極値とは **What is an extremum ?**

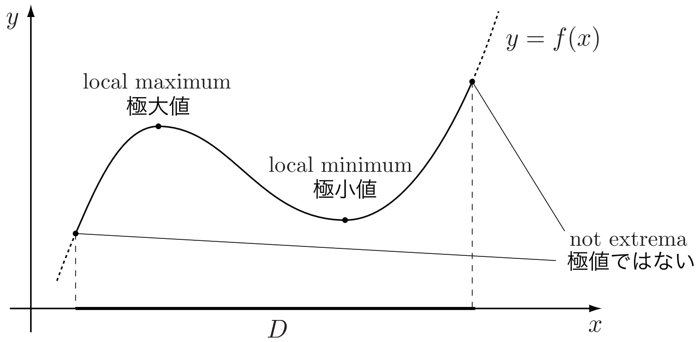
$f(x)$：領域 D を定義域とする微分可能関数 Let $f(x)$ be a differentiable function on D.

$\boxed{\text{極大値}}$：局所的な最大値 (local maxima)

「$f(x)$ は $x = a$ で極大値をとる」“A function f has a local maximum at $x = a$”

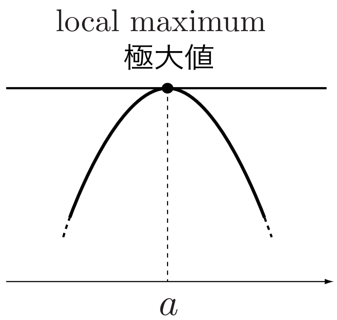
$\Longleftrightarrow$ a を含むある開近傍（開区間）$I \subset D$ が存在し，任意の $b \in I$ にたいし
There exists an open neighborhood (interval) I $(\subset D)$ including a such that for any $b \in I$

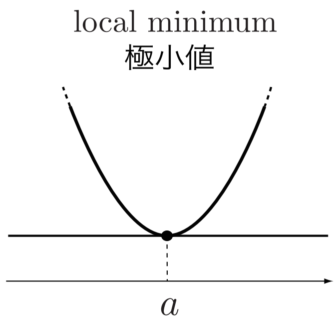
$$f(a) \geq f(b).$$



極値の判定：**1** 変数関数の場合 **Finding extrema**

<u>1st Step</u>：$f'(a) = 0$ となる点 a を求める． Finding a point $a \in D$ such that $f'(a) = 0$.





極値をとる点 a での接線の傾きは 0
(The slope of the tangent line at a) $= 0$

<u>2nd Step</u>：$f''(a)$ の符号を調べる． $x = a$ の近傍での $f(x)$ の 2 次近似は
Check the sign of $f''(a)$. The two-degree Taylor polynomial near a is

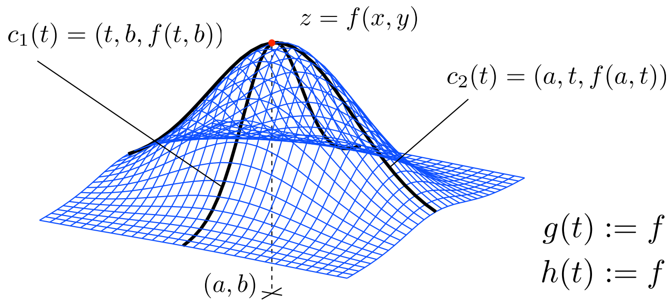
$$f(x) \simeq f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \begin{cases} \leq f(a) & \text{if } \ f''(a) < 0 \\ \geq f(a) & \text{if } \ f''(a) > 0 \end{cases}$$

したがって，$f''(a) < 0$ のとき $x = a$ で極大値をとる．
Hence, if $f''(a) < 0$ then f has a local maximum at a.

極値の判定：2 変数関数の場合 **Functions of two variables**

1st Step：$f_x(a,b)=0$ かつ $f_y(a,b)=0$ を満たす点 (a,b) を求める.

Finding a point (a,b) such that $f_x(a,b)=0$ and $f_y(a,b)=0$.



$g(t):=f(t,b)$ has an extremum at $t=a$.
$h(t):=f(a,t)$ has an extremum at $t=b$.

2nd Step：2 次の項の符号を調べる. Check the sign of the two-degree Taylor polynomial.

$$f(x,y)\simeq f(a,b)+f_x(a,b)\,(x-a)+f_y(a,b)\,(y-b)$$
$$+\frac{1}{2}\underbrace{\left\{f_{xx}(a,b)\,(x-a)^2+2f_{xy}(a,b)\,(x-a)(y-b)+f_{yy}(a,b)\,(y-b)^2\right\}}_{\text{正 positive or 負 negative ?}}$$

極値の判定：2 変数関数の場合 **"The Second-Derivative Test"**

$$F(x,y)=f_{xx}(a,b)\,(x-a)^2+2f_{xy}(a,b)\,(x-a)(y-b)+f_{yy}(a,b)\,(y-b)^2$$

($A:=f_{xx}(a,b)$, $B:=f_{xy}(a,b)$, $C:=f_{yy}(a,b)$ とおく)

$$=A\left\{\underbrace{\left((x-a)+\frac{C}{A}(y-b)\right)^2}_{\geq 0}+\left(AC-B^2\right)\underbrace{\left(\frac{y-b}{A}\right)^2}_{\geq 0}\right\}$$

$$H=\begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b)\end{pmatrix}$$：f の (a,b) におけるヘッセ行列.
the Hessian matrix of f at (a,b).

