Conformal Vector Fields on Complex Hyperbolic Space[1]

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September 8, 2025

Differential Geometry and its Applications

[1]Based on joint work with Hemangi M. Shah (HRI, India), arXiv:2506.09710 and arXiv:2507.1813.

Outline

This Section

What is a conformal vector field?

Definition

Let (M, g) be a Riemannian manifold and ξ a smooth vector field on M. Then, ξ is called a *conformal vector field* if

$$\mathcal{L}_\xi g = 2\rho g,$$

for some function $\rho \in C^\infty(M)$. Here, $\mathcal{L}_\xi g$ denotes the Lie derivative of the metric g along ξ, and ρ is called the *potential function* of ξ.

- **Killing**: $\rho \equiv 0 \quad (\Leftrightarrow \ \mathcal{L}_\xi g = 0)$.
- **Homothetic**: $\rho \equiv c$ for a constant $c \neq 0$
- In general, a conformal vector field generates only a *local* conformal flow.
- On a complete manifold, every Killing vector field generates a *global* one-parameter group of isometries.

Main Theorems

Theorem 1.1 (Complex hyperbolic case)

Let ξ be a conformal vector field on the complex hyperbolic space $\mathbb{C}H^n$ with $n \geq 2$. Then ξ must be a Killing vector field.

This phenomenon also holds for a wider family of homogeneous spaces:

Theorem 1.2 (General case)

Let (M, g) be a Damek–Ricci space. Then any conformal vector field on M is Killing.

- **Note:** These results also follow from classification results by Tashiro (1965) and Kanai (1983), which will be mentioned later.
- **Our contribution:** We give a direct and constructive proof, which will be explained in the following sections.

This Section

How We Arrived at This Problem (I)

Our interest began with **harmonic manifolds**.

Definition

A Riemannian manifold (M, g) is harmonic if the **volume density function** $\sqrt{\det g_{ij}}$ in normal coordinates centered at $\forall p \in M$ depends only on the radius (i.e., it is a radial function).

- Classical examples: Euclidean spaces, spheres, and hyperbolic spaces.
- Every harmonic manifold is an **Einstein manifold**.

Conjecture (Lichnerowicz conjecture)

Every harmonic manifold is either Euclidean space or a rank-one symmetric space.

- Counterexamples: **Damek–Ricci spaces** (discovered in the 1990s) are harmonic and they include non-symmetric examples.

How We Arrived at This Problem (II)

- In this context, we asked the question:

  Do non-Killing conformal vector fields exist on harmonic manifolds?
- As a preliminary step, we first checked how conformal vector fields are constructed on the **real hyperbolic space** $\mathbb{R}H^n$.
- Then, we studied the **complex hyperbolic plane** $\mathbb{C}H^2$, and later extended the method to $\mathbb{C}H^n$ and to Damek–Ricci spaces.
- After completing the main part of our work, we realized that our results also follows from classical results, especially by Tashiro (1965) and Kanai (1983).
- However, our approach is different: it gives a **direct proof using differential equations**, and provides **explicit expressions** for the vector fields.
- This method may lead to further developments and give a new point of view on conformal rigidity.

Background: Conformal Rigidity

Conjecture (Lichnerowicz)

If a compact Riemannian manifold has an *essential* conformal transformation group, then it must be conformally equivalent to $\mathbb{S}^n$ or $\mathbb{E}^n$.

- Here, *essential* means that the conformal group cannot be reduced to the isometry group under any conformal change of the metric.
- Resolved in many cases:
  - Compact case (Lelong-Ferrand, Obata, Ledger),
  - Complete Einstein case (Yano, Nagano),
  - General case (Alekseevskii, Lelong-Ferrand).
- Classical proofs often rely on:
  - global group actions, or
  - Bochner-type vanishing results.

