

From Coverage Testing to Topological Scalar Field Analysis

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1 Abstract

The topological coverage criterion (TCC) can be used to test whether an underlying space is sufficiently well covered by a given data set. Given a sufficiently dense sample, topological scalar field analysis (SFA) can give a summary of the shape of a real-valued function on a space. The goal of this paper is to put these theories together so that one can test coverage with the TCC and then compute a summary with SFA. The challenge is that the TCC requires a well-defined boundary that is not generally available in the SFA settings. To overcome this, we show how the scalar field itself can be used to define a boundary that can then be used to confirm coverage. This requires an interpretation of the TCC that resolves one of the major barriers to wider use. It also extends SFA methods to the setting in which coverage is only confirmed in a subset of the domain. We show how the intersection of these two theories can be used to approximate the persistent homology relative to a static sublevel set. We then discuss how this persistent relative homology relates to that of the scalar field as a whole.

2012 ACM Subject Classification Replace ccsdesc macro with valid one

Keywords and phrases Dummy keyword

Funding Kirk P. Gardner: [funding]

Donald R. Sheehy: [funding]

14 1 Introduction

In the topological analysis of scalar fields (SFA), one computes a topological summary capturing qualitative and quantitative shape information from a set of points endowed with a metric and a real-valued function. That is, we have points with distances and a real number assigned to each point. More generally, it suffices to have a neighborhood graph on the points identifying the pairs of points within some distance. The topological computation uses persistent homology to integrate local information from the function into global information about its *behavior* as whole. In prior work, Chazal et al. [3] showed that for sufficiently dense samples on sufficiently smooth spaces, the persistence diagram can be computed with some guarantees. In followup work, Buchet et al. [1] extended this result to show how to work with noisy inputs. A fundamental assumption required to have strong guarantees on the output of these methods is that the underlying space be sufficiently well-sampled. In this paper, we show how to combine scalar field analysis with the theory of topological coverage testing to simultaneously compute the persistence diagram and also to test that the underlying space is sufficiently well-sampled.

Initiated by De Silva and Christ [6, 4, 5], the theory of homological sensor networks addresses the problem of testing coverage of a bounded domain by a collection of sensors without coordinates. The main result is the topological coverage criterion, which, in its most general form, states that under reasonable geometric assumptions, the d -dimensional homology of a pair of simplicial complexes built on the neighborhood graph will be nontrivial

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42nd Conference on Very Important Topics (CVIT 2016).
Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:22

 Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

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34 if and only if there is sufficient coverage (see Section 3 for the precise statements). This
35 relative persistent homology test is called the Topological Coverage Criterion (TCC).

36 Superficially, the methods of SFA and TCC are very similar. Both construct similar
37 complexes and compute the persistent homology of the homological image of a complex on
38 one scale into that of a larger scale. They even overlap on some common techniques in their
39 analysis including the use of the Nerve theorem and the Rips-Čech interleaving. However,
40 they differ in some fundamental way that makes it difficult to combine them into a single
41 technique. The main difference is that the TCC requires a clearly defined boundary. Not
42 only must the underlying space be a bounded subset of \mathbb{R}^d , the data must also be labeled to
43 indicate which input points are close to the boundary. This requirement is perhaps the main
44 reason why the TCC can so rarely be applied in practice.

45 In applications to data analysis it is more natural to assume that our data measures
46 some unknown function. By requiring that our function is related to the metric of the space
47 we can replace this requirement with assumptions about the function itself. Indeed, these
48 assumptions could relate the behavior of the function to the topological boundary of the
49 space. However, the generalized approach by Cavanna et al. [2] allows much more freedom
50 in how the boundary is defined.

51 We consider the case in which we have incomplete data from a particular sublevel set
52 of our function. Our goal is to isolate this data so we can analyze the function only in the
53 verified region. From this perspective, the TCC confirms that we not only have coverage,
54 but that the sample we have is topologically representative of the region near, and above
55 this sublevel set. We can then re-use the same machinery to analyze a *part* of the function
56 in a specific way.

57 Contribution

58 We will re-cast the TCC as a way to verify that the persistent homology of a scalar field
59 can be *partially* approximated by a given sample. Specifically, we will relate the persistent
60 homology of a function relative to a *static* sublevel set to a *truncation* of the full diagram.
61 That is, beyond a certain point the full diagram remains unchanged, allowing for possible
62 reconstruction. This is in comparison with the *restricted* diagram obtained by simply ignoring
63 part of the domain. We therefore present relative persistent homology as an alternative to
64 restriction in a way that extends the TCC to the analysis of scalar fields.

65 We will first provide some background on important topological, geometric, and algebraic
66 structures required for our re-formulation of the TCC, and the approximation of the relative
67 diagram. Section 2 establishes notation and provides an overview of our main results in
68 Sections 3 and 4. In Section 5 we introduce an interpretation of the relative diagram as a
69 truncation of the full diagram that is motivated by a number of experiments in Section 6.

70 2 Summary

71 Let \mathbb{X} denote an orientable d -manifold and $D \subset \mathbb{X}$ a compact subspace. For a c -Lipschitz
72 function $f : D \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$ let $B_\alpha := f^{-1}((-\infty, \alpha])$ denote the α -sublevel set of f . Our
73 sample will be denoted P , and the subset of points sampling B_α will be denoted $Q_\alpha := P \cap B_\alpha$.
74 For ease of exposition let

$$75 D_{\lfloor \alpha \rfloor_w} := B_\alpha \cup B_w$$

76 to be the *truncated* α sublevel set and

$$77 P_{\lfloor \alpha \rfloor_w} := Q_\alpha \cup Q_w$$

78 denote is sampled counterpart for all $\alpha, w \in \mathbb{R}$.

79 We will select a sublevel set B_ω to serve as our boundary. Specifically, we require that
80 B_ω surrounds D , where the notion of a surrounding set is defined formally in Section 3. This
81 distinction allows us to generalize the standard proof of the TCC to properties of surrounding
82 pairs.

83 Relative, Truncated, and Restricted Persistence Diagrams

84 For fixed $\omega \in \mathbb{R}$ we will refer to the persistence diagram associated with the filtration
85 $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ as the **relative diagram** of f . In Section 5 we relate the relative diagram
86 to the *full* diagram of the sublevel set filtration $\{B_\alpha\}_{\alpha \in \mathbb{R}}$. Specifically, we define the
87 **truncated diagram** to be the subdiagram consisting of features born *after* ω in the full
88 diagram. In Section 6 we compare the relative and truncated diagrams to the **restricted
89 diagram**, defined to be that of the sublevel set filtration of $f|_{D \setminus B_\omega}$.

90 Note that the truncated sublevel sets $D_{\lfloor \alpha \rfloor \omega}$ are equal to the union of B_ω and the restricted
91 sublevel sets. It is in this sense that B_ω is *static* throughout—it is contained in every sublevel
92 set of the relative filtration. As we will not have verified coverage in B_ω we cannot analyze
93 the function in this region directly. We therefore have two alternatives: *restrict* the domain
94 of the function to $D \setminus B_\omega$, or use relative homology to analyze the function *relative* to this
95 region using excision.

96 Results

97 Suppose B_ω surrounds D in \mathbb{X} and $\delta < \varrho/4$. As a minimal assumption we require that every
98 component of $D \setminus B_\omega$ contains a point in P . We also make additional technical assumptions
99 on P and δ with respect to the pair (D, B_ω) (see Section ?? and Lemma 28 of Appendix B).

100 Theorem 6 If

- 101 I. $H_0(D \setminus B_{\omega+5c\delta} \hookrightarrow D \setminus B_\omega)$ is *surjective*,
- 102 II. $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-3c\delta})$ is *injective*,

103 and

$$104 \quad \text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$$

107 then $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D . ¹

108 This formulation of the TCC states that our approximation by a nested pair of Rips
109 complexes not only covers, but captures the homology of the pair (D, B_ω) in a specific
110 way. We use this fact to interleave our sample with the relative diagram of the filtration
111 $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$. This is done by generalizing our regularity assumptions near $D \setminus B_\omega$ in a
112 way that allows us to interleave persistence modules with static components.

113 **Theorem 15** Suppose P satisfies the the TCC: $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in
114 D . If

- 115 I. $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is *surjective* and
- 116 II. $H_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an *isomorphism*

¹ We state this result using constants that will be used to prove the interleaving. The statement of
106 Theorem 6 parameterizes the region around ω in terms of $\zeta \geq \delta$ as $[\omega - c(\delta + \zeta), \omega + c(\delta + \zeta)]$.

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117 for all k the persistent homology modules of

$$118 \quad \{\mathcal{R}^{2\delta}(P_{\lfloor \alpha \rfloor_{\omega-2c\delta}}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor \alpha \rfloor_{\omega+c\delta}}, Q_{\omega+c\delta})\}_{\alpha \in \mathbb{R}}$$

119 and $\{(D_{\lfloor \alpha \rfloor_{\omega}}, B_{\omega})\}_{\alpha \in \mathbb{R}}$ are $4c\delta$ interleaved.

120 The main challenges we face come from the fact that the sublevel set B_{ω} and our
 121 approximation by the inclusion $\mathcal{R}^{2\delta}(Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(Q_{\omega+c\delta})$ remain *static* throughout.
 122 Using the fact that $Q_{\omega-2c\delta}^{\delta}$ surrounds P^{δ} in D we define an *extension* $(D, \mathcal{E}Q_{\omega-2c\delta}^{\delta})$ of the
 123 pair $(P^{\delta}, Q_{\omega-2c\delta}^{\delta})$ that has isomorphic relative homology by excision. These extensions give
 124 us a sequence of inclusion maps

$$125 \quad B_{\omega-3c\delta} \hookrightarrow \mathcal{E}Q_{\omega-2c\delta}^{2\delta} \hookrightarrow B_{\omega} \hookrightarrow \mathcal{E}Q_{\omega+c\delta}^{4\delta} \hookrightarrow B_{\omega+5c\delta}$$

126 that can be used along with our regularity assumptions to prove the interleaving.

127 Outline of Sections 3 and 4

128 We will begin with our reformulation of the TCC in Section 3. This requires the introduction
 129 of a surrounding set before proving the Geometric TCC (Theorem ??), followed by the
 130 computable Algorithmic TCC (Theorem 6). Section 4 formally introduces extensions and
 131 partial interleavings of image modules which will be used in the proof of Theorem 15.

132 3 The Topological Coverage Criterion (TCC)

133 The TCC uses the top-dimensional relative homology of a space D with respect to its
 134 boundary B to provide a computable condition for coverage. Under certain conditions
 135 Alexander Duality can be used to relate this to the number of connected components of its
 136 complement in some larger space \mathbb{X} . One can then check if a collection of subsets covers the
 137 space by comparing the number of connected components of its complement to that of the
 138 space—holes in the cover will appear as additional components in the complement space. As
 139 we cannot compute the number components of the complement space from a sample directly,
 140 the TCC uses duality to recover it from the top dimensional relative homology. However,
 141 this requires that we have a subset Q of our cover to serve as the boundary. That is, a
 142 positive result indicates that we not only have coverage, but also that we have a pair of
 143 spaces that reflects the pair (D, B) topologically. We call such a pair a *surrounding pair*
 144 defined in terms of separating sets. It has been shown that the TCC can be stated in terms
 145 of these surrounding pairs [2], which allows us enough flexibility to define our surrounding set
 146 as a sublevel set of a c -Lipschitz function f . Moreover, this work made assumptions directly
 147 in terms of the *zero dimensional* persistent homology of the domain close to the boundary.

148 ▶ **Definition 1** (Surrounding Pair). *Let X be a topological space and (D, B) a pair in X . The
 149 set B surrounds D in X if B separates X with the pair $(D \setminus B, X \setminus D)$. We will refer to
 150 such a pair as a **surrounding pair in X** .*

151 Let (D, B) be a surrounding pair in X and $U \subseteq D$, $V \subseteq U \cap B$ be subsets. Let
 152 $\ell : H_0(X \setminus B, X \setminus D) \rightarrow H_0(X \setminus V, X \setminus U)$ be induced by inclusion. The following lemma
 153 generalize the proof of the TCC as properties of surrounding sets, its proof can be found in
 154 the Appendix.

