

# Volume entropy of harmonic ~~Hadamard~~ manifolds of hypergeometric type

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# Harmonic manifolds

## Definition

A harmonic manifold is a complete Riemannian manifold  $(X, g)$  whose volume density function  $\sqrt{\det(g_{ij})}$  is a radial function, that is,  $\sqrt{\det(g_{ij})}(x)$  depends only on the distance  $d(o, x)$ .

Here  $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$  are components of the Riemannian metric  $g$  with respect to a normal coordinate system  $\{x_1, x_2, \dots, x_n\}$  at an arbitrary point  $o \in X$

## Remark 1.1

- Every geodesic sphere  $S(p; r)$  in a harmonic manifold has constant mean curvature  $\sigma(r)$ .
- Harmonic manifolds are always Einstein, i.e.,  $\text{Ric}_g = \kappa g$ .

# What is known for harmonic manifolds (1)

- In 1944, Lichnerowicz conjectured that harmonic 4-manifolds are flat or rank one symmetric spaces.
- In 1949, above conjecture was proved by Walker.
- Their work was generalized to the following conjecture;

## Lichnerowicz Conjecture

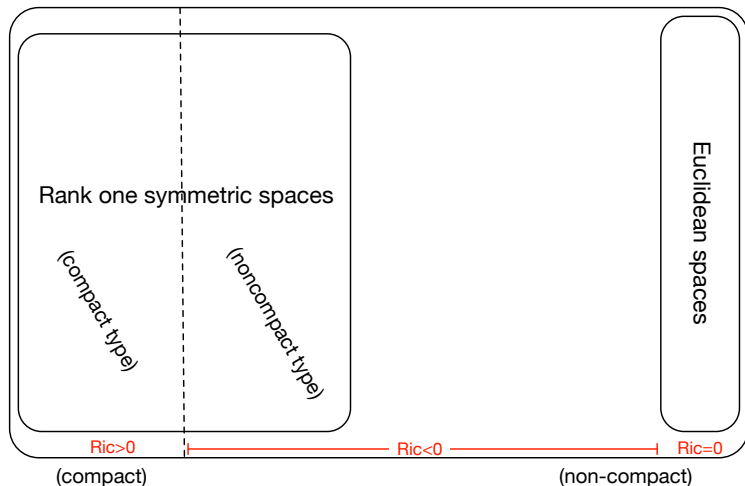
Every simply connected harmonic manifold is flat or a rank-one symmetric space.

## What is known for harmonic manifolds (2)

- In 1990, Szabó proved the Lichnerowicz conjecture for simply connected **compact** harmonic manifolds.
- In 1992, Damek and Ricci gave **counter examples for the Lichnerowicz conjecture**, which is a class of harmonic, homogeneous Hadamard manifolds, called the **Damek-Ricci space**, including rank one symmetric spaces of non-compact type (except  $\mathbb{R}H^n$ ).
- In 2006, Heber proved that a simply connected, **homogeneous** harmonic manifold is isometric to a Euclidean spaces, a rank-one symmetric space, or a Damek-Ricci space.

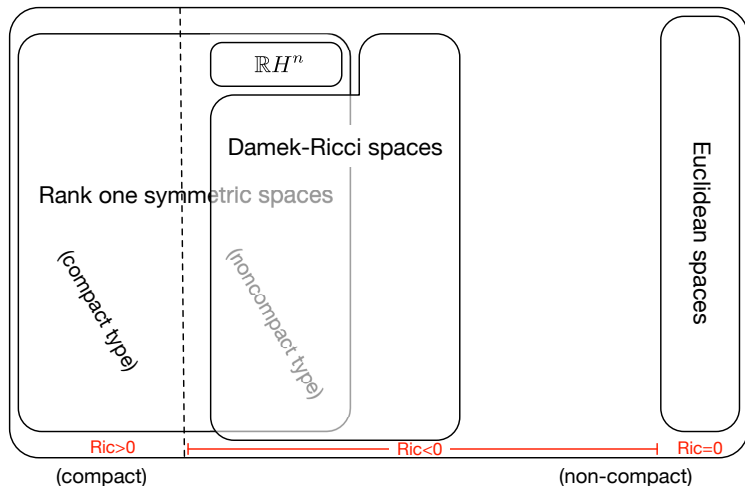
# What is known for harmonic manifolds (3)

