The horosphere version of the Osserman conjecture and related topics

(based on the joint work with Mitsuhiro Itoh)

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§0.1) Horospheres

Definition 1.1

Let M be an n-dimensional non-compact, simply connected, complete Riemannian manifold with no conjugate points.
Let $\gamma : \mathbb{R} \to M$ be a geodesic with unit speed.

- Associated γ, we can define the Busemann function $b_\gamma : M \to \mathbb{R}$ by

$$b_\gamma(x) = \lim_{t \to \infty} \{d(x, \gamma(t)) - t\}$$

  Here d is the Riemannian distance on M.
- A level hypersurface of b_γ is called a horosphere.

§0.2) Theorem A

Theorem A (Itoh-S, Kyushu J.Math. 67 (2013))

Let M be an n-dimensional Hadamard manifold.
Then M is an n-dimensional real space form of constant curvature $-k^2$ if and only if there is a constant k such that all horospheres of M are totally umbilic with constant principal curvature k.

Remark 1.2

A Busemann function on a Hadamard manifold is at least C^2-class.
Replacing the non-positive sectional curvature condition by the condition for a manifold to be without focal points.

See

- [J.-H.Eschenburg, Math.Z. **153**(1977)]
- [N.Innami, Adv.Stud.Pure Math. **3**(1984)]

§0.3) No conjugate points

Let M be a complete Riemannian manifold.

Definition 1.3

- M is said to have no conjugate points if any points $p, q \in M$ are not conjugate.
- Suppose $p, q \in M$ and γ is a geodesic connecting p and q. Then p and q are conjugate along γ if there exists a non-trivial Jacobi field $Y(t)$ along γ that vanishes at p and q.

Lemma 1.4

M has no conjugate points if and only if $\langle Y(t), Y(t)\rangle > 0$ for $t > 0$ where Y is any nontrivial Jacobi field along any geodesic with $Y(0) = 0$.

§0.4) No focal points

Let M be a complete Riemannian manifold.

Definition 1.5

- M is said to have no focal points if every geodesic $\gamma : \mathbb{R} \to M$ has no focal point as a 1-dimensional submanifold in M.
- Let $\gamma : \mathbb{R} \to M$ and $c : [0, \ell] \to M$ be geodesics satisfying $c(0) = \gamma(a)$ for a certain $a \in \mathbb{R}$ and $\langle \gamma'(a), c'(0) \rangle = 0$. For $k \in [0, \ell]$, $c(k)$ is called a focal point of γ along c if there exists a non-trivial Jacobi field $Y(t)$ along c with $Y(0)$ is parallel to $\gamma'(a)$ and $Y(k) = 0$.

Lemma 1.6

M has no focal points if and only if $\langle Y(t), Y(t) \rangle' > 0$ for $t > 0$ where Y is any nontrivial Jacobi field along any geodesic with $Y(0) = 0$.

We remark that if M has no focal point, then M has no conjugate points.

§0.5) Horosphere version of Osserman conjecture

We can rise the following conjecture.

Conjecture (Itoh-S, Kyushu J.Math. 67 (2013))

Let M be an n-dimensional non-compact, simply connected, complete Riemannian manifold with no focal points.
Then, M must be a rank-one symmetric space of non-compact type or a flat Euclidean space, provided that $(*)$ every horosphere has constant principal curvature and these constants are common for all horospheres together with the common multiplicities.

The above is motivated by the *Osserman conjecture*.

§0.6) Osserman manifold and Osserman conjecture

Let M be a Riemannian manifold.

- R denotes the Riemannian curvature tensor of M.
- For $X \in T_pM$, the Jacobi operator $R_X : T_pM \to T_pM$ is defined by $R_X(Y) := R(Y, X)X$.

Definition 1.7

1. M is called *pointwise Osserman* if the eigenvalues of the Jacobi operator R_X do not depend on the choice of a unit tangent vector $X \in T_pM$ at each point $p \in M$.
2. M is called *globally Osserman* if M is pointwise Osserman and the eigenvalues of the Jacobi operator R_X are constant on M.

§0.6) Osserman manifold and Osserman conjecture

Osserman Conjecture (R.Osserman, Amer.Math.Monthly 97(1990))

A globally Osserman manifold is two-point homogeneous, that is, flat or rank-one symmetric.