155 ▶ **Lemma 2.** *If ℓ is injective then $D \setminus B \subseteq U$ and V surrounds U in D .*

We now combine these results on the homology of surrounding pairs with information about both \mathbb{X} as a metric space and our function. Let (\mathbb{X}, \mathbf{d}) be a metric space and $D \subseteq \mathbb{X}$ be a compact subspace. Let $f : D \rightarrow \mathbb{R}$ be a c -Lipschitz function and $B_\alpha = f^{-1}((-\infty, \alpha])$ denote the α -sublevel set of f for $\alpha \in \mathbb{R}$. We introduce a constant ω as a threshold that defines our “boundary” as a sub-levelset of the function f . Let P be a finite subset of D and $Q_\alpha := P \cap B_\alpha$ for $\alpha \in \mathbb{R}$. Let $\zeta \geq \delta > 0$ and $\omega \in \mathbb{R}$ be constants such that $P^\delta \subseteq D$. Here, δ will serve as our communication radius where ζ is reserved for use in Section 4.²

► **Lemma 3.** *Let $i : H_0(\overline{Q_{\omega+c\delta}^\delta}, \overline{P^\delta}) \rightarrow H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$.*

If B_ω surrounds D in \mathbb{X} then $\dim H_0(\overline{B_\omega}, \overline{D}) \geq \text{rk } i$.

Proof. Choose a basis for $\text{im } i$ such that each basis element is represented by a point in $P^\delta \setminus Q_{\omega+c\delta}^\delta$. Let $x \in P^\delta \setminus Q_{\omega+c\delta}^\delta$ be such that $i[x] \neq 0$. So there exists some $p \in P$ such that $\mathbf{d}(p, x) < \delta$ and $p \notin Q_{\omega+c\delta}$, otherwise $x \in Q_{\omega+c\delta}^\delta$. Therefore, because f is c -Lipschitz,

$$f(x) \geq f(p) - c\mathbf{d}(x, p) > \omega + c\delta - c\delta = \omega.$$

So $x \in \overline{B_\omega}$ and, because $x \in P^\delta \subseteq D$, $x \in D \setminus B_\omega$. Because i and $\ell : H_0(\overline{B_\omega}, \overline{D}) \rightarrow H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$ are induced by inclusion $\ell[x] = i[x] \neq 0$ in $H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$. That is, every element of $\text{im } i$ has a preimage in $H_0(\overline{B_\omega}, \overline{D})$, so we may conclude that $\dim H_0(\overline{B_\omega}, \overline{D}) \geq \text{rk } i$. ◀

Note that, while there is a surjective map from $H_0(\overline{B_\omega}, \overline{D})$ to $\text{im } i$ this map is not necessarily induced by inclusion, as $Q_{\omega+c\delta}^\delta \not\subseteq B_\omega$. We therefore must introduce a larger space $B_{\omega+c(\delta+\zeta)}$ that contains $Q_{\omega+c\delta}^\delta$ in order to provide a criteria for the injectivity of $\ell : H_0(\overline{B_\omega}, \overline{D}) \rightarrow H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$ in terms of $\text{rk } i$.

$$\begin{array}{ccc} (P^\delta, Q_{\omega-c\zeta}^\delta) & \longrightarrow & (\overline{B_{\omega+c(\delta+\zeta)}}, \overline{D}) & \longrightarrow & (\overline{B_\omega}, \overline{D}) \\ \downarrow & & \downarrow & & \downarrow \\ (D, B_\omega) & \longrightarrow & (D, B_{\omega+c(\delta+\zeta)}), & \longrightarrow & (\overline{Q_{\omega+c\delta}^\delta}, \overline{P^\delta}) & \longrightarrow & (\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta}). \end{array}$$

$$\begin{array}{ccc} H_0(\overline{B_{\omega+c(\delta+\zeta)}}, \overline{D}) & \xrightarrow{j} & H_0(\overline{B_\omega}, \overline{D}) \\ \downarrow m & & \downarrow \ell \\ H_0(\overline{Q_{\omega+c\delta}^\delta}, \overline{P^\delta}) & \xrightarrow{i} & H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta}). \end{array} \tag{1}$$

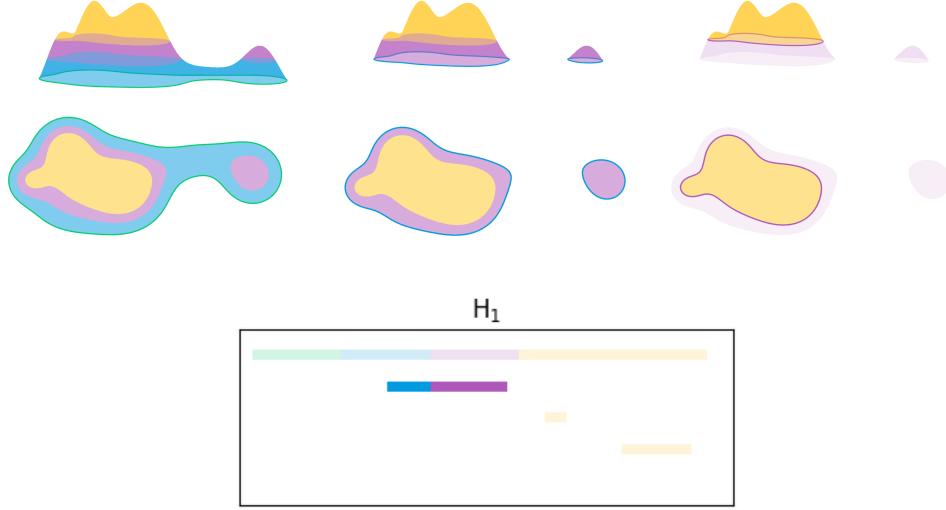
Assumptions

We will first require the map $H_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ to be *surjective*—as we approach ω from *above* no components *appear*. This ensures that the rank of the map j is equal to the dimension of $\dim H_0(\overline{B_\omega}, \overline{D})$ so our map ℓ induced by inclusion depends only on $H_0(\overline{B_\omega}, \overline{D})$ and $\text{im } i$.

We also assume that $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is *injective*—as we move away from ω moving *down* no components *disappear*. Lemma 4 uses Assumption 2 to provide a computable upper bound on $\text{rk } j$, its proof can be found in the Appendix.

² We will set $\zeta = 2\delta$ in the proof of our interleaving with Rips complexes but the TCC holds for all $\zeta \geq \delta$.

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188 ■ **Figure 1** The blue level set does not satisfy either assumption as the smaller component is not in
 189 the inclusion from blue to green and it “pinched out” in the yellow region. This can be seen in the
 190 barcode shown as a feature that is born in the blue region and dies in the purple region.

191 ► **Lemma 4.** If $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega+c(\delta+\zeta)})$ is injective and each component of $D \setminus B_\omega$
 192 contains a point in P then $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$.

193 The Appendix details how to construct the following isomorphism using the Nerve Theorem
 194 along with Alexander Duality and the Universal Coefficient Theorem.

$$195 \xi \mathcal{N}_w^{\varepsilon,k} : H_d(\check{C}^\varepsilon(P, Q_w)) \rightarrow H_0(D \setminus Q_w^\varepsilon, D \setminus P^\varepsilon).$$

196 This isomorphism holds in the specific case when $P^\varepsilon \subseteq \text{int}_{\mathbb{X}}(D)$ and $D \setminus P^\varepsilon, D \setminus Q_w^\varepsilon$ are
 197 locally contractible. We therefore provide the following definition for ease of exposition

198 ► **Definition 5** ((δ, ζ, ω)-Sublevel Sample). For $\zeta \geq \delta > 0$, $\omega \in \mathbb{R}$, and a c -Lipschitz function
 199 $f : D \rightarrow \mathbb{R}$ a finite point set $P \subset D$ is said to be a (δ, ζ, ω) -sublevel sample of f if every
 200 component of $D \setminus B_\omega$ contains a point in P , $P^\delta \subset \text{int}_{\mathbb{X}}(D)$, and $D \setminus P^\delta, D \setminus Q_{\omega-c\zeta}^\delta$, and
 201 $D \setminus Q_{\omega+c\delta}^\delta$ are locally path connected in \mathbb{X} .

202 ► **Theorem 6** (Algorithmic TCC). Let \mathbb{X} be an orientable d -manifold and let D be a compact
 203 subset of \mathbb{X} . Let $f : D \rightarrow \mathbb{R}$ be c -Lipschitz function and $\omega \in \mathbb{R}$ and $\delta \leq \zeta < \varrho_D$ be constants
 204 such that $P \subset D$ is a (δ, ζ, ω) -sublevel sample of f and $B_{\omega-c(\zeta+\delta)}$ surrounds D in \mathbb{X} .

205 If $H_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ is surjective, $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega+c(\delta+\zeta)})$ is injective,
 206 and $\text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-c\zeta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta}))$ then $D \setminus B_\omega \subseteq P^\delta$
 207 and $Q_{\omega-c\zeta}^\delta$ surrounds P^δ in D .

208 **Proof.** Because P is a (δ, ζ, ω) -sublevel sample we have isomorphisms $\xi \mathcal{N}_{\omega-c\zeta}^\delta$ and $\xi \mathcal{N}_{\omega+c\delta}^\delta$
 209 that commute with $q_{\check{C}}$ and $i : H_0(D \setminus Q_{\omega+c\delta}^\delta, D \setminus P^\delta) \rightarrow H_0(D \setminus Q_{\omega-c\zeta}^\delta, D \setminus P^\delta)$. Let
 210 $q_{\mathcal{R}} : H_d(\mathcal{R}^\delta(P, Q_{\omega-c\zeta})) \rightarrow H_d(\mathcal{R}^{2\delta}(P, Q_{\omega+c\delta}))$ be induced by inclusion. Then $\text{rk } q_{\check{C}} \geq \text{rk } q_{\mathcal{R}}$
 211 as $q_{\mathcal{R}}$ factors through $q_{\check{C}}$. As we have assumed $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\zeta+\delta)})$ Lemma 4
 212 implies $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$. It follows that, whenever $\text{rk } q_{\mathcal{R}} \geq$
 213 $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta}))$, we have

$$214 \text{rk } i = \text{rk } q_{\check{C}} \geq \text{rk } q_{\mathcal{R}} \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega).$$

Because j is surjective by hypothesis $\text{rk } j = \dim H_0(\overline{B_\omega}, \overline{D}) = \dim H_0(D \setminus B_\omega)$ so $\text{rk } j \geq \text{rk } i$ by Lemma 3. Therefore, $\text{rk } j = \text{rk } i$ as we have shown $\text{rk } i \geq \dim H_0(D \setminus B_\omega)$. Because P is a finite point set we know that $\text{im } i$ is finite-dimensional and, because $\text{rk } i = \text{rk } j$, $\text{im } j = H_0(\overline{B_\omega}, \overline{D})$ is finite dimensional as well. So $\text{im } j$ is isomorphic to $\text{im } i$ as a subspace of $H_0(Q_{\omega-c\zeta}^\delta, P^\delta)$ which, because j is surjective, requires the map ℓ induced by inclusion to be injective. Therefore $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-c\zeta}^\delta$ surrounds P^δ in D by Lemma 2. \blacktriangleleft

4 From Coverage Testing to the Analysis of Scalar Fields

Because the TCC only confirms coverage of a *superlevel* set $D \setminus B_\omega$, we cannot guarantee coverage of the entire domain. Indeed, we could compute the persistent homology of the *restriction* of f to the superlevel set we cover in the standard way [3]. Instead, we will approximate the persistent homology of the sublevel set filtration *relative* to the sublevel set B_ω .



Figure 2 Full, restricted, and relative barcodes of the function (left).

We will first introduce the notion of an extension which will provide us with maps on relative homology induced by inclusion via excision. However, even then, a map that factors through our pair (D, B_ω) is not enough to prove an interleaving of persistence modules by inclusion directly. To address this we impose conditions on sublevel sets near B_ω which generalize the assumptions made in the TCC on maps induced by the inclusions

$$D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)}$$

on 0-dimensional homology, to assumptions on maps induced by the corresponding inclusions

$$B_{\omega-c(\delta+\zeta)} \hookrightarrow B_\omega \hookrightarrow B_{\omega+c(\delta+\zeta)}$$

on homology in all dimensions k .

4.1 Extensions and Image Persistence Modules

Suppose D is a subspace of X . We define the extension of a surrounding pair in D to a surrounding pair in X with isomorphic relative homology.

► **Definition 7** (Extension). If V surrounds U in a subspace D of X let $\mathcal{EV} := V \sqcup (D \setminus U)$ denote the (disjoint) union of the separating set V with the complement of U in D . The **extension of** (U, V) **in** D is the pair $(D, \mathcal{EV}) = (U \sqcup (D \setminus U), V \sqcup (D \setminus U))$.

