

From Coverage Testing to Topological Scalar Field Analysis

Kirk P. Gardner 

North Carolina State University, United States
kpgardn2@ncsu.edu

Donald R. Sheehy 

North Carolina State University, United States
don.r.sheehy@gmail.com

1 Abstract

The topological coverage criterion (TCC) can be used to test whether an underlying space is sufficiently well covered by a given data set. Given a sufficiently dense sample, topological scalar field analysis (SFA) can give a summary of the shape of a real-valued function on a space. The goal of this paper is to put these theories together so that one can test coverage with the TCC and then compute a summary with SFA. The challenge is that the TCC requires a well-defined boundary that is not generally available in the setting of SFA. To overcome this, we show how the scalar field itself can be used to define a boundary that can then be used to confirm coverage. This requires an interpretation of the TCC that resolves one of the major barriers to wider use. It also extends SFA methods to the setting in which coverage is only confirmed in a subset of the domain.

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11 1 Introduction

In the topological analysis of scalar fields (SFA), one computes a topological summary capturing qualitative and quantitative shape information from a set of points endowed with a metric and a real-valued function. That is, we have points with distances and a real number assigned to each point. More generally, it suffices to have a neighborhood graph on the points identifying the pairs of points within some distance. The topological computation uses persistent homology to integrate local information from the function into global information about its *behavior* as whole. In prior work, Chazal et al. [3] showed that for sufficiently dense samples on sufficiently smooth spaces, the persistence diagram can be computed with some guarantees. In followup work, Buchet et al. [1] extended this result to show how to work with noisy inputs. A fundamental assumption required to have strong guarantees on the output of these methods is that the underlying space be sufficiently well-sampled. In this paper, we show how to combine scalar field analysis with the theory of topological coverage testing to simultaneously compute the persistence diagram while testing that the underlying space is sufficiently well-sampled.

Initiated by De Silva and Ghrist [10, 6, 7], the theory of homological sensor networks addresses the problem of testing coverage of a bounded domain by a collection of sensors without coordinates. The main result is the topological coverage criterion, which, in its most general form, states that under reasonable geometric assumptions, the d -dimensional homology of a pair of simplicial complexes built on the neighborhood graph will be nontrivial if and only if there is sufficient coverage (see Section 3 for the precise statements). This relative persistent homology test is called the Topological Coverage Criterion (TCC).

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33 Superficially, the methods of SFA and TCC are very similar. Both construct similar
34 complexes and compute the persistent homology of the homological image of a complex on
35 one scale into that of a larger scale. They even overlap on some common techniques in their
36 analysis including the use of the Nerve theorem and the Rips-Čech interleaving. However,
37 they differ in some fundamental way that makes it difficult to combine them into a single
38 technique. The main difference is that the TCC requires a clearly defined boundary. Not
39 only must the underlying space be a bounded subset of \mathbb{R}^d , the data must also be labeled to
40 indicate which input points are close to the boundary. This requirement is perhaps the main
41 reason why the TCC can so rarely be applied in practice.

42 In applications to data analysis it is more natural to assume that our data measures
43 some unknown function. By requiring that our function is related to the metric of the space
44 we can replace this requirement with assumptions about the function itself. Indeed, these
45 assumptions could relate the behavior of the function to the topological boundary of the
46 space. However, the generalized approach by Cavanna et al. [2] allows much more freedom
47 in how the boundary is defined.

48 We consider the case in which we have incomplete data from a particular sublevel set
49 of our function. Our goal is to isolate this data so we can analyze the function in only the
50 verified region. From this perspective, the TCC confirms that we not only have coverage,
51 but that the sample we have is topologically representative of the region near, and above
52 this sublevel set. We can then re-use the same machinery to analyze a *part* of the function
53 in a specific way.

54 Contribution

55 We will re-cast the TCC as a way to verify that the persistent homology of a scalar field
56 can be *partially* approximated by a given sample. Specifically, we will relate the persistent
57 homology of a function relative to a *static* sublevel set to a *truncation* of the full diagram.
58 That is, beyond a certain point the full diagram remains unchanged, allowing for possible
59 reconstruction. This is in comparison with the *restricted* diagram obtained by simply ignoring
60 part of the domain. We therefore present relative persistent homology as an alternative to
61 restriction in a way that extends the TCC to the analysis of scalar fields.

62 Section 2 establishes notation and provides an overview of our main results in Sections 3
63 and 4. In Section 5 we introduce an interpretation of the relative diagram as a truncation of
64 the full diagram that is motivated by a number of experiments in Section 6.

65 2 Summary

66 Let \mathbb{X} denote an orientable d -manifold and $D \subset \mathbb{X}$ a compact subspace. For a c -Lipschitz
67 function $f : D \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$ let $B_\alpha := f^{-1}((-\infty, \alpha])$ denote the α -sublevel set of f . Our
68 sample will be denoted P , and the subset of points sampling B_α will be denoted $Q_\alpha := P \cap B_\alpha$.
69 For ease of exposition let

$$70 D_{\lfloor \alpha \rfloor_w} := B_\alpha \cup B_w$$

71 denote the *truncated* α sublevel set and

$$72 P_{\lfloor \alpha \rfloor_w} := Q_\alpha \cup Q_w$$

73 denote its sampled counterpart for all $\alpha, w \in \mathbb{R}$.

74 We will select a sublevel set B_ω to serve as our boundary. Specifically, we require that
75 B_ω surrounds D , where the notion of a surrounding set is defined formally in Section 3. This

⁷⁶ distinction allows us to generalize the standard proof of the TCC to properties of surrounding
⁷⁷ pairs.

⁷⁸ **Results**

⁷⁹ Suppose B_ω surrounds D in \mathbb{X} and $\delta < \varrho/4$. As a minimal assumption we require that every
⁸⁰ component of $D \setminus B_\omega$ contains a point in P . We also make additional technical assumptions
⁸¹ on P and δ with respect to the pair (D, B_ω) (see Section 3 and Lemma 27 of the Appendix).

⁸² **Theorem 6** If

- ⁸³ I. $H_0(D \setminus B_{\omega+5c\delta} \hookrightarrow D \setminus B_\omega)$ is *surjective*,
- ⁸⁴ II. $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-3c\delta})$ is *injective*,

⁸⁵ and

⁸⁶ $\text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$

⁸⁹ then $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D . ¹

⁹⁰ This formulation of the TCC states that our approximation by a nested pair of Rips
⁹¹ complexes captures the homology of the pair (D, B_ω) in a specific way. We use this fact
⁹² to interleave our sample with the relative diagram of the filtration $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$. This
⁹³ is done by generalizing our regularity assumptions near $D \setminus B_\omega$ in a way that allows us to
⁹⁴ interleave persistence modules relative to static sublevels.

⁹⁵ **Theorem 17** Suppose $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D . If

- ⁹⁶ I. $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is *surjective* and
- ⁹⁷ II. $H_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an *isomorphism*

⁹⁸ for all k then the persistent homology modules of

⁹⁹ $\{\mathcal{R}^{2\delta}(P_{\lfloor \alpha \rfloor \omega-2c\delta}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor \alpha \rfloor \omega+c\delta}, Q_{\omega+c\delta})\}_{\alpha \in \mathbb{R}}$

¹⁰⁰ and $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ are $4c\delta$ interleaved.

¹⁰¹ The main challenges we face come from the fact that the sublevel set B_ω and our
¹⁰² approximation by the inclusion $\mathcal{R}^{2\delta}(Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(Q_{\omega+c\delta})$ remain *static* throughout.
¹⁰³ Using the fact that $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D we define an *extension* $(D, \mathcal{E}Q_{\omega-2c\delta}^\delta)$ of the
¹⁰⁴ pair $(P^\delta, Q_{\omega-2c\delta}^\delta)$ that has isomorphic relative homology by excision. These extensions give
¹⁰⁵ us a sequence of inclusion maps

¹⁰⁶ $B_{\omega-3c\delta} \hookrightarrow \mathcal{E}Q_{\omega-2c\delta}^{2\delta} \hookrightarrow B_\omega \hookrightarrow \mathcal{E}Q_{\omega+c\delta}^{4\delta} \hookrightarrow B_{\omega+5c\delta}$

¹⁰⁷ that can be used along with our regularity assumptions to prove the interleaving.

⁸⁷ ⁸⁸ We state this result using constants that will be used to prove the interleaving. The statement of Theorem 6 parameterizes the region around ω in terms of $\zeta \geq \delta$ as $[\omega - c(\delta + \zeta), \omega + c(\delta + \zeta)]$.

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108 Relative, Truncated, and Restricted Persistence Diagrams

109 For fixed $\omega \in \mathbb{R}$ we will refer to the persistence diagram associated with the filtration
110 $\{(D_{[\alpha]\omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ as the **relative diagram** of f . In Section 5 we relate the relative diagram
111 to the *full* diagram of the sublevel set filtration $\{B_\alpha\}_{\alpha \in \mathbb{R}}$. Specifically, we define the
112 **truncated diagram** to be the subdiagram consisting of features born *after* ω in the full.
113 In Section 6 we compare the relative and truncated diagrams to the **restricted diagram**,
114 defined to be that of the sublevel set filtration of $f|_{D \setminus B_\omega}$.

115 Note that the truncated sublevel sets $D_{[\alpha]\omega}$ are equal to the union of B_ω and the restricted
116 sublevel sets. It is in this sense that B_ω is *static* throughout—it is contained in every sublevel
117 set of the relative filtration. As we will not have verified coverage in B_ω we cannot analyze
118 the function in this region directly. We therefore have two alternatives: *restrict* the domain
119 of the function to $D \setminus B_\omega$, or use relative homology to analyze the function *relative* to this
120 region using excision.

121 Outline of Sections 3 and 4

122 We will begin with our reformulation of the TCC in Section 3. This requires the introduction
123 of a surrounding set before proving the TCC (Theorem 6). Section 4 formally introduces
124 extensions and partial interleavings of image modules which will be used in the proof of
125 Theorem 17.

126 3 The Topological Coverage Criterion (TCC)

127 A positive result from the TCC requires that we have a subset of our cover to serve as the
128 boundary. That is, the condition not only checks that we have coverage, but also that
129 we have a pair of spaces that reflects the pair (D, B) topologically. We call such a pair a
130 *surrounding pair* defined in terms of separating sets. It has been shown that the TCC can
131 be stated in terms of these surrounding pairs [2]. Moreover, this work made assumptions
132 directly in terms of the *zero dimensional* persistent homology of the domain close to the
133 boundary. This allows us enough flexibility to define our surrounding set as a sublevel
134 of a c -Lipschitz function f and state our assumptions in terms of its persistent homology.

135 ▶ **Definition 1** (Surrounding Pair). *Let X be a topological space and (D, B) a pair in X . The
136 set B surrounds D in X if B separates X with the pair $(D \setminus B, X \setminus D)$. We will refer to
137 such a pair as a **surrounding pair in X** .*

138 The following lemma generalizes the proof of the TCC as a property of surrounding sets.

139 ▶ **Lemma 2.** *Let (D, B) be a surrounding pair in X and $U \subseteq D, V \subseteq U \cap B$ be subsets. Let
140 $\ell : H_0(X \setminus B, X \setminus D) \rightarrow H_0(X \setminus V, X \setminus U)$ be induced by inclusion.*

141 *If ℓ is injective then $D \setminus B \subseteq U$ and V surrounds U in D .*

143 We now combine these results on the homology of surrounding pairs with information
144 about both \mathbb{X} as a metric space and our function. Let (\mathbb{X}, \mathbf{d}) be a metric space and $D \subseteq \mathbb{X}$
145 be a compact subspace. For a c -Lipschitz function $f : D \rightarrow \mathbb{R}$ we introduce a constant ω as
146 a threshold that defines our “boundary” as a sublevel set B_ω of the function f . Let P be
147 a finite subset of D and $\zeta \geq \delta > 0$ and be constants such that $P^\delta \subseteq \text{int}_{\mathbb{X}}(D)$. Here, δ will
148 serve as our communication radius where ζ is reserved for use in Section 4. ²

142 ² We will set $\zeta = 2\delta$ in the proof of our interleaving with Rips complexes but the TCC holds for all $\zeta \geq \delta$.

¹⁴⁹ ► **Lemma 3.** Let $i : H_0(\overline{Q_{\omega+c\delta}^\delta}, \overline{P^\delta}) \rightarrow H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$.
¹⁵⁰ If B_ω surrounds D in \mathbb{X} then $\dim H_0(\overline{B_\omega}, \overline{D}) \geq \text{rk } i$.

