1 Some Useful Theorems

Lemma 1 (Splitting Lemma (Hatcher p. 147))**.** *For a short exact sequence*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

of abelian groups the following statements are equivalent

1. *There is a homomorphism $p : B \to A$ such that $p \circ i = \mathbf{Id}_A$.*
2. *There is a homomorphism $s : C \to B$ such that $j \circ s = \mathbf{Id}_C$.*
3. *There is an isomorphism $B \cong A \oplus C$ making the commutative diagram below, where the maps in the lower row are the obvious ones $a \mapsto (a, 0)$ and $(a, c) \mapsto c$.*

$$\begin{tikzcd}
 & & B \arrow[dd, "\cong"] \arrow[dr, "j"] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr] & & C \arrow[r] & 0 \\
 & & A \oplus C \arrow[ur] & &
\end{tikzcd}$$

Lemma 2 (The Five-Lemma (Hatcher p. 129))**.** *In a commutative diagram of abelian groups as below, if the two rows are exact and α, β, δ, and ε are isomorphisms then γ is an isomorphism.*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "\ell"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "\ell'"] & E'
\end{tikzcd}$$

Theorem 1 (Alexander Duality)**.** *If D is a compact, locally contractible, nonempty, proper subspace of S^d then for all k there is an isomorphism*

$$\Gamma^k_D : \tilde{\mathrm{H}}_k(D) \to \tilde{\mathrm{H}}^{d-k-1}(S^d \setminus D).$$

If (D, B) is a pair of such subspaces of S^d then for all k there is an isomorphism

$$\Gamma^k_{(D,B)} : \tilde{\mathrm{H}}_k(D, B) \to \tilde{\mathrm{H}}^{d-k}(S^d \setminus B, S^d \setminus D).$$

Lemma 3 (Lemma 3.2 from [**?**])**.** *Given a sequence $A \to B \to C \to D \to E \to F$ of homomorphisms between finite-dimensional vector spaces, if $\mathbf{rk}(A \to F) = \mathbf{rk}(C \to D)$ then this quantity also equals the rank of $B \to E$. Similarly, if $A \to B \to C \to E \to F$ is a sequence of homomorphisms such that $\mathbf{rk}(A \to F) = \dim\ C$ then $\mathbf{rk}(B \to E) = \dim\ C$.*

TODO

- **Excision**

2 Separation

2.1 Separation

Definition 1 (Separated)**.** *Two subsets U, V of a topological space X are **separated in** X if each is disjoint from the other's closure: $\mathbf{cl}(U) \cap V = \emptyset$ and $U \cap \mathbf{cl}(V) = \emptyset$.*

Definition 2 (Separation)**.** *We say that a subset B **separates** a topological space X with the pair (U, V) if B, U, V partitions X, U, V are separated in X, and any path from U to V in X intersects B.*

Lemma 4. *If B separates X with the pair (U, V) then for all k the short exact sequence*

$$0 \to \mathrm{H}_k(V) \xrightarrow{i_*} \mathrm{H}_k(X \setminus B) \xrightarrow{j_*} \mathrm{H}_k(U) \to 0$$

splits.

Proof. Because $X \setminus B$ is the disjoint union of U and V we know that $i_* : \mathrm{H}_k(V) \to \mathrm{H}_k(X \setminus B)$ is the map induced by inclusion and $p_* : \mathrm{H}_k(X \setminus B) \to \mathrm{H}_k(V)$ is induced by the restriction of the identity on $X \setminus B$ to V. Thus $p_* \circ i_* = \mathbf{Id}_{\mathrm{H}_k(V)}$ and therefore, by Lemma 1 the sequence splits. □

Corollary 1. *If B separates X with the pair (U, V) then for all k*

$$\mathrm{H}_k(X \setminus B) \cong H_k(U) \oplus \mathrm{H}_k(V).$$

Lemma 5. *If B separates X with the pair (U, V) then for all k*

$$\mathrm{H}_k(U) \cong \mathrm{H}_k(X \setminus B, V).$$

Proof. First note that the short exact sequence

$$0 \to \mathrm{H}_k(V) \to \mathrm{H}_k(U) \oplus \mathrm{H}_k(V) \to \mathrm{H}_k(U) \to 0$$

extends to a long exact sequence with the zero map $\partial_*^k : \mathrm{H}_k(U) \to \mathrm{H}_k(V)$ as $\mathbf{im}\ j_*^k = \mathrm{H}_k(U) = \mathbf{ker}\ \partial_*^k$ and $\mathbf{im}\ \partial_*^k = \mathbf{ker}\ i_*^{k-1} = \mathbf{0}_{\mathrm{H}_{k-1}(V)}$. Consider the following commutative diagram where the bottom row is the long exact sequence of the pair $(X \setminus B, V)$

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{H}_k(V) \arrow[r, "i_*^k"] \arrow[d, "f_*^k"] & \mathrm{H}_k(U) \oplus \mathrm{H}_k(V) \arrow[r, "j_*^k"] \arrow[d, "g_*^k"] & \mathrm{H}_k(U) \arrow[r, "\partial_*^k"] \arrow[d, "h_*^k"] & \mathrm{H}_{k-1}(V) \arrow[r, "i_*^{k-1}"] \arrow[d, "f_*^{k-1}"] & \mathrm{H}_{k-1}(U) \oplus \mathrm{H}_{k-1}(V) \arrow[r] \arrow[d, "g_*^{k-1}"] & \ldots \\
\ldots \arrow[r] & \mathrm{H}_k(V) \arrow[r, "\widehat{i_*^k}"] & \mathrm{H}_k(X \setminus B) \arrow[r, "\widehat{j_*^k}"] & \mathrm{H}_k(U) \arrow[r, "\widehat{\partial_*^k}"] & \mathrm{H}_{k-1}(V) \arrow[r, "\widehat{i_*^{k-1}}"] & \mathrm{H}_{k-1}(X \setminus B) \arrow[r] & \ldots
\end{tikzcd}$$

As f_*^k is the identity map and, by Corollary 1, g_*^k is an isomorphism for all k it follows that h_*^k is an isomorphism for all k by Lemma 2. □

Naturally the same results hold for (reduced) cohomology.

