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From Coverage Testing to Topological Scalar Field Analysis

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Abstract

The topological coverage criterion (TCC) can be used to test whether an underlying space is sufficiently well covered by a given data set. Given a sufficiently dense sample, topological scalar field analysis (SFA) can give a summary of the shape of a real-valued function on a space. The goal of this paper is to put these theories together so that one can test coverage with the TCC and then compute a summary with SFA. The challenge is that the TCC requires a well-defined boundary that is not generally available in the SFA settings. To overcome this, we show how the scalar field itself can be used to define a boundary that can then be used to confirm coverage. This requires an interpretation of the TCC that resolves one of the major barriers to wider use. It also extends SFA methods to the setting in which coverage is only confirmed in a subset of the domain. We show how the intersection of these two theories can be used to approximate the persistent homology relative to a static sublevel set. We then discuss how this persistent relative homology relates to that of the scalar field as a whole.

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1 Introduction

In the topological analysis of scalar fields (SFA), one computes a topological summary capturing qualitative and quantitative shape information from a set of points endowed with a metric and a real-valued function. That is, we have points with distances and a real number assigned to each point. More generally, it suffices to have a neighborhood graph on the points identifying the pairs of points within some distance. The topological computation uses persistent homology to integrate local information from the function into global information about its *behavior* as whole. In prior work, Chazal et al. [3] showed that for sufficiently dense samples on sufficiently smooth spaces, the persistence diagram can be computed with some guarantees. In followup work, Buchet et al. [1] extended this result to show how to work with noisy inputs. A fundamental assumption required to have strong guarantees on the output of these methods is that the underlying space be sufficiently well-sampled. In this paper, we show how to combine scalar field analysis with the theory of topological coverage testing to simultaneously compute the persistence diagram and also to test that the underlying space is sufficiently well-sampled.

Initiated by De Silva and Ghrist [6, 4, 5], the theory of homological sensor networks addresses the problem of testing coverage of a bounded domain by a collection of sensors without coordinates. The main result is the topological coverage criterion, which, in its most general form, states that under reasonable geometric assumptions, the d-dimensional homology of a pair of simplicial complexes built on the neighborhood graph will be nontrivial

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if and only if there is sufficient coverage (see Section 3 for the precise statements). This relative persistent homology test is called the Topological Coverage Criterion (TCC).

Superficially, the methods of SFA and TCC are very similar. Both construct similar complexes and compute the persistent homology of the homological image of a complex on one scale into that of a larger scale. They even overlap on some common techniques in their analysis including the use of the Nerve theorem and the Rips-Čech interleaving. However, they differ in some fundamental way that makes it difficult to combine them into a single technique. The main difference is that the TCC requires a clearly defined boundary. Not only must the underlying space be a bounded subset of $\mathbb{R}^d$, the data must also be labeled to indicate which input points are close to the boundary. This requirement is perhaps the main reason why the TCC can so rarely be applied in practice.

In applications to data analysis it is more natural to assume that our data measures some unknown function. By requiring that our function is related to the metric of the space we can replace this requirement with assumptions about the function itself. Indeed, these assumptions could relate the behavior of the function to the topological boundary of the space. However, the generalized approach by Cavanna et al. [2] allows much more freedom in how the boundary is defined.

We consider the case in which we have incomplete data from a particular sublevel set of our function. Our goal is to isolate this data so we can analyze the function only in the verified region. From this perspective, the TCC confirms that we not only have coverage, but that the sample we have is topologically representative of the region near, and above this sublevel set. We can then re-use the same machinery to analyze a *part* of the function in a specific way.

1.0.0.1 Contribution

We will re-cast the TCC as a way to verify that the persistent homology of a scalar field can be *partially* approximated by a given sample. Specifically, we will relate the persistent homology of a function relative to a *static* sublevel set to a *truncation* of the full diagram. That is, beyond a certain point the full diagram remains unchanged, allowing for possible reconstruction. This is in comparison with the *restricted* diagram obtained by simply ignoring part of the domain. We therefore present relative persistent homology as an alternative to restriction in a way that extends the TCC to the analysis of scalar fields.

We will first provide some background on important topological, geometric, and algebraic structures required for our re-formulation of the TCC, and the approximation of the relative diagram. Section 2 establishes notation and provides an overview of our main results in Sections 3 and 4.In Section 5 we introduce an interpretation of the relative diagram as a truncation of the full diagram that is motivated by a number of experiments in Section **??**.