¹⁵¹ **Proof.** Choose a basis for $\text{im } i$ such that each basis element is represented by a point in $P^\delta \setminus Q_{\omega+c\delta}^\delta$. Let $x \in P^\delta \setminus Q_{\omega+c\delta}^\delta$ be such that $i[x] \neq 0$. So there exists some $p \in P$ such that $d(p, x) < \delta$ and $p \notin Q_{\omega+c\delta}^\delta$, otherwise $x \in Q_{\omega+c\delta}^\delta$. Therefore, because f is c -Lipschitz,

$$\text{154} \quad f(x) \geq f(p) - cd(x, p) > \omega + c\delta - c\delta = \omega.$$

¹⁵⁵ So $x \in \overline{B_\omega}$ and, because $x \in P^\delta \subseteq D$, $x \in D \setminus B_\omega$. Because i and $\ell : H_0(\overline{B_\omega}, \overline{D}) \rightarrow$
¹⁵⁶ $H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$ are induced by inclusion $\ell[x] = i[x] \neq 0$ in $H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$. That is, every
¹⁵⁷ element of $\text{im } i$ has a preimage in $H_0(\overline{B_\omega}, \overline{D})$, so we may conclude that $\dim H_0(\overline{B_\omega}, \overline{D}) \geq$
¹⁵⁸ $\text{rk } i$. ◀

¹⁵⁹ Note that, while there is a surjective map from $H_0(\overline{B_\omega}, \overline{D})$ to $\text{im } i$ this map is not
¹⁶⁰ necessarily induced by inclusion. We therefore must introduce a larger space $B_{\omega+c(\delta+\zeta)}$
¹⁶¹ that contains $Q_{\omega+c\delta}^\delta$ in order to provide a criteria for the injectivity of $\ell : H_0(\overline{B_\omega}, \overline{D}) \rightarrow$
¹⁶² $H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$ in terms of $\text{rk } i$. We have the following commutative diagrams of inclusion
¹⁶³ maps the induced maps between complements in \mathbb{X} .

$$\begin{array}{ccc} (P^\delta, Q_{\omega-c\zeta}^\delta) & \xhookrightarrow{\quad} & H_0(\overline{B_{\omega+c(\delta+\zeta)}}, \overline{D}) \xrightarrow{j} H_0(\overline{B_\omega}, \overline{D}) \\ \downarrow & & \downarrow m & \downarrow \ell \\ (D, B_\omega) & \xhookrightarrow{\quad} & H_0(\overline{Q_{\omega+c\delta}^\delta}, \overline{P^\delta}) \xrightarrow{i} H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta}). \end{array} \quad (1)$$

¹⁶⁵ Assumptions

¹⁶⁶ We will first require the map $H_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ to be *surjective*—as we approach
¹⁶⁷ ω from *above* no components *appear*. This ensures that the rank of the map j is equal to the
¹⁶⁸ dimension of $\dim H_0(\overline{B_\omega}, \overline{D})$ so our map ℓ induced by inclusion depends only on $H_0(\overline{B_\omega}, \overline{D})$
¹⁶⁹ and $\text{im } i$.

¹⁷⁰ We also assume that $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is *injective*—as we move away from ω
¹⁷¹ moving *down* no components *disappear*. Lemma 4 uses Assumption 2 to provide a computable
¹⁷² upper bound on $\text{rk } j$.

¹⁷⁶ ► **Lemma 4.** If $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega+c(\delta+\zeta)})$ is *injective* and each component of $D \setminus B_\omega$
¹⁷⁷ contains a point in P then $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta}^\delta)) \geq \dim H_0(D \setminus B_\omega)$.

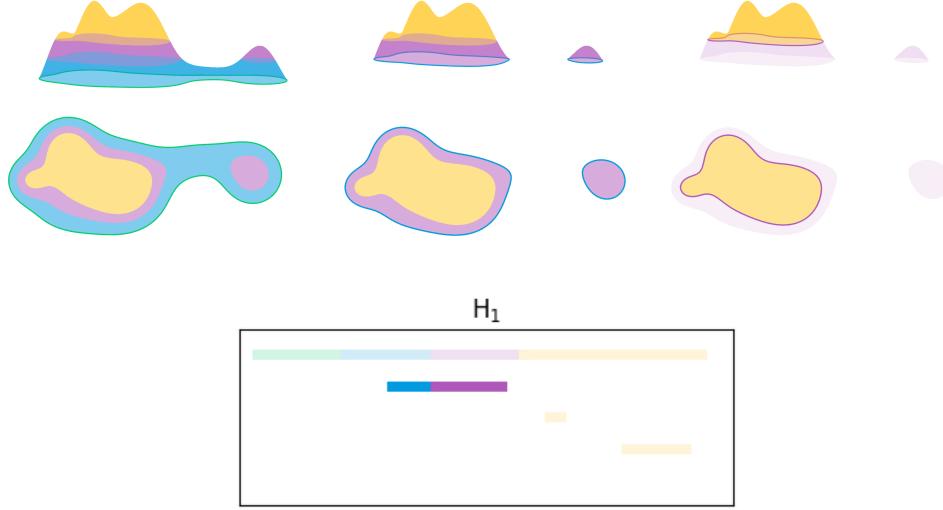
¹⁷⁸ Nerves and Duality

¹⁸¹ Recall that the Nerve Theorem states that for a good open cover \mathcal{U} of a space X the inclusion
¹⁸² map from the *Nerve* of the cover to the space $\mathcal{N}(\mathcal{U}) \hookrightarrow X$ is a homotopy equivalence.³ The
¹⁸³ Persistent Nerve Lemma [4] states that this homotopy equivalence commutes with inclusion
¹⁸⁴ on the level of homology. We note that the standard proof of the Nerve Theorem [9], and
¹⁸⁵ therefore the Persistent Nerve Lemma [4], extends directly to pairs of good open covers $(\mathcal{U}, \mathcal{V})$
¹⁸⁶ of pairs (X, Y) such that \mathcal{V} is a subcover of \mathcal{U} .⁴

¹⁷⁹ ³ In a good open cover every nonempty intersection of sets in the cover is contractible.

¹⁸⁰ ⁴ $\{V_i\}_{i \in I}$ is a subcover of $\{U_i\}_{i \in I}$ if $V_i \subseteq U_i$ for all $i \in I$.

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173 **Figure 1** The blue level set does not satisfy either assumption as the smaller component is not in
 174 the inclusion from blue to green and it “pinched out” in the yellow region. This can be seen in the
 175 barcode shown as a feature that is born in the blue region and dies in the purple region.

187 Recalling the definition of the strong convexity radius ϱ_D (see Chazal et al. [3]) \mathcal{U} is a
 188 good open cover whenever $\varrho_D > \varepsilon$. As the Čech complex is the Nerve of a cover by a union
 189 of balls we will let $\mathcal{N}_w^\varepsilon : H_k(\check{\mathcal{C}}^\varepsilon(P, Q_w)) \rightarrow H_k(P^\varepsilon, Q_w^\varepsilon)$ denote the isomorphism on homology
 190 provided by the Nerve Theorem for all $k, w \in \mathbb{R}$ and $\varepsilon < \varrho_D$.

192 Under certain conditions Alexander Duality provides an isomorphism between the k
 193 relative cohomology of a compact pair in an orientable d -manifold \mathbb{X} with the $d-k$ dimensional
 194 homology of their complements in \mathbb{X} (see Spanier [11]). For finitely generated (co)homology
 195 over a field the Universal Coefficient Theorem can be used with Alexander Duality to give
 196 a natural isomorphism $\xi_w^\varepsilon : H_d(P^\varepsilon, Q_w^\varepsilon) \rightarrow H_0(D \setminus Q_w^\varepsilon, D \setminus P^\varepsilon)$.⁵ This isomorphism holds
 197 in the specific case when $P^\varepsilon \subseteq \text{int}_{\mathbb{X}}(D)$ and $D \setminus P^\varepsilon, D \setminus Q_w^\varepsilon$ are locally contractible. We
 198 therefore provide the following definition for ease of exposition.

199 ▶ **Definition 5** ((δ, ζ, ω)-Sublevel Sample). *For $\zeta \geq \delta > 0$, $\omega \in \mathbb{R}$, and a c -Lipschitz function
 200 $f : D \rightarrow \mathbb{R}$ a finite point set $P \subset D$ is said to be a (δ, ζ, ω) -sublevel sample of f if every
 201 component of $D \setminus B_\omega$ contains a point in P , $P^\delta \subset \text{int}_{\mathbb{X}}(D)$, and $D \setminus P^\delta, D \setminus Q_{\omega-c\zeta}^\delta$, and
 202 $D \setminus Q_{\omega+c\delta}^\delta$ are locally path connected in \mathbb{X} .*

203 Because this isomorphism is natural and the isomorphism provided by the Nerve Theorem
 204 commutes with maps induced by inclusion the composition $\xi \mathcal{N}_w^\varepsilon := \xi_w^\varepsilon \circ \mathcal{N}_w^\varepsilon$ gives an
 205 isomorphism that commutes with maps induced by inclusion for all $w \in \mathbb{R}$ and $\varepsilon < \varrho_D$.

206 ▶ **Theorem 6** (Algorithmic TCC). *Let \mathbb{X} be an orientable d -manifold and let D be a compact
 207 subset of \mathbb{X} . Let $f : D \rightarrow \mathbb{R}$ be c -Lipschitz function and $\omega \in \mathbb{R}$ and $\delta \leq \zeta < \varrho_D$ be constants
 208 such that $P \subset D$ is a (δ, ζ, ω) -sublevel sample of f and $B_{\omega-c(\zeta+\delta)}$ surrounds D in \mathbb{X} .*

209 *If $H_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ is surjective, $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega+c(\delta+\zeta)})$ is injective,
 210 and $\text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-c\zeta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta}))$ then $D \setminus B_\omega \subseteq P^\delta$
 211 and $Q_{\omega-c\zeta}^\delta$ surrounds P^δ in D .*

191 ⁵ For the construction of this isomorphism see the Appendix.

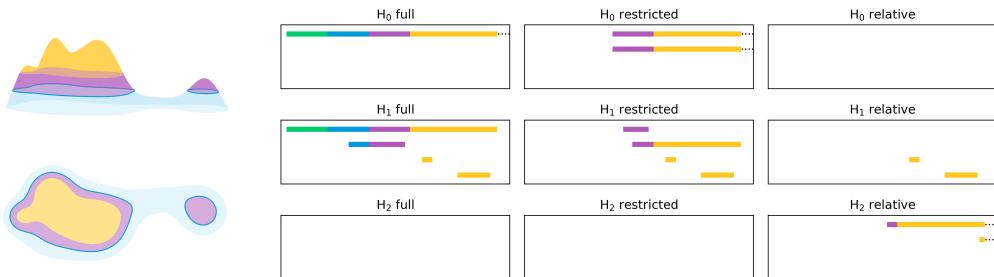
212 **Proof.** Because P is a (δ, ζ, ω) -sublevel sample we have isomorphisms $\xi\mathcal{N}_{\omega-c\zeta}^\delta$ and $\xi\mathcal{N}_{\omega+c\delta}^\delta$
213 that commute with $q_{\check{C}} : H_d(\check{C}^\delta(P, Q_{\omega-c\zeta})) \rightarrow H_d(\check{C}^{2\delta}(P, Q_{\omega+c\delta}))$ and $i : H_0(D \setminus Q_{\omega+c\delta}^\delta, D \setminus$
214 $P^\delta) \rightarrow H_0(D \setminus Q_{\omega-c\zeta}^\delta, D \setminus P^\delta)$. Let $q_{\mathcal{R}} : H_d(\mathcal{R}^\delta(P, Q_{\omega-c\zeta})) \rightarrow H_d(\mathcal{R}^{2\delta}(P, Q_{\omega+c\delta}))$ be
215 induced by inclusion. Then $\text{rk } q_{\check{C}} \geq \text{rk } q_{\mathcal{R}}$ as $q_{\mathcal{R}}$ factors through $q_{\check{C}}$. As we have assumed
216 $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ Lemma 4 implies $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$.
217 It follows that, whenever $\text{rk } q_{\mathcal{R}} \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta}))$, we have

218 $\text{rk } i = \text{rk } q_{\check{C}} \geq \text{rk } q_{\mathcal{R}} \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega).$

219 Because j is surjective by hypothesis $\text{rk } j = \dim H_0(\overline{B_\omega}, \overline{D}) = \dim H_0(D \setminus B_\omega)$ so
220 $\text{rk } j \geq \text{rk } i$ by Lemma 3. As we have shown $\text{rk } i \geq \dim H_0(D \setminus B_\omega)$ it follows that
221 $\text{rk } j = \text{rk } i$. Because P is a finite point set we know that $\text{im } i$ is finite-dimensional and,
222 because $\text{rk } i = \text{rk } j$, $\text{im } j = \overline{H_0(B_\omega, D)}$ is finite dimensional as well. So $\text{im } j$ is isomorphic
223 to $\text{im } i$ as a subspace of $H_0(Q_{\omega-c\zeta}^\delta, P^\delta)$ which, because j is surjective, requires the map ℓ to
224 be injective. Therefore $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-c\zeta}^\delta$ surrounds P^δ in D by Lemma 2. \blacktriangleleft

225 4 From Coverage Testing to the Analysis of Scalar Fields

226 Because the TCC only confirms coverage of a *superlevel* set $D \setminus B_\omega$, we cannot guarantee
227 coverage of the entire domain. Indeed, we could compute the persistent homology of the
228 *restriction* of f to the superlevel set we cover in the standard way [3]. Instead, we will
229 approximate the persistent homology of the sublevel set filtration *relative to* the sublevel
230 set B_ω . In the next section we will discuss an interpretation of the relative diagram that is
231 motivated by examples in Section 6.