Corollary 2. *If B separates X with the pair (U, V) then for all k*

$$\tilde{\mathrm{H}}^k(X \setminus B) \cong \tilde{\mathrm{H}}^k(U) \oplus \tilde{\mathrm{H}}^k(V)$$

and

$$\tilde{\mathrm{H}}^k(U) \cong \tilde{\mathrm{H}}^k(X \setminus B, V).$$

2.2 Surrounding

Definition 3 (Surrounding)**.** *We say that $B \subset D$ **surrounds** $D \subset X$ in X if B separates X with the pair $(D \setminus B, X \setminus D)$. We will refer to such a pair (D, B) as a **surrounding pair** in X.*

We define the *extension* of a surrounding pair (D, B) in X to be the pair

$$(X, \hat{B}) = (D \cup (X \setminus D), B \cup (X \setminus D)).$$

Note that, for $X \subset \mathcal{X}$ the extension of a surrounding pair in X is a surrounding pair in $\mathcal{X}$.

Lemma 6. *If (D, B) is a surrounding pair of open sets in a topological space X then for all k*

$$\mathrm{H}_k(D, B) \cong \mathrm{H}_k(X, \hat{B}).$$

Proof. Because B surrounds D in X and $\hat{B} = B \sqcup (X \setminus D)$ we have $X = (D \setminus B) \sqcup B \sqcup (X \setminus D)$ so $X \setminus \hat{B} = D \setminus B$. Moreover, because D is open $\overline{D}$ is closed in X, so $\mathbf{cl}(\overline{D}) = \overline{D}$. Therefore

$$\begin{aligned} X \setminus (\mathbf{int}(D) \cup \mathbf{int}(\hat{B})) &= X \cap \overline{(\mathbf{int}(D) \cup \mathbf{int}(\hat{B}))} \\ &= X \cap \mathbf{cl}(\overline{D}) \cap \mathbf{cl}(\overline{\hat{B}})) \\ &= (X \setminus D) \cap \mathbf{cl}(D \setminus B). \end{aligned}$$

Because (D, B) is a surrounding pair in X we have that B separates X with the pair $(D \setminus B, X \setminus D)$. So $(X \setminus D)$ and $D \setminus B$ are separated in X, thus

$$X \setminus (\mathbf{int}(D) \cup \mathbf{int}(\hat{B})) = (X \setminus D) \cap \mathbf{cl}(D \setminus B) = \emptyset.$$

It follows that $\mathbf{int}(D) \cup \mathbf{int}(\hat{B}) = X$ and therefore that

$$\mathrm{H}_k(D, B) = \mathrm{H}_k(D, D \cap \hat{B}) \cong \mathrm{H}_k(X, \hat{B})$$

by excision. $\square$

The following is a corollary of Theorem 1 (Alexander Duality).

Corollary 3. *If (D, B) is a surrounding pair of locally contractible, nonempty, proper subspaces in S^d then for all k*

$$\tilde{\mathrm{H}}_k(D, B) \cong \tilde{\mathrm{H}}^{d-k}(D \setminus B).$$

In the following we will assume that (D, B) is a surrounding pair of locally contractible, nonempty, proper subspaces in S^d. Let $(\overline{B}, \overline{D}) = (S^d \setminus B, S^d \setminus D)$ denote the complement of the pair (D, B) in S^d.

Lemma 7. *If (D, B) is a surrounding pair of locally contractible, nonempty, proper subspaces of S^d then*

$$i_*^k : \tilde{\mathrm{H}}_{k+1}(D, B) \to \tilde{\mathrm{H}}_k(B)$$

is injective and

$$j_*^k : \tilde{\mathrm{H}}_k(B) \to \tilde{\mathrm{H}}_k(D)$$

is surjective for all k.

Proof. We have the following commutative diagram of long exact sequences of the pairs (D,B) and $(\overline{B},\overline{D})$.

$$\begin{tikzcd}
\tilde{\mathrm{H}}_{k+1}(D,B) \arrow[r, "\partial_*^{k+1}"] \arrow[d, "\Gamma^{k+1}_{(D,B)}"] & \tilde{\mathrm{H}}_k(B) \arrow[r, "i_*^k"] \arrow[d, "\Gamma^k_B"] & \tilde{\mathrm{H}}_k(D) \arrow[d, "\Gamma^k_D"] \\
\tilde{\mathrm{H}}^{d-k-1}(\overline{B},\overline{D}) \arrow[r, "\overline{j_*^{d-k-1}}"] & \tilde{\mathrm{H}}^{d-k-1}(\overline{B}) \arrow[r, "\overline{i_*^{d-k-1}}"] & \tilde{\mathrm{H}}^{d-k-1}(\overline{D})
\end{tikzcd} \tag{1}$$

Because B surrounds D we have that

$$\tilde{\mathrm{H}}^{d-k-1}(\overline{B}) \cong \tilde{\mathrm{H}}^{d-k-1}(D \setminus B) \oplus \mathrm{H}_k(\overline{D})$$

and

$$\tilde{\mathrm{H}}^{d-k-1}(D \setminus B) \cong \tilde{\mathrm{H}}^{d-k-1}(\overline{B},\overline{D})$$

by Corollary 2. It follows that $\overline{j_*^{d-k-1}}$ is injective and $\overline{i_*^{d-k-1}}$ is surjective.

