Volume entropy of harmonic ~~Hadamard~~ manifolds of hypergeometric type

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Differential Geometry and its Applications

Harmonic manifolds

Definition

A harmonic manifold is a complete Riemannian manifold (X, g) whose volume density function $\sqrt{\det(g_{ij})}$ is a radial function, that is, $\sqrt{\det(g_{ij})}(x)$ depends only on the distance $d(o, x)$.
Here $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ are components of the Riemannian metric g with respect to a normal coordinate system $\{x_1, x_2, \dots, x_n\}$ at an arbitrary point $o \in X$

Remark 1.1

- Every geodesic sphere $S(p; r)$ in a harmonic manifold has constant mean curvature $\sigma(r)$.
- Harmonic manifolds are always Einstein, i.e., $\mathrm{Ric}_g = \kappa g$.

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What is known for harmonic manifolds (1)

- In 1944, Lichnerowicz conjectured that harmonic 4-manifolds are flat or rank one symmetric spaces.
- In 1949, above conjecture was proved by Walker.
- Their work was generalized to the following conjecture;

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- In 1992, Damek and Ricci gave **counter examples for the Lichnerowicz conjecture**, which is a class of harmonic, homogeneous Hadamard manifolds, called the **Damek-Ricci space**, including rank one symmetric spaces of non-compact type (except $\mathbb{R}H^n$).
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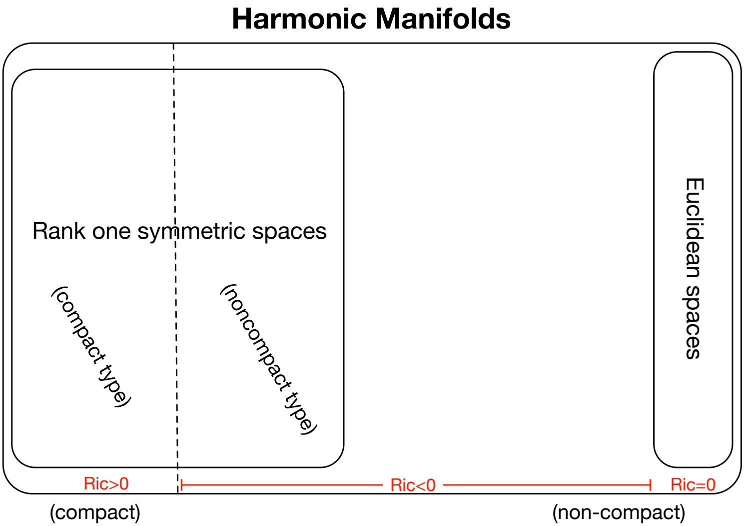

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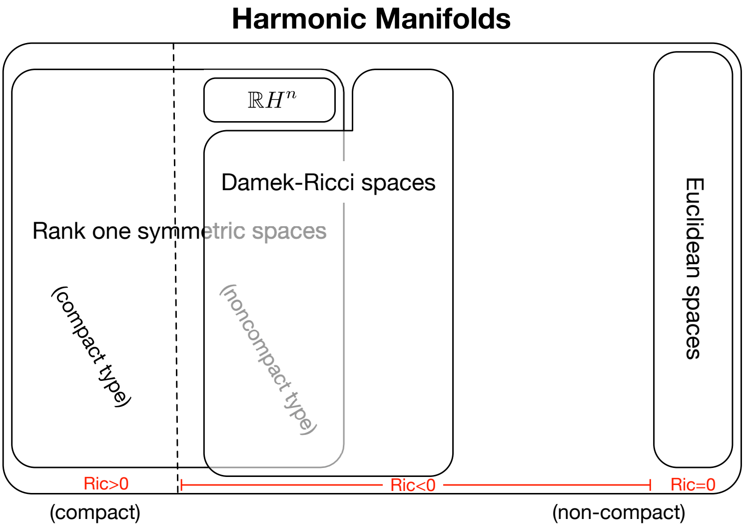

