

From Coverage Testing to Topological Scalar Field Analysis

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1 Abstract

The topological coverage criterion (TCC) can be used to test whether an underlying space is sufficiently well covered by a given data set. Given a sufficiently dense sample, topological scalar field analysis (SFA) can give a summary of the shape of a real-valued function on its domain. The goal of this paper is to put these theories together so that one can test coverage with the TCC while computing a summary with SFA. The challenge is that the TCC requires a well-defined boundary that is not generally available in the setting of SFA. To overcome this, we show how the scalar field itself can be used to define a boundary that can be used to confirm coverage. This requires an interpretation of the TCC that resolves one of the major barriers to wider use. It also extends SFA methods to the setting in which coverage is only confirmed in a subset of the domain.

2012 ACM Subject Classification Replace ccsdesc macro with valid one

Keywords and phrases Dummy keyword

Funding Kirk P. Gardner: [funding]

Donald R. Sheehy: [funding]

11 1 Introduction

In the topological analysis of scalar fields (SFA), one computes a topological summary capturing qualitative and quantitative shape information from a set of points endowed with a metric and a real-valued function. That is, we have points with distances and a real number assigned to each point. More generally, it suffices to have a neighborhood graph identifying the pairs of points within some distance. The topological computation often takes the form of persistent homology and integrates local information about the function into global information about its *behavior* as whole. In prior work, Chazal et al. [3] showed that for sufficiently dense samples on sufficiently smooth spaces, the persistence diagram can be computed with some guarantees. In followup work, Buchet et al. [1] extended this result to show how to work with noisy inputs. A fundamental assumption required to have strong guarantees is that the underlying space is sufficiently well-sampled. In this paper, we show how to combine scalar field analysis with the theory of topological coverage testing to simultaneously compute the persistence diagram while testing that the underlying space is sufficiently well-sampled.

Initiated by De Silva and Ghrist [12, 7, 8], the theory of homological sensor networks addresses the problem of testing coverage of a bounded domain by a collection of sensors without coordinates. The main result is the topological coverage criterion, which, in its most general form, states that under reasonable geometric assumptions, the d -dimensional homology of a pair of simplicial complexes built on the neighborhood graph will be nontrivial if and only if there is sufficient coverage (see Section 3 for the precise statements). This relative persistent homology test is called the Topological Coverage Criterion (TCC).



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42nd Conference on Very Important Topics (CVIT 2016).

Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:21



Lipics Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

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33 Superficially, the methods of SFA and TCC are very similar. Both construct similar
34 complexes and compute the persistent homology of the homological image of a complex on
35 one scale into that of a larger scale. They even overlap on some common techniques in their
36 analysis such as the use of the Nerve theorem and the Rips-Čech interleaving. However,
37 they differ in some fundamental way that makes it difficult to combine them into a single
38 technique. The main difference is that the TCC requires a clearly defined boundary. Not
39 only must the underlying space be a bounded subset of \mathbb{R}^d , the data must also be labeled to
40 indicate which input points are close to the boundary. This requirement is perhaps the main
41 reason why the TCC can so rarely be applied in practice.

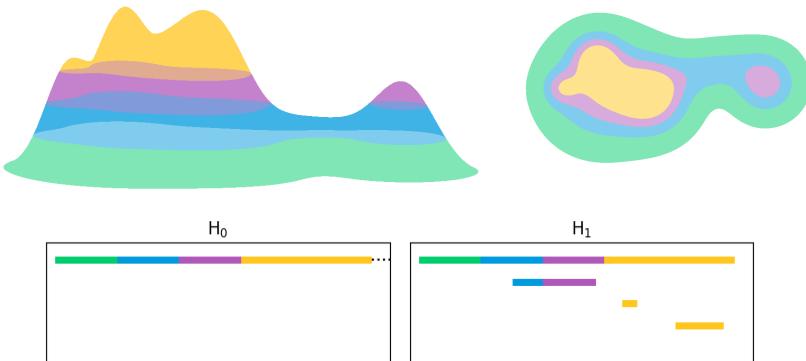
42 In applications to data analysis it is more natural to assume that the data measures
43 some unknown function. We can then replace this requirement with assumptions about the
44 function itself. Indeed, these assumptions could relate the behavior of the function to the
45 topological boundary of the space. However, the generalized approach by Cavanna et al. [2]
46 allows much more freedom in how the boundary is defined.

47 We consider the case in which we have incomplete data from a particular sublevel set
48 of our function. Our goal is to isolate this data so we can analyze the function in only the
49 verified region. From this perspective, the TCC confirms that we not only have coverage,
50 but that the sample we have is topologically representative of the region near, and above
51 this sublevel set. We can then re-use the same machinery to analyze a *part* of the function
52 in a specific way.

53 Contribution

54 We will re-cast the TCC as a way to verify that the persistent homology of a scalar field
55 can be *partially* approximated by a given sample. Specifically, we will relate the persistent
56 homology of a function relative to a *static* sublevel set to a *truncation* of the full diagram.
57 That is, beyond a certain point the full diagram remains unchanged, allowing for possible
58 reconstruction. This is in comparison with the *restricted* diagram obtained by simply ignoring
59 part of the domain. We therefore present relative persistent homology as an alternative to
60 restriction in a way that extends the TCC to the analysis of scalar fields.

61 Section 2 establishes notation and provides an overview of our main results in Sections 3
62 and 4. In Section 5 we introduce an interpretation of the relative diagram as a truncation of
63 the full diagram that is motivated by a number of experiments in Section 6.



64 2 Summary

65 Let \mathbb{X} denote an orientable d -manifold and $D \subset \mathbb{X}$ a compact subspace. For a c -Lipschitz
 66 function $f : D \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$ let $B_\alpha := f^{-1}((-\infty, \alpha])$ denote the α -sublevel set of f . Our
 67 sample will be denoted P , and the subset of points sampling B_α will be denoted $Q_\alpha := P \cap B_\alpha$.
 68 For $\varepsilon > 0$ let P^ε denote the union of open metric balls centered at points in P . For ease of
 69 exposition let

70 $D_{\lfloor \alpha \rfloor z} := B_\alpha \cup B_z$

71 denote the z -truncated sublevel sets of f and

72 $P_{\lfloor \alpha \rfloor z} := Q_\alpha \cup Q_z$

73 for all $z, \alpha \in \mathbb{R}$.¹²

74 We will select a sublevel set B_ω of f that surrounds D to serve as our boundary. Given a
 75 sample of f at a finite number of points P in D we would like to confirm P^δ not only covers
 76 the interior $D \setminus B_\omega$, but also that Q^δ surrounds P^δ for some $Q \subset P$. That is, we would like
 77 to verify that a pair (P^δ, Q^δ) is representative of the pair (D, B_ω) in homology. Our goal is
 78 to use this fact to approximate the persistence of f relative to B_ω .

85 Results

86 Our approximation will be by a nested pair of (Vietoris-)Rips complexes, denoted $\mathcal{R}^\varepsilon(P, Q) =$
 87 $(\mathcal{R}^\varepsilon(P), \mathcal{R}^\varepsilon(Q))$ for $\varepsilon > 0$. Under mild regularity assumptions it can be shown that

88 $\text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$

89 implies $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D . Proof of this fact generalizing the
 90 proof of the TCC to boundaries defined in terms of a function f , eliminating unnatural
 91 assumptions made in previous work. Not only are our subsamples $Q_{\omega-2c\delta}$ and $Q_{\omega+c\delta}$ defined
 92 in terms of their function values, but our regularity assumptions can be stated directly in
 93 terms of the persistent homology of f .

94 Given a sample P that satisfies the TCC we can approximate the persistent homology
 95 of f in a specific way. The nested pair of Rips complexes used to confirm coverage can be
 96 extended to a filtration

97 $\{\mathcal{R}^{2\delta}(P_{\lfloor \alpha \rfloor \omega-2c\delta}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor \alpha \rfloor \omega+c\delta}, Q_{\omega+c\delta})\}_{\alpha \in \mathbb{R}}$

98 that can be used to approximate the persistent homology of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$. Indeed, we
 99 could use existing methods to approximate the persistent homology of f restricted to the
 100 subspace $D \setminus B_\omega$ that we cover. However, the question of what this would approximate is
 101 important to consider. **Restricting the domain of the function can not only introduce noise**
 102 **close to the boundary, but also perturb global structure in our signature.**³ As an

73 ¹ **I'm starting to think:** For ease of exposition let pairs (D_α, B_z) denote $(B_{\max\{\alpha, z\}}, B_z)$ so that $B_z \subseteq D_\alpha$
 74 for all $\alpha \in \mathbb{R}$. Outside of a pair, we will refer to D_α as $\underline{}$. Similarly, let $(P_\alpha^\varepsilon, Q_z^\varepsilon)$ denote $(Q_{\max\{\alpha, z\}}, Q_z)$.

75 ² **Options:** $(P_{\lfloor \alpha \rfloor z+c\varepsilon}^\varepsilon, Q_{z+c\varepsilon}^\varepsilon)$, $(P_{\alpha|z+c\varepsilon}^\varepsilon, Q_{z+c\varepsilon}^\varepsilon)$, $(P_{\max\{\alpha, z+c\varepsilon\}}^\varepsilon, Q_{z+c\varepsilon}^\varepsilon)$, $(P_{\alpha>z+c\varepsilon}^\varepsilon, Q_{z+c\varepsilon}^\varepsilon)$, $(P_{\alpha;z+c\varepsilon}^\varepsilon, Q_{z+c\varepsilon}^\varepsilon)$,
 76 $(P_{z+c\varepsilon,\alpha}^\varepsilon, Q_{z+c\varepsilon}^\varepsilon)$, $((Q_\alpha \cup Q_{z+c\varepsilon})^\varepsilon, Q_{z+c\varepsilon}^\varepsilon)$; Throughout, let $(P_\alpha^\varepsilon, Q_{z+c\varepsilon}^\varepsilon)$ denote the pair
 77 $(Q_{\max\{\alpha, z+c\varepsilon\}}^\varepsilon, Q_{z+c\varepsilon}^\varepsilon)$ so that $Q_{z+c\varepsilon}^\varepsilon \subseteq P_\alpha^\varepsilon$ for all $\alpha \in \mathbb{R}$; Define filtrations for $\alpha \geq z + c\varepsilon$ and
 78 handle all of the edge cases by hand (there are a lot and it's gross).

79 ³ close.

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104 alternative, we approximate the persistence of f relative to the sublevel set B_ω . This is not
105 only to eliminate noise introduced by the restriction, but also to *truncate* the persistence of
106 f in a way that isolates global structure.

107 **Outline**

108 We will begin with our statement of the TCC in Section 3. Part of the proof of the TCC
109 will be generalized to properties of *surrounding pairs*, simplifying our reformulation of the
110 TCC in Theorem 6. Section 4 introduces extensions of surrounding pairs, as well as partial
111 interleavings of image modules. This is to show that a positive result from the TCC verifies
112 that a surrounding pair of samples can be used to approximate the persistence of a function
113 relative to a sublevel set in Theorem 18. In Section 5 we provide an interpretation of this
114 relative persistence as a truncation of the full diagram that is motivated by examples in
115 Section 6.

116 3 The Topological Coverage Criterion (TCC)

117 A positive result from the TCC requires that we have a subset of our cover to serve as the boundary. That is, the condition not only checks that we have coverage, but also that we have a pair of spaces that reflects the pair (D, B) topologically. We call such a pair a *surrounding pair* defined in terms of separating sets. It has been shown that the TCC can be stated in terms of these surrounding pairs [2]. Moreover, this work made assumptions directly in terms of the *zero dimensional* persistent homology of the domain close to the boundary. This allows us enough flexibility to define our surrounding set as a sublevel of a c -Lipschitz function f and state our assumptions in terms of its persistent homology.

125 ► **Definition 1 (Surrounding Pair).** Let X be a topological space and (D, B) a pair X . The set B surrounds D in X if B separates X with the pair $(D \setminus B, X \setminus D)$. We will refer to such a pair as a *surrounding pair in X* .