Background: Tashiro–Kanai Results

Known classification

By results of **Tashiro (1965)** and **Kanai (1983)**, if a complete Einstein manifold (M^n, g) admits a non-homothetic conformal vector field, then (M, g) must be isometric to one of the following:

- $k > 0$: the round sphere $\mathbb{S}^n(1/\sqrt{k})$,
- $k = 0$: the Euclidean space $\mathbb{E}^n$,
- $k < 0$: a warped product $(N, h) \times_f (\mathbb{R}, g_0)$, where (N, h) is a complete Einstein manifold of non-positive scalar curvature.

- In particular, any Einstein manifold with negative Ricci curvature that is **not a warped product** admits only Killing vector fields.
- Hence, our theorems for $\mathbb{C}H^n$ and Damek–Ricci spaces *follow from these classical results*.
- However, our approach gives a **direct, constructive proof** using the conformal Killing equation, offering a new perspective.

Our Approach: PDE and Function Theory

- We study conformal vector fields *locally*, by directly solving the conformal Killing equation: $\mathcal{L}_\xi g = 2\rho g$.
- **Key idea**: view $\mathbb{C}H^2$ as a special case of a Damek-Ricci solvable Lie group with left-invariant metric.
- Then, use:
  - an explicit orthonormal frame on the group,
  - coordinate representation of ξ in this frame,
  - and translate the conformal Killing equation into a PDE system.
- Part of the system yields a Cauchy–Riemann type relation.
  $\Rightarrow$ Some coefficients must be (real, imaginary parts of) **holomorphic functions**.
- Expanding these functions in terms of **harmonic polynomials** and substituting into the remaining PDEs, we obtain an identity between **linearly independent** functions.
  $\Rightarrow$ Almost all expansion coefficients vanish, forcing $\rho \equiv 0$.

Our Approach: From $\mathbb{C}H^2$ to Higher Dimensions

- The PDE method developed for $\mathbb{C}H^2$ also works for general $\mathbb{C}H^n$ and even Damek–Ricci spaces.
- **Key reason:** The conformal Killing equation system in $\mathbb{C}H^n$ (or DR spaces) contains the $\mathbb{C}H^2$ system as a natural subsystem.
- The Cauchy–Riemann-type equations reappear **unchanged** in higher dimensions.
- The only difference is in some derivative terms — the differential operators become more complicated, but the essential analytic structure remains the same.

This Section

Conformal Vector Fields on Real Hyperbolic Space $\mathbb{R}H^n$

Theorem 3.1

Every non-Killing conformal vector field ξ on $\mathbb{R}H^n(-1)$ ($n \geq 3$) can be written as a linear combination of gradient fields:

$$\xi = \sum c_i \nabla e^{b_i},$$

where each b_i is a Busemann function associated with a boundary point.

Lemma 3.2

Let (M, g) be a Riemannian manifold. If a smooth function f satisfies $\mathrm{Hess}(f) = g - df \otimes df$, then the gradient field $\nabla(e^f)$ is conformal.

- These vector fields form an $(n+1)$-dimensional family of non-Killing conformal vector fields.
- Each ∇e^b is *incomplete*: its flow reaches boundary in finite time.
- Although the conformal Lie algebra is larger than the isometry algebra, we have $\mathrm{Conf}(\mathbb{R}H^n) = \mathrm{Isom}(\mathbb{R}H^n)$.

Why the Real Hyperbolic Construction Fails

- The construction via Busemann functions that works in $\mathbb{R}H^n$ fails in $\mathbb{C}H^n$.
- **Key obstruction:** The Hessian of a Busemann function has *two distinct nonzero eigenvalues* in directions orthogonal to its kernel.
- Therefore, we turned to a direct PDE approach, analyzing the conformal Killing equation in a natural left-invariant frame.

In the compact Káhler case, the following result is known:

Theorem 3.3 (Lichnerowicz)

Any conformal vector field on a compact Kähler manifold of real dimension at least four must be Killing.