Theorem 1.8 (Y.Nikolayevsky, Math.Ann. 331(2005))

A globally Osserman manifold of dimension $n \neq 8, 16$ *is two-point homogeneous.*

See also

- [Q.-S.Chi, Pacific J.Math. **150** (1991)]
- [P.Gilkey-A.Swann-L.Vanhecke, Quart.J.Math.Oxford(2) **46**(1995)]

§0.7) Similarities between the Osserman conjecture and our conjecture (horosphere version)

Suppose that we have a linear transformation $\mathcal{R}_X : X^\perp \to X^\perp$ determined by a unit tangent vector $X \in T_pM$.

Both of these conjectures can be interpreted as conjecture that *if the eigenvalues of a map $\mathcal{R}_X$ are constant and do not depend on the choice of a unit tangent vector $X \in T_pM$ and $p \in M$, M is two-point homogeneous*.

§0.5) Horosphere version of Osserman conjecture

Definition 1.9

Let M be an n-dimensional non-compact, simply connected, complete Riemannian manifold with no focal points.
If $(*)$ every horosphere of M has constant principal curvature and these constants are common for all horospheres together with the common multiplicities, then we call M an <u>H-Osserman manifold</u>.

Conjecture (Horosphere version of Osserman conjecture)

An H-Osserman manifold is two-point homogeneous, that is, flat or rank-one symmetric.

In this talk ...

§0 Introduction

§1 Horospheres and the shape operator

§2 Proof of our Theorems

Theorem B

If M is an H-Osserman manifold and satisfies a certain curvature condition (detailed later), then M is a rank-one symmetric space of non-compact type or flat Euclidean space.

§3 Geometry of asymptotically harmonic manifolds

Theorem C

If M is an H-Osserman manifold and homogeneous, then M is a rank-one symmetric space of non-compact type or flat Euclidean space.

§4 Osserman manifolds and Clifford structures

§1.1) The Busemann function and horospheres

Let M be an n-dimensional non-compact, complete, simply connected Riemannian manifold with no conjugate points.
Let γ be an arbitrary geodesic with unit speed in M.

- The Busemann function b_γ is a C^1-convex function satisfying $|\nabla b_\gamma| = 1$, where ∇b_γ is the gradient vector field of b_γ.
- For any $p \in M$, there exits a geodesic $\sigma : \mathbb{R} \to M$ such that
    1. σ passes through p,
    2. $\nabla b_\gamma(\sigma(t)) = -\sigma'(t)$.

  Then, we call σ an asymptote to γ.

- For $c \in \mathbb{R}$, we denote a horosphere by $\mathcal{H}_\gamma(c) = b_\gamma^{-1}(-c)$. Note that $\gamma(c) \in \mathcal{H}_\gamma(c)$.

§1.2) Horospheres and the shape operator

Let M be an n-dimensional non-compact, complete, simply connected Riemannian manifold with no focal points.
Let γ be an arbitrary geodesic with unit speed in M.

- The Busemann function b_γ is a C^2-convex function satisfying $|\nabla b_\gamma| = 1$, where ∇b_γ is the gradient vector field of b_γ.
- Write $\boldsymbol{n} := -\nabla b_\gamma$.
- Restricting $\boldsymbol{n}$ to a horosphere $\mathcal{H}_\gamma(c)$ defines a (outward) unit normal vector field on it.
- The shape operator $S_{\boldsymbol{n}}$ of $\mathcal{H}_\gamma(c)$ with respect to $\boldsymbol{n}$ is defined by

$$S_{\boldsymbol{n}} : T_p\mathcal{H}_\gamma(c) \to T_p\mathcal{H}_\gamma(c); \quad S_{\boldsymbol{n}}(V) = \nabla_V \boldsymbol{n},$$

  which is a self-adjoint endomorphism of $T_p\mathcal{H}_\gamma(c) = \boldsymbol{n}^\perp \subset T_pM$.
- $\langle S_{\boldsymbol{n}}(V), W\rangle = -\nabla db_\gamma(V, W)$ holds for $V, W \in T_p\mathcal{H}_\gamma(c)$.

§1.3) The Riccati equation

Lemma 2.1

Regarding the shape operator $S_{\boldsymbol{n}}$ as a $(1,1)$-tensor field on M, we have

$$\nabla_{\boldsymbol{n}} S_{\boldsymbol{n}} + S_{\boldsymbol{n}} \circ S_{\boldsymbol{n}} + R_{\boldsymbol{n}} = O,$$

where $R_{\boldsymbol{n}} := R(\,\cdot\,, \boldsymbol{n})\boldsymbol{n}$ is the Jacobi operator.