By commutativity of Diagram 1 and because $\Gamma^{k+1}_{(D,B)}, \Gamma^k_B$ and Γ^k_D are isomorphisms we have that

$$\partial_*^{k+1} = (\Gamma^k_B)^{-1} \circ \overline{j_*^{d-k-1}} \circ \Gamma^{k+1}_{(D,B)}$$

is injective and

$$i_*^k = (\Gamma^k_D)^{-1} \circ \overline{i_*^{d-k-1}} \circ \Gamma^{k+1}_B$$

is surjective. □

We note that this implies the following for non-reduced homology[1] for subsets of $\mathbb{R}^d$.[2]

Corollary 4. *If (D,B) is a surrounding pair of locally contractible, nonempty, proper subspaces of $\mathbb{R}^d$ then*

$$i_*^k : \mathrm{H}_{k+1}(D,B) \to \mathrm{H}_k(B)$$

is injective and

$$j_*^k : \mathrm{H}_k(B) \to \mathrm{H}_k(D)$$

is surjective for all k.

[1]**TODO reasoning:**

- **consider** $\mathrm{H}_1(D,B) \to \mathrm{H}_0(B)$.
- $\tilde{\mathrm{H}}_0(B) \to \tilde{\mathrm{H}}_0(D)$ **surjective implies** $\mathrm{H}_0(B) \to \mathrm{H}_0(D)$ **surjective (right?).**

[2]**TODO reasoning:**

- $S^d \cong \mathbb{R}^d \cup \{\infty\}$.
- **Only requires spaces *and* complements remain compact?**

3 Surrounding Covers

In the following let $\mathbf{d}(x,y) = \|x-y\|$ denote the euclidean distance between points $x, y \in \mathbb{R}^d$. For $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ let

$$\mathbf{d}_A(x) = \min_{a \in A} \mathbf{d}(x,a)$$

denote the distance from x to the set A. In the following, we will use open metric balls

$$\mathbf{ball}_\varepsilon(x) = \{y \in \mathbb{R}^d \mid \mathbf{d}(x,y) < \varepsilon\}$$

and offsets

$$A^\varepsilon = \mathbf{d}_A^{-1}[0,\varepsilon) = \{x \in \mathbb{R}^d \mid \mathbf{d}_A(x) < \varepsilon\}.$$

Definition 4 (Surrounding Cover)**.** *For $\delta > 0$, $\gamma > \delta$, and finite subsets $P \subset D$, $Q \subset P \cap B$ we say that (P,Q) is an* ***(open) surrounding*** *(δ,γ)**-cover** of a surrounding pair (D,B) if*

(a) ***(Covers)*** *$D \setminus B \subseteq P^\delta$,*

(b) ***(Surrounds)*** *Q^δ surrounds P^δ in D, and*

(c) ***(Interleaves)*** *$\hat{Q}^\delta \subseteq B \subseteq \hat{Q}^\gamma$.*

Theorem 2. *Let $(\mathcal{D},\mathcal{B})$ be a surrounding pair of open subsets of $\mathbb{R}^d$ and let $P \subset \mathcal{D}$ be a finite subset of $\mathcal{D}$. Let (P,Q) be an (open) surrounding (δ,γ)-cover of $(\mathcal{D},\mathcal{B})$ for $\gamma > \delta > 0$. Let (D_0,B_0) and (D_1,B_1) be surrounding pairs of nonempty, compact subsets of $\mathbb{R}^d$ such that $B_0 \subseteq \hat{Q}^\delta$, $\hat{Q}^\gamma \subseteq B_1$, and*

$$(D_0,B_0) \subset (\mathcal{D},\mathcal{B}) \subset (D_1,B_1).$$

If

$$\mathbf{im}\ \mathrm{H}_k((D_0,B_0) \hookrightarrow (D_1,B_1)) \cong \mathrm{H}_k(\mathcal{D},\mathcal{B})$$

then

$$\mathbf{im}\ \mathrm{H}_k((P^\delta,Q^\delta) \hookrightarrow (P^\gamma,Q^\gamma)) \cong \mathrm{H}_k(\mathcal{D},\mathcal{B}).$$

Proof. In the following let

$$\begin{aligned}\eta_B^k &: \mathrm{H}_k(B_0) \to \mathrm{H}_k(B_1),\\ \eta_D^k &: \mathrm{H}_k(D_0) \to \mathrm{H}_k(D_1), \text{ and}\\ \eta^k &: \mathrm{H}_k(D_0,B_0) \to \mathrm{H}_k(D_1,B_1).\end{aligned}$$

Consider the commutative diagram of long exact sequences of the pairs (D_0,B_0), $(\mathcal{D},\mathcal{B})$ and (D_1,B_1).

