# From Coverage Testing to Topological Scalar Field Analysis

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## Abstract

The topological coverage criterion (TCC) can be used to test whether an underlying space is sufficiently well covered by a given data set. Given a sufficiently dense sample, topological scalar field analysis (SFA) can give a summary of the shape of a real-valued function on its domain. The goal of this paper is to put these theories together so that one can test coverage with the TCC while computing a summary with SFA. The challenge is that the TCC requires a well-defined boundary that is not generally available in the setting of SFA. To overcome this, we show how the scalar field itself can be used to define a boundary that can be used to confirm coverage. This requires an interpretation of the TCC that resolves one of the major barriers to wider use. It also extends SFA methods to the setting in which coverage is only confirmed in a subset of the domain.

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## 1 Introduction

In the topological analysis of scalar fields (SFA), one computes a topological summary capturing qualitative and quantitative shape information from a set of points endowed with a metric and a real-valued function. That is, we have points with distances and a real number assigned to each point. More generally, it suffices to have a neighborhood graph identifying the pairs of points within some distance. The topological computation often takes the form of persistent homology and integrates local information about the function into global information about its *behavior* as whole. In prior work, Chazal et al. [3] showed that for sufficiently dense samples on sufficiently smooth spaces, the persistence diagram can be computed with some guarantees. In followup work, Buchet et al. [1] extended this result to show how to work with noisy inputs. A fundamental assumption required to have strong guarantees is that the underlying space is sufficiently well-sampled. In this paper, we show how to combine scalar field analysis with the theory of topological coverage testing to simultaneously compute the persistence diagram while testing that the underlying space is sufficiently well-sampled.

Initiated by De Silva and Ghrist [10, 6, 7], the theory of homological sensor networks addresses the problem of testing coverage of a bounded domain by a collection of sensors without coordinates. The main result is the topological coverage criterion, which, in its most general form, states that under reasonable geometric assumptions, the $d$-dimensional homology of a pair of simplicial complexes built on the neighborhood graph will be nontrivial if and only if there is sufficient coverage (see Section 3 for the precise statements). This relative persistent homology test is called the Topological Coverage Criterion (TCC).


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Superficially, the methods of SFA and TCC are very similar. Both construct similar complexes and compute the persistent homology of the homological image of a complex on one scale into that of a larger scale. They even overlap on some common techniques in their analysis such as the use of the Nerve theorem and the Rips-Čech interleaving. However, they differ in some fundamental way that makes it difficult to combine them into a single technique. The main difference is that the TCC requires a clearly defined boundary. Not only must the underlying space be a bounded subset of $\mathbb{R}^d$, the data must also be labeled to indicate which input points are close to the boundary. This requirement is perhaps the main reason why the TCC can so rarely be applied in practice.

In applications to data analysis it is more natural to assume that the data measures some unknown function. We can then replace this requirement with assumptions about the function itself. Indeed, these assumptions could relate the behavior of the function to the topological boundary of the space. However, the generalized approach by Cavanna et al. [2] allows much more freedom in how the boundary is defined.

We consider the case in which we have incomplete data from a particular sublevel set of our function. Our goal is to isolate this data so we can analyze the function in only the verified region. From this perspective, the TCC confirms that we not only have coverage, but that the sample we have is topologically representative of the region near, and above this sublevel set. We can then re-use the same machinery to analyze a *part* of the function in a specific way.

### Contribution

We will re-cast the TCC as a way to verify that the persistent homology of a scalar field can be *partially* approximated by a given sample. Specifically, we will relate the persistent homology of a function relative to a *static* sublevel set to a *truncation* of the full diagram. That is, beyond a certain point the full diagram remains unchanged, allowing for possible reconstruction. This is in comparison with the *restricted* diagram obtained by simply ignoring part of the domain. We therefore present relative persistent homology as an alternative to restriction in a way that extends the TCC to the analysis of scalar fields.

Section 2 establishes notation and provides an overview of our main results in Sections 3 and 4. In Section 5 we introduce an interpretation of the relative diagram as a truncation of the full diagram that is motivated by a number of experiments in Section 6.

## 2 Summary

Let $\mathbb{X}$ denote an orientable $d$-manifold and $D \subset \mathbb{X}$ a compact subspace. For a $c$-Lipschitz functon $f : D \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ let $B_\alpha := f^{-1}((-\infty, \alpha])$ denote the $\alpha$-sublevel set of $f$. Our sample will be denoted $P$, and the subset of points sampling $B_\alpha$ will be denoted $Q_\alpha := P \cap B_\alpha$. For ease of exposition let

$$D_{\lfloor \alpha \rfloor_w} := B_\alpha \cup B_w$$

denote the *truncated* $\alpha$ sublevel set and

$$P_{\lfloor \alpha \rfloor_w} := Q_\alpha \cup Q_w$$

denote its sampled counterpart for all $\alpha, w \in \mathbb{R}$.

We will select a sublevel set $B_\omega$ to serve as our boundary. Specifically, we require that $B_\omega$ *surrounds* $D$, where the notion of a surrounding set is defined formally in Section 3. This distinction allows us to generalize the standard proof of the geometric TCC as properties of surrounding pairs.

### Results

Suppose $B_\omega$ surrounds $D$ in $\mathbb{X}$ and $\delta < \varrho_D/4$, where $\varrho_D$ denotes the *strong convexity radius* of $D$ (see Chazal et al. [3]). As a minimal assumption we require that every component of $D \setminus B_\omega$ contains a point in $P$. We also make additional technical assumptions on $P$ and $\delta$ with respect to the pair $(D, B_\omega)$ (see Section 3 and Lemma 25 of the Appendix).

**Theorem 6** If

I. $\mathrm{H}_0(D \setminus B_{\omega+5c\delta} \hookrightarrow D \setminus B_\omega)$ is *surjective*,

II. $\mathrm{H}_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-3c\delta})$ is *injective*,

and

$$\mathbf{rk}\ \mathrm{H}_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$$

then $D \setminus B_\omega \subseteq P^\delta$ and $Q^\delta_{\omega-2c\delta}$ surrounds $P^\delta$ in $D$. [1]

This formulation of the TCC states that our approximation by a nested pair of Rips complexes captures the homology of the pair $(D, B_\omega)$ in a specific way. We use this fact to interleave our sample with the relative diagram of the filtration $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$. This is done by generalizing our regularity assumptions near $D \setminus B_\omega$ in a way that allows us to interleave persistence modules relative to static sublevels.

**Theorem 17** Suppose $D \setminus B_\omega \subseteq P^\delta$ and $Q^\delta_{\omega-2c\delta}$ surrounds $P^\delta$ in $D$. If

I. $\mathrm{H}_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is *surjective* and

II. $\mathrm{H}_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an *isomorphism*

for all $k$ then the persistent homology modules of

$$\{\mathcal{R}^{2\delta}(P_{\lfloor\alpha\rfloor_{\omega-2c\delta}}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor\alpha\rfloor_{\omega+c\delta}}, Q_{\omega+c\delta})\}_{\alpha\in\mathbb{R}}$$

and $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$ are $4c\delta$ interleaved.

The main challenges we face come from the fact that the sublevel set $B_\omega$ and our approximation by the inclusion $\mathcal{R}^{2\delta}(Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(Q_{\omega+c\delta})$ remain *static* throughout. Using the fact that $Q^\delta_{\omega-2c\delta}$ surrounds $P^\delta$ in $D$ we define an *extension* $(D, \mathcal{E}Q^\delta_{\omega-2c\delta})$ of the pair $(P^\delta, Q^\delta_{\omega-2c\delta})$ that has isomorphic relative homology by excision. These extensions give us a sequence of inclusion maps

$$B_{\omega-3c\delta} \hookrightarrow \mathcal{E}Q^{2\delta}_{\omega-2c\delta} \hookrightarrow B_\omega \hookrightarrow \mathcal{E}Q^{4\delta}_{\omega+c\delta} \hookrightarrow B_{\omega+5c\delta}$$

that can be used along with our regularity assumptions to prove the interleaving.

[1] We state this result using constants that will be used to prove the interleaving. The statement of Theorem 6 parameterizes the region around $\omega$ in terms of $\zeta \geq \delta$ as $[\omega - c(\delta+\zeta), \omega + c(\delta+\zeta)]$.

### Relative, Truncated, and Restricted Persistence Diagrams

For fixed $\omega \in \mathbb{R}$ we will refer to the persistence diagram associated with the filtration $\{(D_{\lfloor \alpha \rfloor_\omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ as the **relative diagram** of $f$. In Section 5 we relate the relative diagram to the *full* diagram of the sublevel set filtration $\{B_\alpha\}_{\alpha\in\mathbb{R}}$. Specifically, we define the **truncated diagram** to be the subdiagram consisting of features born *after* $\omega$ in the full. In Section 6 we compare the relative and truncated diagrams to the **restricted diagram**, defined to be that of the sublevel set filtration of $f|_{D\setminus B_\omega}$.

Note that the truncated sublevel sets $D_{\lfloor \alpha \rfloor_\omega}$ are equal to the union of $B_\omega$ and the restricted sublevel sets. It is in this sense that $B_\omega$ is *static* throughout—it is contained in every sublevel set of the relative filtration. As we will not have verified coverage in $B_\omega$ we cannot analyze the function in this region directly. We therefore have two alternatives: *restrict* the domain of the function to $D \setminus B_\omega$, or use relative homology to analyze the function *relative* to this region using excision.

### Outline of Sections 3 and 4

We will begin with our statement of the TCC in Section 3. This requires the introduction of surrounding pairs before proving our reformulation of the TCC (Theorem 6). Section 4 formally introduces extensions and partial interleavings of image modules which will be used to interleave our approximation with the relative diagram (Theorem 17).

## 3 The Topological Coverage Criterion (TCC)

A positive result from the TCC requires that we have a subset of our cover to serve as the boundary. That is, the the condition not only checks that we have coverage, but also that we have a pair of spaces that reflects the pair $(D, B)$ topologically. We call such a pair a *surrounding pair* defined in terms of separating sets. It has been shown that the TCC can be stated in terms of these surrounding pairs [2]. Moreover, this work made assumptions directly in terms of the *zero dimensional* persistent homology of the domain close to the boundary. This allows allows us enough flexibility to define our surrounding set as a sublevel of a $c$-Lipschitz function $f$ and state our assumptions in terms of its persistent homology.

▶ **Definition 1** (Surrounding Pair). *Let $X$ be a topological space and $(D, B)$ a pair in a topological space $X$. The set $B$ **surrounds** $D$ **in** $X$ if $B$ separates $X$ with the pair $(D \setminus B, X \setminus D)$. We will refer to such a pair as a **surrounding pair in** $X$.*

The following lemma generalizes the proof of the TCC as a property of surrounding sets.We will then combine these results on the homology of surrounding pairs with information about both $\mathbb{X}$ as a metric space and our function.

▶ **Lemma 2.** *Let $(D, B)$ be a surrounding pair in $X$ and $U \subseteq D$, $V \subseteq U \cap B$ be subsets. Let $\ell : \mathrm{H}_0(X \setminus B, X \setminus D) \to \mathrm{H}_0(X \setminus V, X \setminus U)$ be induced by inclusion.*

*If $\ell$ is injective then $D \setminus B \subseteq U$ and $V$ surrounds $U$ in $D$.*

Let $(\mathbb{X}, \mathbf{d})$ be a metric space and $D \subseteq \mathbb{X}$ be a compact subspace. For a $c$-Lipschitz function $f : D \to \mathbb{R}$ we introduce a constant $\omega$ as a threshold that defines our "boundary" as a sublevel set $B_\omega$ of the function $f$. Let $P$ be a finite subset of $D$ and $\zeta \geq \delta > 0$ and be constants such that $P^\delta \subseteq \mathbf{int}_{\mathbb{X}}(D)$. Here, $\delta$ will serve as our communication radius where $\zeta$ is reserved for use in Section 4. [2]

[2] We will set $\zeta = 2\delta$ in the proof of our interleaving with Rips complexes but the TCC holds for all $\zeta \geq \delta$.

▶ **Lemma 3.** *Let* $i : \mathrm{H}_0(\overline{Q^\delta_{\omega+c\delta}}, \overline{P^\delta}) \to \mathrm{H}_0(\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta})$.
*If* $B_\omega$ *surrounds* $D$ *in* $\mathbb{X}$ *then* $\mathbf{dim}\ \mathrm{H}_0(\overline{B_\omega}, \overline{D}) \geq \mathbf{rk}\ i$.