232 **Figure 2** Full, restricted, and relative barcodes of the function (left).

233 We will first introduce the notion of an extension which will provide us with maps on
234 relative homology induced by inclusion via excision. However, even then, a map that factors
235 through our pair (D, B_ω) is not enough to prove an interleaving of persistence modules by
236 inclusion directly. To address this we impose conditions on sublevel sets near B_ω which
237 generalize the assumptions made in the TCC.

238 4.1 Extensions and Image Persistence Modules

239 Suppose D is a subspace of X . We define the extension of a surrounding pair in D to a
240 surrounding pair in X with isomorphic relative homology.

241 ► **Definition 7** (Extension). If V surrounds U in a subspace D of X let $\mathcal{EV} := V \sqcup (D \setminus U)$
242 denote the (disjoint) union of the separating set V with the complement of U in D . The
243 **extension of (U, V) in D** is the pair $(D, \mathcal{EV}) = (U \sqcup (D \setminus U), V \sqcup (D \setminus U))$.

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244 Lemma 8 states that we can use these extensions to interleave a pair (U, V) with a
 245 sequence of subsets of (D, B) . Lemma ?? we can apply excision to the relative homology
 246 groups in order to get equivalent maps on homology that are induced by inclusions.

247 ▶ **Lemma 8.** Suppose V surrounds U in D and $B' \subseteq B \subset D$.

248 If $D \setminus B \subseteq U$ and $U \cap B' \subseteq V \subseteq B'$ then $B' \subseteq \mathcal{E}V \subseteq B$.

249 ▶ **Lemma 9.** Let (U, V) be an open surrounding pair in a subspace D of X .

250 Then $H_k((U \cap A, V) \hookrightarrow (A, \mathcal{E}V))$ is an isomorphism for all k and $A \subseteq D$ with $\mathcal{E}V \subset A$.

251 The TCC uses a nested pair of spaces in order to filter out noise introduced by the sample.
 252 This same technique is used to approximate the persistent homology of a scalar fields [3]. As
 253 modules, these nested pairs are the images of homomorphisms between homology groups
 254 induced by inclusion, which we refer to as image persistence modules.

255 ▶ **Definition 10** (Image Persistence Module). The *image persistence module* of a homomorphism $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$ is the family of subspaces $\{\Gamma_\alpha := \mathbf{im} \gamma_\alpha\}$ in \mathbb{V} along with linear maps $\{\gamma_\alpha^\beta := v_\alpha^\beta|_{\mathbf{im} \gamma_\alpha} : \Gamma_\alpha \rightarrow \Gamma_\beta\}$ and will be denoted by $\mathbf{im} \Gamma$.

258 While we will primarily work with homomorphisms of persistence modules induced by
 259 inclusions, in general, defining homomorphisms between images simply as subspaces of the
 260 codomain is not sufficient. Instead, we require that homomorphisms between image modules
 261 commute not only with shifts in scale, but also with the functions themselves.

264 ▶ **Definition 11** (Image Module Homomorphism). Given $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$ and $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$
 265 along with $(F, G) \in \text{Hom}^\delta(\mathbb{U}, \mathbb{S}) \times \text{Hom}^\delta(\mathbb{V}, \mathbb{T})$ let $\Phi(F, G) : \mathbf{im} \Gamma \rightarrow \mathbf{im} \Lambda$ denote the family
 266 of linear maps $\{\phi_\alpha := g_\alpha|_{\Gamma_\alpha} : \Gamma_\alpha \rightarrow \Lambda_{\alpha+\delta}\}$. $\Phi(F, G)$ is an *image module homomorphism*
 267 of degree δ if the following diagram commutes for all $\alpha \leq \beta$.⁶

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\gamma_\alpha[\beta-\alpha]} & V_\beta \\ \downarrow f_\alpha & & \downarrow g_\beta \\ S_{\alpha+\delta} & \xrightarrow{\lambda_{\alpha+\delta}[\beta-\alpha]} & T_{\beta+\delta} \end{array} \quad (2)$$

269 The space of image module homomorphisms of degree δ between $\mathbf{im} \Gamma$ and $\mathbf{im} \Lambda$ will be
 270 denoted $\text{Hom}^\delta(\mathbf{im} \Gamma, \mathbf{im} \Lambda)$.

271 The composition of image module homomorphisms are image module homomorphisms. Proof
 272 of this fact can be found in the Appendix.

273 Partial Interleavings of Image Modules

274 Image module homomorphisms introduce a direction to the traditional notion of interleaving.
 275 As we will see, our interleaving via Lemma 13 involves partially interleaving an image module
 276 to two other image modules whose composition is isomorphic to our target.

277 ▶ **Definition 12** (Partial Interleaving of Image Modules). An image module homomorphism
 278 $\Phi(F, G)$ is a *partial δ -interleaving of image modules*, and denoted $\Phi_M(F, G)$, if there
 279 exists $M \in \text{Hom}^\delta(\mathbb{S}, \mathbb{V})$ such that $\Gamma[2\delta] = M \circ F$ and $\Lambda[2\delta] = G \circ M$.

262 ⁶ We use the notation $\gamma_\alpha[\beta-\alpha] = v_\alpha^\beta \circ \gamma_\alpha$, $\lambda_\alpha[\beta-\alpha] = t_\alpha^\beta \circ \lambda_\alpha$ to denote the composition of homomorphisms
 263 between persistence modules and shifts in scale.

Lemma 13 uses partial interleavings of a map Λ with $\mathbb{U} \rightarrow \mathbb{V}$ and $\mathbb{V} \rightarrow \mathbb{W}$ along with the hypothesis that $\mathbb{U} \rightarrow \mathbb{W}$ is isomorphic to \mathbb{V} to interleave $\mathbf{im} \Lambda$ with \mathbb{V} . When applied, this hypothesis will be satisfied by assumptions on our sublevel set similar to those made in the TCC.

► **Lemma 13.** Suppose $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$, $\Pi \in \text{Hom}(\mathbb{V}, \mathbb{W})$, and $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$.

If $\Phi_M(F, G) \in \text{Hom}^\delta(\mathbf{im} \Gamma, \mathbf{im} \Lambda)$ and $\Psi_G(M, N) \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbf{im} \Pi)$ are partial δ -interleavings of image modules such that Γ is a epimorphism and Π is a monomorphism then $\mathbf{im} \Lambda$ is δ -interleaved with \mathbb{V} .

4.2 Proof of the Interleaving

For $w, \alpha \in \mathbb{R}$ let \mathbb{D}_w^k denote the k th persistent (relative) homology module of the filtration $\{(D_{\lfloor \alpha \rfloor w}, B_w)\}_{\alpha \in \mathbb{R}}$ with respect to B_w , and let $\mathbb{P}_w^{\varepsilon, k}$ denote the k th persistent (relative) homology module of $\{(P_{\lfloor \alpha \rfloor w}^\varepsilon, Q_w^\varepsilon)\}_{\alpha \in \mathbb{R}}$. Similarly, let $\check{C}\mathbb{P}_w^{\varepsilon, k}$ and $\mathcal{R}\mathbb{P}_w^{\varepsilon, k}$ denote the corresponding Čech and Vietoris-Rips filtrations, respectively. We will omit the dimension k and write \mathbb{D}_w (resp. \mathbb{P}_w^ε) if a statement holds for all dimensions.

If Q_w^ε surrounds P^ε in D let $\mathcal{EP}_w^\varepsilon$ denote the k th persistent homology module of the filtration of extensions $\{(\mathcal{EP}_{\lfloor \alpha \rfloor w}^\varepsilon, \mathcal{EQ}_w^\varepsilon)\}$, where $\mathcal{EP}_{\lfloor \alpha \rfloor w}^\varepsilon = P_{\lfloor \alpha \rfloor w}^\varepsilon \cup (D \setminus P^\varepsilon)$. The following lemmas extend Lemmas 9 and 8 to persistence modules. Lemma 16 applies these results to obtain partial interleavings required for our proof.

► **Lemma 14.** If Q_w^ε surrounds P^ε in D then there is an isomorphism $\mathcal{E}_w^\varepsilon \in \text{Hom}(\mathbb{P}_w^\varepsilon, \mathcal{EP}_w^\varepsilon)$.

► **Lemma 15.** If Q_w^ε surrounds P^ε in D and $D \setminus B_{w+c\varepsilon} \subseteq P^\varepsilon$ then we have a sequence $\mathbb{D}_{w-c\varepsilon} \xrightarrow{F} \mathcal{EP}_w^\varepsilon \xrightarrow{M} \mathbb{D}_{w+c\varepsilon}$ of homomorphisms of degree $c\varepsilon$ induced by inclusions.

► **Lemma 16.** Let $s + 3c\delta \leq t + 2c\delta \leq u \leq v - c\delta \leq w - 5c\delta$ and suppose $\delta < \varrho_D$. Let $\Gamma \in \text{Hom}(\mathbb{D}_s, \mathbb{D}_u)$, $\Pi \in \text{Hom}(\mathbb{D}_u, \mathbb{D}_w)$, and $\Lambda \in \text{Hom}(\mathcal{R}\mathbb{P}_t^{2\delta}, \mathcal{R}\mathbb{P}_v^{4\delta})$ be induced by inclusions.

If Q_t^δ surrounds P^δ in D and $D \setminus B_u \subseteq P^\delta$ then there is a partial $2c\delta$ interleaving $\Phi^* \in \text{Hom}^{2c\delta}(\mathbf{im} \Gamma, \mathbf{im} \Lambda)$ and a partial $4c\delta$ interleaving $\Psi^* \in \text{Hom}^{4c\delta}(\mathbf{im} \Lambda, \mathbf{im} \Pi)$.

Proof. Let $\mathcal{EP}_t^\delta \xrightarrow{A} \mathcal{EP}_t^{2\delta} \xrightarrow{C} \mathcal{EP}_v^{2\delta} \xrightarrow{E} \mathcal{EP}_v^{4\delta}$ be maps induced by inclusion. Because f is c -Lipschitz, $B_s \cap P^\delta \subseteq Q_t^\delta$ and $B_u \cap P^\delta \subseteq Q_v^{2\delta}$. Similarly, $Q_t^{2\delta} \subseteq B_u$ and $Q_v^{4\delta} \subseteq B_w$. Therefore, by Lemma 8 $B_s \subseteq \mathcal{EQ}_t^\delta \subseteq \mathcal{EQ}_t^{2\delta} \subseteq B_u \subseteq \mathcal{EQ}_v^{2\delta} \subseteq \mathcal{EQ}_v^{4\delta} \subseteq B_w$. It follows that the following diagrams commute for all $\alpha \leq \beta$ by Lemma 20.

$$\begin{array}{ccc}
 H_k(D_{\lfloor \alpha - 2c\delta \rfloor s}, B_s) & \xrightarrow{f_{\alpha-2c\delta}} & H_k(\mathcal{EP}_{\lfloor \alpha \rfloor t}^\delta, B_t) \\
 \downarrow \gamma_{\alpha-2c\delta}[\beta-\alpha] & & \downarrow c_\alpha[\beta-\alpha] \circ a_\alpha \\
 H_k(D_{\lfloor \beta - 2c\delta \rfloor u}, B_u) & \xrightarrow{g_{\beta-2c\delta}} & H_k(\mathcal{EP}_{\lfloor \beta \rfloor v}^{2\delta}, B_v)
 \end{array}
 \quad
 \begin{array}{ccc}
 H_k(\mathcal{EP}_{\lfloor \alpha \rfloor t}^{2\delta}, B_t) & \xrightarrow{m_\alpha} & H_k(D_{\lfloor \alpha + 4c\delta \rfloor u}, B_u) \\
 \downarrow e_\beta \circ c_\alpha[\beta-\alpha] & & \downarrow \gamma_{\alpha+4c\delta}[\beta-\alpha] \\
 H_k(\mathcal{EP}_{\lfloor \beta \rfloor v}^{4\delta}, B_v) & \xrightarrow{n_\beta} & H_k(D_{\lfloor \beta + 4c\delta \rfloor w}, B_w)
 \end{array}
 \tag{3}$$

So we have we have image module homomorphisms $\Phi(F, G) \in \text{Hom}^{2c\delta}(\mathbf{im} \Gamma, \mathbf{im} C \circ A)$ and $\Psi(M, N) \in \text{Hom}^{4c\delta}(\mathbf{im} E \circ C, \mathbf{im} \Pi)$.

