問 1.

$$
\begin{aligned}
\frac{\partial^2 f^*}{\partial s^2} =& \frac{\partial}{\partial s}\left(\frac{\partial f^*}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\right) \\
=& \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial x}\cdot\frac{\partial^2 x}{\partial s^2} + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial^2 y}{\partial s^2} \\
=& \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial s} + \frac{\partial f}{\partial x}\cdot\frac{\partial^2 x}{\partial s^2} \\
& + \left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial s}\right)\frac{\partial y}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial^2 y}{\partial s^2}.
\end{aligned}
$$

t に関する偏微分も同様にすると

$$
\begin{aligned}
\frac{\partial^2 f^*}{\partial t^2} =& \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \frac{\partial f}{\partial x}\cdot\frac{\partial^2 x}{\partial t^2} \\
& + \left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial^2 y}{\partial t^2}.
\end{aligned}
$$

ここで,

$$
\begin{aligned}
&\frac{\partial x}{\partial s} = \frac{\partial x^2}{\partial s^2} = x, \qquad \frac{\partial x}{\partial t} = -y, \qquad \frac{\partial x^2}{\partial t^2} = -x, \\
&\frac{\partial y}{\partial s} = \frac{\partial y^2}{\partial s^2} = y, \qquad \frac{\partial y}{\partial t} = x, \qquad \frac{\partial y^2}{\partial t^2} = -y.
\end{aligned}
$$

したがって,

$$
\begin{aligned}
\frac{\partial^2 f^*}{\partial s^2} =& \frac{\partial^2 f}{\partial x^2}x^2 + \frac{\partial^2 f}{\partial x\partial y}xy + \frac{\partial f}{\partial x}x + \frac{\partial^2 f}{\partial x\partial y}xy + \frac{\partial^2 f}{\partial y^2}y^2 + \frac{\partial f}{\partial y}y \\
=& \frac{\partial^2 f}{\partial x^2}x^2 + 2\frac{\partial^2 f}{\partial x\partial y}xy + \frac{\partial f}{\partial x}x + \frac{\partial^2 f}{\partial y^2}y^2 + \frac{\partial f}{\partial y}y, \\
\frac{\partial^2 f^*}{\partial t^2} =& \frac{\partial^2 f}{\partial x^2}y^2 - \frac{\partial^2 f}{\partial x\partial y}xy - \frac{\partial f}{\partial x}x - \frac{\partial^2 f}{\partial x\partial y}xy + \frac{\partial^2 f}{\partial y^2}x^2 - \frac{\partial f}{\partial y}y \\
=& \frac{\partial^2 f}{\partial x^2}y^2 - 2\frac{\partial^2 f}{\partial x\partial y}xy - \frac{\partial f}{\partial x}x + \frac{\partial^2 f}{\partial y^2}x^2 - \frac{\partial f}{\partial y}y.
\end{aligned}
$$

$x^2 + y^2 = e^{2s}$ より,

$$
\frac{\partial^2 f^*}{\partial s^2} + \frac{\partial^2 f^*}{\partial t^2} = e^{2s}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).
$$

問 2. (1) $f_x(x,y) = 2x^2 + 2x - y^2$, $f_y(x,y) = -2yx + 2y$, $f_{xx}(x,y) = 4x + 2$, $f_{xy}(x,y) = -2y$, $f_{yy}(x,y) = -2x + 2$.

(2) $f_y(a,b) = 0$ より，$b = 0$ または $a = 1$．$b = 0$ のとき，$f_x(a,0) = 2a^2 + 2a = 0$ より $a = 0$ または -1．$a = 1$ のとき，$f_x(1,b) = 4 - b^2 = 0$ より $b = \pm 2$．したがって，極値をとる候補の点は $\mathbf{(0,0)}, \mathbf{(-1,0)}, \mathbf{(1,\pm 2)}$ の 4 点である．

$(0,0)$, $(-1,0)$, $(1,\pm 2)$ におけるヘッセ行列 $H_{(0,0)}$, $H_{(-1,0)}$, $H_{(1,\pm 2)}$ は

$$H_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad H_{(-1,0)} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}, \quad H_{(1,\pm 2)} = \begin{pmatrix} 6 & \mp 4 \\ \mp 4 & 0 \end{pmatrix}.$$

この中で行列式が正になるのは $(0,0)$ のみ．また，$f_{xx}(0,0) = 2 > 0$．以上のことから極値をとるのは $\mathbf{(0,0)}$ のみで，この点で極小値 $\mathbf{0}$ をとる．

問 3.　$\int_0^2 \left(\int_{\frac{x^2}{4}}^{3-x} f(x,y)\,dy \right) dx = \int_0^1 \left(\int_0^{2\sqrt{y}} f(x,y)\,dx \right) dy + \int_1^3 \left(\int_0^{3-y} f(x,y)\,dx \right) dy$

問 4.　求めるものは $\iint_D (x^2+y^2)\,dxdy$．ただし D は xy-平面上の円 $x^2+y^2=2x$ の内部．極座標変換 $(x,y)=(r\cos\theta, r\sin\theta)$ により，領域 D は

$$E = \left\{ (r,\theta) \;\middle|\; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta \right\}$$

に移る．したがって，

$$\begin{aligned}
\iint_D (x^2+y^2)\,dxdy &= \iint_E r^2 \cdot r\,drd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\theta} r^3\,dr \right) d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{4} r^4 \right]_0^{2\cos\theta} d\theta = 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^4\,d\theta \\
&= 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos(2\theta)}{2} \right)^2 d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ 1 + 2\cos(2\theta) + \cos^2(2\theta) \right\} d\theta \\
&= \left[\theta + \sin(2\theta) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(4\theta)}{2} d\theta \\
&= \pi + \frac{1}{2} \left[\theta + \frac{1}{4}\sin(4\theta) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi + \frac{\pi}{2} = \boldsymbol{\frac{3\pi}{2}}.
\end{aligned}$$