**Proof.** Choose a basis for $\mathbf{im}\ i$ such that each basis element is represented by a point in $P^\delta \setminus Q^\delta_{\omega+c\delta}$. Let $x \in P^\delta \setminus Q^\delta_{\omega+c\delta}$ be such that $i[x] \neq 0$. So there exits some $p \in P$ such that $\mathbf{d}(p, x) < \delta$ and $p \notin Q_{\omega+c\delta}$, otherwise $x \in Q^\delta_{\omega+c\delta}$. Therefore, because $f$ is $c$-Lipschitz,

$$f(x) \geq f(p) - c\mathbf{d}(x, p) > \omega + c\delta - c\delta = \omega.$$

So $x \in \overline{B_\omega}$ and, because $x \in P^\delta \subseteq D$, $x \in D \setminus B_\omega$. Because $i$ and $\ell : \mathrm{H}_0(\overline{B_\omega}, \overline{D}) \to \mathrm{H}_0(\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta})$ are induced by inclusion $\ell[x] = i[x] \neq 0$ in $\mathrm{H}_0(\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta})$. That is, every element of $\mathbf{im}\ i$ has a preimage in $\mathrm{H}_0(\overline{B_\omega}, \overline{D})$, so we may conclude that $\mathbf{dim}\ \mathrm{H}_0(\overline{B_\omega}, \overline{D}) \geq \mathbf{rk}\ i$. ◀

Note that, while there is a surjective map from $\mathrm{H}_0(\overline{B_\omega}, \overline{D})$ to $\mathbf{im}\ i$ this map is not necessarily induced by inclusion. We therefore must introduce a larger space $B_{\omega+c(\delta+\zeta)}$ that contains $Q^\delta_{\omega+c\delta}$ in order to provide a criteria for the injectivity of $\ell : \mathrm{H}_0(\overline{B_\omega}, \overline{D}) \to \mathrm{H}_0(\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta})$ in terms of $\mathbf{rk}\ i$. We have the following commutative diagrams of inclusion maps the induced maps between complements in $\mathbb{X}$.

$$\begin{tikzcd}
(P^\delta, Q^\delta_{\omega-c\zeta}) \arrow[r, hook] \arrow[d, hook] & (P^\delta, Q^\delta_{\omega+c\delta}) \arrow[d, hook] \\
(D, B_\omega) \arrow[r, hook] & (D, B_{\omega+c(\delta+\zeta)}),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{H}_0(\overline{B_{\omega+c(\delta+\zeta)}}, \overline{D}) \arrow[r, "j"] \arrow[d, "m"] & \mathrm{H}_0(\overline{B_\omega}, \overline{D}) \arrow[d, "\ell"] \\
\mathrm{H}_0(\overline{Q^\delta_{\omega+c\delta}}, \overline{P^\delta}) \arrow[r, "i"] & \mathrm{H}_0(\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta}).
\end{tikzcd} \tag{1}$$

## Assumptions

We will first require the map $\mathrm{H}_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ to be *surjective*—as we approach $\omega$ from *above* no components *appear*. This ensures that the rank of the map $j$ is equal to the dimension of $\mathbf{dim}\ \mathrm{H}_0(\overline{B_\omega}, \overline{D})$ so $\ell$ depends only on $\mathrm{H}_0(\overline{B_\omega}, \overline{D})$ and $\mathbf{im}\ i$.

We also assume that $\mathrm{H}_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is *injective*—as we move away from $\omega$ moving *down* no components *disappear*. Lemma 4 uses Assumption 2 to provide a computable upper bound on $\mathbf{rk}\ j$.

▶ **Lemma 4.** *If* $\mathrm{H}_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega+c(\delta+\zeta)})$ *is injective and each component of* $D \setminus B_\omega$ *contains a point in* $P$ *then* $\mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \mathbf{dim}\ \mathrm{H}_0(D \setminus B_\omega)$.

## Nerves and Duality

Recall that the Nerve Theorem states that for a good open cover$\mathcal{U}$ of a space $X$ the inclusion map from the *Nerve* of the cover to the space $\mathcal{N}(\mathcal{U}) \hookrightarrow X$ is a homotopy equivalence.[3] The Persistent Nerve Lemma [4] states that this homotopy equivalence commutes with inclusion on the level of homology. We note that the standard proof of the Nerve Theorem [9], and therefore the Persistent Nerve Lemma [4], extends directly to pairs of good open covers $(\mathcal{U}, \mathcal{V})$ of pairs $(X, Y)$ such that $\mathcal{V}$ is a subcover of $\mathcal{U}$.[4]

[3] In a good open cover every nonempty intersection of sets in the cover is contractible.
[4] $\{V_i\}_{i\in I}$ is a subcover of $\{U_i\}_{i\in I}$ if $V_i \subseteq U_i$ for all $i \in I$.

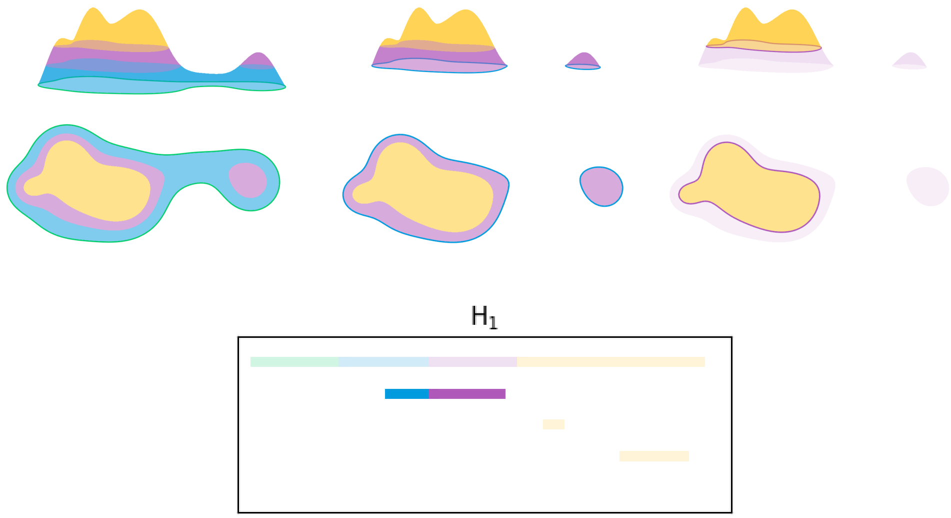


**Figure 1** The blue level set does not satisfy either assumption as the smaller component is not in the inclusion from blue to green and it "pinched out" in the yellow region. This can be seen in the barcode shown as a feature that is born in the blue region and dies in the purple region.

Recalling the definition of the strong convexity radius $\varrho_D$ (see Chazal et al. [3]) $\mathcal{U}$ is a good open cover whenever $\varrho_D > \varepsilon$. As the Čech complex is the Nerve of a cover by a union of balls we will let $\mathcal{N}_w^\varepsilon : \mathrm{H}_k(\check{\mathcal{C}}^\varepsilon(P, Q_w)) \to \mathrm{H}_k(P^\varepsilon, Q_w^\varepsilon)$ denote the isomorphism on homology provided by the Nerve Theorem for all $k$, $w \in \mathbb{R}$ and $\varepsilon < \varrho_D$.

Under certain conditions Alexander Duality provides an isomorphism between the $k$ relative cohomology of a compact pair in an orientable $d$-manifold $\mathbb{X}$ with the $d-k$ dimensional homology of their complements in $\mathbb{X}$ (see Spanier [11]). For finitely generated (co)homology over a field the Universal Coefficient Theorem can be used with Alexander Duality to give a natural isomorphism $\xi_w^\varepsilon : \mathrm{H}_d(P^\varepsilon, Q_w^\varepsilon) \to \mathrm{H}_0(D \setminus Q_w^\varepsilon, D \setminus P^\varepsilon)$.[5] This isomorphism holds in the specific case when $P^\varepsilon \subseteq \mathbf{int}_{\mathbb{X}}(D)$ and $D \setminus P^\varepsilon$, $D \setminus Q_w^\varepsilon$ are locally contractible. We therefore provide the following definition for ease of exposition.

▶ **Definition 5** ($(\delta, \zeta, \omega)$-Sublevel Sample)**.** *For $\zeta \geq \delta > 0$, $\omega \in \mathbb{R}$, and a c-Lipschitz function $f : D \to \mathbb{R}$ a finite point set $P \subset D$ is said to be a $(\delta, \zeta, \omega)$-**sublevel sample** of $f$ if every component of $D \setminus B_\omega$ contains a point in $P$, $P^\delta \subset \mathbf{int}_{\mathbb{X}}(D)$, and $D \setminus P^\delta$, $D \setminus Q_{\omega - c\zeta}^\delta$, and $D \setminus Q_{\omega + c\delta}^\delta$ are locally path connected in $\mathbb{X}$.*

Because this isomorphism is natural and the isomorphism provided by the Nerve Theorem commutes with maps induced by inclusion the composition $\xi\mathcal{N}_w^\varepsilon := \xi_w^\varepsilon \circ \mathcal{N}_w^\varepsilon$ gives an isomorphism that commutes with maps induced by inclusion for all $w \in \mathbb{R}$ and $\varepsilon < \varrho_D$.

▶ **Theorem 6** (Algorithmic TCC)**.** *Let $\mathbb{X}$ be an orientable $d$-manifold and let $D$ be a compact subset of $\mathbb{X}$. Let $f : D \to \mathbb{R}$ be c-Lipschitz function and $\omega \in \mathbb{R}$ and $\delta \leq \zeta < \varrho_D$ be constants such that $P \subset D$ is a $(\delta, \zeta, \omega)$-sublevel sample of $f$ and $B_{\omega - c(\zeta + \delta)}$ surrounds $D$ in $\mathbb{X}$.*

*If $\mathrm{H}_0(D \setminus B_{\omega + c(\delta + \zeta)} \hookrightarrow D \setminus B_\omega)$ is surjective, $\mathrm{H}_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega - c(\delta + \zeta)})$ is injective, and $\mathbf{rk}\ \mathrm{H}_d(\mathcal{R}^\delta(P, Q_{\omega - c\zeta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega + c\delta})) \geq \mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega - c\zeta}))$ then $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega - c\zeta}^\delta$ surrounds $P^\delta$ in $D$.*

[5] For the construction of this isomorphism see the Appendix.

**Proof.** Because $P$ is a $(\delta, \zeta, \omega)$-sublevel sample we have isomorphisms $\xi\mathcal{N}^{\delta}_{\omega-c\zeta}$ and $\xi\mathcal{N}^{\delta}_{\omega+c\delta}$ that commute with $q_{\check{\mathcal{C}}} : \mathrm{H}_d(\check{\mathcal{C}}^{\delta}(P, Q_{\omega-c\zeta})) \to \mathrm{H}_d(\check{\mathcal{C}}^{2\delta}(P, Q_{\omega+c\delta}))$ and $i : \mathrm{H}_0(D \setminus Q^{\delta}_{\omega+c\delta}, D \setminus P^{\delta}) \to \mathrm{H}_0(D \setminus Q^{\delta}_{\omega-c\zeta}, D \setminus P^{\delta})$. Let $q_{\mathcal{R}} : \mathrm{H}_d(\mathcal{R}^{\delta}(P, Q_{\omega-c\zeta})) \to \mathrm{H}_d(\mathcal{R}^{2\delta}(P, Q_{\omega+c\delta}))$ be induced by inclusion. Then $\mathbf{rk}\ q_{\check{\mathcal{C}}} \geq \mathbf{rk}\ q_{\mathcal{R}}$ as $q_{\mathcal{R}}$ factors through $q_{\check{\mathcal{C}}}$. As we have assumed $\mathrm{H}_0(D \setminus B_{\omega} \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ Lemma 4 implies $\mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^{\delta}(P \setminus Q_{\omega-c\zeta})) \geq \mathbf{dim}\ \mathrm{H}_0(D \setminus B_{\omega})$. It follows that, whenever $\mathbf{rk}\ q_{\mathcal{R}} \geq \mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^{\delta}(P \setminus Q_{\omega-c\zeta}))$, we have

$$\mathbf{rk}\ i = \mathbf{rk}\ q_{\check{\mathcal{C}}} \geq \mathbf{rk}\ q_{\mathcal{R}} \geq \mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^{\delta}(P \setminus Q_{\omega-c\zeta})) \geq \mathbf{dim}\ \mathrm{H}_0(D \setminus B_{\omega}).$$

Because $j$ is surjective by hypothesis $\mathbf{rk}\ j = \mathbf{dim}\ \mathrm{H}_0(\overline{B_{\omega}}, \overline{D}) = \mathbf{dim}\ \mathrm{H}_0(D \setminus B_{\omega})$ so $\mathbf{rk}\ j \geq \mathbf{rk}\ i$ by Lemma 3. As we have shown $\mathbf{rk}\ i \geq \mathbf{dim}\ \mathrm{H}_0(D \setminus B_{\omega})$ it follows that $\mathbf{rk}\ j = \mathbf{rk}\ i$. Because $P$ is a finite point set we know that $\mathbf{im}\ i$ is finite-dimensional and, because $\mathbf{rk}\ i = \mathbf{rk}\ j$, $\mathbf{im}\ j = \mathrm{H}_0(\overline{B_{\omega}}, \overline{D})$ is finite dimensional as well. So $\mathbf{im}\ j$ is isomorphic to $\mathbf{im}\ i$ as a subspace of $\mathrm{H}_0(\overline{Q^{\delta}_{\omega-c\zeta}}, \overline{P^{\delta}})$ which, because $j$ is surjective, requires the map $\ell$ to be injective. Therefore $D \setminus B_{\omega} \subseteq P^{\delta}$ and $Q^{\delta}_{\omega-c\zeta}$ surrounds $P^{\delta}$ in $D$ by Lemma 2. ◀

# 4 From Coverage Testing to the Analysis of Scalar Fields

Because the TCC only confirms coverage of a *superlevel* set $D \setminus B_{\omega}$, we cannot guarantee coverage of the entire domain. Indeed, we could compute the persistent homology of the *restriction* of $f$ to the superlevel set we cover in the standard way [3]. Instead, we will approximate the persistent homology of the sublevel set filtration *relative to* the sublevel set $B_{\omega}$. In the next section we will discuss an interpretation of the relative diagram that is motivated by examples in Section 6.


