

Volume entropy of harmonic ~~Hadamard~~ manifolds of hypergeometric type

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Harmonic manifolds

Definition

A harmonic manifold is a complete Riemannian manifold (X, g) whose volume density function $\sqrt{\det(g_{ij})}$ is a radial function, that is, $\sqrt{\det(g_{ij})}(x)$ depends only on the distance $d(o, x)$.

Here $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ are components of the Riemannian metric g with respect to a normal coordinate system $\{x_1, x_2, \dots, x_n\}$ at an arbitrary point $o \in X$

Remark 1.1

- Every geodesic sphere $S(p; r)$ in a harmonic manifold has constant mean curvature $\sigma(r)$.
- Harmonic manifolds are always Einstein, i.e., $\text{Ric}_g = \kappa g$.

What is known for harmonic manifolds (1)

- In 1944, Lichnerowicz conjectured that harmonic 4-manifolds are flat or rank one symmetric spaces.
- In 1949, above conjecture was proved by Walker.
- Their work was generalized to the following conjecture;

Lichnerowicz Conjecture

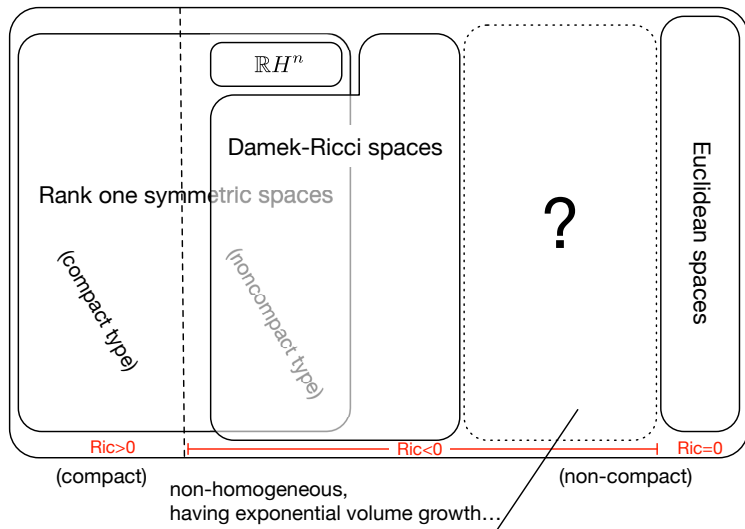
Every simply connected harmonic manifold is flat or a rank-one symmetric space.

What is known for harmonic manifolds (2)

- In 1990, Szabó proved the Lichnerowicz conjecture for simply connected **compact** harmonic manifolds.
- In 1992, Damek and Ricci gave **counter examples for the Lichnerowicz conjecture**, which is a class of harmonic, homogeneous Hadamard manifolds, called the **Damek-Ricci space**, including rank one symmetric spaces of non-compact type (except $\mathbb{R}H^n$).
- In 2006, Heber proved that a simply connected, **homogeneous** harmonic manifold is isometric to a Euclidean spaces, a rank-one symmetric space, or a Damek-Ricci space.

What is known for harmonic manifolds (3)

Harmonic Manifolds



The aim of this talk

- We define a new class of non-compact harmonic manifolds, including all Damek-Ricci spaces and rank-one symmetric spaces of non-compact type.
- We call such a space, harmonic manifold **of hypergeometric type**.
- It is motivated to develop the theory of the spherical Fourier transform on harmonic manifolds (cf. Itoh's talk on Monday).

We show the following fact;

Main Theorem (Itoh-S. in preparation)

Let (X, g) be an n -dimensional harmonic manifold of hypergeometric type, whose metric g is normalized as $\text{Ric}_g = -(n-1)g$. Then, the volume entropy Q_g of (X, g) satisfies

$$\frac{2\sqrt{2}(n-1)}{3} \leq Q_g \leq n-1. \quad (1)$$

1 Introduction

2 Harmonic manifolds of hypergeometric type and its properties

- Motivation
- Definition
- Properties

3 Proof of Main Theorem

- Bishop volume comparison theorem
- Estimates of volume entropy
- Harmonic manifolds that attain the maximum or minimum value of Q_g

Spherical Fourier transform on Damek-Ricci spaces

Anker-Damek-Yacoub (1996), see also Rouvière (2003)

- They developed **the spherical Fourier transform** on a Damek-Ricci space by reducing it to a special case of the Jacobi transform of Jacobi functions.
- The spherical Fourier transform $f \mapsto \hat{f}$ for a smooth radial function $f(x)$ on a harmonic manifold is defined by

$$\hat{f}(\lambda) = \int_{x \in X} f(x) \varphi_\lambda(x) dv_g = \omega_{n-1} \int_0^\infty f(r) \varphi_\lambda(r) \Theta(r) dr,$$

where $\Theta(r)$ is the volume density of a geodesic sphere $S(o; r)$.