128 For a surrounding pair (D, B) in \mathbb{X} the complement $\overline{B} = \mathbb{X} \setminus B$ is the union of disconnected sets $\mathbb{X} \setminus D$ and $D \setminus B$. Therefore, $H_k(\overline{B}) \cong H_k(\overline{D}) \oplus H_k(D \setminus B)$ thus $H_k(\overline{B}, \overline{D}) \cong H_k(D \setminus B)$ for all k . The following lemma generalizes the proof of the TCC as a property of surrounding sets. We will then combine these results on the homology of surrounding pairs with information about both \mathbb{X} as a metric space and our function.

133 ► **Lemma 2.** Let (D, B) be a surrounding pair in X and $U \subseteq D, V \subseteq U \cap B$ be subsets. Let $\ell : H_0(X \setminus B, X \setminus D) \rightarrow H_0(X \setminus V, X \setminus U)$ be induced by inclusion.

135 If ℓ is injective then $D \setminus B \subseteq U$ and V surrounds U in D .

137 Let (\mathbb{X}, \mathbf{d}) be a metric space and $D \subseteq \mathbb{X}$ be a compact subspace. For a c -Lipschitz function $f : D \rightarrow \mathbb{R}$ we introduce a constant ω as a threshold that defines our “boundary” as a sublevel set B_ω of the function f . Let P be a finite subset of D and $\zeta \geq \delta > 0$ be constants such that $P^\delta \subseteq \text{int}_{\mathbb{X}}(D)$. Here, δ will serve as our communication radius where ζ is reserved for use in Section 4.⁴

142 ► **Lemma 3.** Let $i : H_0(\overline{Q_{\omega+c\delta}^\delta}, \overline{P^\delta}) \rightarrow H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$.

143 If B_ω surrounds D in \mathbb{X} then $\dim H_0(\overline{B_\omega}, \overline{D}) \geq \text{rk } i$.

136 ⁴ We will set $\zeta = 2\delta$ in the proof of our interleaving with Rips complexes but the TCC holds for all $\zeta \geq \delta$.

¹⁴⁴ **Proof.** Choose a basis for $\mathbf{im} \ i$ such that each basis element is represented by a point in $P^\delta \setminus Q_{\omega+c\delta}^\delta$. Let $x \in P^\delta \setminus Q_{\omega+c\delta}^\delta$ be such that $i[x] \neq 0$. So there exists some $p \in P$ such that ¹⁴⁵ $\mathbf{d}(p, x) < \delta$ and $p \notin Q_{\omega+c\delta}$, otherwise $x \in Q_{\omega+c\delta}^\delta$. Therefore, because f is c -Lipschitz,

$$\text{147} \quad f(x) \geq f(p) - c\mathbf{d}(x, p) > \omega.$$

¹⁴⁸ So $x \in \overline{B_\omega}$ and, because $x \in P^\delta \subseteq D$ it follows that $x \in D \setminus B_\omega$. Because i and ¹⁴⁹ $\ell : H_0(\overline{B_\omega}, \overline{D}) \rightarrow H_0(Q_{\omega-c\zeta}^\delta, \overline{P^\delta})$ are induced by inclusion $\ell[x] = i[x] \neq 0$ in $H_0(Q_{\omega-c\zeta}^\delta, \overline{P^\delta})$. ¹⁵⁰ That is, every element of $\mathbf{im} \ i$ has a preimage in $H_0(\overline{B_\omega}, \overline{D})$, so we may conclude that ¹⁵¹ $\dim H_0(\overline{B_\omega}, \overline{D}) \geq \mathbf{rk} \ i$. \blacktriangleleft

¹⁵² While there is a surjective map from $H_0(\overline{B_\omega}, \overline{D})$ to $\mathbf{im} \ i$ this map is not necessarily ¹⁵³ induced by inclusion. We will therefore introduce a larger space $B_{\omega+c(\delta+\zeta)}$ that contains ¹⁵⁴ $Q_{\omega+c\delta}^\delta$ in order to provide a criteria for the injectivity of $\ell : H_0(\overline{B_\omega}, \overline{D}) \rightarrow H_0(Q_{\omega-c\zeta}^\delta, \overline{P^\delta})$ in ¹⁵⁵ terms of $\mathbf{rk} \ i$. We have the following commutative diagrams of inclusion maps and maps ¹⁵⁶ induced by inclusion between complements in \mathbb{X} .

$$\begin{array}{ccc} (P^\delta, Q_{\omega-c\zeta}^\delta) & \hookrightarrow & H_0(\overline{B_{\omega+c(\delta+\zeta)}}, \overline{D}) \xrightarrow{j} H_0(\overline{B_\omega}, \overline{D}) \\ \downarrow & & \downarrow m \\ (D, B_\omega) & \hookrightarrow & H_0(\overline{Q_{\omega+c\delta}^\delta}, \overline{P^\delta}) \xrightarrow{i} H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta}). \end{array} \quad (1)$$

158 Assumptions

¹⁵⁹ We will first require the map $H_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ to be *surjective*—as we approach ¹⁶⁰ ω from *above* no components *appear*. This ensures that the rank of the map j is equal to the ¹⁶¹ dimension of $\dim H_0(\overline{B_\omega}, \overline{D})$ so ℓ depends only on $H_0(\overline{B_\omega}, \overline{D})$ and $\mathbf{im} \ i$.

¹⁶² We also assume that $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is *injective*—as we move away from ω ¹⁶³ moving *down* no components *disappear*. Lemma 4 uses Assumption 2 to provide a computable ¹⁶⁴ upper bound on $\mathbf{rk} \ j$.

¹⁷⁰ ▶ **Lemma 4.** *If $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega+c(\delta+\zeta)})$ is injective and each component of $D \setminus B_\omega$ ¹⁷¹ contains a point in P then $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$.*

172 Nerves and Duality

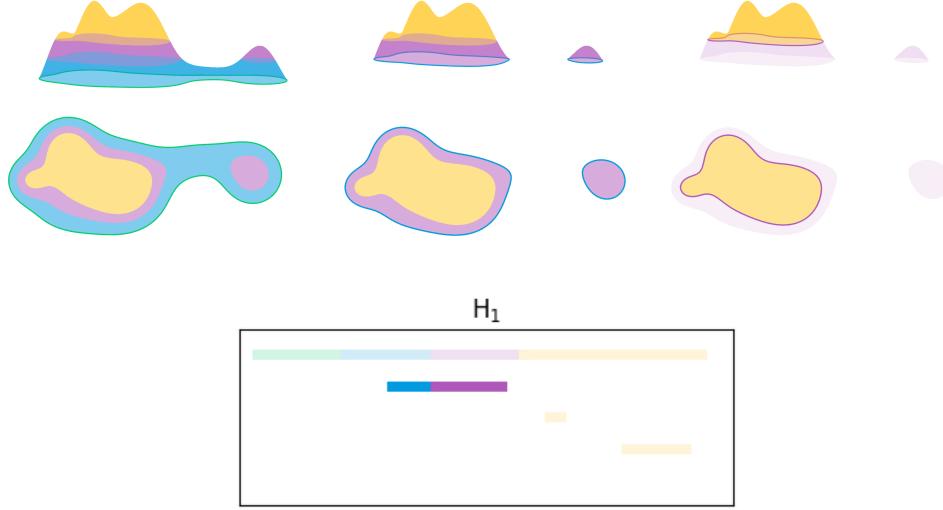
¹⁷⁵ Recall that the Nerve Theorem states that for a good open cover \mathcal{U} of a space X the inclusion ¹⁷⁶ map from the *Nerve* of the cover to the space $\mathcal{N}(\mathcal{U}) \hookrightarrow X$ is a homotopy equivalence.⁵ The ¹⁷⁷ Persistent Nerve Lemma [5] states that this homotopy equivalence commutes with inclusion ¹⁷⁸ on the level of homology. The standard proof of the Nerve Theorem [10], and therefore the ¹⁷⁹ Persistent Nerve Lemma [5], extends directly to pairs of good open covers $(\mathcal{U}, \mathcal{V})$ of pairs ¹⁸⁰ (X, Y) such that \mathcal{V} is a subcover of \mathcal{U} .⁶

¹⁸¹ Recalling the definition of the strong convexity radius ϱ_D (see Chazal et al. [3]) \mathcal{U} is a ¹⁸² good open cover whenever $\varrho_D > \varepsilon$. As the Čech complex is the Nerve of a cover by a union ¹⁸³ of balls we will let $\mathcal{N}_z^\varepsilon : H_k(\check{\mathcal{C}}^\varepsilon(P, Q_z)) \rightarrow H_k(P^\varepsilon, Q_z^\varepsilon)$ denote the isomorphism on homology ¹⁸⁴ provided by the Nerve Theorem for all $k, z \in \mathbb{R}$ and $\varepsilon < \varrho_D$.

¹⁷³ ⁵ In a good open cover every nonempty intersection of sets in the cover is contractible.

¹⁷⁴ ⁶ $\{V_i\}_{i \in I}$ is a subcover of $\{U_i\}_{i \in I}$ if $V_i \subseteq U_i$ for all $i \in I$.

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165 **Figure 1** The blue level set in the middle does not satisfy either assumption. The inclusion from
 166 the right is not *surjective* as the smaller component appears in the middle (in the sublevel barcode,
 167 a H_{d-1} feature dies in the purple region). The inclusion to the left is not *injective* as the smaller
 168 component is merged with the large (in the sublevel barcode, a H_{d-1} feature is born in the blue
 169 region).

185 Under certain conditions Alexander Duality provides an isomorphism between the k
 186 relative cohomology of a compact pair in an orientable d -manifold \mathbb{X} with the $d-k$ dimensional
 187 homology of their complements in \mathbb{X} (see Spanier [13]). For finitely generated (co)homology
 188 over a field the Universal Coefficient Theorem can be used with Alexander Duality to
 189 show $H_d(P^\varepsilon, Q_z^\varepsilon) \cong H_0(D \setminus Q_z^\varepsilon, D \setminus P^\varepsilon)$. This isomorphism holds in the specific case when
 190 $P^\varepsilon \subseteq \text{int}_{\mathbb{X}}(D)$ and $D \setminus P^\varepsilon, D \setminus Q_z^\varepsilon$ are locally contractible. We therefore provide the following
 191 definition for ease of exposition.

192 ▶ **Definition 5** ((ω, δ, ζ)-Sample). For $\zeta \geq \delta > 0$, $\omega \in \mathbb{R}$, and a c -Lipschitz function
 193 $f : D \rightarrow \mathbb{R}$ a finite point set $P \subset D$ is said to be an **(ω, δ, ζ) -sublevel sample** of f if

- 194
 - $P^\delta \subset \text{int}_{\mathbb{X}}(D)$ and
 - $D \setminus P^\delta, D \setminus Q_{\omega-c\zeta}^\delta$, and $D \setminus Q_{\omega+c\delta}^\delta$ are locally path connected in \mathbb{X} .

196 ▶ **Theorem 6** (Algorithmic TCC). Let \mathbb{X} be an orientable d -manifold and let D be a compact
 197 subset of \mathbb{X} . Let $f : D \rightarrow \mathbb{R}$ be c -Lipschitz function and $\omega \in \mathbb{R}$, $\delta \leq \zeta < \varrho_D$ be constants
 198 such that $B_{\omega-c(\zeta+\delta)}$ surrounds D in \mathbb{X} . Let P be an (ω, δ, ζ) -sample of f such that every
 199 component of $D \setminus B_\omega$ contains a point in P . Suppose $H_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ is
 200 surjective and $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is injective.

201 If $\text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-c\zeta})) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta}) \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta}))$ then $D \setminus B_\omega \subseteq P^\delta$
 202 and $Q_{\omega-c\zeta}^\delta$ surrounds P^δ in D .