**Figure 2** Full, restricted, and relative barcodes of the function (left).

We will first introduce the notion of an extension which will provide us with maps on relative homology induced by inclusion via excision. However, even then, a map that factors through our pair $(D, B_{\omega})$ is not enough to prove an interleaving of persistence modules by inclusion directly. To address this we impose conditions on sublevel sets near $B_{\omega}$ which generalize the assumptions made in the TCC.

## 4.1 Extensions and Image Persistence Modules

Suppose $D$ is a subspace of $X$. We define the extension of a surrounding pair in $D$ to a surrounding pair in $X$ with isomorphic relative homology.

▶ **Definition 7** (Extension). *If $V$ surrounds $U$ in a subspace $D$ of $X$ let $\mathcal{E}V := V \sqcup (D \setminus U)$ denote the (disjoint) union of the separating set $V$ with the complement of $U$ in $D$. The* ***extension of*** *$(U, V)$* ***in*** *$D$ is the pair $(D, \mathcal{E}V) = (U \sqcup (D \setminus U), V \sqcup (D \setminus U))$.*

Lemma 8 states that we can use these extensions to interleave a pair $(U, V)$ with a sequence of subsets of $(D, B)$. Lemma 9 states that we can apply excision to the relative homology groups in order to get equivalent maps on homology that are induced by inclusions.

▶ **Lemma 8.** *Suppose $V$ surrounds $U$ in $D$ and $B' \subseteq B \subset D$.*
*If $D \setminus B \subseteq U$ and $U \cap B' \subseteq V \subseteq B'$ then $B' \subseteq \mathcal{E}V \subseteq B$.*

▶ **Lemma 9.** *Let $(U, V)$ be an open surrounding pair in a subspace $D$ of $X$.*
*Then $\mathrm{H}_k((U \cap A, V) \hookrightarrow (A, \mathcal{E}V))$ is an isomorphism for all $k$ and $A \subseteq D$ with $\mathcal{E}V \subset A$.*

The TCC uses a nested pair of spaces in order to filter out noise introduced by the sample. This same technique is used to approximate the persistent homology of a scalar fields [3]. As modules, these nested pairs are the images of homomorphisms between homology groups induced by inclusion, which we refer to as image persistence modules.

▶ **Definition 10** (Image Persistence Module)**.** *The **image persistence module** of a homomorphism $\Gamma \in \mathrm{Hom}(\mathbb{U}, \mathbb{V})$ is the family of subspaces $\{\Gamma_\alpha := \mathbf{im}\ \gamma_\alpha\}$ in $\mathbb{V}$ along with linear maps $\{\gamma_\alpha^\beta := v_\alpha^\beta|_{\mathbf{im}\ \gamma_\alpha} : \Gamma_\alpha \to \Gamma_\beta\}$ and will be denoted by $\mathbf{im}\ \Gamma$.*

While we will primarily work with homomorphisms of persistence modules induced by inclusions, in general, defining homomorphisms between images simply as subspaces of the codomain is not sufficient. Instead, we require that homomorphisms between image modules commute not only with shifts in scale, but also with the functions themselves.

▶ **Definition 11** (Image Module Homomorphism)**.** *Given $\Gamma \in \mathrm{Hom}(\mathbb{U}, \mathbb{V})$ and $\Lambda \in \mathrm{Hom}(\mathbb{S}, \mathbb{T})$ along with $(F, G) \in \mathrm{Hom}^\delta(\mathbb{U}, \mathbb{S}) \times \mathrm{Hom}^\delta(\mathbb{V}, \mathbb{T})$ let $\Phi(F, G) : \mathbf{im}\ \Gamma \to \mathbf{im}\ \Lambda$ denote the family of linear maps $\{\phi_\alpha := g_\alpha|_{\Gamma_\alpha} : \Gamma_\alpha \to \Lambda_{\alpha+\delta}\}$. $\Phi(F, G)$ is an **image module homomorphism of degree** $\delta$ if the following diagram commutes for all $\alpha \leq \beta$.*[6]

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\gamma_\alpha[\beta-\alpha]} & V_\beta \\ \downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle g_\beta} \\ S_{\alpha+\delta} & \xrightarrow{\lambda_{\alpha+\delta}[\beta-\alpha]} & T_{\beta+\delta} \end{array} \tag{2}$$

*The space of image module homomorphisms of degree $\delta$ between $\mathbf{im}\ \Gamma$ and $\mathbf{im}\ \Lambda$ will be denoted $\mathrm{Hom}^\delta(\mathbf{im}\ \Gamma, \mathbf{im}\ \Lambda)$.*

The composition of image module homomorphisms are image module homomorphisms. Proof of this fact can be found in the Appendix.

## Partial Interleavings of Image Modules

Image module homomorphisms introduce a direction to the traditional notion of interleaving. As we will see, our interleaving via Lemma 13 involves partially interleaving an image module to two other image modules whose composition is isomorphic to our target.

▶ **Definition 12** (Partial Interleaving of Image Modules)**.** *An image module homomorphism $\Phi(F, G)$ is a **partial $\delta$-interleaving of image modules**, and denoted $\Phi_M(F, G)$, if there exists $M \in \mathrm{Hom}^\delta(\mathbb{S}, \mathbb{V})$ such that $\Gamma[2\delta] = M \circ F$ and $\Lambda[2\delta] = G \circ M$.*

[6] We use the notation $\gamma_\alpha[\beta-\alpha] = v_\alpha^\beta \circ \gamma_\alpha$, $\lambda_\alpha[\beta-\alpha] = t_\alpha^\beta \circ \lambda_\alpha$ to denote the composition of homomorphisms between persistence modules and shifts in scale.

Lemma 13 uses partial interleavings of a map $\Lambda$ with $\mathbb{U} \to \mathbb{V}$ and $\mathbb{V} \to \mathbb{W}$ along with the hypothesis that $\mathbb{U} \to \mathbb{W}$ is isomorphic to $\mathbb{V}$ to interleave $\mathbf{im}\ \Lambda$ with $\mathbb{V}$. When applied, this hypothesis will be satisfied by assumptions on our sublevel set similar to those made in the TCC.

▶ **Lemma 13.** *Suppose* $\Gamma \in \mathrm{Hom}(\mathbb{U}, \mathbb{V})$, $\Pi \in \mathrm{Hom}(\mathbb{V}, \mathbb{W})$, *and* $\Lambda \in \mathrm{Hom}(\mathbb{S}, \mathbb{T})$.

*If* $\Phi_M(F, G) \in \mathrm{Hom}^{\delta}(\mathbf{im}\ \Gamma, \mathbf{im}\ \Lambda)$ *and* $\Psi_G(M, N) \in \mathrm{Hom}^{\delta}(\mathbf{im}\ \Lambda, \mathbf{im}\ \Pi)$ *are partial* $\delta$*-interleavings of image modules such that* $\Gamma$ *is a epimorphism and* $\Pi$ *is a monomorphism then* $\mathbf{im}\ \Lambda$ *is* $\delta$*-interleaved with* $\mathbb{V}$.

## 4.2 Proof of the Interleaving

For $w, \alpha \in \mathbb{R}$ let $\mathbb{D}_w^k$ denote the $k$th persistent (relative) homology module of the filtration $\{(D_{\lfloor \alpha \rfloor_w}, B_w)\}_{\alpha \in \mathbb{R}}$ with respect to $B_w$, and let $\mathbb{P}_w^{\varepsilon;k}$ denote the $k$th persistent (relative) homology module of $\{(P^{\varepsilon}_{\lfloor \alpha \rfloor_w}, Q^{\varepsilon}_w)\}_{\alpha \in \mathbb{R}}$. Similarly, let $\check{\mathcal{C}}\mathbb{P}_w^{\varepsilon;k}$ and $\mathcal{R}\mathbb{P}_w^{\varepsilon;k}$ denote the corresponding Čech and Vietoris-Rips filtrations, respectively. We will omit the dimension $k$ and write $\mathbb{D}_w$ (resp. $\mathbb{P}^{\varepsilon}_w$) if a statement holds for all dimensions.

If $Q^{\delta}_w$ surrounds $P^{\delta}$ in $D$ let $\mathcal{E}\mathbb{P}^{\varepsilon}_w$ denote the $k$th persistent homology module of the filtration of extensions $\{(\mathcal{E}P^{\varepsilon}_{\lfloor \alpha \rfloor_w}, \mathcal{E}Q^{\varepsilon}_w)\}$ for any $\varepsilon \geq \delta$, where $\mathcal{E}P^{\varepsilon}_{\lfloor \alpha \rfloor_w} = P^{\varepsilon}_{\lfloor \alpha \rfloor_w} \cup (D \setminus P^{\delta})$.

Lemma 14 follows directly from the definition of truncated sublevel sets. This is used to extend Lemma 8 to persistence modules in Lemma 15 to provide a sequence of shifted homomorphisms $\mathbb{D}_{\omega - 3c\delta} \xrightarrow{F} \mathcal{E}\mathbb{P}^{\varepsilon}_{\omega - 2c\delta} \xrightarrow{M} \mathbb{D}_{\omega} \xrightarrow{G} \mathcal{E}\mathbb{P}^{2\varepsilon}_{\omega + c\delta} \xrightarrow{N} \mathbb{D}_{\omega + 5c\delta}$ of varying degree for $\varepsilon \in [\delta, 2\delta]$. These homomorphisms can then be combined with homomorphisms given by the Nerve theorem and the Rips-Čech interleaving in Lemma 16 to obtain the partial interleavings required for our proof of Theorem 17.

▶ **Lemma 14.** *If* $\delta \leq \varepsilon$ *and* $t, \alpha \in \mathbb{R}$ *then* $P^{\delta} \cap D_{\lfloor \alpha - c\varepsilon \rfloor_{t - c\varepsilon}} \subseteq P^{\varepsilon}_{\lfloor \alpha \rfloor_t} \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor_{t + c\varepsilon}}$.

▶ **Lemma 15.** *Let* $s + 3c\delta \leq t + 2c\delta \leq u \leq v - c\delta \leq w - 5c\delta$ *and* $\varepsilon \in [\delta, 2\delta]$. *If* $Q^{\delta}_t$ *surrounds* $P^{\delta}$ *in* $D$ *and* $D \setminus B_u \subseteq P^{\delta}$ *then there exist shifted homomorphisms induced by inclusions:*

$$(F, G) \in \mathrm{Hom}^{c\delta}(\mathbb{D}_s, \mathcal{E}\mathbb{P}^{\varepsilon}_t) \times \mathrm{Hom}^{2c\delta}(\mathbb{D}_u, \mathcal{E}\mathbb{P}^{2\varepsilon}_v),\ (M, N) \in \mathrm{Hom}^{c\varepsilon}(\mathcal{E}\mathbb{P}^{\varepsilon}_t, \mathbb{D}_u) \times \mathrm{Hom}^{2c\varepsilon}(\mathcal{E}\mathbb{P}^{2\varepsilon}_v, \mathbb{D}_w).$$

▶ **Lemma 16.** *For* $\delta < \varrho_D$ *let* $\Gamma \in \mathrm{Hom}(\mathbb{D}_s, \mathbb{D}_u)$, $\Pi \in \mathrm{Hom}(\mathbb{D}_u, \mathbb{D}_w)$, *and* $\Lambda \in \mathrm{Hom}(\mathcal{R}\mathbb{P}^{2\delta}_t, \mathcal{R}\mathbb{P}^{4\delta}_v)$ *be induced by inclusions for* $s + 3c\delta \leq t + 2c\delta \leq u \leq v - c\delta \leq w - 5c\delta$.