Damek–Ricci Spaces: Definition and Basic Properties

- A **Damek–Ricci space** S is a solvable Lie group with Lie algebra $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, where:
  - $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is 2-step nilpotent, with $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$ and $[\mathfrak{v}, \mathfrak{z}] = 0$. (Here $\mathfrak{z}$ is the center.)
  - For $Z \in \mathfrak{z}$, define $J_Z : \mathfrak{v} \to \mathfrak{v}$ by $\langle J_Z V, W\rangle = \langle Z, [V, W]\rangle$.
  - **Additional condition:** For unit $Z \in \mathfrak{z}$, one requires $J_Z^2 = -|Z|^2 \mathrm{Id}_{\mathfrak{v}}$.
  - $\mathfrak{a} = \mathbb{R}A$ is 1-dimensional, acting on $\mathfrak{n}$ by

$$[A, V] = \tfrac{1}{2} V, \quad [A, Z] = Z \quad (V \in \mathfrak{v},\ Z \in \mathfrak{z}).$$

  - The left-invariant metric extends the inner product on $\mathfrak{n}$, with A unit and orthogonal to $\mathfrak{n}$.

- These are harmonic, noncompact Riemannian manifolds that include all noncompact rank-one symmetric spaces except for $\mathbb{R}H^n$.

- **In particular:** $\mathbb{C}H^n$ is a special case of a Damek–Ricci space ($\dim \mathfrak{z} = 1$).

$\mathbb{C}H^2$ as a Damek–Ricci Space

We realize $\mathbb{C}H^2$ as a Damek–Ricci space S.

- $\mathbb{C}H^2$ corresponds to the case $\dim \mathfrak{v} = 2$ and $\dim \mathfrak{z} = 1$.
- The Lie algebra $\mathfrak{s}$ has orthonormal basis:

$$\{V,\, J_Z V,\, Z,\, A\},$$

  with nontrivial brackets:

$$[V, J_Z V] = Z, \quad [A, V] = \frac{1}{2}V, \quad [A, J_Z V] = \frac{1}{2}J_Z V, \quad [A, Z] = Z,$$

  and all other brackets vanish.
- These define a left-invariant frame on the group $S\,(= \mathbb{C}H^2)$.
- By the exponential map, we introduce natural coordinates:

$$(x, y, z, a) \in \mathbb{R}^4 \cong S.$$

Setup for the Conformal Killing System

- A general vector field ξ on $\mathbb{C}H^2 \cong S$ is expressed as:

$$\xi = f_1 V + f_2 J_Z V + f_3 Z + f_4 A,$$

where each f_i is a smooth function of (x, y, z, a).

- In these coordinates, the left-invariant vector fields are given by:

$$\begin{aligned} V &= e^{a/2}\left(\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}\right), & J_Z V &= e^{a/2}\left(\frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}\right), \\ Z &= e^{a}\frac{\partial}{\partial z}, & A &= \frac{\partial}{\partial a}. \end{aligned}$$

- We compute the conformal Killing equation

$$\mathcal{L}_\xi g = 2\rho g,$$

with respect to this left-invariant frame.

- This leads to a system of PDEs for f_1, f_2, f_3, f_4 and ρ.

Conformal Killing Equation in Coordinates (I)