Lemma states that we can use these extensions to interleave a pair (U, V) with a sequence of subsets of (D, B) . Lemma we can apply excision to the relative homology groups in order

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245 to get equivalent maps on homology that are induced by inclusions. Proof of these facts, and
 246 other extensions to homomorphisms of persistence modules in the next section, can be found
 247 in the Appendix.

248 ▶ **Lemma 8.** Suppose V surrounds U in D and $B' \subseteq B \subset D$.

249 If $D \setminus B \subseteq U$ and $U \cap B' \subseteq V \subseteq B'$ then $B' \subseteq \mathcal{E}V \subseteq B$.

250 ▶ **Lemma 9.** Let (U, V) be an open surrounding pair in a subspace D of X .

251 Then $H_k(U \cap A, V) \hookrightarrow (A, \mathcal{E}V)$ is an isomorphism for all k and $A \subseteq D$ with $\mathcal{E}V \subset A$.

252 ▶ **Definition 10 (Image Persistence Module).** The **image persistence module** of a homomorphism $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$ is the family of subspaces $\{\Gamma_\alpha := \text{im } \gamma_\alpha\}$ in \mathbb{V} along with linear maps $\{\gamma_\alpha^\beta := v_\alpha^\beta|_{\text{im } \gamma_\alpha} : \Gamma_\alpha \rightarrow \Gamma_\beta\}$ and will be denoted by $\text{im } \Gamma$.

256 ▶ **Definition 11 (Image Module Homomorphism).** Given $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$ and $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$
 257 along with $(F, G) \in \text{Hom}^\delta(\mathbb{U}, \mathbb{S}) \times \text{Hom}^\delta(\mathbb{V}, \mathbb{T})$ let $\Phi(F, G) : \text{im } \Gamma \rightarrow \text{im } \Lambda$ denote the family
 258 of linear maps $\{\phi_\alpha := g_\alpha|_{\Gamma_\alpha} : \Gamma_\alpha \rightarrow \Lambda_{\alpha+\delta}\}$. $\Phi(F, G)$ is an **image module homomorphism**
 259 of degree δ if the following diagram commutes for all $\alpha \leq \beta$.³

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\gamma_\alpha[\beta-\alpha]} & V_\beta \\ \downarrow f_\alpha & & \downarrow g_\beta \\ S_{\alpha+\delta} & \xrightarrow{\lambda_{\alpha+\delta}[\beta-\alpha]} & T_{\beta+\delta} \end{array} \quad (2)$$

261 The space of image module homomorphisms of degree δ between $\text{im } \Gamma$ and $\text{im } \Lambda$ will be
 262 denoted $\text{Hom}^\delta(\text{im } \Gamma, \text{im } \Lambda)$.

263 The composition of image module homomorphisms are image module homomorphisms. Proof
 264 of this fact can be found in the Appendix.

265 Partial Interleavings of Image Modules

266 Image module homomorphisms introduce a direction to the traditional notion of interleaving.
 267 As we will see, our interleaving via Lemma 13 involves partially interleaving an image module
 268 to two other image modules whose composition is isomorphic to our target.

269 ▶ **Definition 12 (Partial Interleaving of Image Modules).** An image module homomorphism
 270 $\Phi(F, G)$ is a **partial δ -interleaving of image modules**, and denoted $\Phi_M(F, G)$, if there
 271 exists $M \in \text{Hom}^\delta(\mathbb{S}, \mathbb{V})$ such that $\Gamma[2\delta] = M \circ F$ and $\Lambda[2\delta] = G \circ M$.

272 uses partial interleavings surrounding a module \mathbb{V} to prove an interleaving of an image
 273 module with \mathbb{V} . Its proof is straightforward and can be found in the Appendix. It uses
 274 partial interleavings of a map Λ with $\mathbb{U} \rightarrow \mathbb{V}$ and $\mathbb{V} \rightarrow \mathbb{W}$ along with the hypothesis that
 275 $\mathbb{U} \rightarrow \mathbb{W}$ is isomorphic to \mathbb{V} to interleave $\text{im } \Lambda$ with \mathbb{V} . When applied, this hypothesis will be
 276 satisfied by assumptions on our sublevel set similar to those made in the TCC.

277 ▶ **Lemma 13.** Suppose $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$, $\Pi \in \text{Hom}(\mathbb{V}, \mathbb{W})$, and $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$.

278 If $\Phi_M(F, G) \in \text{Hom}^\delta(\text{im } \Gamma, \text{im } \Lambda)$ and $\Psi_G(M, N) \in \text{Hom}^\delta(\text{im } \Lambda, \text{im } \Pi)$ are partial
 279 δ -interleavings of image modules such that Γ is a epimorphism and Π is a monomorphism
 280 then $\text{im } \Lambda$ is δ -interleaved with \mathbb{V} .

255 ³ Recall that $\gamma_\alpha[\beta - \alpha] = v_\alpha^\beta \circ \gamma_\alpha$ and $\lambda_\alpha[\beta - \alpha] = t_\alpha^\beta \circ \lambda_\alpha$.

281 **4.1.0.1 Proof of the Interleaving**

282 For $w, \alpha \in \mathbb{R}$ let \mathbb{D}_w^k denote the k th persistent (relative) homology module of the filtration
 283 $\{(D_{\lfloor \alpha \rfloor w}, B_w)\}_{\alpha \in \mathbb{R}}$ with respect to B_w , and let $\mathbb{P}_w^{\varepsilon, k}$ denote the k th persistent (relative) homology module of $\{(P_{\lfloor \alpha \rfloor w}^\varepsilon, Q_w^\varepsilon)\}_{\alpha \in \mathbb{R}}$. Similarly, let $\check{C}\mathbb{P}_w^{\varepsilon, k}$ and $R\mathbb{P}_w^{\varepsilon, k}$ denote the corresponding Čech and Vietoris-Rips filtrations, respectively. We will omit the dimension k and write \mathbb{D}_w
 285 (resp. \mathbb{P}_w^ε) if a statement holds for all dimensions.

286 If Q_w^ε surrounds P^ε in D let $\mathcal{E}\mathbb{P}_w^\varepsilon$ denote the k th persistent homology module of the filtration of extensions $\{(\mathcal{E}P_{\lfloor \alpha \rfloor w}^\varepsilon, \mathcal{E}Q_w^\varepsilon)\}$, where $\mathcal{E}P_{\lfloor \alpha \rfloor w}^\varepsilon = P_{\lfloor \alpha \rfloor w}^\varepsilon \cup (D \setminus P^\varepsilon)$. Lemma 9 can be extended to show that we have isomorphisms $\mathcal{E}_w^\varepsilon \in \text{Hom}(\mathbb{P}_w^\varepsilon, \mathcal{E}\mathbb{P}_w^\varepsilon)$ of persistence modules induced by inclusions. If $\varepsilon < \varrho_D$ then for any $\alpha \in \mathbb{R}$ the inclusion $\check{C}^\varepsilon(P_{\lfloor \alpha \rfloor w}, Q_w) \hookrightarrow (P_{\lfloor \alpha \rfloor w}^\varepsilon, Q_w^\varepsilon)$ is a homotopy equivalence by the Nerve Theorem. As the module homomorphisms of $\check{C}\mathbb{P}_w^\varepsilon$ and \mathbb{P}_w^ε are induced by inclusion we have an isomorphism $\mathcal{N}_w^\varepsilon \in \text{Hom}(\check{C}\mathbb{P}_w^\varepsilon, \mathbb{P}_w^\varepsilon)$ of persistence modules that commutes with maps induced by inclusions by the Persistent Nerve Lemma. As the isomorphisms of $\mathcal{E}_w^\varepsilon$ are given by excision they are induced by inclusions, so the composition $\mathcal{E}\mathcal{N}_w^\varepsilon := \mathcal{E}_w^\varepsilon \circ \mathcal{N}_w^\varepsilon$ is an isomorphism that commutes with maps induced by inclusion as well. The following lemma uses these isomorphisms along with inclusions $\mathcal{I}_w^\varepsilon \in \text{Hom}(\check{C}\mathbb{P}_w^\varepsilon, R\mathbb{P}_w^{2\varepsilon})$ and $\mathcal{J}_w^\varepsilon \in \text{Hom}(R\mathbb{P}_w^\varepsilon, \check{C}\mathbb{P}_w^\varepsilon)$ to establish image module homomorphisms by maps $\Sigma_w^\varepsilon \in \text{Hom}(\mathbb{P}_w^\varepsilon, R\mathbb{P}_w^{2\varepsilon})$ and $\Upsilon_w^\varepsilon \in \text{Hom}(R\mathbb{P}_w^\varepsilon, \mathbb{P}_w^\varepsilon)$. Proof of this lemma, along with the existence of the maps $\mathcal{E}\mathcal{N}_w^\varepsilon$ can be found in the Appendix.

300 **► Lemma 14.** For $w \in \mathbb{R}$ and $\varepsilon \leq \varrho_D/4$ let $\Lambda^\varepsilon \in \text{Hom}(\mathcal{E}\mathbb{P}_w^\varepsilon, \mathcal{E}\mathbb{P}_z^{2\varepsilon})$ and $R\Lambda \in \text{Hom}(R\mathbb{P}_w^{2\varepsilon}, R\mathbb{P}_z^{4\varepsilon})$.
 301 Then $\tilde{\Phi}(\Sigma_w^\varepsilon, \Sigma_z^{2\varepsilon}) \in \text{Hom}(\text{im } \Lambda^\varepsilon, \text{im } R\Lambda)$ and $\tilde{\Psi}(\Upsilon_w^{2\varepsilon}, \Upsilon_z^{4\varepsilon}) \in \text{Hom}(\text{im } R\Lambda, \text{im } \Lambda^{2\varepsilon})$ are image
 302 module homomorphisms.

303 Suppose $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D and $D \setminus B_\omega \subseteq P^\delta$. Then, because f is c -Lipschitz,
 304 $B_{\omega-3c\delta} \cap P^\delta \subseteq Q_{\omega-2c\delta}^\delta$ and $B_\omega \cap P^\delta \subseteq Q_{\omega+c\delta}^{2\delta}$. Similarly, $Q_{\omega-2c\delta}^{2\delta} \subseteq B_\omega$ and $Q_{\omega+c\delta}^{4\delta} \subseteq B_{\omega+5c\delta}$.
 305 Therefore, by Lemma 8

$$B_{\omega-3c\delta} \subseteq \mathcal{E}Q_{\omega-2c\delta}^\delta \subseteq \mathcal{E}Q_{\omega-2c\delta}^{2\delta} \subseteq B_\omega \subseteq \mathcal{E}Q_{\omega+c\delta}^{2\delta} \subseteq \mathcal{E}Q_{\omega+c\delta}^{4\delta} \subseteq B_{\omega+5c\delta}.$$

307 We have the following commutative diagrams of persistence modules where all maps are
 308 induced by inclusions. Proof that inclusions given by Lemma 8 extend to maps (F, G) and
 309 (M, N) of persistence modules can be found in the Appendix.