203 **Proof.** Let $q : H_d(P^\delta, Q_{\omega-c\zeta}^\delta) \rightarrow H_d(P^\delta, Q_{\omega+c\delta}^\delta)$, $q_{\check{C}} : H_d(\check{C}^\delta(P, Q_{\omega-c\zeta})) \rightarrow H_d(\check{C}^\delta(P, Q_{\omega+c\delta}))$,
 204 and $q_{\mathcal{R}} : H_d(\mathcal{R}^\delta(P, Q_{\omega-c\zeta})) \rightarrow H_d(\mathcal{R}^{2\delta}(P, Q_{\omega+c\delta}))$ be induced by inclusion. Then $\text{rk } q_{\check{C}} \geq$
 205 $\text{rk } q_{\mathcal{R}}$ as $q_{\mathcal{R}}$ factors through $q_{\check{C}}$ by the Rips-Čech interleaving. Moreover, $\text{rk } q = \text{rk } q_{\check{C}}$ by the
 206 persistent nerve lemma, so $\text{rk } q \geq \text{rk } q_{\mathcal{R}}$. As we have assumed $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$
 207 Lemma 4 implies $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$. Because P is an (ω, δ, ζ) -sample
 208 we have $H_d(P^\delta, Q_{\omega-c\zeta}^\delta) \cong H_0(D \setminus Q_{\omega-c\zeta}^\delta, D \setminus P^\delta)$ and $H_d(P^\delta, Q_{\omega+c\delta}^\delta) \cong H_0(D \setminus Q_{\omega-2c\delta}^\delta, D \setminus P^\delta)$

so $\text{rk } i \geq \text{rk } q$ by Alexander Duality and the Universal Coefficient Theorem. So, by our hypothesis that $\text{rk } q_{\mathcal{R}} \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta}))$ we have $\text{rk } i \geq \dim H_0(D \setminus B_\omega)$.

Because $j : H_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ is surjective by hypothesis $\text{rk } j = \dim H_0(D \setminus B_\omega)$ so $\text{rk } j \geq \text{rk } i$ by Lemma 3. As we have shown $\text{rk } i \geq \dim H_0(D \setminus B_\omega)$ it follows that $\text{rk } j = \text{rk } i$. Because P is a finite point set we know that $\text{im } i$ is finite-dimensional and, because $\text{rk } i = \text{rk } j$, $\text{im } j = \overline{H_0(B_\omega, D)}$ is finite dimensional as well. So $\text{im } j$ is isomorphic to $\text{im } i$ as a subspace of $H_0(Q_{\omega-c\zeta}^\delta, P^\delta)$ which, because j is surjective, requires the map ℓ to be injective. Therefore $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-c\zeta}^\delta$ surrounds P^δ in D by Lemma 2. ◀

4 From Coverage Testing to the Analysis of Scalar Fields

Because the TCC only confirms coverage of a *superlevel* set $D \setminus B_\omega$, we cannot guarantee coverage of the entire domain. Indeed, we could compute the persistent homology of the *restriction* of f to the superlevel set we cover in the standard way [3]. Instead, we will approximate the persistent homology of the sublevel set filtration *relative to* the sublevel set B_ω . In the next section we will discuss an interpretation of the relative diagram that is motivated by examples in Section 6.

We will first introduce the notion of an extension which will provide us with maps on relative homology induced by inclusion via excision. However, even then, a map that factors through our pair (D, B_ω) is not enough to prove an interleaving of persistence modules by inclusion directly. To address this we impose conditions on sublevel sets near B_ω which generalize the assumptions made in the TCC.

4.1 Extensions and Image Persistence Modules

Suppose D is a subspace of X . We define the extension of a surrounding pair in D to a surrounding pair in X with isomorphic relative homology.

► **Definition 7** (Extension). *If V surrounds U in a subspace D of X let $\mathcal{EV} := V \sqcup (D \setminus U)$ denote the (disjoint) union of the separating set V with the complement of U in D . The extension of (U, V) in D is the pair $(D, \mathcal{EV}) = (U \sqcup (D \setminus U), V \sqcup (D \setminus U))$.*

Lemma 8 states that we can use these extensions to interleave a pair (U, V) with a sequence of subsets of (D, B) . Lemma 9 states that we can apply excision to the relative homology groups in order to get equivalent maps on homology that are induced by inclusions.

► **Lemma 8.** *Suppose V surrounds U in D and $B' \subseteq B \subset D$. If $D \setminus B \subseteq U$ and $U \cap B' \subseteq V \subseteq B'$ then $B' \subseteq \mathcal{EV} \subseteq B$.*

► **Lemma 9.** *Let (U, V) be an open surrounding pair in a subspace D of X . Then $H_k((U \cap A, V) \hookrightarrow (A, \mathcal{EV}))$ is an isomorphism for all k and $A \subseteq D$ with $\mathcal{EV} \subset A$.*

The TCC uses a nested pair of spaces in order to filter out noise introduced by the sample. This same technique is used to approximate the persistent homology of a scalar fields [3]. As modules, these nested pairs are the images of homomorphisms between homology groups induced by inclusion, which we refer to as image persistence modules. For a full background on persistence modules, shifted homomorphisms, and interleavings of persistence modules see Chazal et al. [4].

► **Definition 10** (Image Persistence Module). *The **image persistence module** of a homomorphism $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$ is the family of subspaces $\{\Gamma_\alpha := \text{im } \gamma_\alpha\}$ in \mathbb{V} along with linear maps $\{\gamma_\alpha^\beta := v_\alpha^\beta|_{\text{im } \gamma_\alpha} : \Gamma_\alpha \rightarrow \Gamma_\beta\}$ and will be denoted by $\text{im } \Gamma$.*

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For a homomorphism $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$ let $\Gamma[\delta] \in \text{Hom}^\delta(\mathbb{U}, \mathbb{V})$ denote the shifted homomorphism defined to be the family of linear maps $\{\gamma_\alpha[\delta] := v_\alpha^\delta \circ \gamma_\alpha : U_\alpha \rightarrow V_{\alpha+\delta}\}$. While we will primarily work with homomorphisms of persistence modules induced by inclusions, in general, defining homomorphisms between images simply as subspaces of the codomain is not sufficient. Instead, we require that homomorphisms between image modules commute not only with shifts in scale, but also with the functions themselves.

► **Definition 11** (Image Module Homomorphism). *Given $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$ and $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$ along with $(F, G) \in \text{Hom}^\delta(\mathbb{U}, \mathbb{S}) \times \text{Hom}^\delta(\mathbb{V}, \mathbb{T})$ let $\Phi(F, G) : \text{im } \Gamma \rightarrow \text{im } \Lambda$ denote the family of linear maps $\{\phi_\alpha := g_\alpha|_{\Gamma_\alpha} : \Gamma_\alpha \rightarrow \Lambda_{\alpha+\delta}\}$. $\Phi(F, G)$ is an **image module homomorphism of degree δ** if the following diagram commutes for all $\alpha \leq \beta$.⁷*

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\gamma_\alpha[\beta-\alpha]} & V_\beta \\ \downarrow f_\alpha & & \downarrow g_\beta \\ S_{\alpha+\delta} & \xrightarrow{\lambda_{\alpha+\delta}[\beta-\alpha]} & T_{\beta+\delta} \end{array} \quad (2)$$

The space of image module homomorphisms of degree δ between $\text{im } \Gamma$ and $\text{im } \Lambda$ will be denoted $\text{Hom}^\delta(\text{im } \Gamma, \text{im } \Lambda)$.

► **Lemma 12.** *Suppose $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$, $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$, and $\Lambda' \in \text{Hom}(\mathbb{S}', \mathbb{T}')$. If $\Phi(F, G) \in \text{Hom}^\delta(\text{im } \Gamma, \text{im } \Lambda)$ and $\Phi'(F', G') \in \text{Hom}^{\delta'}(\text{im } \Lambda, \text{im } \Lambda')$ then $\Phi''(F' \circ F, G' \circ G) := \Phi' \circ \Phi \in \text{Hom}^{\delta+\delta'}(\text{im } \Gamma, \text{im } \Lambda')$.*

269 Partial Interleavings of Image Modules

Image module homomorphisms introduce a direction to the traditional notion of interleaving. As we will see, our interleaving via Lemma 14 involves partially interleaving an image module to two other image modules whose composition is isomorphic to our target.

► **Definition 13** (Partial Interleaving of Image Modules). *An image module homomorphism $\Phi(F, G)$ is a **partial δ -interleaving of image modules**, and denoted $\Phi_M(F, G)$, if there exists $M \in \text{Hom}^\delta(\mathbb{S}, \mathbb{V})$ such that $\Gamma[2\delta] = M \circ F$ and $\Lambda[2\delta] = G \circ M$.*

Lemma 14 uses partial interleavings of a map Λ with $\mathbb{U} \rightarrow \mathbb{V}$ and $\mathbb{V} \rightarrow \mathbb{W}$ along with the hypothesis that $\mathbb{U} \rightarrow \mathbb{W}$ is isomorphic to \mathbb{V} to interleave $\text{im } \Lambda$ with \mathbb{V} . When applied, this hypothesis will be satisfied by assumptions on our sublevel set similar to those made in the TCC.

► **Lemma 14.** *Suppose $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$, $\Pi \in \text{Hom}(\mathbb{V}, \mathbb{W})$, and $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$.*

If $\Phi_M(F, G) \in \text{Hom}^\delta(\text{im } \Gamma, \text{im } \Lambda)$ and $\Psi_G(M, N) \in \text{Hom}^\delta(\text{im } \Lambda, \text{im } \Pi)$ are partial δ -interleavings of image modules such that Γ is a epimorphism and Π is a monomorphism then $\text{im } \Lambda$ is δ -interleaved with \mathbb{V} .

284 4.2 Proof of the Interleaving

For $z, \alpha \in \mathbb{R}$ let \mathbb{D}_z^k denote the k th persistent (relative) homology module of the filtration $\{(D_{\lfloor \alpha \rfloor z}, B_z)\}_{\alpha \in \mathbb{R}}$ with respect to B_z , and let $\mathbb{P}_z^{e,k}$ denote the k th persistent (relative) homology module of $\{(P_{\lfloor \alpha \rfloor z}^e, Q_z^e)\}_{\alpha \in \mathbb{R}}$. Similarly, let $\check{C}\mathbb{P}_z^{e,k}$ and $\mathcal{R}\mathbb{P}_z^{e,k}$ denote the corresponding

⁷ We use the notation $\gamma_\alpha[\beta-\alpha] = v_\alpha^\beta \circ \gamma_\alpha$, $\lambda_\alpha[\beta-\alpha] = t_\alpha^\beta \circ \lambda_\alpha$ to denote the composition of homomorphisms between persistence modules and shifts in scale.

288 Čech and Vietoris-Rips filtrations, respectively. We will omit the dimension k and write \mathbb{D}_z
 289 (resp. \mathbb{P}_z^ε) if a statement holds for all dimensions. If Q_z^δ surrounds P^δ in D let $\mathcal{EP}_z^\varepsilon$ denote
 290 the k th persistent homology module of the filtration of extensions $\{(\mathcal{EP}_{\lfloor \alpha \rfloor z}^\varepsilon, \mathcal{EQ}_z^\varepsilon)\}$ for any
 291 $\varepsilon \geq \delta$, where $\mathcal{EP}_{\lfloor \alpha \rfloor z}^\varepsilon = P_{\lfloor \alpha \rfloor z}^\varepsilon \cup (D \setminus P^\delta)$.

292 Lemma 15 follows directly from the definition of truncated sublevel sets. This is used
 293 to extend Lemma 8 to persistence modules in Lemma 16 in order to provide a sequence of
 294 shifted homomorphisms $\mathbb{D}_{\omega-3c\delta} \xrightarrow{F} \mathcal{EP}_{\omega-2c\delta}^\varepsilon \xrightarrow{M} \mathbb{D}_\omega \xrightarrow{G} \mathcal{EP}_{\omega+c\delta}^{2\varepsilon} \xrightarrow{N} \mathbb{D}_{\omega+5c\delta}$ of varying degree.
 295 These homomorphisms are then combined with those given by the Nerve theorem and the
 296 Rips-Čech interleaving in Lemma 17 to obtain partial interleavings required for our proof of
 297 Theorem 18.