- Here φ_λ is an eigenfunction of the radial part of the Laplace operator Δ with eigenvalue $\frac{Q_g^2}{4} + \lambda^2$.

Definition

The **volume entropy** of (X, g) ; $Q_g = \lim_{r \rightarrow \infty} \frac{1}{r} \log \text{Vol} B(o; r)$.

Spherical Fourier transform on Damek-Ricci spaces

Anker-Damek-Yacoub (1996), see also Rouvière (2003)

- They developed **the spherical Fourier transform** on a Damek-Ricci space by reducing it to a special case of the Jacobi transform of Jacobi functions.
- The essence of their method is that **the eigenfunction equation of the Laplace operator is transformed into a hypergeometric differential equation** by a variable transformation $z = -\sinh^2(r/2)$;

$$\frac{d^2 f}{dr^2}(r) + \sigma(r) \cdot \frac{df}{dr}(r) + \left(\frac{Q_g^2}{4} + \lambda^2 \right) f(r) = 0 \quad (2)$$

$$\Downarrow z = -\sinh^2 \frac{r}{2}$$

$$z(1-z)u''(z) + \{c - (a+b+1)z\}u'(z) - abu(z) = 0 \quad (3)$$

- Therefore, φ_λ is described by using the hypergeometric function ${}_2F_1(a, b, c; z)$.

Definition of “hypergeometric type”

Definition (Itoh's talk)

Let (X, g) be a non-compact harmonic manifold. When the equation (2) turns into (3) by a variable transformation $z = -\sinh^2(r/2)$, we call (X, g) a harmonic manifold of **hypergeometric type**.

Theorem A

Let (X, g) be a non-compact harmonic manifold. If for a variable transformation $z = z(r)$ the equation (2) turns into (3), then it holds only for $z(r) = -\sinh^2(\ell r)$ for some $\ell > 0$.

Definition of “hypergeometric type”

Definition (In this talk)

Let (X, g) be a non-compact harmonic manifold. When the equation (2) turns into (3) by a variable transformation $z = -\sinh^2(\ell r)$ for some $\ell > 0$, we call (X, g) a harmonic manifold of **hypergeometric type**.

Why change the definition?

- The main subject of this talk is **the volume entropy Q_g** which is **not scale invariant**.
- Since we want to normalize the metric and discuss estimates of Q_g , we modify the definition as above (these are essentially the same).

Characterization of “hypergeometric type”

Theorem B

A non-compact harmonic manifold (X, g) is of hypergeometric type, if and only if its volume density function $\Theta(r)$ of a geodesic sphere $S(o; r)$ is given by

$$\Theta(r) = K_g \sinh^{n-1}(\ell r) \cosh^{Q_g/\ell - (n-1)}(\ell r). \quad (4)$$

Theorem C

If the volume density function of a geodesic sphere $S(o; r)$ in a harmonic manifold is expressed by (4), then the constant K_g is given by

$$K_g = -\frac{1}{\ell^n} \cdot \frac{\text{Ric}_g}{3Q_g - 2(n-1)\ell}.$$

Remark 2.1

In 2005, Nikolayevsky proved that the volume density function $\Theta(r)$ of $S(o; r)$ in a harmonic manifold is an **exponential polynomial**.

Tools to prove Theorem A, B and C

Let (X, g) be a non-compact harmonic manifold. Let $\Theta(r)$ and $\sigma(r)$ be the volume density function and the mean curvature of $S(o; r)$, respectively.

Lemma 2.2

- ① $\sigma(r) = \frac{\Theta'(r)}{\Theta(r)}$
- ② $\lim_{r \rightarrow \infty} \sigma(r) = Q_g$
- ③ $\sigma(r)$ is non-negative.
- ④ $\lim_{r \rightarrow 0} r \sigma(r) = n - 1$

Lemma 2.3 (Ledger formula)

$$\frac{d^2}{dr^2} \left(\frac{\Theta(r)}{r^{n-1}} \right) \Big|_{r=0} = -\frac{1}{3} \text{Ric}_g$$

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Bishop volume comparison theorem

Theorem 3.1 (see [Sakai, Theorem 3.1(2), IV])

Let (X, g) be an n -dimensional complete Riemannian manifold satisfying $\text{Ric}_g \geq (n-1)\delta$ and $\gamma : [0, \infty] \rightarrow X$ be a geodesic satisfying $\gamma(0) = p \in X$ and $\gamma'(0) = u \in T_p X, |u| = 1$.