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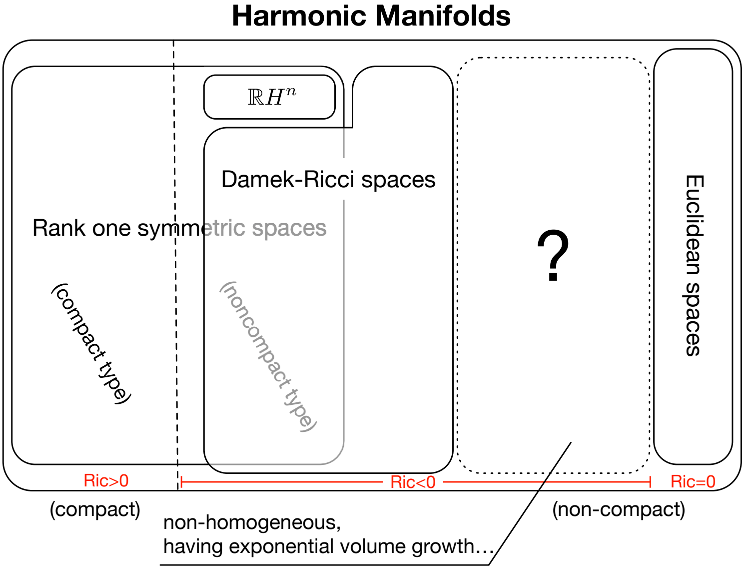

The aim of this talk

- We define a new class of non-compact harmonic manifolds, including all Damek-Ricci spaces and rank-one symmetric spaces of non-compact type.
- We call such a space, harmonic manifold **of hypergeometric type**.
- It is motivated to develop the theory of the spherical Fourier transform on harmonic manifolds (cf. Itoh's talk on Monday).

We show the following fact;

Main Theorem (Itoh-S. in preparation)

Let (X, g) be an n-dimensional harmonic manifold of hypergeometric type, whose metric g is normalized as $\mathrm{Ric}_g = -(n-1)g$. Then, the volume entropy Q_g of (X, g) satisfies

$$\frac{2\sqrt{2}(n-1)}{3} \leq Q_g \leq n-1. \tag{1}$$

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- Motivation
- Definition
- Properties

3 Proof of Main Theorem
- Bishop volume comparison theorem
- Estimates of volume entropy
- Harmonic manifolds that attain the maximum or minimum value of Q_g

Spherical Fourier transform on Damek-Ricci spaces

Anker-Damek-Yacoub (1996), see also Rouvière (2003)

- They developed **the spherical Fourier transform** on a Damek-Ricci space by reducing it to a special case of the Jacobi transform of Jacobi functions.
- The spherical Fourier transform $f \mapsto \hat{f}$ for a smooth radial function $f(x)$ on a harmonic manifold is defined by

$$\hat{f}(\lambda) = \int_{x \in X} f(x)\, \varphi_\lambda(x)\, dv_g = \omega_{n-1} \int_0^\infty f(r)\, \varphi_\lambda(r)\, \Theta(r)\, dr,$$

where $\Theta(r)$ is the volume density of a geodesic sphere $S(o;r)$.
- Here φ_λ is an eigenfunction of the radial part of the Laplace operator Δ with eigenvalue $\frac{Q_g^2}{4} + \lambda^2$.

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The **volume entropy** of (X,g); $Q_g = \lim_{r \to \infty} \frac{1}{r} \log \mathrm{Vol} B(o;r)$.

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- They developed **the spherical Fourier transform** on a Damek-Ricci space by reducing it to a special case of the Jacobi transform of Jacobi functions.
- The essence of their method is that the eigenfunction equation of the Laplace operator is transformed into a **hypergeometric differential equation** by a variable transformation $z = -\sinh^2(r/2)$;

$$\frac{d^2 f}{dr^2}(r) + \sigma(r) \cdot \frac{df}{dr}(r) + \left(\frac{Q_g^2}{4} + \lambda^2 \right) f(r) = 0 \qquad (2)$$

$$\Downarrow \; z = -\sinh^2 \tfrac{r}{2}$$

$$z(1-z)u''(z) + \{c - (a+b+1)z\}u'(z) - abu(z) = 0 \qquad (3)$$

- Therefore, φ_λ is described by using the hypergeometric function ${}_2F_1(a, b, c; z)$.