For any $\varepsilon < \varrho_D$, $z \in \mathbb{R}$ we have isomorphisms $\mathcal{N}_z^\varepsilon \in \text{Hom}(\check{C}\mathbb{P}_z^\varepsilon, \mathbb{P}_z^\varepsilon)$ that commutes with maps induced by inclusions by the Persistent Nerve Lemma. As the isomorphisms of $\mathcal{E}_z^\varepsilon$ are given by excision they are induced by inclusions, so the composition $\mathcal{E}_z^\varepsilon \circ \mathcal{N}_z^\varepsilon$ is an isomorphism that commutes with maps induced by inclusion as well. Using these compositions and the Rips-Čech interleaving let $\mathcal{EP}_t^\delta \xrightarrow{F'} \mathcal{R}\mathbb{P}_t^{2\delta} \xrightarrow{M'} \mathcal{EP}_t^{2\delta}$ and $\mathcal{EP}_v^{2\delta} \xrightarrow{G'} \mathcal{R}\mathbb{P}_v^{4\delta} \xrightarrow{N'} \mathcal{EP}_v^{4\delta}$ be maps

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317 that commute with maps induced by inclusions. As all maps are induced by inclusions or
 318 commute with maps induced by inclusions we have the following commutative diagram.

$$\begin{array}{ccccccc}
 \mathcal{E}\mathbb{P}_t^\delta & \xrightarrow{A} & \mathcal{E}\mathbb{P}_t^{2\delta} & \xrightarrow{C} & \mathcal{E}\mathbb{P}_v^{2\delta} & \xrightarrow{E} & \mathcal{E}\mathbb{P}_v^{4\delta} \\
 \searrow F' & & \nearrow M' & & \searrow G' & & \nearrow N' \\
 \mathcal{R}\mathbb{P}_t^{2\delta} & \xrightarrow{\Lambda} & \mathcal{R}\mathbb{P}_v^{4\delta} & & & &
 \end{array} \tag{4}$$

320 That is, we have image module homomorphisms $\Phi'(F', G')$ and $\Psi'(M', N')$ such that $A =$
 321 $M' \circ F'$, $E = N' \circ G'$, and $\Lambda = G' \circ C \circ M'$. Because image module homomorphisms compose
 322 we have we have $\Phi^* = \Phi' \circ \Phi \in \text{Hom}^{2c\delta}(\text{im } \Gamma, \text{im } \Lambda)$ and $\Psi^* = \Psi \circ \Psi' \in \text{Hom}^{4c\delta}(\text{im } \Lambda, \text{im } \Pi)$.

323 Because all maps are induced by inclusions $C[3c\delta] = G \circ M$ so $\Lambda[3c\delta] = G' \circ C[3c\delta] \circ M' =$
 324 $G' \circ (G \circ M) \circ M'$ as G', M' commute with maps induced by inclusions. In the same way,
 325 $\Gamma[3c\delta] = M \circ (A \circ F) = M \circ (M' \circ F') \circ F$ and $\Pi[5c\delta] = N \circ (E \circ G) = N \circ (N' \circ G') \circ G$.
 326 Let $F^* := F' \circ F$, $G^* := G' \circ G$, $M^* := M' \circ M$, and $N^* := N' \circ N$. So $\Phi_{M^*}^*$ is a partial $2c\delta$
 327 interleaving as $\Gamma[3c\delta] = M^* \circ F^*$ and $\Lambda[3c\delta] = G^* \circ M^*$, and $\Psi_{G^*}^*$ is a partial $4c\delta$ interleaving
 328 as $\Lambda[3c\delta] = G^* \circ M^*$ and $\Pi[5c\delta] = N^* \circ G^*$. \blacktriangleleft

329 The partial interleavings given by Lemma 16 along with assumptions that imply $\text{im}(\mathbb{D}_{\omega-3c\delta} \rightarrow$
 330 $\mathbb{D}_{\omega+5c\delta}) \cong \mathbb{D}_\omega$ provide the proof of Theorem 17 by Lemma 13.

331 ▶ **Theorem 17.** *Let \mathbb{X} be a d-manifold, $D \subset \mathbb{X}$ and $f : D \rightarrow \mathbb{R}$ be a c-Lipschitz function.
 332 Let $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $B_{\omega-3c\delta}$ surrounds D in \mathbb{X} . Let $P \subset D$ be
 333 a finite subset and suppose $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $H_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an
 334 isomorphism for all k .*

335 *If $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D then the kth persistent homology
 336 module of $\{\mathcal{R}^{2\delta}(P_{\lfloor \alpha \rfloor \omega-2c\delta}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor \alpha \rfloor \omega+2c\delta}, Q_{\omega+2c\delta})\}_{\alpha \in \mathbb{R}}$ is $4c\delta$ -interleaved with that
 337 of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$.*

338 **Proof.** Let $\Lambda \in \text{Hom}(\mathcal{R}\mathbb{P}_{\omega-2c\delta}^{2c\delta}, \mathcal{R}\mathbb{P}_{\omega+2c\delta}^{4c\delta})$, $\Gamma \in \text{Hom}(\mathbb{D}_{\omega-3c\delta}, \mathbb{D}_\omega)$, and $\Pi \in \text{Hom}(\mathbb{D}_\omega, \mathbb{D}_{\omega+5c\delta})$
 339 be induced by inclusions. Because $\delta < \varrho_D/4$, $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D
 340 we have a partial $2c\delta$ interleaving $\Phi^* \in \text{Hom}^{2c\delta}(\text{im } \Gamma, \text{im } \Lambda)$ and a partial $4c\delta$ interleaving
 341 $\Psi^* \in \text{Hom}^{4c\delta}(\text{im } \Lambda, \text{im } \Pi)$ by Lemma 16. As we have assumed that $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$
 342 is surjective and $H_k(B_\omega) \cong H_k(B_{\omega+5c\delta})$ the five-lemma implies γ_α is surjective and π_α is
 343 an isomorphism (and therefore injective) for all α . So Γ is an epimorphism and Π is a
 344 monomorphism, thus $\text{im } \Lambda$ is $4c\delta$ -interleaved with \mathbb{D}_ω by Lemma 13 as desired. \blacktriangleleft

345 5 Approximation of the Truncated Diagram

346 In this section we will relate the relative persistence diagram that we have approximated in
 347 the previous section to a truncation of the full diagram. Let \mathbb{L}^k denote the k th persistent
 348 homology module of the sublevel set filtration $\{B_\alpha\}_{\alpha \in \mathbb{R}}$. As in the previous section, let \mathbb{D}_ω^k
 349 denote the k th persistent (relative) homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$. Throughout we
 350 will assume that we are taking homology in a field \mathbb{F} and that the homology groups $H_k(B_\alpha)$
 351 and $H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega)$ are finite dimensional vector spaces for all k and $\alpha \in \mathbb{R}$. We will use the
 352 interval decomposition of \mathbb{L}^k to give a decomposition of the relative module \mathbb{D}_ω^k in terms of a
 353 truncation of \mathbb{L}^k . Recall, the *truncated diagram* is defined to be that of \mathbb{L}^k consisting only of
 354 those features born after ω . For fixed $\omega \in \mathbb{R}$ we will define the truncation \mathbb{T}_ω^k of \mathbb{L}^k in terms
 355 of the intervals decomposing \mathbb{L}^k that are in $[\omega, \infty)$.

356 **Truncated Interval Modules**

357 For an interval $I = [s, t] \subseteq \mathbb{R}$ let $I_+ := [t, \infty)$ and $I_- := (-\infty, s]$. For $\omega \in \mathbb{R}$ let \mathbb{F}_ω^I denote the
 358 interval module consisting of vector spaces $\{F_{\lfloor \alpha \rfloor \omega}^I\}_{\alpha \in \mathbb{R}}$ and linear maps $\{f_{\lfloor \alpha, \beta \rfloor \omega}^I : F_{\lfloor \alpha \rfloor \omega}^I \rightarrow F_{\lfloor \beta \rfloor \omega}^I\}_{\alpha \leq \beta}$ where

$$360 \quad F_{\lfloor \alpha \rfloor \omega}^I := \begin{cases} F_\alpha^I & \text{if } \omega \in I_- \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{\lfloor \alpha, \beta \rfloor \omega}^I := \begin{cases} f_{\alpha, \beta}^I & \text{if } \omega \in I_- \\ 0 & \text{otherwise.} \end{cases}$$

361 For a collection \mathcal{I} of intervals let $\mathcal{I}_\omega := \{I \in \mathcal{I} \mid \omega \in I\}$.

362 ► **Lemma 18.** Suppose \mathcal{I}^k and \mathcal{I}^{k-1} are collections of intervals that decompose \mathbb{L}^k and \mathbb{L}^{k-1} ,
 363 respectively. Then the k th persistent homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ is equal to

$$364 \quad \bigoplus_{I \in \mathcal{I}^k} \mathbb{F}_\omega^I \oplus \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} \mathbb{F}^{I+}$$

365 for all k .

366 **Proof.** (See Appendix A) ◀

367 **Main Theorem**

368 Let \mathbb{L}^k denote the k th persistent homology module of the sublevel set filtration $\{B_\alpha\}_{\alpha \in \mathbb{R}}$
 369 of f and let \mathcal{I}^k denote the decomposing intervals of \mathbb{L}^k for all k . For a fixed $\omega \in \mathbb{R}$ let \mathbb{D}_ω^k
 370 denote the k th persistent (relative) homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$. Let

$$371 \quad \mathbb{T}_\omega^k := \bigoplus_{I \in \mathcal{I}^k} \mathbb{F}_\omega^I$$

372 denote the ω -truncated k th persistent homology module of \mathbb{L}^k . Let

$$373 \quad \mathbb{L}_\omega^{k-1} := \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} \mathbb{F}^{I+}.$$

374 denote the submodule of \mathbb{D}_ω^k consisting of intervals $[\beta, \infty)$ corresponding to features $[\alpha, \beta)$
 375 in \mathbb{L}^{k-1} such that $\alpha \leq \omega < \beta$. Now, by Lemma 18 the k th persistent (relative) homology
 376 module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ is

$$377 \quad \mathbb{D}_\omega^k = \mathbb{T}_\omega^k \oplus \mathbb{L}_\omega^{k-1}.$$

378 Our main theorem combines this decomposition with our coverage and interleaving results of
 379 Theorems 6 and 17.

380 ► **Theorem 19.** Let \mathbb{X} be an orientable d -manifold and let D be a compact subset of \mathbb{X} . Let
 381 $f : D \rightarrow \mathbb{R}$ be a c -Lipschitz function and $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $P \subset D$ is a
 382 $(\delta, 2\delta, \omega)$ -sublevel sample of f and $B_{\omega-3c\delta}$ surrounds D in \mathbb{X} .

383 Suppose $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $H_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an isomorphism for
 384 all k . If

$$385 \quad \text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$$

386 then the k th (relative) homology module of

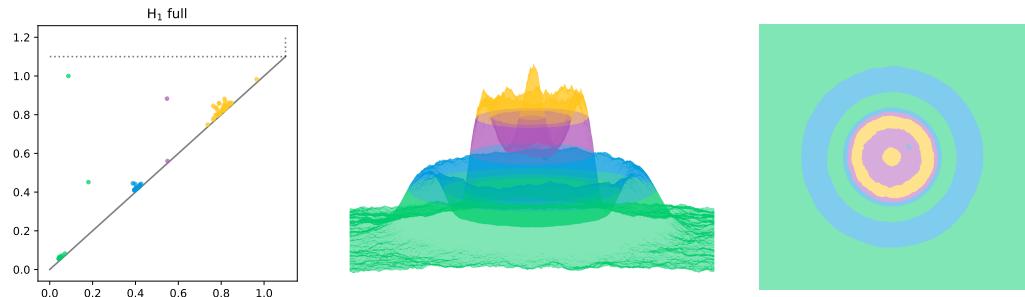
$$387 \quad \{\mathcal{R}^{2\delta}(P_{\lfloor \alpha \rfloor \omega-2c\delta}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor \alpha \rfloor \omega+c\delta}, Q_{\omega+c\delta})\}_{\alpha \in \mathbb{R}}$$

388 is $4c\delta$ -interleaved with $\mathbb{T}_\omega^k \oplus \mathbb{L}_\omega^{k-1}$: the k th persistent homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$.