- If $\delta < 0$, then we have

$$\Theta_p(r, u) \leq \left(\frac{1}{\sqrt{|\delta|}} \sinh \left(\sqrt{|\delta|} r \right) \right)^{n-1}, \quad 0 < r < t_0(\gamma),$$

where $\Theta_p(r, u)$ is volume density of $S(p; r)$ at $\exp_p(ru)$ and $t_0(\gamma)$ attains the first conjugate point of p along γ .

- If equality holds at $T \leq t_0(\gamma)$, then the equality holds for any $0 \leq r \leq T$ and the sectional curvature of any plane spanned by $\gamma'(r)$ and a unit vector perpendicular to $\gamma(r)$ is constant δ .

Estimates of volume entropy

- Let (X, g) be a harmonic manifold of hypergeometric type and $\text{Ric}_g = (n-1)\delta$, $\delta < 0$.
- From the Bishop volume comparison theorem, we have

$$\Theta(r) = K_g \sinh^{n-1}(\ell r) \cosh^{\frac{Q_g}{\ell} - (n-1)}(\ell r) \leq \left(\frac{1}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|}r) \right)^{n-1}.$$

- Expanding above inequality into power-series, we have

$$K_g \left\{ \ell^{n-1} + \left(\frac{Q_g}{2\ell} - \frac{n-1}{3} \right) \ell^{n+1} r^2 + O(r^4) \right\} \leq 1 + \frac{n-1}{3!} |\delta| r^2 + O(r^4).$$

- When $r \rightarrow 0$, we have $K_g \ell^{n-1} \leq 1$.
- From Theorem C, we have

$$\frac{n-1}{3} \left(\frac{|\delta|}{\ell} + 2\ell \right) \leq Q_g.$$

Estimates of volume entropy

- Let (X, g) be a harmonic manifold of hypergeometric type and $\text{Ric}_g = (n-1)\delta$, $\delta < 0$.
- From the Bishop volume comparison theorem, we have

$$\Theta(r) = K_g \sinh^{n-1}(\ell r) \cosh^{\frac{Q_g}{\ell} - (n-1)}(\ell r) \leq \left(\frac{1}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|}r) \right)^{n-1}.$$

- On the other hand, from the definition of volume entropy, we have

$$\begin{aligned} Q_g &= \lim_{r \rightarrow \infty} \frac{\log \text{Vol} B(o; r)}{r} = \lim_{r \rightarrow \infty} \frac{\log \int_0^r \Theta(r) dr}{r} \\ &\leq \lim_{r \rightarrow \infty} \frac{\log \int_0^r \left(\frac{1}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|}r) \right)^{n-1} dr}{r} = \dots = (n-1)\sqrt{|\delta|}. \end{aligned}$$

- Remark : above inequality holds for any harmonic manifolds.

Estimates of volume entropy

- Hence, we obtain

$$\frac{n-1}{3} \left(\frac{|\delta|}{\ell} + 2\ell \right) \leq Q_g \leq (n-1)\sqrt{|\delta|}$$

- From the famous theorem of the arithmetic and geometric means, we have

$$\frac{2\sqrt{2|\delta|}(n-1)}{3} \leq Q_g \leq (n-1)\sqrt{|\delta|}$$

- We normalize the metric g satisfying $\text{Ric}_g = -(n-1)g$, i.e., $\delta = -1$, we have

$$\therefore \frac{2\sqrt{2}(n-1)}{3} \leq Q_g \leq (n-1).$$

Q.E.D

Harmonic manifolds that attain the maximum or minimum value of Q_g

- In the case of $Q_g = (n - 1)$, we have $\Theta(r) = \sinh^{n-1} r$.
- From the Bishop volume comparison theorem, we find that (X, g) is an n -dimensional **real hyperbolic space** of constant sectional curvature -1 .
- In the case of $Q_g = \frac{2\sqrt{2}(n-1)}{3}$, it is not clear what properties such a harmonic manifold generally carries.
- We check whether **there are harmonic manifold of hypergeometric type, homothetic to a Damek-Ricci space, whose volume entropy $Q_g = \frac{2\sqrt{2}(n-1)}{3}$.**