## Harmonic Manifolds



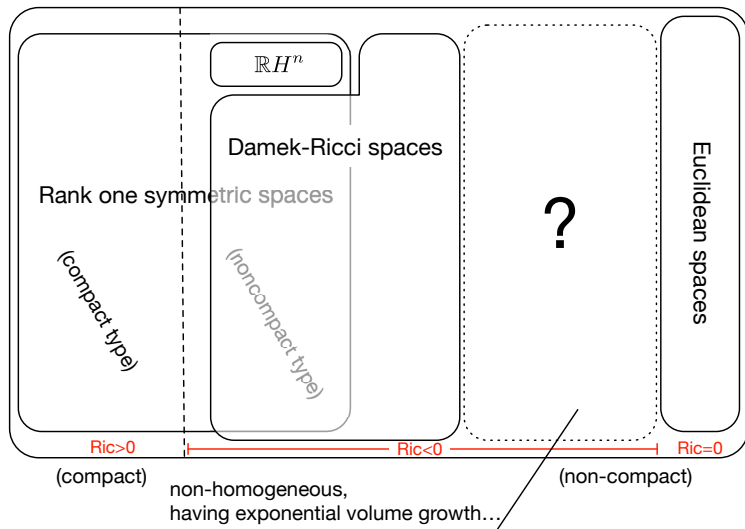
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## Harmonic Manifolds



# What is known for harmonic manifolds (3)

## Harmonic Manifolds



# The aim of this talk

- We define a new class of non-compact harmonic manifolds, including all Damek-Ricci spaces and rank-one symmetric spaces of non-compact type.
- We call such a space, harmonic manifold **of hypergeometric type**.
- It is motivated to develop the theory of the spherical Fourier transform on harmonic manifolds (cf. Itoh's talk on Monday).

We show the following fact;

## Main Theorem (Itoh-S. in preparation)

Let  $(X, g)$  be an  $n$ -dimensional harmonic manifold of hypergeometric type, whose metric  $g$  is normalized as  $\text{Ric}_g = -(n-1)g$ . Then, the volume entropy  $Q_g$  of  $(X, g)$  satisfies

$$\frac{2\sqrt{2}(n-1)}{3} \leq Q_g \leq n-1. \quad (1)$$



## 1 Introduction

## 2 Harmonic manifolds of hypergeometric type and its properties

- Motivation
- Definition
- Properties

## 3 Proof of Main Theorem

- Bishop volume comparison theorem
- Estimates of volume entropy
- Harmonic manifolds that attain the maximum or minimum value of  $Q_g$

# Spherical Fourier transform on Damek-Ricci spaces

Anker-Damek-Yacoub (1996), see also Rouvière (2003)

- They developed **the spherical Fourier transform** on a Damek-Ricci space by reducing it to a special case of the Jacobi transform of Jacobi functions.
- The spherical Fourier transform  $f \mapsto \hat{f}$  for a smooth radial function  $f(x)$  on a harmonic manifold is defined by

$$\hat{f}(\lambda) = \int_{x \in X} f(x) \varphi_\lambda(x) dv_g = \omega_{n-1} \int_0^\infty f(r) \varphi_\lambda(r) \Theta(r) dr,$$

where  $\Theta(r)$  is the volume density of a geodesic sphere  $S(o; r)$ .

- Here  $\varphi_\lambda$  is an eigenfunction of the radial part of the Laplace operator  $\Delta$  with eigenvalue  $\frac{Q_g^2}{4} + \lambda^2$ .

## Definition

The **volume entropy** of  $(X, g)$ ;  $Q_g = \lim_{r \rightarrow \infty} \frac{1}{r} \log \text{Vol} B(o; r)$ .

