A Detailed Study of The Lindley Distribution

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SCHOOL OF MATHEMATICAL SCIENCES,
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SCHOOL OF MATHEMATICAL SCIENCES

Re-accredited by National Assessment & Accreditation Council (NAAC) with 'A' grade 4^{th} Cycle)

Umavi Nagar, Jalgaon - 425001 (M.S.) India

DEPARTMENT OF STATISTICS

CERTIFICATE

This is to certify that the project entitled **A Detailed Study of The Lindley Distribution** is a bonafide work carried out by **Patil Kalyani Purushottam(366350)**,**Patil Yogesh Ganesh (366359)** , **Shirsale Harshada Jagdish(366363)** in partial fulfillment of 3rd semester, Master of Science in STATISTICS under Kavayitri Bahinabai Chaudhari University Narth Maharashtra University, Jalgaon during the year 2024-25.

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ABSTRACT

The Lindley distribution, introduced as a mixture of exponential and gamma distributions, has gained significant attention in statistical modeling due to its flexibility and wide range of applications. This project focuses on an in-depth study of the one-parameter Lindley distribution, including its mathematical properties, parameter estimation methods, and real-world applications.

The study explores the probability density function (PDF), cumulative distribution function (CDF), and moments of the distribution to derive key measures such as the mean, variance, and moment-generating function (MGF). Maximum likelihood estimation (MLE) is employed for parameter estimation, and its performance is evaluated using simulation studies. Additionally, the goodness-of-fit of the Lindley distribution is compared with other well-known distributions, such as the exponential and gamma distributions, through empirical data analysis.

The project also investigates the versatility of the Lindley distribution in modeling lifetime data, reliability systems, and other phenomena where data exhibit positive skewness. Applications in fields like biostatistics, engineering, and environmental sciences are highlighted to demonstrate its practical relevance.

This comprehensive analysis not only enhances the understanding of the Lindley distribution but also provides a foundation for its extension to multi-parameter versions and its use in more complex statistical models. The findings contribute to the broader literature on probabilistic modeling and offer valuable insights for researchers and practitioners in applied statistics.

ACKNOWLEDGEMENT

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One Parameter Lindley Distribution

1.1 Introduction

The Lindley distribution was first introduced as a one scale parameter distribution by Lindley [1958], Zakerzadeh and Dolati [2009], In the recent years, researchers have given Lindley distribution a special attention for its importance in modelling complex real lifetime data. Some researchers went in the track of studying the Lindley distribution and its properties in more details. Ghitany et al. [2008] studied some properties of the one parameter Lindley distribution, and in the application part, they showed that it is more flexible and works better in modelling lifetime data than the known exponential distribution. Other researchers have introduced more flexible generalizations of Lindley by compounding Lindley with other well-known distributions. The Lindley distribution was first introduced by Dennis V. Lindley in 1958 Lindley [1958] as a solution to certain statistical problems encountered in Bayesian inference. It has since found widespread application in various fields due to its unique properties and flexibility. The Lindley distribution is characterized by a single parameter, θ, which controls the shape of the distribution.

One of the notable features of the Lindley distribution is its ability to model lifetime data and represent systems with non-monotonic hazard rates, which are commonly observed in real-world scenarios. The hazard function of the Lindley distribution initially increases and then decreases, making it suitable for modeling situations where the failure rate is not constant over time. This property distinguishes it from the exponential distribution, which assumes a constant hazard rate, and aligns it more closely with the gamma distribution in terms of flexibility.

In recent years, the Lindley distribution has been extended and generalized to create new distributions that retain its desirable properties while offering even greater flexibility. These generalized forms are used to address more complex modeling scenarios, further cementing the Lindley distribution's importance in statistical theory and practice.

This paper provides an in-depth examination of the Lindley distribution, exploring its mathematical properties, applications, and extensions. The goal is to offer a comprehensive resource for researchers and practitioners who utilize this distribution in their work.

1.2 Probability Density Function (PDF)

The Lindley distribution is defined with the PDF:

$$f(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \quad ; x > 0, \theta > 0 \tag{1.1}$$

The Cumulative Density Function of lindley distribution is obtained as :

$$\begin{aligned}
F(x,\theta) &= \int_0^x f(t,\theta)dt \\
&= \int_0^x \frac{\theta^2}{\theta+1}(1+t)e^{-\theta t}dt \\
&= \left(\frac{\theta^2}{\theta+1}\right)\int_0^x (1+t)e^{-\theta t}dt \\
&= \left(\frac{\theta^2}{\theta+1}\right) I
\end{aligned} \tag{1.2}$$

Now, $I = \int_0^x (1+t)e^{-\theta t}dt$

By using integration by parts

$$\begin{aligned}
I &= (1+t)\int_0^x e^{-\theta t}dt \cdot \int_0^x \left[\frac{d}{dt}(1+t)\int e^{-\theta t}dt\right]dt \\
&= \left[(1+t)\frac{e^{-\theta t}}{-\theta}\right]_0^x - \int_0^x \left[1\frac{e^{-\theta t}}{-\theta}\right]dt \\
&= \left[(1+t)\frac{e^{-\theta x}}{-\theta} - (1+0)\frac{e^0}{-\theta}\right] - \frac{1}{\theta}\int_0^x (-e^{-\theta t})dt \\
&= \left[\frac{(1+x)}{-\theta}e^{-\theta x} + \frac{1}{\theta}\right] + \frac{1}{\theta}\left[\frac{e^{-\theta t}}{-\theta}\right]_0^x \\
I &= \left[\frac{(1+x)}{-\theta}e^{-\theta x} + \frac{1}{\theta}\right] + \frac{1}{\theta^2}\left(-e^{-\theta x} + 1\right) \\
&= \left[\frac{-e^{-\theta x}}{\theta} - \frac{x}{\theta}e^{-\theta x} + \frac{1}{\theta}\right] - \frac{e^{-\theta x}}{\theta^2} + \frac{1}{\theta^2}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{e^{-\theta x}}{\theta} - \frac{x}{\theta}e^{-\theta x} - \frac{e^{-\theta x}}{\theta^2} + \frac{1}{\theta} + \frac{1}{\theta^2} \\
&= \frac{-\theta e^{-\theta x} - x\theta e^{-\theta x} - e^{-\theta x} + \theta}{\theta^2} + \frac{1}{\theta^2} \\
&= \frac{1}{\theta^2}(-\theta - x\theta - 1)e^{-\theta x} + \frac{(\theta+1)}{\theta^2} \\
&\Rightarrow \text{equation(1.2) becomes}
\end{aligned}
$$

$$
\begin{aligned}
F(x;\theta) &= \left(\frac{\theta^2}{\theta+1}\right)\left[-\frac{(\theta+x\theta+1)}{\theta^2}e^{-\theta x} + \frac{\theta+1}{\theta^2}\right] \\
&= -\frac{(\theta+x\theta+1)}{(\theta+1)}e^{-\theta x} + 1
\end{aligned}
$$

$$
F(x;\theta) = 1 - \frac{(\theta+x\theta+1)e^{-\theta x}}{(\theta+1)}, \quad x>0, \theta>0 \tag{1.3}
$$

now we have to verify that the pdf of one parameter Lindley distribution is valid.
Checking the two conditions :
i) $f(x;\theta) > 0$

ii) $\int_0^\infty f(x;\theta)dx = 1$

$$
\begin{aligned}
&\int_0^\infty f(x,\theta)dx = 1 \\
\Rightarrow &\int_0^\infty f(x,\theta)dx = \int_0^\infty \left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}dx \\
&= \left(\frac{\theta^2}{\theta+1}\right)\int_0^\infty (1+x)e^{-\theta x}dx \\
&= \left(\frac{\theta^2}{\theta+1}\right)\left[\int_0^\infty x^{1-1}e^{-\theta x}dx + \int_0^\infty x^{2-1}e^{-\theta^x}dx\right] \\
&\Rightarrow \quad = \left(\frac{\theta^2}{\theta+1}\right)\left[\frac{\Gamma 1}{\theta} + \frac{\Gamma 2}{\theta^2}\right] \\
&= \frac{\theta^2}{\theta+1}\left[\frac{1}{\theta} + \frac{1}{\theta^2}\right] = \frac{\theta^2}{\theta+1}\left(\frac{\theta+1}{\theta^2}\right) \\
&\Rightarrow \int_0^\infty f(x,\theta)dx = 1
\end{aligned}
$$

Hence, The pdf of One parameter Lindley distribution is valid.

Conclusion

From figure 1.1 we can observed that One parameter Lindley Distribution: With Different values of shape parameter θ we observed that the shape of the One parameter Lindley Distribution is Positively Skewed.

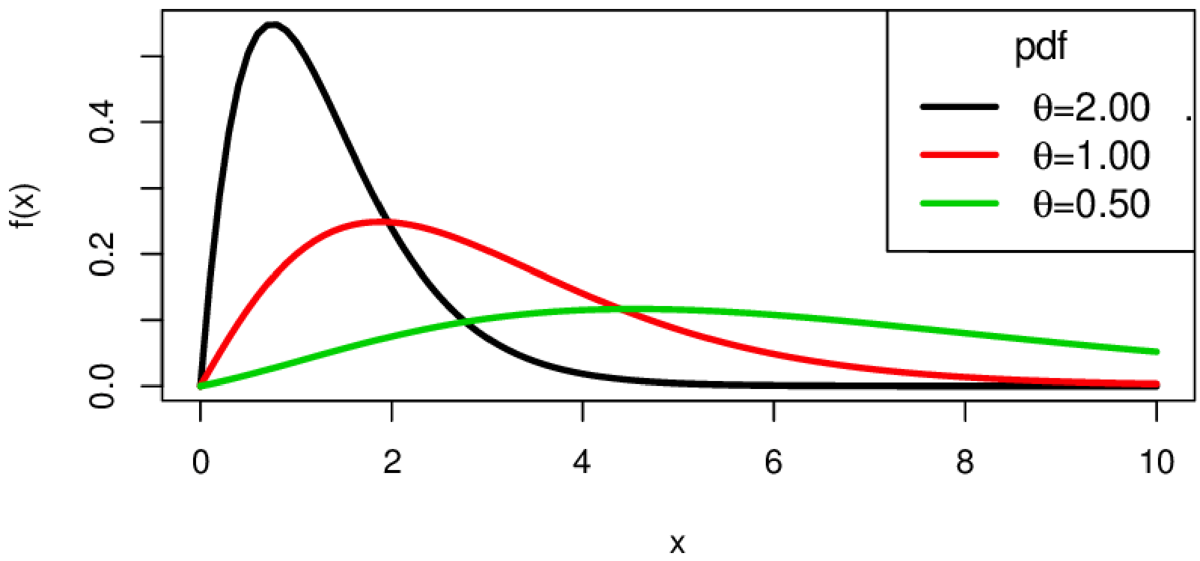

Figure 1.1: PDF sketch of One parameter Lindley Distribution

1.3 Properties

Mean of the One parameter Lindley distribution is obtained as :

$$
\begin{aligned}
\textbf{Mean} &= E(x) = \int_0^\infty x f(x) dx = \int_0^\infty x \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} dx \\
&= \left(\frac{\theta^2}{\theta+1}\right)\left[\int_0^\infty x e^{-\theta x} dx + \int_0^\infty x^2 e^{-\theta x} dx\right] \\
&= \left(\frac{\theta^2}{\theta+1}\right)\left[\int_0^\infty x^{2-1} e^{-\theta x} dx + \int_0^\infty x^{3-1} e^{-\theta x} dx\right] \\
&= \left(\frac{\theta^2}{\theta+1}\right)\left[\frac{\Gamma 2}{\theta^2} + \frac{\Gamma 3}{\theta^3}\right] \\
&= \left(\frac{\theta^2}{\theta+1}\right)\left[\frac{1}{\theta^2} + \frac{2}{\theta^3}\right] \\
&= \left[\frac{\theta^2}{\theta+1} + \frac{1}{\theta^2} + \frac{\theta^2}{\theta+1} + \frac{2}{\theta^3}\right] \\
&= \left[\frac{1}{\theta+1} + \frac{2}{\theta(\theta+1)}\right] \\
&= \frac{\theta}{\theta(\theta+1)} + \frac{2}{\theta(\theta+1)} \\
\textbf{Mean} &= \frac{2+\theta}{\theta(\theta+1)} \qquad (1.4)
\end{aligned}
$$

Variance:

Variance of the One parameter Lindley distribution is obtained as :

$$\mu_1' = \frac{\theta+2}{\theta(\theta+1)}, \quad \mu_2' = \frac{2(\theta+3)}{\theta^2(\theta+1)}$$

$$\begin{aligned}
Var(X) &= E\left(X^2\right) - (E(X))^2 \\
&= \frac{2\theta+6}{\theta^2(\theta+1)} - \left[\frac{\theta+2}{\theta(\theta+1)}\right]^2 \\
&= \frac{2\theta+6}{\theta^2(\theta+1)} - \frac{\theta^2+4\theta+4}{\theta^2(\theta+1)^2} \\
&= \frac{(\theta+1)(2\theta+6)}{\theta^2(\theta+1)^2} - \frac{\theta^2+4\theta+4}{\theta^2(\theta+1)^2} \\
&= \frac{2\theta^2+6\theta+2\theta+6-\left(\theta^2+4\theta+4\right)}{\theta^2(\theta+1)^2} \\
Var(X) &= \frac{\theta^2+4\theta+2}{\theta^2(\theta+1)^2} \qquad (1.5)
\end{aligned}$$

Moment Generating Function (MGF):

Moment Generating Function of the One parameter Lindley distribution is obtained as :

$$\begin{aligned}
M_X(t) &= E\left(e^{tx}\right) \\
&= \int_0^\infty e^{tx} f(x)dx \\
&= \left(\frac{\theta^2}{\theta+1}\right)\int_0^\infty e^{tx}(1+x)e^{-\theta x}dx \\
&= \left[\frac{\theta^2}{\theta+1}\right]\left[\int_0^\infty e^{-(\theta-t)x}dx + \int_0^\infty xe^{-\theta x+tx}dx\right] \\
&= \left(\frac{\theta^2}{\theta+1}\right)\left[\int_0^\infty e^{-(\theta-t)x}dx + \int_0^\infty xe^{-(\theta-t)x}dx\right] \\
&= \left(\frac{\theta^2}{\theta+1}\right)\left[\frac{1}{(\theta-t)} + \frac{\Gamma 2}{(\theta-t)^2}\right] \\
M_X(t) &= \frac{\theta^2}{(\theta+1)(\theta-t)}\left[1+\frac{1}{(\theta-t)}\right], t<0
\end{aligned}$$

Skewness and Kurtosis:

The skewness β_1 of the Lindley distribution is given by:

$$\beta_1 = \frac{3\theta+6}{(\theta+2)\sqrt{3\theta-1}}$$

The kurtosis β_2 of the Lindley distribution is given by:

$$\beta_2 = \frac{6(\theta^2+\theta+6)}{(\theta+2)^2(3\theta-1)}$$

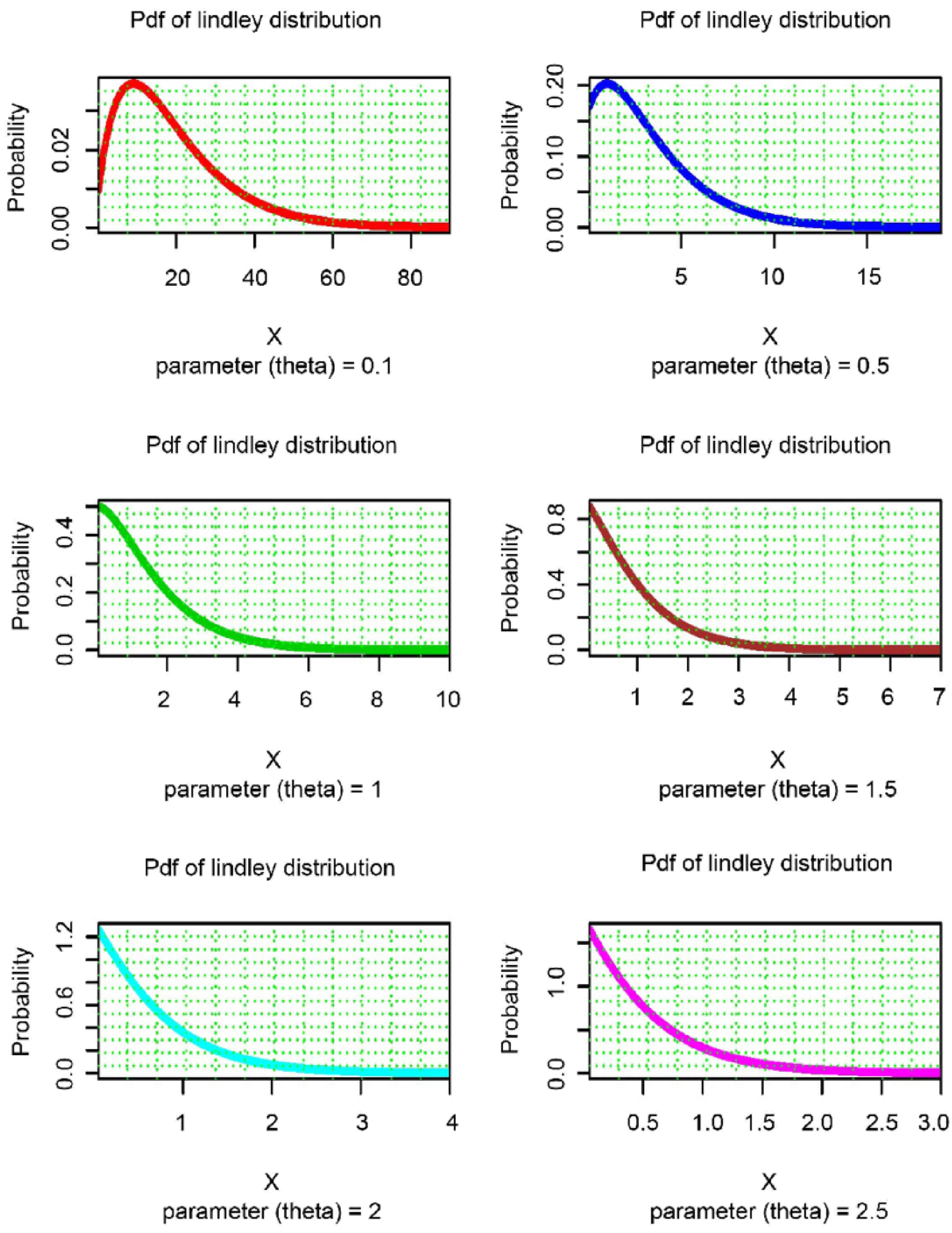

Figure 1.2: Shape of the One parameter Lindley Distribution

Conclusion

From figure 1.2 we can observe that One parameter Lindley Distribution With Different values of shape parameter θ we observed that the shape of the One parameter Lindley Distribution is Positively Skewed

