極値とは What is an extremum ?

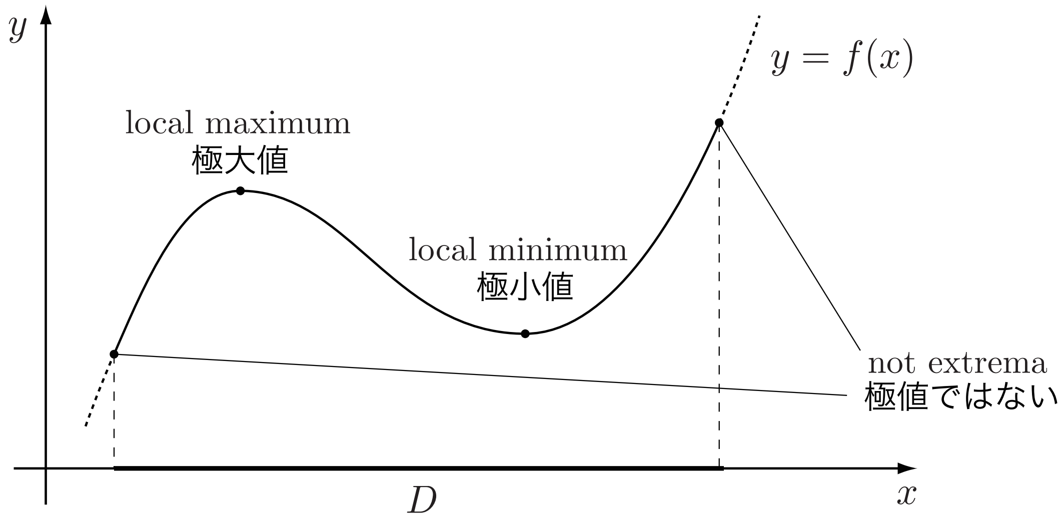
$f(x)$：領域 D を定義域とする微分可能関数 Let $f(x)$ be a differentiable function on D.

$\boxed{\text{極大値}}$：局所的な最大値 (local maxima)

「$f(x)$ は $x = a$ で極大値をとる」“A function f has a local maximum at $x = a$”

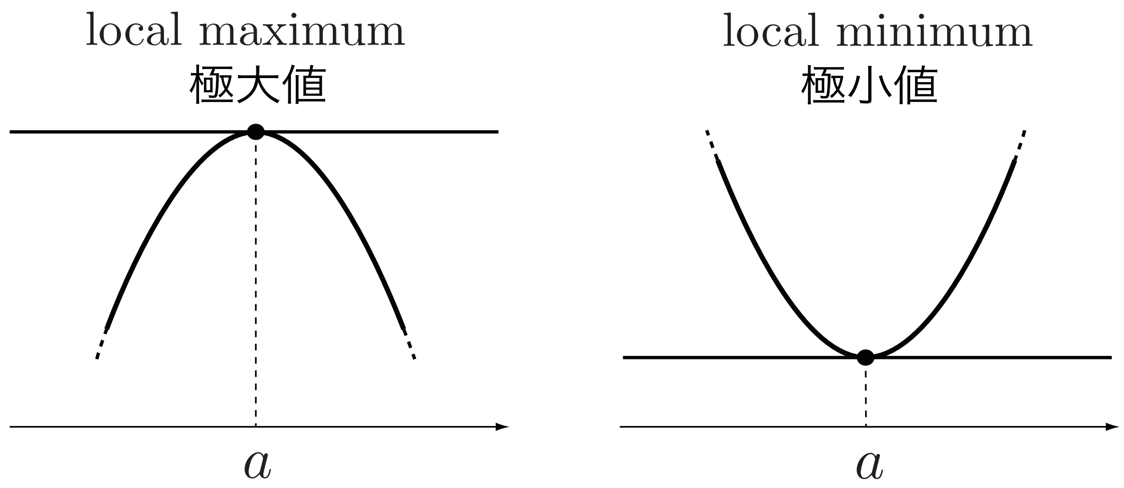
$\Longleftrightarrow$ a を含むある開近傍（開区間）$I \subset D$ が存在し，任意の $b \in I$ にたいし
There exists an open neighborhood (interval) I ($\subset D$) including a such that for any $b \in I$

$$f(a) \geq f(b).$$



極値の判定：1 変数関数の場合 **Finding extrema**

1st Step：$f'(a) = 0$ となる点 a を求める． Finding a point $a \in D$ such that $f'(a) = 0$.