Definition of "hypergeometric type"

Definition (Itoh's talk)

Let (X, g) be a non-compact harmonic manifold. When the equation (2) turns into (3) by a variable transformation $z = -\sinh^2(r/2)$, we call (X, g) a harmonic manifold of **hypergeometric type**.

Theorem A

Let (X, g) be a non-compact harmonic manifold. If for a variable transformation $z = z(r)$ the equation (2) turns into (3), then it holds only for $z(r) = -\sinh^2(\ell r)$ for some $\ell > 0$.

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Why change the definition?

- The main subject of this talk is **the volume entropy Q_g** which is not scale invariant.
- Since we want to normalize the metric and discuss estimates of Q_g, we modify the definition as above (these are essentially the same).

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Characterization of "hypergeometric type"

Theorem B

A non-compact harmonic manifold (X, g) is of hypergeometric type, if and only if its volume density function $\Theta(r)$ of a geodesic sphere $S(o;r)$ is given by

$$\Theta(r) = K_g \sinh^{n-1}(\ell r) \cosh^{Q_g/\ell-(n-1)}(\ell r). \tag{4}$$

Theorem C

If the volume density function of a geodesic sphere $S(o;r)$ in a harmonic manifold is expressed by (4), then the constant K_g is given by

$$K_g = -\frac{1}{\ell^n} \cdot \frac{\mathrm{Ric}_g}{3Q_g - 2(n-1)\ell}.$$

Remark 2.1

In 2005, Nikolayevsky proved that the volume density function $\Theta(r)$ of $S(o;r)$ in a harmonic manifold is an exponential polynomial.

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Tools to prove Theorem A, B and C

Let (X, g) be a non-compact harmonic manifold. Let $\Theta(r)$ and $\sigma(r)$ be the volume density function and the mean curvature of $S(o; r)$, respectively.

Lemma 2.2

1. $\sigma(r) = \dfrac{\Theta'(r)}{\Theta(r)}$
2. $\lim_{r\to\infty} \sigma(r) = Q_g$
3. $\sigma(r)$ *is non-negative.*
4. $\lim_{r\to 0} r\,\sigma(r) = n-1$

Lemma 2.3 (Ledger formula)

$$\frac{d^2}{dr^2}\left(\frac{\Theta(r)}{r^{n-1}}\right)\bigg|_{r=0} = -\frac{1}{3}\mathrm{Ric}_g$$

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Bishop volume comparison theorem

Theorem 3.1 (see [Sakai, Theorem 3.1(2), IV])

Let (X, g) *be an n-dimensional complete Riemannian manifold satisfying* $\mathrm{Ric}_g \geq (n-1)\delta$ *and* $\gamma : [0, \infty] \to X$ *be a geodesic satisfying* $\gamma(0) = p \in X$ *and* $\gamma'(0) = u \in T_pX, |u| = 1$.

- *If* $\delta < 0$*, then we have*

$$\Theta_p(r, u) \leq \left(\frac{1}{\sqrt{|\delta|}} \sinh\left(\sqrt{|\delta|}r\right) \right)^{n-1}, \quad 0 < r < t_0(\gamma),$$

  where $\Theta_p(r, u)$ *is volume density of* $S(p; r)$ *at* $\exp_p(ru)$ *and* $t_0(\gamma)$ *attains the first conjugate point of* p *along* γ.

- *If equality holds at* $T \leq t_0(\gamma)$*, then the equality holds for any* $0 \leq r \leq T$ *and the sectional curvature of any plane spanned by* $\gamma'(r)$ *and a unit vector perpendicular to* $\gamma(r)$ *is constant* δ.