298 ▶ **Lemma 15.** *If $\delta \leq \varepsilon$ and $t, \alpha \in \mathbb{R}$ then $P^\delta \cap D_{\lfloor \alpha - c\varepsilon \rfloor t - c\varepsilon} \subseteq P_{\lfloor \alpha \rfloor t}^\varepsilon \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor t + c\varepsilon}$.*

299 ▶ **Lemma 16.** *Let $s + 3c\delta \leq t + 2c\delta \leq u \leq v - c\delta \leq w - 5c\delta$ and $\varepsilon \in [\delta, 2\delta]$. If Q_t^δ surrounds
 300 P^δ in D and $D \setminus B_u \subseteq P^\delta$ then the following homomorphisms are induced by inclusions:*

$$301 (F, G) \in \text{Hom}^{c\delta}(\mathbb{D}_s, \mathcal{EP}_t^\varepsilon) \times \text{Hom}^{2c\delta}(\mathbb{D}_u, \mathcal{EP}_v^{2\varepsilon}), (M, N) \in \text{Hom}^{c\varepsilon}(\mathcal{EP}_t^\varepsilon, \mathbb{D}_u) \times \text{Hom}^{2c\varepsilon}(\mathcal{EP}_v^{2\varepsilon}, \mathbb{D}_w).$$

302 ▶ **Lemma 17.** *For $\delta < \varrho_D$ and $s + 3c\delta \leq t + 2c\delta \leq u \leq v - c\delta \leq w - 5c\delta$ let $\Gamma \in \text{Hom}(\mathbb{D}_s, \mathbb{D}_u)$,
 303 $\Pi \in \text{Hom}(\mathbb{D}_u, \mathbb{D}_w)$, and $\Lambda \in \text{Hom}(\mathcal{RP}_t^{2\delta}, \mathcal{RP}_v^{4\delta})$ be induced by inclusions.*

304 *If Q_t^δ surrounds P^δ in D and $D \setminus B_u \subseteq P^\delta$ then there is a partial $2c\delta$ interleaving
 305 $\Phi^* \in \text{Hom}^{2c\delta}(\mathbf{im} \Gamma, \mathbf{im} \Lambda)$ and a partial $4c\delta$ interleaving $\Psi^* \in \text{Hom}^{4c\delta}(\mathbf{im} \Lambda, \mathbf{im} \Pi)$.*

306 **Proof.** Because the shifted homomorphisms provided by Lemma 16 are all induced by
 307 inclusions the following diagram commutes for all $\alpha \leq \beta$. So we have image module
 308 homomorphisms $\Phi(F, G) \in \text{Hom}^{2c\delta}(\mathbf{im} \Gamma, \mathbf{im} C \circ A)$ and $\Psi(M, N) \in \text{Hom}^{4c\delta}(\mathbf{im} E \circ C, \mathbf{im} \Pi)$.

$$\begin{array}{ccccc} H_k(D_{\lfloor \alpha - 2c\delta \rfloor s}, B_s) & \xrightarrow{f_{\alpha-2c\delta}} & H_k(\mathcal{EP}_{\lfloor \alpha \rfloor t}^\delta, \mathcal{EQ}_t^\delta) & H_k(\mathcal{EP}_{\lfloor \alpha \rfloor t}^{2\delta}, \mathcal{EQ}_t^{2\delta}) & \xrightarrow{m_\alpha} H_k(D_{\lfloor \alpha + 4c\delta \rfloor u}, B_u) \\ \downarrow \gamma_{\alpha-2c\delta}[\beta-\alpha] & & \downarrow c_\alpha[\beta-\alpha] \circ a_\alpha & \downarrow e_\beta \circ c_\alpha[\beta-\alpha] & \downarrow \gamma_{\alpha+4c\delta}[\beta-\alpha] \\ H_k(D_{\lfloor \beta - 2c\delta \rfloor u}, B_u) & \xrightarrow{g_{\beta-2c\delta}} & H_k(\mathcal{EP}_{\lfloor \beta \rfloor v}^{2\delta}, \mathcal{EQ}_v^{2\delta}) & H_k(\mathcal{EP}_{\lfloor \beta \rfloor v}^{4\delta}, \mathcal{EQ}_v^{4\delta}) & \xrightarrow{n_\beta} H_k(D_{\lfloor \beta + 4c\delta \rfloor w}, B_w) \end{array}$$

310 Because the isomorphisms provided by Lemma 9 are given by excision they are induced
 311 by inclusion, and therefore give isomorphisms $\mathcal{E}_z^\varepsilon \in \text{Hom}(\mathbb{P}_z^\varepsilon, \mathcal{EP}_z^\varepsilon)$ for any $z \in \mathbb{R}$ such that Q_z^ε
 312 surrounds P^δ in D . For any $\varepsilon < \varrho_D$ we have isomorphisms $\mathcal{N}_z^\varepsilon \in \text{Hom}(\check{\mathcal{C}}\mathbb{P}_z^\varepsilon, \mathbb{P}_z^\varepsilon)$ that commute
 313 with maps induced by inclusions by the Persistent Nerve Lemma. So the compositions
 314 $\mathcal{E}_z^\varepsilon \circ \mathcal{N}_z^\varepsilon$ are isomorphisms that commute with maps induced by inclusion as well. These
 315 compositions, along with the Rips-Čech interleaving, provide maps $\mathcal{EP}_t^\delta \xrightarrow{F'} \mathcal{RP}_t^{2\delta} \xrightarrow{M'} \mathcal{EP}_t^{2\delta}$
 316 and $\mathcal{EP}_v^{2\delta} \xrightarrow{G'} \mathcal{RP}_v^{4\delta} \xrightarrow{N'} \mathcal{EP}_v^{4\delta}$ that commute with maps induced by inclusions. So we have
 317 the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{EP}_t^\delta & \xrightarrow{A} & \mathcal{EP}_t^{2\delta} & \xrightarrow{C} & \mathcal{EP}_v^{2\delta} & \xrightarrow{E} & \mathcal{EP}_v^{4\delta} \\ \searrow F' & \nearrow M' & \nearrow & & \searrow G' & \nearrow N' & \nearrow \\ \mathcal{RP}_t^{2\delta} & \xrightarrow{\Lambda} & \mathcal{RP}_v^{4\delta} & & & & \end{array} \quad (3)$$

319 That is, we have image module homomorphisms $\Phi'(F', G')$ and $\Psi'(M', N')$ such that $A =$
 320 $M' \circ F'$, $E = N' \circ G'$, and $\Lambda = G' \circ C \circ M'$. Because image module homomorphisms compose
 321 we have we have $\Phi^* = \Phi' \circ \Phi \in \text{Hom}^{2c\delta}(\mathbf{im} \Gamma, \mathbf{im} \Lambda)$ and $\Psi^* = \Psi \circ \Psi' \in \text{Hom}^{4c\delta}(\mathbf{im} \Lambda, \mathbf{im} \Pi)$.

322 Because G, M, C are induced by inclusions $C[3c\delta] = G \circ M$, so $\Lambda[3c\delta] = G' \circ C[3c\delta] \circ M' =$
 323 $G' \circ (G \circ M) \circ M'$ as G', M' commute with maps induced by inclusions. In the same way,
 324 $\Gamma[3c\delta] = M \circ (A \circ F) = M \circ (M' \circ F') \circ F$ and $\Pi[5c\delta] = N \circ (E \circ G) = N \circ (N' \circ G') \circ G$.

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325 Let $F^* := F' \circ F$, $G^* := G' \circ G$, $M^* := M' \circ M$, and $N^* := N' \circ N$. So $\Phi_{M^*}^*$ is a
 326 partial $2c\delta$ interleaving as $\Gamma[3c\delta] = M^* \circ F^*$ and $\Lambda[3c\delta] = G^* \circ M^*$, and $\Psi_{G^*}^*$ is a partial $4c\delta$
 327 interleaving as $\Lambda[3c\delta] = G^* \circ M^*$ and $\Pi[5c\delta] = N^* \circ G^*$. \blacktriangleleft

328 The partial interleavings given by Lemma 17, along with assumptions that imply
 329 $\text{im}(\mathbb{D}_{\omega-3c\delta} \rightarrow \mathbb{D}_{\omega+5c\delta}) \cong \mathbb{D}_\omega$, provide the proof of Theorem 18 by Lemma 14.

330 ▶ **Theorem 18.** *Let \mathbb{X} be a d -manifold, $D \subset \mathbb{X}$ and $f : D \rightarrow \mathbb{R}$ be a c -Lipschitz function.
 331 Let $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $B_{\omega-3c\delta}$ surrounds D in \mathbb{X} . Let $P \subset D$ be
 332 a finite subset and suppose $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $H_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an
 333 isomorphism for all k .*

334 *If $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D then the k th persistent homology
 335 module of $\{\mathcal{R}^{2\delta}(P_{\lfloor \alpha \rfloor \omega-2c\delta}, Q_{\omega-2c\delta}) \rightarrow \mathcal{R}^{4\delta}(P_{\lfloor \alpha \rfloor \omega+c\delta}, Q_{\omega+c\delta})\}_{\alpha \in \mathbb{R}}$ is $4c\delta$ -interleaved with that
 336 of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$.*

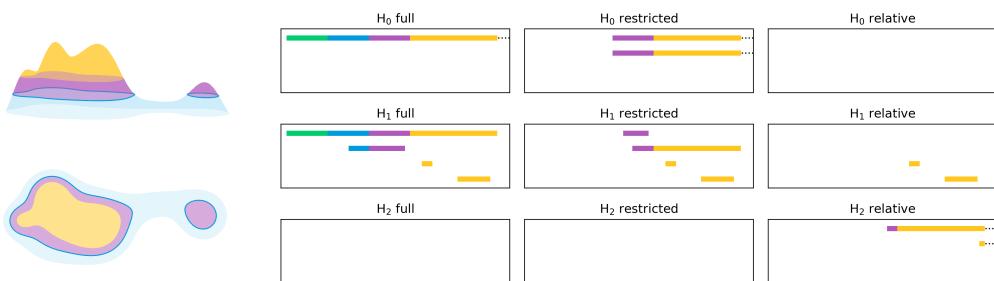
337 **Proof.** Let $\Lambda \in \text{Hom}(\mathcal{R}\mathbb{P}_{\omega-2c\delta}^{2c\delta}, \mathcal{R}\mathbb{P}_{\omega+c\delta}^{4c\delta})$, $\Gamma \in \text{Hom}(\mathbb{D}_{\omega-3c\delta}, \mathbb{D}_\omega)$, and $\Pi \in \text{Hom}(\mathbb{D}_\omega, \mathbb{D}_{\omega+5c\delta})$
 338 be induced by inclusions. Because $\delta < \varrho_D/4$, $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D
 339 we have a partial $2c\delta$ interleaving $\Phi^* \in \text{Hom}^{2c\delta}(\text{im } \Gamma, \text{im } \Lambda)$ and a partial $4c\delta$ interleaving
 340 $\Psi^* \in \text{Hom}^{4c\delta}(\text{im } \Lambda, \text{im } \Pi)$ by Lemma 17. As we have assumed that $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$
 341 is surjective and $H_k(B_\omega) \cong H_k(B_{\omega+5c\delta})$ the five-lemma implies γ_α is surjective and π_α is
 342 an isomorphism (and therefore injective) for all α . So Γ is an epimorphism and Π is a
 343 monomorphism, thus $\text{im } \Lambda$ is $4c\delta$ -interleaved with \mathbb{D}_ω by Lemma 14 as desired. \blacktriangleleft

344 5 Approximation of the Truncated Diagram

345 Relative, Truncated, and Restricted Persistence Diagrams

346 For fixed $\omega \in \mathbb{R}$ we will refer to the persistence diagram associated with the filtration
 347 $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ as the **relative diagram** of f . In this section we will relate the relative
 348 diagram to the *full* diagram of the sublevel set filtration $\{B_\alpha\}_{\alpha \in \mathbb{R}}$. Specifically, we define
 349 the **truncated diagram** to be the subdiagram consisting of features born *after* ω in the
 350 full. In the following section we will compare the relative and truncated diagrams to the
 351 **restricted diagram**, defined to be that of the sublevel set filtration of $f|_{D \setminus B_\omega}$.

352 Note that the truncated sublevel sets $D_{\lfloor \alpha \rfloor \omega}$ are equal to the union of B_ω and the restricted
 353 sublevel sets. It is in this sense that B_ω is *static* throughout—it is contained in every sublevel
 354 set of the relative filtration. As we will not have verified coverage in B_ω we cannot analyze
 355 the function in this region directly. We therefore have two alternatives: *restrict* the domain
 356 of the function to $D \setminus B_\omega$, or use relative homology to analyze the function *relative* to this
 357 region using excision.



358 ■ **Figure 2** Full, restricted, and relative barcodes of the function (left).