*If* $Q^{\delta}_t$ *surrounds* $P^{\delta}$ *in* $D$ *and* $D \setminus B_u \subseteq P^{\delta}$ *then there is a partial* $2c\delta$ *interleaving* $\Phi^* \in \mathrm{Hom}^{2c\delta}(\mathbf{im}\ \Gamma, \mathbf{im}\ \Lambda)$ *and a partial* $4c\delta$ *interleaving* $\Psi^* \in \mathrm{Hom}^{4c\delta}(\mathbf{im}\ \Lambda, \mathbf{im}\ \Pi)$.

**Proof.** Because the shifted homomorphisms provided by Lemma 15 are all induced by inclusions the following diagram commutes for all $\alpha \leq \beta$.

$$\begin{array}{ccccccc}
\mathrm{H}_k(D_{\lfloor \alpha - 2c\delta \rfloor_s}, B_s) & \xrightarrow{f_{\alpha - 2c\delta}} & \mathrm{H}_k(\mathcal{E}P^{\delta}_{\lfloor \alpha \rfloor_t}, B_t) & & \mathrm{H}_k(\mathcal{E}P^{2\delta}_{\lfloor \alpha \rfloor_t}, B_t) & \xrightarrow{m_{\alpha}} & \mathrm{H}_k(D_{\lfloor \alpha + 4c\delta \rfloor_u}, B_u) \\
\Big\downarrow{\scriptstyle \gamma_{\alpha - 2c\delta}[\beta - \alpha]} & & \Big\downarrow{\scriptstyle c_{\alpha}[\beta - \alpha] \circ a_{\alpha}} & & \Big\downarrow{\scriptstyle e_{\beta} \circ c_{\alpha}[\beta - \alpha]} & & \Big\downarrow{\scriptstyle \gamma_{\alpha + 4c\delta}[\beta - \alpha]} \\
\mathrm{H}_k(D_{\lfloor \beta - 2c\delta \rfloor_u}, B_u) & \xrightarrow{g_{\beta - 2c\delta}} & \mathrm{H}_k(\mathcal{E}P^{2\delta}_{\lfloor \beta \rfloor_v}, B_v) & & \mathrm{H}_k(\mathcal{E}P^{4\delta}_{\lfloor \beta \rfloor_v}, B_v) & \xrightarrow{n_{\beta}} & \mathrm{H}_k(D_{\lfloor \beta + 4c\delta \rfloor_w}, B_w)
\end{array} \tag{3}$$

So we have image module homomorphisms $\Phi(F, G) \in \mathrm{Hom}^{2c\delta}(\mathbf{im}\ \Gamma, \mathbf{im}\ C \circ A)$ and $\Psi(M, N) \in \mathrm{Hom}^{4c\delta}(\mathbf{im}\ E \circ C, \mathbf{im}\ \Pi)$.

Because the isomorphisms provided by Lemma 9 are given by excision they are induced by inclusion, and therefore give isomorphisms $\mathcal{E}^{\varepsilon}_z \in \mathrm{Hom}(\mathbb{P}^{\varepsilon}_z, \mathcal{E}\mathbb{P}^{\varepsilon}_z)$ of persistence modules for any $Q_z^\varepsilon$ surrounding $P^\delta$ in $D$. Moreover, for any $\varepsilon < \varrho_D$, $z \in \mathbb{R}$ we have isomorphisms $\mathcal{N}_z^\varepsilon \in \mathrm{Hom}(\check{\mathcal{C}}\mathbb{P}_z^\varepsilon, \mathbb{P}_z^\varepsilon)$ that commute with maps induced by inclusions by the Persistent Nerve Lemma. Therefore, the composition $\mathcal{E}_z^\varepsilon \circ \mathcal{N}_z^\varepsilon$ is an isomorphism that commutes with maps induced by inclusion as well. These compositions and the Rips-Čech interleaving provide maps $\mathcal{E}\mathbb{P}_t^\delta \xrightarrow{F'} \mathcal{R}\mathbb{P}_t^{2\delta} \xrightarrow{M'} \mathcal{E}\mathbb{P}_t^{2\delta}$ and $\mathcal{E}\mathbb{P}_v^{2\delta} \xrightarrow{G'} \mathcal{R}\mathbb{P}_v^{4\delta} \xrightarrow{N'} \mathcal{E}\mathbb{P}_v^{4\delta}$ that commute with maps induced by inclusions. As all maps are induced by inclusions or commute with maps induced by inclusions we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{E}\mathbb{P}_t^\delta \arrow[rr, "A"] \arrow[dr, "F'"] & & \mathcal{E}\mathbb{P}_t^{2\delta} \arrow[r, "C"] & \mathcal{E}\mathbb{P}_v^{2\delta} \arrow[rr, "E"] \arrow[dr, "G'"] & & \mathcal{E}\mathbb{P}_v^{4\delta} \\
& \mathcal{R}\mathbb{P}_t^{2\delta} \arrow[ur, "M'"] \arrow[rrr, "\Lambda"] & & & \mathcal{R}\mathbb{P}_v^{4\delta} \arrow[ur, "N'"] &
\end{tikzcd} \tag{4}$$

That is, we have image module homomorphisms $\Phi'(F', G')$ and $\Psi'(M', N')$ such that $A = M' \circ F'$, $E = N' \circ G'$, and $\Lambda = G' \circ C \circ M'$. Because image module homomorphisms compose we have we have $\Phi^* = \Phi' \circ \Phi \in \mathrm{Hom}^{2c\delta}(\mathbf{im}\ \Gamma, \mathbf{im}\ \Lambda)$ and $\Psi^* = \Psi \circ \Psi' \in \mathrm{Hom}^{4c\delta}(\mathbf{im}\ \Lambda, \mathbf{im}\ \Pi)$.

Because all maps are induced by inclusions $C[3c\delta] = G \circ M$ so $\Lambda[3c\delta] = G' \circ C[3c\delta] \circ M' = G' \circ (G \circ M) \circ M'$ as $G', M'$ commute with maps induced by inclusions. In the same way, $\Gamma[3c\delta] = M \circ (A \circ F) = M \circ (M' \circ F') \circ F$ and $\Pi[5c\delta] = N \circ (E \circ G) = N \circ (N' \circ G') \circ G$. Let $F^* := F' \circ F$, $G^* := G' \circ G$, $M^* := M' \circ M$, and $N^* := N' \circ N$. So $\Phi_{M^*}^*$ is a partial $2c\delta$ interleaving as $\Gamma[3c\delta] = M^* \circ F^*$ and $\Lambda[3c\delta] = G^* \circ M^*$, and $\Psi_{G^*}^*$ is a partial $4c\delta$ interleaving as $\Lambda[3c\delta] = G^* \circ M^*$ and $\Pi[5c\delta] = N^* \circ G^*$. ◀

The partial interleavings given by Lemma 16 along with assumptions that imply $\mathbf{im}(\mathbb{D}_{\omega-3c\delta} \to \mathbb{D}_{\omega+5c\delta}) \cong \mathbb{D}_\omega$ provide the proof of Theorem 17 by Lemma 13.

▶ **Theorem 17.** *Let $\mathbb{X}$ be a $d$-manifold, $D \subset \mathbb{X}$ and $f : D \to \mathbb{R}$ be a $c$-Lipschitz function. Let $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $B_{\omega-3c\delta}$ surrounds $D$ in $\mathbb{X}$. Let $P \subset D$ be a finite subset and suppose $\mathrm{H}_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $\mathrm{H}_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an isomorphism for all $k$.*

*If $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds $P^\delta$ in $D$ then the $k$th persistent homology module of $\{\mathcal{R}^{2\delta}(P_{\lfloor\alpha\rfloor_{\omega-2c\delta}}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor\alpha\rfloor_{\omega+c\delta}}, Q_{\omega+c\delta})\}_{\alpha\in\mathbb{R}}$ is $4c\delta$-interleaved with that of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$.*

**Proof.** Let $\Lambda \in \mathrm{Hom}(\mathcal{R}\mathbb{P}_{\omega-2c\delta}^{2c\delta}, \mathcal{R}\mathbb{P}_{\omega+c\delta}^{4c\delta})$, $\Gamma \in \mathrm{Hom}(\mathbb{D}_{\omega-3c\delta}, \mathbb{D}_\omega)$, and $\Pi \in \mathrm{Hom}(\mathbb{D}_\omega, \mathbb{D}_{\omega+5c\delta})$ be induced by inclusions. Because $\delta < \varrho_D/4$, $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds $P^\delta$ in $D$ we have a partial $2c\delta$ interleaving $\Phi^* \in \mathrm{Hom}^{2c\delta}(\mathbf{im}\ \Gamma, \mathbf{im}\ \Lambda)$ and a partial $4c\delta$ interleaving $\Psi^* \in \mathrm{Hom}^{4c\delta}(\mathbf{im}\ \Lambda, \mathbf{im}\ \Pi)$ by Lemma 16. As we have assumed that $\mathrm{H}_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $\mathrm{H}_k(B_\omega) \cong \mathrm{H}_k(B_{\omega+5c\delta})$ the five-lemma implies $\gamma_\alpha$ is surjective and $\pi_\alpha$ is an isomorphism (and therefore injective) for all $\alpha$. So $\Gamma$ is an epimorphism and $\Pi$ is a monomorphism, thus $\mathbf{im}\ \Lambda$ is $4c\delta$-interleaved with $\mathbb{D}_\omega$ by Lemma 13 as desired. ◀

## 5 Approximation of the Truncated Diagram

We will relate the relative persistence diagram that we have approximated in the previous section to a truncation of the full diagram. Let $\mathbb{L}^k$ denote the $k$th persistent homology module of the sublevel set filtration $\{B_\alpha\}_{\alpha\in\mathbb{R}}$. As in the previous section, let $\mathbb{D}_\omega^k$ denote the $k$th persistent (relative) homology module of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$. Throughout we will assume that we are taking homology in a field $\mathbb{F}$ and that the homology groups $\mathrm{H}_k(B_\alpha)$ and $\mathrm{H}_k(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)$ are finite dimensional vector spaces for all $k$ and $\alpha \in \mathbb{R}$. We will use the interval decomposition of $\mathbb{L}^k$ to give a decomposition of the relative module $\mathbb{D}^k_\omega$ in terms of a *truncation* of $\mathbb{L}^k$. Recall, the *truncated diagram* is defined to be that of $\mathbb{L}^k$ consisting only of those features born after $\omega$. For fixed $\omega \in \mathbb{R}$ we will define the truncation $\mathbb{T}^k_\omega$ of $\mathbb{L}^k$ in terms of the intervals decomposing $\mathbb{L}^k$ that are in $[\omega, \infty)$.

### Truncated Interval Modules

For an interval $I = [s, t) \subseteq \mathbb{R}$ let $I_+ := [t, \infty)$ and $I_- := (-\infty, s]$. For $\omega \in \mathbb{R}$ let $\mathbb{F}^I_\omega$ denote the interval module consisting of vector spaces $\{F^I_{\lfloor\alpha\rfloor_\omega}\}_{\alpha\in\mathbb{R}}$ and linear maps $\{f^I_{\lfloor\alpha,\beta\rfloor_\omega} : F^I_{\lfloor\alpha\rfloor_\omega} \to F^I_{\lfloor\beta\rfloor_\omega}\}_{\alpha\le\beta}$ where

$$F^I_{\lfloor\alpha\rfloor_\omega} := \begin{cases} F^I_\alpha & \text{if } \omega \in I_- \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f^I_{\lfloor\alpha,\beta\rfloor_\omega} := \begin{cases} f^I_{\alpha,\beta} & \text{if } \omega \in I_- \\ 0 & \text{otherwise.} \end{cases}$$

For a collection $\mathcal{I}$ of intervals let $\mathcal{I}_\omega := \{I \in \mathcal{I} \mid \omega \in I\}$.