310

$$\begin{array}{ccc} \mathbb{D}_{\omega-3c\delta} & \xrightarrow{\Gamma} & \mathbb{D}_\omega \\ \downarrow F & & \downarrow G \\ \mathcal{E}\mathbb{P}_{\omega-2c\delta}^\delta & \xrightarrow{\Lambda} & \mathcal{E}\mathbb{P}_{\omega+c\delta}^{2\delta} \end{array} \quad (3a) \quad \begin{array}{ccc} \mathcal{E}\mathbb{P}_{\omega-2c\delta}^{2\delta} & \xrightarrow{\Lambda'} & \mathcal{E}\mathbb{P}_{\omega+c\delta}^{4\delta} \\ \downarrow M & & \downarrow N \\ \mathbb{D}_\omega & \xrightarrow{\Pi} & \mathbb{D}_{\omega+5c\delta} \end{array} \quad (3b)$$

312 In the following let $R\Lambda \in \text{Hom}(R\mathbb{P}_{\omega-2c\delta}^{2\delta}, R\mathbb{P}_{\omega+c\delta}^{4\delta})$ be induced by inclusion. Clearly,
 313 $\Phi(F, G)$ is an image module homomorphism of degree $2c\delta$ and $\Psi(M, N)$ is an image module
 314 homomorphism of degree $4c\delta$. By Lemma 14 we have image module homomorphisms
 315 $\tilde{\Phi}(\Sigma_{\omega-2c\delta}^\delta, \Sigma_{\omega+c\delta}^{2\delta})$ and $\tilde{\Psi}(\Upsilon_{\omega-2c\delta}^{2\delta}, \Upsilon_{\omega+c\delta}^{4\delta})$. Therefore, as the composition of image module
 316 homomorphisms are image module homomorphisms we have

$$317 \quad \mathcal{R}\Phi := \tilde{\Phi} \circ \Phi \in \text{Hom}^{2c\delta}(\text{im } \Gamma, \text{im } R\Lambda) \quad \text{and} \quad \mathcal{R}\Psi := \tilde{\Psi} \circ \Psi \in \text{Hom}^{4c\delta}(\text{im } R\Lambda, \text{im } \Pi)$$

318 given by the compositions

$$319 \quad \mathcal{R}\Phi(\mathcal{R}F, \mathcal{R}G) := (\Sigma_{\omega-2c\delta}^\delta \circ F, \Sigma_{\omega+c\delta}^{2\delta} \circ G) \quad \text{and} \quad \mathcal{R}\Psi(\mathcal{R}M, \mathcal{R}N) := (M \circ \Upsilon_{\omega-2c\delta}^{2\delta}, N \circ \Upsilon_{\omega+c\delta}^{4\delta}).$$

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Because all maps are induced by inclusions, or commute with maps induced by inclusions it can be shown that $\mathcal{R}\Phi_{RM}$ is a partial $2c\delta$ -interleaving of image modules and $\mathcal{R}\Psi_{RG}$ is a partial $4c\delta$ -interleaving of image modules by a straightforward diagram chasing argument. Proof of these facts can be found in the Appendix. These maps, along with assumptions that imply $\text{im}(\mathbb{D}_{\omega-3c\delta} \rightarrow \mathbb{D}_{\omega+5c\delta}) \cong \mathbb{D}_\omega$ provide the proof of Theorem 15 by Lemma 13.

► **Theorem 15.** Let \mathbb{X} be a d -manifold, $D \subset \mathbb{X}$ and $f : D \rightarrow \mathbb{R}$ be a c -Lipschitz function. Let $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $B_{\omega-3c\delta}$ surrounds D in \mathbb{X} . Let $P \subset D$ be a finite subset and suppose $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $H_k(B_\omega) \cong H_k(B_{\omega+5c\delta})$ for all k .

If $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D then the k th persistent homology module of $\{\mathcal{R}^{2\delta}(P_{[\alpha]\omega-2c\delta}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{[\alpha]\omega+c\delta}, Q_{\omega+c\delta})\}_{\alpha \in \mathbb{R}}$ is $4c\delta$ -interleaved with that of $\{(D_{[\alpha]\omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$.

5 Approximation of the Truncated Diagram

In this section we will relate the relative persistence diagram that we have approximated in the previous section to a truncation of the full diagram. Let \mathbb{L}^k denote the k th persistent homology module of the sublevel set filtration $\{B_\alpha\}_{\alpha \in \mathbb{R}}$. As in the previous section, let \mathbb{D}_ω^k denote the k th persistent (relative) homology module of $\{(D_{[\alpha]\omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$. Throughout we will assume that we are taking homology in a field \mathbb{F} and that the homology groups $H_k(B_\alpha)$ and $H_k(D_{[\alpha]\omega}, B_\omega)$ are finite dimensional vector spaces for all k and $\alpha \in \mathbb{R}$. We will use the interval decomposition of \mathbb{L}^k to give a decomposition of the relative module \mathbb{D}_ω^k in terms of a truncation of \mathbb{L}^k . Recall, the *truncated diagram* is defined to be that of \mathbb{L}^k consisting only of those features born after ω . For fixed $\omega \in \mathbb{R}$ we will define the truncation \mathbb{T}_ω^k of \mathbb{L}^k in terms of the intervals decomposing \mathbb{L}^k that are in $[\omega, \infty)$.

Truncated Interval Modules

For an interval $I = [s, t] \subseteq \mathbb{R}$ let $I_+ := [t, \infty)$ and $I_- := (-\infty, s]$. For $\omega \in \mathbb{R}$ let \mathbb{F}_ω^I denote the interval module consisting of vector spaces $\{F_{[\alpha]\omega}^I\}_{\alpha \in \mathbb{R}}$ and linear maps $\{f_{[\alpha], [\beta]\omega}^I : F_{[\alpha]\omega}^I \rightarrow F_{[\beta]\omega}^I\}_{\alpha \leq \beta}$ where

$$F_{[\alpha]\omega}^I := \begin{cases} F_\alpha^I & \text{if } \omega \in I_- \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{[\alpha], [\beta]\omega}^I := \begin{cases} f_{\alpha, \beta}^I & \text{if } \omega \in I_- \\ 0 & \text{otherwise.} \end{cases}$$

For a collection \mathcal{I} of intervals let $\mathcal{I}_\omega := \{I \in \mathcal{I} \mid \omega \in I\}$.

► **Lemma 16.** Suppose \mathcal{I}^k and \mathcal{I}^{k-1} are collections of intervals that decompose \mathbb{L}^k and \mathbb{L}^{k-1} , respectively. Then the k th persistent homology module of $\{(D_{[\alpha]\omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ is equal to

$$\bigoplus_{I \in \mathcal{I}^k} \mathbb{F}_\omega^I \oplus \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} \mathbb{F}^{I+}$$

for all k .

Proof. (See Appendix A) ◀

Main Theorem

Let \mathbb{L}^k denote the k th persistent homology module of the sublevel set filtration $\{B_\alpha\}_{\alpha \in \mathbb{R}}$ of f and let \mathcal{I}^k denote the decomposing intervals of \mathbb{L}^k for all k . For a fixed $\omega \in \mathbb{R}$ let \mathbb{D}_ω^k

356 denote the k th persistent (relative) homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$. Let

$$357 \quad \mathbb{T}_\omega^k := \bigoplus_{I \in \mathcal{I}^k} \mathbb{F}_\omega^I$$

358 denote the ω -truncated k th persistent homology module of \mathbb{L}^k . Let

$$359 \quad \mathbb{L}_\omega^{k-1} := \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} \mathbb{F}^{I+}.$$

360 denote the submodule of \mathbb{D}_ω^k consisting of intervals $[\beta, \infty)$ corresponding to features $[\alpha, \beta]$
 361 in \mathbb{L}^{k-1} such that $\alpha \leq \omega < \beta$. Now, by Lemma 16 the k th persistent (relative) homology
 362 module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ is

$$363 \quad \mathbb{D}_\omega^k = \mathbb{T}_\omega^k \oplus \mathbb{L}_\omega^{k-1}.$$

364 Our main theorem combines this decomposition with our coverage and interleaving results of
 365 Theorems 6 and 15.

366 ▶ **Theorem 17.** *Let \mathbb{X} be an orientable d -manifold and let D be a compact subset of \mathbb{X} . Let
 367 $f : D \rightarrow \mathbb{R}$ be a c -Lipschitz function and $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $P \subset D$ is a
 368 $(\delta, 2\delta, \omega)$ -sublevel sample of f and $B_{\omega-3c\delta}$ surrounds D in \mathbb{X} .*

369 Suppose $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $H_k(B_\omega) \cong H_k(B_{\omega+5c\delta})$ for all k . If

$$370 \quad \text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$$

371 then the k th (relative) homology module of

$$372 \quad \{\mathcal{R}^{2\delta}(P_{\lfloor \alpha \rfloor \omega-2c\delta}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor \alpha \rfloor \omega+c\delta}, Q_{\omega+c\delta})\}_{\alpha \in \mathbb{R}}$$

373 is $4c\delta$ -interleaved with $\mathbb{T}_\omega^k \oplus \mathbb{L}_\omega^{k-1}$: the k th persistent homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$.

374 6 Experiments

375 In this section we will discuss a number of experiments which illustrate the benefit of
 376 truncated diagrams, and their approximation by relative diagrams, in comparison to their
 377 restricted counterparts. We will focus on the persistent homology of functions on a square
 378 2d grid. We chose as our function a radially symmetric damped sinusoid with random noise,
 379 depicted in Figure 3, as it has prominent persistent homology in dimension one.

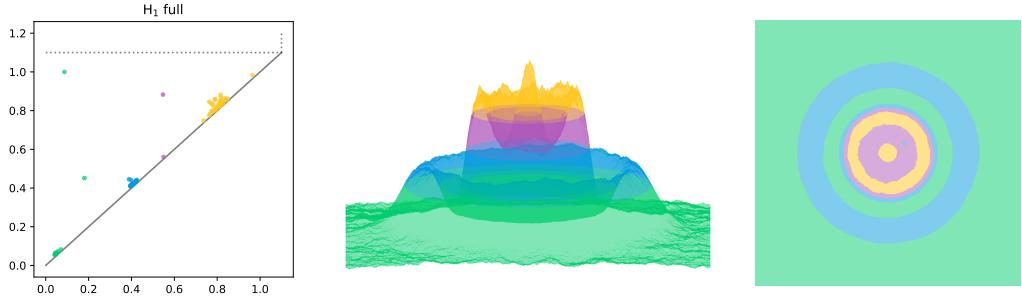
380 Experimental setup.

384 Throughout, the four interlevel sets shown correspond to the ranges $[0, 0.3)$, $[0.3, 0.5)$, $[0.5, 0.7)$,
 385 and $[0.7, 1)$, respectively. Our persistent homology computations were done primarily with
 386 Dionysus augmented with custom software for computing representative cycles of infinite
 387 features.⁴ The persistent homology of our function was computed with the lower-star
 388 filtration of the Freudenthal triangulation on an $N \times N$ grid over $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. We
 389 take this filtration as $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ where P is the set of grid points and $\delta = \sqrt{2}/N$.

390 We note that the purpose of these experiments is not to demonstrate the effectiveness of our
 391 approximation by Rips complexes, but to demonstrate the relationships between restricted,

383⁴ 3D figures were made with Mayavi, all other figures were made with Matplotlib.

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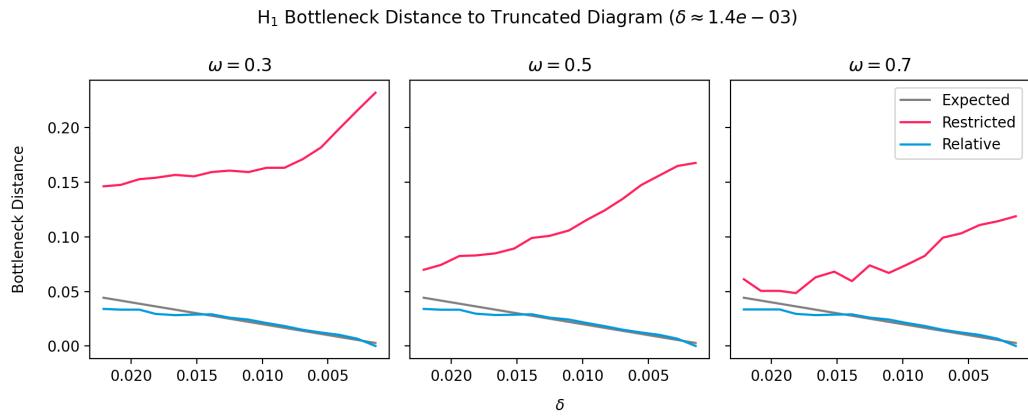
381 ■ **Figure 3** The H_1 persistence diagram of the sinusoidal function pictured to the right. Features
382 are colored by birth time, infinite features are drawn above the dotted line.

392 relative, and truncated diagrams. Therefore, for simplicity, we will omit the inclusion
393 $\mathcal{R}^{2\delta}(P_\alpha) \hookrightarrow \mathcal{R}^{4\delta}(P_\alpha)$ and take the persistent homology of $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ with sufficiently small
394 δ as our ground-truth.

395 In the following we will take $N = 1024$, so $\delta \approx 1.4 \times 10^{-3}$, as our ground-truth. Figure 3
396 shows the *full diagram* of our function with features colored by birth time. Therefore, for
397 $\omega = 0.3, 0.5, 0.7$ the *truncated diagram* is obtained by successively removing features in
398 each interlevel set. Recall the *restricted diagram* is that of the function restricted to the ω
399 *super-levelset filtration*, and computed with $\{\mathcal{R}^{2\delta}(P_\alpha \setminus Q_\omega)\}$. We will compare this restricted
400 diagram with the *relative diagram*, computed as the relative persistent homology of the
401 filtration of pairs $\{\mathcal{R}^{2\delta}(P_\alpha, Q_\omega)\}$.