# Spherical Fourier transform on Damek-Ricci spaces

Anker-Damek-Yacoub (1996), see also Rouvière (2003)

- They developed **the spherical Fourier transform** on a Damek-Ricci space by reducing it to a special case of the Jacobi transform of Jacobi functions.
- The essence of their method is that **the eigenfunction equation of the Laplace operator is transformed into a hypergeometric differential equation** by a variable transformation  $z = -\sinh^2(r/2)$ ;

$$\frac{d^2 f}{dr^2}(r) + \sigma(r) \cdot \frac{df}{dr}(r) + \left( \frac{Q_g^2}{4} + \lambda^2 \right) f(r) = 0 \quad (2)$$

$$\Downarrow z = -\sinh^2 \frac{r}{2}$$

$$z(1-z)u''(z) + \{c - (a+b+1)z\}u'(z) - abu(z) = 0 \quad (3)$$

- Therefore,  $\varphi_\lambda$  is described by using the hypergeometric function  ${}_2F_1(a, b, c; z)$ .

# Definition of “hypergeometric type”

## Definition (Itoh's talk)

Let  $(X, g)$  be a non-compact harmonic manifold. When the equation (2) turns into (3) by a variable transformation  $z = -\sinh^2(r/2)$ , we call  $(X, g)$  a harmonic manifold of **hypergeometric type**.

## Theorem A

*Let  $(X, g)$  be a non-compact harmonic manifold. If for a variable transformation  $z = z(r)$  the equation (2) turns into (3), then it holds only for  $z(r) = -\sinh^2(\ell r)$  for some  $\ell > 0$ .*

# Definition of “hypergeometric type”

## Definition (In this talk)

Let  $(X, g)$  be a non-compact harmonic manifold. When the equation (2) turns into (3) by a variable transformation  $z = -\sinh^2(\ell r)$  for some  $\ell > 0$ , we call  $(X, g)$  a harmonic manifold of **hypergeometric type**.

Why change the definition?

- The main subject of this talk is **the volume entropy  $Q_g$**  which is **not scale invariant**.
- Since we want to normalize the metric and discuss estimates of  $Q_g$ , we modify the definition as above (these are essentially the same).

# Characterization of “hypergeometric type”

## Theorem B

*A non-compact harmonic manifold  $(X, g)$  is of hypergeometric type, if and only if its volume density function  $\Theta(r)$  of a geodesic sphere  $S(o; r)$  is given by*

$$\Theta(r) = K_g \sinh^{n-1}(\ell r) \cosh^{Q_g/\ell - (n-1)}(\ell r). \quad (4)$$

## Theorem C

*If the volume density function of a geodesic sphere  $S(o; r)$  in a harmonic manifold is expressed by (4), then the constant  $K_g$  is given by*

$$K_g = -\frac{1}{\ell^n} \cdot \frac{\text{Ric}_g}{3Q_g - 2(n-1)\ell}.$$

## Remark 2.1

In 2005, Nikolayevsky proved that the volume density function  $\Theta(r)$  of  $S(o; r)$  in a harmonic manifold is an **exponential polynomial**.

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# Tools to prove Theorem A, B and C

Let  $(X, g)$  be a non-compact harmonic manifold. Let  $\Theta(r)$  and  $\sigma(r)$  be the volume density function and the mean curvature of  $S(o; r)$ , respectively.

## Lemma 2.2

- ①  $\sigma(r) = \frac{\Theta'(r)}{\Theta(r)}$
- ②  $\lim_{r \rightarrow \infty} \sigma(r) = Q_g$
- ③  $\sigma(r)$  is non-negative.
- ④  $\lim_{r \rightarrow 0} r \sigma(r) = n - 1$

## Lemma 2.3 (Ledger formula)

$$\frac{d^2}{dr^2} \left( \frac{\Theta(r)}{r^{n-1}} \right) \Big|_{r=0} = -\frac{1}{3} \text{Ric}_g$$



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# Bishop volume comparison theorem

## Theorem 3.1 (see [Sakai, Theorem 3.1(2), IV])

Let  $(X, g)$  be an  $n$ -dimensional complete Riemannian manifold satisfying  $\text{Ric}_g \geq (n-1)\delta$  and  $\gamma : [0, \infty] \rightarrow X$  be a geodesic satisfying  $\gamma(0) = p \in X$  and  $\gamma'(0) = u \in T_p X, |u| = 1$ .