Let \mathbb{L}^k denote the k th persistent homology module of the sublevel set filtration $\{B_\alpha\}_{\alpha \in \mathbb{R}}$. As in the previous section, let \mathbb{D}_ω^k denote the k th persistent (relative) homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$. Throughout we will assume that we are taking homology in a field \mathbb{F} and that the homology groups $H_k(B_\alpha)$ and $H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega)$ are finite dimensional vector spaces for all k and $\alpha \in \mathbb{R}$. We will use the interval decomposition of \mathbb{L}^k to give a decomposition of the relative module \mathbb{D}_ω^k in terms of a *truncation* of \mathbb{L}^k . Recall, the *truncated diagram* is defined to be that of \mathbb{L}^k consisting only of those features born after ω . For fixed $\omega \in \mathbb{R}$ we will define the truncation \mathbb{T}_ω^k of \mathbb{L}^k in terms of the intervals decomposing \mathbb{L}^k that are in $[\omega, \infty)$.

Truncated Interval Modules

For an interval $I = [s, t] \subseteq \mathbb{R}$ let $I_+ := [t, \infty)$ and $I_- := (-\infty, s]$. For $\omega \in \mathbb{R}$ let \mathbb{F}_ω^I denote the interval module consisting of vector spaces $\{F_{\lfloor \alpha \rfloor \omega}^I\}_{\alpha \in \mathbb{R}}$ and linear maps $\{f_{\lfloor \alpha, \beta \rfloor \omega}^I : F_{\lfloor \alpha \rfloor \omega}^I \rightarrow F_{\lfloor \beta \rfloor \omega}^I\}_{\alpha \leq \beta}$ where

$$F_{\lfloor \alpha \rfloor \omega}^I := \begin{cases} F_\alpha^I & \text{if } \omega \in I_- \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{\lfloor \alpha, \beta \rfloor \omega}^I := \begin{cases} f_{\alpha, \beta}^I & \text{if } \omega \in I_- \\ 0 & \text{otherwise.} \end{cases}$$

For a collection \mathcal{I} of intervals let $\mathcal{I}_\omega := \{I \in \mathcal{I} \mid \omega \in I\}$.

► **Lemma 19.** Suppose \mathcal{I}^k and \mathcal{I}^{k-1} are collections of intervals that decompose \mathbb{L}^k and \mathbb{L}^{k-1} , respectively. Then for all k the k th persistent homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ is equal to

$$\bigoplus_{I \in \mathcal{I}^k} \mathbb{F}_\omega^I \oplus \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} \mathbb{F}^{I+}.$$

Letting \mathcal{I}^k denote the decomposing intervals of \mathbb{L}^k for all k we can define the ω -truncated k th persistent homology module of \mathbb{L}^k as

$$\mathbb{T}_\omega^k := \bigoplus_{I \in \mathcal{I}^k} \mathbb{F}_\omega^I \quad \text{and let} \quad \mathbb{L}_\omega^{k-1} := \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} \mathbb{F}^{I+}$$

denote the submodule of \mathbb{D}_ω^k consisting of intervals $[\beta, \infty)$ corresponding to features $[\alpha, \beta)$ in \mathbb{L}^{k-1} such that $\alpha \leq \omega < \beta$. Now, by Lemma 19 the k th persistent (relative) homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ is $\mathbb{D}_\omega^k = \mathbb{T}_\omega^k \oplus \mathbb{L}_\omega^{k-1}$. Theorems 6 and 18 can then be used to show that

$$\{R^{2\delta}(P_{\lfloor \alpha \rfloor \omega - 2c\delta}, Q_{\omega - 2c\delta}) \hookrightarrow R^{4\delta}(P_{\lfloor \alpha \rfloor \omega + c\delta}, Q_{\omega + c\delta})\}_{\alpha \in \mathbb{R}}$$

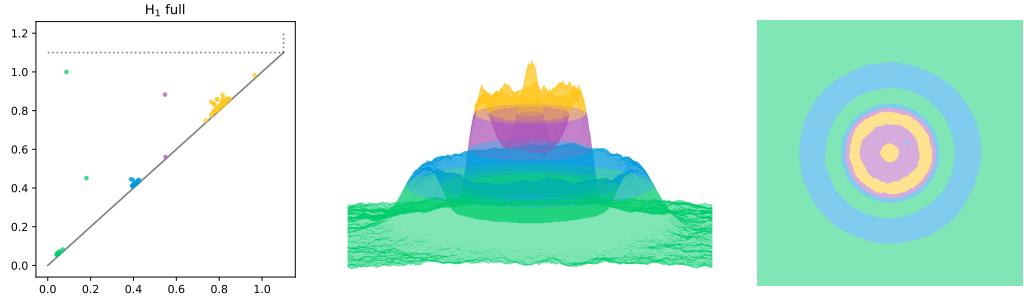
is $4c\delta$ interleaved with $\mathbb{T}_\omega^k \oplus \mathbb{L}_\omega^{k-1}$ whenever

$$\text{rk } H_d(R^\delta(P, Q_{\omega - 2c\delta}) \hookrightarrow R^{2\delta}(P, Q_{\omega + c\delta})) \geq \dim H_0(R^\delta(P \setminus Q_{\omega - 2c\delta})).$$

6 Experiments

In this section we will discuss a number of experiments which illustrate the benefit of truncated diagrams, and their approximation by relative diagrams, in comparison to their restricted counterparts. We will focus on the persistent homology of functions on a square 2d grid. We chose as our function a radially symmetric damped sinusoid with random noise, depicted in Figure 3, as it has prominent persistent homology in dimension one.

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394 ■ **Figure 3** The H_1 persistence diagram of the sinusoidal function pictured to the right. Features
395 are colored by birth time, infinite features are drawn above the dotted line.

393 **Experimental setup.**

397 Throughout, the four interlevel sets shown correspond to the ranges $[0, 0.3]$, $[0.3, 0.5]$, $[0.5, 0.7]$,
398 and $[0.7, 1]$, respectively. Our persistent homology computations were done primarily with
399 Dionysus augmented with custom software for computing representative cycles of infinite
400 features.⁸ The persistent homology of our function was computed with the lower-star
401 filtration of the Freudenthal triangulation on an $N \times N$ grid over $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. We
402 take this filtration as $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ where P is the set of grid points and $\delta = \sqrt{2}/N$.

403 We note that the purpose of these experiments is not to demonstrate the effectiveness of our
404 approximation by Rips complexes, but to demonstrate the relationships between restricted,
405 relative, and truncated diagrams. Therefore, for simplicity, we will omit the inclusion
406 $\mathcal{R}^{2\delta}(P_\alpha) \hookrightarrow \mathcal{R}^{4\delta}(P_\alpha)$ and take the persistent homology of $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ with sufficiently small
407 δ as our ground-truth.

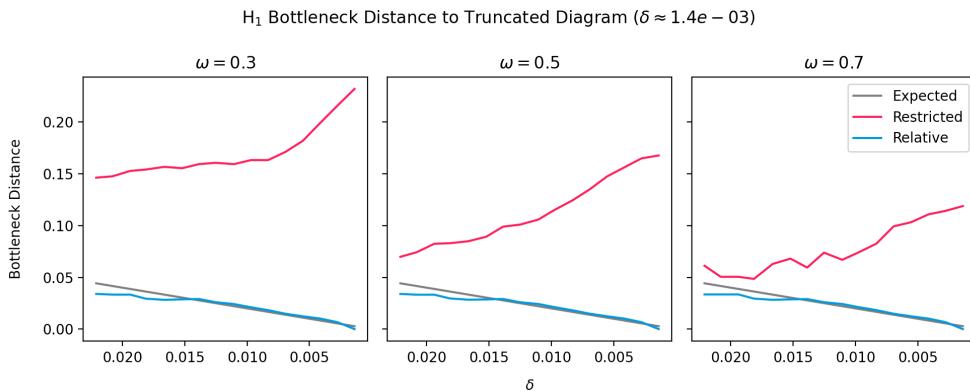
408 In the following we will take $N = 1024$, so $\delta \approx 1.4 \times 10^{-3}$, as our ground-truth. Figure 3
409 shows the *full diagram* of our function with features colored by birth time. Therefore, for
410 $\omega = 0.3, 0.5, 0.7$ the *truncated diagram* is obtained by successively removing features in
411 each interlevel set. Recall the *restricted diagram* is that of the function restricted to the ω
412 *super-levelset* filtration, and computed with $\{\mathcal{R}^{2\delta}(P_\alpha \setminus Q_\omega)\}$. We will compare this restricted
413 diagram with the *relative diagram*, computed as the relative persistent homology of the
414 filtration of pairs $\{\mathcal{R}^{2\delta}(P_\alpha, Q_\omega)\}$.

415 **The issue with restricted diagrams.**

416 Figure 4 shows the bottleneck distance from the truncated diagram at full resolution ($N =$
417 1024) to both the relative and restricted diagrams with varying resolution. Specifically, the
418 function on a 1024×1024 grid is down-sampled to grids ranging from 64×64 to 1024×1024 .
419 We also show the expected bottleneck distance to the true truncated diagram given by the
420 interleaving in Theorem 18 in black.

423 As we can see, the relative diagram clearly performs better than the restricted diagram,
424 which diverges with increasing resolution. Recall that 1-dimensional features that are born
425 before ω and die after ω become infinite 2-dimensional features in the relative diagram, with
426 birth time equal to the death time of the corresponding feature in the full diagram. These
427 same features remain 1-dimensional figures in the restricted diagram, but with their birth

396 ⁸ 3D figures were made with Mayavi, all other figures were made with Matplotlib.



421 ■ **Figure 4** Comparison of the bottleneck distance between the truncated diagram and those of the
422 restricted and relative diagrams with increasing resolution.

428 times shifted to ω .

433 Figure 5 shows this distance for a feature that persists throughout the diagram. As the
434 restricted diagram in full resolution the restricted filtration is a subset of the full filtration,
435 so these features can be matched by their death simplices. For illustrative purposes we also
436 show the representative cycles associated with these features.

437 **Relative diagrams and reconstruction.**

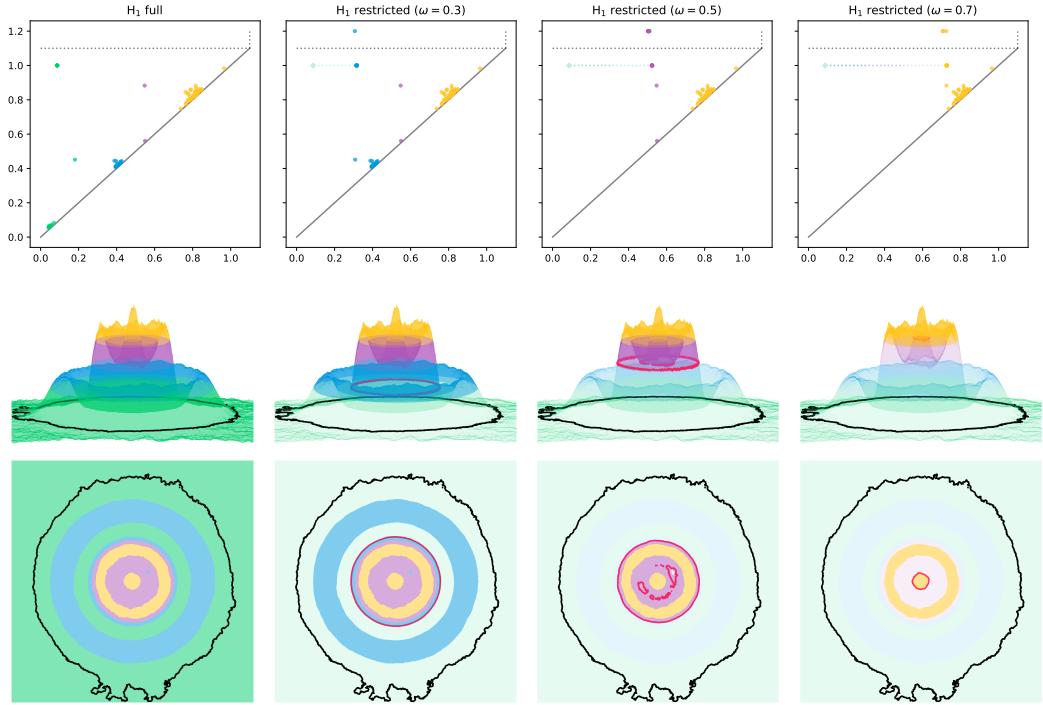
443 Now, imagine we obtain the persistence diagram of our sub-levelset B_ω . That is, we now
444 know that we cover B_ω , or some subset, and do not want to re-compute the diagram above
445 ω . If we compute the persistence diagram of the function restricted to the *sub*-levelset B_ω
446 any 1-dimensional features born before ω that die after ω will remain infinite features in
447 this restricted (below) diagram. Indeed, we could match these infinite 1-features with the
448 corresponding shifted finite 1-features in the restricted (above) diagram, as shown in Figure 5.
449 However, that would require sorting through all finite features that are born near ω and
450 deciding if they are in fact features of the full diagram that have been shifted.