$\det(H)>0,\ f_{xx}(a,b)>0 \implies F(x,y)\geq 0$（$f$ は (a,b) で極小値をとる）
f has a local minimum at (a,b).

$\det(H)>0,\ f_{xx}(a,b)<0 \implies F(x,y)\leq 0$（$f$ は (a,b) で極大値をとる）
f has a local maximum at (a,b).

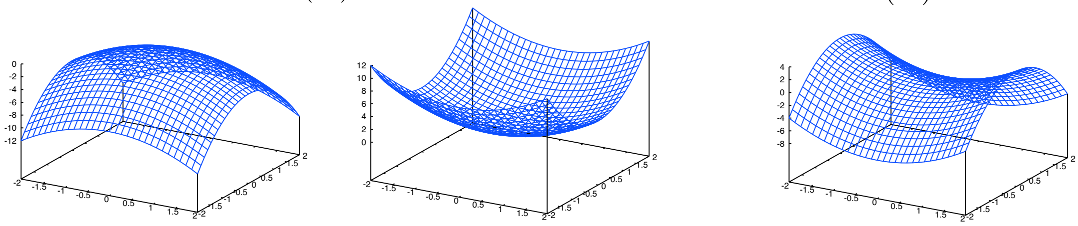
$\det(H)<0 \implies F(x,y)$ は正にも負にもなる（極値をとらない）
f does not have an extremum at (a,b).

$\det(H)=0 \implies$ 極値をとる場合もとらない場合もある
The test tells us nothing.

例 Examples

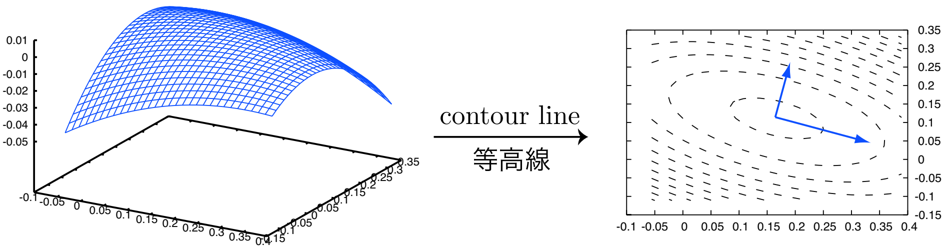
典型例 typical examples：$f(x,y)=\alpha(x-a)^2+\beta(y-b)^2$

$\det(H)>0$ $\det(H)<0$



$\alpha,\beta<0$ $\alpha,\beta>0$ $(\alpha\beta<0)$

問題 3.18 (2) $f(x,y)\simeq\frac{1}{162}-\frac{2}{9}\left(x-\frac{1}{6}\right)^2-\frac{1}{3}\left(y-\frac{1}{9}\right)\left(x-\frac{1}{6}\right)-\frac{1}{2}\left(y-\frac{1}{9}\right)^2+\cdots$



$F(x,y)$ の評価（もう一つの考え方）“Coordinate Rotation”

(a,b) を中心に座標を回転：$\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x-a\\y-b\end{pmatrix}$ として

$$F(x,y)=\begin{pmatrix}x-a & y-b\end{pmatrix}\begin{pmatrix}f_{xx}(a,b) & f_{xy}(a,b)\\ f_{xy}(a,b) & f_{yy}(a,b)\end{pmatrix}\begin{pmatrix}x-a\\y-b\end{pmatrix}$$

$$=\begin{pmatrix}X & Y\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}=\alpha X^2+\beta Y^2$$ (diagonalizaton of H)

となるようにできないだろうか？ $\Longrightarrow$ 可能（H の対角化．α,β は H の固有値）
We can choose an angle θ which satisfies above equation. We call above α and β eigenvalues of H.

H の固有値がすべて正 $\Longrightarrow$ f は (a,b) で極小値をとる
If all eigenvalues of H are positive, then f has a local minimum at (a,b).

H の固有値がすべて負 $\Longrightarrow$ f は (a,b) で極大値をとる
If all eigenvalues of H are negative, then f has a local maximum at (a,b).

H は零固有値をもつ $\Longrightarrow$ 極値をとる場合もとらない場合もある
If H has 0 eigenvalues, then, the test tells us nothing.

上記以外 $\Longrightarrow$ 極値をとらない
otherwise f does not have an extremum at (a,b).