▶ **Lemma 18.** *Suppose $\mathcal{I}^k$ and $\mathcal{I}^{k-1}$ are collections of intervals that decompose $\mathbb{L}^k$ and $\mathbb{L}^{k-1}$, respectively. Then for all $k$ the $k$th persistent homology module of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$ is equal to*

$$\bigoplus_{I\in\mathcal{I}^k} \mathbb{F}^I_\omega \oplus \bigoplus_{I\in\mathcal{I}^{k-1}_\omega} \mathbb{F}^{I_+}.$$

## Main Theorem

Let $\mathcal{I}^k$ denote the decomposing intervals of $\mathbb{L}^k$ for all $k$. Let

$$\mathbb{T}^k_\omega := \bigoplus_{I\in\mathcal{I}^k} \mathbb{F}^I_\omega$$

denote the **$\omega$-truncated $k$th persistent homology module** of $\mathbb{L}^k$ and

$$\mathbb{L}^{k-1}_\omega := \bigoplus_{I\in\mathcal{I}^{k-1}_\omega} \mathbb{F}^{I_+}.$$

denote the submodule of $\mathbb{D}^k_\omega$ consisting of intervals $[\beta, \infty)$ corresponding to features $[\alpha, \beta)$ in $\mathbb{L}^{k-1}$ such that $\alpha \le \omega < \beta$. Now, by Lemma 18 the $k$th persistent (relative) homology module of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$ is $\mathbb{D}^k_\omega = \mathbb{T}^k_\omega \oplus \mathbb{L}^{k-1}_\omega$. Our main theorem combines this decomposition with our coverage and interleaving results of Theorems 6 and 17.

▶ **Theorem 19.** *Let $\mathbb{X}$ be an orientable $d$-manifold and let $D$ be a compact subset of $\mathbb{X}$. Let $f : D \to \mathbb{R}$ be a $c$-Lipschitz function and $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $P \subset D$ is a $(\delta, 2\delta, \omega)$-sublevel sample of $f$ and $B_{\omega-3c\delta}$ surrounds $D$ in $\mathbb{X}$.*

*Suppose $\mathrm{H}_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $\mathrm{H}_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an isomorphism for all $k$. If*

$$\mathbf{rk}\ \mathrm{H}_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \ge \mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$$

*then the $k$th (relative) homology module of*

$$\{\mathcal{R}^{2\delta}(P_{\lfloor\alpha\rfloor_{\omega-2c\delta}}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor\alpha\rfloor_{\omega+c\delta}}, Q_{\omega+c\delta})\}_{\alpha\in\mathbb{R}}$$

*is $4c\delta$-interleaved with $\mathbb{T}^k_\omega \oplus \mathbb{L}^{k-1}_\omega$: the $k$th persistent homology module of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$.*

## 6 Experiments

In this section we will discuss a number of experiments which illustrate the benefit of truncated diagrams, and their approximation by relative diagrams, in comparison to their restricted counterparts. We will focus on the persistent homology of functions on a square 2d grid.We chose as our function a radially symmetric damped sinusoid with random noise, depicted in Figure 3, as it has prominent persistent homology in dimension one.

### Experimental setup.

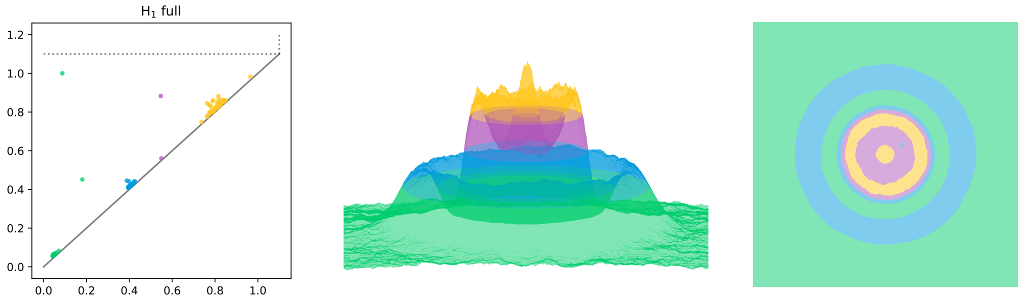


**Figure 3** The $H_1$ persistence diagram of the sinusoidal function pictured to the right. Features are colored by birth time, infinite features are drawn above the dotted line.

Throughout, the four interlevel sets shown correspond to the ranges $[0, 0.3)$, $[0.3, 0.5)$, $[0.5, 0.7)$, and $[0.7, 1)$, respectively. Our persistent homology computations were done primarily with Dionysus augmented with custom software for computing representative cycles of infinite features. [7] The persistent homology of our function was computed with the lower-star filtration of the Freudenthal triangulation on an $N \times N$ grid over $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. We take this filtration as $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ where $P$ is the set of grid points and $\delta = \sqrt{2}/N$.

We note that the purpose of these experiments is not to demonstrate the effectiveness of our approximation by Rips complexes, but to demonstrate the relationships between restricted, relative, and truncated diagrams. Therefore, for simplicity, we will omit the inclusion $\mathcal{R}^{2\delta}(P_\alpha) \hookrightarrow \mathcal{R}^{4\delta}(P_\alpha)$ and take the persistent homology of $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ with sufficiently small $\delta$ as our ground-truth.

In the following we will take $N = 1024$, so $\delta \approx 1.4 \times 10^{-3}$, as our ground-truth. Figure 3 shows the *full diagram* of our function with features colored by birth time. Therefore, for $\omega = 0.3, 0.5, 0.7$ the *truncated diagram* is obtained by successively removing features in each interlevel set. Recall the *restricted diagram* is that of the function restricted to the $\omega$ *super*-levelset filtration, and computed with $\{\mathcal{R}^{2\delta}(P_\alpha \setminus Q_\omega)\}$. We will compare this restricted diagram with the *relative diagram*, computed as the relative persistent homology of the filtration of pairs $\{\mathcal{R}^{2\delta}(P_\alpha, Q_\omega)\}$.

### The issue with restricted diagrams.

Figure **??** shows the bottleneck distance from the truncated diagram at full resolution ($N = 1024$) to both the relative and restricted diagrams with varying resolution. Specifically,

[7] 3D figures were made with MayAvi, all other figures were made with Matplotlib.

the function on a $1024 \times 1024$ grid is down-sampled to grids ranging from $64 \times 64$ to $1024 \times 1024$. We also show the expected bottleneck distance to the true truncated diagram given by the interleaving in Theorem 17 in black.

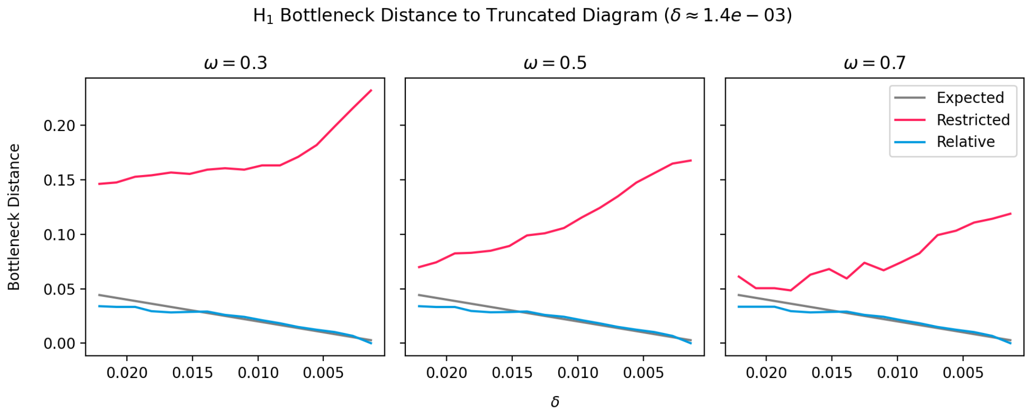


**Figure 4** Comparison of the bottleneck distance between the truncated diagram and those of the restricted and relative diagrams with increasing resolution.

As we can see, the relative diagram clearly performs better than the restricted diagram, which diverges with increasing resolution. Recall that 1-dimensional features that are born before $\omega$ and die after $\omega$ become infinite 2-dimensional features in the relative diagram, with birth time equal to the death time of the corresponding feature in the full diagram. These same features remain 1-dimensional figures in the restricted diagram, but with their birth times shifted to $\omega$.

Figure 5 shows this distance for a feature that persists throughout the diagram. As the restricted diagram in full resolution the restricted filtration is a subset of the full filtration, so these features can be matched by their death simplices. For illustrative purposes we also show the representative cycles associated with these features.

### Relative diagrams and reconstruction.

Now, imagine we obtain the persistence diagram of our sub-levelset $B_\omega$. That is, we now know that we cover $B_\omega$, or some subset, and do not want to re-compute the diagram above $\omega$. If we compute the persistence diagram of the function restricted to the *sub*-levelset $B_\omega$ any 1-dimensional features born before $\omega$ that die after $\omega$ will remain infinite features in this restricted (below) diagram. Indeed, we could match these infinite 1-features with the corresponding shifted finite 1-features in the restricted (above) diagram, as shown in Figure 5. However, that would require sorting through all finite features that are born near $\omega$ and deciding if they are in fact features of the full diagram that have been shifted.

Recalling that these same features become infinite 2-features in the relative diagram, we can use the relative diagram instead and match infinite 1-features of the diagram restricted below to infinite 2-features in the relative diagram, as shown in Figures 6 and 7. For this example the matching is given by sorting the 1-features by ascending and the 2-features by descending birth time. How to construct this matching in general, especially in the presence of infinite features in the full diagram, is the subject of future research.

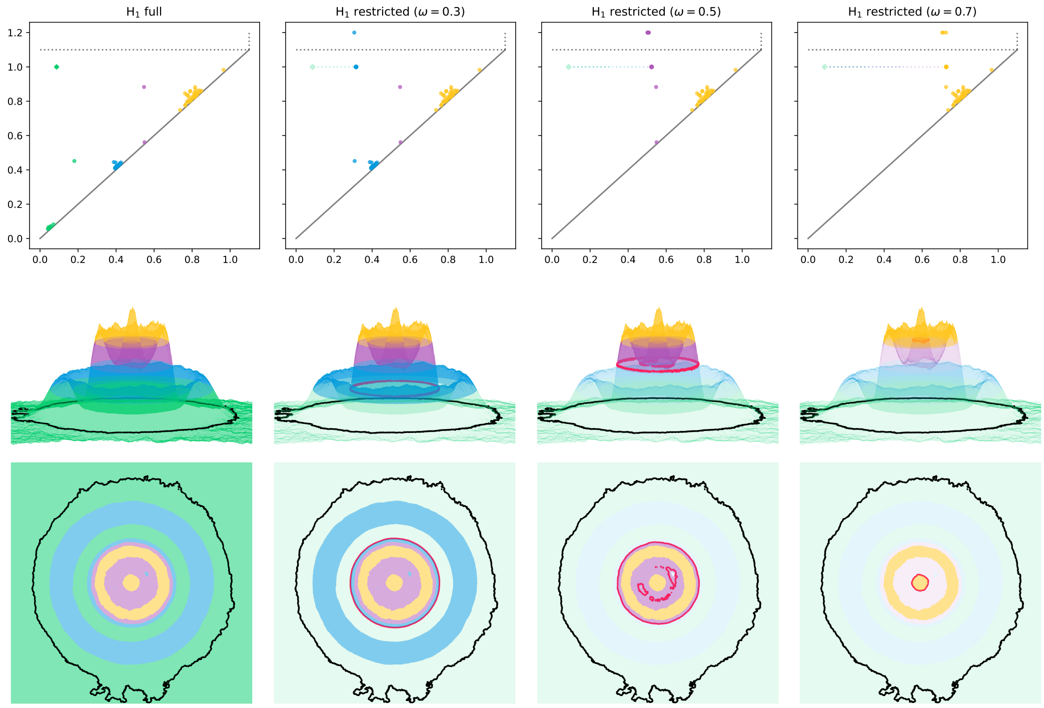


**Figure 5** (Top) $H_1$ persistence diagrams of the function depicted in Figure 3 restricted to *super*-levelsets at $\omega = 0.3, 0.5$, and $0.7$ (on a $1024 \times 1024$ grid). The matching is shown between a feature in the full diagram (marked with a diamond) with its representative cycle in black. The corresponding representative cycle in the restricted diagram is pictured in red.