Damek-Ricci spaces

- A **Damek-Ricci space** $S = AN$ is a solvable Lie group which is a one-dimensional extension of a generalized Heisenberg group N having a certain left-invariant metric g .
- A generalized Heisenberg group is a 2-step nilpotent group which satisfies a certain condition.
- Let \mathfrak{z} be the center of the Lie algebra \mathfrak{n} of N and \mathfrak{v} is the orthogonal complement of \mathfrak{z} in \mathfrak{n} ($m_{\mathfrak{v}} = \dim \mathfrak{v}$, $m_{\mathfrak{z}} = \dim \mathfrak{z}$).
- $S \simeq \mathbb{R} \times \mathfrak{v} \times \mathfrak{z} \simeq \mathbb{R}^{m_{\mathfrak{v}}+m_{\mathfrak{z}}+1}$
- It is known that for each $m \in \mathbb{N}$ there exist an infinite number of non-isomorphic generalized Heisenberg groups with $m_{\mathfrak{z}} = m$.
- Moreover, the Ricci curvature tensor Ric_g and the volume entropy Q_g of a Damek-Ricci space (S, g) are given by

$$\text{Ric}_g = - \left(m_{\mathfrak{z}} + \frac{m_{\mathfrak{v}}}{4} \right) g, \quad Q_g = m_{\mathfrak{z}} + \frac{m_{\mathfrak{v}}}{2}.$$

In the case of $Q_g = \frac{2\sqrt{2}(n-1)}{3}$ on Damek-Ricci spaces

- Since $n = \dim S = m_3 + m_v + 1$, we have

$$\operatorname{Ric}_g = -\left(m_3 + \frac{m_v}{4}\right)g = -(n-1) \cdot \frac{(m_3 + m_v/4)}{m_3 + m_v}g.$$

- By constant rescaling g as k^2g where $k = \sqrt{\frac{(m_3 + m_v/4)}{m_3 + m_v}}$, we obtain $\operatorname{Ric}_{k^2g} = -(n-1)(k^2g)$ and

$$Q_{k^2g} = \frac{Q_g}{k} = \left(m_3 + \frac{m_v}{2}\right) \sqrt{\frac{m_3 + m_v}{(m_3 + m_v/4)}}.$$

- Solving the equation

$$\left(m_3 + \frac{m_v}{2}\right) \sqrt{\frac{m_3 + m_v}{(m_3 + m_v/4)}} = \frac{2\sqrt{2}(n-1)}{3} = \frac{2\sqrt{2}(m_3 + m_v)}{3},$$

we have $m_v = 2m_3$.

In the case of $Q_g = \frac{2\sqrt{2}(n-1)}{3}$ on Damek-Ricci spaces

From the classification of generalized Heisenberg groups, we find that Damek-Ricci spaces satisfying $m_0 = 2m_3$ are only in the following 4 cases;

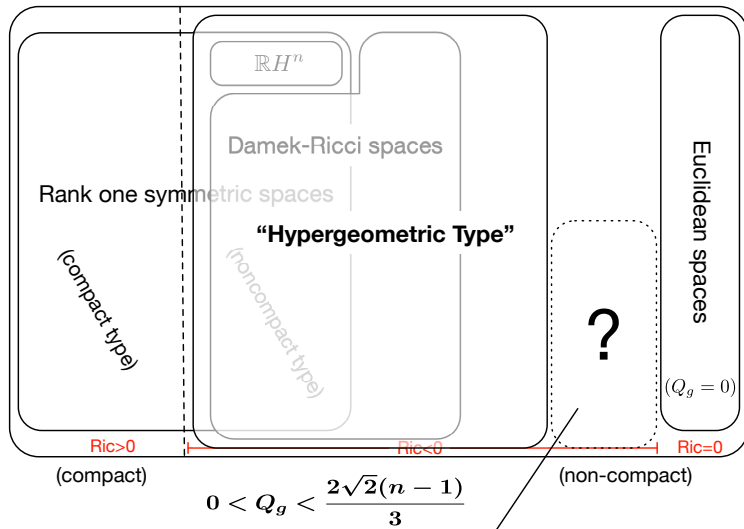
- ① $m_3 = 1$ ($n = 4$)
- ② $m_3 = 2$ ($n = 7$)
- ③ $m_3 = 4$ ($n = 13$)
- ④ $m_3 = 8$ ($n = 25$)

Remark 3.2

In the case of i), S is isometric to a 4-dimensional complex hyperbolic space $\mathbb{C}H^2$. In the case of ii), S has 7-dimension which is smallest dimension among non-symmetric Damek-Ricci spaces.

What is known for harmonic manifolds (4)

Harmonic Manifolds



Future work

- Characterize harmonic manifolds of hypergeometric type whose volume entropy satisfies $Q_g = \frac{2\sqrt{2}(n-1)}{3}$.
- Find a harmonic manifold of hypergeometric type which is not a Damek-Ricci space.
- Show the existence of a non-compact harmonic manifold whose volume entropy satisfies $0 < Q_g < \frac{2\sqrt{2}(n-1)}{3}$.

Thank you for your attention.

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