(Proof)

- Extend a vector field V on $\mathcal{H}_\gamma(c)$ to a vector field on M by parallel transport along a geodesic σ which is an asymptote to γ.
- Then we have $\nabla_{\boldsymbol{n}} \boldsymbol{n} = 0$ and $\nabla_{\boldsymbol{n}} V = 0$.
- Calculate $R_{\boldsymbol{n}}(V) = \nabla_V(\nabla_{\boldsymbol{n}} \boldsymbol{n}) - \nabla_{\boldsymbol{n}}(\nabla_V \boldsymbol{n}) - \nabla_{[V,\boldsymbol{n}]} \boldsymbol{n}$.

$\square$

§1.3) The Riccati equation

Remark 2.2

We write

- the shape operator of $\mathcal{H}_\gamma(t)$ at $\gamma(t)$ by $S(t)$, and
- the Jacobi operator $R_{\gamma'(t)}$ at $\gamma(t)$ by $R(t)$.

Then, the Riccati equation along γ can be expressed as

$$S'(t) + S(t)^2 + R(t) = O.$$

Lemma 2.3

For any geodesic γ, the shape operator $S(t)$ of $\mathcal{H}_\gamma(t)$ at $\gamma(t)$ satisfies

$$S'(t) + S(t)^2 + R(t) = O$$

§2.1) One approach to solve our conjecture

Riccati Equation

$$S'(t) + S(t)^2 + R(t) = O \quad \text{along any geodesic } \gamma \tag{RE}$$

Suppose that M is H-Osserman.

- If $S'(t) = O$ holds for any γ, we find that M is <u>globally Osserman</u>.
- Then, $R'(t) = -\left(S(t)^2\right)' = -S'(t) \circ S(t) - S(t) \circ S'(t) = O$.
- $R'(t) = O$ for any γ means that $(\nabla_X R)(Y, X)X = 0$ holds for any orthogonal vectors X, Y, which is equivalent to $\nabla R = O$, that is, M is <u>locally symmetric</u>.
  See [L.Vanhecke-T.J.Willmore, Math.Ann. 263 (1983)].

Lemma 2.4 (Gilkey-Swann-Vanhecke (1995))

If M is pointwise Osserman and also locally symmetric, then M is flat or locally rank-one symmetric.

§2.2) Theorem A

Theorem A

Let M be an n-dimensional non-compact, simply connected, complete Riemannian manifold with no focal points.
Then M is an n-dimensional real space form of constant curvature $-k^2$ if and only if there is a constant k such that all horospheres of M are totally umbilic with constant principal curvature k.

(Proof of "$\Longleftarrow$")

- The assumption means $S(t) = k\,\mathrm{Id}_{\gamma'(t)^\perp}$ for any geodesic γ.
  Hence, we have $S'(t) = O$.
- Moreover, $R(t) = -k^2\,\mathrm{Id}_{\gamma'(t)^\perp}$ for any geodesic γ, which means that M has constant sectional curvature $-k^2$.

$\square$

§2.3) Theorem B

Theorem B

Let M be an n-dimensional non-compact, simply connected, complete Riemannian manifold with no focal points.
Suppose that

1. *M is H-Osserman, that is, every horosphere of M carries constant distinct principal curvatures $\lambda_1 < \cdots < \lambda_\ell (\leq 0)$, and these principal curvatures with the common multiplicity are independent of the choice of horospheres.*
2. *for any geodesic γ, if $u \in T_{\gamma(0)}\mathcal{H}_{\gamma(0)}$ is a unit eigenvector of the shape operator with eigenvalue λ_i, then*
$$\langle R(t)u(t), u(t)\rangle \geq -\lambda_i^2$$
holds for any $t \geq 0$ $(i = 1, \ldots, \ell - 1)$.
Here $u(t)$ is the parallel extension of u along γ.

Then M is a flat Euclidean space or rank-one symmetric space of non-compact type.

§2.3) Theorem B : Outline of Proof

(Proof)

- For λ_i we set $T_{\lambda_i}(t) = S(t) - \lambda_i \mathrm{Id}$ and $Q_{\lambda_i}(t) = R(t) + \lambda_i^2 \mathrm{Id}$.
- Then, $\langle R(t)u(t), u(t)\rangle \geq -\lambda_i^2$ means $\langle Q_{\lambda_i}(t)u(t), u(t)\rangle \geq 0$.
- The Riccati equation (RE) can be rewritten to

$$T'_{\lambda_i}(t) + T_{\lambda_i}(t)^2 + 2\lambda_i\, T_{\lambda_i}(t) + Q_{\lambda_i}(t) = O. \qquad \text{(MRE)}$$

- By differentiating (MRE) j-times along γ, we obtain the following formula.