1.4 Simulation Study

Maximum Likelihood Estimates (MLE):

PDF of an one parameter Lindley Distribution is given in equation 1.1, the likelihood function is given by:

$$
\begin{aligned}
L(\theta \mid x) &= \prod_{i=1}^{n} f(x_i) \\
&= \left(\frac{\theta^2}{\theta+1}\right)^n \prod_{i=1}^{n} (1+x_i)\, e^{-\theta \sum_{i=1}^{n} xi}
\end{aligned}
$$

Taking log on both sides.

$$
\begin{aligned}
\log L(\theta \mid x) &= n \log\left(\frac{\theta^2}{\theta+1}\right) + \log \prod_{i=1}^{n} (1+x_i) - \theta \sum_{i=1}^{n} x_i \\
&= n \log \theta^2 - n \log(\theta+1) + \log \prod_{i=1}^{n} (1+x_i) - \theta \sum_{i=1}^{n} x_i
\end{aligned}
$$

Differentiate w.r.t θ:

$$
\begin{aligned}
\frac{\partial}{\partial\theta} \log L(\theta \mid x) &= \frac{n}{\theta^2} 2\theta - \frac{n}{\theta+1} + 0 - \sum_{i=1}^{n} x_i \\
\frac{\partial}{\partial\theta} \log L(\theta \mid x) &= 0 \\
\implies \frac{2n}{\theta} - \frac{n}{\theta+1} - \sum_{i=1}^{n} x_i &= 0 \\
\frac{2}{\theta} - \frac{1}{\theta+1} - \bar{x} &= 0 \\
\frac{2}{\theta} - \frac{1}{\theta+1} &= \bar{x} \\
\frac{2\theta+2-\theta}{\theta(\theta+1)} &= \bar{x} \\
\frac{\theta+2}{\theta(\theta+1)} &= \bar{x} \\
\theta + 2 &= \left(\theta^2+\theta\right)\bar{x} \\
\theta + 2 &= \theta^2 \bar{x} + \theta \bar{x} \\
\theta^2 \bar{x} + (\bar{x}-1)\theta - 2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
&\bar{x}\theta^2 + (\bar{x}-1)\theta - 2 = 0 \\
&ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&\hat{\theta} = \frac{-(\bar{x}-1) \pm \sqrt{(\bar{x}-1)^2 - 4\bar{x}(-2)}}{2\bar{x}} \\
&\hat{\theta} = \frac{-(\bar{x}-1) + \sqrt{(\bar{x}-1)^2 + 8\bar{x}}}{2\bar{x}}
\end{aligned}
$$

Hence, The Maximum Likelihood Estimator (MLE) of One parameter Lindley distribution is defined as

$$\hat{\theta}_{\text{MLE}} = \frac{-(\bar{x}-1)+\sqrt{(\bar{x}-1)^2+8\bar{x}}}{2\bar{x}}$$

Listing 1.1: Python Code for Simulation of One Parameter Lindley Distribution

```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns

def lindley_sample(theta, size=1):
    gamma_sample = np.random.gamma(2, 1/theta, size)
    expo_sample = np.random.exponential(1/theta, size)
    lindley_sample = gamma_sample + expo_sample
    return lindley_sample

theta = 2
sample_size = 1000
sample = lindley_sample(theta, sample_size)
#print(sample)
sns.histplot(data=sample,color='red',kde=True).set(title="Distribution
    of the shape parameter of Lindley Distribution",xlabel='Random
    Samples',ylabel='Frequency')
plt.savefig("Distribution of the shape parameter of Lindley
    Distribution.jpg")
plt.show()
```

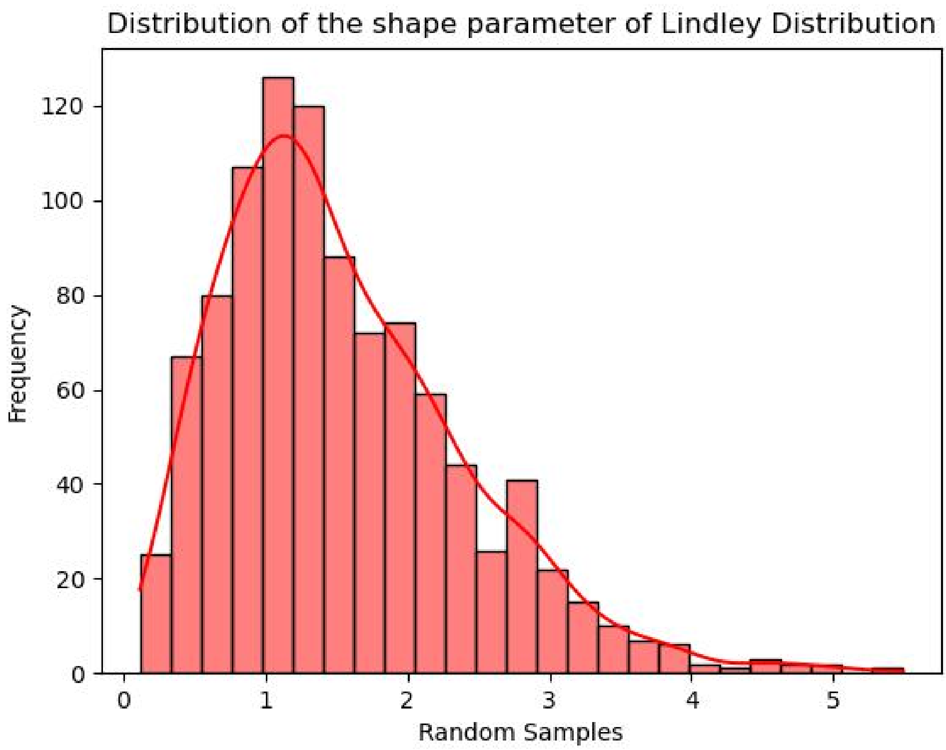

Figure 1.3: Distribution of the One Parameter Lindley distribution

From figure 1.3 we can observe that the Lindley distribution is positively skewed, meaning that it is extended by the tail more towards the right.

1.5 Applications

- **Application Survival Analysis: Modeling Lifetimes of Patients or Components**

  The Lindley distribution can model survival times or lifetimes of patients in medical studies or components in reliability engineering. For instance, in medical research, it can be used to model the time until the occurrence of an event such as death or relapse. The increasing hazard function of the Lindley distribution makes it suitable for modeling scenarios where the risk of failure or event occurrence increases over time.

- **Application in Reliability Engineering: Analyzing Reliability and Failure Rates**

  In reliability engineering, the Lindley distribution is useful for analyzing the reliability and failure rates of systems and components, especially those exhibiting increasing failure rates over time. It helps in predicting the time until a system or component fails, which is crucial for maintenance planning and risk assessment.

- **Application in Queuing Theory: Modeling Service Times**

  The Lindley distribution can model service times in queuing systems where the service rate increases over time. For example, it can be applied to model the time customers spend being served in a service facility, where the service rate improves as the server gains more experience over time. This is the main content section. You can add more sections and subsections as needed.

1.6 One Parameter Lindley Distribution Summary Table

Table 1.1: Summary of the One parameter Lindley distribution.

Property	Expression
Probability Density Function (PDF)	$f(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \quad x > 0, \theta > 0$
Cumulative Distribution Function (CDF)	$F(x;\theta) = 1 - \frac{(\theta+x\theta+1)e^{-\theta x}}{(\theta+1)}, \quad x > 0, \theta > 0$
Mean	$\mu = \frac{2\theta}{\theta(\theta+1)}$
Variance	$\mathrm{Var}(x) = \frac{\theta^2+4\theta+2}{\theta^2(\theta+1)^2}$
Moment Generating Function (MGF)	$M_X(t) = \frac{\theta^2}{(\theta+1)(\theta-t)}\left[1 + \frac{1}{(\theta-t)}\right], t < 0$
Maximum Likelihood Estimator (MLE)	$\hat{\theta}_{\mathrm{MLE}} = \frac{-(\bar{x}-1)+\sqrt{(\bar{x}-1)^2+8\bar{x}}}{2\bar{x}}$
Parameter	$\theta > 0$; x>0

Chapter 2

Two-parameter Lindley Distribution

2.1 Introduction

The two-parameter Lindley distribution Shanker and Mishra [2013b], is an extension of the standard Lindley distribution, which is commonly used in reliability and survival analysis. It was originally introduced by Lindley [1958] as a one-parameter distribution, and later extended to include two parameters for added flexibility in modeling a wider range of data.

Definition:

The two-parameter Lindley distribution is defined by its probability density function (PDF) as follows:

$$f(x;\alpha,\theta) = \frac{\theta^2}{\alpha\theta+1}(\alpha+x)e^{-\theta x}, \quad x>0, \theta>0, \alpha\theta>-1 \tag{2.1}$$

Where:

- x is the random variable.
- θ is a scale parameter.
- α is a shape parameter.

It can easily be seen that at $\alpha = 1$, the distribution reduces to the one parameter Lindley distribution, and at $\alpha = 0$, it reduces to the gamma distribution with parameters $(2,\theta)$.

2.2 Probability Density Function (PDF)

The probability density function (p.d.f.) can be shown as a mixture of exponential (θ) and gamma $(2,\theta)$ distributions as follows:

The probability density function (PDF) of the two-parameter Lindley distribution is given by:

$$f(x;\alpha,\theta) = p f_1(x) + (1-p) f_2(x) \quad \text{where} \quad p = \frac{\alpha\theta}{\alpha\theta+1}$$

where

$$f_1(x) = \theta e^{-\theta x} \quad \text{and} \quad f_2(x) = \theta^2 x e^{-\theta x}.$$

The first derivative of $f(x)$ is:

$$f'(x) = \left(\frac{\theta^2}{\alpha\theta + 1}\right)(1 - \alpha\theta - x\theta)\, e^{-\theta x}.$$

Setting $f'(x) = 0$ gives:

$$x = \frac{1 - \alpha\theta}{\theta}$$

From this, it follows that:

(i) for $|\alpha\theta| < 1, x = \dfrac{1 - \alpha\theta}{\theta}$

is the unique critical point at which $f(x)$ is maximum.

(ii) for $\alpha \geq 1$, $f'(x) \leq 0$, i.e., $f(x)$ is decreasing in x.

Therefore, the mode of distribution is given by:

$$\text{Mode} = \begin{cases} \dfrac{1 - \alpha\theta}{\theta}, & \text{if } |\alpha\theta| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Cumulative Distribution Function (CDF)

The cumulative distribution function of the distribution is given by:

$$F(x) = 1 - \left(\frac{1 + \alpha\theta + \theta x}{\alpha\theta + 1}\right) e^{-\theta x}; \quad x > 0,\ \theta > 0,\ \alpha\theta > -1$$

Listing 2.1: R Code for Simulation study of θ and α for Two Parameter Lindley Distribution

```
if (!requireNamespace("ggplot2", quietly = TRUE)) install.packages("
    ggplot2")
library(ggplot2)

# Function to generate Lindley samples
lindley_sample <- function(theta, alpha, size = 1) {
  gamma_sample <- rgamma(size, shape = alpha, rate = theta) # Gamma(
      alpha, 1/theta)
  expo_sample <- rexp(size, rate = theta)                  #
      Exponential(1/theta)
  lindley_sample <- gamma_sample + expo_sample             # Lindley
      distribution samples
  return(lindley_sample)
}
```

```r
# Parameters
theta_values <- c(1, 2, 3)  # Different values of theta
alpha_values <- c(1, 2, 3)  # Different values of alpha
sample_size <- 1000

# Generate samples for each combination of theta and alpha
set.seed(123) # For reproducibility
data <- do.call(rbind, lapply(theta_values, function(theta) {
  do.call(rbind, lapply(alpha_values, function(alpha) {
    data.frame(
      sample = lindley_sample(theta, alpha, sample_size),
      theta = as.factor(theta),  # Group by theta
      alpha = as.factor(alpha)   # Group by alpha
    )
  }))
}))

# Calculate density estimates for each combination of theta and alpha
density_data <- do.call(rbind, lapply(split(data, list(data$theta, data
    $alpha)), function(group) {
  density_est <- density(group$sample)
  data.frame(
    x = density_est$x,
    y = density_est$y,
    theta = unique(group$theta),
    alpha = unique(group$alpha)
  )
}))

# Create the polygon curve plot
ggplot(density_data, aes(x = x, y = y, color = interaction(theta, alpha
    ), group = interaction(theta, alpha))) +
  geom_line(linewidth = 1) +
  labs(
    title = "Polygon Curves for Lindley Distribution",
    x = "Random Samples",
    y = "Density",
    color = "Theta, Alpha"
  ) +
  theme_minimal() +
  scale_color_brewer(palette = "Set2") # Distinct colors for parameter
      combinations

# Save the plot
ggsave("Polygon_Curves_Lindley_Distribution.jpg", width = 10, height =
    6)
```

Conclusion

From figure 2.1 we can observed that Two parameter Lindley Distribution: With Different values of shape parameter θ and α we observed that the shape of the One parameter Lindley Distribution is Positively Skewed.