- If  $\delta < 0$ , then we have

$$\Theta_p(r, u) \leq \left( \frac{1}{\sqrt{|\delta|}} \sinh \left( \sqrt{|\delta|} r \right) \right)^{n-1}, \quad 0 < r < t_0(\gamma),$$

where  $\Theta_p(r, u)$  is volume density of  $S(p; r)$  at  $\exp_p(ru)$  and  $t_0(\gamma)$  attains the first conjugate point of  $p$  along  $\gamma$ .

- If equality holds at  $T \leq t_0(\gamma)$ , then the equality holds for any  $0 \leq r \leq T$  and the sectional curvature of any plane spanned by  $\gamma'(r)$  and a unit vector perpendicular to  $\gamma(r)$  is constant  $\delta$ .

## Estimates of volume entropy

- Let  $(X, g)$  be a harmonic manifold of hypergeometric type and  $\text{Ric}_g = (n-1)\delta$ ,  $\delta < 0$ .
- From the Bishop volume comparison theorem, we have

$$\Theta(r) = K_g \sinh^{n-1}(\ell r) \cosh^{\frac{Q_g}{\ell} - (n-1)}(\ell r) \leq \left( \frac{1}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|}r) \right)^{n-1}.$$

- Expanding above inequality into power-series, we have

$$K_g \left\{ \ell^{n-1} + \left( \frac{Q_g}{2\ell} - \frac{n-1}{3} \right) \ell^{n+1} r^2 + O(r^4) \right\} \leq 1 + \frac{n-1}{3!} |\delta| r^2 + O(r^4).$$

- When  $r \rightarrow 0$ , we have  $K_g \ell^{n-1} \leq 1$ .
- From Theorem C, we have

$$\frac{n-1}{3} \left( \frac{|\delta|}{\ell} + 2\ell \right) \leq Q_g.$$

## Estimates of volume entropy

- Let  $(X, g)$  be a harmonic manifold of hypergeometric type and  $\text{Ric}_g = (n-1)\delta$ ,  $\delta < 0$ .
- From the Bishop volume comparison theorem, we have

$$\Theta(r) = K_g \sinh^{n-1}(\ell r) \cosh^{\frac{Q_g}{\ell} - (n-1)}(\ell r) \leq \left( \frac{1}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|}r) \right)^{n-1}.$$

- On the other hand, from the definition of volume entropy, we have

$$\begin{aligned} Q_g &= \lim_{r \rightarrow \infty} \frac{\log \text{Vol} B(o; r)}{r} = \lim_{r \rightarrow \infty} \frac{\log \int_0^r \Theta(r) dr}{r} \\ &\leq \lim_{r \rightarrow \infty} \frac{\log \int_0^r \left( \frac{1}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|}r) \right)^{n-1} dr}{r} = \dots = (n-1)\sqrt{|\delta|}. \end{aligned}$$

- Remark : above inequality holds for any harmonic manifolds.

# Estimates of volume entropy

- Hence, we obtain

$$\frac{n-1}{3} \left( \frac{|\delta|}{\ell} + 2\ell \right) \leq Q_g \leq (n-1)\sqrt{|\delta|}$$