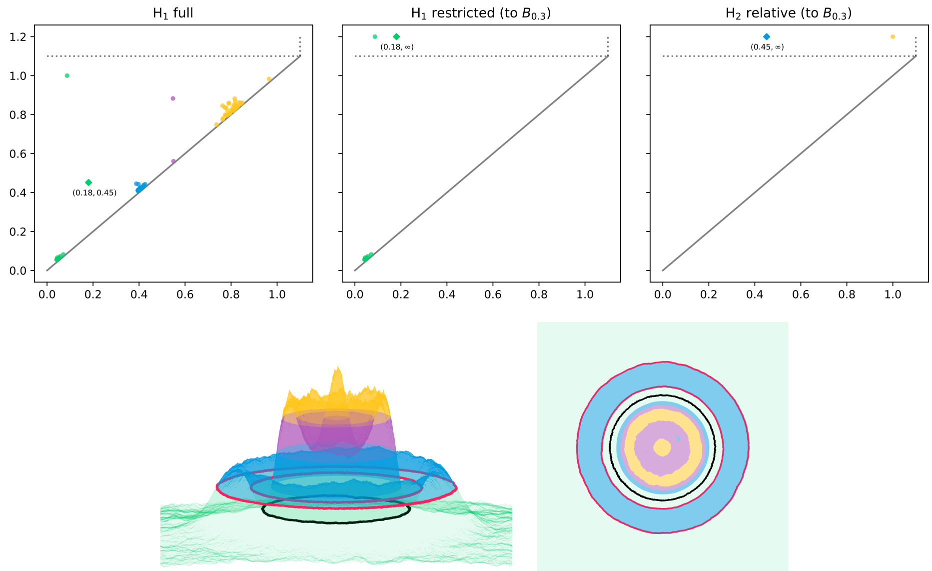


**Figure 6** (Top) The indicated infinite features in the restricted and relative diagrams correspond to the birth and death of the 1-feature $(0.18, 0.45)$ in the full diagram. (Bottom) In black, the representative cycle of the infinite 1-feature born at 0.18 in the restricted diagram is shown in black. In red, the *boundary* of the representative *relative* 2-cycle born at 0.45 in the relative diagram is shown in red.

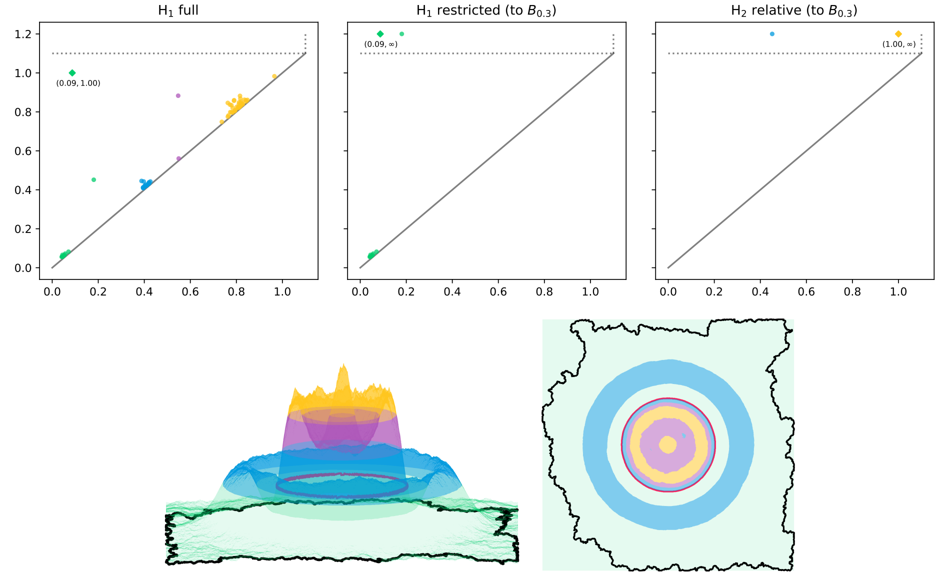


**Figure 7** The infinite 1-features of the restricted diagram can be matched with the infinite 2-features of the relative diagrams. The sequence birth times of relative 2-features in *decreasing* order correspond to the deaths of restricted 1-features in *increasing* order.

## 7 Conclusion

We have extended the Topological Coverage Criterion to the setting of Topological Scalar Field Analysis. By defining the boundary in terms of a sublevel set of a scalar field we provide an interpretation of the TCC that applies more naturally to data coverage. We then showed how the assumptions and machinery of the TCC can be used to approximate the persistent homology of the scalar field relative to a static sublevel set. This relative persistent homology is shown to be related to a truncation of that of the scalar field as whole, and therefore provides a way to approximate a part of its persistence diagram in the presence of un-verified data.

There are a number of unanswered questions and directions for future work. From the theoretical perspective, our understanding of duality limited us in providing a more elegant extension of the TCC. A better understanding of when and how duality can be applied would allow us to give a more rigorous statement of our assumptions. Moreover, as duality plays a central role in the TCC it is natural to investigate its role in the analysis of scalar fields. This would not only allow us to apply duality to persistent homology [8], but also allow us to provide a rigorous comparison between the relative approach and the persistent homology of the superlevel set filtration and explore connections with Extended Persistence [5].

From a computational perspective, we interested in exploring how to recover the full diagram as discussed in Section 6. Our statements in terms of sublevel sets can be generalized to disjoint unions of sub and superlevel sets, where coverage is confirmed in an *interlevel* set. This, along with a better understanding of the relationship between sub and superlevel sets could lead to an iterative approach in which the persistent homology of a scalar field is constructed as data becomes available. We are also interested in finding efficient ways to compute the image persistent (relative) homology that vary in both scalar and scale.

The problem of relaxing our assumptions on the boundary can be approached from both a theoretical and computational perspective. Ways to avoid the isomorphism we require could be investigated in theory, and the interaction of relative persistent homology and the Persistent Nerve Lemma may be used tighten our assumptions. We would also like to conduct a more rigorous investigation on the effect of these assumptions in practice.

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## A Omitted Proofs

**Proof of Lemma 2.** This proof is in two parts.

**$\ell$ injective $\implies D \setminus B \subseteq U$** Suppose, for the sake of contradiction, that $p$ is injective and there exists a point $x \in (D \setminus B) \setminus U$. Because $B$ surrounds $D$ in $X$ the pair $(D \setminus B, \overline{D})$ forms a separation of $\overline{B}$. Therefore, $\mathrm{H}_0(\overline{B}) \cong \mathrm{H}_0(D \setminus B) \oplus \mathrm{H}_0(\overline{D})$ so

$$\mathrm{H}_0(\overline{B}, \overline{D}) \cong \mathrm{H}_0(D \setminus B).$$

So $[x]$ is non-trivial in $\mathrm{H}_0(\overline{B}, \overline{D}) \cong \mathrm{H}_0(D \setminus B)$ as $x$ is in some connected component of $D \setminus B$. So we have the following sequence of maps induced by inclusions

$$\mathrm{H}_0(\overline{B}, \overline{D}) \xrightarrow{f} \mathrm{H}_0(\overline{B}, \overline{D} \cup \{x\}) \xrightarrow{g} \mathrm{H}_0(\overline{V}, \overline{U}).$$

As $f[x]$ is trivial in $\mathrm{H}_0(\overline{B}, \overline{D} \cup \{x\})$ we have that $\ell[x] = (g \circ f)[x]$ is trivial, contradicting our hypothesis that $\ell$ is injective.

**$\ell$ injective $\implies$ $V$ surrounds $U$ in $D$.** Suppose, for the sake of contradiction, that $V$ does not surround $U$ in $D$. Then there exists a path $\gamma : [0,1] \to \overline{V}$ with $\gamma(0) \in U \setminus V$ and $\gamma(1) \in D \setminus U$. As we have shown, $D \setminus B \subseteq U$, so $D \setminus B \subseteq U \setminus V$.

Choose $x \in D \setminus B$ and $z \in \overline{D}$ such that there exist paths $\xi : [0,1] \to U \setminus V$ with $\xi(0) = x$, $\xi(1) = \gamma(0)$ and $\zeta : [0,1] \to \overline{D} \cup (D \setminus U)$ with $\zeta(0) = z$, $\zeta(1) = \gamma(1)$. $\xi, \gamma$ and $\zeta$ all generate chains in $C_1(\overline{V}, \overline{U})$ and $\xi + \gamma + \zeta = \gamma^* \in C_1(\overline{V}, \overline{U})$ with $\partial\gamma^* = x + z$. Moreover, $z$ generates a chain in $C_0(\overline{U})$ as $\overline{D} \subseteq \overline{U}$. So $x = \partial\gamma^* + z$ is a relative boundary in $C_0(\overline{V}, \overline{U})$, thus $\ell[x] = \ell[z]$ in $\mathrm{H}_0(\overline{V}, \overline{L})$. However, because $B$ surrounds $D$, $[x] \neq [y]$ in $\mathrm{H}_0(\overline{B}, \overline{D})$ contradicting our assumption that $\ell$ is injective.

◀

**Proof of Lemma 4.** Assume there exist $p, q \in P \setminus Q_{\omega - c\zeta}$ such that $p$ and $q$ are connected in $\mathcal{R}^\delta(P \setminus Q_{\omega - c\zeta})$ but not in $D \setminus B_\omega$. So the shortest path from $p, q$ is a subset of $(P \setminus Q_{\omega - c\zeta})^\delta$. For any $x \in (P \setminus Q_{\omega - c\zeta})^\delta$ there exists some $p \in P$ such that $f(p) > \omega - c\zeta$ and $\mathbf{d}(p, x) < \delta$. Because $f$ is $c$-Lipschitz

$$f(x) \geq f(p) - c\mathbf{d}(x, p) > \omega - c(\delta + \zeta)$$

so there is a path from $p$ to $q$ in $D \setminus B_{\omega - c(\delta+\zeta)}$, thus $[p] = [q]$ in $\mathrm{H}_0(D \setminus B_{\omega - c(\delta+\zeta)})$.

But we have assumed that $[p] \neq [q]$ in $\mathrm{H}_0(D \setminus B_\omega)$, contradicting our assumption that $\mathrm{H}_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega - c(\delta+\zeta)})$ is injective, so any $p, q$ connected in $\mathcal{R}^\delta(P \setminus Q_{\omega - c\zeta})$ are connected in $D \setminus B_\omega$. That is, $\mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega - c\zeta})) \geq \mathbf{dim}\ \mathrm{H}_0(D \setminus B_\omega)$. ◀

## A.1 Extensions

**Proof of Lemma 8.** Note that $B' \setminus (D \setminus U) = B' \cap U \subseteq V$ implies $B' \subseteq V \sqcup (D \setminus U) = \mathcal{E}V$. Moreover, because $V \subseteq B$ and $D \setminus B \subseteq U$ implies $D \setminus U \subset D \setminus (D \setminus B) = B$, we have

$$\mathcal{E}V = V \sqcup (D \setminus U) \subseteq B \cup (D \setminus U) = B.$$

So $B' \subseteq \mathcal{E}V \subseteq B$ as desired. ◀

**Proof of Lemma 9.** Because $V$ surrounds $U$ in $D$, $(U \setminus V, D \setminus U)$ is a separation of $D \setminus V$, a subspace of $D$. So $\mathbf{cl}_D(U \setminus V) \setminus U = \mathbf{cl}_D(U \setminus V) \cap (D \setminus U) = \emptyset$ which implies $\mathbf{cl}_D(U \setminus V) \subseteq U = \mathbf{int}_D(U)$ as $U$ is open in $D$. Therefore,

$$\begin{aligned}
\mathbf{cl}_D(D \setminus U) &= D \setminus \mathbf{int}_D(U) \\
&\subseteq D \setminus \mathbf{cl}_D(U \setminus V) \\
&= \mathbf{int}_D(D \setminus (U \setminus V)) \\
&= \mathbf{int}_D(\mathcal{E}V).
\end{aligned}$$

so,

$$\begin{aligned}
\mathrm{H}_k(U \cap A, V) &= \mathrm{H}_k(A \setminus (D \setminus U), \mathcal{E}V \setminus (D \setminus U)) \\
&\cong \mathrm{H}_k(A, \mathcal{E}V)
\end{aligned}$$

for all $k$ and any $A \subseteq D$ such that $\mathcal{E}V \subset A$ by Excision. ◀

## A.2 Image Modules

▶ **Lemma 20.** *Suppose* $\Gamma \in \mathrm{Hom}(\mathbb{U}, \mathbb{V})$, $\Lambda \in \mathrm{Hom}(\mathbb{S}, \mathbb{T})$, *and* $\Lambda' \in \mathrm{Hom}(\mathbb{S}', \mathbb{T}')$. *If* $\Phi(F, G) \in \mathrm{Hom}^\delta(\mathbf{im}\ \Gamma, \mathbf{im}\ \Lambda)$ *and* $\Phi'(F', G') \in \mathrm{Hom}^{\delta'}(\mathbf{im}\ \Lambda, \mathbf{im}\ \Lambda')$ *then* $\Phi''(F' \circ F, G' \circ G) := \Phi' \circ \Phi \in \mathrm{Hom}^{\delta+\delta'}(\mathbf{im}\ \Gamma, \mathbf{im}\ \Lambda')$.