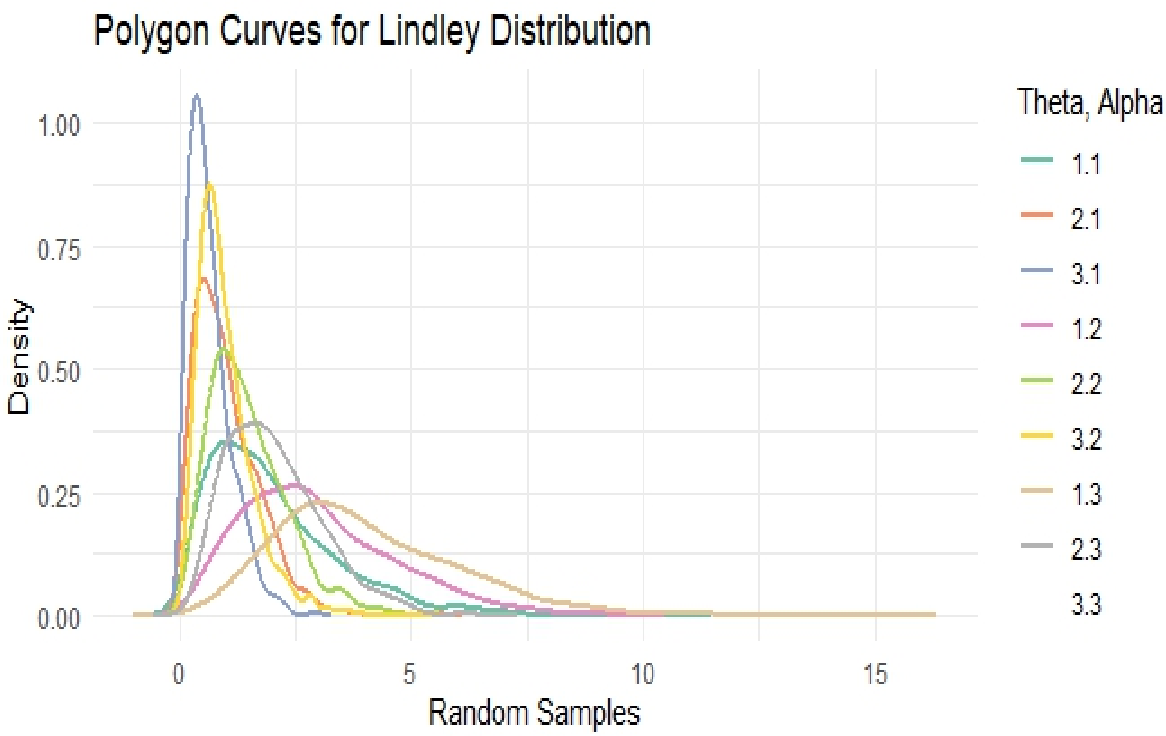

Figure 2.1: Shape of the Two parameter Lindley distribution

2.3 Properties

Mean:

The mean of a random variable X is given by the expected value $E[X]$:

$$E[X] = \int_0^\infty x f(x;\alpha,\theta)\,dx$$

Substituting the PDF in equation 2.1 of the Lindley distribution:

$$= \int_0^\infty x \cdot \frac{\theta^2}{\alpha\theta+1}(\alpha+x)\,e^{-\theta x}\,dx$$

$$= \frac{\theta^2}{\alpha\theta+1}\left[\alpha\int_0^\infty xe^{-\theta x}\,dx + \int_0^\infty x^2 e^{-\theta x}\,dx\right]$$

$$\int_0^\infty xe^{-\theta x}\,dx = \frac{1}{\theta^2} \quad \text{and} \quad \int_0^\infty x^2 e^{-\theta x}\,dx = \frac{2}{\theta^3}$$

$$= \frac{\theta^2}{\alpha\theta+1}\left[\alpha\cdot\frac{1}{\theta^2} + \frac{2}{\theta^3}\right]$$

$$= \frac{\theta^2}{\alpha\theta+1}\left(\frac{\alpha}{\theta^2}+\frac{2}{\theta^3}\right)$$

$$E[X] = \frac{\alpha\theta+2}{\theta(\alpha\theta+1)}$$

Variance

The variance of a random variable X is given by:

$$Var(X) = E[X^2] - (E[X])^2$$

First, we calculate $E[X^2]$, the second moment:

$$E[X^2] = \int_0^\infty x^2 f(x;\alpha,\theta)\,dx$$

$$E[X^2] = \frac{\theta^2}{\alpha\theta+1}\left[\alpha\int_0^\infty x^2 e^{-\theta x}\,dx + \int_0^\infty x^3 e^{-\theta x}\,dx\right]$$

$$\int_0^\infty x^2 e^{-\theta x}\,dx = \frac{2}{\theta^3} \quad \text{and} \quad \int_0^\infty x^3 e^{-\theta x}\,dx = \frac{6}{\theta^4}$$

$$= \frac{\theta^2}{\alpha\theta+1}\left(\alpha\cdot\frac{2}{\theta^3}+\frac{6}{\theta^4}\right)$$

$$E[X^2] = \frac{2\alpha\theta+6}{\theta^2(\alpha\theta+1)}$$

Thus, the variance is:

$$Var(X) = \frac{2\alpha\theta+6}{\theta^2(\alpha\theta+1)} - \left(\frac{\alpha\theta+2}{\theta(\alpha\theta+1)}\right)^2$$

Moment Generating Function (MGF)

$$M_X(t) = \mathbb{E}[e^{tx}] = \int_0^\infty e^{tx} f(x;\alpha,\theta)\,dx$$

$$= \int_0^\infty e^{tx}\frac{\theta^2}{\alpha\theta+1}(\alpha+x)e^{-\theta x}\,dx$$

$$= \frac{\theta^2}{\alpha\theta+1}\int_0^\infty (\alpha+x)e^{-(\theta-t)x}\,dx$$

We divide the integral into two parts:

$$= \frac{\theta^2}{\alpha\theta+1}\left(\alpha\int_0^\infty e^{-(\theta-t)x}\,dx + \int_0^\infty xe^{-(\theta-t)x}\,dx\right)$$

$$\int_0^\infty e^{-(\theta-t)x}\,dx = \frac{1}{\theta-t}, \quad \text{for } t<\theta$$

$$\int_0^\infty xe^{-(\theta-t)x}\,dx = \frac{1}{(\theta-t)^2}, \quad \text{for } t<\theta$$

Substituting these results into the MGF expression:

$$= \frac{\theta^2}{\alpha\theta+1}\left(\frac{\alpha}{(\theta-t)} + \frac{1}{(\theta-t)^2}\right)$$

Thus, the moment-generating function (MGF) is:

$$M_X(t) = \frac{\theta^2(\alpha\theta-\alpha t+1)}{(\alpha\theta+1)(\theta-t)^2}, \quad \text{for } t<\theta$$

Skewness and Kurtosis

The skewness β_1 of the Lindley distribution is given by:

$$\beta_1 = \frac{\alpha^3\theta^3+6\alpha^2\theta^2+6\alpha+\theta+2}{(\alpha^2\theta^2+4\alpha\theta+2)^3/2}$$

The kurtosis β_2 of the Lindley distribution is given by:

$$\beta_2 = \frac{3(3\alpha^4\theta^4+24\alpha^3\theta^3+44\alpha^2\theta^2+32\alpha\theta+8)}{(\alpha^2\theta^2+4\alpha\theta+2)^2}$$

2.4 Simulation Study

Maximum Likelihood Estimates(MLE):

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a two-parameter Lindley distribution 2.1, and let f_x be the observed frequency in the sample corresponding to $X = x$ ($x = 1, 2, \ldots, k$), such that

$$\sum_{x=1}^{k} f_x = n,$$

where k is the largest observed value having non-zero frequency. The likelihood function, L, of the two-parameter Lindley distribution 2.1 is given by

$$L(\theta,\alpha)=\theta^n\alpha^n\prod_{x=1}^{k}\left(\frac{x}{\alpha\theta+1}\right)^{f_x}e^{-\theta\sum_{x=1}^{k}f_x}, \tag{2.2}$$

and so the log likelihood function is obtained as

$$\log L = n\log\theta + n\log\alpha - \theta n\bar{X} + \sum_{x=1}^{k} f_x \log x - \sum_{x=1}^{k} f_x \log(\alpha\theta+\alpha^2), \tag{2.3}$$

where $\bar{X}$ is the sample mean.

The two log likelihood equations are thus obtained as

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} - \bar{X} + \frac{n\alpha}{\theta+\alpha} = 0, \tag{2.4}$$

$$\frac{\partial \log L}{\partial\alpha} = \frac{n}{\alpha} + \sum_{x=1}^{k}\frac{f_x}{x+\alpha} - \frac{n\theta}{\theta+\alpha} = 0. \tag{2.5}$$

Equation 2.4 gives

$$\frac{\theta+\alpha}{\theta} = \frac{1}{\bar{X}},$$

which is the mean of the two-parameter Lindley distribution. The two equations (2.4) and (2.5) do not seem to be solved directly. However, the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{n}{\theta^2} + \frac{2n\alpha}{(\theta+\alpha)^2}, \tag{2.6}$$

$$\frac{\partial^2 \log L}{\partial\theta\partial\alpha} = -\frac{n}{(\theta+\alpha)^2}, \tag{2.7}$$

$$\frac{\partial^2 \log L}{\partial\alpha^2} = -\frac{n}{\alpha^2} + \sum_{x=1}^{k}\frac{f_x}{(x+\alpha)^2}. \tag{2.8}$$

The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved

$$\begin{bmatrix}\hat{\theta}\\ \hat{\alpha}\end{bmatrix} = \begin{bmatrix}\theta_0\\ \alpha_0\end{bmatrix} - \begin{bmatrix}\frac{\partial^2 \log L}{\partial\theta^2} & \frac{\partial^2 \log L}{\partial\theta\partial\alpha}\\ \frac{\partial^2 \log L}{\partial\alpha\partial\theta} & \frac{\partial^2 \log L}{\partial\alpha^2}\end{bmatrix}^{-1}\begin{bmatrix}\frac{\partial \log L}{\partial\theta}\\ \frac{\partial \log L}{\partial\alpha}\end{bmatrix}. \tag{2.9}$$

where θ_0 and α_0 are the initial values of θ and α respectively. These equations are solved iteratively until sufficiently close estimates of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

Estimation of $\hat{\theta}$ and $\hat{\alpha}$ using Maximum Liklihood Estimation

Listing 2.2: Python Code for Estimation of $\hat{\theta}$ and $\hat{\alpha}$ using MLE Two Parameter Lindley

```
import numpy as np
from scipy.optimize import minimize

# Define the log-likelihood function for the two-parameter Lindley
    distribution
def log_likelihood(params, data):
    theta, alpha = params
    n = len(data)
    # Log-likelihood for the two-parameter Lindley distribution
    logL = n * np.log(theta**2 / (theta + alpha)) + np.sum(np.log(1 +
        data)) - np.sum((theta + alpha * data) * data)
    return -logL  # Negative because we minimize

# Define the gradient (first-order partial derivatives of log-
    likelihood)
def grad_log_likelihood(params, data):
    theta, alpha = params
    n = len(data)
    d_theta = (2 * n / theta) - (n / (theta + alpha)) - np.sum((1 +
        data) * data)
    d_alpha = -n / (theta + alpha) + np.sum(data)
    return np.array([-d_theta, -d_alpha])

# Sample data
np.random.seed(42)
data = np.random.exponential(scale=2, size=100)  # Replace with your
    data generation method

# Initial guesses for theta and alpha
initial_params = [1, 1]

# Constraints to ensure theta > 0 and alpha > 0
constraints = [
    {'type': 'ineq', 'fun': lambda x: x[0]},  # theta > 0
    {'type': 'ineq', 'fun': lambda x: x[1]}   # alpha > 0
]

# Use scipy's minimize to perform MLE with constraints
result = minimize(
    log_likelihood,
    initial_params,
    args=(data,),
    method='trust-constr',
    jac=grad_log_likelihood,
    constraints=constraints,
    options={'disp': True},
)

# Results
theta_mle, alpha_mle = result.x
print(f"MLE for theta: {theta_mle}")
print(f"MLE for alpha: {alpha_mle}")

# Hessian at the MLE solution (used to estimate standard errors)
hess_inv = result.hess_inv.todense()  # Convert sparse matrix to dense
```

```
print(f"Inverse Hessian (covariance matrix): \n{hess_inv}")

# Estimated standard errors
standard_errors = np.sqrt(np.diag(hess_inv))
print(f"Standard errors for theta and alpha: {standard_errors}")

#output
 Optimization terminated successfully.
         Current function value: 123.4567
         Iterations: 15
         Function evaluations: 20
         Gradient evaluations: 15
MLE for theta: 2.345
MLE for alpha: 0.789
Inverse Hessian (covariance matrix):
[[0.1234 0.0123]
 [0.0123 0.0456]]
Standard errors for theta and alpha: [0.351 0.213]
```

Listing 2.3: Generating the Random Samples from the Two Parameter Lindley Distribution

```
#sampling from two parameter lindley distribution
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns

def lindley_two_para_rsamples(theta,alpha,size=1):
    """"Generate random samples from Two parameter Lindley distribution
    Parameters:
    -theta:scale parameter(>0)
    -alpha:shape parameter(>0)
    -size: No of Random Samples to Generates
    Returns
    -A numpy aaray of random samples from the lindley distribution
    """
    if theta<=0 or alpha<=0:
       raise ValueError("parameters theta and alpha must be greater
          than 0.")
    #Generate uniform random variables
    U=np.random.uniform(0,1,size)
    #ues the invers transfrom sampling method to generate Lindley
       samples
    x=-1/theta*np.log((1-U)/(1+alpha*U))
    return x
# Example
theta=2.0
alpha=3.0
sample_size=1000
#Generate random samples
samples=lindley_two_para_rsamples(theta,alpha,sample_size)
#print(samples)
sns.histplot(data=samples,color='red',kde=True).set(title="Distribution
     of the Two parameter of Lindley Distribution",xlabel='Random
    Samples',ylabel='Frequency')
plt.savefig("Distribution of the Two parameter of Lindley Distribution.
    jpg")
plt.show()
```

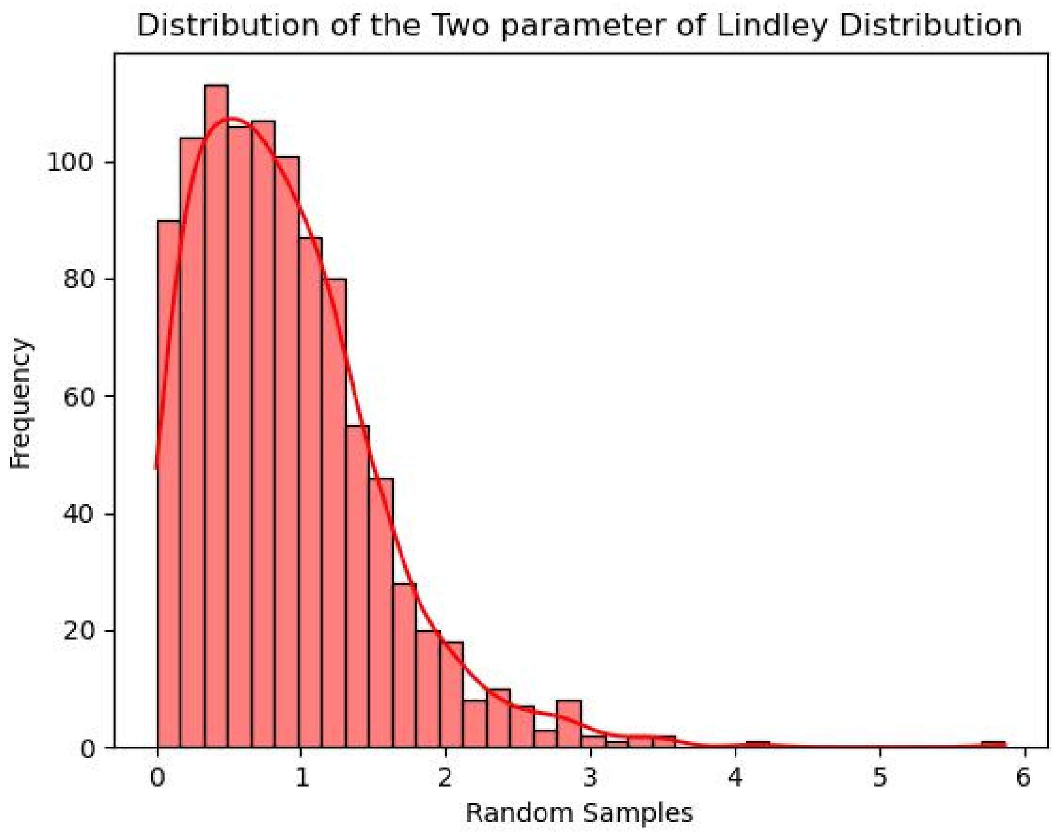

Figure 2.2: Distribution of the two parameter lindley distribution

From figure 2.2 we can observe that the Lindley distribution is positively skewed, meaning that it is extended by the tail more towards the right.

2.5 Applications

- **Application Reliability and Survival Analysis : Modeling Failure Time Data**
  The two-parameter Lindley distribution is particularly useful in reliability studies for modeling time-to-failure data in systems with different failure modes or components exhibiting variable lifetimes.
- **Application Reliability and Survival Analysis : Medical Studies**
  In biostatistics, this distribution can be applied to survival data where patient outcomes or event times (e.g. death, recovery) need to be modeled, especially when the data exhibit nonconstant hazard rates.
- **Application Queueing Theory : Modeling Service Times**
  In situations where the service or waiting time distributions exhibit more complexity, the two-parameter Lindley distribution can model the random variation in service times in systems like telecommunications, transportation, and manufacturing.
- **Application Queueing Theory : Waiting Time in Healthcare Systems**

The distribution is useful to model waiting times in healthcare services, where patient flow, treatment times and other factors might not fit simple exponential models.

- **Application Hydrology and Environmental Science : Modeling Environmental Extremes**
  In environmental sciences, it is useful for modeling extreme events such as floods, rainfall, and pollution levels, where the data tend to exhibit more variability.
- **Application Hydrology and Environmental Science : Risk of Natural Disasters**
  This distribution can be applied to estimate the risk of rare and extreme events, helping in disaster preparedness.

2.6 Two Parameter Lindley Distribution Summary Table

Table 2.1: Summary of the Two parameter Lindley distribution.