- From the famous theorem of the arithmetic and geometric means, we have

$$\frac{2\sqrt{2|\delta|}(n-1)}{3} \leq Q_g \leq (n-1)\sqrt{|\delta|}$$

- We normalize the metric  $g$  satisfying  $\text{Ric}_g = -(n-1)g$ , i.e.,  $\delta = -1$ , we have

$$\therefore \frac{2\sqrt{2}(n-1)}{3} \leq Q_g \leq (n-1).$$

Q.E.D

# Harmonic manifolds that attain the maximum or minimum value of $Q_g$

- In the case of  $Q_g = (n - 1)$ , we have  $\Theta(r) = \sinh^{n-1} r$ .
- From the Bishop volume comparison theorem, we find that  $(X, g)$  is an  $n$ -dimensional **real hyperbolic space** of constant sectional curvature  $-1$ .
- In the case of  $Q_g = \frac{2\sqrt{2}(n-1)}{3}$ , it is not clear what properties such a harmonic manifold generally carries.
- We check whether **there are harmonic manifold of hypergeometric type, homothetic to a Damek-Ricci space, whose volume entropy  $Q_g = \frac{2\sqrt{2}(n-1)}{3}$ .**

# Damek-Ricci spaces

- A **Damek-Ricci space**  $S = AN$  is a solvable Lie group which is a one-dimensional extension of a generalized Heisenberg group  $N$  having a certain left-invariant metric  $g$ .
- A generalized Heisenberg group is a 2-step nilpotent group which satisfies a certain condition.
- Let  $\mathfrak{z}$  be the center of the Lie algebra  $\mathfrak{n}$  of  $N$  and  $\mathfrak{v}$  is the orthogonal complement of  $\mathfrak{z}$  in  $\mathfrak{n}$  ( $m_{\mathfrak{v}} = \dim \mathfrak{v}$ ,  $m_{\mathfrak{z}} = \dim \mathfrak{z}$ ).
- $S \simeq \mathbb{R} \times \mathfrak{v} \times \mathfrak{z} \simeq \mathbb{R}^{m_{\mathfrak{v}}+m_{\mathfrak{z}}+1}$
- It is known that for each  $m \in \mathbb{N}$  there exist an infinite number of non-isomorphic generalized Heisenberg groups with  $m_{\mathfrak{z}} = m$ .
- Moreover, the Ricci curvature tensor  $\text{Ric}_g$  and the volume entropy  $Q_g$  of a Damek-Ricci space  $(S, g)$  are given by

$$\text{Ric}_g = - \left( m_{\mathfrak{z}} + \frac{m_{\mathfrak{v}}}{4} \right) g, \quad Q_g = m_{\mathfrak{z}} + \frac{m_{\mathfrak{v}}}{2}.$$

In the case of  $Q_g = \frac{2\sqrt{2}(n-1)}{3}$  on Damek-Ricci spaces

- Since  $n = \dim S = m_3 + m_v + 1$ , we have

$$\operatorname{Ric}_g = -\left(m_3 + \frac{m_v}{4}\right)g = -(n-1) \cdot \frac{(m_3 + m_v/4)}{m_3 + m_v}g.$$

- By constant rescaling  $g$  as  $k^2g$  where  $k = \sqrt{\frac{(m_3 + m_v/4)}{m_3 + m_v}}$ , we obtain  $\operatorname{Ric}_{k^2g} = -(n-1)(k^2g)$  and

$$Q_{k^2g} = \frac{Q_g}{k} = \left(m_3 + \frac{m_v}{2}\right) \sqrt{\frac{m_3 + m_v}{(m_3 + m_v/4)}}.$$

- Solving the equation

$$\left(m_3 + \frac{m_v}{2}\right) \sqrt{\frac{m_3 + m_v}{(m_3 + m_v/4)}} = \frac{2\sqrt{2}(n-1)}{3} = \frac{2\sqrt{2}(m_3 + m_v)}{3},$$

we have  $m_v = 2m_3$ .