2 Summary

Let $\mathbb{X}$ denote an orientable d-manifold and $D \subset \mathbb{X}$ a compact subspace. For a c-Lipschitz functon $f : D \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ let $B_\alpha := f^{-1}((-\infty, \alpha])$ denote the α-sublevel set of f. Our sample will be denoted P, and the subset of points sampling B_α will be denoted $Q_\alpha := P \cap B_\alpha$. For ease of exposition let

$$D_{\lfloor \alpha \rfloor_w} := B_\alpha \cup B_w$$

to be the *truncated* α sublevel set and

$$P_{\lfloor \alpha \rfloor_w} := Q_\alpha \cup Q_w$$

denote is sampled counterpart for all $\alpha, w \in \mathbb{R}$.

We will select a sublevel set B_ω to serve as our boundary. Specifically, we require that B_ω *surrounds* D, where the notion of a surrounding set is defined formally in Section 3. This distinction allows us to generalize the standard proof of the TCC to properties of surrounding pairs.

2.0.0.1 Relative, Truncated, and Restricted Persistence Diagrams

For fixed $\omega \in \mathbb{R}$ we will refer to the persistence diagram associated with the filtration $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$ as the **relative diagram** of f. In Section 5 we relate the relative diagram to the *full* diagram of the sublevel set filtration $\{B_\alpha\}_{\alpha\in\mathbb{R}}$. Specifically, we define the **truncated diagram** to be the subdiagram consisting of features born *after* ω in the full diagram. In Section **??** we compare the relative and truncated diagrams to the **restricted diagram**, defined to be that of the sublevel set filtration of $f|_{D\setminus B_\omega}$.

Note that the truncated sublevel sets $D_{\lfloor\alpha\rfloor_\omega}$ are equal to the union of B_ω and the restricted sublevel sets. It is in this sense that B_ω is *static* throughout—it is contained in every sublevel set of the relative filtration. As we will not have verified coverage in B_ω we cannot analyze the function in this region directly. We therefore have two alternatives: *restrict* the domain of the function to $D \setminus B_\omega$, or use relative homology to analyze the function *relative* to this region using excision.

2.0.0.2 Results

Suppose B_ω surrounds D in $\mathbb{X}$ and $\delta < \varrho/4$. As a minimal assumption we require that every component of $D \setminus B_\omega$ contains a point in P. We also make additional technical assumptions on P and δ with respect to the pair (D, B_ω) (see Section **??** and Lemma 24 of Appendix A).

Theorem 8 If

I. $\mathrm{H}_0(D \setminus B_{\omega+5c\delta} \hookrightarrow D \setminus B_\omega)$ is *surjective*,
II. $\mathrm{H}_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-3c\delta})$ is *injective*,

and

$$\mathbf{rk}\ \mathrm{H}_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$$

then $D \setminus B_\omega \subseteq P^\delta$ and $Q^\delta_{\omega-2c\delta}$ surrounds P^δ in D. [1]

This formulation of the TCC states that our approximation by a nested pair of Rips complexes not only covers, but captures the homology of the pair (D, B_ω) in a specific way. We use this fact to interleave our sample with the relative diagram of the filtration $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$. This is done by generalizing our regularity assumptions near $D \setminus B_\omega$ in a way that allows us to interleave persistence modules with static components.

Theorem 17 Suppose P satisfies the the TCC: $D \setminus B_\omega \subseteq P^\delta$ and $Q^\delta_{\omega-2c\delta}$ surrounds P^δ in D. If

I. $\mathrm{H}_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is *surjective* and
II. $\mathrm{H}_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an *isomorphism*

[1] We state this result using constants that will be used to prove the interleaving. The statement of Theorem 8 parameterizes the region around ω in terms of $\zeta \geq \delta$ as $[\omega - c(\delta+\zeta), \omega + c(\delta+\zeta)]$.

for all k the persistent homology modules of

$$\{\mathcal{R}^{2\delta}(P_{\lfloor\alpha\rfloor_{\omega-2c\delta}}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor\alpha\rfloor_{\omega+c\delta}}, Q_{\omega+c\delta})\}_{\alpha\in\mathbb{R}}$$

and $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$ are $4c\delta$ interleaved.