402 The issue with restricted diagrams.

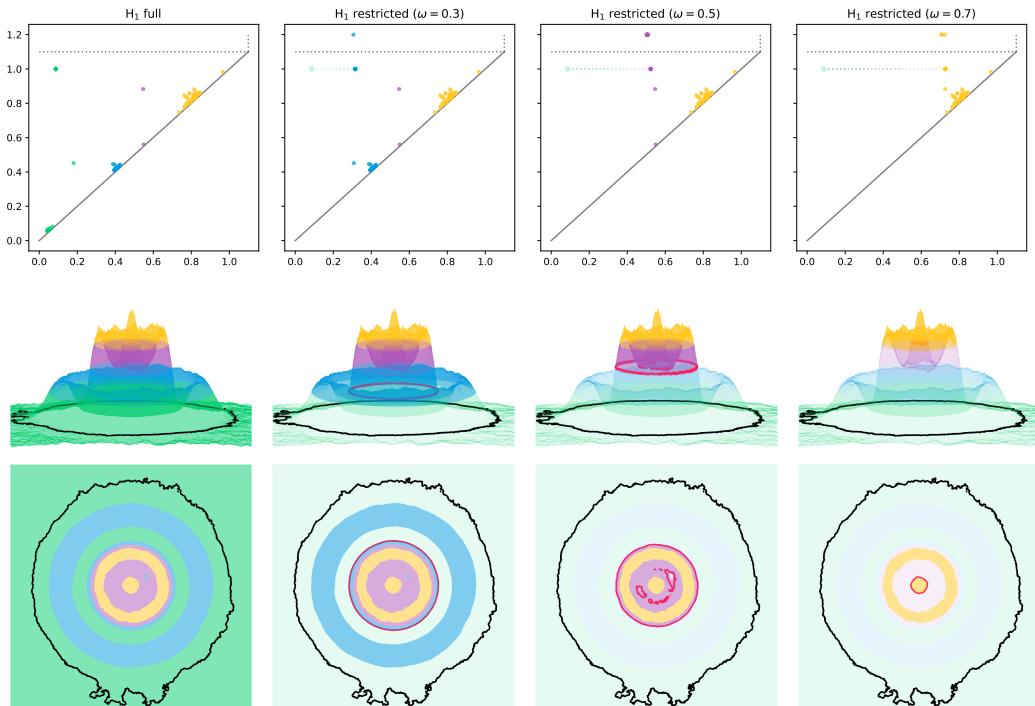
403 Figure ?? shows the bottleneck distance from the truncated diagram at full resolution
404 ($N = 1024$) to both the relative and restricted diagrams with varying resolution. Specifically,
405 the function on a 1024×1024 grid is down-sampled to grids ranging from 64×64 to 1024×1024 .
406 We also show the expected bottleneck distance to the true truncated diagram given by the
407 interleaving in Theorem 15 in black.



408 ■ **Figure 4** Comparison of the bottleneck distance between the truncated diagram and those of the
409 restricted and relative diagrams with increasing resolution.

410 As we can see, the relative diagram clearly performs better than the restricted diagram,

411 which diverges with increasing resolution. Recall that 1-dimensional features that are born
 412 before ω and die after ω become infinite 2-dimensional features in the relative diagram, with
 413 birth time equal to the death time of the corresponding feature in the full diagram. These
 414 same features remain 1-dimensional figures in the restricted diagram, but with their birth
 415 times shifted to ω .



416 ■ **Figure 5** (Top) H_1 persistence diagrams of the function depicted in Figure 3 restricted to *super-*
 417 *levelsets* at $\omega = 0.3, 0.5$, and 0.7 (on a 1024×1024 grid). The matching is shown between a feature in
 418 the full diagram (marked with a diamond) with its representative cycle in black. The corresponding
 419 representative cycle in the restricted diagram is pictured in red.

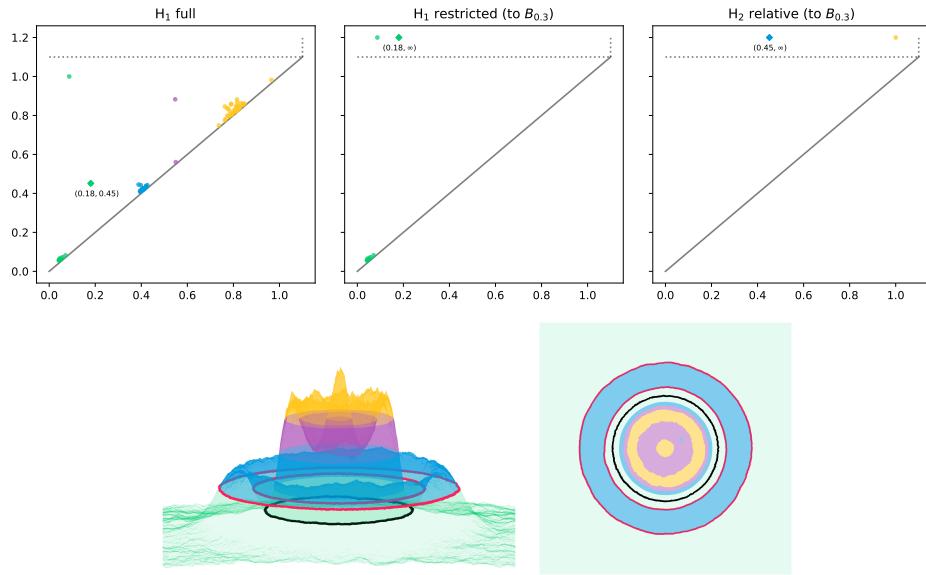
420 Figure 5 shows this distance for a feature that persists throughout the diagram. As the
 421 restricted diagram in full resolution the restricted filtration is a subset of the full filtration,
 422 so these features can be matched by their death simplices. For illustrative purposes we also
 423 show the representative cycles associated with these features.

424 Relative diagrams and reconstruction.

425 Now, imagine we obtain the persistence diagram of our sub-levelset B_ω . That is, we now
 426 know that we cover B_ω , or some subset, and do not want to re-compute the diagram above
 427 ω . If we compute the persistence diagram of the function restricted to the *sub*-levelset B_ω
 428 any 1-dimensional features born before ω that die after ω will remain infinite features in
 429 this restricted (below) diagram. Indeed, we could match these infinite 1-features with the
 430 corresponding shifted finite 1-features in the restricted (above) diagram, as shown in Figure 5.
 431 However, that would require sorting through all finite features that are born near ω and
 432 deciding if they are in fact features of the full diagram that have been shifted.

433 Recalling that these same features become infinite 2-features in the relative diagram, we
 434 can use the relative diagram instead and match infinite 1-features of the diagram restricted

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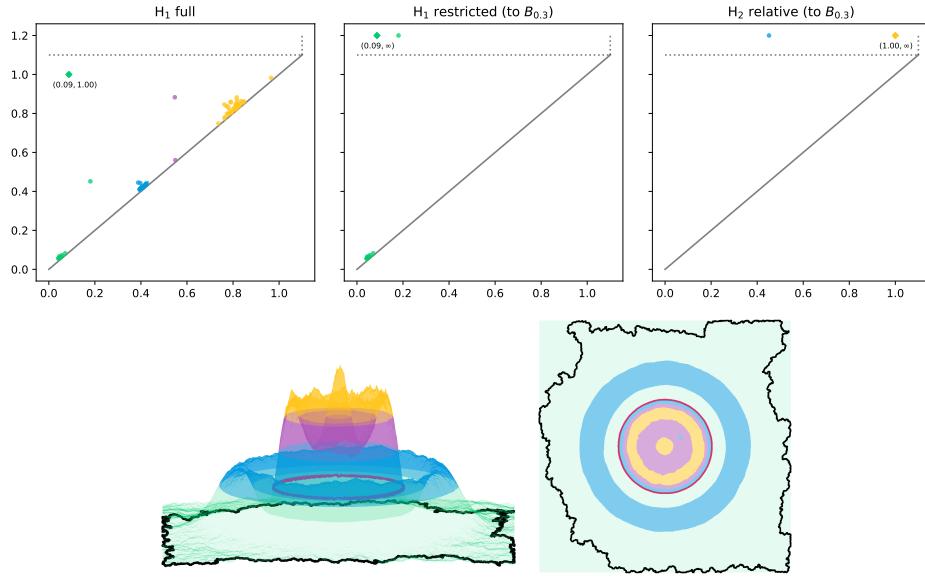


425 **Figure 6** (Top) The indicated infinite features in the restricted and relative diagrams correspond
 426 to the birth and death of the 1-feature $(0.18, 0.45)$ in the full diagram. (Bottom) In black, the
 427 representative cycle of the infinite 1-feature born at 0.18 in the restricted diagram is shown in black.
 428 In red, the *boundary* of the representative *relative* 2-cycle born at 0.45 in the relative diagram is
 429 shown in red.

440 below to infinite 2-features in the relative diagram, as shown in Figures 6 and 7. For this
 441 example the matching is given by sorting the 1-features by ascending and the 2-features by
 442 descending birth time. How to construct this matching in general, especially in the presence
 443 of infinite features in the full diagram, is the subject of future research.

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444 ■ **Figure 7** The infinite 1-features of the restricted diagram can be matched with the infinite
 445 2-features of the relative diagrams. The sequence birth times of relative 2-features in *decreasing*
 446 order correspond to the deaths of restricted 1-features in *increasing* order.

466 A Omitted Proofs

467 **Proof of Lemma 2.** This proof is in two parts.

468 **ℓ injective $\implies D \setminus B \subseteq U$** Suppose, for the sake of contradiction, that p is injective and
 469 there exists a point $x \in (D \setminus B) \setminus U$. Because B surrounds D in X the pair $(D \setminus B, \overline{D})$
 470 forms a separation of \overline{B} . Therefore, $H_0(\overline{B}) \cong H_0(D \setminus B) \oplus H_0(\overline{D})$ so

$$471 H_0(\overline{B}, \overline{D}) \cong H_0(D \setminus B).$$

472 So $[x]$ is non-trivial in $H_0(\overline{B}, \overline{D}) \cong H_0(D \setminus B)$ as x is in some connected component of
 473 $D \setminus B$. So we have the following sequence of maps induced by inclusions

$$474 H_0(\overline{B}, \overline{D}) \xrightarrow{f} H_0(\overline{B}, \overline{D} \cup \{x\}) \xrightarrow{g} H_0(\overline{V}, \overline{U}).$$

475 As $f[x]$ is trivial in $H_0(\overline{B}, \overline{D} \cup \{x\})$ we have that $\ell[x] = (g \circ f)[x]$ is trivial, contradicting
 476 our hypothesis that ℓ is injective.

477 **ℓ injective $\implies V$ surrounds U in D .** Suppose, for the sake of contradiction, that V does
 478 not surround U in D . Then there exists a path $\gamma : [0, 1] \rightarrow \overline{V}$ with $\gamma(0) \in U \setminus V$ and
 479 $\gamma(1) \in D \setminus U$. By Lemma 2 we know that $D \setminus B \subseteq U$, so $D \setminus B \subseteq U \setminus V$.

480 Choose $x \in D \setminus B$ and $z \in \overline{D}$ such that there exist paths $\xi : [0, 1] \rightarrow U \setminus V$ with $\xi(0) = x$,
 481 $\xi(1) = \gamma(0)$ and $\zeta : [0, 1] \rightarrow \overline{D} \cup (D \setminus U)$ with $\zeta(0) = z$, $\zeta(1) = \gamma(1)$. ξ, γ and ζ all
 482 generate chains in $C_1(\overline{V}, \overline{U})$ and $\xi + \gamma + \zeta = \gamma^* \in C_1(\overline{V}, \overline{U})$ with $\partial \gamma^* = x + z$. Moreover, z
 483 generates a chain in $C_0(\overline{U})$ as $\overline{D} \subseteq \overline{U}$. So $x = \partial \gamma^* + z$ is a relative boundary in $C_0(\overline{V}, \overline{U})$,
 484 thus $\ell[x] = \ell[z]$ in $H_0(\overline{V}, \overline{L})$. However, because B surrounds D , $[x] \neq [y]$ in $H_0(\overline{B}, \overline{D})$
 485 contradicting our assumption that ℓ is injective.

486

◀

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487 **Proof of Lemma 4.** Assume there exist $p, q \in P \setminus Q_{\omega-c\zeta}$ such that p and q are connected in
 488 $\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})$ but not in $D \setminus B_\omega$. So the shortest path from p, q is a subset of $(P \setminus Q_{\omega-c\zeta})^\delta$.
 489 For any $x \in (P \setminus Q_{\omega-c\zeta})^\delta$ there exists some $p \in P$ such that $f(p) > \omega - c\zeta$ and $\mathbf{d}(p, x) < \delta$.
 490 Because f is c -Lipschitz

$$491 \quad f(x) \geq f(p) - c\mathbf{d}(x, p) > \omega - c(\delta + \zeta)$$

492 so there is a path from p to q in $D \setminus B_{\omega-c(\delta+\zeta)}$, thus $[p] = [q]$ in $H_0(D \setminus B_{\omega-c(\delta+\zeta)})$.