In the case of  $Q_g = \frac{2\sqrt{2}(n-1)}{3}$  on Damek-Ricci spaces

From the classification of generalized Heisenberg groups, we find that Damek-Ricci spaces satisfying  $m_0 = 2m_3$  are only in the following 4 cases;

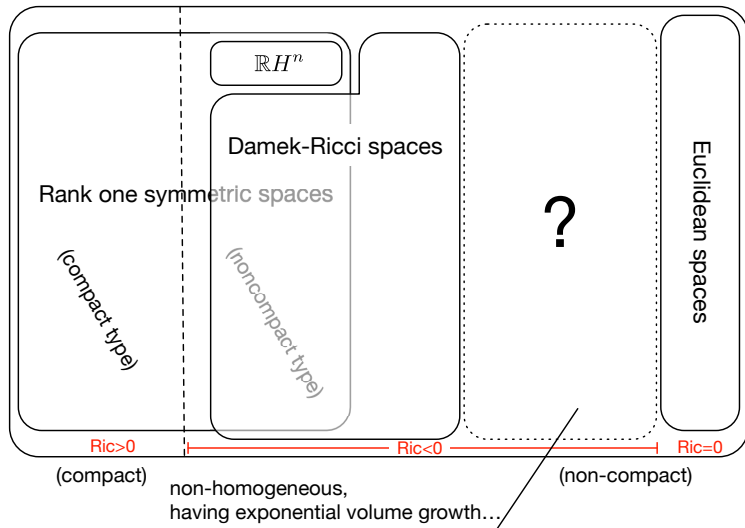
- ①  $m_3 = 1$  ( $n = 4$ )
- ②  $m_3 = 2$  ( $n = 7$ )
- ③  $m_3 = 4$  ( $n = 13$ )
- ④  $m_3 = 8$  ( $n = 25$ )

### Remark 3.2

In the case of i),  $S$  is isometric to a 4-dimensional complex hyperbolic space  $\mathbb{C}H^2$ . In the case of ii),  $S$  has 7-dimension which is the smallest dimension among non-symmetric Damek-Ricci spaces.

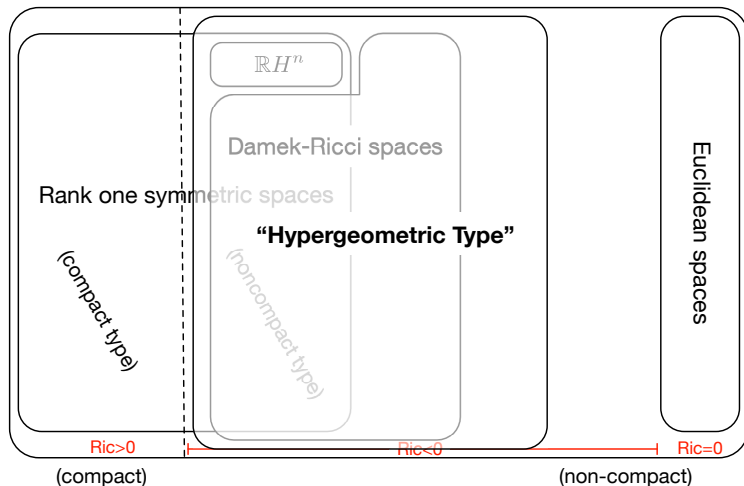
# What is known for harmonic manifolds (4)

## Harmonic Manifolds



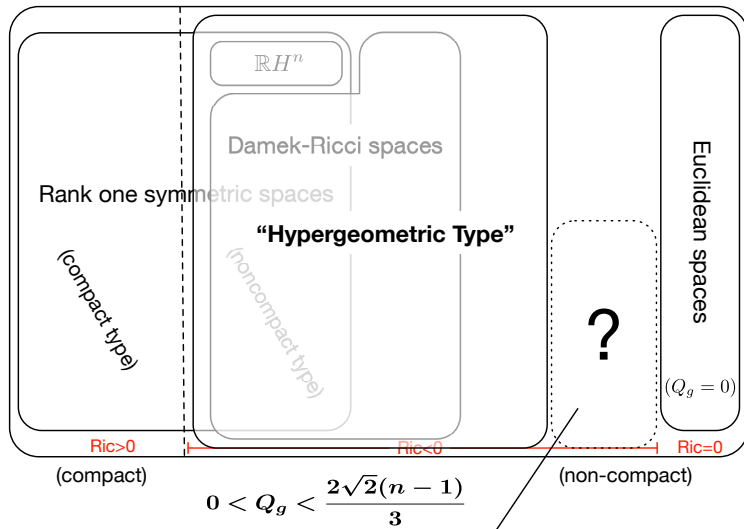
# What is known for harmonic manifolds (4)