$$\mathcal{L}_\xi g = 2\rho g \quad \Longleftrightarrow \quad \begin{cases} 2e^{a/2}\left(\dfrac{\partial f_1}{\partial x} - \dfrac{y}{2}\dfrac{\partial f_1}{\partial z}\right) - f_4 = 2\rho, & (1) \\ e^{a/2}\left(\dfrac{\partial f_1}{\partial y} + \dfrac{x}{2}\dfrac{\partial f_1}{\partial z}\right) + e^{a/2}\left(\dfrac{\partial f_2}{\partial x} - \dfrac{y}{2}\dfrac{\partial f_2}{\partial z}\right) = 0, & (2) \\ 2e^{a/2}\left(\dfrac{\partial f_2}{\partial y} + \dfrac{x}{2}\dfrac{\partial f_2}{\partial z}\right) - f_4 = 2\rho, & (3) \\ e^{a}\dfrac{\partial f_1}{\partial z} + e^{a/2}\left(\dfrac{\partial f_3}{\partial x} - \dfrac{y}{2}\dfrac{\partial f_3}{\partial z}\right) + f_2 = 0, & (4) \\ e^{a}\dfrac{\partial f_2}{\partial z} + e^{a/2}\left(\dfrac{\partial f_3}{\partial y} + \dfrac{x}{2}\dfrac{\partial f_3}{\partial z}\right) - f_1 = 0, & (5) \\ 2e^{a}\dfrac{\partial f_3}{\partial z} - 2f_4 = 2\rho, & (6) \\ \dfrac{\partial f_1}{\partial a} + e^{a/2}\left(\dfrac{\partial f_4}{\partial x} - \dfrac{y}{2}\dfrac{\partial f_4}{\partial z}\right) + \dfrac{1}{2}f_1 = 0, & (7) \\ \dfrac{\partial f_2}{\partial a} + e^{a/2}\left(\dfrac{\partial f_4}{\partial y} + \dfrac{x}{2}\dfrac{\partial f_4}{\partial z}\right) + \dfrac{1}{2}f_2 = 0, & (8) \\ \dfrac{\partial f_3}{\partial a} + e^{a}\dfrac{\partial f_4}{\partial z} + f_3 = 0, & (9) \\ 2\dfrac{\partial f_4}{\partial a} = 2\rho. & (10) \end{cases}$$

Conformal Killing Equation in Coordinates (II)

$$
\mathcal{L}_\xi g = 2\rho g \quad \Longrightarrow \begin{cases}
\left(\frac{\partial f_1}{\partial x} - \frac{y}{2}\frac{\partial f_1}{\partial z}\right) = e^{-a}\frac{\partial}{\partial a}\left(e^{a/2} f_4\right), & (1) \\
\left(\frac{\partial f_1}{\partial y} + \frac{x}{2}\frac{\partial f_1}{\partial z}\right) + \left(\frac{\partial f_2}{\partial x} - \frac{y}{2}\frac{\partial f_2}{\partial z}\right) = 0, & (2) \\
\left(\frac{\partial f_2}{\partial y} + \frac{x}{2}\frac{\partial f_2}{\partial z}\right) = e^{-a}\frac{\partial}{\partial a}\left(e^{a/2} f_4\right), & (3) \\
e^{a}\frac{\partial f_1}{\partial z} + e^{a/2}\left(\frac{\partial f_3}{\partial x} - \frac{y}{2}\frac{\partial f_3}{\partial z}\right) + f_2 = 0, & (4) \\
e^{a}\frac{\partial f_2}{\partial z} + e^{a/2}\left(\frac{\partial f_3}{\partial y} + \frac{x}{2}\frac{\partial f_3}{\partial z}\right) - f_1 = 0, & (5) \\
\frac{\partial}{\partial z}\left(e^{a} f_3\right) = e^{-a}\frac{\partial}{\partial a}\left(e^{a} f_4\right), & (6) \\
e^{a}\left(\frac{\partial f_4}{\partial x} - \frac{y}{2}\frac{\partial f_4}{\partial z}\right) + \frac{\partial}{\partial a}\left(e^{a/2} f_1\right) = 0, & (7) \\
e^{a}\left(\frac{\partial f_4}{\partial y} + \frac{x}{2}\frac{\partial f_4}{\partial z}\right) + \frac{\partial}{\partial a}\left(e^{a/2} f_2\right) = 0, & (8) \\
e^{-a}\frac{\partial}{\partial a}\left(e^{a} f_3\right) = -\frac{\partial}{\partial z}\left(e^{a} f_4\right). & (9)
\end{cases}
$$