493 But we have assumed that $[p] \neq [q]$ in $H_0(D \setminus B_\omega)$, contradicting our assumption that
 494 $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is injective, so any p, q connected in $\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})$ are
 495 connected in $D \setminus B_\omega$. That is, $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$. \blacktriangleleft

496 A.1 Extensions

497 **Proof of Lemma 8.** Note that $B' \setminus (D \setminus U) = B' \cap U \subseteq V$ implies $B' \subseteq V \sqcup (D \setminus U) = \mathcal{E}V$.
 498 Moreover, because $V \subseteq B$ and $D \setminus B \subseteq U$ implies $D \setminus U \subset D \setminus (D \setminus B) = B$, we have

$$499 \quad \mathcal{E}V = V \sqcup (D \setminus U) \subseteq B \cup (D \setminus U) = B.$$

500 So $B' \subseteq \mathcal{E}V \subseteq B$ as desired. \blacktriangleleft

501 **► Lemma 18.** If Q_w^ε surrounds P^ε in D and $D \setminus B_{w+\varepsilon} \subseteq P^\varepsilon$ then we have the following
 502 sequence of homomorphisms of degree $c\varepsilon$ induced by inclusions

$$503 \quad \mathbb{D}_{w-c\varepsilon} \xrightarrow{F} \mathcal{EP}_w^\varepsilon \xrightarrow{M} \mathbb{D}_{w+c\varepsilon}.$$

504 **Proof.** Suppose $x \in (P^\varepsilon \cap B_{\lfloor \alpha - c\varepsilon \rfloor w - c\varepsilon}) \setminus B_{w+\varepsilon}$. Because $B_{w-\varepsilon} \subset B_{w+\varepsilon}$ we know $x \notin B_{w-\varepsilon}$
 505 so $w + c\varepsilon < f(x) \leq \alpha - c\varepsilon$ and there exists some $p \in P$ such that $\mathbf{d}(x, p) < \varepsilon$. Because f is
 506 c -Lipschitz it follows

$$507 \quad f(p) \leq f(x) + c\mathbf{d}(x, p) < \alpha - c\varepsilon + c\varepsilon = \alpha$$

508 and

$$509 \quad f(p) \geq f(x) - c\mathbf{d}(x, p) > w + c\varepsilon - c\varepsilon = w.$$

510 So $x \in P_{\lfloor \alpha \rfloor w}^\varepsilon$.

511 Now, suppose $x \in P_{\lfloor \alpha \rfloor w}^\varepsilon \setminus B_{w+c\varepsilon}$. So $w + c\varepsilon < f(x)$ and there exists some $p \in P_{\lfloor \alpha \rfloor w}$ such
 512 that $\mathbf{d}(x, p) < \varepsilon$. Because f is c -Lipschitz it follows

$$513 \quad f(x) \leq f(p) + c\mathbf{d}(x, p) < a + c\varepsilon.$$

514 So $x \in B_{\lfloor \alpha + c\varepsilon \rfloor w + c\varepsilon} \setminus B_{w+c\varepsilon}$.

515 Because $D \setminus B_{w+c\varepsilon} \subseteq P^\varepsilon$ we know that $D \setminus P^\varepsilon \subseteq B_{w+c\varepsilon}$, so

$$516 \quad D_{\lfloor \alpha - c\varepsilon \rfloor w - c\varepsilon} \setminus B_{w+c\varepsilon} \subseteq P_{\lfloor \alpha \rfloor w}^\varepsilon \setminus B_{w+c\varepsilon} \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor w + c\varepsilon} \setminus B_{w+c\varepsilon}$$

517 implies

$$518 \quad D_{\lfloor \alpha - c\varepsilon \rfloor w - c\varepsilon} \subseteq P_{\lfloor \alpha \rfloor w}^\varepsilon \cup (D \setminus P^\varepsilon) = \mathcal{EP}_{\lfloor \alpha \rfloor w}^\varepsilon \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor w + c\varepsilon}$$

519 as desired.

520 Because f is c -Lipschitz, $B_{w-c\varepsilon} \cap P^\delta \subseteq Q_w^\varepsilon$ so $B_{w-c\varepsilon} \subseteq \mathcal{E}Q_w^\varepsilon \subseteq B_{w+c\varepsilon}$ by Lemma 8. It
 521 follows that we have homomorphisms $F \in \text{Hom}^{c\varepsilon}(\mathbb{D}_{w-c\varepsilon}, \mathcal{EP}_w^\varepsilon)$ and $M \in \text{Hom}^{c\varepsilon}(\mathcal{EP}_w^\varepsilon, \mathbb{D}_{w+c\varepsilon})$
 522 induced by inclusions. \blacktriangleleft

523 **Proof of Lemma 9.** Because V surrounds U in D , $(U \setminus V, D \setminus U)$ is a separation of $D \setminus V$, a
 524 subspace of D . So $\text{cl}_D(U \setminus V) \setminus U = \text{cl}_D(U \setminus V) \cap (D \setminus U) = \emptyset$ which implies $\text{cl}_D(U \setminus V) \subseteq$
 525 $U = \text{int}_D(U)$ as U is open in D . Therefore,

$$\begin{aligned} 526 \quad \text{cl}_D(D \setminus U) &= D \setminus \text{int}_D(U) \\ 527 \quad &\subseteq D \setminus \text{cl}_D(U \setminus V) \\ 528 \quad &= \text{int}_D(D \setminus (U \setminus V)) \\ 529 \quad &= \text{int}_D(\mathcal{E}V). \end{aligned}$$

530 SO,

$$\begin{aligned} 531 \quad H_k(U \cap A, V) &= H_k(A \setminus (D \setminus U), \mathcal{E}V \setminus (D \setminus U)) \\ 532 \quad &\cong H_k(A, \mathcal{E}V) \end{aligned}$$

533 for all k and any $A \subseteq D$ such that $\mathcal{E}V \subset A$ by Excision. \blacktriangleleft

534 **► Lemma 19.** If Q_w^ε surrounds P^ε in D then there is an isomorphism $\mathcal{E}Q_w^\varepsilon \in \text{Hom}(\mathbb{P}_w^\varepsilon, \mathcal{E}\mathbb{P}_w^\varepsilon)$.

535 **Proof.** Because $P_{\lfloor a \rfloor w} := P \cap D_{\lfloor a \rfloor w}$ and $B_w \subseteq D_{\lfloor a \rfloor w}$ we know $Q_w = P \cap B_w \subseteq P_{\lfloor a \rfloor w}$ for all
 536 $a \in \mathbb{R}$. So

$$537 \quad \mathcal{E}Q_a^\varepsilon = Q_a^\varepsilon \cup (D \setminus P^\varepsilon) \subseteq P_{\lfloor a \rfloor w}^\varepsilon \cup (D \setminus P^\varepsilon) = \mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon.$$

538 As $(P^\varepsilon, Q_w^\varepsilon)$ is a surrounding pair in D , P^ε is open in D and $\mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon \subseteq D$ is such that
 539 $\mathcal{E}Q_a^\varepsilon \subseteq \mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon$ it follows that

$$540 \quad H_k(P_{\lfloor a \rfloor w}^\varepsilon, Q_a^\varepsilon) = H_k(P^\varepsilon \cap \mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon, Q_a^\varepsilon) \cong H_k(\mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon, \mathcal{E}Q_a^\varepsilon)$$

541 by Lemma 9.

542 Because these isomorphisms commute with inclusions we have an isomorphism $\mathcal{E}_{\lfloor \cdot \rfloor w}^\varepsilon \in$
 543 $\text{Hom}(\mathbb{P}_w^\varepsilon, \mathcal{E}\mathbb{P}_w^\varepsilon)$ defined to be the family $\{\mathcal{E}_{\lfloor \alpha \rfloor w}^\varepsilon : \mathcal{P}_{\lfloor \alpha \rfloor w}^\varepsilon \rightarrow \mathcal{E}\mathcal{P}_{\lfloor \alpha \rfloor w}^\varepsilon\}$. \blacktriangleleft

544 A.2 Image Modules

545 **► Lemma 20.** Suppose $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$, $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$, and $\Lambda' \in \text{Hom}(\mathbb{S}', \mathbb{T}')$. If $\Phi(F, G) \in$
 546 $\text{Hom}^\delta(\text{im } \Gamma, \text{im } \Lambda)$ and $\Phi'(F', G') \in \text{Hom}^{\delta'}(\text{im } \Lambda, \text{im } \Lambda')$ then $\Phi''(F' \circ F, G' \circ G) := \Phi' \circ \Phi \in$
 547 $\text{Hom}^{\delta+\delta'}(\text{im } \Gamma, \text{im } \Lambda')$.

548 **Proof.** Because $\Phi(F, G)$ is an image module homomorphism of degree δ we have $g_{\beta-\delta} \circ$
 549 $\gamma_{\alpha-\delta}[\beta - \alpha] = \lambda_\alpha[\beta - \alpha] \circ f_{\alpha-\delta}$. Similarly, $g'_\beta \circ \lambda_\alpha[\beta - \alpha] = \lambda'_{\alpha+\delta'}[\beta - \alpha] \circ f'_\alpha$. So $\Phi''(F' \circ$
 550 $F, G' \circ G) \in \text{Hom}^{\delta+\delta'}(\text{im } \Gamma, \text{im } \Lambda')$ as

$$551 \quad g'_\beta \circ (g_{\beta-\delta} \circ \gamma_{\alpha-\delta}[\beta - \alpha]) = (g'_\beta \circ \lambda_\alpha[\beta - \alpha]) \circ f_{\alpha-\delta} = \lambda_{\alpha+\delta'}[\beta - \alpha] \circ f'_\alpha \circ f_{\alpha-\delta}$$

552 for all $\alpha \leq \beta$. \blacktriangleleft

553 **Proof of Lemma 13.** For ease of notation let Φ denote $\Phi_M(F, G)$ and Ψ denote $\Psi_G(M, N)$.

554 If Γ is an epimorphism γ_α is surjective so $\Gamma_\alpha = V_\alpha$ and $\phi_\alpha = g_\alpha|_{\Gamma_\alpha} = g_\alpha$ for all α . So
 555 $\text{im } \Gamma = \mathbb{V}$ and $\Phi \in \text{Hom}^\delta(\mathbb{V}, \text{im } \Lambda)$.

556 If Π is a monomorphism then π_α is injective so we can define a natural isomorphism
 557 $\pi_\alpha^{-1} : \Pi_\alpha \rightarrow V_\alpha$ for all α . Let Ψ^* be defined as the family of linear maps $\{\psi_\alpha^* := \pi_\alpha^{-1} \circ \psi_\alpha :$

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558 $\Lambda_\alpha \rightarrow V_{\alpha+\delta}\}$. Because Ψ is a partial δ -interleaving of image modules, $n_\alpha \circ \lambda_\alpha = \pi_{\alpha+\delta} \circ m_\alpha$.
559 So, because $\psi_\alpha = n_\alpha|_{\Lambda_\alpha}$ for all α ,

$$\begin{aligned} 560 \quad \mathbf{im} \psi_\alpha^* &= \mathbf{im} \pi_{\alpha+\delta}^{-1} \circ \psi_\alpha \\ 561 \quad &= \mathbf{im} \pi^{-1} \circ (n_\alpha \circ \lambda_\alpha) \\ 562 \quad &= \mathbf{im} \pi^{-1} \circ (\pi_{\alpha+\delta} \circ m_\alpha) \\ 563 \quad &= \mathbf{im} m_\alpha. \end{aligned}$$

564 It follows that $\mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ \psi_\alpha^* = \mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha$

565 Similarly, because Ψ is a δ -interleaving of image modules $n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha = w_{\alpha+\delta}^{\beta+\delta} \circ \pi_{\alpha+\delta} \circ m_\alpha$.