## Harmonic Manifolds



# What is known for harmonic manifolds (4)

## Harmonic Manifolds



## Future work

- Characterize harmonic manifolds of hypergeometric type whose volume entropy satisfies  $Q_g = \frac{2\sqrt{2}(n-1)}{3}$ .
- Show the existence of a harmonic manifold of hypergeometric type which is not a Damek-Ricci space.
- Show the existence of a non-compact harmonic manifold whose volume entropy satisfies  $0 < Q_g < \frac{2\sqrt{2}(n-1)}{3}$ .

**Thank you for your attention.**

# References I



Anker, J.-P., Damek, E., and Yacoub, C., *Spherical Analysis on Harmonic AN Groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. Ser.4, **23** (1996), 643-679.



Berndt, J., Tricerri, F. and Vanhecke, L., Generalized Heisenberg groups and Damek-Ricci harmonic spaces, Lecture Notes in Math. **1598**, Springer-Verlag, Berlin, 1995.



Besse A. L., Manifolds all of whose Geodesics are Closed, Springer-Verlag, Berlin, 1978.



Damek, E. and Ricci, F., *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. **27** (1992), 139-142.



Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, Francesco G., Higher Transcendental Functions, Vol. I, Robert E. Krieger Publishing Co., Inc., Melbourne, 1981.



Gray, A. and Vanhecke, L., Riemannian geometry as determined by the volumes of small geodesic balls, Acta Math. **11** (1979), 157-198.



Heber, J., *On harmonic and asymptotically harmonic homogeneous spaces*, Geom. Funct. Anal., **16** (2006), 869-890.

# References II



Itoh, M. and Satoh, H., *Harmonic Hadamard manifolds and Gauss hypergeometric differential equations*, Publ. Res. Inst. Math. Sci. **55** (2019), 531-564.



Itoh, M. and Satoh, H., *Spherical Fourier Transform on harmonic manifolds of hypergeometric type and Plancherel Theorem*, preprint.



Knieper, G., *New results on noncompact harmonic manifolds*, Comment. Math. Helv. **87** (2012), 669-703.



Knieper, G., *A survey on noncompact harmonic and asymptotically harmonic manifolds*, Geometry, topology, and dynamics in negative curvature, London Math. Soc. Lecture Note Ser. **425** (2016), 146-197, Cambridge, Cambridge Univ. Press.



Koornwinder, T. H., *Jacobi functions and analysis on noncompact semisimple Lie groups*, In "Special functions: Group theoretical aspects and applications", R. A. Askey et al. (eds.), Reidel, 1984, 1-85.



Lichnerowicz, A., *Sur les espaces riemanniens completement harmoniques*, Bull. Soc. Math. France, **72** (1944), 146-168.

# References III



Nikolayevsky, Y., *Two theorems on harmonic manifolds*, Comment. Math. Helv. **80** (2005), 29-50.



Ranjan, A. and Shah, H., *Harmonic manifolds with minimal horospheres*, J. Geom. Anal. **12** (2002), 683-694.



Ranjan, A. and Shah, H., *Busemann functions in a harmonic manifold*, Geom. Dedicata **101** (2003), 167-183.



Rouvière, F., *Espaces de Damek-Ricci*, Geometrie et Analyse, Séminaires et Congrès **7** (2003), 45-100.



Sakai, T., Riemannian geometry, Transl. Math. Monogr. **149**, Amer. Math. Soc., 201 Charles St., Providence, RI, 1996.



Szabó, Z. I., *The Lichnerowicz conjecture on harmonic manifolds*, J. Differential Geom., **31** (1990), 1-28.



Walker, A. G., *On Lichnerowicz's conjecture for harmonic 4-spaces*, J. London Math. Soc. **24** (1949), 2128.