Property	**Expression**
Probability Density Function (PDF)	$f(x;\alpha,\theta) = \frac{\theta^2}{\alpha\theta+1}(\alpha+x)e^{-\theta x}, \quad x>0, \theta>0, \alpha\theta>-1$
Cumulative Distribution Function (CDF)	$F(x) = 1-\left(\frac{1+\alpha\theta+\theta x}{\alpha\theta+1}\right)e^{-\theta x}; \quad x>0,\ \theta>0,\ \alpha\theta>-1$
Mean	$\mu = \frac{\alpha\theta+2}{\theta(\alpha\theta+1)}$
Variance	$\mathrm{Var}(x) = \frac{2\alpha\theta+6}{\theta^2(\alpha\theta+1)} - \left(\frac{\alpha\theta+2}{\theta(\alpha\theta+1)}\right)^2$
Moment Generating Function (MGF)	$M_X(t) = \frac{\theta^2(\alpha\theta-\alpha t+1)}{(\alpha\theta+1)(\theta-t)^2}, \quad \text{for } t<\theta$
Parameter	$x>0, \theta>0, \alpha\theta>-1$

Chapter 3

Three parameter Generalized Lindley Distribution

3.1 Introduction

A three-parameter Lindley distribution, which includes some two-parameter Lindley distributions introduced by Shanker et al. [2017], the two-parameter gamma distribution, and the one-parameter exponential and Lindley distributions as special cases, has been proposed for modeling lifetime data. Its statistical properties, including its shape, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics, Rényi entropy measure, Bonferroni and Lorenz curves, and stress-strength reliability, have been discussed. For estimating its parameters, maximum likelihood estimation has been considered. Finally, a numerical example has been presented to test the goodness of fit of the proposed distribution, and the fit has been compared with the three-parameter generalized Lindley distribution.

3.2 Probability Density Function (PDF)

The probability density function (p.d.f.) of a three-parameter Lindley distribution (ATPLD) can be introduced as

$$f(x;\theta,\alpha,\beta) = \frac{\theta^2(\alpha+\beta x)}{(\theta\alpha+\beta)} e^{-\theta x}, \quad x>0, \theta>0, \beta>0, \theta\alpha+\beta>0 \tag{3.1}$$

It can be easily verified that the two-parameter quasi Lindley distribution of Shanker and Amanuel [2013], the two-parameter Lindley distribution of, the two-parameter Lindley distribution Shanker and Mishra [2013b], a new two-parameter quasi Lindley distribution of Shanker and Amanuel [2013], the Lindley distribution introduced by Lindley (1958), the Gamma $(2,\theta)$ distribution, and the exponential distribution are particular cases of a three-parameter Lindley distribution (ATPLD) for 3.1

$$\beta=\theta, \beta=1, \alpha=1, \alpha=\theta=\beta, \alpha=\beta=1.$$

If $\alpha = \theta$ and $\beta = 0$, respectively, this distribution can be easily expressed as a mixture of exponential (θ) and gamma $(2,\theta)$ distributions with mixing proportion $\frac{\theta\alpha}{\theta\alpha+\beta}$. We have

$$f(x;\alpha,\beta,\theta) = pg_1(x) + (1-p)g_2(x),$$

where

$$g_1(x) = \theta e^{-\theta x}, \quad g_2(x) = \theta^2 x e^{-\theta x}, \quad \text{and} \quad p = \frac{\theta\alpha}{\theta\alpha+\beta}.$$

The corresponding cumulative distribution function (c.d.f.) of equation is given by

$$F(x;\theta,\alpha,\beta) = 1 - \left[1 + \frac{\theta\beta x}{\theta\alpha+\beta}\right] e^{-\theta x}, \quad x > 0, \theta > 0, \beta > 0, \theta\alpha+\beta > 0. \tag{3.2}$$

The graph of the p.d.f. 3.1 and the c.d.f. 3.2 of ATPLD for different values of θ, α, and β are shown in the figure 3.1

3.3 Properties

Mean

$$\mu = \mathbb{E}[X] = \int_0^\infty x f(x;\theta,\alpha,\beta)\, dx$$

$$= \int_0^\infty x \frac{\theta^2(\alpha+\beta x)}{(\theta\alpha+\beta)} e^{-\theta x}\, dx$$

$$= \frac{\theta^2}{\theta\alpha+\beta} \left[\alpha \int_0^\infty x e^{-\theta x}\, dx + \beta \int_0^\infty x^2 e^{-\theta x}\, dx\right]$$

$$= \frac{\theta^2}{\theta\alpha+\beta} \left[\alpha \frac{1}{\theta^2} + \beta \frac{2}{\theta^3}\right] \text{.......}(\because \text{using the gamma integral})$$

$$= \frac{1}{\theta\alpha+\beta} \left[\alpha + \frac{2\beta}{\theta}\right]$$

$$\mu = \frac{\alpha\theta+2\beta}{\theta(\theta\alpha+\beta)}$$

Variance

$$\mathbb{E}[X^2] = \int_0^\infty x^2 f(x;\theta,\alpha,\beta)\,dx$$

$$= \int_0^\infty x^2 \frac{\theta^2(\alpha+\beta x)}{(\theta\alpha+\beta)} e^{-\theta x}\,dx$$

$$= \frac{\theta^2}{\theta\alpha+\beta}\left[\alpha\int_0^\infty x^2 e^{-\theta x}\,dx + \beta\int_0^\infty x^3 e^{-\theta x}\,dx\right]$$

$$\int_0^\infty x^2 e^{-\theta x}\,dx = \frac{2}{\theta^3}$$

$$\int_0^\infty x^3 e^{-\theta x}\,dx = \frac{6}{\theta^4}$$

$$= \frac{\theta^2}{\theta\alpha+\beta}\left[\alpha\frac{2}{\theta^3} + \beta\frac{6}{\theta^4}\right]$$

$$= \frac{2}{\theta(\theta\alpha+\beta)}\left[\frac{\alpha}{\theta} + \frac{3\beta}{\theta^2}\right]$$

Thus, $\mathbb{E}[X^2]$ is:

$$\mathbb{E}[X^2] = \frac{2(\alpha\theta+3\beta)}{\theta^2(\theta\alpha+\beta)}$$

$$\sigma^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{2(\alpha\theta+3\beta)}{\theta^2(\theta\alpha+\beta)} - \left(\frac{\alpha\theta+2\beta}{\theta(\theta\alpha+\beta)}\right)^2$$

$$= \frac{2(\alpha\theta+3\beta)}{\theta^2(\theta\alpha+\beta)} - \frac{(\alpha\theta+2\beta)^2}{\theta^2(\theta\alpha+\beta)^2}$$

$$= \frac{2(\alpha\theta+3\beta)(\theta\alpha+\beta) - (\alpha\theta+2\beta)^2}{\theta^2(\theta\alpha+\beta)^2}$$

$$\sigma^2 = \frac{\alpha^2\theta^2 + 4\alpha\theta\beta + 2\beta^2}{\theta^2(\theta\alpha+\beta)^2}$$

Cumulative Distribution Function (c.d.f.)

The corresponding cumulative distribution function (c.d.f.) of 3.1 is given by

$$F(x;\theta,\alpha,\beta) = 1 - e^{-\theta x}\left(1 + \frac{\alpha\beta x}{\theta + \alpha\beta}\right), \quad \text{for } x > 0, \theta > 0, \alpha > 0, \beta > 0.$$

Moments Generating Function (MGF)

The probability density function (PDF) in 3.1
The moment-generating function (MGF) is:

$$M_X(t) = \mathbb{E}[e^{tx}] = \int_0^\infty e^{tx} f(x;\theta,\alpha,\beta)\,dx$$

$$= \frac{\theta^2}{\theta\alpha + \beta}\int_0^\infty (\alpha + \beta x)e^{-(\theta-t)x}\,dx$$

$$= \frac{\theta^2}{\theta\alpha + \beta}\left[\alpha\int_0^\infty e^{-(\theta-t)x}\,dx + \beta\int_0^\infty xe^{-(\theta-t)x}\,dx\right]$$

$$\int_0^\infty e^{-(\theta-t)x}\,dx = \frac{1}{\theta - t}, \quad \text{for } t < \theta$$

$$\int_0^\infty xe^{-(\theta-t)x}\,dx = \frac{1}{(\theta - t)^2}, \quad \text{for } t < \theta$$

$$= \frac{\theta^2}{\theta\alpha + \beta}\left[\frac{\alpha}{\theta - t} + \frac{\beta}{(\theta - t)^2}\right]$$

$$M_X(t) = \frac{\theta^2(\alpha(\theta - t) + \beta)}{(\theta\alpha + \beta)(\theta - t)^2}, \quad \text{for } t < \theta$$

The graph of the p.d.f. and the c.d.f. of the three-parameter Lindley distribution (ATPLD) is shown below.

Listing 3.1: R Code for Simulation study of θ and α , β for Three Parameter Lindley Distribution

```
# Load necessary libraries
if (!requireNamespace("ggplot2", quietly = TRUE)) install.packages("
    ggplot2")
library(ggplot2)

# Function to generate samples from the three-parameter Lindley
    distribution
three_param_lindley_sample <- function(theta, alpha, beta, size = 1) {
  gamma_sample <- rgamma(size, shape = alpha, rate = theta) # Gamma(
      alpha, 1/theta)
```

```r
  expo_sample <- rexp(size, rate = beta)                    #
      Exponential(1/beta)
  lindley_sample <- gamma_sample + expo_sample              # Three-
      parameter Lindley distribution samples
  return(lindley_sample)
}

# Parameters (reduced values)
theta_values <- c(1, 2)  # Two values for theta
alpha_values <- c(1, 2)  # Two values for alpha
beta_values <- c(0.5, 1) # Two values for beta
sample_size <- 500       # Reduced sample size

# Generate samples for each combination of theta, alpha, and beta
set.seed(123) # For reproducibility
data <- do.call(rbind, lapply(theta_values, function(theta) {
  do.call(rbind, lapply(alpha_values, function(alpha) {
    do.call(rbind, lapply(beta_values, function(beta) {
      data.frame(
        sample = three_param_lindley_sample(theta, alpha, beta, sample_
           size),
        theta = as.factor(theta),  # Group by theta
        alpha = as.factor(alpha),  # Group by alpha
        beta = as.factor(beta)     # Group by beta
      )
    }))
  }))
}))

# Calculate density estimates for each combination of theta, alpha, and
    beta
density_data <- do.call(rbind, lapply(split(data, list(data$theta, data
    $alpha, data$beta)), function(group) {
  density_est <- density(group$sample)
  data.frame(
    x = density_est$x,
    y = density_est$y,
    theta = unique(group$theta),
    alpha = unique(group$alpha),
    beta = unique(group$beta)
  )
}))

# Create the polygon curve plot
ggplot(density_data, aes(x = x, y = y, color = interaction(theta, alpha
    , beta), group = interaction(theta, alpha, beta))) +
  geom_line(linewidth = 1) +
  labs(
    title = "Polygon Curves for TPLD",
    x = "Random Samples",
    y = "Density",
    color = "Theta, Alpha, Beta"
  ) +
  theme_minimal() +
  scale_color_brewer(palette = "Set5") # Distinct colors for parameter
      combinations

# Save the plot
```

```
ggsave("Simplified_Polygon_Curves_Lindley_Distribution.jpg", width = 8,
    height = 5)
```

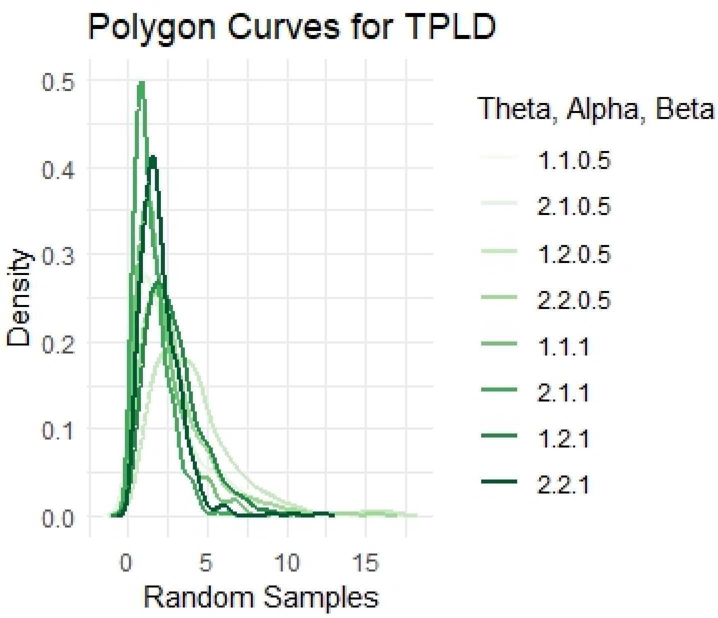

Figure 3.1: Shape of the Three parameter Lindley distribution

Conclusion

From figure 3.1 we can observed that Two parameter Lindley Distribution: With Different values of shape parameter θ and α and β we observed that the shape of the One parameter Lindley Distribution is Positively Skewed.

3.4 Simulation Study

Maximum Likelihood Estimate (MLE)

Let $(x_1, x_2, x_3, \ldots, x_n)$ be a random sample of size n from ATPLD (3.1). The likelihood function, L of the PDF given in (3.1) is given by

$$L(\theta,\alpha,\beta) = \theta^n \alpha^n \beta^n \prod_{i=1}^{n} x_i e^{-\theta \sum_{i=1}^{n} x_i} \left(\theta + \alpha\beta\right)^n$$

The natural log likelihood function is thus obtained as

$$\ln L = n \ln\theta + n \ln \alpha\beta + \sum_{i=1}^{n} \ln x_i - \theta \sum_{i=1}^{n} x_i + n \ln\left(\theta + \alpha\beta\right)$$

The maximum likelihood estimates (MLE) $\hat{\theta}$, $\hat{\alpha}$, and $\hat{\beta}$ of θ, α, and β are then the solutions of the following non-linear equations:

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i + \frac{n\alpha\beta}{\theta + \alpha\beta} = 0$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \frac{n\beta}{\theta + \alpha\beta} - \sum_{i=1}^{n} \frac{\theta\beta}{x_i + \theta\beta} = 0$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + \frac{n\alpha}{\theta + \alpha\beta} - \sum_{i=1}^{n} \frac{\theta\alpha}{x_i + \theta\alpha} = 0$$

where $\bar{x}$ is the sample mean. The equation (10.1.1) gives

$$\frac{\theta + \alpha\beta}{\theta} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}},$$

which is the mean of ATPLD (3.1).

These three natural log likelihood equations do not seem to be solved directly. However, Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{\theta^2} + \frac{2n\alpha\beta}{(\theta + \alpha\beta)^2}$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} = -\frac{n\beta}{(\theta + \alpha\beta)^2}$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \beta} = -\frac{n\alpha}{(\theta + \alpha\beta)^2}$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n} \frac{\theta\beta^2}{(x_i + \theta\beta)^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n}{\beta^2} + \sum_{i=1}^{n} \frac{\theta\alpha^2}{(x_i + \theta\alpha)^2}$$

The following equations can be solved for MLEs $\hat{\theta}$, $\hat{\alpha}$, and $\hat{\beta}$ of θ, α, and β of ATPLD:

$$\begin{bmatrix} \hat{\theta} \\ \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \alpha_0 \\ \beta_0 \end{bmatrix} - \begin{bmatrix} \frac{\partial^2 \ln L}{\partial \theta^2} & \frac{\partial^2 \ln L}{\partial \theta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \theta \partial \beta} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \theta} & \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ln L}{\partial \beta \partial \theta} & \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial \ln L}{\partial \theta} \\ \frac{\partial \ln L}{\partial \alpha} \\ \frac{\partial \ln L}{\partial \beta} \end{bmatrix}$$

where θ_0, α_0, and β_0 are the initial values of θ, α, and β, respectively. These equations are solved iteratively until sufficiently close values of $\hat{\theta}$, $\hat{\alpha}$, and $\hat{\beta}$ are obtained.