Estimates of volume entropy

- Let (X, g) be a harmonic manifold of hypergeometric type and $\mathrm{Ric}_g = (n-1)\delta$, $\delta < 0$.
- From the Bishop volume comparison theorem, we have

$$\Theta(r) = K_g \sinh^{n-1}(\ell r) \cosh^{\frac{Q_g}{\ell}-(n-1)}(\ell r) \leq \left(\frac{1}{\sqrt{|\delta|}} \sinh\left(\sqrt{|\delta|} r\right) \right)^{n-1}.$$

- Expanding above inequality into power-series, we have

$$K_g \left\{ \ell^{n-1} + \left(\frac{Q_g}{2\ell} - \frac{n-1}{3} \right) \ell^{n+1} r^2 + O(r^4) \right\} \leq 1 + \frac{n-1}{3!} |\delta| r^2 + O(r^4).$$

- When $r \to 0$, we have $K_g \ell^{n-1} \leq 1$.
- From Theorem C, we have

$$\frac{n-1}{3} \left(\frac{|\delta|}{\ell} + 2\ell \right) \leq Q_g.$$

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- On the other hand, from the definition of volume entropy, we have

$$\begin{aligned} Q_g &= \lim_{r\to\infty} \frac{\log \mathrm{Vol} B(o; r)}{r} = \lim_{r\to\infty} \frac{\log \int_0^r \Theta(r)\, dr}{r} \\ &\leq \lim_{r\to\infty} \frac{\log \displaystyle\int_0^r \left(\frac{1}{\sqrt{|\delta|}} \sinh\left(\sqrt{|\delta|} r\right) \right)^{n-1} dr}{r} = \cdots = (n-1)\sqrt{|\delta|}. \end{aligned}$$

- Remark : above inequality holds for any harmonic manifolds.

Estimates of volume entropy

- Hence, we obtain

$$\frac{n-1}{3}\left(\frac{|\delta|}{\ell}+2\ell\right) \leq Q_g \leq (n-1)\sqrt{|\delta|}$$

- From the famous theorem of the arithmetic and geometric means, we have

$$\frac{2\sqrt{2|\delta|}(n-1)}{3} \leq Q_g \leq (n-1)\sqrt{|\delta|}$$

- We normalize the metric g satisfying $\mathrm{Ric}_g = -(n-1)g$, i.e., $\delta = -1$, we have

$$\therefore\ \frac{2\sqrt{2}(n-1)}{3} \leq Q_g \leq (n-1).$$

Q.E.D

Harmonic manifolds that attain the maximum or minimum value of Q_g

- In the case of $Q_g = (n-1)$, we have $\Theta(r) = \sinh^{n-1} r$.
- From the Bishop volume comparison theorem, we find that (X, g) is an n-dimensional **real hyperbolic space** of constant sectional curvature -1.

- In the case of $Q_g = \frac{2\sqrt{2}(n-1)}{3}$, it is not clear what properties such a harmonic manifold generally carries.
- We check whether there are harmonic manifold of hypergeometric type, homothetic to a Damek-Ricci space, whose volume entropy $Q_g = \frac{2\sqrt{2}(n-1)}{3}$.

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Damek-Ricci spaces

- A **Damek-Ricci space** $S = AN$ is a solvable Lie group which is a one-dimensional extension of a generalized Heisenberg group N having a certain left-invariant metric g.
- A generalized Heisenberg group is a 2-step nilpotent group which satisfies a certain condition.
- Let $\mathfrak{z}$ be the center of the Lie algebra $\mathfrak{n}$ of N and $\mathfrak{v}$ is the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ ($m_{\mathfrak{v}} = \dim \mathfrak{v}$, $m_{\mathfrak{z}} = \dim \mathfrak{z}$).
- $S \simeq \mathbb{R} \times \mathfrak{v} \times \mathfrak{z} \simeq \mathbb{R}^{m_{\mathfrak{v}}+m_{\mathfrak{z}}+1}$
- It is known that for each $m \in \mathbb{N}$ there exit an infinite number of non-isomorphic generalized Heisenberg groups with $m_{\mathfrak{z}} = m$.
- Moreover, the Ricci curvature tensor Ric_g and the volume entropy Q_g of a Damek-Ricci space (S, g) are given by

$$\mathrm{Ric}_g = -\left(m_{\mathfrak{z}} + \frac{m_{\mathfrak{v}}}{4}\right) g, \quad Q_g = m_{\mathfrak{z}} + \frac{m_{\mathfrak{v}}}{2}.$$