451 Recalling that these same features become infinite 2-features in the relative diagram, we
452 can use the relative diagram instead and match infinite 1-features of the diagram restricted
453 below to infinite 2-features in the relative diagram, as shown in Figures 6 and 7. For this
454 example the matching is given by sorting the 1-features by ascending and the 2-features by
455 descending birth time. How to construct this matching in general, especially in the presence
456 of infinite features in the full diagram, is the subject of future research.

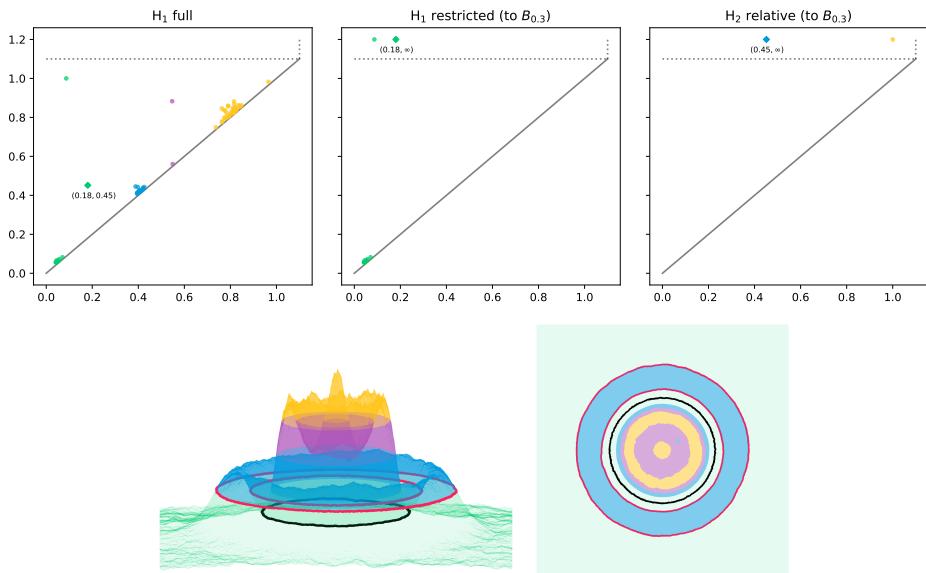
460 **7 Conclusion**

461 We have extended the Topological Coverage Criterion to the setting of Topological Scalar
462 Field Analysis. By defining the boundary in terms of a sublevel set of a scalar field we
463 provide an interpretation of the TCC that applies more naturally to data coverage. We then
464 showed how the assumptions and machinery of the TCC can be used to approximate the
465 persistent homology of the scalar field relative to a static sublevel set. This relative persistent
466 homology is shown to be related to a truncation of that of the scalar field as whole, and
467 therefore provides a way to approximate a part of its persistence diagram in the presence of
468 un-verified data.

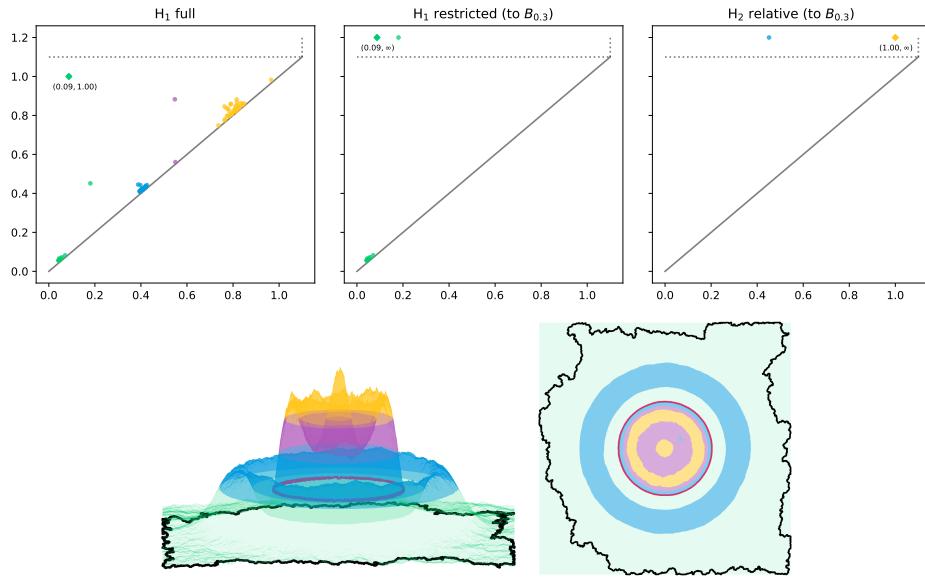
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429 ■ **Figure 5** (Top) H_1 persistence diagrams of the function depicted in Figure 3 restricted to *super-*
430 *levelsets* at $\omega = 0.3, 0.5$, and 0.7 (on a 1024×1024 grid). The matching is shown between a feature in
431 the full diagram (marked with a diamond) with its representative cycle in black. The corresponding
432 representative cycle in the restricted diagram is pictured in red.



438 ■ **Figure 6** (Top) The indicated infinite features in the restricted and relative diagrams correspond
439 to the birth and death of the 1-feature $(0.18, 0.45)$ in the full diagram. (Bottom) In black, the
440 representative cycle of the infinite 1-feature born at 0.18 in the restricted diagram is shown in black.
441 In red, the *boundary* of the representative relative 2-cycle born at 0.45 in the relative diagram is
442 shown in red.



457 ■ **Figure 7** The infinite 1-features of the restricted diagram can be matched with the infinite
 458 2-features of the relative diagrams. The sequence birth times of relative 2-features in *decreasing*
 459 order correspond to the deaths of restricted 1-features in *increasing* order.

469 There are a number of unanswered questions and directions for future work. From the
 470 theoretical perspective, our understanding of duality limited us in providing a more elegant
 471 extension of the TCC. A better understanding of when and how duality can be applied would
 472 allow us to give a more rigorous statement of our assumptions. Moreover, as duality plays
 473 a central role in the TCC it is natural to investigate its role in the analysis of scalar fields.
 474 This would not only allow us to apply duality to persistent homology [9], but also allow us
 475 to provide a rigorous comparison between the relative approach and the persistent homology
 476 of the superlevel set filtration and explore connections with Extended Persistence [6].

477 From a computational perspective, we interested in exploring how to recover the full
 478 diagram as discussed in Section 6. Our statements in terms of sublevel sets can be generalized
 479 to disjoint unions of sub and superlevel sets, where coverage is confirmed in an *interlevel*
 480 set. This, along with a better understanding of the relationship between sub and superlevel
 481 sets could lead to an iterative approach in which the persistent homology of a scalar field is
 482 constructed as data becomes available. We are also interested in finding efficient ways to
 483 compute the image persistent (relative) homology that vary in both scalar and scale.

484 The problem of relaxing our assumptions on the boundary can be approached from both
 485 a theoretical and computational perspective. Ways to avoid the isomorphism we require
 486 could be investigated in theory, and the interaction of relative persistent homology and the
 487 Persistent Nerve Lemma may be used tighten our assumptions. We would also like to conduct
 488 a more rigorous investigation on the effect of these assumptions in practice.

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523 **A** Omitted Proofs

524 **Proof of Lemma 2.** This proof is in two parts.

525 **ℓ injective $\implies D \setminus B \subseteq U$** Suppose, for the sake of contradiction, that p is injective and
 526 there exists a point $x \in (D \setminus B) \setminus U$. Because B surrounds D in X the pair $(D \setminus B, \overline{D})$
 527 forms a separation of \overline{B} . Therefore, $H_0(\overline{B}) \cong H_0(D \setminus B) \oplus H_0(\overline{D})$ so

$$528 \quad H_0(\overline{B}, \overline{D}) \cong H_0(D \setminus B).$$

529 So $[x]$ is non-trivial in $H_0(\overline{B}, \overline{D}) \cong H_0(D \setminus B)$ as x is in some connected component of
 530 $D \setminus B$. So we have the following sequence of maps induced by inclusions

$$531 \quad H_0(\overline{B}, \overline{D}) \xrightarrow{f} H_0(\overline{B}, \overline{D} \cup \{x\}) \xrightarrow{g} H_0(\overline{V}, \overline{U}).$$

532 As $f[x]$ is trivial in $H_0(\overline{B}, \overline{D} \cup \{x\})$ we have that $\ell[x] = (g \circ f)[x]$ is trivial, contradicting
 533 our hypothesis that ℓ is injective.

534 **ℓ injective $\implies V$ surrounds U in D .** Suppose, for the sake of contradiction, that V does
 535 not surround U in D . Then there exists a path $\gamma : [0, 1] \rightarrow \overline{V}$ with $\gamma(0) \in U \setminus V$ and
 536 $\gamma(1) \in D \setminus U$. As we have shown, $D \setminus B \subseteq U$, so $D \setminus B \subseteq U \setminus V$.

537 Choose $x \in D \setminus B$ and $z \in \overline{D}$ such that there exist paths $\xi : [0, 1] \rightarrow U \setminus V$ with $\xi(0) = x$,
 538 $\xi(1) = \gamma(0)$ and $\zeta : [0, 1] \rightarrow \overline{D} \cup (D \setminus U)$ with $\zeta(0) = z$, $\zeta(1) = \gamma(1)$. ξ, γ and ζ all
 539 generate chains in $C_1(\overline{V}, \overline{U})$ and $\xi + \gamma + \zeta = \gamma^* \in C_1(\overline{V}, \overline{U})$ with $\partial\gamma^* = x + z$. Moreover, z
 540 generates a chain in $C_0(\overline{U})$ as $\overline{D} \subseteq \overline{U}$. So $x = \partial\gamma^* + z$ is a relative boundary in $C_0(\overline{V}, \overline{U})$,
 541 thus $\ell[x] = \ell[z]$ in $H_0(\overline{V}, \overline{L})$. However, because B surrounds D , $[x] \neq [y]$ in $H_0(\overline{B}, \overline{D})$
 542 contradicting our assumption that ℓ is injective.

543

544 **Proof of Lemma 4.** Assume there exist $p, q \in P \setminus Q_{\omega-c\zeta}$ such that p and q are connected in
 545 $\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})$ but not in $D \setminus B_\omega$. So the shortest path from p, q is a subset of $(P \setminus Q_{\omega-c\zeta})^\delta$.
 546 For any $x \in (P \setminus Q_{\omega-c\zeta})^\delta$ there exists some $p \in P$ such that $f(p) > \omega - c\zeta$ and $d(p, x) < \delta$.
 547 Because f is c -Lipschitz

$$548 \quad f(x) \geq f(p) - cd(x, p) > \omega - c(\delta + \zeta)$$

549 so there is a path from p to q in $D \setminus B_{\omega-c(\delta+\zeta)}$, thus $[p] = [q]$ in $H_0(D \setminus B_{\omega-c(\delta+\zeta)})$.

550 But we have assumed that $[p] \neq [q]$ in $H_0(D \setminus B_\omega)$, contradicting our assumption that
 551 $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is injective, so any p, q connected in $\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})$ are
 552 connected in $D \setminus B_\omega$. That is, $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$.