$$\begin{tikzcd}
\mathrm{H}_k(B_0) \arrow[r, "i_0^k"] \arrow[d, "a^k"] & \mathrm{H}_k(D_0) \arrow[r, "j_0^k"] \arrow[d, "b^k"] & \mathrm{H}_k(D_0,B_0) \arrow[r, "\partial_0^k"] \arrow[d, "d^k"] & \mathrm{H}_{k-1}(B_0) \arrow[r, "i_0^{k-1}"] \arrow[d, "a^{k-1}"] & \mathrm{H}_{k-1}(D_0) \arrow[d, "b^{k-1}"] \\
\mathrm{H}_k(\mathcal{B}) \arrow[r, "i_*^k"] \arrow[d, "f^k"] & \mathrm{H}_k(\mathcal{D}) \arrow[r, "j_*^k"] \arrow[d, "g^k"] & \mathrm{H}_k(\mathcal{D},\mathcal{B}) \arrow[r, "\partial_*^k"] \arrow[d, "h^k"] & \mathrm{H}_{k-1}(\mathcal{B}) \arrow[r, "i_*^{k-1}"] \arrow[d, "f^{k-1}"] & \mathrm{H}_{k-1}(\mathcal{D}) \arrow[d, "g^{k-1}"] \\
\mathrm{H}_k(B_1) \arrow[r, "i_1^k"] & \mathrm{H}_k(D_1) \arrow[r, "j_1^k"] & \mathrm{H}_k(D_1,B_1) \arrow[r, "\partial_1^k"] & \mathrm{H}_{k-1}(B_1) \arrow[r, "i_1^{k-1}"] & \mathrm{H}_{k-1}(D_1)
\end{tikzcd} \tag{2}$$

Because (D_0, B_0) and (D_1, B_1) are nonempty, compact surrounding pairs of $\mathbb{R}^d$ we can embed them in $S^d \cong \mathbb{R}^d \cup \{\infty\}$ in order to show that i_0^k and i_1^k are surjective by Lemma 7. So, by exactness, $\mathbf{im}\ i_0^k = \mathbf{ker}\ j_0^k = \mathrm{H}_k(D_0)$ and $\mathbf{im}\ i_1^k = \mathbf{ker}\ j_1^k = \mathrm{H}_k(D_1)$. It follows that for any $[y] \in \mathrm{H}_k(D_0, B_0)$ such that $\eta^k[y]$ is nonzero we must have that $\eta^k[y] \in \mathbf{cok}\ j_1^k$ and $[y] \in \mathbf{cok}\ j_0^k$.

That is, $\partial_0^k[y]$ is nonzero in $\mathrm{H}_{k-1}(B_0)$ and $\partial_1^k \circ \eta^k[y] = \eta_B^{k-1} \circ \partial_0^k[y]$ is nonzero in $\mathrm{H}_{k-1}(B_1)$. Moreover, because η^k factors through $\mathrm{H}_k(\mathcal{D}, \mathcal{B})$ as $\eta^k = h^k \circ d^k$ we have that $d^k[y]$ is nonzero in $\mathrm{H}_k(\mathcal{D}, \mathcal{B})$. Similarly, because η_B^{k-1} factors through $\mathrm{H}_{k-1}(\mathcal{B})$ as $\eta_B^{k-1} = f^{k-1} \circ a^{k-1}$ we have that $\partial_*^k \circ d^k[y] = a^{k-1} \circ \partial_0^k[y]$ is nonzero in $\mathrm{H}_{k-1}(\mathcal{B})$.

Consider the long exact sequence of the pair $(\mathcal{D}, \hat{Q}^\delta)$

$$\ldots \to \mathrm{H}_k(\mathcal{D}, \hat{Q}^\delta) \xrightarrow{\partial_\delta^k} \mathrm{H}_{k-1}(\hat{Q}^\delta) \xrightarrow{p_\delta^{k-1}} \mathrm{H}_{k-1}(\mathcal{D}) \to \ldots,$$

and the following commutative diagrams taken from the long exact sequences of the pairs $(\mathcal{D}, \hat{Q}^\delta)$ and $(\mathcal{D}, \mathcal{B})$.

$$\begin{tikzcd} \mathrm{H}_{k-1}(\hat{Q}^\delta) \arrow[rr, "p_\delta^{k-1}"] \arrow[dr, "\psi_\delta^{k-1}"] & & \mathrm{H}_{k-1}(\mathcal{D}) \\ & \mathrm{H}_{k-1}(\mathcal{B}) \arrow[ur, "i_*^{k-1}"] & \end{tikzcd} \tag{3}$$

where $\psi_\delta^{k-1} : \mathrm{H}_{k-1}(\hat{Q}^\delta) \to \mathrm{H}_{k-1}(\mathcal{B})$ is induced by inclusion.

Letting $\phi_0^{k-1} : \mathrm{H}_{k-1}(B_0) \to \mathrm{H}_{k-1}(\hat{Q}^\delta)$ be the homomorphism induced by inclusion we have that $a^{k-1} = \psi_\delta^{k-1} \circ \phi_0^{k-1}$ so, because

$$\eta_B^{k-1} = f^{k-1} \circ a^{k-1} = f^{k-1} \circ \psi_\delta^{k-1} \circ \phi_0^{k-1}$$

and $\partial_1^k \circ \eta^k = \partial_0^k \circ \eta_B^{k-1}$ it follows that $\phi_0^{k-1} \circ \partial_0^k[y]$ is nonzero in $\mathrm{H}_{k-1}(\hat{Q}^\delta)$. Moreover, because $a^{k-1} \circ \partial_0^k[y] \in \mathbf{im}\ \partial_*^k$ we have that $a^{k-1} \circ \partial_0^k[y] \in \mathbf{ker}\ i_*^{k-1}$ by exactness, thus $\phi_0^{k-1} \circ \partial_0^k[y] \in \mathbf{ker}\ p_\partial^{k-1}$ as $p_\delta^{k-1} = i_*^{k-1} \circ \psi_\delta^{k-1}$. That is, $\phi_0^{k-1} \circ \partial_0^k[y] \in \mathbf{im}\ \partial_\delta^k$ by exactness. We can therefore construct a homomorphism $\mu^k : \mathrm{H}_k(D_0, B_0) \to \mathrm{H}_k(P^\delta, Q^\delta)$ for $[y] \in \mathrm{H}_k(D_0, B_0)$ as the preimage of $\phi_0^{k-1}\partial_0^k[y]$ in $\mathrm{H}_k(P^\delta, Q^\delta)$ for nonzero $[y''] = \eta^k[y]$, 0 otherwise.