$$T_{\lambda_i}^{(j+1)}(t) + \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} T_{\lambda_i}^{(r)}(t) \circ T_{\lambda_i}^{(j-r)}(t) + 2\lambda_i\, T_{\lambda_i}^{(j)}(t) + Q_{\lambda_i}^{(j)}(t) = O. \qquad \text{(dMRE)}$$

$\vdots$

§2.3) Theorem B : Outline of Proof

(Proof)

$\vdots$

- By induction on $i = 1, \ldots, \ell - 1$ and $j = 0, 1, 2, \ldots$ together with some basic lemmas, we get $T_{\lambda_i}^{(j+1)}(0)u = \mathbf{0}$ for any eigenvector $u \in T_{\gamma(0)}\mathcal{H}_\gamma(0)$ of the Shape operator $S(0)$ with eigenvalue λ_i.
- In our situation, M is analytic and the Busemann function is also analytic.
  See [A.Ranjan-H.Shah, Geom.Dedicata **101** (2003)]
- Hence we obtain $T'_{\lambda_i}(t)\, u(t) = \mathbf{0}$ along γ, where $u(t)$ is the parallel extension of u.
- From the above argument, we derive $S'(t) = O$ for any geodesic.

□

§3.1) Harmonic and asymptotically harmonic manifolds

Definition 3.1

- A complete Riemannian manifold M is called a harmonic manifold if every geodesic sphere of sufficiently small radii is of constant mean curvature.
- A complete, noncompact Riemannian manifold M with no focal points is called a asymptotically harmonic manifold if every horosphere is of constant mean curvature.

Remark 3.2

- Every non-compact harmonic manifold is an asymptotically harmonic manifold.
- Every harmonic manifold is an Einstein manifold.

§3.1) Harmonic and asymptotically harmonic manifolds

Lichnerowicz Conjecture (Bull.Soc.Math.France 72 (1944))

Any harmonic manifold is two-point homogeneous.

- The Lichnerowicz conjecture is true for compact manifolds with finite fundamental group.
  See [Z.I.Szabó, J.Differential Geom. **31** (1990)].
- There exists a class of non-compact harmonic spaces with non-positive sectional curvatures, which are homogeneous but non-symmetric (called "Damek-Ricci space").
  See [E.Damek-F.Ricci, Bull.Amer.Math.Soc. **27** (1992)]

§3.1) Harmonic and asymptotically harmonic manifolds

Theorem 3.3 (J.Heber, Geom.Funct.Anal. 16 (2006))

Let M be a non-compact, simply connected homogeneous space. Then the following are equivalent:

1. *M is asymptotically harmonic and Einstein.*
2. *M is flat, or rank-one symmetric of non-compact type, or a non-symmetric Damek-Ricci space.*

From this theorem, we immediately obtain the following fact.

§3.2) Theorem C

Theorem C

If M is H-Osserman and homogeneous, then M is a flat Euclidean space or rank-one symmetric space of non-compact type.

(Proof) It is enough to show that an H-Osserman manifold is

- asymptotically harmonic,
- Einstein and
- non-symmetric Damek-Ricci spaces are not H-Osserman.

Taking a trace of the Riccati equation $S'(t) + S(t)^2 + R(t) = O$.
If M is H-Osserman,

- then $\mathrm{Tr}\,(S'(t)) = 0$ holds.
- Hence we have

$$\mathrm{Ric}(\boldsymbol{n}, \boldsymbol{n}) = \mathrm{Tr}\,(R(t)) = -\mathrm{Tr}\,\left(S(t)^2\right) = \text{Constant}$$

along any geodesic γ.

□

§4.1) Osserman manifolds and Clifford structures

Gilky-Swann-Vanhecke suggested the following approach to the Osserman conjecture:

1. show that the Riemannian curvature tensor of an Osserman manifold has a Clifford structure at every point.
2. classify Riemannian manifolds having curvature tensor as in (CS).

Definition 4.1

The Riemannian curvature tensor R has a $\mathrm{Cliff}(\nu)$-structure at $p \in M$ if

$$R(X,Y)Z = \lambda_0(\langle X,Z\rangle Y - \langle Y,Z\rangle X) + \frac{1}{3}\sum_{i=1}^{\nu}(\mu_i - \lambda_0)(2\langle J_iX,Y\rangle J_iZ + \langle J_iZ,Y\rangle J_iX - \langle J_iZ,X\rangle J_iY), \quad \text{(CS)}$$

where $J_1,\dots,J_\nu$ are skew-symmetric orthogonal operators satisfying the Hurwitz relations $J_iJ_i + J_jJ_i = -2\delta_{ij}\mathrm{Id}_{T_pM}$ and $\mu_i \neq \lambda_0$.