極値をとる点 a での接線の傾きは 0
(The slope of the tangent line at a) $= 0$

2nd Step：$f''(a)$ の符号を調べる． $x = a$ の近傍での $f(x)$ の 2 次近似は

Check the sign of $f''(a)$. The two-degree Taylor polynomial near a is

$$f(x) \simeq f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \begin{cases} \leq f(a) & \text{if } f''(a) < 0 \\ \geq f(a) & \text{if } f''(a) > 0 \end{cases}$$

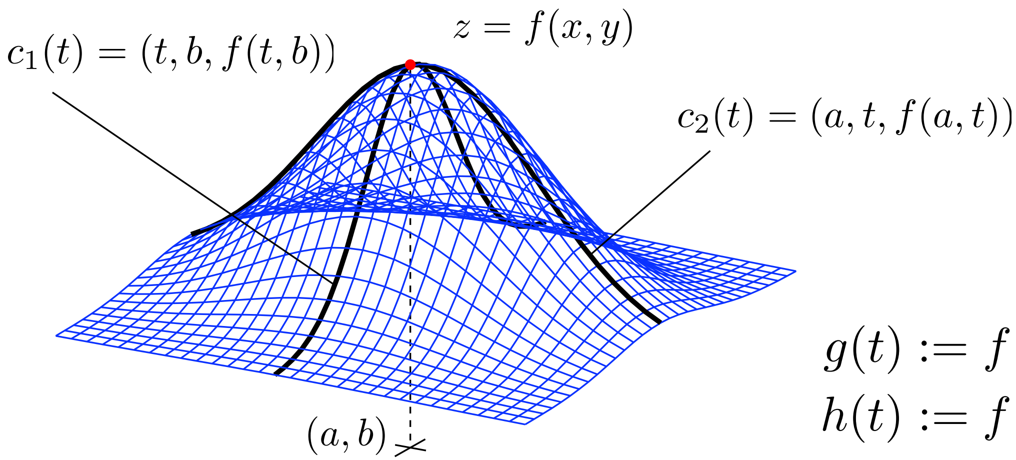
したがって，$f''(a) < 0$ のとき $x = a$ で極大値をとる．

Hence, if $f''(a) < 0$ then f has a local maximum at a.

極値の判定：2 変数関数の場合 **Functions of two variables**

1st Step：$f_x(a,b)=0$ かつ $f_y(a,b)=0$ を満たす点 (a,b) を求める.

Finding a point (a,b) such that $f_x(a,b)=0$ and $f_y(a,b)=0$.



$g(t) := f(t,b)$ has an extremum at $t=a$.

$h(t) := f(a,t)$ has an extremum at $t=b$.

2nd Step：2 次の項の符号を調べる. Check the sign of the two-degree Taylor polynomial.

$$f(x,y) \simeq f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$+\frac{1}{2}\underbrace{\left\{f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right\}}_{\text{正 positive or 負 negative ?}}$$

極値の判定：2 変数関数の場合 **“The Second-Derivative Test”**

$$F(x,y) = f_{xx}(a,b)\,(x-a)^2 + 2f_{xy}(a,b)\,(x-a)(y-b) + f_{yy}(a,b)\,(y-b)^2$$

（$A := f_{xx}(a,b),\ B := f_{xy}(a,b),\ C := f_{yy}(a,b)$ とおく）

$$= A\left\{\underbrace{\left((x-a) + \frac{C}{A}(y-b)\right)^2}_{\geq 0} + \left(AC - B^2\right)\underbrace{\left(\frac{y-b}{A}\right)^2}_{\geq 0}\right\}$$

$$H = \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{pmatrix}$$

：f の (a,b) におけるヘッセ行列.
the Hessian matrix of f at (a,b).

$\det(H) > 0,\ f_{xx}(a,b) > 0 \implies F(x,y) \geq 0$（$f$ は (a,b) で極小値をとる）
f has a local minimum at (a,b).

$\det(H) > 0,\ f_{xx}(a,b) < 0 \implies F(x,y) \leq 0$（$f$ は (a,b) で極大値をとる）
f has a local maximum at (a,b).

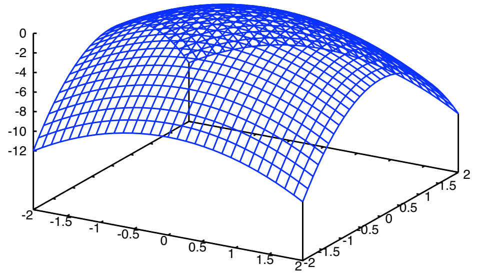
$\det(H) < 0 \implies F(x,y)$ は正にも負にもなる（極値をとらない）
f does not have an extremum at (a,b).