Now, consider the long exact sequence of the pair $(\mathcal{D}, \hat{Q}^\gamma)$

$$\ldots \to \mathrm{H}_k(\mathcal{D}, \hat{Q}^\gamma) \xrightarrow{\partial_\gamma^k} \mathrm{H}_{k-1}(\hat{Q}^\gamma) \xrightarrow{p_\gamma^{k-1}} \mathrm{H}_{k-1}(\mathcal{D}) \to \ldots.$$

We have the following commutative diagrams

$$\begin{tikzcd} \mathrm{H}_{k-1}(\mathcal{B}) \arrow[rr, "i_*^{k-1}"] \arrow[dr, "\psi_\gamma^{k-1}"] & & \mathrm{H}_{k-1}(\mathcal{D}) \\ & \mathrm{H}_{k-1}(\hat{Q}^\gamma) \arrow[ur, "p_\gamma^{k-1}"] & \end{tikzcd} \tag{4a}$$

$$\begin{tikzcd} \mathrm{H}_k(\mathcal{D}, \hat{Q}^\gamma) \arrow[r, "\partial_\gamma^k"] \arrow[d, "\nu^k"] & \mathrm{H}_{k-1}(\hat{Q}^\gamma) \arrow[d, "\phi_1^{k-1}"] \\ \mathrm{H}_k(D_1, B_1) \arrow[r, "\partial_1^k"] & \mathrm{H}_{k-1}(B_1) \end{tikzcd} \tag{4b}$$

where $\psi_\gamma^{k-1}, \sigma_\gamma^{k-1}, \phi_1^{k-1}$ and ν^k are induced by inclusion.

For brevity, let $[y'] = d^k[y]$ denote the nonzero image of $[y]$ in $\mathrm{H}_k(\mathcal{D},\mathcal{B})$ and recall that $\partial_*^k[y'] = a^{k-1}\circ\partial_0^k[y]$ is nonzero in $\mathrm{H}_{k-1}(\mathcal{B})$. Because $\partial_*^k[y']$ is nonzero $\partial_*^k[y'] \in \mathbf{ker}\ i_*^{k-1}$ by exactness and, by commutativity of diagram 4a, $i_*^{k-1} = p_\gamma^{k-1}\circ\psi_\gamma^{k-1}$. So $\partial_*^k[y'] \in \mathbf{ker}\ p_\gamma^{k-1}\circ\psi_\gamma^{k-1}$. Noting that f^{k-1} factors through $\mathrm{H}_{k-1}(\hat{Q}^\gamma)$ as $f^{k-1} = \phi_1^{k-1}\circ\psi_\gamma^{k-1}$ we have that $\psi_\gamma^{k-1}\circ\partial_*^k[y']$ is nonzero in $\mathrm{H}_{k-1}(Q^\gamma)$. So $\psi_\gamma^{k-1}\circ\partial_*^k[y'] \in \mathbf{ker}\ p_\gamma^{k-1}$ thus $\psi_\gamma^{k-1}\circ\partial_*^k[y'] \in \mathbf{im}\ \mathrm{H}_k(\mathcal{D},\hat{Q}^\gamma)$ by exactness.

So we may conclude that η^k factors through $\hat{\tau}^k : \mathrm{H}_k(\mathcal{D},\hat{Q}^\delta)\to\mathrm{H}_k(\mathcal{D}),\hat{Q}^\gamma)$ with the maps $\mu^k = (\partial_\delta^k)^{-1}\circ\phi_0^{k-1}\circ\partial_0^k$ and $\nu^k : \mathrm{H}_k(\mathcal{D},\hat{Q}^\gamma)\to\mathrm{H}_k(D_1,B_1)$. We therefore have the following sequence of homomorphisms

$$\mathrm{H}_k(D_0,B_0)\xrightarrow{\mu^k}\mathrm{H}_k(\mathcal{D},\hat{Q}^\delta)\to\mathrm{H}_k(\mathcal{D},\mathcal{B})\to\mathrm{H}_k(\mathcal{D},\hat{Q}^\gamma)\xrightarrow{\nu^k}\mathrm{H}_k(D_1,B_1).$$

So, by Lemma 3, $\mathbf{im}\ \eta^k \cong \mathrm{H}_k(\mathcal{D},\mathcal{B})$ implies $\mathbf{im}\ \hat{\tau}^k \cong \mathrm{H}_k(\mathcal{D},\mathcal{B})$. The result follows from Lemma 6 as

$$\mathbf{im}\ \mathrm{H}_k((P^\delta,Q^\delta)\hookrightarrow(P^\gamma,Q^\gamma))\cong\mathbf{im}\ \hat{\tau}^k\cong\mathrm{H}_k(\mathcal{D},\mathcal{B}).$$

□

4 Connection with the TCC

4.1 Assumptions

Let $P\subset\mathcal{D}$ be a finite collection of sensors p with the following capabilities.