Cauchy–Riemann Type Structure

$$
\begin{cases}
\dfrac{\partial}{\partial z}\left(e^{a} f_3\right) = e^{-a}\dfrac{\partial}{\partial a}\left(e^{a} f_4\right), & (6)\\[2ex]
e^{-a}\dfrac{\partial}{\partial a}\left(e^{a} f_3\right) = -\dfrac{\partial}{\partial z}\left(e^{a} f_4\right). & (9)
\end{cases}
$$

- Let $w = e^{a}$, and define new functions $F_3 = w\, f_3$, $F_4 = w\, f_4$.
- Then equations (6) and (9) become:

$$
\begin{cases}
\dfrac{\partial F_3}{\partial z} = \dfrac{\partial F_4}{\partial w},\\[2ex]
\dfrac{\partial F_3}{\partial w} = -\,\dfrac{\partial F_4}{\partial z}.
\end{cases}
$$

  which is a Cauchy–Riemann system on the (z, w)-plane.
- So, $F_3 + i\, F_4$ must be a holomorphic function on the (z, w)-plane, depending on parameters x, y.

Harmonic Polynomial Expansion

- Since F_3, F_4 are harmonic in (z, w), they are real-analytic.
- Hence, around the origin they admit expansions into homogeneous harmonic polynomials (See [1, Cor. 5.34]) :

$$F_3 = \sum_{m=1}^{\infty} F_3^{[m]} + C_3, \quad F_4 = \sum_{m=1}^{\infty} F_4^{[m]} + C_4.$$

- Each component:

$$F_3^{[m]} + iF_4^{[m]} = \left(C_1^{[m]} + iC_2^{[m]}\right)(z + iw)^m.$$

  Here, $C_1^{[m]}, C_2^{[m]}, C_3$ and C_4 are smooth in the parameters x, y.
- This reduces the PDE system to algebraic conditions on the coefficients.

Expression of $F_3^{[m]}$ and $F_4^{[m]}$

- For each $m \geq 1$,

$$F_3^{[m]} + iF_4^{[m]} = \left(C_1^{[m]} + i\,C_2^{[m]}\right)(z + iw)^m, \qquad C_j^{[m]} = C_j^{[m]}(x, y).$$

- Using

$$(z + iw)^m = \sum_{k=0}^{m} \binom{m}{k} i^{\,k} z^{\,m-k} w^k \qquad \text{and} \qquad i^{\,k} = \cos\frac{\pi k}{2} + i \sin\frac{\pi k}{2},$$

we obtain the real formulas:

$$\begin{cases} F_3^{[m]}(z, w) = \displaystyle\sum_{k=0}^{m} \binom{m}{k} \left(C_1^{[m]} \cos\frac{\pi k}{2} - C_2^{[m]} \sin\frac{\pi k}{2}\right) z^{\,m-k} w^k, \\ F_4^{[m]}(z, w) = \displaystyle\sum_{k=0}^{m} \binom{m}{k} \left(C_1^{[m]} \sin\frac{\pi k}{2} + C_2^{[m]} \cos\frac{\pi k}{2}\right) z^{\,m-k} w^k. \end{cases}$$

Analytic Remarks

- The expansion is *local*, valid near the origin (other centers also possible).
- Convergence is uniform on small neighbourhoods, together with derivatives.
- Therefore, termwise differentiation and substitution into the PDE system are justified.
- Integration and differentiation can also be interchanged with the series.

These facts allow us to compare coefficients in the PDEs rigorously.

This Section

Proof Outline of Theorem 1.1 ($n = 2$) – Step 1

- Recall equation (10):
$$\frac{\partial f_4}{\partial a} = \rho.$$
  If f_4 is independent of a, then $\rho \equiv 0$.

- Recall $F_3 = w\, f_3,\ F_4 = w\, f_4$ with $w = e^a$.
  - $F_3 + iF_4$ is holomorphic in (z, w).
  - Hence F_3, F_4 are harmonic and admit expansions into homogeneous harmonic polynomials.

- By substituting the expansion into the remaining PDEs and comparing coefficients, we find that almost all coefficients must vanish:
  - $C_4(x, y) = 0$
  - $C_2^{[m]}(x, y) = 0$ for any positive integer m.
  - $C_1^{[m]}(x, y) = 0$ for $m \geq 3$.

