left[\frac{5}{36} + \frac{4}{36}\right]$$

$$= \frac{3}{4}.$$

Q. 4 fair coins are tossed simultaneously, find probability that at least one head and one tail, turn up.

P(at least one hed & one tail)

$$= P(X=1) + P(X=2) + P(X=3)$$

$$= ({}^4C_1 + {}^4C_2 + {}^4C_3) \cdot \left(\frac{1}{2}\right)^4$$

$$= \frac{4+6+4}{16} \qquad \because P(X=k) = {}^nC_k \cdot p^n$$

$$= \frac{7}{8}$$

or

$$P(E) = 1 - [P(X=4) + P(X=0)]$$

$$= 1 - \left[({}^4C_4 + {}^4C_0)\left(\frac{1}{2}\right)^4\right]$$

$$= 1 - \left(\frac{1+1}{16}\right) = \frac{7}{8}$$

5 fair coins are tossed simultaneously. Find probability that at least one head turn up.

$P(E) = P(X =$

$P(X \geq 1) = 1 - P(X=0)$

$= 1 - {}^5C_0 \cdot (\frac{1}{2})^5 = 1 - \frac{1}{32}$

$= \frac{31}{32}$

Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys & 2 girls

(ii) at least 1 boy

(iii) at most 2 girls.

Family cannot be descriminated - can't go for poisson's distribution. (go for binomial)

$n = 4 \qquad p = q = \frac{1}{2}$

(i) $P(X=2) = {}^4C_2 \cdot (\frac{1}{2})^4 = 6 \times \frac{1}{16} \times 800 = 300$

(ii) $P(X \geq 1) = 1 - P(X=0)$

$= 1 - {}^4C_0 (\frac{1}{2})^4 = 1 - \frac{1}{16}$

$= \frac{15}{16} \times 800 = 750$

(iii) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= ({}^4C_0 + {}^4C_1 + {}^4C_2) \cdot (\frac{1}{2})^4$

$= \frac{1+4+6}{16} = \frac{11}{16} \times 800 = 550$

Q. Suppose that a book of 300 pages containing 30 printing mistakes. Assuming that these errors are randomely distributed throughout the book & X is the no. of errors per page has a poisson distribution. What is the probability that 20 pages selected at random will be free of errors.

$$p = \frac{30}{300} = \frac{1}{10}$$

$$n = 20$$

$$\lambda = n \cdot p$$

$$= 20 \times \frac{1}{10} = 2.$$

No error is there, $p(X=0)$

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= e^{-2} = 0.135.$$

Q. A box contains 3 blue balls & 4 red balls. Another identical box contains 2 blue balls & 5 red balls. One ball is picked at random from one of the two boxes & it is red. Probability that the ball came from first box is

For identical boxes $= p(E_1) = p(E_2) = \frac{1}{2}$ for 2 boxes

$= \frac{1}{3}$ for 3 boxes

p(drawing red ball from 1st box) $= \frac{4}{7}$

p(drawing red ball from 2nd box) $= \frac{5}{7}$

$$p(\text{red ball from } 1^{st} \text{ box}) = \frac{p(E_1) \cdot p(A/E_1)}{p(E_1) \cdot p\left(\frac{A}{E_1}\right) + p(E_2) \cdot p\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{2}(4/7)}{\frac{1}{2}(4/7) + \frac{1}{2}(5/7)} = \frac{4}{9}$$