Sensor Capabilities

a. **(Communication Radii)** detect the presence, but not location or distance, of sensors within distances $\delta > 0$ and $\gamma \geq 3\delta$, and discriminate between sensors within each scale,

b. **(Coverage Radius)** cover a radially symmetric subset of the domain with radius δ,

We will refer to the following preliminary assumptions about pairs (D_0,B_0) and (D_1,B_1) for $\delta > 0$ and $\gamma \geq 3\delta$.

Geometric Assumptions

1. **(Domain)** (D_0,B_0) and (D_1,B_1) are surrounding pairs of nonempty, compact subsets of $\mathbb{R}^d$ with $(D_0^{\delta+\gamma},B_0^{\delta+\gamma})\subset(D_1,B_1)$.

2. **(Boundary)** $\mathrm{H}_0(D_1\setminus B_1\hookrightarrow D_0\setminus B_0^{2\delta})$ is surjective.

In the following let $Q = P \cap B_0^\delta$ and $(\mathcal{D}, \mathcal{B}) = (D_0^{2\delta}, B_0^{2\delta})$.

4.2 Proof of the TCC

In the following let $\overline{X} = \mathbb{R}^d \setminus X$ for all subsets $X \subset \mathbb{R}^d$. We have the following commutative diagrams of inclusions between the pairs (P, Q) and $(\mathcal{D}, \mathcal{B})$ and their complements *in* $\mathbb{R}^d$ with increasing scale.

$$\begin{array}{ccccccc} (P^\delta, Q^\delta) & \hookrightarrow & (P^\gamma, Q^\gamma) & & (\overline{B_1}, \overline{D_1}) & \xhookrightarrow{j} & (\overline{\mathcal{B}}, \overline{\mathcal{D}}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ (\mathcal{D}, \mathcal{B}) & \hookrightarrow & (D_1, B_1), & & (\overline{Q^\gamma}, \overline{P^\gamma}) & \xhookrightarrow{i} & (\overline{Q^\delta}, \overline{P^\delta}). \end{array}$$

The following diagram is formed by applying the homology functor.

$$\begin{array}{ccc} \mathrm{H}_0(\overline{B_1}, \overline{D_1}) & \xrightarrow{j_*} & \mathrm{H}_0(\overline{\mathcal{B}}, \overline{\mathcal{D}}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{H}_0(\overline{Q^\gamma}, \overline{P^\gamma}) & \xrightarrow{i_*} & \mathrm{H}_0(\overline{Q^\delta}, \overline{P^\delta}). \end{array} \tag{5}$$

Let $p_* : \mathbf{im}\ j_* \to \mathbf{im}\ i_*$.

Lemma 8. *Given assumptions 1 & 2, the map p_* is surjective.*

Proof. Choose a basis for $\mathbf{im}\ i_*$ such that each basis element is represented by a point in $P^\delta \setminus Q^\gamma$. Let $x \in P^\delta \setminus Q^\gamma$ be such that $[x]$ is non-trivial in $\mathbf{im}\ i_*$. Suppose $x \in \mathcal{B}$ and let $y \in B_0$ so that $\mathbf{d}(x, y) < 2\delta$.

Now, because $x \in \overline{Q^\gamma}$ by hypothesis $\mathbf{d}(x, q) \geq \gamma$ for all $q \in Q$. For any z in the shortest path between x and y we have $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) < 2\delta$, so the following inequality holds for all $q \in Q$

$$\begin{aligned} \mathbf{d}(x, q) &\geq \mathbf{d}(x, q) - \mathbf{d}(x, z) \\ &> \gamma - 2\delta \\ &\geq \delta. \end{aligned}$$

So $z \in \overline{Q^\delta}$ for all z in the shortest path from x to y. In particular, $x, y \in \overline{Q^\delta}$.

Now, suppose $y \in P^\delta$. So there exists some $p \in P$ such that $\mathbf{d}(p, y) < \delta$. So $\mathbf{d}(p, y) < \delta$ which implies $p \in Q$ thus $y \in Q^\delta$. But we have shown that $y \in \overline{Q^\delta}$, a contradiction, so we may assume that $y \in \overline{P^\delta}$.

Because $x, y \in \overline{Q^\delta}$ we have corresponding chains $x, y \in C_0(\overline{Q^\delta})$ as well as $y \in \overline{P^\delta}$ generating a chain $y \in C_0(\overline{P^\delta})$. As we have shown that $x \in \mathcal{B}$ implies that the shortest path from x to y is contained in $\overline{Q^\delta}$ there exists a path $h : [0, 1] \to \overline{Q^\delta}$ with $h(0) = x$ and $h(1) = y$ that generates a chain $h \in C_1(\overline{Q^\delta})$. So for $h \in C_1(\overline{Q^\delta}, \overline{P^\delta})$ with $\partial h = x + y$ we have that $x = \partial h + y$. Thus $[x]$ is a relative boundary and is therefore trivial in $\mathrm{H}_0(\overline{P^\delta}, \overline{Q^\delta})$, a contradiction, as we have assumed $[x]$ is non-trivial in $\mathbf{im}\ i_*$. So we may conclude that $x \notin \mathcal{B}$.