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In the case of $Q_g = \frac{2\sqrt{2}(n-1)}{3}$ on Damek-Ricci spaces

- Since $n = \dim S = m_{\mathfrak{z}} + m_{\mathfrak{v}} + 1$, we have

$$\mathrm{Ric}_g = -\left(m_{\mathfrak{z}} + \frac{m_{\mathfrak{v}}}{4}\right) g = -(n-1) \cdot \frac{(m_{\mathfrak{z}} + m_{\mathfrak{v}}/4)}{m_{\mathfrak{z}} + m_{\mathfrak{v}}} g.$$

- By constant rescaling g as $k^2 g$ where $k = \sqrt{\frac{(m_{\mathfrak{z}} + m_{\mathfrak{v}}/4)}{m_{\mathfrak{z}} + m_{\mathfrak{v}}}}$, we obtain $\mathrm{Ric}_{k^2 g} = -(n-1)(k^2 g)$ and

$$Q_{k^2 g} = \frac{Q_g}{k} = \left(m_{\mathfrak{z}} + \frac{m_{\mathfrak{v}}}{2}\right) \sqrt{\frac{m_{\mathfrak{z}} + m_{\mathfrak{v}}}{(m_{\mathfrak{z}} + m_{\mathfrak{v}}/4)}}.$$

- Solving the equation

$$\left(m_{\mathfrak{z}} + \frac{m_{\mathfrak{v}}}{2}\right) \sqrt{\frac{m_{\mathfrak{z}} + m_{\mathfrak{v}}}{(m_{\mathfrak{z}} + m_{\mathfrak{v}}/4)}} = \frac{2\sqrt{2}(n-1)}{3} = \frac{2\sqrt{2}(m_{\mathfrak{z}} + m_{\mathfrak{v}})}{3},$$

we have $m_{\mathfrak{v}} = 2m_{\mathfrak{z}}$.

In the case of $Q_g = \frac{2\sqrt{2}(n-1)}{3}$ on Damek-Ricci spaces

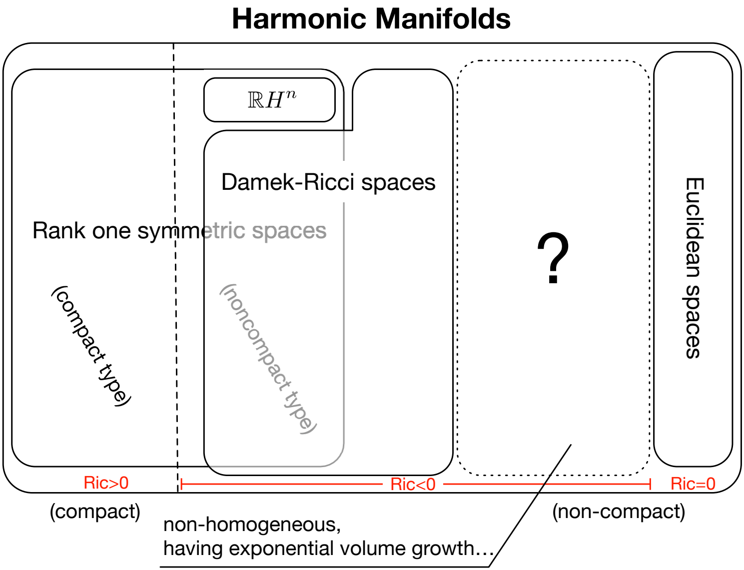
From the classification of generalized Heisenberg groups, we find that Damek-Ricci spaces satisfying $m_{\mathfrak{v}} = 2m_{\mathfrak{z}}$ are only in the following 4 cases;

1. $m_{\mathfrak{z}} = 1$ $(n = 4)$
2. $m_{\mathfrak{z}} = 2$ $(n = 7)$
3. $m_{\mathfrak{z}} = 4$ $(n = 13)$
4. $m_{\mathfrak{z}} = 8$ $(n = 25)$

Remark 3.2

In the case of i), S is isometric to a 4-dimensional complex hyperbolic space $\mathbb{C}H^2$. In the case of ii), S has 7-dimension which is the smallest dimension among non-symmetric Damek-Ricci spaces.