553 A.1 Extensions

554 **Proof of Lemma 8.** Note that $B' \setminus (D \setminus U) = B' \cap U \subseteq V$ implies $B' \subseteq V \sqcup (D \setminus U) = \mathcal{E}V$.
 555 Moreover, because $V \subseteq B$ and $D \setminus B \subseteq U$ implies $D \setminus U \subset D \setminus (D \setminus B) = B$ we have that
 556 $\mathcal{E}V = V \sqcup (D \setminus U) \subseteq B \cup (D \setminus U) = B$. So $B' \subseteq \mathcal{E}V \subseteq B$ as desired. ◀

557 **Proof of Lemma 9.** Because V surrounds U in D , $(U \setminus V, D \setminus U)$ is a separation of $D \setminus V$, a
 558 subspace of D . Recall, $(U \setminus V, D \setminus U)$ is a *separation* of $D \setminus V$ in D if $\text{cl}_D(U \setminus V) \cap (D \setminus U) = \emptyset$
 559 and $(U \setminus V) \cap \text{cl}_D(D \setminus U) = \emptyset$ (see Munkres [11], Lemma 23.1). So $\text{cl}_D(U \setminus V) \setminus U =$
 560 $\text{cl}_D(U \setminus V) \cap (D \setminus U) = \emptyset$ which implies $\text{cl}_D(U \setminus V) \subseteq U = \text{int}_D(U)$ as U is open in D .
 561 Therefore, $\text{cl}_D(D \setminus U) = D \setminus \text{int}_D(U) \subseteq D \setminus \text{cl}_D(U \setminus V) = \text{int}_D(\mathcal{E}V)$ so for all k and any
 562 $A \subseteq D$ such that $\mathcal{E}V \subset A$ we have $H_k(U \cap A, V) = H_k(A \setminus (D \setminus U), \mathcal{E}V \setminus (D \setminus U)) \cong H_k(A, \mathcal{E}V)$
 563 by excision. ◀

564 A.2 Image Modules

565 **Proof of Lemma 12.** Because $\Phi(F, G)$ is an image module homomorphism of degree δ we
 566 have $g_{\beta-\delta} \circ \gamma_{\alpha-\delta}[\beta - \alpha] = \lambda_\alpha[\beta - \alpha] \circ f_{\alpha-\delta}$. Similarly, $g'_\beta \circ \lambda_\alpha[\beta - \alpha] = \lambda'_{\alpha+\delta'}[\beta - \alpha] \circ f'_\alpha$.
 567 So $\Phi''(F' \circ F, G' \circ G) \in \text{Hom}^{\delta+\delta'}(\mathbf{im} \Gamma, \mathbf{im} \Lambda')$ as

$$568 \quad g'_\beta \circ (g_{\beta-\delta} \circ \gamma_{\alpha-\delta}[\beta - \alpha]) = (g'_\beta \circ \lambda_\alpha[\beta - \alpha]) \circ f_{\alpha-\delta} = \lambda_{\alpha+\delta'}[\beta - \alpha] \circ f'_\alpha \circ f_{\alpha-\delta}$$

569 for all $\alpha \leq \beta$. ◀

570 **Proof of Lemma 14.** If Γ is an epimorphism γ_α is surjective so $\Gamma_\alpha = V_\alpha$ and $\phi_\alpha = g_\alpha|_{\Gamma_\alpha} =$
 571 g_α for all α . So $\mathbf{im} \Gamma = \mathbb{V}$ and $\Phi \in \text{Hom}^\delta(\mathbb{V}, \mathbf{im} \Lambda)$.

572 If Π is a monomorphism then π_α is injective so we can define an isomorphism $\pi_\alpha^{-1} : \Pi_\alpha \rightarrow$
 573 V_α for all α . Let Ψ^* be defined as the family of linear maps $\{\psi_\alpha^* := \pi_\alpha^{-1} \circ \psi_\alpha : \Lambda_\alpha \rightarrow V_{\alpha+\delta}\}$.
 574 Because Ψ is a partial δ -interleaving of image modules, $n_\alpha \circ \lambda_\alpha = \pi_{\alpha+\delta} \circ m_\alpha$. So, because
 575 $\psi_\alpha = n_\alpha|_{\Lambda_\alpha}$ for all α ,

$$576 \quad \mathbf{im} \psi_\alpha^* = \mathbf{im} \pi_{\alpha+\delta}^{-1} \circ \psi_\alpha = \mathbf{im} \pi_{\alpha+\delta}^{-1} \circ (n_\alpha \circ \lambda_\alpha) = \mathbf{im} \pi_{\alpha+\delta}^{-1} \circ (\pi_{\alpha+\delta} \circ m_\alpha) = \mathbf{im} m_\alpha.$$

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577 It follows that $\text{im } v_{\alpha+\delta}^{\beta+\delta} \circ \psi_\alpha^* = \text{im } v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha$

578 Similarly, because Ψ is a δ -interleaving of image modules $n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha = w_{\alpha+\delta}^{\beta+\delta} \circ \pi_{\alpha+\delta} \circ m_\alpha$.

579 Moreover, because Π is a homomorphism of persistence modules, $w_{\alpha+\delta}^{\beta+\delta} \circ \pi_{\alpha+\delta} = \pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta}$,

580 so $n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha = \pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha$. As $\psi_\beta \circ \lambda_\alpha^\beta = n_\beta \circ \lambda_\alpha^\beta = n_\beta \circ t_\alpha^\beta|_{\Lambda_\alpha}$ it follows

$$\begin{aligned} 581 \quad \text{im } \psi_\beta^* \circ \lambda_\alpha^\beta &= \text{im } \pi_{\beta+\delta}^{-1} \circ (n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha) \\ 582 &= \text{im } \pi_{\beta+\delta}^{-1} \circ (\pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta}) \circ m_\alpha \\ 583 &= \text{im } v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha \\ 584 &= \text{im } v_{\alpha+\delta}^{\beta+\delta} \circ \psi_\alpha^*. \end{aligned}$$

585 So we may conclude that $\Psi^* \in \text{Hom}^\delta(\text{im } \Lambda, \mathbb{V})$.

586 So $\Phi \in \text{Hom}^\delta(\mathbb{V}, \text{im } \Lambda)$ and $\Psi_G^* \in \text{Hom}^\delta(\text{im } \Lambda, \mathbb{V})$. As we have shown, $\text{im } \psi_{\alpha-\delta}^* =$

587 $\text{im } m_{\alpha-\delta}$ so $\text{im } \phi_\alpha \circ \psi_{\alpha-\delta}^* = \text{im } \phi_\alpha \circ m_{\alpha-\delta}$. Moreover, because γ_α is surjective $\phi_\alpha = g_\alpha$ and,

588 because Φ is a partial δ -interleaving of image modules, $g_\alpha \circ m_{\alpha-\delta} = t_{\alpha-\delta}^{\alpha+\delta} \circ \lambda_{\alpha-\delta}$. As $\lambda_{\alpha-\delta}^{\alpha+\delta} =$

589 $t_{\alpha-\delta}^{\alpha+\delta}|_{\text{im } \lambda_{\alpha-\delta}}$ it follows that the following diagram commutes as $\text{im } \phi_\alpha \circ \psi_{\alpha-\delta}^* = \text{im } \lambda_{\alpha-\delta}^{\alpha+\delta}$:

$$590 \quad \begin{array}{ccc} & V_\alpha & \\ \psi_{\alpha-\delta}^* \nearrow & \nearrow \phi_\alpha & \\ \Lambda_{\alpha-\delta} & \xrightarrow{\lambda_{\alpha-\delta}^{\alpha+\delta}} & \Lambda_{\alpha+\delta}. \end{array} \quad (4)$$

591 Finally, $\psi_\alpha^* \circ \phi_\alpha = \pi_{\alpha+\delta}^{-1} \circ n_\alpha \circ g_{\alpha-\delta}$ where, because Ψ is a partial δ -interleaving of image

592 modules, $n_\alpha \circ g_{\alpha-\delta} = w_{\alpha-\delta}^{\alpha+\delta} \circ \pi_{\alpha-\delta}$. Because Π is a homomorphism of persistence modules

593 $w_{\alpha-\delta}^{\alpha+\delta} \circ \pi_{\alpha-\delta} = \pi_{\alpha+\delta} \circ v_{\alpha-\delta}^{\alpha+\delta}$. Therefore,

$$594 \quad \psi_\alpha^* \circ \phi_{\alpha-\delta} = \pi_{\alpha+\delta}^{-1} \circ n_\alpha \circ g_{\alpha-\delta} = \pi_{\alpha+\delta}^{-1} \circ (\pi_{\alpha+\delta} \circ v_{\alpha-\delta}^{\alpha+\delta}) = v_{\alpha-\delta}^{\alpha+\delta}$$

595 so the following diagram commutes

$$596 \quad \begin{array}{ccc} V_{\alpha-\delta} & \xrightarrow{v_{\alpha-\delta}^{\alpha+\delta}} & V_{\alpha+\delta} \\ \phi_\alpha \searrow & \swarrow \psi_\alpha^* & \\ & \Lambda_\alpha & \end{array} \quad (5)$$

597 Because $\Phi \in \text{Hom}^\delta(\mathbb{V}, \text{im } \Lambda)$, $\Psi^* \in \text{Hom}^\delta(\text{im } \Lambda, \mathbb{V})$, and Diagrams 4 and 5 commute we
598 may conclude that $\text{im } \Lambda$ and \mathbb{V} are δ -interleaved.

599

◀

600 A.3 Partial Interleavings

601 **Proof of Lemma 15.** Suppose $x \in P^\delta \cap D_{\lfloor \alpha - c\varepsilon \rfloor t - c\varepsilon}$. Because x in P^δ there exists some
602 $p \in P$ such that $d(x, p) < \delta$. Because f is c -Lipschitz $f(p) \leq f(x) + c\delta < f(x) + c\delta$.

603 If $\alpha \leq t$ then $x \in B_{t-c\varepsilon}$ implies $f(p) < t - c\varepsilon + c\delta \leq t$ so $x \in Q_t^\varepsilon$ as $\delta \leq \varepsilon$. If $\alpha \geq t$ then
604 $x \in B_{\alpha-c\varepsilon}$ which implies $f(p) \leq \alpha$ $x \in Q_\alpha^\varepsilon$. So $P^\delta \cap D_{\lfloor \alpha - c\varepsilon \rfloor t - c\varepsilon} \subseteq P_{\lfloor \alpha \rfloor t}^\varepsilon$ as $P_{\lfloor \alpha \rfloor t} = Q_t^\varepsilon \cup Q_\alpha^\varepsilon$.

605 Now, suppose $x \in P_{\lfloor \alpha \rfloor t}^\varepsilon$. If $\alpha \leq t$ then $x \in Q_t^\varepsilon \subseteq B_{t+c\varepsilon}$ because f is c -Lipschitz. Similarly,
606 $\alpha > t$ implies $x \in Q_\alpha^\varepsilon \subseteq B_{\alpha+c\varepsilon}$, so $P_{\lfloor \alpha \rfloor t}^\varepsilon \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor t + c\varepsilon}$ as $D_{\lfloor \alpha + c\varepsilon \rfloor t + c\varepsilon} = B_{t+c\varepsilon} \cup B_{\alpha+c\varepsilon}$. ◀

607 **Proof of Lemma 16.** Because Q_t^δ surrounds P^δ in D and $\delta \leq \varepsilon$, $t < v$ we know Q_t^ε and Q_v^ε
608 surround P^δ in D . As $P^\delta \cap B_s \subseteq Q_t^\varepsilon$ and $P^\delta \cap B_u \subseteq Q_v^{2\varepsilon}$ for all $\varepsilon \in [\delta, 2\delta]$ Lemma 8 implies
609 that we have a sequence of inclusions $B_s \subseteq \mathcal{E}Q_t^\varepsilon \subseteq B_u \subseteq \mathcal{E}Q_v^{2\varepsilon} \subseteq B_w$.

610 For any $\alpha \in \mathbb{R}$ we know that $D \setminus P^\delta \subseteq \mathcal{E}P_{\lfloor \alpha \rfloor_t}^\varepsilon$ by the definition of $\mathcal{E}P_{\lfloor \alpha \rfloor_t}^\varepsilon$. Moreover,
 611 $D \setminus P^\delta \subseteq D_{\lfloor \alpha \rfloor_u}$ because $D \setminus B_u \subseteq P^\delta$. Lemma 15 therefore implies $D_{\lfloor \alpha - c\delta \rfloor_s} \subseteq \mathcal{E}P_{\lfloor \alpha \rfloor_t}^\varepsilon \subseteq$
 612 $D_{\lfloor \alpha + c\varepsilon \rfloor_u}$ as $s + c\delta \leq t \leq u - c\varepsilon$. So the inclusions $(D_{\lfloor \alpha - c\delta \rfloor_s}, B_s) \subseteq (\mathcal{E}P_{\lfloor \alpha \rfloor_t}^\varepsilon, \mathcal{E}Q_t^\varepsilon)$ induce
 613 $F \in \text{Hom}^{c\delta}(\mathbb{D}_s, \mathcal{E}\mathbb{P}_t^\varepsilon)$ and $(\mathcal{E}P_{\lfloor \alpha \rfloor_t}^\varepsilon, \mathcal{E}Q_t^\varepsilon) \subseteq (D_{\lfloor \alpha + c\varepsilon \rfloor_u}, B_u)$ induce $M \in \text{Hom}^{c\varepsilon}(\mathcal{E}\mathbb{P}_t^\varepsilon, \mathbb{D}_u)$.