566 Moreover, because Π is a homomorphism of persistence modules, $w_{\alpha+\delta}^{\beta+\delta} \circ \pi_{\alpha+\delta} = \pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta}$,
567 so

$$568 \quad n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha = \pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha.$$

569 As $\psi_\beta \circ \lambda_\alpha^\beta = n_\beta \circ \lambda_\alpha^\beta = n_\beta \circ t_\alpha^\beta|_{\Lambda_\alpha}$ it follows

$$\begin{aligned} 570 \quad \mathbf{im} \psi_\beta^* \circ \lambda_\alpha^\beta &= \mathbf{im} \pi_{\beta+\delta}^{-1} \circ (n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha) \\ 571 \quad &= \mathbf{im} \pi_{\beta+\delta}^{-1} \circ (\pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta}) \circ m_\alpha \\ 572 \quad &= \mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha \\ 573 \quad &= \mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ \psi_\alpha^*. \end{aligned}$$

574 So we may conclude that $\Psi^* \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbb{V})$.

575 So $\Phi \in \text{Hom}^\delta(\mathbb{V}, \mathbf{im} \Lambda)$ and $\Psi_G^* \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbb{V})$. As we have shown, $\mathbf{im} \psi_{\alpha-\delta}^* =$
576 $\mathbf{im} m_{\alpha-\delta}$ so $\mathbf{im} \phi_\alpha \circ \psi_{\alpha-\delta}^* = \mathbf{im} \phi_\alpha \circ m_{\alpha-\delta}$. Moreover, because γ_α is surjective $\phi_\alpha = g_\alpha$
577 and, because Φ is a partial δ -interleaving of image modules, $g_\alpha \circ m_{\alpha-\delta} = t_{\alpha-\delta}^{\alpha+\delta} \circ \lambda_{\alpha-\delta}$. As
578 $\lambda_{\alpha-\delta}^{\alpha+\delta} = t_{\alpha-\delta}^{\alpha+\delta}|_{\mathbf{im} \lambda_{\alpha-\delta}}$ it follows that $\mathbf{im} \phi_\alpha \circ \psi_{\alpha-\delta}^* = \mathbf{im} \lambda_{\alpha-\delta}^{\alpha+\delta}$.

579 Finally, $\psi_\alpha^* \circ \phi_\alpha = \pi_{\alpha+\delta}^{-1} \circ n_\alpha \circ g_{\alpha-\delta}$ where, because Ψ is a partial δ -interleaving of image
580 modules, $n_\alpha \circ g_{\alpha-\delta} = w_{\alpha-\delta}^{\alpha+\delta} \circ \pi_{\alpha-\delta}$. Because Π is a homomorphism of persistence modules
581 $w_{\alpha-\delta}^{\alpha+\delta} \circ \pi_{\alpha-\delta} = \pi_{\alpha+\delta} \circ v_{\alpha-\delta}^{\alpha+\delta}$. Therefore,

$$\begin{aligned} 582 \quad \psi_\alpha^* \circ \phi_\alpha &= \pi_{\alpha+\delta}^{-1} \circ n_\alpha \circ g_{\alpha-\delta} \\ 583 \quad &= \pi_{\alpha+\delta}^{-1} \circ (\pi_{\alpha+\delta} \circ v_{\alpha-\delta}^{\alpha+\delta}) \\ 584 \quad &= v_{\alpha-\delta}^{\alpha+\delta} \end{aligned}$$

585 which, along with $\phi_\alpha \circ \mathbf{im} \psi_{\alpha-\delta}^* = \lambda_{\alpha-\delta}^{\alpha+\delta}$ implies Diagrams ?? and ?? commute with
586 $\Phi \in \text{Hom}^\delta(\mathbb{V}, \mathbf{im} \Lambda)$ and $\Psi^* \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbb{V})$. We may therefore conclude that $\mathbf{im} \Lambda$ and
587 \mathbb{V} are δ -interleaved. \blacktriangleleft

588 A.3 Partial Interleavings

589 For all $w \in \mathbb{R}$ and $\varepsilon < \varrho_D$ let $\mathcal{I}_w^\varepsilon \in \text{Hom}(\check{\mathcal{C}}\mathbb{P}_w^\varepsilon, \mathcal{R}\mathbb{P}_w^{2\varepsilon})$ and $\mathcal{J}_w^\varepsilon \in \text{Hom}(\mathcal{R}\mathbb{P}_w^\varepsilon, \check{\mathcal{C}}\mathbb{P}_w^\varepsilon)$ be induced
590 by the inclusions

$$591 \quad \check{\mathcal{C}}^\varepsilon(P_{\lfloor \alpha \rfloor w}, Q_w) \subseteq \mathcal{R}^{2\varepsilon}(P_{\lfloor \alpha \rfloor w}, Q_w) \subseteq \check{\mathcal{C}}^{2\varepsilon}(P_{\lfloor \alpha \rfloor w}, Q_w)$$

592 and define the composite maps

$$593 \quad \Sigma_w^\varepsilon := \mathcal{I}_w^\varepsilon \circ (\mathcal{E}\mathcal{N}_w^\varepsilon)^{-1} \in \text{Hom}(\mathbb{P}_w^\varepsilon, \mathcal{R}\mathbb{P}_w^{2\varepsilon}) \quad \text{and} \quad \Upsilon_w^\varepsilon := \mathcal{E}\mathcal{N}_w^\varepsilon \circ \mathcal{J}_w^\varepsilon \in \text{Hom}(\mathcal{R}\mathbb{P}_w^\varepsilon, \mathbb{P}_w^\varepsilon).$$

594 **Proof of Lemma 14.** By the Persistent Nerve Lemma we have $\check{C}\Lambda \circ (\mathcal{EN}_w^\varepsilon)^{-1} = (\mathcal{EN}_z^\eta)^{-1} \circ \Lambda$
 595 for $\check{C}\Lambda \in \text{Hom}(\check{\mathbb{CP}}_w^\varepsilon, \check{\mathbb{CP}}_z^\eta)$ induced by inclusions. As $\mathcal{R}\Lambda \circ \mathcal{I}_w^\varepsilon = \mathcal{I}_z^\eta \circ \check{C}\Lambda$

$$596 \quad \mathcal{R}\Lambda \circ \mathcal{I}_w^\varepsilon \circ (\mathcal{EN}_w^\varepsilon)^{-1} = \mathcal{I}_z^\eta \circ \check{C}\Lambda \circ (\mathcal{EN}_w^\varepsilon)^{-1} = \mathcal{I}_z^\eta \circ (\mathcal{EN}_z^\eta)^{-1} \circ \Lambda.$$

597 It follows that $\mathcal{R}\Lambda \circ \Sigma_w^\varepsilon = \Sigma_z^\eta \circ \Lambda$ by the definition of Σ . So Diagram 2 commutes and we
 598 may therefore conclude that $\tilde{\Phi}(\Sigma_w^\varepsilon, \Sigma_z^\eta)$ is an image module homomorphism.

599 By the Persistent Nerve Lemma we have $\mathcal{EN}_z^{2\eta} \circ \check{C}\Lambda' = \check{C}\Lambda \circ \mathcal{EN}_w^{2\varepsilon}$ for $\check{C}\Lambda' \in \text{Hom}(\check{\mathbb{CP}}_w^{2\varepsilon}, \check{\mathbb{CP}}_z^{2\eta})$
 600 induced by inclusions. As $\mathcal{J}_z^\eta \circ \mathcal{R}\Lambda = \check{C}\Lambda' \circ \mathcal{J}_w^\varepsilon$

$$601 \quad \mathcal{EN}_z^{2\eta} \circ \mathcal{J}_z^\eta \circ \mathcal{R}\Lambda = \mathcal{EN}_z^{2\eta} \circ \check{C}\Lambda' \circ \mathcal{J}_w^\varepsilon = \check{C}\Lambda \circ \mathcal{EN}_w^{2\varepsilon} \circ \mathcal{J}_w^\varepsilon.$$

602 Once again, Diagram 2 commutes by the definition of Υ , so $\tilde{\Psi}(\Upsilon_w^{2\varepsilon}, \Upsilon_z^{2\eta})$ is an image module
 603 homomorphism. \blacktriangleleft

604 **► Lemma 21.** *The pair (RM, RG) factors $\mathcal{R}\Lambda[4c\delta]$ through \mathbb{D}_ω .*

605 **Proof.** Let $\Theta \in \text{Hom}(\mathcal{EP}_{\omega-2c\delta}^{2\delta}, \mathcal{EP}_{\omega+c\delta}^{2\delta})$ and $\check{C}\Theta \in \text{Hom}(\check{\mathbb{CP}}_{\omega-2c\delta}^{2\delta}, \check{\mathbb{CP}}_{\omega+c\delta}^{2\delta})$ be induced by
 606 inclusions so that $\Theta[4c\delta] = G \circ M$ and $\mathcal{R}\Lambda = \mathcal{I}_{\omega+c\delta}^{2\delta} \circ \check{C}\Theta \circ \mathcal{J}_{\omega-2c\delta}^{2\delta}$. So $\check{C}\Theta$ factors through Θ
 607 with the pair $(\mathcal{EN}_{\omega-2c\delta}^{2\delta}, (\mathcal{EN}_{\omega+c\delta}^{2\delta})^{-1})$ by Lemma ???. That is,

$$\begin{aligned} 608 \quad \mathcal{R}\Lambda &= \mathcal{I}_{\omega+c\delta}^{2\delta} \circ \check{C}\Theta \circ \mathcal{J}_{\omega-2c\delta}^{2\delta} \\ 609 &= (\mathcal{I}_{\omega+c\delta}^{2\delta} \circ (\mathcal{EN}_{\omega+c\delta}^{2\delta})^{-1}) \circ \Theta \circ (\mathcal{EN}_{\omega-2c\delta}^{2\delta} \circ \mathcal{J}_{\omega-2c\delta}^{2\delta}) \\ 610 &= \Sigma_{\omega+c\delta}^{2\delta} \circ \Theta \circ \Upsilon_{\omega-2c\delta}^{2\delta} \end{aligned}$$

611

612 As $\Theta[4c\delta] = G \circ M$ the result follows from the definition

$$613 \quad \mathcal{R}\Lambda[4c\delta] = (\Sigma_{\omega+c\delta}^{2\delta} \circ G) \circ (M \circ \Upsilon_{\omega-2c\delta}^{2\delta}) = RG \circ RM.$$

614 \blacktriangleleft

615 **► Corollary 22.** $\mathcal{R}\Phi_{RM} := \tilde{\Phi} \circ \Phi \in \text{Hom}^{2c\delta}(\text{im } \Gamma, \text{im } \mathcal{R}\Lambda)$ is a partial $2c\delta$ -interleaving of
 616 image modules.

617 **Proof.** Because F, M are induced by inclusions and $\Upsilon_{\omega-2c\delta}^{2\delta} \circ \Sigma_{\omega-2c\delta}^\delta$ commutes with inclusion
 618 it follows that

$$619 \quad \Gamma[3c\delta] = M \circ (\Upsilon_{\omega-2c\delta}^{2\delta} \circ \Sigma_{\omega-2c\delta}^\delta) \circ F = RM \circ RF.$$

620 So $\mathcal{R}\Phi$ with RM is a left $2c\delta$ -interleaving of image modules. As Lemma 21 implies $\mathcal{R}\Phi$
 621 (with RM) is a right $2c\delta$ -interleaving of image modules it follows that $\mathcal{R}\Phi_{RM}$ is a partial
 622 $2c\delta$ -interleaving of image modules. \blacktriangleleft

623 The proof of Corollary 23 is identical to that of Corollary 22.

624 **► Corollary 23.** $\mathcal{R}\Psi_{RG} := \Psi \circ \tilde{\Psi} \in \text{Hom}^{4c\delta}(\text{im } \mathcal{R}\Lambda, \text{im } \Pi)$ is a partial $4c\delta$ -interleaving of
 625 image modules.

626 **Proof.** This proof is identical to that of Corollary 22. Because G, N are induced by inclusions
 627 and $\Upsilon_{\omega+c\delta}^{4\delta} \circ \Sigma_{\omega+c\delta}^{2\delta}$ commutes with inclusion

$$628 \quad \Pi[6c\delta] = N \circ (\Upsilon_{\omega+c\delta}^{4\delta} \circ \Sigma_{\omega+c\delta}^{2\delta}) \circ G = RN \circ RG.$$

629 So $\mathcal{R}\Psi$ with RG is a right $4c\delta$ -interleaving of image modules. As Lemma 21 implies $\mathcal{R}\Psi$
 630 (with RG) is a left $2c\delta$ -interleaving of image modules it follows that $\mathcal{R}\Psi_{RG}$ is a partial
 631 $4c\delta$ -interleaving of image modules. \blacktriangleleft

632 **Proof of Theorem 15.** Let $\Lambda \in \text{Hom}(\mathcal{RP}_{\omega-2c\delta}^{2\delta}, \mathcal{RP}_{\omega+c\delta}^{4\delta})$ be induced by inclusions. Because
 633 $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D Diagrams 3a and 3b commute as all maps are
 634 induced by inclusions. Moreover, because $\delta < \varrho_D/4$ the isomorphisms provided by the Nerve
 635 Theorem commute with inclusions by Lemma ??.