389 **6 Experiments**

390 In this section we will discuss a number of experiments which illustrate the benefit of
 391 truncated diagrams, and their approximation by relative diagrams, in comparison to their
 392 restricted counterparts. We will focus on the persistent homology of functions on a square
 393 2d grid. We chose as our function a radially symmetric damped sinusoid with random noise,
 394 depicted in Figure 3, as it has prominent persistent homology in dimension one.

395 **Experimental setup.**



396 **Figure 3** The H_1 persistence diagram of the sinusoidal function pictured to the right. Features
 397 are colored by birth time, infinite features are drawn above the dotted line.

399 Throughout, the four interlevel sets shown correspond to the ranges $[0, 0.3)$, $[0.3, 0.5)$,
 400 $[0.5, 0.7)$, and $[0.7, 1)$, respectively. Our persistent homology computations were done primarily
 401 with Dionysus augmented with custom software for computing representative cycles of
 402 infinite features.⁷ The persistent homology of our function was computed with the lower-star
 403 filtration of the Freudenthal triangulation on an $N \times N$ grid over $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. We
 404 take this filtration as $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ where P is the set of grid points and $\delta = \sqrt{2}/N$.

405 We note that the purpose of these experiments is not to demonstrate the effectiveness of our
 406 approximation by Rips complexes, but to demonstrate the relationships between restricted,
 407 relative, and truncated diagrams. Therefore, for simplicity, we will omit the inclusion
 408 $\mathcal{R}^{2\delta}(P_\alpha) \hookrightarrow \mathcal{R}^{4\delta}(P_\alpha)$ and take the persistent homology of $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ with sufficiently small
 409 δ as our ground-truth.

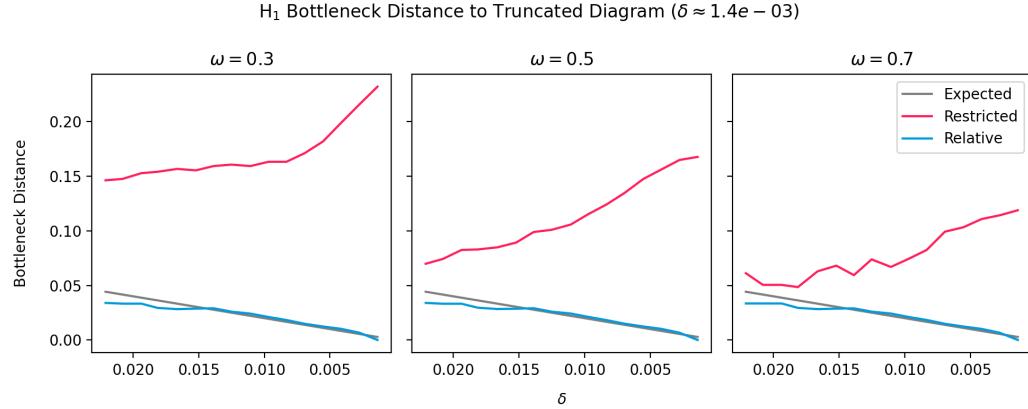
410 In the following we will take $N = 1024$, so $\delta \approx 1.4 \times 10^{-3}$, as our ground-truth. Figure 3
 411 shows the *full diagram* of our function with features colored by birth time. Therefore, for
 412 $\omega = 0.3, 0.5, 0.7$ the *truncated diagram* is obtained by successively removing features in
 413 each interlevel set. Recall the *restricted diagram* is that of the function restricted to the ω
 414 *super-levelset* filtration, and computed with $\{\mathcal{R}^{2\delta}(P_\alpha \setminus Q_\omega)\}$. We will compare this restricted
 415 diagram with the *relative diagram*, computed as the relative persistent homology of the
 416 filtration of pairs $\{\mathcal{R}^{2\delta}(P_\alpha, Q_\omega)\}$.

417 **The issue with restricted diagrams.**

418 Figure ?? shows the bottleneck distance from the truncated diagram at full resolution
 419 ($N = 1024$) to both the relative and restricted diagrams with varying resolution. Specifically,

398 ⁷ 3D figures were made with Mayavi, all other figures were made with Matplotlib.

420 the function on a 1024×1024 grid is down-sampled to grids ranging from 64×64 to 1024×1024 .
 421 We also show the expected bottleneck distance to the true truncated diagram given by the
 422 interleaving in Theorem 17 in black.



423 ■ **Figure 4** Comparison of the bottleneck distance between the truncated diagram and those of the
 424 restricted and relative diagrams with increasing resolution.

425 As we can see, the relative diagram clearly performs better than the restricted diagram,
 426 which diverges with increasing resolution. Recall that 1-dimensional features that are born
 427 before ω and die after ω become infinite 2-dimensional features in the relative diagram, with
 428 birth time equal to the death time of the corresponding feature in the full diagram. These
 429 same features remain 1-dimensional figures in the restricted diagram, but with their birth
 430 times shifted to ω .

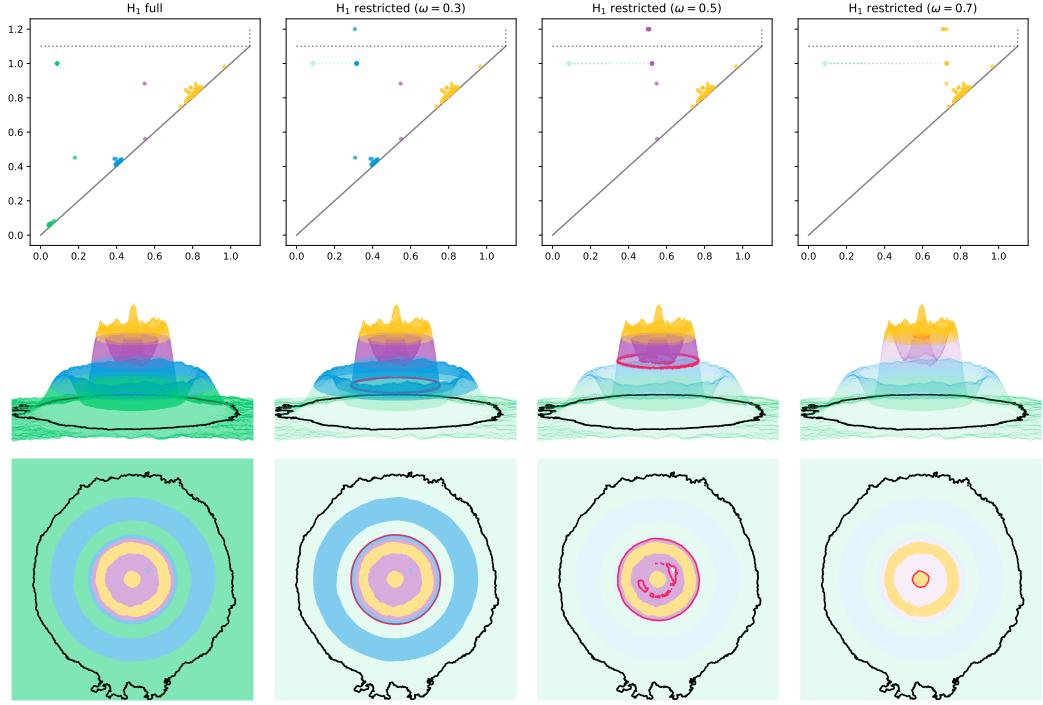
435 Figure 5 shows this distance for a feature that persists throughout the diagram. As the
 436 restricted diagram in full resolution the restricted filtration is a subset of the full filtration,
 437 so these features can be matched by their death simplices. For illustrative purposes we also
 438 show the representative cycles associated with these features.

439 Relative diagrams and reconstruction.

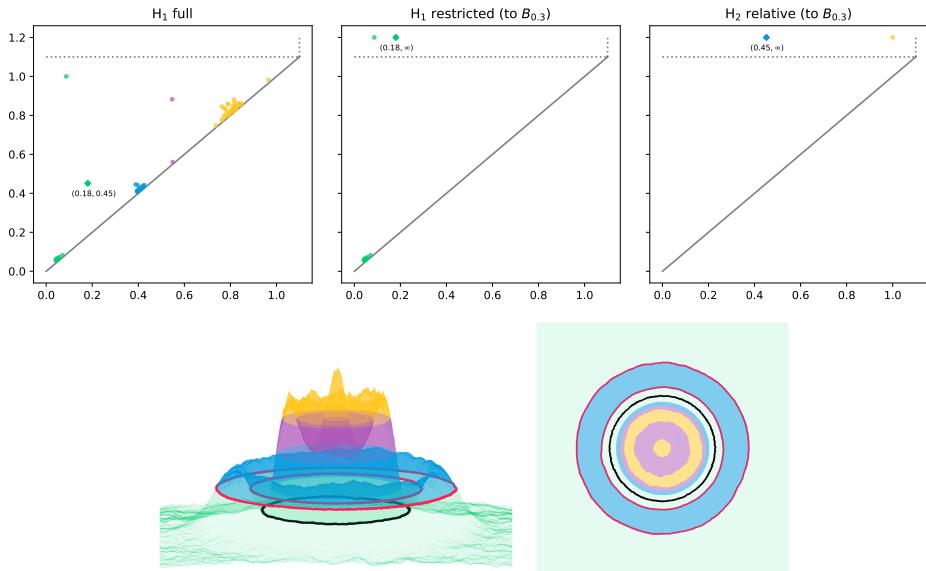
445 Now, imagine we obtain the persistence diagram of our sub-levelset B_ω . That is, we now
 446 know that we cover B_ω , or some subset, and do not want to re-compute the diagram above
 447 ω . If we compute the persistence diagram of the function restricted to the sub-levelset B_ω
 448 any 1-dimensional features born before ω that die after ω will remain infinite features in
 449 this restricted (below) diagram. Indeed, we could match these infinite 1-features with the
 450 corresponding shifted finite 1-features in the restricted (above) diagram, as shown in Figure 5.
 451 However, that would require sorting through all finite features that are born near ω and
 452 deciding if they are in fact features of the full diagram that have been shifted.

453 Recalling that these same features become infinite 2-features in the relative diagram, we
 454 can use the relative diagram instead and match infinite 1-features of the diagram restricted
 455 below to infinite 2-features in the relative diagram, as shown in Figures 6 and 7. For this
 456 example the matching is given by sorting the 1-features by ascending and the 2-features by
 457 descending birth time. How to construct this matching in general, especially in the presence
 458 of infinite features in the full diagram, is the subject of future research.

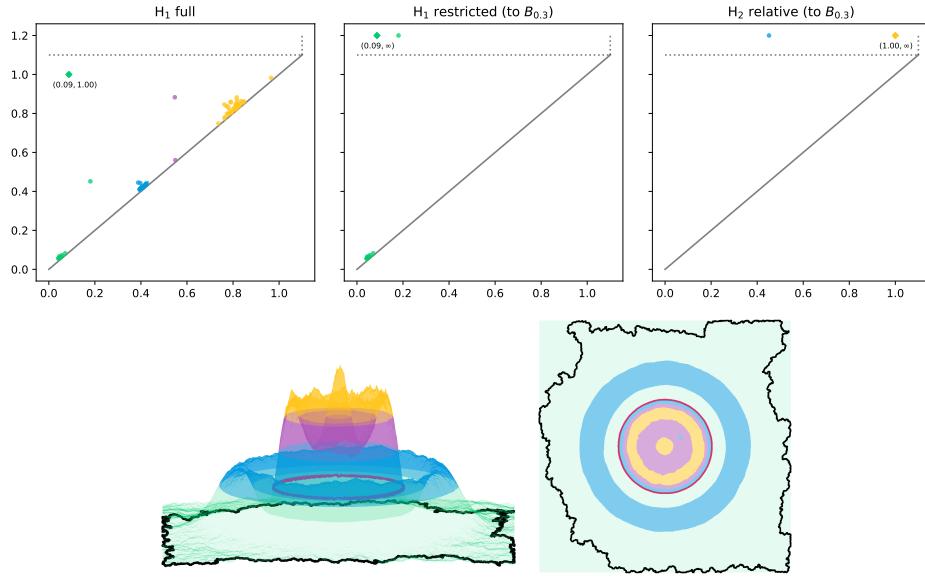
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431 ■ **Figure 5** (Top) H_1 persistence diagrams of the function depicted in Figure 3 restricted to *super-*
432 levelsets at $\omega = 0.3, 0.5$, and 0.7 (on a 1024×1024 grid). The matching is shown between a feature in
433 the full diagram (marked with a diamond) with its representative cycle in black. The corresponding
434 representative cycle in the restricted diagram is pictured in red.