614 By an identical argument Lemma 15 implies $D_{\lfloor \alpha - 2c\delta \rfloor_u} \subseteq \mathcal{E}P_{\lfloor \alpha \rfloor_v}^\varepsilon \subseteq D_{\lfloor \alpha + 2c\varepsilon \rfloor_w}$ as $u + c\delta \leq$
 615 $v \leq w - 4c\delta$. So $(D_{\lfloor \alpha - 2c\delta \rfloor_u}, B_u) \subseteq (\mathcal{E}P_{\lfloor \alpha \rfloor_v}^\varepsilon, \mathcal{E}Q_v^{2\varepsilon})$ induce $G \in \text{Hom}^{2c\delta}(\mathbb{D}_u, \mathcal{E}\mathbb{P}_v^{2\varepsilon})$ and
 616 $(\mathcal{E}P_{\lfloor \alpha \rfloor_v}^\varepsilon, \mathcal{E}Q_v^{2\varepsilon}) \subseteq (D_{\lfloor \alpha + 2c\varepsilon \rfloor_w}, B_w)$ induce $N \in \text{Hom}^{2c\varepsilon}(\mathcal{E}\mathbb{P}_v^{2\varepsilon}, \mathbb{D}_w)$. \blacktriangleleft

A.4 Truncated Interval Modules

618 **Proof of Lemma 19.** Suppose $\alpha \leq \omega$. So $H_k(D_{\lfloor \alpha \rfloor_\omega}, B_\omega) = 0$ as $D_{\lfloor \alpha \rfloor_\omega} = B_\omega \cup B_\alpha$ and
 619 $\mathbb{T}_\omega^k = 0$ as $F_\alpha^I = 0$ for any $I \in \mathcal{I}^k$ such that $\omega \in I_-$. Moreover, $\omega \in I$ for all $I \in \mathcal{I}_\omega^{k-1}$, thus
 620 $F_\alpha^{I+} = 0$ for all $\alpha \leq \omega$. So it suffices to assume $\omega < \alpha$.

621 Consider the long exact sequence of the pair $H_k(D_{\lfloor \alpha \rfloor_\omega}, B_\omega) = H_k(B_\alpha, B_\omega)$

$$\dots \rightarrow H_k(B_\omega) \xrightarrow{p_\alpha^k} H_k(B_\alpha) \xrightarrow{q_\alpha^k} H_k(D_{\lfloor \alpha \rfloor_\omega}, B_\omega) \xrightarrow{r_\alpha^k} H_{k-1}(B_\omega) \xrightarrow{p_\alpha^{k-1}} H_{k-1}(B_\alpha) \rightarrow \dots$$

623 where $H_k(B_\alpha) = \bigoplus_{I \in \mathcal{I}^k} F_\alpha^I$, $H_k(B_\omega) = \bigoplus_{I \in \mathcal{I}^k} F_\omega^I$, and $p_\alpha^k = \bigoplus_{I \in \mathcal{I}^k} f_{\omega, \alpha}^I$.

624 Noting that $\text{im } q_\alpha^k \cong H_k(B_\alpha)/\ker q_\alpha^k$ where $\ker q_\alpha^k = \text{im } p_\alpha^k$ by exactness we have
 625 $\ker r_\alpha^k \cong H_k(B_\alpha)/\text{im } p_\alpha^k$. By the definition of F_α^I and $f_{\omega, \alpha}^I$ we know $\text{im } f_{\omega, \alpha}^I$ is F_α^I if $\omega \in I$
 626 and 0 otherwise. As $\text{im } p_\alpha^k$ is equal to the direct sum of images $\text{im } f_{\omega, \alpha}^I$ over $I \in \mathcal{I}^k$ it follows
 627 that $\text{im } p_\alpha^k$ is the direct sum of those F_α^I over those $I \in \mathcal{I}^k$ such that $\omega \in I$. Now, because
 628 $H_k(B_\alpha) = \bigoplus_{I \in \mathcal{I}^k} F_\alpha^I$ and each F_α^I is either 0 or \mathbb{F} the quotient $H_k(B_\alpha)/\text{im } p_\alpha^k$ is the direct
 629 sum of those F_α^I such that $\omega \notin I$. Therefore, by the definition of $F_{\lfloor \alpha \rfloor_\omega}^I$ we have

$$630 \quad \ker r_\alpha^k = \bigoplus_{I \in \mathcal{I}_\omega^k} F_{\lfloor \alpha \rfloor_\omega}^I.$$

631 Similarly, $\text{im } r_\alpha^k = \ker p_\alpha^{k-1}$ by exactness where $\ker p_\alpha^{k-1}$ is the direct sum of kernels
 632 $\ker f_{\omega, \alpha}^I$ over $I \in \mathcal{I}^{k-1}$. By the definition of F_α^I and $f_{\omega, \alpha}^I$ we know that $\ker f_{\omega, \alpha}^I$ is F_α^I if
 633 $\omega \notin I$ and 0 otherwise. Noting that $\ker f_{\omega, \alpha}^I = 0$ for any $I \in \mathcal{I}^{k-1}$ such that $\omega \notin I$ it suffices
 634 to consider only those $I \in \mathcal{I}_\omega^{k-1}$. It follows that $\ker f_{\omega, \alpha}^I = F_\alpha^{I+}$ for any I containing ω as
 635 $\omega < \alpha$. Therefore,

$$636 \quad \text{im } r_\alpha^k = \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} F_\alpha^{I+}.$$

637 We have the following split exact sequence associated with r_α^k

$$638 \quad 0 \rightarrow \ker r_\alpha^k \rightarrow H_k(D_{\lfloor \alpha \rfloor_\omega}, B_\omega) \rightarrow \text{im } r_\alpha^k \rightarrow 0.$$

639 The desired result follows from the fact that for all $\alpha \in \mathbb{R}$

$$640 \quad H_k(D_{\lfloor \alpha \rfloor_\omega}, B_\omega) \cong \ker r_\alpha^k \oplus \text{im } r_\alpha^k = \bigoplus_{I \in \mathcal{I}^k} F_{\lfloor \alpha \rfloor_\omega}^I \oplus \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} F_\alpha^{I+}.$$

642 **B Duality**

643 For a pair (A, B) in a topological space X and any R module G let $H^k(A, B; G)$ denote
 644 the **singular cohomology** of (A, B) (with coefficients in G). Let $H_c^k(A, B; G)$ denote
 645 the corresponding **singular cohomology with compact support**, where $H_c^k(A, B; G) \cong$
 646 $H^k(A, B; G)$ for any compact pair (A, B) .

647 The following corollary follows from the Universal Coefficient Theorem for singular
 648 homology (and cohomology) as vector spaces over a field \mathbb{F} , as the dual vector space
 649 $\text{Hom}(H_k(A, B), \mathbb{F})$ is isomorphic to $H_k(A, B; \mathbb{F})$ for any finitely generated $H_k(A, B)$.

650 ▶ **Corollary 20.** *For a topological pair (A, B) and a field \mathbb{F} such that $H_k(A, B)$ is finitely
 651 generated there is a natural isomorphism*

$$652 \quad \nu : H^k(A, B; \mathbb{F}) \rightarrow H_k(A, B; \mathbb{F}).$$

653 Let $\overline{H}^k(A, B; G)$ be the **Alexander-Spanier cohomology** of the pair (A, B) , defined
 654 as the limit of the direct system of neighborhoods (U, V) of the pair (A, B) . Let $\overline{H}_c^k(A, B; G)$
 655 denote the corresponding **Alexander-Spanier cohomology with compact support**
 656 where $\overline{H}_c^k(A, B; G) \cong \overline{H}^k(A, B; G)$ for any compact pair (A, B) .

657 ▶ **Theorem 21 (Alexander-Poincaré-Lefschetz Duality** (Spanier [13], Theorem 6.2.17)). *Let
 658 X be an orientable d -manifold and (A, B) be a compact pair in X . Then for all k and R
 659 modules G there is a (natural) isomorphism*

$$660 \quad \lambda : H_k(X \setminus B, X \setminus A; G) \rightarrow \overline{H}^{d-k}(A, B; G).$$

661 A space X is said to be **homologically locally connected in dimension n** if for
 662 every $x \in X$ and neighborhood U of x there exists a neighborhood V of x in U such that
 663 $\tilde{H}_n(V) \rightarrow \tilde{H}_n(U)$ is trivial for $k \leq n$.

664 ▶ **Lemma 22** (Spanier p. 341, Corollary 6.9.6). *Let A be a closed subset, homologically
 665 locally connected in dimension n , of a Hausdorff space X , homologically locally connected in
 666 dimension n . If X has the property that every open subset is paracompact, $\mu : \overline{H}_c^k(X, A; G) \rightarrow$
 667 $H_c^k(X, A; G)$ is an isomorphism for $k \leq n$ and a monomorphism for $q = n + 1$.*

668 In the following we will assume homology (and cohomology) over a field \mathbb{F} .

669 ▶ **Lemma 23.** *Let X be an orientable d -manifold and (A, B) a compact pair of locally path
 670 connected subspaces in X . Then*

$$671 \quad \xi : H_d(X \setminus B, X \setminus A) \rightarrow H_0(A, B)$$

672 *is a natural isomorphism.*

673 **Proof.** Because X is orientable and (A, B) are compact $\lambda : H_d(X \setminus B, X \setminus A) \rightarrow \overline{H}^0(A, B)$
 674 is an isomorphism by Theorem 21. Note that Moreover, because every subset of X is
 675 (hereditarily) paracompact every open set in A , with the subspace topology, is paracompact.
 676 For any neighborhood U of a point x in a locally path connected space there must exist some
 677 neighborhood $V \subset U$ of x that is path connected in the subspace topology. As $\tilde{H}_0(V) = 0$
 678 for any nonempty, path connected topological space V (see Spanier p. 175, Lemma 4.4.7)
 679 it follows that A (resp. B) are homologically locally connected in dimension 0. Because
 680 (A, B) is a compact pair the singular and Alexander-Spanier cohomology modules of (A, B)
 681 with compact support are isomorphic to those without, thus $\mu : \overline{H}^0(A, B) \rightarrow H^0(A, B)$ is an
 682 isomorphism. By Corollary 20 we have a natural isomorphism $\nu : H^0(A, B) \rightarrow H_0(A, B)$ thus
 683 the composition $\xi := \nu \circ \mu \circ \lambda : H_d(X \setminus B, X \setminus A) \rightarrow H_0(A, B)$ is a natural isomorphism. ◀

684 ► **Lemma 24.** Let \mathbb{X} be an orientable d -manifold let D be a compact subset of \mathbb{X} . Let P be
685 a finite subset of D such that $P^\varepsilon \subset \text{int}_{\mathbb{X}}(D)$ and $Q \subseteq P$.

686 If $D \setminus Q^\varepsilon$ and $D \setminus P^\varepsilon$ are locally path connected then there is a natural isomorphism

687 $\xi : H_d(P^\varepsilon, Q^\varepsilon) \rightarrow H_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon).$

688 **Proof.** Because Q^ε and P^ε are open in D and D is compact in \mathbb{X} the complement $D \setminus Q^\varepsilon$
689 is closed in D , and therefore compact in \mathbb{X} . Moreover, because $P^\varepsilon \subset \text{int}_{\mathbb{X}}(D)$, $H_d(\mathbb{X} \setminus (D \setminus
690 P^\varepsilon), \mathbb{X} \setminus (D \setminus Q^\varepsilon)) = H_d(P^\varepsilon, Q^\varepsilon)$. As we have assumed these complements are locally path
691 connected by assumption we have a natural isomorphism $\xi : H_d(P^\varepsilon, Q^\varepsilon) \rightarrow H_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon)$
692 by Lemma 23. ◀