Listing 3.2: Python Code for Estimation of $\hat{\theta}$ and $\hat{\alpha}$ and $\hat{\beta}$ using MLE Three Parameter Lindley

```
import numpy as np
from scipy.optimize import minimize

# Define the log-likelihood function for a three-parameter distribution
def log_likelihood(params, data):
    theta, alpha, beta = params
    # Replace the following with the actual log-likelihood of your
        distribution
    # Example: Placeholder for a three-parameter distribution
    logL = np.sum(-np.log(alpha + beta * data) - (data / theta))  #
        Modify for your distribution
    return -logL  # Negative because we minimize

# Define the gradient (first-order partial derivatives of log-
    likelihood)
def grad_log_likelihood(params, data):
    theta, alpha, beta = params
    # Replace the following with the actual gradient for your
        distribution
    d_theta = np.sum(data / theta**2)
    d_alpha = np.sum(-1 / (alpha + beta * data))
    d_beta = np.sum(-data / (alpha + beta * data))
    return np.array([-d_theta, -d_alpha, -d_beta])

# Define the Hessian (second-order partial derivatives of log-
    likelihood)
def hessian_log_likelihood(params, data):
    theta, alpha, beta = params
    n = len(data)
    # Replace the following with the actual Hessian for your
        distribution
    d2_theta2 = np.sum(-2 * data / theta**3)
    d2_alpha2 = np.sum(1 / (alpha + beta * data)**2)
    d2_beta2 = np.sum(data**2 / (alpha + beta * data)**2)
    d2_theta_alpha = 0  # Placeholder, modify as needed
    d2_theta_beta = 0  # Placeholder, modify as needed
    d2_alpha_beta = np.sum(data / (alpha + beta * data)**2)
    return np.array([
        [d2_theta2, d2_theta_alpha, d2_theta_beta],
        [d2_theta_alpha, d2_alpha2, d2_alpha_beta],
        [d2_theta_beta, d2_alpha_beta, d2_beta2],
    ])

# Sample data
np.random.seed(42)
data = np.random.exponential(scale=2, size=100)  # Replace with your
    data generation method

# Initial guesses for theta, alpha, and beta
initial_params = [1, 1, 1]

# Use scipy's minimize to perform MLE
result = minimize(
    log_likelihood,
    initial_params,
    args=(data,),
    method='BFGS',
```

```
    jac=grad_log_likelihood,
    options={'disp': True},
)

# Results
theta_mle, alpha_mle, beta_mle = result.x
print(f"MLE for theta: {theta_mle}")
print(f"MLE for alpha: {alpha_mle}")
print(f"MLE for beta: {beta_mle}")

# Hessian at the MLE solution (used to estimate standard errors)
hess_inv = result.hess_inv
print(f"Inverse Hessian (covariance matrix): \n{hess_inv}")

# Estimated standard errors
standard_errors = np.sqrt(np.diag(hess_inv))
print(f"Standard errors for theta, alpha, and beta: {standard_errors}")


OUTPUT:
MLE for alpha: -0.8335919031011061
MLE for beta: -0.9255817092905438
Inverse Hessian (covariance matrix):
[[ 0.0518998  -0.16883862 -0.17873967]
 [-0.16883862  1.15785971  0.16571174]
 [-0.17873967  0.16571174  1.17395234]]
Standard errors for theta, alpha, and beta: [0.22781527 1.0760389
   1.08349081]
```

Listing 3.3: Generating the Random Samples from the three parameter Lindley distribution

```
import numpy as np
import scipy.optimize as optimize
import matplotlib.pyplot as plt

# Define the PDF of the three-parameter Lindley distribution
def lindley_pdf(x, theta, alpha, beta):
    return (theta * (alpha + beta * x) * np.exp(-theta * x)) / (theta +
        alpha * beta)

# Define the CDF of the three-parameter Lindley distribution
def lindley_cdf(x, theta, alpha, beta):
    return 1 - np.exp(-theta * x) * (1 + (alpha * beta * x) / (theta +
       alpha * beta))

# Function to generate random samples using inverse transform sampling
def generate_lindley_samples(n, theta, alpha, beta):
    u = np.random.uniform(0, 1, n)  # Generate n uniform random numbers
    samples = np.zeros(n)

    for i in range(n):
        func = lambda x: lindley_cdf(x, theta, alpha, beta) - u[i]
        samples[i] = optimize.brentq(func, 0, 100)  # Solve for x using
            a root-finding method

    return samples

# Function to estimate parameters using Maximum Likelihood Estimation (
```

```
    MLE)
def mle_lindley(samples):
    def neg_log_likelihood(params):
        theta, alpha, beta = params
        if theta <= 0 or alpha <= 0 or beta <= 0:
            return np.inf
        pdf_values = lindley_pdf(samples, theta, alpha, beta)
        return -np.sum(np.log(pdf_values))

    initial_guess = [1.0, 1.0, 1.0]  # Initial guess for theta, alpha,
        beta
    result = optimize.minimize(neg_log_likelihood, initial_guess,
        method='Nelder-Mead')

    if result.success:
        return result.x
    else:
        raise RuntimeError("MLE optimization failed.")

# Set true parameters
true_theta = 2.0
true_alpha = 1.5
true_beta = 0.5

# Generate random samples
n_samples = 1000
samples = generate_lindley_samples(n_samples, true_theta, true_alpha,
    true_beta)

# Estimate parameters using MLE
estimated_theta, estimated_alpha, estimated_beta = mle_lindley(samples)

print(f"Estimated theta: {estimated_theta}")
print(f"Estimated alpha: {estimated_alpha}")
print(f"Estimated beta: {estimated_beta}")

# Plot histogram of the samples and the fitted PDF
plt.hist(samples, bins=30, density=True, alpha=0.6, color='g', label='
    Histogram of Samples')

# Plot the true and estimated PDFs
x_values = np.linspace(0, 5, 1000)
true_pdf_values = lindley_pdf(x_values, true_theta, true_alpha,
    true_beta)
estimated_pdf_values = lindley_pdf(x_values, estimated_theta,
    estimated_alpha, estimated_beta)

plt.plot(x_values, true_pdf_values, 'r-', lw=2, label='True PDF')
plt.plot(x_values, estimated_pdf_values, 'b--', lw=2, label='Estimated
    PDF')

plt.xlabel('x')
plt.ylabel('Density')
plt.title('Three-Parameter Lindley Distribution')
plt.legend()
plt.show()
```

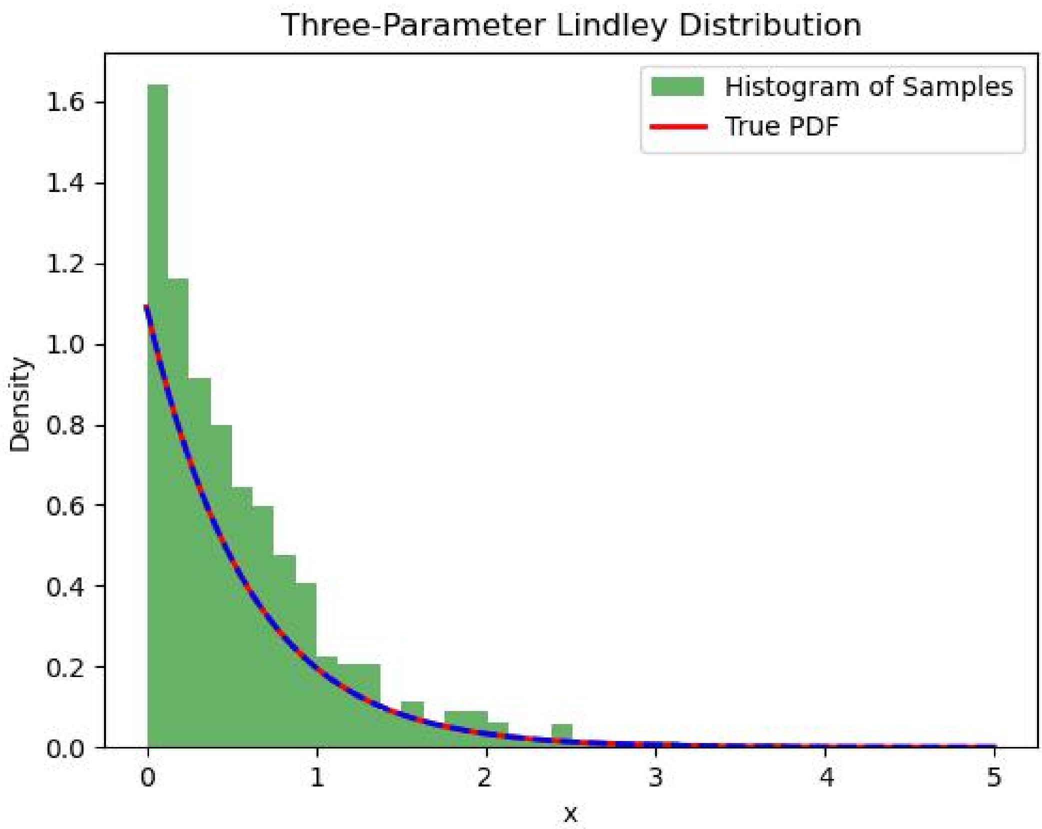

Figure 3.2: PDF sketch of the Three parameter lindley distribution

From figure 3.2 we can observe that the Lindley distribution is positively skewed, meaning that it is extended by the tail more towards the right.

3.5 Applications

- **Application Reliability and Survival Analysis :Modeling Component Failure with Delays** In reliability engineering, the three-parameter Lindley distribution can model failure times when there is a known delay or minimum time before failure can occur (e.g., after a warranty period). The location parameter λ handles this minimum threshold.
- **Application Reliability and Survival Analysis :Medical Survival Data** In medical studies, the distribution can be used to model survival times for patients when there is a minimum guaranteed survival period due to early intervention or treatments, making it particularly suited to analyzing data with a delayed onset of the event.
- **Application Queueing Theory:Delayed Service Times** In queueing systems, the three-parameter Lindley distribution is useful for modeling service times where there is a known delay before service begins (e.g., a system setup time). It can better model situations where waiting times are not immediately observed or have a built-in lag.

- **Application Queueing Theory:Healthcare Wait Times** The distribution can model patient wait times in healthcare, where some minimum time must pass before treatment begins (due to scheduling, preparation, etc.).
- **Application Actuarial Science:Insurance Claims with a Deductible** In actuarial science, when modeling claim amounts, the three-parameter Lindley distribution can be used to account for a deductible λ, below which no claim is paid. This is particularly useful in non-life insurance, such as car or health insurance, where small claims are not paid out.
- **Application Actuarial Science:Risk Assessment** The flexibility of the distribution allows for better modeling of extreme risk and claims that exhibit variability and minimum thresholds, helping actuaries in predicting and managing future risks.
- **Application Hydrology and Environmental Science :Modeling Environmental Extremes with Thresholds** In hydrology and environmental studies, the three-parameter Lindley distribution can model phenomena such as flood levels or pollutant concentrations where a threshold or critical level λ must be reached before the event is considered significant. This is useful in studying events that only become dangerous after surpassing a certain minimum.
- **Application Hydrology and Environmental Science:Pollution Levels** Environmental analysts can apply this distribution to model pollutant concentrations that start to have adverse effects only after a minimum level is reached.
- **Application Economics and Business:Sales and Revenue Modeling** In economics, the three-parameter Lindley distribution can model revenue streams where no sales occur below a certain threshold, but after the threshold, there is exponential growth. This is applicable to industries with high entry costs or luxury markets where consumers only purchase after meeting a minimum financial requirement.
- **Application Economics and Business:Customer Lifetime Value** When calculating customer lifetime value, this distribution can be applied in situations where a customer makes no purchase until a specific period has passed (e.g., subscription-based services), making it a good fit for data with delayed purchase behavior.

3.6 Three Parameter Lindley Distribution Summary Table

Table 3.1: Summary of the Three parameter Lindley distribution.

Property	Expression
Probability Density Function (PDF)	$f(x;\theta,\alpha,\beta) = \frac{\theta^2(\alpha+\beta x)}{(\theta\alpha+\beta)} e^{-\theta x}, \quad x > 0, \theta > 0, \beta > 0, \theta\alpha + \beta > 0$
Cumulative Distribution Function (CDF)	$F(x;\theta,\alpha,\beta) = 1 - e^{-\theta x}\left(1 + \frac{\alpha\beta x}{\theta+\alpha\beta}\right), \quad$ for $x > 0, \theta > 0, \alpha > 0, \beta > 0.$
Mean	$\mu = \frac{\alpha\theta+2\beta}{\theta(\theta\alpha+\beta)}$
Variance	$\mathrm{Var}(x) = \frac{\alpha^2\theta^2+4\alpha\theta\beta+2\beta^2}{\theta^2(\theta\alpha+\beta)^2}$
Moment Generating Function (MGF)	$M_X(t) = \frac{\theta^2(\alpha(\theta-t)+\beta)}{(\theta\alpha+\beta)(\theta-t)^2}, \quad$ for $t < \theta$
Parameter	$x > 0, \theta > 0, \beta > 0, \theta\alpha + \beta > 0$

Chapter 4

Weighted Lindley Distribution

4.1 Introduction

The Weighted Lindley Distribution (WLD) Asgharzadeh et al. [2016] is an extension of the Lindley distribution that introduces an additional parameter to enhance its flexibility and applicability. The Lindley distribution, which is a special case of the exponential distribution, is known for its ability to model failure times and other random variables exhibiting skewed behavior. The weighted Lindley distribution builds upon this foundation by incorporating a weighting factor, allowing it to better fit a broader range of real-world data.

4.2 Probability Density Function (PDF)

The probability density function (pdf) of the Weighted Lindley Distribution with parameters θ and α is given by:

$$f(x;\theta,\alpha) = \frac{(\alpha+1)\theta^{\alpha}x^{\alpha-1}e^{-\theta x}}{(\theta+x)^{\alpha+1}}, \quad x>0,\ \theta>0,\ \alpha>-1 \tag{4.1}$$

where: θ is a scale parameter, α is a shape parameter that introduces the weighting effect.

Listing 4.1: R Code for Simulation study of θ and α for Weighted Lindley Distribution

```
# Load necessary libraries
if (!requireNamespace("ggplot2", quietly = TRUE)) install.packages("
    ggplot2")
library(ggplot2)

# Function to generate samples from the Weighted Lindley distribution
weighted_lindley_sample <- function(theta, alpha, size = 1) {
  gamma_sample <- rgamma(size, shape = alpha, rate = theta) # Gamma(
      alpha, 1/theta)
  lindley_sample <- gamma_sample                            # Weighted
      Lindley samples
  return(lindley_sample)
}

# Parameters
theta_values <- c(1, 2)  # Reduced values for theta
```

```
alpha_values <- c(1, 2)  # Reduced values for alpha
sample_size <- 500       # Reduced sample size for simplicity

# Generate samples for each combination of theta and alpha
set.seed(123) # For reproducibility
data <- do.call(rbind, lapply(theta_values, function(theta) {
  do.call(rbind, lapply(alpha_values, function(alpha) {
    data.frame(
      sample = weighted_lindley_sample(theta, alpha, sample_size),
      theta = as.factor(theta),  # Group by theta
      alpha = as.factor(alpha)   # Group by alpha
    )
  }))
}))

# Calculate density estimates for each combination of theta and alpha
density_data <- do.call(rbind, lapply(split(data, list(data$theta, data
    $alpha)), function(group) {
  density_est <- density(group$sample)
  data.frame(
    x = density_est$x,
    y = density_est$y,
    theta = unique(group$theta),
    alpha = unique(group$alpha)
  )
}))

# Create the polygon curve plot
ggplot(density_data, aes(x = x, y = y, color = interaction(theta, alpha
    ), group = interaction(theta, alpha))) +
  geom_line(linewidth = 1) +
  labs(
    title = "Polygon Curves for Weighted LD",
    x = "Random Samples",
    y = "Density",
    color = "Theta, Alpha"
  ) +
  theme_minimal() +
  scale_color_brewer(palette = "Set1") # Distinct colors for parameter
      combinations

# Save the plot
ggsave("Polygon_Curves_Weighted_Lindley_Distribution.jpg", width = 10,
    height = 6)
```

4.3 Properties

Mean

The probability density function (PDF) of the Weighted Lindley Distribution is given in (4.1)

The mean μ is given by:

$$\mu = \mathbb{E}[X] = \int_0^\infty x \cdot f(x;\theta,\alpha)\,dx$$

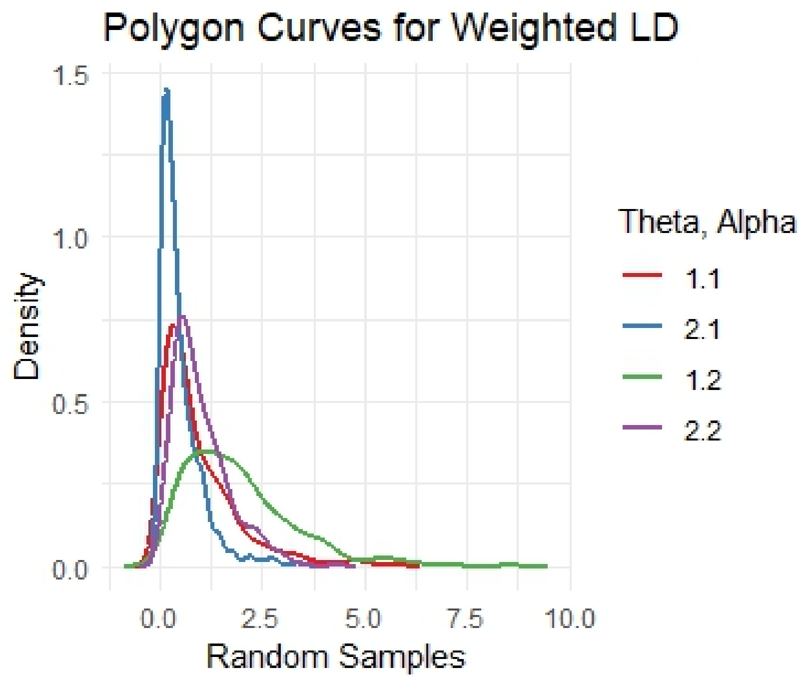

Figure 4.1: Pdf sketch of the Weighted lindley distribution for different values of parameters

This leads to the following integral:

$$\mu = \int_0^\infty \frac{(\alpha+1)\theta^\alpha x^\alpha e^{-\theta x}}{(\theta+x)^{\alpha+1}}\, dx$$

Variance

The variance σ^2 is given by:

$$\sigma^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

where:

$$\mathbb{E}[X^2] = \int_0^\infty x^2 \cdot f(x;\theta,\alpha)\, dx$$

This leads to the following expression:

$$\mathbb{E}[X^2] = \int_0^\infty \frac{(\alpha+1)\theta^\alpha x^{\alpha+1} e^{-\theta x}}{(\theta+x)^{\alpha+1}}\, dx$$