So $x \in \overline{\mathcal{B}}$ and $x \in \mathcal{D} \setminus \mathcal{B}$. So $[x]$ is non-trivial in $\mathrm{H}_0(\overline{\mathcal{B}}, \overline{\mathcal{D}})$ and, because j_* is surjective, $\mathbf{im}\ j_* = \mathrm{H}_0(\overline{\mathcal{B}}, \overline{\mathcal{D}})$. So p_* is surjective as $p_*[x] = [x] \in \mathbf{im}\ p_*$ for all non-trivial $[x] \in \mathbf{im}\ i_*$. $\square$

Lemma 9. *Given assumptions 1 & 2, if p_* is injective then $\mathcal{D} \setminus \mathcal{B} \subseteq P^\delta$.*

Proof. Suppose, for the sake of contradiction, that p_* is injective and there exists a point $x \in (\mathcal{D} \setminus \mathcal{B}) \setminus P^\delta$. So $[x]$ is non-trivial in $\mathrm{H}_0(\overline{\mathcal{B}}, \overline{\mathcal{D}}) = \mathbf{im}\ j_*$ as x is in some connected component of $\mathcal{D} \setminus \mathcal{B}$ and j_* is surjective. So we have the following sequence of maps induced by inclusions

$$\mathrm{H}_0(\overline{\mathcal{B}}, \overline{\mathcal{D}}) \xrightarrow{f_*} \mathrm{H}_0(\overline{\mathcal{B}}, \overline{\mathcal{D}} \cup \{x\}) \xrightarrow{g_*} \mathrm{H}_0(\overline{Q^\delta}, \overline{P^\delta}).$$

As $f_*[x]$ is trivial in $\mathrm{H}_0(\overline{\mathcal{B}}, \overline{\mathcal{D}} \cup \{x\})$ we have that $p_*[x] = (g_* \circ f_*)[x]$ is trivial, contradicting our hypothesis that p_* is injective. □

Lemma 10. *Given assumptions 1 & 2, if the map p_* is injective then Q^δ surrounds P^δ in $\mathcal{D}$.*

Proof. Suppose, for the sake of contradiction, that Q^δ does not surround P^δ in $\mathcal{D}$. Then there exists a path $\pi : [0,1] \to \overline{Q^\delta}$ with $\pi(0) \in P^\delta \setminus Q^\delta$ and $\pi(1) \in \mathcal{D} \setminus P^\delta$. By Lemma 9 we know that $\mathcal{D} \setminus \mathcal{B} \subset P^\delta$ and, because $Q^\delta \subset \mathcal{B}$ it follows that $\mathcal{D} \setminus \mathcal{B} \subset P^\delta \setminus Q^\delta$. Choose $x \in \mathcal{D} \setminus \mathcal{B}$ and $y \in \overline{\mathcal{D}}$ such that there exist paths $\pi_x : [0,1] \to P^\delta \setminus Q^\delta$ with $\pi_x(0) = x$, $\pi_x(1) = \pi(0)$ and $\pi_y : [0,1] \to \overline{D} \cup (\mathcal{D} \setminus P^\delta)$ with $\pi_y(0) = y$, $\pi_y(1) = \pi(1)$. π_x, π_y and π all generate chains in $C_1(\overline{Q^\delta}, \overline{P^\delta})$ and $\pi_x + \pi + \pi_y = \pi^* \in C_1(\overline{Q^\delta}, \overline{P^\delta})$ with $\partial\pi^* = x + y$. Moreover, y generates a chain in $C_0(\overline{P^\delta})$ as $\overline{\mathcal{D}^{2\delta}} \subseteq \overline{P^\delta}$. So $x = \partial\pi^* + y$ is a relative boundary in $C_0(\overline{Q^\delta}, \overline{P^\delta})$ thus $[x] = 0 = [y]$ in $\mathrm{H}_0(\overline{Q^\delta}, \overline{P^\delta})$ and therefore $[x] = [y]$ in $\mathbf{im}\ i_*$. However, because $\mathcal{B}$ surrounds $\mathcal{D}$ we know that $[x] \neq [y]$ in $\mathrm{H}_0(\overline{\mathcal{B}}, \overline{\mathcal{D}}) \cong \mathbf{im}\ j_*$, contradicting our assumption that p_* is injective. □

Recall that Q^δ surrounding P^δ in $\mathcal{D}$ implies that $\hat{Q}^\delta$, and therefore $\hat{Q}^\gamma$, surrounds $\mathcal{D}$ in $\mathbb{R}^d$.

Lemma 11. *Given assumptions 1 & 2, if p_* is injective then then $B_0 \subseteq \hat{Q}^\delta$.*

Proof. Given assumptions 1 & 2 and p_* injective we have that $\mathcal{D} \setminus \mathcal{B} \subseteq P^\delta$ and Q^δ surrounds P^δ in $\mathcal{D}$ by Lemmas 9 and 10. Recalling that $\mathcal{B} = B_0^{2\delta}$, $P \subset D_0$ and $Q = P \cap B_0^\delta$ we first note that $B_0 \cap P^\delta \subseteq Q^\delta$ as $x \in B_0 \cap P^\delta$ implies there exists some $p \in P$ such that $\mathbf{d}(x,p) < \delta$ which, for $x \in B_0$ implies that $p \in Q = P \cap B_0^\delta$, and therefore that $x \in Q^\delta$. It follows that

$$B_0 \cap (P^\delta \setminus Q^\delta) = B_0 \cap P^\delta \cap \overline{Q^\delta} \subseteq Q^\delta \cap \overline{Q^\delta} = \emptyset.$$

As Q^δ surrounds P^δ in $\mathcal{D}$ we have

$$B_0 \subset \mathcal{D} = (P^\delta \setminus Q^\delta) \sqcup Q^\delta \sqcup (\mathcal{D} \setminus P^\delta)$$

where $B_0 \cap (P^\delta \setminus Q^\delta) = \emptyset$. It therefore follows that $B_0 \subseteq Q^\delta \sqcup (\mathcal{D} \setminus P^\delta) = \hat{Q}^\delta$. □

Lemma 12. *Given assumptions 1 & 2, if p_* is injective then $\mathcal{B} \subseteq \hat{Q}^\gamma$.*

Proof. As $\mathcal{B} = B_0^{2\delta}$ we know that for all $x \in \mathcal{B}$ there exists some $y \in B_0$ such that $\mathbf{d}(x,y) < 2\delta$. By Lemma 11 we know that $B_0 \subseteq \hat{Q}^\delta = Q^\delta \sqcup (\mathcal{D} \setminus P^\delta)$ so either $y \in \mathcal{D} \setminus P^\delta$ or $y \in Q^\delta$.