What is known for harmonic manifolds (4)



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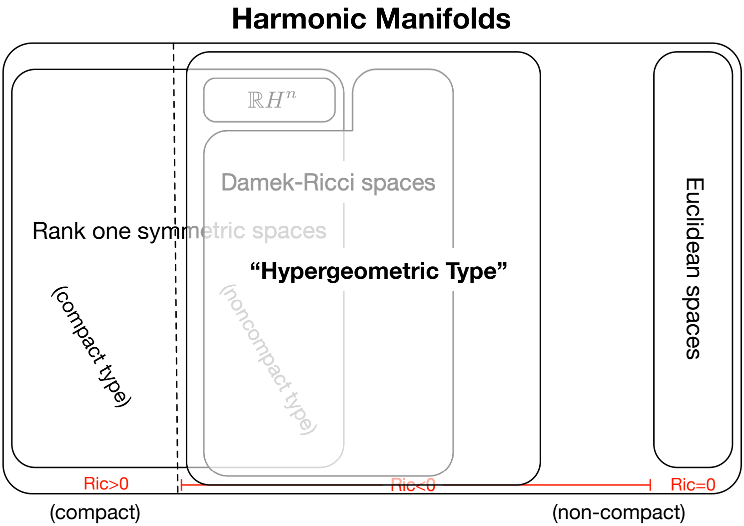

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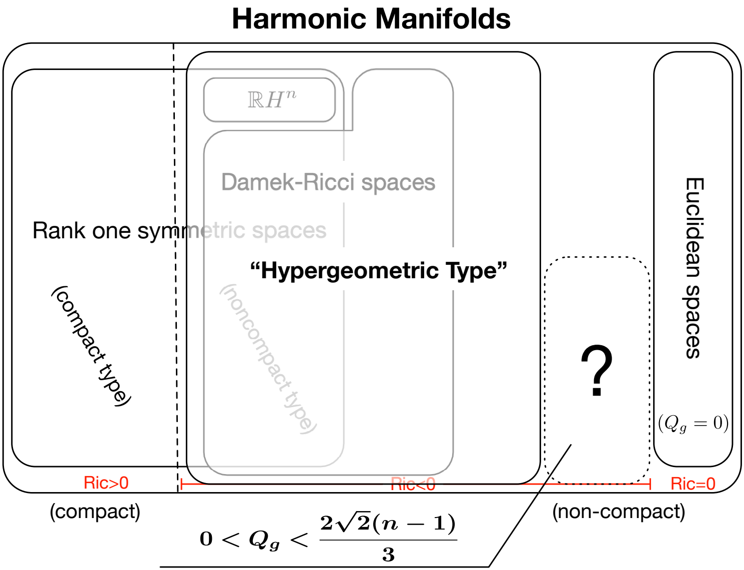

Future work

- Characterize harmonic manifolds of hypergeometric type whose volume entropy satisfies $Q_g = \frac{2\sqrt{2}(n-1)}{3}$.
- Show the existence of a harmonic manifold of hypergeometric type which is not a Damek-Ricci space.
- Show the existence of a non-compact harmonic manifold whose volume entropy satisfies $0 < Q_g < \frac{2\sqrt{2}(n-1)}{3}$.

Thank you for your attention.

Future work

- Characterize harmonic manifolds of hypergeometric type whose volume entropy satisfies $Q_g = \frac{2\sqrt{2}(n-1)}{3}$.
- Show the existence of a harmonic manifold of hypergeometric type which is not a Damek-Ricci space.
- Show the existence of a non-compact harmonic manifold whose volume entropy satisfies $0 < Q_g < \frac{2\sqrt{2}(n-1)}{3}$.

Thank you for your attention.

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