微積分 II 演習（試験問題の解答）

問 1.

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) =& g'(y+cx)+h'(y-cx),\\ \frac{\partial^2 f}{\partial y^2}(x,y) =& \frac{\partial}{\partial y}\left(g'(y+cx)+h'(y-cx)\right) = g''(y+cx)+h''(y-cx).\end{aligned}$$

一方，

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) =& g'(y+cx)\cdot c + h'(y-cx)\cdot(-c)\\ =& cg'(y+cx)-ch'(y-cx),\\ \frac{\partial^2 f}{\partial x^2}(x,y) =& \frac{\partial}{\partial x}\left(cg'(y+cx)-ch'(y-cx)\right)\\ =& c^2g''(y+cx)+(-c)^2h''(y-cx)\\ =& c^2\left(g''(y+cx)+h''(y-cx)\right)\\ =& c^2\frac{\partial^2 f}{\partial x^2}(x,y).\end{aligned}$$

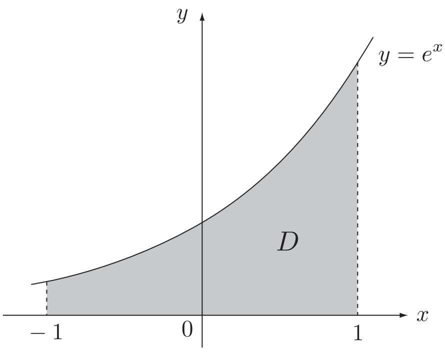
問 2. $f_x(x,y)=3(x^2-y)$, $f_y(x,y)=3(-x+y^2)$．したがって，$f_x=f_y=0$ を満たすのは $(0,0)$ と $(1,1)$.

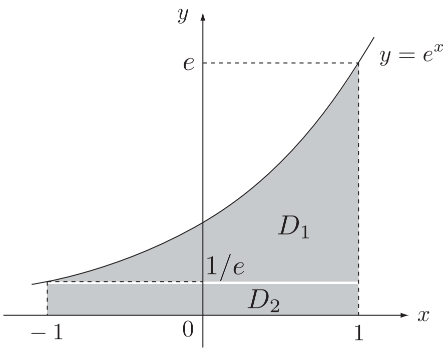
$$\mathrm{Hess}(f)_{(x,y)} = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y)\end{pmatrix} = \begin{pmatrix} 6x & -3 \\ -3 & 6y\end{pmatrix}$$

(i) $\det(\mathrm{Hess}(f)_{(0,0)}) = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0$ より，$(0,0)$ では極値をとらない．

(ii) $\det(\mathrm{Hess}(f)_{(1,1)}) = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 27 > 0$，$f_{xx}(1,1)=6>0$ より，f は $(1,1)$ で極小値をとる．

問 3. 積分領域を D とする．このとき $D=\{(x,y)\,|\,0\le y\le e^x,\ -1\le x\le 1\}$.





図のように D を D_1 と D_2 に分割すると

$$
\begin{aligned}
D_1 &= \{(x,y)\,|\,\log y \le x \le 1,\ 1/e \le y \le e\},\\
D_1 &= \{(x,y)\,|\,-1 \le x \le 1,\ 0 \le y \le 1/e\}.
\end{aligned}
$$

したがって，

$$
\iint_D f(x,y)\,dydx = \int_{1/e}^{e}\left(\int_{\log y}^{1} f(x,y)\,dx\right)dy + \int_0^{1/e}\left(\int_{-1}^{1} f(x,y\,dx)\right)dy.
$$

$\boxed{\text{問 } \mathbf{4.}}$ $D = \{(x,y)\,|\,x^2+y^2 \le 2x\}$ とおくと，求めるものは領域 D 上で 2 つの曲面 $z=\sqrt{4-x^2-y^2}$ と $z=-\sqrt{4-x^2-y^2}$ に囲まれた部分の体積 V である;

$$
V = 2\iint_D \sqrt{4-x^2-y^2}\,dxdy.
$$

原点を中心とする極座標変換により D は領域

$$
E = \left\{(r,\theta)\ \middle|\ 0 \le r \le 2\cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}
$$

に移る．したがって，

$$
\begin{aligned}
V =& 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^{2\cos\theta} r\sqrt{4-r^2}\,dr\right)d\theta = -\frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\Big[(4-r^2)^{\frac{3}{2}}\Big]_0^{2\cos\theta}\right)d\theta\\
=& -\frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\left(4-4\cos^2\theta\right)^{\frac{3}{2}}-8\right\}d\theta = -\frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{8\left(\sin^2\theta\right)^{\frac{3}{2}}-8\right\}d\theta\\
=& -\frac{16}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(|\sin\theta|^3-1\right)d\theta\\
=& -\frac{16}{3}\left\{\int_{-\frac{\pi}{2}}^{0}(-\sin^3\theta)\,d\theta + \int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\right\}\\
=& -\frac{16}{3}\left\{2\int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta - \pi\right\} = -\frac{16}{3}\left\{\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}} - \pi\right\}\\
=& \boldsymbol{\frac{16}{3}\left(\pi-\frac{2}{3}\right)}.
\end{aligned}
$$