$\det(H) = 0 \implies$ 極値をとる場合もとらない場合もある
The test tells us nothing.

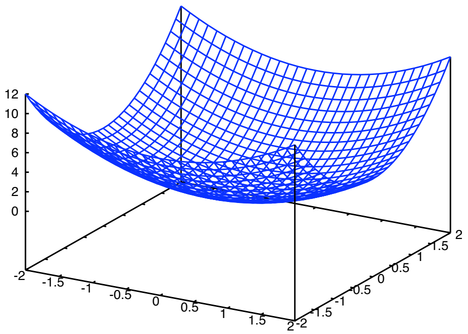
例 **Examples**

典型例 typical examples：$f(x,y) = \alpha(x-a)^2 + \beta(y-b)^2$

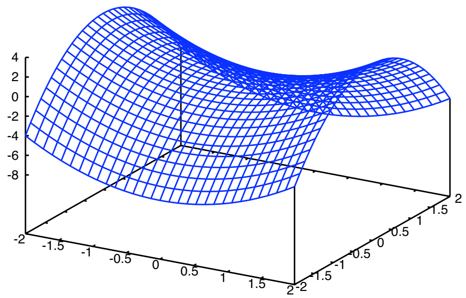
$\det(H) > 0$ $\qquad\qquad\qquad\qquad$ $\det(H) < 0$



$\alpha, \beta < 0$

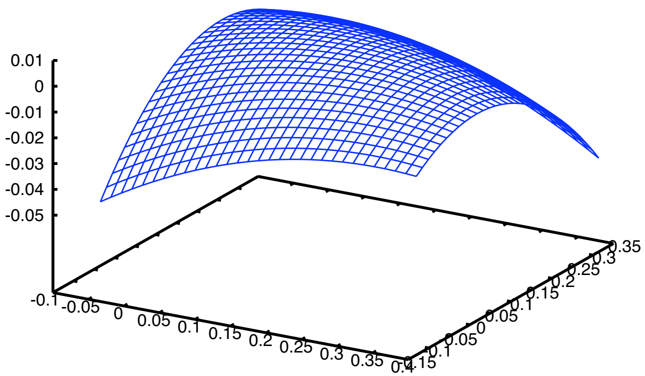


$\alpha, \beta > 0$

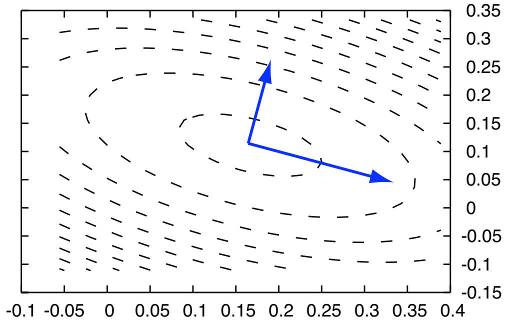


$(\alpha\beta < 0)$

問題 3.18 (2) $f(x,y) \simeq \frac{1}{162} - \frac{2}{9}\left(x - \frac{1}{6}\right)^2 - \frac{1}{3}\left(y - \frac{1}{9}\right)\left(x - \frac{1}{6}\right) - \frac{1}{2}\left(y - \frac{1}{9}\right)^2 + \cdots$



$\xrightarrow[\text{等高線}]{\text{contour line}}$



$F(x,y)$ の評価（もう一つの考え方）"Coordinate Rotation"

(a,b) を中心に座標を回転：$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}$ として

$$F(x,y) = \begin{pmatrix} x-a & y-b \end{pmatrix} \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

$$= \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \alpha X^2 + \beta Y^2$$ (diagonalizaton of H)

となるようにできないだろうか？ $\Longrightarrow$ 可能（H の対角化．α,β は H の固有値）

We can choose an angle θ which satisfies above equation. We call above α and β eigenvalues of H.

H の固有値がすべて正 If all eigenvalues of H are positive,	$\Longrightarrow$	f は (a,b) で極小値をとる then f has a local minimum at (a,b).
H の固有値がすべて負 If all eigenvalues of H are negative,	$\Longrightarrow$	f は (a,b) で極大値をとる then f has a local maximum at (a,b).
H は零固有値をもつ If H has 0 eigenvalues,	$\Longrightarrow$	極値をとる場合もとらない場合もある then, the test tells us nothing.
上記以外 otherwise	$\Longrightarrow$	極値をとらない f does not have an extremum at (a,b).