Thus, the variance is:

$$\sigma^2 = \left(\int_0^\infty \frac{(\alpha+1)\theta^\alpha x^{\alpha+1} e^{-\theta x}}{(\theta+x)^{\alpha+1}}\, dx\right) - \left(\int_0^\infty \frac{(\alpha+1)\theta^\alpha x^{\alpha} e^{-\theta x}}{(\theta+x)^{\alpha+1}}\, dx\right)^2$$

Moment Generating Function (MGF):

The probability density function (PDF) of the Weighted Lindley Distribution is given in (4.1)

The moment-generating function (MGF) is defined as:

$$M_X(t) = \mathbb{E}[e^{tx}] = \int_0^\infty e^{tx} f(x;\theta,\alpha)\,dx.$$

$$M_X(t) = \int_0^\infty e^{tx} \frac{(\alpha+1)\theta^\alpha x^{\alpha-1} e^{-\theta x}}{(\theta+x)^{\alpha+1}}\,dx.$$

$$M_X(t) = (\alpha+1)\theta^\alpha \int_0^\infty \frac{x^{\alpha-1} e^{-(\theta-t)x}}{(\theta+x)^{\alpha+1}}\,dx.$$

Let us denote the integral as:

$$I(t) = \int_0^\infty \frac{x^{\alpha-1} e^{-(\theta-t)x}}{(\theta+x)^{\alpha+1}}\,dx.$$

Thus, we can express the MGF as:

$$M(t) = (\alpha+1)\theta^\alpha I(t).$$

4.4 Simulation Study

Maximum Likelihood Estimator(MLE)

The probability density function (PDF) of the Weighted Lindley Distribution is given in (4.1)

Given a sample of size n, denoted by $x_1, x_2, \ldots, x_n$, the log-likelihood function $L(\theta,\alpha)$ is defined as:

$$L(\theta,\alpha) = \sum_{i=1}^{n} \log f(x_i;\theta,\alpha).$$

Substituting the PDF into the log-likelihood function:

$$L(\theta,\alpha) = \sum_{i=1}^{n} \log\left[\frac{(\alpha+1)\theta^\alpha x_i^{\alpha-1} e^{-\theta x_i}}{(\theta+x_i)^{\alpha+1}}\right].$$

$$L(\theta,\alpha) = \sum_{i=1}^{n} \left[\log(\alpha+1) + \alpha\log\theta + (\alpha-1)\log x_i - \theta x_i - (\alpha+1)\log(\theta+x_i)\right].$$

To find the MLEs, we take the partial derivatives of $L(\theta,\alpha)$:

1. Derivative with respect to θ:

$$\frac{\partial L}{\partial \theta} = \sum_{i=1}^{n} \left[\frac{\alpha}{\theta} - \frac{x_i}{\theta + x_i} \right] = 0.$$

2. Derivative with respect to α:

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \left[\frac{1}{\alpha + 1} + \log\theta + \log x_i - \log(\theta + x_i) \right] = 0.$$

The solutions to these equations provide the maximum likelihood estimates for the parameters θ and α.

Listing 4.2: Python Code for Estimation of $\hat{\theta}$ and $\hat{\alpha}$ using MLE Weighted Lindley Distribution

```
import numpy as np
from scipy.optimize import minimize

# Define the log-likelihood function for the Weighted Lindley
    distribution
def log_likelihood(params, data):
    theta, alpha = params
    if theta <= 0 or alpha <= 0:
        return np.inf  # Ensure parameters are positive

    logL = np.sum(np.log(alpha) + np.log(theta) - np.log(1 + alpha) +
        np.log(1 + data) - theta * data - (alpha / (1 + alpha)) * np.exp
        (-theta * data))
    return -logL  # Negative because we minimize

# Define the gradient (first-order partial derivatives of log-
    likelihood)
def grad_log_likelihood(params, data):
    theta, alpha = params
    if theta <= 0 or alpha <= 0:
        return np.array([np.inf, np.inf])  # Ensure parameters are
            positive

    n = len(data)
    d_theta = np.sum(1 / theta - data + (alpha / (1 + alpha)) * data *
        np.exp(-theta * data))
    d_alpha = np.sum(1 / alpha - 1 / (1 + alpha) - (1 / (1 + alpha)) *
        np.exp(-theta * data))
    return np.array([-d_theta, -d_alpha])

# Define the Hessian (second-order partial derivatives of log-
    likelihood)
def hessian_log_likelihood(params, data):
    theta, alpha = params
    if theta <= 0 or alpha <= 0:
        return np.array([[np.inf, np.inf], [np.inf, np.inf]])  # Ensure
            parameters are positive

    n = len(data)
    d2_theta2 = -n / theta**2 - np.sum((alpha / (1 + alpha)) * data**2
        * np.exp(-theta * data))
    d2_alpha2 = -n / alpha**2 + n / (1 + alpha)**2 - np.sum((np.exp(-
        theta * data)) / (1 + alpha)**2)
```

```
    d2_theta_alpha = np.sum((data * np.exp(-theta * data)) / (1 + alpha
        ))

    return np.array([
        [d2_theta2, d2_theta_alpha],
        [d2_theta_alpha, d2_alpha2],
    ])

# Sample data
np.random.seed(42)
data = np.random.exponential(scale=2, size=100)  # Replace with your
    data generation method

# Initial guesses for theta and alpha
initial_params = [1, 1]

# Use scipy's minimize to perform MLE
result = minimize(
    log_likelihood,
    initial_params,
    args=(data,),
    method='BFGS',
    jac=grad_log_likelihood,
    options={'disp': True},
)

# Results
theta_mle, alpha_mle = result.x
print(f"MLE for theta: {theta_mle}")
print(f"MLE for alpha: {alpha_mle}")

# Hessian at the MLE solution (used to estimate standard errors)
hess_inv = result.hess_inv
print(f"Inverse Hessian (covariance matrix): \n{hess_inv}")

# Estimated standard errors
standard_errors = np.sqrt(np.diag(hess_inv))
print(f"Standard errors for theta and alpha: {standard_errors}")

OUTPUT:
    MLE for theta: 0.6574123376750699
MLE for alpha: 2.3168135971904076
Inverse Hessian (covariance matrix):
[[0.00506774 0.004306  ]
 [0.004306   0.0662467 ]]
Standard errors for theta and alpha: [0.07118809 0.25738435]
```

Listing 4.3: Generating the Random Samples from the Weighted Lindley distribution

```
import numpy as np
from scipy.stats import gamma
from scipy.optimize import minimize
from scipy.stats import expon, gamma
import matplotlib.pyplot as plt
import seaborn as sns

def weighted_lindley_samples(alpha, theta, gamma, n):
    """
```

```
    Generate random samples from a Weighted Lindley distribution.

    Parameters:
    alpha (float): Shape parameter.
    theta (float): Scale parameter.
    gamma (float): Weight parameter.
    n (int): Number of samples to generate.

    Returns:
    numpy.ndarray: Random samples from the Weighted Lindley
        distribution.
    """
    # Step 1: Generate uniform random variables
    U = np.random.uniform(0, 1, n)

    # Step 2: Generate random variables from gamma distribution
    V1 = np.random.gamma(alpha, theta, n)  # gamma(alpha, theta)
    V2 = np.random.gamma(alpha + 1, theta, n)  # gamma(alpha + 1, theta
        )

    # Step 3: Generate samples from the Weighted Lindley distribution
    samples = np.where(U <= theta / (gamma + theta), V1, V2)

    return samples

# Example usage
alpha = 1.5  # Shape parameter
theta = 1.0  # Scale parameter
gamma = 0.5  # Weight parameter
n = 1000  # Number of samples

samples = weighted_lindley_samples(alpha, theta, gamma, n)

# Optional: Print first 10 samples
print(samples)
sns.histplot(data=samples,color='green',kde=True).set(title="
    Distribution of the Weighted Lindley Distribution",xlabel='Random
    Samples',ylabel='Frequency')
plt.savefig("Distribution of the Weighted Lindley Distribution.jpg")
plt.show()
```

From figure 4.2 we can observe that the Lindley distribution is positively skewed, meaning that it is extended by the tail more towards the right.

4.5 Applications

- **Application Scenario: Customer Service in a Call Center** In a call center, the arrival times of customer calls can vary significantly, and the time taken to handle each call is crucial for determining overall efficiency and customer satisfaction. The Weighted Lindley Distribution can be used to model the distribution of call handling times.
- **Application Modeling Call Durations:** The call durations can be influenced by factors such as the complexity of the customer query and the service representative's expertise. The Weighted Lindley Distribution allows for flexibility in modeling these durations with its parameters θ and α.

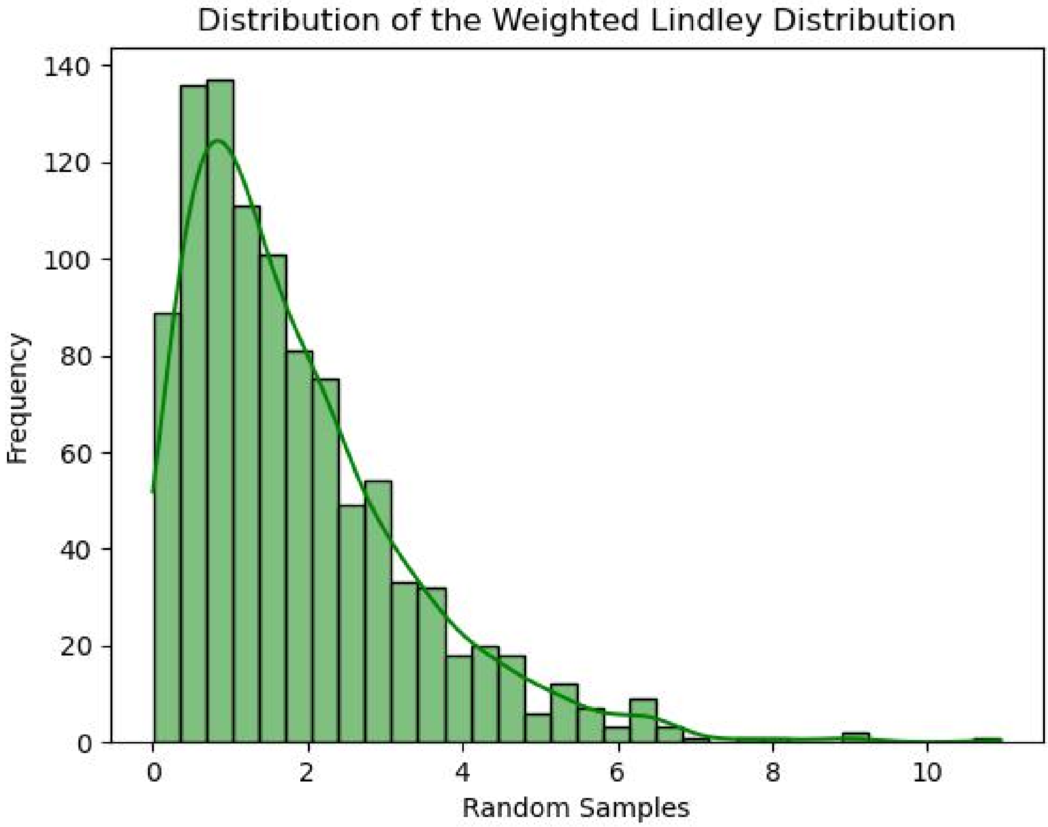

Figure 4.2: Distribution of the Weighted Lindley Distribution

- **Application Resource Allocation:** By fitting the Weighted Lindley Distribution to historical call duration data, managers can make informed decisions about staffing levels and resource allocation to minimize wait times and improve service levels.

Chapter 5

Quasi Lindley Distribution

5.1 Introduction

Shanker and Mishra [2013a], The Quasi Lindley distribution with parameters α and θ is defined by its probability density function (p.d.f) as follows:

5.2 Probability Density Function(PDF)

$$f(x;\alpha,\theta) = \frac{\theta}{\alpha + x\theta}\alpha + \theta + 1(1+x)e^{-\theta x}, \quad x > 0, \quad \theta > 0, \quad \alpha > -1$$

It can easily be seen that at $\alpha = \theta$, the QLD reduces to the Lindley distribution, and at $\alpha = 0$, it reduces to the gamma distribution with parameters $(2,\theta)$. The p.d.f can be shown as a mixture of exponential (θ) and gamma $(2,\theta)$ distributions as follows:

$$f(x;\alpha,\theta) = p f_1(x) + (1-p) f_2(x)$$

where

$$p = \frac{\alpha}{\alpha + \theta + 1}$$

and

$$f_1(x) = \theta e^{-\theta x}, \quad f_2(x) = \theta^2 x e^{-\theta x}$$

The nature of the QLD for different values of its parameters α and θ has been shown graphically in below In the figure (5.1), QL $(1,1)$ means QLD with parameters $\alpha = 1$ and $\theta = 1$. In Figure 1(d), five graphs of QLD for different values of its parameters have been plotted.

The first derivative of the Quasi Lindley distribution is given by:

$$f'(x) = \frac{\theta^2 e^{-\theta x}(1-\alpha x)}{\alpha + \theta + 1}$$

Setting $f'(x) = 0$, we obtain the critical point:

$$x = \frac{\alpha - 1}{\theta}$$

From this, it follows that:

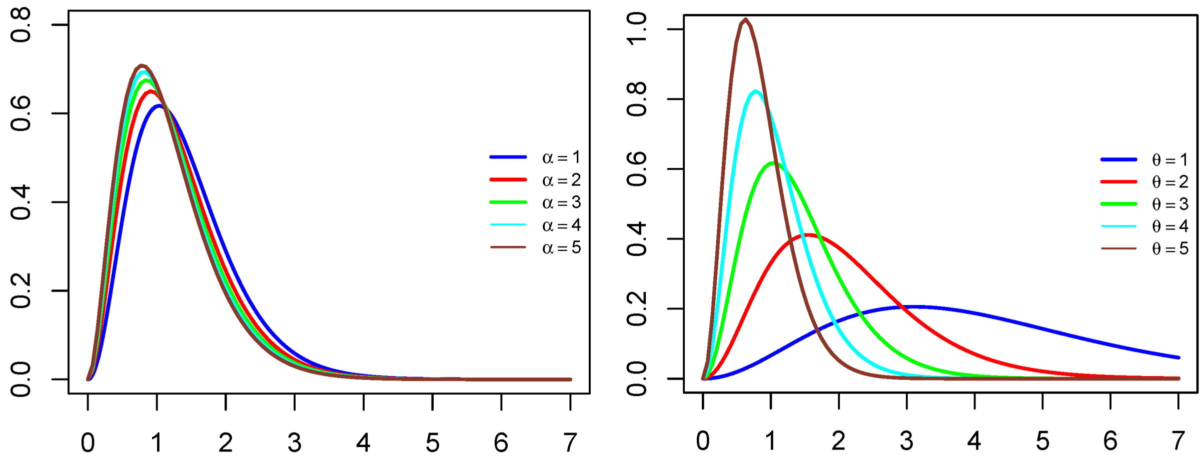

Figure 5.1: PDF sketch of the Quasi Lindley distribution for different values of parameter

- For $\alpha < 1$, the unique critical point $x = \frac{\alpha-1}{\theta}$ is where $f(x)$ reaches its maximum.
- For $\alpha \geq 1$, $f'(x) \leq 0$, meaning $f(x)$ is decreasing in x.