Proof Outline of Theorem 1.1 ($n = 2$) – Step 2

- With the previous vanishing, all $C_2^{[m]}$ vanish, and $C_1^{[m]}$ remains only when $m = 1, 2$.
- From the definition, the two remaining terms can be written as:

$$F_4^{[1]} = e^a \, C_1^{[1]}(x, y), \qquad F_4^{[2]} = 2z \, e^a \, C_1^{[2]}(x, y),$$

  while $C_4 = 0$.
- Hence

$$F_4 = w \left(C_1^{[1]}(x, y) + 2z \, C_1^{[2]}(x, y) \right),$$

$$\Rightarrow \quad f_4 = \frac{F_4}{w} = C_1^{[1]}(x, y) + 2z \, C_1^{[2]}(x, y),$$

  which is independent of a.
- Therefore, by (10) we obtain $\rho \equiv 0$.
- By analyticity (identity theorem), this polynomial form of F_4 holds globally, not only near the origin.

From $\mathbb{C}H^2$ to General $\mathbb{C}H^n$ and Damek–Ricci Spaces S

- Let $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}A$ be the Lie algebra of S.
- Choose unit vectors $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$.
- Take an orthonormal basis

$$\{V, J_Z V, V_3, \ldots, Z, Z_2, \ldots, A\},$$

  and introduce natural coordinates $(x, y, v_3, \ldots, z, z_2, \ldots, a)$.
- Consider the conformal Killing equation

$$\mathcal{L}_\xi g = 2\rho g.$$

- Focus on the $\{V, J_Z V, Z, A\}$-block of equations.
  - This subsystem is almost identical to the 10 equations obtained in $\mathbb{C}H^2$.
  - The only difference: differential operators in x, y variables are slightly modified.
  - But the essential structure of the equations remains the same.

Conclusion of the General Case

- The PDE subsystem for $\{V, J_Z V, Z, A\}$ has the same analytic structure as in $\mathbb{C}H^2$:
  - Cauchy–Riemann type equations reappear unchanged.
  - Harmonic polynomial expansion applies in the same way.
- Therefore, the $\mathbb{C}H^2$ argument extends directly to $\mathbb{C}H^n$ and to any Damek–Ricci space.
- We obtain

$$\rho \equiv 0,$$

  i.e., every conformal vector field on S is Killing.

Geometric Interpretation and Future Direction

- In a general Damek–Ricci space S, for any unit vectors $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, the set $\{V, J_Z V, Z, A\}$ forms a Lie subalgebra of $\mathfrak{s}$, and the subgroup it generates is a complex hyperbolic plane.
- Thus, **any Damek–Ricci space S contains a totally geodesic submanifold isometric to $\mathbb{C}H^2$**.
- Our proof can be viewed as showing that the **infinitesimal conformal rigidity** (non-existence of nontrivial conformal vector fields) of this totally geodesic submanifold *propagates to the ambient space*.
- This suggests the following problem ...

Remarks on the Proposed Problem

Q1 If N is a totally geodesic submanifold of M, and N has no nontrivial conformal vector fields, then does M also have none?

$\Rightarrow$ False.

- Example: $M = N \times_f \mathbb{R}$, where f is a non-constant positive function satisfying $f(0) = 1$. Then $\xi = f(t)\frac{\partial}{\partial t}$ is a nontrivial conformal vector field.

Q2 If M has no nontrivial conformal vector fields, then do all its totally geodesic submanifolds also?

$\Rightarrow$ False.

- Example: $\mathbb{C}H^n$ contains $\mathbb{R}H^k$ totally geodesically, but $\mathbb{R}H^k$ admits many nontrivial conformal vector fields.

Problem 4.1

What are the precise **geometric conditions** under which infinitesimal conformal rigidity propagates between ambient spaces and their totally geodesic submanifolds?

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