440 ■ **Figure 6** (Top) The indicated infinite features in the restricted and relative diagrams correspond
441 to the birth and death of the 1-feature $(0.18, 0.45)$ in the full diagram. (Bottom) In black, the
442 representative cycle of the infinite 1-feature born at 0.18 in the restricted diagram is shown in black.
443 In red, the *boundary* of the representative relative 2-cycle born at 0.45 in the relative diagram is
444 shown in red.



459 ■ **Figure 7** The infinite 1-features of the restricted diagram can be matched with the infinite
 460 2-features of the relative diagrams. The sequence birth times of relative 2-features in *decreasing*
 461 order correspond to the deaths of restricted 1-features in *increasing* order.

462 7 Conclusion

463 We have extended the Topological Coverage Criterion to the setting of Topological Scalar
 464 Field Analysis. By defining the boundary in terms of a sublevel set of a scalar field we
 465 provide an interpretation of the TCC that applies more naturally to data coverage. We then
 466 showed how the assumptions and machinery of the TCC can be used to approximate the
 467 persistent homology of the scalar field relative to a static sublevel set. This relative persistent
 468 homology is shown to be related to a truncation of that of the scalar field as whole, and
 469 therefore provides a way to approximate a part of its persistence diagram in the presence of
 470 un-verified data.

471 There are a number of unanswered questions and directions for future work. From the
 472 theoretical perspective, our understanding of duality limited us in providing a more elegant
 473 extension of the TCC. A better understanding of when and how duality can be applied would
 474 allow us to give a more rigorous statement of our assumptions. Moreover, as duality plays
 475 a central role in the TCC it is natural to investigate its role in the analysis of scalar fields.
 476 This would not only allow us to apply duality to persistent homology [8], but also allow us
 477 to provide a rigorous comparison between the relative approach and the persistent homology
 478 of the superlevel set filtration and explore connections with Extended Persistence [5].

479 From a computational perspective, we interested in exploring how to recover the full
 480 diagram as discussed in Section 6. Our statements in terms of sublevel sets can be generalized
 481 to disjoint unions of sub and superlevel sets, where coverage is confirmed in an *interlevel*
 482 set. This, along with a better understanding of the relationship between sub and superlevel
 483 sets could lead to an iterative approach in which the persistent homology of a scalar field is
 484 constructed as data becomes available. We are also interested in finding efficient ways to
 485 compute the image persistent (relative) homology that vary in both scalar and scale.

486 The problem of relaxing our assumptions on the boundary can be approached from both
 487 a theoretical and computational perspective. Ways to avoid the isomorphism we require

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488 could be investigated in theory, and the interaction of relative persistent homology and the
489 Persistent Nerve Lemma may be used to tighten our assumptions. We would also like to conduct
490 a more rigorous investigation on the effect of these assumptions in practice.

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521 A Omitted Proofs

522 **Proof of Lemma 2.** This proof is in two parts.

523 ℓ injective $\implies D \setminus B \subseteq U$ Suppose, for the sake of contradiction, that p is injective and
524 there exists a point $x \in (D \setminus B) \setminus U$. Because B surrounds D in X the pair $(D \setminus B, \overline{D})$
525 forms a separation of \overline{B} . Therefore, $H_0(\overline{B}) \cong H_0(D \setminus B) \oplus H_0(\overline{D})$ so

$$526 H_0(\overline{B}, \overline{D}) \cong H_0(D \setminus B).$$

527 So $[x]$ is non-trivial in $H_0(\overline{B}, \overline{D}) \cong H_0(D \setminus B)$ as x is in some connected component of
528 $D \setminus B$. So we have the following sequence of maps induced by inclusions

$$529 H_0(\overline{B}, \overline{D}) \xrightarrow{f} H_0(\overline{B}, \overline{D} \cup \{x\}) \xrightarrow{g} H_0(\overline{V}, \overline{U}).$$

530 As $f[x]$ is trivial in $H_0(\overline{B}, \overline{D} \cup \{x\})$ we have that $\ell[x] = (g \circ f)[x]$ is trivial, contradicting
531 our hypothesis that ℓ is injective.

532 ℓ injective $\implies V$ surrounds U in D . Suppose, for the sake of contradiction, that V does
 533 not surround U in D . Then there exists a path $\gamma : [0, 1] \rightarrow \overline{V}$ with $\gamma(0) \in U \setminus V$ and
 534 $\gamma(1) \in D \setminus U$. As we have shown, $D \setminus B \subseteq U$, so $D \setminus B \subseteq U \setminus V$.
 535 Choose $x \in D \setminus B$ and $z \in \overline{D}$ such that there exist paths $\xi : [0, 1] \rightarrow U \setminus V$ with $\xi(0) = x$,
 536 $\xi(1) = \gamma(0)$ and $\zeta : [0, 1] \rightarrow \overline{D} \cup (D \setminus U)$ with $\zeta(0) = z$, $\zeta(1) = \gamma(1)$. ξ, γ and ζ all
 537 generate chains in $C_1(\overline{V}, \overline{U})$ and $\xi + \gamma + \zeta = \gamma^* \in C_1(\overline{V}, \overline{U})$ with $\partial\gamma^* = x + z$. Moreover, z
 538 generates a chain in $C_0(\overline{U})$ as $\overline{D} \subseteq \overline{U}$. So $x = \partial\gamma^* + z$ is a relative boundary in $C_0(\overline{V}, \overline{U})$,
 539 thus $\ell[x] = \ell[z]$ in $H_0(\overline{V}, \overline{L})$. However, because B surrounds D , $[x] \neq [y]$ in $H_0(\overline{B}, \overline{D})$
 540 contradicting our assumption that ℓ is injective.

541

542 **Proof of Lemma 4.** Assume there exist $p, q \in P \setminus Q_{\omega-c\zeta}$ such that p and q are connected in
 543 $\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})$ but not in $D \setminus B_\omega$. So the shortest path from p, q is a subset of $(P \setminus Q_{\omega-c\zeta})^\delta$.
 544 For any $x \in (P \setminus Q_{\omega-c\zeta})^\delta$ there exists some $p \in P$ such that $f(p) > \omega - c\zeta$ and $\mathbf{d}(p, x) < \delta$.
 545 Because f is c -Lipschitz

$$546 \quad f(x) \geq f(p) - c\mathbf{d}(x, p) > \omega - c(\delta + \zeta)$$

547 so there is a path from p to q in $D \setminus B_{\omega-c(\delta+\zeta)}$, thus $[p] = [q]$ in $H_0(D \setminus B_{\omega-c(\delta+\zeta)})$.

548 But we have assumed that $[p] \neq [q]$ in $H_0(D \setminus B_\omega)$, contradicting our assumption that
 549 $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is injective, so any p, q connected in $\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})$ are
 550 connected in $D \setminus B_\omega$. That is, $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$. \blacktriangleleft

551 A.1 Extensions

552 **Proof of Lemma 8.** Note that $B' \setminus (D \setminus U) = B' \cap U \subseteq V$ implies $B' \subseteq V \sqcup (D \setminus U) = \mathcal{E}V$.
 553 Moreover, because $V \subseteq B$ and $D \setminus B \subseteq U$ implies $D \setminus U \subset D \setminus (D \setminus B) = B$, we have

$$554 \quad \mathcal{E}V = V \sqcup (D \setminus U) \subseteq B \cup (D \setminus U) = B.$$

555 So $B' \subseteq \mathcal{E}V \subseteq B$ as desired. \blacktriangleleft

556 **► Lemma 20.** If Q_w^ε surrounds P^ε in D and $D \setminus B_{w+\varepsilon} \subseteq P^\varepsilon$ then we have the following
 557 sequence of homomorphisms of degree $c\varepsilon$ induced by inclusions

$$558 \quad \mathbb{D}_{w-c\varepsilon} \xrightarrow{F} \mathcal{EP}_w^\varepsilon \xrightarrow{M} \mathbb{D}_{w+c\varepsilon}.$$

559 **Proof.** Suppose $x \in (P^\varepsilon \cap B_{\lfloor \alpha - c\varepsilon \rfloor w - c\varepsilon}) \setminus B_{w+\varepsilon}$. Because $B_{w-\varepsilon} \subset B_{w+\varepsilon}$ we know $x \notin B_{w-\varepsilon}$
 560 so $w + c\varepsilon < f(x) \leq \alpha - c\varepsilon$ and there exists some $p \in P$ such that $\mathbf{d}(x, p) < \varepsilon$. Because f is
 561 c -Lipschitz it follows

$$562 \quad f(p) \leq f(x) + c\mathbf{d}(x, p) < \alpha - c\varepsilon + c\varepsilon = \alpha$$

563 and

$$564 \quad f(p) \geq f(x) - c\mathbf{d}(x, p) > w + c\varepsilon - c\varepsilon = w.$$

565 So $x \in P_{\lfloor \alpha \rfloor w}^\varepsilon$.

566 Now, suppose $x \in P_{\lfloor \alpha \rfloor w}^\varepsilon \setminus B_{w+c\varepsilon}$. So $w + c\varepsilon < f(x)$ and there exists some $p \in P_{\lfloor \alpha \rfloor w}$ such
 567 that $\mathbf{d}(x, p) < \varepsilon$. Because f is c -Lipschitz it follows

$$568 \quad f(x) \leq f(p) + c\mathbf{d}(x, p) < a + c\varepsilon.$$

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569 So $x \in B_{\lfloor \alpha + c\varepsilon \rfloor w + c\varepsilon} \setminus B_{w+c\varepsilon}$.

570 Because $D \setminus B_{w+c\varepsilon} \subseteq P^\varepsilon$ we know that $D \setminus P^\varepsilon \subseteq B_{w+c\varepsilon}$, so

$$571 D_{\lfloor \alpha - c\varepsilon \rfloor w - c\varepsilon} \setminus B_{w+c\varepsilon} \subseteq P_{\lfloor \alpha \rfloor w}^\varepsilon \setminus B_{w+c\varepsilon} \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor w + c\varepsilon} \setminus B_{w+c\varepsilon}$$

572 implies

$$573 D_{\lfloor \alpha - c\varepsilon \rfloor w - c\varepsilon} \subseteq P_{\lfloor \alpha \rfloor w}^\varepsilon \cup (D \setminus P^\varepsilon) = \mathcal{E}P_{\lfloor \alpha \rfloor w}^\varepsilon \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor w + c\varepsilon}$$

574 as desired.

575 Because f is c -Lipschitz, $B_{w-c\varepsilon} \cap P^\delta \subseteq Q_w^\varepsilon$ so $B_{w-c\varepsilon} \subseteq \mathcal{E}Q_w^\varepsilon \subseteq B_{w+c\varepsilon}$ by Lemma 8. It
576 follows that we have homomorphisms $F \in \text{Hom}^{c\varepsilon}(\mathbb{D}_{w-c\varepsilon}, \mathcal{E}\mathbb{P}_w^\varepsilon)$ and $M \in \text{Hom}^{c\varepsilon}(\mathcal{E}\mathbb{P}_w^\varepsilon, \mathbb{D}_{w+c\varepsilon})$
577 induced by inclusions. ◀

578 **Proof of Lemma 9.** Because V surrounds U in D , $(U \setminus V, D \setminus U)$ is a separation of $D \setminus V$, a
579 subspace of D . So $\text{cl}_D(U \setminus V) \setminus U = \text{cl}_D(U \setminus V) \cap (D \setminus U) = \emptyset$ which implies $\text{cl}_D(U \setminus V) \subseteq$
580 $U = \text{int}_D(U)$ as U is open in D . Therefore,

$$\begin{aligned} 581 \text{cl}_D(D \setminus U) &= D \setminus \text{int}_D(U) \\ 582 &\subseteq D \setminus \text{cl}_D(U \setminus V) \\ 583 &= \text{int}_D(D \setminus (U \setminus V)) \\ 584 &= \text{int}_D(\mathcal{E}V). \end{aligned}$$

585 SO,

$$\begin{aligned} 586 H_k(U \cap A, V) &= H_k(A \setminus (D \setminus U), \mathcal{E}V \setminus (D \setminus U)) \\ 587 &\cong H_k(A, \mathcal{E}V) \end{aligned}$$

588 for all k and any $A \subseteq D$ such that $\mathcal{E}V \subset A$ by Excision. ◀

589 ▶ **Lemma 21.** If Q_w^ε surrounds P^ε in D then there is an isomorphism $\mathcal{E}_w^\varepsilon \in \text{Hom}(\mathbb{P}_w^\varepsilon, \mathcal{E}\mathbb{P}_w^\varepsilon)$.