**Proof.** Because $\Phi(F, G)$ is an image module homomorphism of degree $\delta$ we have $g_{\beta-\delta} \circ \gamma_{\alpha-\delta}[\beta - \alpha] = \lambda_\alpha[\beta - \alpha] \circ f_{\alpha-\delta}$. Similarly, $g'_\beta \circ \lambda_\alpha[\beta - \alpha] = \lambda'_{\alpha+\delta'}[\beta - \alpha] \circ f'_\alpha$. So $\Phi''(F' \circ F, G' \circ G) \in \mathrm{Hom}^{\delta+\delta'}(\mathbf{im}\ \Gamma, \mathbf{im}\ \Lambda')$ as

$$g'_\beta \circ (g_{\beta-\delta} \circ \gamma_{\alpha-\delta}[\beta - \alpha]) = (g'_\beta \circ \lambda_\alpha[\beta - \alpha]) \circ f_{\alpha-\delta} = \lambda_{\alpha+\delta'}[\beta - \alpha] \circ f'_\alpha \circ f_{\alpha-\delta}$$

for all $\alpha \le \beta$. ◀

**Proof of Lemma 13.** For ease of notation let $\Phi$ denote $\Phi_M(F, G)$ and $\Psi$ denote $\Psi_G(M, N)$.

If $\Gamma$ is an epimorphism $\gamma_\alpha$ is surjective so $\Gamma_\alpha = V_\alpha$ and $\phi_\alpha = g_\alpha|_{\Gamma_\alpha} = g_\alpha$ for all $\alpha$. So $\mathbf{im}\ \Gamma = \mathbb{V}$ and $\Phi \in \mathrm{Hom}^\delta(\mathbb{V}, \mathbf{im}\ \Lambda)$.

If $\Pi$ is a monomorphism then $\pi_\alpha$ is injective so we can define a natural isomorphism $\pi_\alpha^{-1} : \Pi_\alpha \to V_\alpha$ for all $\alpha$. Let $\Psi^*$ be defined as the family of linear maps $\{\psi^*_\alpha := \pi_\alpha^{-1} \circ \psi_\alpha : \Lambda_\alpha \to V_{\alpha+\delta}\}$. Because $\Psi$ is a partial $\delta$-interleaving of image modules, $n_\alpha \circ \lambda_\alpha = \pi_{\alpha+\delta} \circ m_\alpha$. So, because $\psi_\alpha = n_\alpha|_{\Lambda_\alpha}$ for all $\alpha$,

$$\begin{aligned}
\mathbf{im}\ \psi^*_\alpha &= \mathbf{im}\ \pi^{-1}_{\alpha+\delta} \circ \psi_\alpha \\
&= \mathbf{im}\ \pi^{-1} \circ (n_\alpha \circ \lambda_\alpha) \\
&= \mathbf{im}\ \pi^{-1} \circ (\pi_{\alpha+\delta} \circ m_\alpha) \\
&= \mathbf{im}\ m_\alpha.
\end{aligned}$$

It follows that $\mathbf{im}\ v^{\beta+\delta}_{\alpha+\delta} \circ \psi^*_\alpha = \mathbf{im}\ v^{\beta+\delta}_{\alpha+\delta} \circ m_\alpha$

Similarly, because $\Psi$ is a $\delta$-interleaving of image modules $n_\beta \circ t^\beta_\alpha \circ \lambda_\alpha = w^{\beta+\delta}_{\alpha+\delta} \circ \pi_{\alpha+\delta} \circ m_\alpha$. Moreover, because $\Pi$ is a homomorphism of persistence modules, $w^{\beta+\delta}_{\alpha+\delta} \circ \pi_{\alpha+\delta} = \pi_{\beta+\delta} \circ v^{\beta+\delta}_{\alpha+\delta}$, so

$$n_\beta \circ t^\beta_\alpha \circ \lambda_\alpha = \pi_{\beta+\delta} \circ v^{\beta+\delta}_{\alpha+\delta} \circ m_\alpha.$$

As $\psi_\beta \circ \lambda^\beta_\alpha = n_\beta \circ \lambda^\beta_\alpha = n_\beta \circ t^\beta_\alpha|_{\Lambda_\alpha}$ it follows

$$\begin{aligned}
\mathbf{im}\ \psi^*_\beta \circ \lambda^\beta_\alpha &= \mathbf{im}\ \pi^{-1}_{\beta+\delta} \circ (n_\beta \circ t^\beta_\alpha \circ \lambda_\alpha) \\
&= \mathbf{im}\ \pi^{-1}_{\beta+\delta} \circ (\pi_{\beta+\delta} \circ v^{\beta+\delta}_{\alpha+\delta}) \circ m_\alpha \\
&= \mathbf{im}\ v^{\beta+\delta}_{\alpha+\delta} \circ m_\alpha \\
&= \mathbf{im}\ v^{\beta+\delta}_{\alpha+\delta} \circ \psi^*_\alpha.
\end{aligned}$$

So we may conclude that $\Psi^* \in \mathrm{Hom}^\delta(\mathbf{im}\ \Lambda, \mathbb{V})$.

So $\Phi \in \mathrm{Hom}^\delta(\mathbb{V}, \mathbf{im}\ \Lambda)$ and $\Psi^*_G \in \mathrm{Hom}^\delta(\mathbf{im}\ \Lambda, \mathbb{V})$. As we have shown, $\mathbf{im}\ \psi^*_{\alpha-\delta} = \mathbf{im}\ m_{\alpha-\delta}$ so $\mathbf{im}\ \phi_\alpha \circ \psi^*_{\alpha-\delta} = \mathbf{im}\ \phi_\alpha \circ m_{\alpha-\delta}$. Moreover, because $\gamma_\alpha$ is surjective $\phi_\alpha = g_\alpha$ and, because $\Phi$ is a partial $\delta$-interleaving of image modules, $g_\alpha \circ m_{\alpha-\delta} = t^{\alpha+\delta}_{\alpha-\delta} \circ \lambda_{\alpha-\delta}$. As $\lambda^{\alpha+\delta}_{\alpha-\delta} = t^{\alpha+\delta}_{\alpha-\delta}|_{\mathbf{im}\ \lambda_{\alpha-\delta}}$ it follows that $\mathbf{im}\ \phi_\alpha \circ \psi^*_{\alpha-\delta} = \mathbf{im}\ \lambda^{\alpha+\delta}_{\alpha-\delta}$.

Finally, $\psi^*_\alpha \circ \phi_\alpha = \pi^{-1}_{\alpha+\delta} \circ n_\alpha \circ g_{\alpha-\delta}$ where, because $\Psi$ is a partial $\delta$-interleaving of image modules, $n_\alpha \circ g_{\alpha-\delta} = w^{\alpha+\delta}_{\alpha-\delta} \circ \pi_{\alpha-\delta}$. Because $\Pi$ is a homomorphism of persistence modules

$w_{\alpha-\delta}^{\alpha+\delta} \circ \pi_{\alpha-\delta} = \pi_{\alpha+\delta} \circ v_{\alpha-\delta}^{\alpha+\delta}$. Therefore,

$$\begin{aligned}\psi_\alpha^* \circ \phi_\alpha &= \pi_{\alpha+\delta}^{-1} \circ n_\alpha \circ g_{\alpha-\delta} \\ &= \pi_{\alpha+\delta}^{-1} \circ (\pi_{\alpha+\delta} \circ v_{\alpha-\delta}^{\alpha+\delta}) \\ &= v_{\alpha-\delta}^{\alpha+\delta}\end{aligned}$$

which, along with $\phi_\alpha \circ \mathbf{im}\ \psi_{\alpha-\delta}^* = \lambda_{\alpha-\delta}^{\alpha+\delta}$ implies Diagrams **??** and **??** commute with $\Phi \in \mathrm{Hom}^\delta(\mathbb{V}, \mathbf{im}\ \Lambda)$ and $\Psi^* \in \mathrm{Hom}^\delta(\mathbf{im}\ \Lambda, \mathbb{V})$. We may therefore conclude that $\mathbf{im}\ \Lambda$ and $\mathbb{V}$ are $\delta$-interleaved. ◀

## A.3 Partial Interleavings

**Proof of Lemma 14.** Suppose $x \in P^\delta \cap D_{\lfloor \alpha - c\varepsilon \rfloor_{t-c\varepsilon}}$. Because $x$ in $P^\delta$ there exists some $p \in P$ such that $\mathbf{d}(x,p) < \delta$. Because $f$ is $c$-Lipschitz $f(p) \le f(x) + c\mathbf{d}(x,p) < f(x) + c\delta$. If $\alpha \le t$ then $x \in B_{t-c\varepsilon}$ implies $f(p) < t - c\varepsilon + c\delta \le t$ so $x \in Q_t^\varepsilon$ as $\delta \le \varepsilon$ If $\alpha \ge t$ then $x \in B_{\alpha - c\varepsilon}$ which implies $f(p) \le \alpha$ $x \in Q_\alpha^\varepsilon$. So $P^\delta \cap D_{\lfloor \alpha - c\varepsilon \rfloor_{t-c\varepsilon}} \subseteq P_{\lfloor \alpha \rfloor_t}^\varepsilon$ as $P_{\lfloor \alpha \rfloor_t} = Q_t^\varepsilon \cup Q_\alpha^\varepsilon$.

Now, suppose $x \in P_{\lfloor \alpha \rfloor_t}^\varepsilon$. If $\alpha \le t$ then $x \in Q_t^\varepsilon \subseteq B_{t+c\varepsilon}$ because $f$ is $c$-Lipschitz. Similarly, $\alpha > t$ implies $x \in Q_\alpha^\varepsilon \subseteq B_{\alpha + c\varepsilon}$, so $P_{\lfloor \alpha \rfloor_t}^\varepsilon \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor_{t+c\varepsilon}}$ as $D_{\lfloor \alpha + c\varepsilon \rfloor_{t+c\varepsilon}} = B_{t+c\varepsilon} \cup B_{\alpha + c\varepsilon}$. ◀

**Proof of Lemma 15.** Because $Q_t^\delta$ surrounds $P^\delta$ in $D$ and $\delta \le \varepsilon$, $t < v$ we know $Q_t^\varepsilon$ and $Q_v^\varepsilon$ surround $P^\delta$ in $D$. As $P^\delta \cap B_s \subseteq Q_t^\varepsilon$ and $P^\delta \cap B_u \subseteq Q_v^{2\varepsilon}$ for all $\varepsilon \in [\delta, 2\delta]$ Lemma 8 implies that we have a sequence of inclusions $B_s \subseteq \mathcal{E}Q_t^\varepsilon \subseteq B_u \subseteq \mathcal{E}Q_v^{2\varepsilon} \subseteq B_w$.

For any $\alpha \in \mathbb{R}$ we know that $D \setminus P^\delta \subseteq \mathcal{E}P_{\lfloor \alpha \rfloor_t}^\varepsilon$ by the definition of $\mathcal{E}P_{\lfloor \alpha \rfloor_t}^\varepsilon$. Moreover, $D \setminus P^\delta \subseteq D_{\lfloor \alpha \rfloor_u}$ because $D \setminus B_u \subseteq P^\delta$. Lemma 14 therefore implies $D_{\lfloor \alpha - c\delta \rfloor_s} \subseteq \mathcal{E}P_{\lfloor \alpha \rfloor_t}^\varepsilon \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor_u}$ as $s + c\delta \le t \le u - c\varepsilon$. So the inclusions $(D_{\lfloor \alpha - c\delta \rfloor_s}, B_s) \subseteq (\mathcal{E}P_{\lfloor \alpha \rfloor_t}^\varepsilon, \mathcal{E}Q_t^\varepsilon)$ induce $F \in \mathrm{Hom}^{c\delta}(\mathbb{D}_s, \mathcal{E}\mathbb{P}_t^\varepsilon)$ and $(\mathcal{E}P_{\lfloor \alpha \rfloor_t}^\varepsilon, \mathcal{E}Q_t^\varepsilon) \subseteq (D_{\lfloor \alpha + c\varepsilon \rfloor_u}, B_u)$ induce $M \in \mathrm{Hom}^{c\varepsilon}(\mathcal{E}\mathbb{P}_t^\varepsilon, \mathbb{D}_u)$.