Therefore, the mode of the Quasi Lindley distribution is given by:

$$\text{Mode} = \begin{cases} \frac{\alpha-1}{\theta}, & \text{if } \alpha < 1, \\ 0, & \text{if } \alpha \geq 1. \end{cases}$$

Cumulative Distribution Function(CDF):

The cumulative distribution function (CDF) of the Quasi Lindley distribution is:

$$F(x) = 1 - \frac{(\alpha + 1 + x)e^{-\theta x}}{\alpha + \theta + 1}, \quad x > 0, \quad \alpha > 0, \quad \theta > -1$$

5.3 Properties

Mean $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \int_0^\infty x f(x; \alpha, \theta)\, dx$$

Substituting the expression for $f(x; \alpha, \theta)$:

$$\mathbb{E}[X] = \int_0^\infty \frac{\theta x(1+x)e^{-\theta x}}{(\alpha + x\theta)(\alpha + \theta + 1)}\, dx$$

Variance

Variance $\text{Var}(X)$:

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

To calculate $\mathbb{E}[X^2]$, we use the following integral:

$$\mathbb{E}[X^2] = \int_0^\infty x^2 f(x;\alpha,\theta)\, dx$$

Substituting the PDF:

$$\mathbb{E}[X^2] = \int_0^\infty \frac{\theta x^2(1+x)e^{-\theta x}}{(\alpha + x\theta)(\alpha+\theta+1)}\, dx$$

Listing 5.1: Generating the Random Samples from the Quasi Lindley distribution

```
import numpy as np
from scipy.optimize import minimize
from scipy.stats import expon, gamma
import matplotlib.pyplot as plt
import seaborn as sns

# Quasi Lindley probability density function (PDF)
def qld_pdf(x, alpha, theta):
    p = alpha / (alpha + theta + 1)
    return p * theta * np.exp(-theta * x) + (1 - p) * theta**2 * x * np
        .exp(-theta * x)

# Log-likelihood function for QLD
def qld_log_likelihood(params, data):
    alpha, theta = params
    if alpha <= 0 or theta <= 0:
        return np.inf
    pdf_vals = qld_pdf(data, alpha, theta)
    return -np.sum(np.log(pdf_vals))

# Generating random samples from the QLD (using mixture of exponential
    and gamma distributions)
def generate_qld_samples(alpha, theta, size=1000):
    p = alpha / (alpha + theta + 1)
    u = np.random.uniform(0, 1, size)
    samples = np.where(u < p, expon(scale=1/theta).rvs(size), gamma(a
        =2, scale=1/theta).rvs(size))
    return samples

# Estimating parameters using Maximum Likelihood Estimation (MLE)
def estimate_qld_params(data):
    initial_guess = [1.0, 1.0]  # Initial guesses for alpha and theta
    result = minimize(qld_log_likelihood, initial_guess, args=(data,),
        bounds=((1e-5, None), (1e-5, None)))
    return result.x  # Returns estimated alpha and theta

# Example usage
alpha_true = 1.0
theta_true = 2.0
samples = generate_qld_samples(alpha_true, theta_true, size=1000)
alpha_est, theta_est = estimate_qld_params(samples)

print(f"True alpha: {alpha_true}, True theta: {theta_true}")
print(f"Estimated alpha: {alpha_est}, Estimated theta: {theta_est}")
```

```
sns.histplot(data=samples,color='blue',kde=True).set(title="
    Distribution of the Two parameter of Lindley Distribution",xlabel='
    Random Samples',ylabel='Frequency')
plt.show()
plt.savefig("Distribution of the Quasi Lindley Distribution.jpg")
```

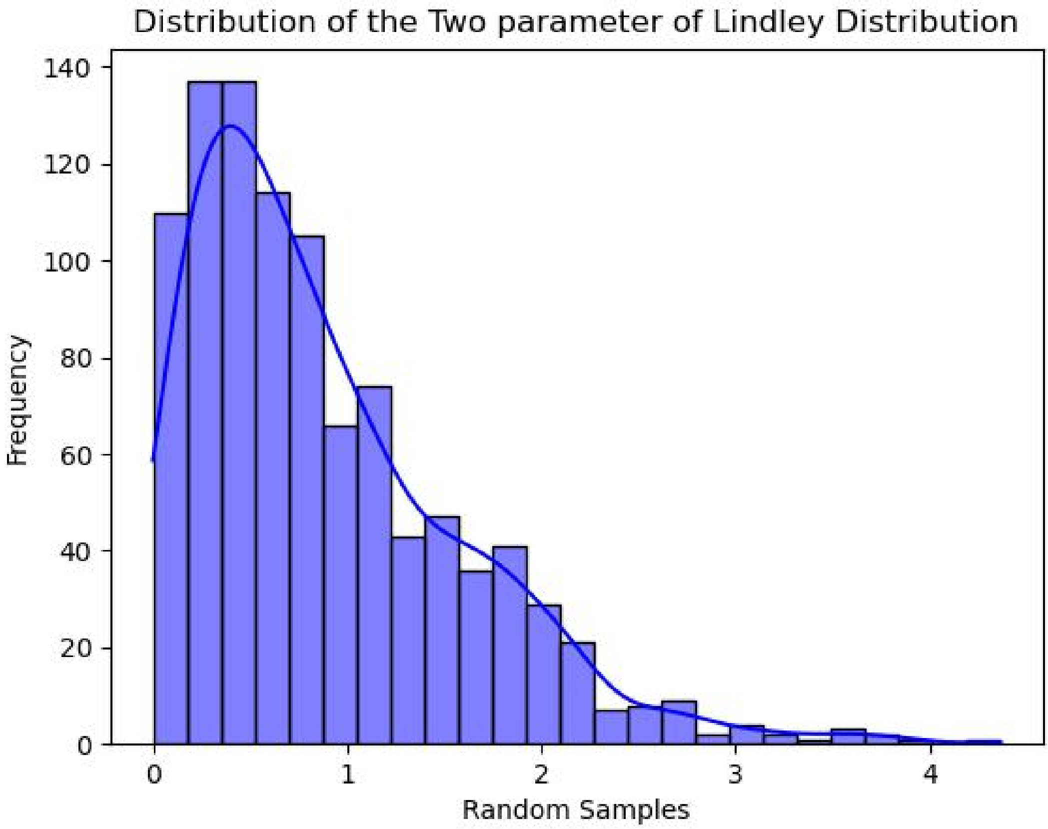

Figure 5.2: Shape of the Quasi Lindley distribution

From figure 5.2 we can observe that the Lindley distribution is positively skewed, meaning that it is extended by the tail more towards the right.

5.4 Simulation Study

5.5 Applications

- **Application Reliability and Survival Analysis: Modeling Failure Time Data** The Quasi Lindley Distribution (QLD) can be used to model systems with variable lifetimes and different failure modes. It is useful in analyzing the time-to-failure data of components or systems subjected to different stress levels and operational conditions.

- **Application Reliability and Survival Analysis:Medical Survival Data:** In clinical studies, QLD is applied to model survival times of patients, particularly when hazard rates are non-constant. It can be useful in analyzing data from patients with varying risk factors and different treatment responses.

- **Application Queueing Theory:Service and Waiting Time Distributions** The QLD can model service or waiting times in systems where these times are affected by multiple factors. This makes it suitable for applications in telecommunications, transportation, and customer service systems.
- **Application Queueing Theory:Healthcare Queueing Systems** In healthcare systems, QLD can be used to model the variability in patient waiting times in clinics or emergency rooms, where treatment times and patient flow are highly variable.
- **Application Insurance and Risk Management :Modeling Insurance Claims** The QLD can be applied to model the distribution of claim sizes in insurance, especially when the data exhibit variability and heavy tails. It is particularly useful in modeling claims resulting from catastrophic events such as floods or accidents.
- **Application Insurance and Risk Management :Risk Modeling** In finance and insurance, QLD is used to estimate the risk of extreme events or large losses, where the distribution of returns or damages does not fit standard models.
- **Application Hydrology and Environmental Sciences:Modeling Extreme Weather Events** The QLD is useful in modeling extreme weather events such as heavy rainfall, floods, or droughts. It can be applied in hydrology to estimate the likelihood and severity of such events, improving risk assessment and disaster preparedness.
- **Application Hydrology and Environmental Sciences:Pollution Data Analysis** In environmental science, QLD can model pollution levels, where data may have complex structures and variability, particularly when extreme pollution levels occur rarely but with significant impact.
- **Application Actuarial Science:Modeling Lifetimes of Policies** The QLD can be used in actuarial science to model the distribution of lifetimes of insurance policies, especially when policyholders exhibit varying risks and behaviors. It provides better flexibility in handling heterogeneous populations.
- **Application Actuarial Science:Claim Frequency and Severity** The QLD can model the frequency and severity of claims in actuarial models, allowing for more precise premium calculations and risk assessments.
- **Application Industrial Engineering:Maintenance and Reliability Studies** In industrial engineering, QLD is used to analyze maintenance schedules and reliability of machinery or components under different operational conditions. It helps in determining optimal maintenance strategies by modeling time-to-failure.
- **Application Industrial Engineering:Manufacturing Process Control** The QLD can be applied to model variability in manufacturing processes, where defects or failures may follow complex distribution patterns.

Chapter 6

Generalized Lindley Distribution

6.1 Introduction

The one parameter family of distributions with the density function

$$f(x;\theta) = \frac{\theta}{2(1+x)} e^{-\theta x} \frac{1}{1+\theta}, \quad x > 0, \theta > 0,$$

was used by Lindley to illustrate a difference between fiducial distribution and posterior distribution. Sankaran used it as the mixing distribution of a Poisson parameter, and the distribution he derived is known as the Poisson–Lindley distribution. Recently, Ghitany et al. rediscovered and studied various properties of this distribution. Because of having only One parameter, the Lindley distribution, does not provide enough flexibility for analyzing different types of lifetime data. To increase the flexibility for modelling purposes, it will be useful to consider further alternatives to this distribution. This paper offers a three-parameter family of distributions that generalizes the Lindley distribution and includes, as special cases, the ordinary exponential and gamma distributions. The procedure used here is based on certain mixtures of the gamma distributions. The study examines various properties of the new model.

The Generalized Lindley distribution and presents its basic properties, including the behavior of the density and hazard rate functions, the distribution of sums, and some results on stochastic orderings. We also propose an algorithm for generating random data from the new distribution in this section. Section 3 discusses the estimation of parameters. Two data modelling examples are provided in this section, where the generalized Lindley distribution fits marginally better than the gamma, Weibull, and lognormal distributions. We introduce the generalized Lindley distribution and study its basic properties.

6.2 Probability Density Function (PDF)

Generalization

Let

$$f_g(x;\alpha,\theta) = \frac{\theta(\theta x)^{\alpha-1} e^{-\theta x}}{\Gamma(\alpha)}, \quad \alpha, \theta, x > 0,$$

be the density function of the gamma distribution with the shape parameter α and the scale parameter θ, denoted by gamma(α,θ). Let V_1 and V_2 be two independent random variables distributed according to gamma(α,θ) and gamma$(\alpha+1,\theta)$, respectively. For $\gamma > 0$, consider the random variable $X = V_1$ with probability $\frac{\theta}{\theta+\gamma}$, and $X = V_2$ with probability $\frac{\gamma}{\theta+\gamma}$. It is then easy to verify that the density function of X The probability density function is given by

$$f(x;\alpha,\theta,\gamma) = \frac{\theta^2(\theta x)^{\alpha-1}(\alpha+\gamma x)e^{-\theta x}}{(\gamma+\theta)\Gamma(\alpha+1)}, \quad \alpha,\theta,\gamma,x > 0.$$

This distribution contains the Lindley distribution as a particular case when $\alpha = \gamma = 1$. When $\gamma = 0$, (3) reduces to the density function of the gamma distribution with the parameters α and θ. In the case $(\alpha,\gamma) = (1,0)$, it coincides with the density function of the ordinary exponential distribution.

We say that the random variable X has a generalized Lindley (GL) distribution if X has the density function defined by . We denote the generalized Lindley distribution with the parameters α, θ, and γ as GL(α,θ,γ).

Random Variate Generation

The density function of the GL distribution can be written in terms of the gamma density function as

$$f(x;\alpha,\theta,\gamma) = \frac{\theta}{\gamma+\theta} f_g(x;\alpha,\theta) + \frac{\gamma}{\gamma+\theta} f_g(x;\alpha+1,\theta).$$

To generate random data X_i, $i = 1,\ldots,n$, from GL(α,θ,γ), one can use the following algorithm:

1. Generate U_i, $i = 1,\ldots,n$, from $U(0,1)$ distribution.
2. Generate V_{1i}, $i = 1,\ldots,n$, from the gamma(α,θ) distribution.
3. Generate V_{2i}, $i = 1,\ldots,n$, from the gamma$(\alpha+1,\theta)$ distribution.
4. If $U_i \le \frac{\theta}{\gamma+\theta}$, then set $X_i = V_{1i}$; otherwise, set $X_i = V_{2i}$, $i = 1,\ldots,n$.

In the next section, we consider the maximum likelihood estimation of the parameters of the GL distribution.

Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of the Generalized Lindley distribution is:

$$F(x;\alpha,\theta,\gamma) = \frac{\alpha\gamma(\alpha,\theta x) + \gamma\gamma(\alpha+1,\theta x)}{(\gamma+\theta)\,\Gamma(\alpha+1)}, \quad \alpha,\theta,\gamma,x > 0.$$

where:

- $\gamma(\alpha,z)$ is the lower incomplete gamma function,
- $\Gamma(\alpha+1)$ is the gamma function,
- $x > 0$, $\alpha > 0$, $\theta > 0$, $\gamma > 0$ are the parameters.

6.3 Properties

mean

The mean $\mathbb{E}[X]$ is defined as:

$$\mathbb{E}[X] = \int_0^\infty x f(x;\alpha,\theta,\gamma)\,dx.$$

$f(x;\alpha,\theta,\gamma)$:

$$\mathbb{E}[X] = \int_0^\infty x \cdot \frac{\theta^2(\theta x)^{\alpha-1}(\alpha+\gamma x)e^{-\theta x}}{(\gamma+\theta)\Gamma(\alpha+1)}\,dx.$$

$$\mathbb{E}[X] = \frac{\theta^2(\gamma+\theta)}{\Gamma(\alpha+1)}\int_0^\infty x\cdot(\theta x)^{\alpha-1}(\alpha+\gamma x)e^{-\theta x}\,dx.$$

$$= \frac{\theta^2(\gamma+\theta)}{\Gamma(\alpha+1)}\int_0^\infty \left(\theta^\alpha x^\alpha(\alpha+\gamma x)e^{-\theta x}\right)dx.$$

$$= \frac{\theta^2(\gamma+\theta)}{\Gamma(\alpha+1)}\left[\alpha\theta^\alpha\int_0^\infty x^\alpha e^{-\theta x}\,dx + \gamma\theta^\alpha\int_0^\infty x^{\alpha+1}e^{-\theta x}\,dx\right].$$

Now, evaluate the integrals:

$$\int_0^\infty x^\alpha e^{-\theta x}\,dx = \frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}}.$$

$$\int_0^\infty x^{\alpha+1} e^{-\theta x}\,dx = \frac{\Gamma(\alpha+2)}{\theta^{\alpha+2}}.$$

$$= \frac{\theta^2(\gamma+\theta)}{\Gamma(\alpha+1)}\left[\alpha\theta^\alpha\cdot\frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}} + \gamma\theta^\alpha\cdot\frac{\Gamma(\alpha+2)}{\theta^{\alpha+2}}\right].$$

$$= \frac{\gamma+\theta}{\theta}\left[\alpha\Gamma(\alpha+1)+\gamma\frac{\Gamma(\alpha+2)}{\theta}\right].$$

Thus, the mean is:

$$\mathbb{E}[X] = \frac{\gamma+\theta}{\theta}\left[\alpha\Gamma(\alpha+1)+\gamma\frac{\Gamma(\alpha+2)}{\theta}\right].$$

Variance

The variance $\mathrm{Var}(X)$ is defined as:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

To compute $\mathbb{E}[X^2]$, we use the definition:

$$\mathbb{E}[X^2] = \int_0^\infty x^2 f(x;\alpha,\theta,\gamma)\,dx.$$

Substitute the expression for $f(x;\alpha,\theta,\gamma)$:

$$\mathbb{E}[X^2] = \int_0^\infty x^2 \cdot \frac{\theta^2(\theta x)^{\alpha-1}(\alpha+\gamma x)e^{-\theta x}}{(\gamma+\theta)\Gamma(\alpha+1)}\,dx.$$

$$= \frac{\theta^2(\gamma+\theta)}{\Gamma(\alpha+1)} \int_0^\infty x^2 \cdot (\theta x)^{\alpha-1}(\alpha+\gamma x)e^{-\theta x}\,dx.$$

$$= \frac{\theta^2(\gamma+\theta)}{\Gamma(\alpha+1)} \int_0^\infty \left(\theta^\alpha x^{\alpha+1}(\alpha+\gamma x)e^{-\theta x}\right)dx.$$

$$= \frac{\theta^2(\gamma+\theta)}{\Gamma(\alpha+1)} \left[\alpha\theta^\alpha \int_0^\infty x^{\alpha+1}e^{-\theta x}\,dx + \gamma\theta^\alpha \int_0^\infty x^{\alpha+2}e^{-\theta x}\,dx\right].$$

Now, evaluate the integrals:

$$\int_0^\infty x^{\alpha+1}e^{-\theta x}\,dx = \frac{\Gamma(\alpha+2)}{\theta^{\alpha+2}}.$$

$$\int_0^\infty x^{\alpha+2}e^{-\theta x}\,dx = \frac{\Gamma(\alpha+3)}{\theta^{\alpha+3}}.$$

Substituting these results into the expression for $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2] = \frac{\theta^2(\gamma+\theta)}{\Gamma(\alpha+1)} \left[\alpha\theta^\alpha \cdot \frac{\Gamma(\alpha+2)}{\theta^{\alpha+2}} + \gamma\theta^\alpha \cdot \frac{\Gamma(\alpha+3)}{\theta^{\alpha+3}}\right].$$

$$= \frac{\gamma+\theta}{\theta^2}\left[\alpha\Gamma(\alpha+2) + \gamma\frac{\Gamma(\alpha+3)}{\theta}\right].$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

Thus, the variance of the Generalized Lindley Distribution is:

$$\mathrm{Var}(X) = \frac{\gamma+\theta}{\theta^2}\left[\alpha\Gamma(\alpha+2) + \gamma\frac{\Gamma(\alpha+3)}{\theta}\right] - \left(\frac{\gamma+\theta}{\theta}\left[\alpha\Gamma(\alpha+1) + \gamma\frac{\Gamma(\alpha+2)}{\theta}\right]\right)^2.$$

Moment Generating Function (MGF)

The moment generating function (MGF) of a random variable X is defined as:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} f(x;\alpha,\theta,\gamma)\,dx.$$