590 **Proof.** Because $P_{\lfloor a \rfloor w} := P \cap D_{\lfloor a \rfloor w}$ and $B_w \subseteq D_{\lfloor a \rfloor w}$ we know $Q_w = P \cap B_w \subseteq P_{\lfloor a \rfloor w}$ for all
591 $a \in \mathbb{R}$. So

$$592 \mathcal{E}Q_a^\varepsilon = Q_a^\varepsilon \cup (D \setminus P^\varepsilon) \subseteq P_{\lfloor a \rfloor w}^\varepsilon \cup (D \setminus P^\varepsilon) = \mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon.$$

593 As $(P^\varepsilon, Q_w^\varepsilon)$ is a surrounding pair in D , P^ε is open in D and $\mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon \subseteq D$ is such that
594 $\mathcal{E}Q_a^\varepsilon \subseteq \mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon$ it follows that

$$595 H_k(P_{\lfloor a \rfloor w}^\varepsilon, Q_a^\varepsilon) = H_k(P^\varepsilon \cap \mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon, Q_a^\varepsilon) \cong H_k(\mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon, \mathcal{E}Q_a^\varepsilon)$$

596 by Lemma 9.

597 Because these isomorphisms commute with inclusions we have an isomorphism $\mathcal{E}_{\lfloor \cdot \rfloor w}^\varepsilon \in$
598 $\text{Hom}(\mathbb{P}_w^\varepsilon, \mathcal{E}\mathbb{P}_w^\varepsilon)$ defined to be the family $\{\mathcal{E}_{\lfloor a \rfloor w}^\varepsilon : \mathcal{P}_{\lfloor a \rfloor w}^\varepsilon \rightarrow \mathcal{E}\mathcal{P}_{\lfloor a \rfloor w}^\varepsilon\}$. ◀

599 A.2 Image Modules

600 ▶ **Lemma 22.** Suppose $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$, $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$, and $\Lambda' \in \text{Hom}(\mathbb{S}', \mathbb{T}')$. If $\Phi(F, G) \in$
601 $\text{Hom}^\delta(\text{im } \Gamma, \text{im } \Lambda)$ and $\Phi'(F', G') \in \text{Hom}^{\delta'}(\text{im } \Lambda, \text{im } \Lambda')$ then $\Phi''(F' \circ F, G' \circ G) := \Phi' \circ \Phi \in$
602 $\text{Hom}^{\delta+\delta'}(\text{im } \Gamma, \text{im } \Lambda')$.

603 **Proof.** Because $\Phi(F, G)$ is an image module homomorphism of degree δ we have $g_{\beta-\delta} \circ$
 604 $\gamma_{\alpha-\delta}[\beta - \alpha] = \lambda_\alpha[\beta - \alpha] \circ f_{\alpha-\delta}$. Similarly, $g'_\beta \circ \lambda_\alpha[\beta - \alpha] = \lambda'_{\alpha+\delta'}[\beta - \alpha] \circ f'_\alpha$. So $\Phi''(F' \circ$
 605 $F, G' \circ G) \in \text{Hom}^{\delta+\delta'}(\mathbf{im} \Gamma, \mathbf{im} \Lambda')$ as

$$606 \quad g'_\beta \circ (g_{\beta-\delta} \circ \gamma_{\alpha-\delta}[\beta - \alpha]) = (g'_\beta \circ \lambda_\alpha[\beta - \alpha]) \circ f_{\alpha-\delta} = \lambda_{\alpha+\delta'}[\beta - \alpha] \circ f'_\alpha \circ f_{\alpha-\delta}$$

607 for all $\alpha \leq \beta$. ◀

608 **Proof of Lemma 13.** For ease of notation let Φ denote $\Phi_M(F, G)$ and Ψ denote $\Psi_G(M, N)$.

609 If Γ is an epimorphism γ_α is surjective so $\Gamma_\alpha = V_\alpha$ and $\phi_\alpha = g_\alpha|_{\Gamma_\alpha} = g_\alpha$ for all α . So
 610 $\mathbf{im} \Gamma = \mathbb{V}$ and $\Phi \in \text{Hom}^\delta(\mathbb{V}, \mathbf{im} \Lambda)$.

611 If Π is a monomorphism then π_α is injective so we can define a natural isomorphism
 612 $\pi_\alpha^{-1} : \Pi_\alpha \rightarrow V_\alpha$ for all α . Let Ψ^* be defined as the family of linear maps $\{\psi_\alpha^* := \pi_\alpha^{-1} \circ \psi_\alpha : \Lambda_\alpha \rightarrow V_{\alpha+\delta}\}$. Because Ψ is a partial δ -interleaving of image modules, $n_\alpha \circ \lambda_\alpha = \pi_{\alpha+\delta} \circ m_\alpha$.
 614 So, because $\psi_\alpha = n_\alpha|_{\Lambda_\alpha}$ for all α ,

$$\begin{aligned} 615 \quad \mathbf{im} \psi_\alpha^* &= \mathbf{im} \pi_{\alpha+\delta}^{-1} \circ \psi_\alpha \\ 616 &= \mathbf{im} \pi^{-1} \circ (n_\alpha \circ \lambda_\alpha) \\ 617 &= \mathbf{im} \pi^{-1} \circ (\pi_{\alpha+\delta} \circ m_\alpha) \\ 618 &= \mathbf{im} m_\alpha. \end{aligned}$$

619 It follows that $\mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ \psi_\alpha^* = \mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha$

620 Similarly, because Ψ is a δ -interleaving of image modules $n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha = w_{\alpha+\delta}^{\beta+\delta} \circ \pi_{\alpha+\delta} \circ m_\alpha$.

621 Moreover, because Π is a homomorphism of persistence modules, $w_{\alpha+\delta}^{\beta+\delta} \circ \pi_{\alpha+\delta} = \pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta}$,
 622 SO

$$623 \quad n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha = \pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha.$$

624 As $\psi_\beta \circ \lambda_\alpha^\beta = n_\beta \circ \lambda_\alpha^\beta = n_\beta \circ t_\alpha^\beta|_{\Lambda_\alpha}$ it follows

$$\begin{aligned} 625 \quad \mathbf{im} \psi_\beta^* \circ \lambda_\alpha^\beta &= \mathbf{im} \pi_{\beta+\delta}^{-1} \circ (n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha) \\ 626 &= \mathbf{im} \pi_{\beta+\delta}^{-1} \circ (\pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta}) \circ m_\alpha \\ 627 &= \mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha \\ 628 &= \mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ \psi_\alpha^*. \end{aligned}$$

629 So we may conclude that $\Psi^* \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbb{V})$.

630 So $\Phi \in \text{Hom}^\delta(\mathbb{V}, \mathbf{im} \Lambda)$ and $\Psi^* \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbb{V})$. As we have shown, $\mathbf{im} \psi_{\alpha-\delta}^* =$
 631 $\mathbf{im} m_{\alpha-\delta}$ so $\mathbf{im} \phi_\alpha \circ \psi_{\alpha-\delta}^* = \mathbf{im} \phi_\alpha \circ m_{\alpha-\delta}$. Moreover, because γ_α is surjective $\phi_\alpha = g_\alpha$
 632 and, because Φ is a partial δ -interleaving of image modules, $g_\alpha \circ m_{\alpha-\delta} = t_{\alpha-\delta}^{\alpha+\delta} \circ \lambda_{\alpha-\delta}$. As
 633 $\lambda_{\alpha-\delta}^{\alpha+\delta} = t_{\alpha-\delta}^{\alpha+\delta}|_{\mathbf{im} \lambda_{\alpha-\delta}}$ it follows that $\mathbf{im} \phi_\alpha \circ \psi_{\alpha-\delta}^* = \mathbf{im} \lambda_{\alpha-\delta}^{\alpha+\delta}$.

634 Finally, $\psi_\alpha^* \circ \phi_\alpha = \pi_{\alpha+\delta}^{-1} \circ n_\alpha \circ g_{\alpha-\delta}$ where, because Ψ is a partial δ -interleaving of image
 635 modules, $n_\alpha \circ g_{\alpha-\delta} = w_{\alpha-\delta}^{\alpha+\delta} \circ \pi_{\alpha-\delta}$. Because Π is a homomorphism of persistence modules
 636 $w_{\alpha-\delta}^{\alpha+\delta} \circ \pi_{\alpha-\delta} = \pi_{\alpha+\delta} \circ v_{\alpha-\delta}^{\alpha+\delta}$. Therefore,

$$\begin{aligned} 637 \quad \psi_\alpha^* \circ \phi_\alpha &= \pi_{\alpha+\delta}^{-1} \circ n_\alpha \circ g_{\alpha-\delta} \\ 638 &= \pi_{\alpha+\delta}^{-1} \circ (\pi_{\alpha+\delta} \circ v_{\alpha-\delta}^{\alpha+\delta}) \\ 639 &= v_{\alpha-\delta}^{\alpha+\delta} \end{aligned}$$

640 which, along with $\phi_\alpha \circ \mathbf{im} \psi_{\alpha-\delta}^* = \lambda_{\alpha-\delta}^{\alpha+\delta}$ implies Diagrams ?? and ?? commute with
 641 $\Phi \in \text{Hom}^\delta(\mathbb{V}, \mathbf{im} \Lambda)$ and $\Psi^* \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbb{V})$. We may therefore conclude that $\mathbf{im} \Lambda$ and
 642 \mathbb{V} are δ -interleaved. ◀

643 A.3 Partial Interleavings

644 **Proof of Lemma 21.** Because $P_{\lfloor a \rfloor w} := P \cap D_{\lfloor a \rfloor w}$ and $B_w \subseteq D_{\lfloor a \rfloor w}$ we know $Q_w = P \cap B_w \subseteq P_{\lfloor a \rfloor w}$ for all $a \in \mathbb{R}$. So

$$646 \quad \mathcal{E}Q_a^\varepsilon = Q_a^\varepsilon \cup (D \setminus P^\varepsilon) \subseteq P_{\lfloor a \rfloor w}^\varepsilon \cup (D \setminus P^\varepsilon) = \mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon.$$

647 As $(P^\varepsilon, Q_w^\varepsilon)$ is a surrounding pair in D , P^ε is open in D and $\mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon \subseteq D$ is such that
648 $\mathcal{E}Q_a^\varepsilon \subseteq \mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon$ it follows that

$$649 \quad H_k(P_{\lfloor a \rfloor w}^\varepsilon, Q_a^\varepsilon) = H_k(P^\varepsilon \cap \mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon, Q_a^\varepsilon) \cong H_k(\mathcal{E}P_{\lfloor a \rfloor w}^\varepsilon, \mathcal{E}Q_a^\varepsilon)$$

650 by Lemma 9.