Following the above scheme, Nikolayevsky proved his theorem.

§4.1) Osserman manifolds and Clifford structures

Remark 4.2

If the Riemannian curvature tensor R has a $\mathrm{Cliff}(\nu)$-structure at $p \in M$, then the Jacobi operator is expressed as the form

$$R_X(Y) = \lambda_0(\|X\|^2 Y - \langle Y, X\rangle X) + \sum_{i=1}^{\nu}(\mu_i - \lambda_0)\langle J_i X, Y\rangle J_i X. \qquad \text{(CS')}$$

The eigenvalues are $\lambda_0, \lambda_1, \ldots, \lambda_\ell$, where $\lambda_1, \ldots, \lambda_\ell$ are the μ_i's without repetitions.

The tensor R in the form (CS) can be reconstructed from (CS') using polarization and the first Bianchi identities.

§4.2) Clifford structure on a horosphere

In our case, we can also define a $\mathrm{Cliff}(\nu)$-structure with respect to the shape operator of a horosphere.

Definition 4.3

Let M be an n-dimensional non-compact, simply connected, complete Riemannian manifold with no focal points.
The shape operator $S_{\boldsymbol{n}}$ of a horosphere $\mathcal{H}_\gamma(c)$ has a $\mathrm{Cliff}(\nu)$-structure if

$$S_{\boldsymbol{n}}(V) = \lambda_0 V + \sum_{i=1}^{\nu}(\mu_i - \lambda_0)\langle J_i \boldsymbol{n}, V\rangle J_i \boldsymbol{n}, \quad V \in T_p\mathcal{H}_\gamma(c). \tag{H-CS}$$

where $J_1, \ldots, J_\nu$ are skew-symmetric orthogonal operators satisfying the Hurwitz relations $J_iJ_i + J_jJ_i = -2\delta_{ij}\mathrm{Id}_{\boldsymbol{n}^\perp}$ and $\mu_i \neq \lambda_0$.

§4.3) The shape operator of a horosphere in a rank-one symmetric space of non-compact type

Remark 4.4

Let M be a hyperbolic space other than the real hyperbolic space, namely, a complex, quaternionic, or octonionic hyperbolic space.
The shape operator of a horosphere in M is completely described as

$$S_{\boldsymbol{n}}(V) = -\left(V + \sum_{i=1}^{d-1} \langle J_i \boldsymbol{n}, V \rangle \, J_i \boldsymbol{n}\right), \quad V \in T_p\mathcal{H}_\gamma(c)$$

in terms of the metric $\langle \ , \ \rangle$ and the associated complex (or quaternionic, octonionic) structure, normalized as holomorphic sectional curvature being -4. Here $d = 2, 4$ and 8 for the complex, quaternionic and octonionic space, respectively.

§4.4) Question 1

Question 1

Will the scheme suggested by Gilkey-Swann-Vanhecke also be useful in our conjecture?

1. Show that the shape operator of a horosphere in an H-Osserman manifold has a Clifford structure
2. Classify manifolds whose horosphere has the shape operator as in (H-CS).

There are two difficulties in proceeding with this scheme in our problem:

1. To describe the relation of shape operators at point p of two horospheres that both contain point p.
2. If M is H-Osserman, do the shape operators satisfy the duality principle?

§4.5) The duality principle

Let $\mathcal{V}$ be a linear space over $\mathbb{R}$.
Let $\mathcal{R} : \mathcal{V} \to \mathrm{End}(\mathcal{V})$ be a map. ($\mathcal{R}_X$ is a linear map on $\mathcal{V}$ for $X \in \mathcal{V}$. However, $X \mapsto \mathcal{R}_X$ does not have to be linear.)

Definition 4.5

We say that a map $\mathcal{R} : \mathcal{V} \to \mathrm{End}(\mathcal{V})$ satisfies the duality principle, if for any unit vector $X, Y \in \mathcal{V}$, the vector Y is an eigenvector of $\mathcal{R}_X$ if and only if the vector X is an eigenvector of $\mathcal{R}_Y$ (with the same eigenvalue).

Theorem 4.6 (Z.Rakić(1999), Y.Nikolayevsky-Z.Rakić(2013))

The following two conditions are equivalent:

1. *The Jacobi operator satisfies the duality principle;*
2. *M is Osserman.*

§4.6) Question 2

Question 2

Are the following two conditions equivalent?

1. The shape operator of any horosphere satisfies the duality principle;
2. M is H-Osserman.