636 As we have assumed that $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $H_k(B_\omega) \cong H_k(B_{\omega+5c\delta})$
 637 the five-lemma implies γ_α is surjective and π_α is an isomorphism (and therefore injective)
 638 for all α . So Γ is an epimorphism and Π is a monomorphism. Because $\mathcal{R}\Phi_{RM}(\mathcal{RF}, \mathcal{RG}) \in$
 639 $\text{Hom}^{2c\delta}(\mathbf{im} \Gamma, \mathbf{im} \mathcal{R}\Lambda)$ is a partial $2c\delta$ -interleaving of image modules and $\mathcal{R}\Psi_{RG}(\mathcal{RM}, \mathcal{RN}) \in$
 640 $\text{Hom}^{4c\delta}(\mathbf{im} \mathcal{R}\Lambda, \mathbf{im} \Pi)$ is a partial $4c\delta$ -interleaving of image modules it follows that $\mathbf{im} \mathcal{R}\Lambda$
 641 is $4c\delta$ -interleaved with \mathbb{D}_ω by Lemma 13. \blacktriangleleft

642 A.4 Truncated Interval Modules

643 **Proof of Lemma 16.** Suppose $\alpha \leq \omega$. So $H_k(D_{[\alpha]_\omega}, B_\omega) = 0$ as $D_{[\alpha]_\omega} = B_\omega \cup B_\alpha$ and
 644 $\mathbb{T}_\omega^k = 0$ as $F_\alpha^I = 0$ for any $I \in \mathcal{I}^k$ such that $\omega \in I_-$. Moreover, $\omega \in I$ for all $I \in \mathcal{I}_\omega^{k-1}$, thus
 645 $F_\alpha^{I+} = 0$ for all $\alpha \leq \omega$. So it suffices to assume $\omega < \alpha$.

646 Consider the long exact sequence of the pair $H_k(D_{[\alpha]_\omega}, B_\omega) = H_k(B_\alpha, B_\omega)$

$$647 \dots \rightarrow H_k(B_\omega) \xrightarrow{p_\alpha^k} H_k(B_\alpha) \xrightarrow{q_\alpha^k} H_k(D_{[\alpha]_\omega}, B_\omega) \xrightarrow{r_\alpha^k} H_{k-1}(B_\omega) \xrightarrow{p_\alpha^{k-1}} H_{k-1}(B_\alpha) \rightarrow \dots$$

648 where $H_k(B_\alpha) = \bigoplus_{I \in \mathcal{I}^k} F_\alpha^I$, $H_k(B_\omega) = \bigoplus_{I \in \mathcal{I}^k} F_\omega^I$, and $p_\alpha^k = \bigoplus_{I \in \mathcal{I}^k} f_{\omega,\alpha}^I$.

649 Noting that $\mathbf{im} q_\alpha^k \cong H_k(B_\alpha)/\ker q_\alpha^k$ where $\ker q_\alpha^k = \mathbf{im} p_\alpha^k$ by exactness we have
 650 $\ker r_\alpha^k \cong H_k(B_\alpha)/\mathbf{im} p_\alpha^k$. By the definition of F_α^I and $f_{\omega,\alpha}^I$ we know $\mathbf{im} f_{\omega,\alpha}^I$ is F_α^I if $\omega \in I$
 651 and 0 otherwise. As $\mathbf{im} p_\alpha^k$ is equal to the direct sum of images $\mathbf{im} f_{\omega,\alpha}^I$ over $I \in \mathcal{I}^k$ it follows
 652 that $\mathbf{im} p_\alpha^k$ is the direct sum of those F_α^I over those $I \in \mathcal{I}^k$ such that $\omega \in I$. Now, because
 653 $H_k(B_\alpha) = \bigoplus_{I \in \mathcal{I}^k} F_\alpha^I$ and each F_α^I is either 0 or \mathbb{F} the quotient $H_k(B_\alpha)/\mathbf{im} p_\alpha^k$ is the direct
 654 sum of those F_α^I such that $\omega \notin I$. Therefore, by the definition of $F_{[\alpha]_\omega}^I$ we have

$$655 \ker r_\alpha^k = \bigoplus_{I \in \mathcal{I}_\omega^k} F_{[\alpha]_\omega}^I.$$

656 Similarly, $\mathbf{im} r_\alpha^k = \ker p_\alpha^{k-1}$ by exactness where $\ker p_\alpha^{k-1}$ is the direct sum of kernels
 657 $\ker f_{\omega,\alpha}^I$ over $I \in \mathcal{I}^{k-1}$. By the definition of F_α^I and $f_{\omega,\alpha}^I$ we know that $\ker f_{\omega,\alpha}^I$ is F_α^I if
 658 $\omega \notin I$ and 0 otherwise. Noting that $\ker f_{\omega,\alpha}^I = 0$ for any $I \in \mathcal{I}^{k-1}$ such that $\omega \notin I$ it suffices
 659 to consider only those $I \in \mathcal{I}_\omega^{k-1}$. It follows that $\ker f_{\omega,\alpha}^I = F_\alpha^{I+}$ for any I containing ω as
 660 $\omega < \alpha$. Therefore,

$$661 \mathbf{im} r_\alpha^k = \bigoplus_{I \in \mathcal{I}^{k-1}} F_\alpha^{I+}.$$

662 We have the following split exact sequence associated with r_α^k

$$663 0 \rightarrow \ker r_\alpha^k \rightarrow H_k(D_{[\alpha]_\omega}, B_\omega) \rightarrow \mathbf{im} r_\alpha^k \rightarrow 0.$$

664 The desired result follows from the fact that for all $\alpha \in \mathbb{R}$

$$665 H_k(D_{[\alpha]_\omega}, B_\omega) \cong \ker r_\alpha^k \oplus \mathbf{im} r_\alpha^k$$

$$666 = \bigoplus_{I \in \mathcal{I}^k} F_{[\alpha]_\omega}^I \oplus \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} F_\alpha^{I+}.$$

668 **B Duality**

669 For a pair (A, B) in a topological space X and any R module G let $H^k(A, B; G)$ denote
 670 the **singular cohomology** of (A, B) (with coefficients in G) as a vector space. Let
 671 $H_c^k(A, B; G)$ denote the corresponding **singular cohomology with compact support**,
 672 where $H_c^k(A, B; G) \cong H^k(A, B; G)$ for any compact pair (A, B) .

673 The following corollary follows from the Universal Coefficient Theorem for singular
 674 homology (and cohomology) as vector spaces over a field \mathbb{F} , as the dual vector space
 675 $\text{Hom}(H_k(A, B), \mathbb{F})$ is isomorphic to $H_k(A, B; \mathbb{F})$ for any finitely generated $H_k(A, B)$.⁵

676 ▶ **Corollary 24.** *For a topological pair (A, B) and a field \mathbb{F} such that $H_0(A, B)$ is finitely
 677 generated there is a natural isomorphism*

678
$$\nu : H^0(A, B; \mathbb{F}) \rightarrow H_0(A, B; \mathbb{F}).$$

679 Let $\overline{H}^k(A, B; G)$ be the **Alexander-Spanier cohomology** of the pair (A, B) , defined
 680 as the limit of the direct system of neighborhoods (U, V) of the pair (A, B) . Let $\overline{H}_c^k(A, B; G)$
 681 denote the corresponding **Alexander-Spanier cohomology with compact support**
 682 where $\overline{H}_c^k(A, B; G) \cong \overline{H}^k(A, B; G)$ for any compact pair (A, B) .

683 ▶ **Theorem 25 (Alexander-Poincaré-Lefschetz Duality** (Spanier [7], Theorem 6.2.17)). *Let
 684 X be an orientable d -manifold and (A, B) be a compact pair in X . Then for all k and R
 685 modules G there is a (natural) isomorphism*

686
$$\lambda : H_k(X \setminus B, X \setminus A; G) \rightarrow \overline{H}^{d-k}(A, B; G).$$

687 A space X is said to be **homologically locally connected in dimension n** if for
 688 every $x \in X$ and neighborhood U of x there exists a neighborhood V of x in U such that
 689 $\tilde{H}_n(V) \rightarrow \tilde{H}_n(U)$ is trivial for $k \leq n$.

690 ▶ **Lemma 26** (Spanier p. 341, Corollary 6.9.6). *Let A be a closed subset, homologically
 691 locally connected in dimension n , of a Hausdorff space X , homologically locally connected in
 692 dimension n . If X has the property that every open subset is paracompact, $\mu : \overline{H}_c^k(X, A; G) \rightarrow$
 693 $H_c^k(X, A; G)$ is an isomorphism for $k \leq n$ and a monomorphism for $k = n + 1$.*

694 In the following we will assume homology (and cohomology) over a field \mathbb{F} .

695 ▶ **Lemma 27.** *Let X be an orientable d -manifold and (A, B) a compact pair of locally path
 696 connected subspaces in X . Then*

697
$$\xi : H_d(X \setminus B, X \setminus A) \rightarrow H_0(A, B)$$

698 is a natural isomorphism.

699 **Proof.** Because X is orientable and (A, B) are compact $\lambda : H_d(X \setminus B, X \setminus A) \rightarrow \overline{H}^0(A, B)$
 700 is an isomorphism by Theorem 25. Note that Moreover, because every subset of X is
 701 (hereditarily) paracompact every open set in A , with the subspace topology, is paracompact.
 702 For any neighborhood U of a point x in a locally path connected space there must exist some
 703 neighborhood $V \subset U$ of x that is path connected in the subspace topology. As $\tilde{H}_0(V) = 0$

673 ⁵ Reference/verify.

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for any nonempty, path connected topological space V (see Spanier p. 175, Lemma 4.4.7) it follows that A (resp. B) are homologically locally connected in dimension 0. Because (A, B) is a compact pair the singular and Alexander-Spanier cohomology modules of (A, B) with compact support are isomorphic to those without, thus $\mu : \overline{H}^0(A, B) \rightarrow H^0(A, B)$ is an isomorphism. By Corollary 24 we have a natural isomorphism $\nu : H^0(A, B) \rightarrow H_0(A, B)$ thus the composition $\xi := \nu \circ \mu \circ \lambda : H_d(X \setminus B, X \setminus A) \rightarrow H_0(A, B)$ is a natural isomorphism. \blacktriangleleft

► **Lemma 28.** *Let \mathbb{X} be an orientable d -manifold let D be a compact subset of \mathbb{X} with strong convexity radius $\varrho_D > \varepsilon$. Let P be a finite subset of D such that $P^\varepsilon \subset \text{int}_{\mathbb{X}}(D)$ and $Q \subseteq P$. If $D \setminus Q^\varepsilon$ and $D \setminus P^\varepsilon$ are locally path connected then there is an isomorphism*

$$\xi \mathcal{N} : H_d(\check{\mathcal{C}}^\varepsilon(P, Q)) \rightarrow H_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon)$$

that commutes with maps induced by inclusions.

Proof. Because Q^ε and P^ε are open in D and D is compact in \mathbb{X} the complement $D \setminus Q^\varepsilon$ is closed in D , and therefore compact in \mathbb{X} . Moreover, because $P^\varepsilon \subset \text{int}_{\mathbb{X}}(D)$, $H_d(\mathbb{X} \setminus (D \setminus P^\varepsilon), \mathbb{X} \setminus (D \setminus Q^\varepsilon)) = H_d(P^\varepsilon, Q^\varepsilon)$. As we have assumed these complements are locally path connected by assumption we have a natural isomorphism $\xi : H_d(P^\varepsilon, Q^\varepsilon) \rightarrow H_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon)$ by Lemma 27.

Because $\varepsilon > \varrho_D$ the covers by metric balls associated with P^ε and Q^ε are good, so we have isomorphisms $\mathcal{N} : H_d(\check{\mathcal{C}}^\varepsilon(P, Q)) \rightarrow H_d(P^\varepsilon, Q^\varepsilon)$ for all $Q \subseteq P$ by the Nerve Theorem. So the composition $\xi \mathcal{N} := \xi \circ \mathcal{N}$ is an isomorphism. Moreover, because ξ is natural and \mathcal{N} commutes with maps induced by inclusions by the persistent nerve lemma the composition $\xi \mathcal{N}$ does as well. \blacktriangleleft