If $y \in Q^\delta$ then there exists some $q \in Q$ such that $\mathbf{d}(y,q) < \delta$. Then

$$\mathbf{d}(x,q) \leq \mathbf{d}(x,y) + \mathbf{d}(y,q) < 2\delta + \delta \leq \gamma$$

which would imply $x \in Q^\gamma \subset \hat{Q}^\gamma$.

Now, suppose $y \in B_0 \cap (\mathcal{D} \setminus P^\delta)$. Because Q^δ surrounds P^δ in $\mathcal{D}$ there is no path from $x \in \mathcal{B} \cap (P^\delta \setminus Q^\gamma) \subset P^\delta \setminus Q^\delta$ to $y \in B_0 \cap (\mathcal{D} \setminus P^\delta) \subset \mathcal{D} \setminus P^\delta$ that does not cross Q^δ. So there must be some point $z \in Q^\delta$ in the shortest path from x to y. That is, there exists some $q \in Q$ such that $\mathbf{d}(q,z) < \delta$ and $\mathbf{d}(z,x) < \mathbf{d}(x,y) < 2\delta$ so

$$\mathbf{d}(q,x) \leq \mathbf{d}(q,z) + \mathbf{d}(z,x) < \delta + 2\delta \leq \gamma.$$

So $y \in B_0 \cap (\mathcal{D} \setminus P^\delta)$ implies $x \in Q^\gamma$. $\square$

Theorem 3 (Geometric TCC)**.** *Let (D_0, B_0) and (D_1, B_1) be surrounding pairs of nonempty, compact subsets of $\mathbb{R}^d$ satisfying assumptions 1 & 2 for $\delta > 0$, and $\gamma > 3\delta$. Let $P \subset D_0$ be a finite collection of sensors and $Q = P \cap B_0^\delta$. Let $(\mathcal{D}, \mathcal{B}) = (D_0^{2\delta}, B_0^{2\delta})$ and $p_* : \mathbf{im}\ j_* \to \mathbf{im}\ i_*$ for j_*, i_* as defined in Diagram 5.*

If $\mathbf{rk}\ i_* \geq \mathbf{rk}\ j_*$ and

$$\mathbf{im}\ \mathrm{H}_k((D_0, B_0) \hookrightarrow (D_1, B_1)) \cong \mathrm{H}_k(\mathcal{D}, \mathcal{B})$$

for all k then

$$\mathbf{im}\ \mathrm{H}_k((P^\delta, Q^\delta) \hookrightarrow (P^\gamma, Q^\gamma)) \cong \mathrm{H}_k(\mathcal{D}, \mathcal{B})$$

for all k.

Proof. Because P is a finite point set we know that $\mathbf{im}\ i_*$ is finite-dimensional. Because $\mathbf{rk}\ i_* \geq \mathbf{rk}\ j_*$ j_* is finite dimensional as well so p_* is injective. Therefore $\mathcal{D} \setminus \mathcal{B} \subseteq P^\delta$ by Lemma 9 and Q^δ surrounds P^δ in $\mathcal{D}$ by Lemma 10. We can extend (P^δ, Q^δ) and (P^γ, Q^γ) to pairs $(\mathcal{D}, \hat{Q}^\delta)$ and $(\mathcal{D}, \hat{Q}^\gamma)$ surrounding $\mathcal{D}$ in $\mathbb{R}^d$.

As $Q = P \cap B_0^\delta$ we have that $Q^\delta \setminus P^\delta \subset B_0^{2\delta}\mathcal{B}$. Moreover, $\mathcal{D} \setminus \mathcal{B} \subset P^\delta$ so $\mathcal{D} \setminus P^\delta \subset \mathcal{D} \setminus (\mathcal{D} \setminus \mathcal{B}) = \mathcal{B}$. So $\hat{Q}^\delta = Q^\delta \cup (\mathcal{D} \setminus P^\delta) \subseteq \mathcal{B}$. Moreover, as p_* is injective $\mathcal{B} \subseteq \hat{Q}^\gamma$ by Lemma 12, so $\hat{Q}^\delta \subseteq \mathcal{B} \subseteq \hat{Q}^\gamma$.

As

(a) **(Covers)** $\mathcal{D} \setminus \mathcal{B} \subseteq P^\delta$,

(b) **(Surrounds)** Q^δ surrounds P^δ in $\mathcal{D}$, and

(c) **(Interleaves)** $\hat{Q}^\delta \subseteq \mathcal{B} \subseteq \hat{Q}^\gamma$.

we may conclude that (P, Q) is an (open) surrounding (δ, γ)-cover of $(\mathcal{D}, \mathcal{B})$. Therefore, as $B_0 \subset \hat{Q}^\delta$ by Lemma 11 and $(D_0^{\delta+\gamma}, B_0^{\delta+\gamma}) \subset (D_1, B_1)$ by assumption 1 we have that $\hat{Q}^\gamma \subset B_1$. The result follows from Theorem 2. $\square$