By an identical argument Lemma 14 implies $D_{\lfloor \alpha - 2c\delta \rfloor_u} \subseteq \mathcal{E}P_{\lfloor \alpha \rfloor_v}^\varepsilon \subseteq D_{\lfloor \alpha + 2c\varepsilon \rfloor_w}$ as $u + c\delta \le v \le w - 4c\delta$. So $(D_{\lfloor \alpha - 2c\delta \rfloor_u}, B_u) \subseteq (\mathcal{E}P_{\lfloor \alpha \rfloor_v}^\varepsilon, \mathcal{E}Q_v^{2\varepsilon})$ induce $G \in \mathrm{Hom}^{2c\delta}(\mathbb{D}_u, \mathcal{E}\mathbb{P}_v^{2\varepsilon})$ and $(\mathcal{E}P_{\lfloor \alpha \rfloor_v}^\varepsilon, \mathcal{E}Q_v^{2\varepsilon}) \subseteq (D_{\lfloor \alpha + 2c\varepsilon \rfloor_w}, B_w)$ induce $N \in \mathrm{Hom}^{2c\varepsilon}(\mathcal{E}\mathbb{P}_v^{2\varepsilon}, \mathbb{D}_u)$. ◀

## A.4 Truncated Interval Modules

**Proof of Lemma 18.** Suppose $\alpha \le \omega$. So $\mathrm{H}_k(D_{\lfloor \alpha \rfloor_\omega}, B_\omega) = 0$ as $D_{\lfloor \alpha \rfloor_\omega} = B_\omega \cup B_\alpha$ and $\mathbb{T}_\omega^k = 0$ as $F_\alpha^I = 0$ for any $I \in \mathcal{I}^k$ such that $\omega \in I_-$. Moreover, $\omega \in I$ for all $I \in \mathcal{I}_\omega^{k-1}$, thus $F_\alpha^{I_+} = 0$ for all $\alpha \le \omega$. So it suffices to assume $\omega < \alpha$.

Consider the long exact sequence of the pair $\mathrm{H}_k(D_{\lfloor \alpha \rfloor_\omega}, B_\omega) = \mathrm{H}_k(B_\alpha, B_\omega)$

$$\ldots \to \mathrm{H}_k(B_\omega) \xrightarrow{p_\alpha^k} \mathrm{H}_k(B_\alpha) \xrightarrow{q_\alpha^k} \mathrm{H}_k(D_{\lfloor \alpha \rfloor_\omega}, B_\omega) \xrightarrow{r_\alpha^k} \mathrm{H}_{k-1}(B_\omega) \xrightarrow{p_\alpha^{k-1}} \mathrm{H}_{k-1}(B_\alpha) \to \ldots$$

where $\mathrm{H}_k(B_\alpha) = \bigoplus_{I \in \mathcal{I}^k} F_\alpha^I$, $\mathrm{H}_k(B_\omega) = \bigoplus_{I \in \mathcal{I}^k} F_\omega^I$, and $p_\alpha^k = \bigoplus_{I \in \mathcal{I}^k} f_{\omega,\alpha}^I$.

Noting that $\mathbf{im}\ q_\alpha^k \cong \mathrm{H}_k(B_\alpha)/\mathbf{ker}\ q_\alpha^k$ where $\mathbf{ker}\ q_\alpha^k = \mathbf{im}\ p_\alpha^k$ by exactness we have $\mathbf{ker}\ r_\alpha^k \cong \mathrm{H}_k(B_\alpha)/\mathbf{im}\ p_\alpha^k$. By the definition of $F_\alpha^I$ and $f_{\omega,\alpha}^I$ we know $\mathbf{im}\ f_{\omega,\alpha}^I$ is $F_\alpha^I$ if $\omega \in I$ and 0 otherwise. As $\mathbf{im}\ p_\alpha^k$ is equal to the direct sum of images $\mathbf{im}\ f_{\omega,\alpha}^I$ over $I \in \mathcal{I}^k$ it follows that $\mathbf{im}\ p_\alpha^k$ is the direct sum of those $F_\alpha^I$ over those $I \in \mathcal{I}^k$ such that $\omega \in I$. Now, because $\mathrm{H}_k(B_\alpha) = \bigoplus_{I \in \mathcal{I}^k} F_\alpha^I$ and each $F_\alpha^I$ is either 0 or $\mathbb{F}$ the quotient $\mathrm{H}_k(B_\alpha)/\mathbf{im}\ p_\alpha^k$ is the direct sum of those $F_\alpha^I$ such that $\omega \notin I$. Therefore, by the definition of $F_{\lfloor \alpha \rfloor_\omega}^I$ we have

$$\mathbf{ker}\ r_\alpha^k = \bigoplus_{I \in \mathcal{I}_\omega^k} F_{\lfloor \alpha \rfloor_\omega}^I.$$

Similarly, $\mathbf{im}\ r^k_\alpha = \mathbf{ker}\ p^{k-1}_\alpha$ by exactness where $\mathbf{ker}\ p^{k-1}_\alpha$ is the direct sum of kernels $\mathbf{ker}\ f^I_{\omega,\alpha}$ over $I \in \mathcal{I}^{k-1}$. By the definition of $F^I_\alpha$ and $f^I_{\omega,\alpha}$ we know that $\mathbf{ker}\ f^I_{\omega,\alpha}$ is $F^I_\alpha$ if $\omega \notin I$ and 0 otherwise. Noting that $\mathbf{ker}\ f^I_{\omega,\alpha} = 0$ for any $I \in \mathcal{I}^{k-1}$ such that $\omega \notin I$ it suffices to consider only those $I \in \mathcal{I}^{k-1}_\omega$. It follows that $\mathbf{ker}\ f^I_{\omega,\alpha} = F^{I+}_\alpha$ for any $I$ containing $\omega$ as $\omega < \alpha$. Therefore,

$$\mathbf{im}\ r^k_\alpha = \bigoplus_{I \in \mathcal{I}^{k-1}} F^{I+}_\alpha.$$

We have the following split exact sequence associated with $r^k_\alpha$

$$0 \to \mathbf{ker}\ r^k_\alpha \to \mathrm{H}_k(D_{\lfloor\alpha\rfloor_\omega}, B_\omega) \to \mathbf{im}\ r^k_\alpha \to 0.$$

The desired result follows from the fact that for all $\alpha \in \mathbb{R}$

$$\begin{aligned}\mathrm{H}_k(D_{\lfloor\alpha\rfloor_\omega}, B_\omega) &\cong \mathbf{ker}\ r^k_\alpha \oplus \mathbf{im}\ r^k_\alpha \\ &= \bigoplus_{I \in \mathcal{I}^k} F^I_{\lfloor\alpha\rfloor_\omega} \oplus \bigoplus_{I \in \mathcal{I}^{k-1}_\omega} F^{I+}_\alpha.\end{aligned}$$

◀

# B Duality

For a pair $(A, B)$ in a topological space $X$ and any $R$ module $G$ let $\mathrm{H}^k(A, B; G)$ denote the **singular cohomology** of $(A, B)$ (with coefficients in $G$). Let $\mathrm{H}^k_c(A, B; G)$ denote the corresponding **singular cohomology with compact support**, where $\mathrm{H}^k_c(A, B; G) \cong \mathrm{H}^k(A, B; G)$ for any compact pair $(A, B)$.

The following corollary follows from the Universal Coefficient Theorem for singular homology (and cohomology) as vector spaces over a field $\mathbb{F}$, as the dual vector space $\mathrm{Hom}(\mathrm{H}_k(A, B), \mathbb{F})$ is isomorphic to $\mathrm{H}_k(A, B; \mathbb{F})$ for any finitely generated $\mathrm{H}_k(A, B)$.

▶ **Corollary 21.** *For a topological pair $(A, B)$ and a field $\mathbb{F}$ such that $\mathrm{H}_0(A, B)$ is finitely generated there is a natural isomorphism*

$$\nu : \mathrm{H}^0(A, B; \mathbb{F}) \to \mathrm{H}_0(A, B; \mathbb{F}).$$

Let $\overline{\mathrm{H}}^k(A, B; G)$ be the **Alexander-Spanier cohomology** of the pair $(A, B)$, defined as the limit of the direct system of neighborhoods $(U, V)$ of the pair $(A, B)$. Let $\overline{\mathrm{H}}^k_c(A, B; G)$ denote the corresponding **Alexander-Spanier cohomology with compact support** where $\overline{\mathrm{H}}^k_c(A, B; G) \cong \overline{\mathrm{H}}^k(A, B; G)$ for any compact pair $(A, B)$.

▶ **Theorem 22** (**Alexander-Poincaré-Lefschetz Duality** (Spanier [11], Theorem 6.2.17))**.** *Let $X$ be an orientable $d$-manifold and $(A, B)$ be a compact pair in $X$. Then for all $k$ and $R$ modules $G$ there is a (natural) isomorphism*

$$\lambda : \mathrm{H}_k(X \setminus B, X \setminus A; G) \to \overline{\mathrm{H}}^{d-k}(A, B; G).$$

A space $X$ is said to be **homologically locally connected in dimension** $n$ if for every $x \in X$ and neighborhood $U$ of $x$ there exists a neighborhood $V$ of $x$ in $U$ such that $\tilde{\mathrm{H}}_n(V) \to \tilde{\mathrm{H}}_n(U)$ is trivial for $k \leq n$.

▶ **Lemma 23** (Spanier p. 341, Corollary 6.9.6)**.** *Let $A$ be a closed subset, homologically locally connected in dimension $n$, of a Hausdorff space $X$, homologically locally connected in dimension $n$. If $X$ has the property that every open subset is paracompact, $\mu : \overline{\mathrm{H}}_c^k(X, A; G) \to \mathrm{H}_c^k(X, A; G)$ is an isomorphism for $k \le n$ and a monomorphism for $q = n + 1$.*

In the following we will assume homology (and cohomology) over a field $\mathbb{F}$.

▶ **Lemma 24.** *Let $X$ be an orientable $d$-manifold and $(A, B)$ a compact pair of locally path connected subspaces in $X$. Then*

$$\xi : \mathrm{H}_d(X \setminus B, X \setminus A) \to \mathrm{H}_0(A, B)$$

*is a natural isomorphism.*

**Proof.** Because $X$ is orientable and $(A, B)$ are compact $\lambda : \mathrm{H}_d(X \setminus B, X \setminus A) \to \overline{\mathrm{H}}^0(A, B)$ is an isomorphism by Theorem 22. Note that Moreover, because every subset of $X$ is (hereditarily) paracompact every open set in $A$, with the subspace topology, is paracompact. For any neighborhood $U$ of a point $x$ in a locally path connected space there must exist some neighborhood $V \subset U$ of $x$ that is path connected in the subspace topology. As $\tilde{\mathrm{H}}_0(V) = 0$ for any nonempty, path connected topological space $V$ (see Spanier p. 175, Lemma 4.4.7) it follows that $A$ (resp. $B$) are homologically locally connected in dimension 0. Because $(A, B)$ is a compact pair the singular and Alexander-spanier cohomology modules of $(A, B)$ with compact support are isomorphic to those without, thus $\mu : \overline{\mathrm{H}}^0(A, B) \to \mathrm{H}^0(A, B)$ is an isomorphism. By Corollary 21 we have a natural isomorphism $\nu : \mathrm{H}^0(A, B) \to \mathrm{H}_0(A, B)$ thus the composition $\xi := \nu \circ \mu \circ \lambda : \mathrm{H}_d(X \setminus B, X \setminus A) \to \mathrm{H}_0(A, B)$ is a natural isomorphism. ◀

▶ **Lemma 25.** *Let $\mathbb{X}$ be an orientable $d$-manifold let $D$ be a compact subset of $\mathbb{X}$. Let $P$ be a finite subset of $D$ such that $P^\varepsilon \subset \mathbf{int}_{\mathbb{X}}(D)$ and $Q \subseteq P$.*

*If $D \setminus Q^\varepsilon$ and $D \setminus P^\varepsilon$ are locally path connected then there is a natural isomorphism*

$$\xi : \mathrm{H}_d(P^\varepsilon, Q^\varepsilon) \to \mathrm{H}_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon).$$

**Proof.** Because $Q^\varepsilon$ and $P^\varepsilon$ are open in $D$ and $D$ is compact in $\mathbb{X}$ the complement $D \setminus Q^\varepsilon$ is closed in $D$, and therefore compact in $\mathbb{X}$. Moreover, because $P^\varepsilon \subset \mathbf{int}_{\mathbb{X}}(D)$, $\mathrm{H}_d(\mathbb{X} \setminus (D \setminus P^\varepsilon), \mathbb{X} \setminus (D \setminus Q^\varepsilon)) = \mathrm{H}_d(P^\varepsilon, Q^\varepsilon)$. As we have assumed these complements are locally path connected by assumption we have a natural isomorphism $\xi : \mathrm{H}_d(P^\varepsilon, Q^\varepsilon) \to \mathrm{H}_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon)$ by Lemma 24. ◀