651 Because these isomorphisms commute with inclusions we have an isomorphism $\mathcal{E}_{\lfloor \cdot \rfloor w}^\varepsilon \in$
652 $\text{Hom}(\mathbb{P}_w^\varepsilon, \mathcal{E}\mathbb{P}_w^\varepsilon)$ defined to be the family $\{\mathcal{E}_{\lfloor a \rfloor w}^\varepsilon : \mathcal{P}_{\lfloor a \rfloor w}^\varepsilon \rightarrow \mathcal{E}\mathcal{P}_{\lfloor a \rfloor w}^\varepsilon\}$. \blacktriangleleft

653 **Proof of Lemma 20.** Suppose $x \in (P^\varepsilon \cap B_{\lfloor \alpha - c\varepsilon \rfloor w - c\varepsilon}) \setminus B_{w+\varepsilon}$. Because $B_{w-\varepsilon} \subset B_{w+\varepsilon}$ we
654 know $x \notin B_{w-\varepsilon}$ so $w + c\varepsilon < f(x) \leq \alpha - c\varepsilon$ and there exists some $p \in P$ such that $\mathbf{d}(x, p) < \varepsilon$.
655 Because f is c -Lipschitz it follows

$$656 \quad f(p) \leq f(x) + c\mathbf{d}(x, p) < \alpha - c\varepsilon + c\varepsilon = \alpha$$

657 and

$$658 \quad f(p) \geq f(x) - c\mathbf{d}(x, p) > w + c\varepsilon - c\varepsilon = w.$$

659 So $x \in P_{\lfloor \alpha \rfloor w}^\varepsilon$.

660 Now, suppose $x \in P_{\lfloor \alpha \rfloor w}^\varepsilon \setminus B_{w+c\varepsilon}$. So $w + c\varepsilon < f(x)$ and there exists some $p \in P_{\lfloor \alpha \rfloor w}$ such
661 that $\mathbf{d}(x, p) < \varepsilon$. Because f is c -Lipschitz it follows

$$662 \quad f(x) \leq f(p) + c\mathbf{d}(x, p) < a + c\varepsilon.$$

663 So $x \in B_{\lfloor \alpha + c\varepsilon \rfloor w + c\varepsilon} \setminus B_{w+c\varepsilon}$.

664 Because $D \setminus B_{w+c\varepsilon} \subseteq P^\varepsilon$ we know that $D \setminus P^\varepsilon \subseteq B_{w+c\varepsilon}$, so

$$665 \quad D_{\lfloor \alpha - c\varepsilon \rfloor w - c\varepsilon} \setminus B_{w+c\varepsilon} \subseteq P_{\lfloor \alpha \rfloor w}^\varepsilon \setminus B_{w+c\varepsilon} \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor w + c\varepsilon} \setminus B_{w+c\varepsilon}$$

666 implies

$$667 \quad D_{\lfloor \alpha - c\varepsilon \rfloor w - c\varepsilon} \subseteq P_{\lfloor \alpha \rfloor w}^\varepsilon \cup (D \setminus P^\varepsilon) = \mathcal{E}P_{\lfloor \alpha \rfloor w}^\varepsilon \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor w + c\varepsilon}$$

668 as desired.

669 Because f is c -Lipschitz, $B_{w-c\varepsilon} \cap P^\delta \subseteq Q_w^\varepsilon$ so $B_{w-c\varepsilon} \subseteq \mathcal{E}Q_w^\varepsilon \subseteq B_{w+c\varepsilon}$ by Lemma 8. It
670 follows that we have homomorphisms $F \in \text{Hom}^{c\varepsilon}(\mathbb{D}_{w-c\varepsilon}, \mathcal{E}\mathbb{P}_w^\varepsilon)$ and $M \in \text{Hom}^{c\varepsilon}(\mathcal{E}\mathbb{P}_w^\varepsilon, \mathbb{D}_{w+c\varepsilon})$
671 induced by inclusions. \blacktriangleleft

672 A.4 Truncated Interval Modules

673 **Proof of Lemma 18.** Suppose $\alpha \leq \omega$. So $H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega) = 0$ as $D_{\lfloor \alpha \rfloor \omega} = B_\omega \cup B_\alpha$ and
674 $\mathbb{T}_\omega^k = 0$ as $F_\alpha^I = 0$ for any $I \in \mathcal{I}^k$ such that $\omega \in I_-$. Moreover, $\omega \in I$ for all $I \in \mathcal{I}_\omega^{k-1}$, thus
675 $F_\alpha^{I,+} = 0$ for all $\alpha \leq \omega$. So it suffices to assume $\omega < \alpha$.

676 Consider the long exact sequence of the pair $H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega) = H_k(B_\alpha, B_\omega)$

$$677 \quad \dots \rightarrow H_k(B_\omega) \xrightarrow{p_\alpha^k} H_k(B_\alpha) \xrightarrow{q_\alpha^k} H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega) \xrightarrow{r_\alpha^k} H_{k-1}(B_\omega) \xrightarrow{p_\alpha^{k-1}} H_{k-1}(B_\alpha) \rightarrow \dots$$

678 where $H_k(B_\alpha) = \bigoplus_{I \in \mathcal{I}^k} F_\alpha^I$, $H_k(B_\omega) = \bigoplus_{I \in \mathcal{I}^k} F_\omega^I$, and $p_\alpha^k = \bigoplus_{I \in \mathcal{I}^k} f_{\omega, \alpha}^I$.

679 Noting that $\text{im } q_\alpha^k \cong H_k(B_\alpha)/\ker q_\alpha^k$ where $\ker q_\alpha^k = \text{im } p_\alpha^k$ by exactness we have
680 $\ker r_\alpha^k \cong H_k(B_\alpha)/\text{im } p_\alpha^k$. By the definition of F_α^I and $f_{\omega, \alpha}^I$ we know $\text{im } f_{\omega, \alpha}^I$ is F_α^I if $\omega \in I$
681 and 0 otherwise. As $\text{im } p_\alpha^k$ is equal to the direct sum of images $\text{im } f_{\omega, \alpha}^I$ over $I \in \mathcal{I}^k$ it follows
682 that $\text{im } p_\alpha^k$ is the direct sum of those F_α^I over those $I \in \mathcal{I}^k$ such that $\omega \in I$. Now, because
683 $H_k(B_\alpha) = \bigoplus_{I \in \mathcal{I}^k} F_\alpha^I$ and each F_α^I is either 0 or \mathbb{F} the quotient $H_k(B_\alpha)/\text{im } p_\alpha^k$ is the direct
684 sum of those F_α^I such that $\omega \notin I$. Therefore, by the definition of $F_{\lfloor \alpha \rfloor \omega}^I$ we have

$$685 \quad \ker r_\alpha^k = \bigoplus_{I \in \mathcal{I}_\omega^k} F_{\lfloor \alpha \rfloor \omega}^I.$$

686 Similarly, $\text{im } r_\alpha^k = \ker p_\alpha^{k-1}$ by exactness where $\ker p_\alpha^{k-1}$ is the direct sum of kernels
687 $\ker f_{\omega, \alpha}^I$ over $I \in \mathcal{I}^{k-1}$. By the definition of F_α^I and $f_{\omega, \alpha}^I$ we know that $\ker f_{\omega, \alpha}^I$ is F_α^I if
688 $\omega \notin I$ and 0 otherwise. Noting that $\ker f_{\omega, \alpha}^I = 0$ for any $I \in \mathcal{I}^{k-1}$ such that $\omega \notin I$ it suffices
689 to consider only those $I \in \mathcal{I}_\omega^{k-1}$. It follows that $\ker f_{\omega, \alpha}^I = F_\alpha^{I+}$ for any I containing ω as
690 $\omega < \alpha$. Therefore,

$$691 \quad \text{im } r_\alpha^k = \bigoplus_{I \in \mathcal{I}^{k-1}} F_\alpha^{I+}.$$

692 We have the following split exact sequence associated with r_α^k

$$693 \quad 0 \rightarrow \ker r_\alpha^k \rightarrow H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega) \rightarrow \text{im } r_\alpha^k \rightarrow 0.$$

694 The desired result follows from the fact that for all $\alpha \in \mathbb{R}$

$$695 \quad H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega) \cong \ker r_\alpha^k \oplus \text{im } r_\alpha^k \\ 696 \quad = \bigoplus_{I \in \mathcal{I}^k} F_{\lfloor \alpha \rfloor \omega}^I \oplus \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} F_\alpha^{I+}.$$

697

698 B Duality

699 For a pair (A, B) in a topological space X and any R module G let $H^k(A, B; G)$ denote
700 the **singular cohomology** of (A, B) (with coefficients in G). Let $H_c^k(A, B; G)$ denote
701 the corresponding **singular cohomology with compact support**, where $H_c^k(A, B; G) \cong$
702 $H^k(A, B; G)$ for any compact pair (A, B) .

703 The following corollary follows from the Universal Coefficient Theorem for singular
704 homology (and cohomology) as vector spaces over a field \mathbb{F} , as the dual vector space
705 $\text{Hom}(H_k(A, B), \mathbb{F})$ is isomorphic to $H_k(A, B; \mathbb{F})$ for any finitely generated $H_k(A, B)$.

706 ▶ **Corollary 23.** *For a topological pair (A, B) and a field \mathbb{F} such that $H_0(A, B)$ is finitely
707 generated there is a natural isomorphism*

$$708 \quad \nu : H^0(A, B; \mathbb{F}) \rightarrow H_0(A, B; \mathbb{F}).$$

709 Let $\overline{H}^k(A, B; G)$ be the **Alexander-Spanier cohomology** of the pair (A, B) , defined
710 as the limit of the direct system of neighborhoods (U, V) of the pair (A, B) . Let $\overline{H}_c^k(A, B; G)$
711 denote the corresponding **Alexander-Spanier cohomology with compact support**
712 where $\overline{H}_c^k(A, B; G) \cong \overline{H}^k(A, B; G)$ for any compact pair (A, B) .



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713 ► **Theorem 24 (Alexander-Poincaré-Lefschetz Duality** (Spanier [11], Theorem 6.2.17)). Let
714 X be an orientable d -manifold and (A, B) be a compact pair in X . Then for all k and R
715 modules G there is a (natural) isomorphism

$$\lambda : H_k(X \setminus B, X \setminus A; G) \rightarrow \bar{H}^{d-k}(A, B; G).$$

717 A space X is said to be **homologically locally connected in dimension n** if for
718 every $x \in X$ and neighborhood U of x there exists a neighborhood V of x in U such that
719 $\tilde{H}_n(V) \rightarrow \tilde{H}_n(U)$ is trivial for $k \leq n$.

720 ► **Lemma 25** (Spanier p. 341, Corollary 6.9.6). Let A be a closed subset, homologically
721 locally connected in dimension n , of a Hausdorff space X , homologically locally connected in
722 dimension n . If X has the property that every open subset is paracompact, $\mu : \bar{H}_c^k(X, A; G) \rightarrow$
723 $H_c^k(X, A; G)$ is an isomorphism for $k \leq n$ and a monomorphism for $k = n + 1$.

724 In the following we will assume homology (and cohomology) over a field \mathbb{F} .

725 ► **Lemma 26.** Let X be an orientable d -manifold and (A, B) a compact pair of locally path
726 connected subspaces in X . Then

$$\xi : H_d(X \setminus B, X \setminus A) \rightarrow H_0(A, B)$$

728 is a natural isomorphism.

729 **Proof.** Because X is orientable and (A, B) are compact $\lambda : H_d(X \setminus B, X \setminus A) \rightarrow \bar{H}^0(A, B)$
730 is an isomorphism by Theorem 24. Note that Moreover, because every subset of X is
731 (hereditarily) paracompact every open set in A , with the subspace topology, is paracompact.
732 For any neighborhood U of a point x in a locally path connected space there must exist some
733 neighborhood $V \subset U$ of x that is path connected in the subspace topology. As $\tilde{H}_0(V) = 0$
734 for any nonempty, path connected topological space V (see Spanier p. 175, Lemma 4.4.7)
735 it follows that A (resp. B) are homologically locally connected in dimension 0. Because
736 (A, B) is a compact pair the singular and Alexander-Spanier cohomology modules of (A, B)
737 with compact support are isomorphic to those without, thus $\mu : \bar{H}^0(A, B) \rightarrow H^0(A, B)$ is an
738 isomorphism. By Corollary 23 we have a natural isomorphism $\nu : H^0(A, B) \rightarrow H_0(A, B)$ thus
739 the composition $\xi := \nu \circ \mu \circ \lambda : H_d(X \setminus B, X \setminus A) \rightarrow H_0(A, B)$ is a natural isomorphism. ◀

740 ► **Lemma 27.** Let \mathbb{X} be an orientable d -manifold let D be a compact subset of \mathbb{X} . Let P be
741 a finite subset of D such that $P^\varepsilon \subset \text{int}_{\mathbb{X}}(D)$ and $Q \subseteq P$.

742 If $D \setminus Q^\varepsilon$ and $D \setminus P^\varepsilon$ are locally path connected then there is a natural isomorphism

$$\xi : H_d(P^\varepsilon, Q^\varepsilon) \rightarrow H_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon).$$

744 **Proof.** Because Q^ε and P^ε are open in D and D is compact in \mathbb{X} the complement $D \setminus Q^\varepsilon$
745 is closed in D , and therefore compact in \mathbb{X} . Moreover, because $P^\varepsilon \subset \text{int}_{\mathbb{X}}(D)$, $H_d(\mathbb{X} \setminus (D \setminus$
746 $P^\varepsilon), \mathbb{X} \setminus (D \setminus Q^\varepsilon)) = H_d(P^\varepsilon, Q^\varepsilon)$. As we have assumed these complements are locally path
747 connected by assumption we have a natural isomorphism $\xi : H_d(P^\varepsilon, Q^\varepsilon) \rightarrow H_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon)$
748 by Lemma 26. ◀