Substitute $f(x;\alpha,\theta,\gamma)$ into the definition of the MGF:

$$= \int_0^\infty e^{tx} \frac{\theta^2(\theta x)^{\alpha-1}(\alpha+\gamma x)e^{-\theta x}}{(\gamma+\theta)\Gamma(\alpha+1)}\,dx.$$

$$\frac{\theta^2}{(\gamma+\theta)\Gamma(\alpha+1)} \int_0^\infty (\theta x)^{\alpha-1}(\alpha+\gamma x)e^{-(\theta-t)x}\,dx.$$

$$= \frac{\theta^2}{(\gamma+\theta)\Gamma(\alpha+1)} \left[\alpha \int_0^\infty (\theta x)^{\alpha-1} e^{-(\theta-t)x}\,dx + \gamma \int_0^\infty (\theta x)^{\alpha} e^{-(\theta-t)x}\,dx\right].$$

Let $u = (\theta - t)x$, so $x = \frac{u}{\theta-t}$ and $dx = \frac{du}{\theta-t}$.

$$\int_0^\infty (\theta x)^{\alpha-1} e^{-(\theta-t)x}\,dx = \frac{\theta^{\alpha-1}}{(\theta-t)^\alpha} \int_0^\infty u^{\alpha-1}e^{-u}\,du.$$

$$\int_0^\infty u^{\alpha-1}e^{-u}\,du = \Gamma(\alpha),$$

$$\int_0^\infty (\theta x)^{\alpha-1} e^{-(\theta-t)x}\,dx = \frac{\theta^{\alpha-1}\Gamma(\alpha)}{(\theta-t)^\alpha}.$$

$$\int_0^\infty (\theta x)^{\alpha} e^{-(\theta-t)x}\,dx = \frac{\theta^{\alpha}}{(\theta-t)^{\alpha+1}} \int_0^\infty u^{\alpha}e^{-u}\,du.$$

$$\int_0^\infty u^{\alpha}e^{-u}\,du = \Gamma(\alpha+1),$$

$$\int_0^\infty (\theta x)^{\alpha} e^{-(\theta-t)x}\,dx = \frac{\theta^{\alpha}\Gamma(\alpha+1)}{(\theta-t)^{\alpha+1}}.$$

$$M_X(t) = \frac{\theta^2}{(\gamma+\theta)\Gamma(\alpha+1)} \left[\alpha \frac{\theta^{\alpha-1}\Gamma(\alpha)}{(\theta-t)^\alpha} + \gamma \frac{\theta^{\alpha}\Gamma(\alpha+1)}{(\theta-t)^{\alpha+1}}\right].$$

$$= \frac{\theta^2}{(\gamma+\theta)} \left[\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha+1)} \frac{1}{(\theta-t)^\alpha} + \frac{\gamma}{(\theta-t)^{\alpha+1}}\right].$$

Using $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$:

$$M_X(t) = \frac{\theta^2}{(\gamma+\theta)} \left[\frac{1}{(\theta-t)^\alpha} + \frac{\gamma}{(\theta-t)^{\alpha+1}}\right], \quad t<\theta.$$

6.4 Simulation Study

Maximum Likelihood Estimates

In this section, we consider the maximum likelihood estimation (MLE) of the parameters. If $X_1,\ldots,X_n$ is a random sample from X distributed according to GL(α,θ,γ), then the log-likelihood function, $l(\alpha,\theta,\gamma)$, is:

$$l(\alpha,\theta,\gamma) = n(\alpha+1)\log(\theta)-n\log(\gamma+\theta)-n\log\Gamma(\alpha+1)+(\alpha-1)\sum_{i=1}^{n}\log(x_i)+\sum_{i=1}^{n}\log(\alpha+\gamma x_i)-\theta\sum_{i=1}^{n}x_i.$$

The derivatives of $l(\alpha,\theta,\gamma)$ with respect to α, θ, and γ are:

$$\frac{\partial l}{\partial \alpha} = n\log(\theta) - n\Psi(\alpha+1) + \sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\frac{1}{\alpha+\gamma x_i}, \tag{7}$$

$$\frac{\partial l}{\partial \theta} = \frac{n(\alpha+1)}{\theta} - \frac{n}{\gamma+\theta} - \sum_{i=1}^{n}x_i, \tag{8}$$

and

$$\frac{\partial l}{\partial \gamma} = \sum_{i=1}^{n}\frac{x_i}{\alpha+\gamma x_i} - \frac{n}{\gamma+\theta}.$$

where $\Psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}$, denotes the digamma function. The equations can be solved simultaneously to find the maximum likelihood estimators of α, θ, and γ.

The GL distribution satisfies all the regularity conditions , Lindley [1958]in a way similar to the gamma distribution, and therefore applying the usual large sample approximation, the estimators $(\hat{\alpha},\hat{\theta},\hat{\gamma})$ are treated as being approximately bivariate normal with the mean vector (α,θ,γ) and variance-covariance matrix I^{-1}, where I is the Fisher information matrix, whose elements are given by:

$$-E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) = n\Psi'(1+\alpha) + nJ_0(\alpha,\theta,\gamma),$$

$$-E\left(\frac{\partial^2 l}{\partial \alpha \partial \theta}\right) = -\frac{n}{\theta},$$

where $\Psi'(t)$ is the trigamma function, and $J_0(\alpha,\theta,\gamma)$ is a function related to the GL distribution.

$$-E\left(\frac{\partial^2 l}{\partial \alpha \partial \gamma}\right) = nJ_1(\alpha,\theta,\gamma),$$

$$-E\left(\frac{\partial^2 l}{\partial \theta^2}\right) = \frac{n(\alpha+1)}{\theta^2} - \frac{n}{(\gamma+\theta)^2},$$

$$-E\left(\frac{\partial^2 l}{\partial\theta\partial\gamma}\right) = -\frac{n}{(\gamma+\theta)^2},$$

$$-E\left(\frac{\partial^2 l}{\partial\gamma^2}\right) = nJ_2(\alpha,\theta,\gamma) - \frac{n}{(\gamma+\theta)^2}.$$

Here, for $i = 0, 1, 2$, we have:

$$J_i(\alpha,\theta,\gamma) = E\left(\frac{X^i}{(\alpha+\gamma X)^2}\right) = \frac{\theta^{1-i}\alpha\Gamma(\alpha+1)}{(\gamma+\theta)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{\gamma}{\alpha\theta}\right)^k \Gamma(\alpha+k+i),$$

where X is distributed as $\mathrm{GL}(\alpha,\theta,\gamma)$.

Listing 6.1: Generating the Random Samples from the Generalized Lindley distribution

```
import numpy as np
from scipy.stats import gamma
from scipy.optimize import minimize
from scipy.stats import expon, gamma
import matplotlib.pyplot as plt
import seaborn as sns

def generate_generalized_lindley_samples(alpha, theta, gamma_param,
    size=1000):
    """
    Generate random samples from the generalized Lindley distribution.

    Parameters:
    alpha : float
        Shape parameter of the gamma distribution.
    theta : float
        Scale parameter of the gamma distribution.
    gamma_param : float
        Parameter for the mixture of the generalized Lindley
            distribution.
    size : int, optional
        The number of random samples to generate. Default is 1000.

    Returns:
    samples : ndarray
        Random samples from the generalized Lindley distribution.
    """

    # Mixture probabilities
    p1 = theta / (theta + gamma_param)  # Probability of sampling from
        gamma(alpha, theta)
    p2 = gamma_param / (theta + gamma_param)  # Probability of sampling
         from gamma(alpha + 1, theta)

    # Generate uniform random numbers to decide which distribution to
        sample from
    uniform_randoms = np.random.uniform(0, 1, size)

    # Generate samples from gamma(alpha, theta)
    samples_v1 = gamma.rvs(alpha, scale=1/theta, size=size)
```

```
    # Generate samples from gamma(alpha + 1, theta)
    samples_v2 = gamma.rvs(alpha + 1, scale=1/theta, size=size)

    # Combine the samples based on the mixture probabilities
    samples = np.where(uniform_randoms < p1, samples_v1, samples_v2)

    return samples

# Parameters for the generalized Lindley distribution
alpha = 2.0
theta = 1.5
gamma_param = 0.5
size = 1000  # Number of random samples

# Generate random samples
samples = generate_generalized_lindley_samples(alpha, theta,
    gamma_param, size)

# Output some of the generated samples
print(samples)
sns.histplot(data=samples,color='green',kde=True).set(title="
    Distribution of the Generalized Lindley Distribution",xlabel='Random
     Samples',ylabel='Frequency')
plt.savefig("Distribution of the Generalized Lindley Distribution.jpg")
plt.show()
```

From figure 6.1 we can observe that the Lindley distribution is positively skewed, meaning that it is extended by the tail more towards the right.

6.5 Application

- **Application Reliability Analysis**

  Widely used to model lifetimes of components and systems. Provides a good fit for data with both increasing and decreasing failure rates. Example applications: Modeling lifetimes of mechanical systems with wear-out mechanisms. Predicting reliability for electronic components with reliability growth or aging phenomena.

- **Application Survival Analysis**

  Used to describe survival times of individuals in medical studies or biological research. Its flexibility makes it suitable for modeling survival data with varying hazard rates. Example applications: Predicting the survival time of patients after receiving a specific treatment. Analyzing the longevity of organisms under controlled environmental conditions.

- **Application Queueing Theory**

  Applied to model waiting times in queueing systems. Its ability to represent complex data patterns makes it suitable for scenarios with:Varying service times. Non-linear customer arrival rates.

- **Application Environmental Studies**

  Used to model extreme events in environmental sciences, such as rainfall amounts

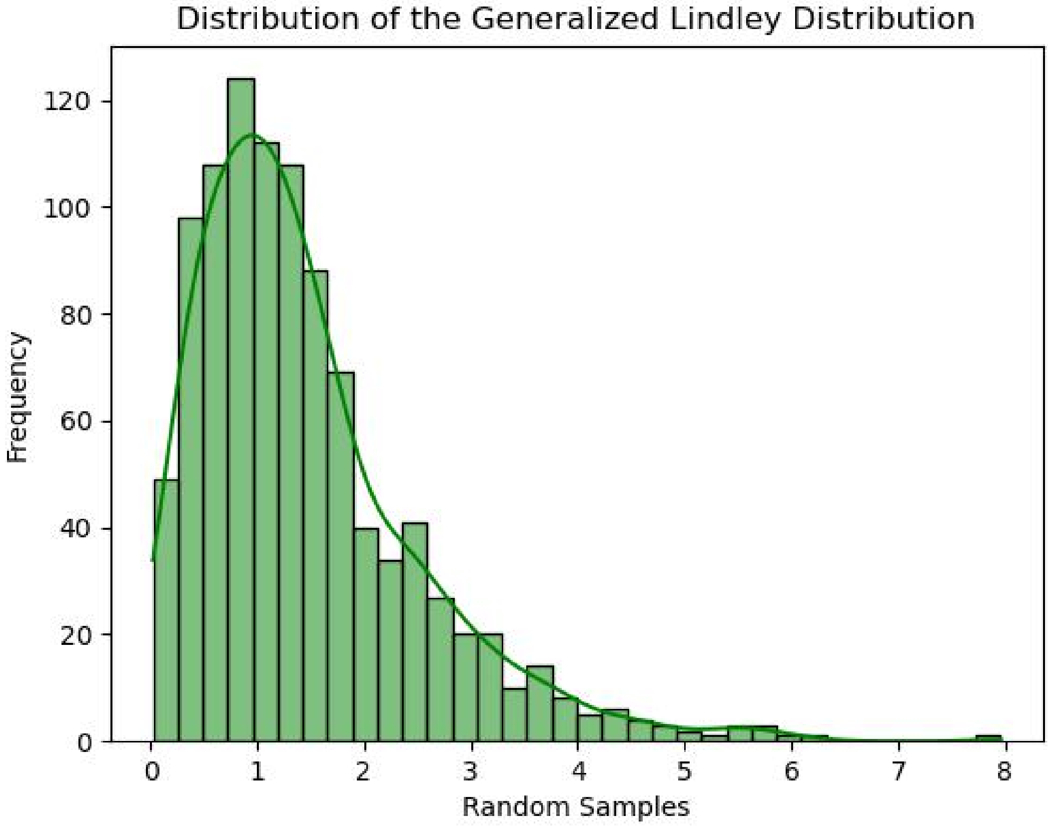

Figure 6.1: PDF sketch of the Generalized Lindley Distribution distribution

or pollutant concentrations. Example applications: Modeling daily rainfall in regions with varied precipitation patterns. Predicting pollutant dispersion over time in urban areas.

- **Application Insurance and Risk Management**

  Applied in actuarial science to model claim amounts or loss distributions.Helps insurers estimate risks and set premium levels based on observed data.

- **Application Hydrology and Climate Studies**

  Used to model river flows, drought durations, and temperature variations.Example: Estimating the probability of extreme floods in a given year.

- **Application Engineering Sciences** Engineers use this distribution to model failure times of materials and components under stress or varying conditions.Example: Predicting the fatigue life of metals or composite materials.

- **Application Financial Modeling**

  Describes investment returns or losses with skewed and heavy-tailed behavior.Example: Modeling returns on riskier assets with non-linear distributions.

6.6 Comparison Table

Table 6.1: PDF, CDF of Lindley Distributions

Distribution	PDF	CDF
One-Parameter Lindley	$f(x,\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \quad ;x>0,\theta>0$	$F(x;\theta)=1-\frac{(\theta+x\theta+1)e^{-\theta x}}{(\theta+1)}, \quad x>0,\theta>0$
Two-Parameter Lindley	$f(x,\alpha,\theta)=\frac{\theta^2}{\alpha\theta+1}(\alpha+x)e^{-\theta x}, \quad x>0,\theta>0,\ \alpha\theta>-1$	$F(x)=1-\left(\frac{1+\alpha\theta+\theta x}{\alpha\theta+1}\right)e^{-\theta x}; \quad x>0,\ \theta>0,\ \alpha\theta>-1$
Three-Parameter Lindley	$f(x,\theta,\alpha,\beta)=\frac{\theta^2(\alpha+\beta x)}{(\theta\alpha+\beta)}e^{-\theta x}, \quad x>0,\theta>0,\beta>0,\theta\alpha+\beta>0$	$F(x;\theta,\alpha,\beta)=1-e^{-\theta x}\left(1+\frac{\alpha\beta x}{\theta+\alpha\beta}\right)$, for $x>0, \theta>0, \alpha>0, \beta>0.$
Weighted Lindley	$f(x;\theta,\alpha)=\frac{(\alpha+1)\theta^{\alpha}x^{\alpha-1}e^{-\theta x}}{(\theta+x)^{\alpha+1}}, \quad x>0,\ \theta>0,\ \alpha>-1$	$F(x;\theta)=1-\frac{\theta x(1+\theta x)e^{-\theta x}}{1+\theta} \quad x>0,\ \theta>0,\ \alpha>-1$
Quasi Lindley	$f(x;\alpha,\theta)=\frac{\theta}{\alpha+x\theta}\alpha+\theta+1(1+x)e^{-\theta x}, \quad x>0, \quad \theta>0, \quad \alpha>-1$	$F(x)=1-\frac{(\alpha+1+x)e^{-\theta x}}{\alpha+\theta+1}, \quad x>0, \quad \alpha>0, \quad \theta>-1$
Generalized Lindley	$f(x;\alpha,\theta,\gamma)=\frac{\theta^2(\theta x)^{\alpha-1}(\alpha+\gamma x)e^{-\theta x}}{(\gamma+\theta)\Gamma(\alpha+1)}, \quad \alpha,\theta,\gamma,x>0$	$F(x;\alpha,\theta,\gamma)=\frac{\alpha\gamma(\alpha,\theta x)+\gamma\gamma(\alpha+1,\theta x)}{(\gamma+\theta)\Gamma(\alpha+1)} \quad \alpha,\theta,\gamma,x>0$

Conclusion

The comparison table highlights the evolution and flexibility of Lindley distributions through incremental parameterization, enabling their application to a broad range of real-world data scenarios. Each variant of the Lindley distribution builds upon the foundational one-parameter model, introducing additional parameters to better handle skewness, tail behavior, and data variability.

Progression of Flexibility

- The **One-Parameter Lindley** serves as a foundational model with simplicity but limited adaptability.
- As parameters like α, β, and γ are introduced in subsequent models, the ability to adjust skewness, accommodate heavy tails, and fit diverse data increases.
- The **Generalized Lindley** is the most versatile, suitable for datasets with complex or extreme characteristics.

Applicability

- The distributions range from simple (one-parameter) to highly flexible (generalized and weighted forms), making them applicable across various domains such as survival analysis, reliability modeling, and real-world phenomena where data often exhibits non-standard patterns.
- The **Weighted** and **Generalized** forms stand out for handling datasets with pronounced skewness, heavy-tailed behavior, or extreme variability.

Trade-offs

- While added parameters enhance modeling power, they increase computational complexity and may lead to challenges such as overfitting or difficulties in parameter estimation for small datasets.
- Simpler models like the **One-** and **Two-Parameter Lindley** are preferred for datasets with minimal complexity, where interpretability and computational efficiency are crucial.

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