The main challenges we face come from the fact that the sublevel set B_ω and our approximation by the inclusion $\mathcal{R}^{2\delta}(Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(Q_{\omega+c\delta})$ remain *static* throughout. Using the fact that $Q^\delta_{\omega-2c\delta}$ surrounds P^δ in D we define an *extension* $(D, \mathcal{E}Q^\delta_{\omega-2c\delta})$ of the pair $(P^\delta, Q^\delta_{\omega-2c\delta})$ that has isomorphic relative homology by excision. These extensions give us a sequence of inclusion maps

$$B_{\omega-3c\delta} \hookrightarrow \mathcal{E}Q^{2\delta}_{\omega-2c\delta} \hookrightarrow B_\omega \hookrightarrow \mathcal{E}Q^{4\delta}_{\omega+c\delta} \hookrightarrow B_{\omega+5c\delta}$$

that can be used along with our regularity assumptions to prove the interleaving.

2.0.0.3 Outline of Sections 3 and 4

We will begin with our reformulation of the TCC in Section 3. This requires the introduction of a surrounding set before proving the Geometric TCC (Theorem **??**), followed by the computable Algorithmic TCC (Theorem 8). Section 4 formally introduces extensions and partial interleavings of image modules which will be used in the proof of Theorem 17.

3 The Topological Coverage Criterion (TCC)

The TCC uses the top-dimensional relative homology of a space D with respect to its boundary B to provide a computable condition for coverage. Under certain conditions Alexander Duality can be used to relate this to the number of connected components of its complement in some larger space $\mathbb{X}$. One can then check if a collection of subsets covers the space by comparing the number of connected components of its complement to that of the space—holes in the cover will appear as additional components in the complement space. As we cannot compute the number components of the complement space from a sample directly, the TCC uses duality to recover it from the top dimensional relative homology. However, this requires that we have a subset Q of our cover to serve as the boundary. That is, a positive result indicates that we not only have coverage, but also that we have a pair of spaces that reflects the pair (D, B) topologically. We call such a pair a *surrounding pair* defined in terms of separating sets.

It has been shown that the TCC can be stated in terms of these surrounding pairs [2], which allows us enough flexibility to define our surrounding set as a sublevel set of a c-Lipschitz function f. Moreover, this work made assumptions directly in terms of the *zero dimensional* persistent homology of the domain close to the boundary. This is in comparison to the original statement of the TCC which required the boundary to be smooth [4]. Now, we can state these assumptions in terms of the persistent homology of the function itself.

This opens the door to a wide array of applications in which we have either a small portion or a very rough approximation of the persistence diagram of a function. We can then either identify or verify a threshold ω for our sublevel set that satisfies our assumptions. As we will see these same assumptions will appear as statements about persistent homology in *all dimensions* in our proof of the interleaving. It is in this sense that the approximation of a truncated diagram can be seen as a generalization of the TCC to the topological analysis of scalar fields with guarantees.

▶ **Definition 1** (Surrounding Pair)**.** *Let X be a topological space and (D, B) a pair in X. The set B **surrounds** D **in** X if B separates X with the pair $(D \setminus B, X \setminus D)$. We will refer to such a pair as a **surrounding pair in** X.*

Unlike the definition of a separating set, which simply breaks a space into disjoint subsets, we make the distinction between interior and exterior explicit by defining the subset B relative D. That is, the set $D \setminus B$ corresponds to the interior of D and $X \setminus D$ corresponds to the complement of D in X. B then serves as a boundary in the sense that there is no path from the "interior" to the "complement," which is sufficient for a homological coverage criterion.

Let (D, B) be a surrounding pair in X and $U \subseteq D$, $V \subseteq U \cap B$ be subsets. Let $\ell : \mathrm{H}_0(X \setminus B, X \setminus D) \to \mathrm{H}_0(X \setminus V, X \setminus U)$ be induced by inclusion. The following lemma generalizes the proof of the TCC as properties of surrounding sets.

▶ **Lemma 2.** *If ℓ is injective then $D \setminus B \subseteq U$ and V surrounds U in D.*

Proof. (See Appendix B) ◀

▶ **Definition 3** (Surrounding Pair)**.** *Let X be a topological space and (D, B) a pair in X. The set B **surrounds** D **in** X if B separates X with the pair $(D \setminus B, X \setminus D)$. We will refer to such a pair as a **surrounding pair in** X.*

Let (D, B) be a surrounding pair in X and $U \subseteq D$, $V \subseteq U \cap B$ be subsets. Let $\ell : \mathrm{H}_0(X \setminus B, X \setminus D) \to \mathrm{H}_0(X \setminus V, X \setminus U)$ be induced by inclusion. The following lemma generalize the proof of the TCC as properties of surrounding sets, its proof can be found in the full version of this paper.

▶ **Lemma 4.** *If ℓ is injective then $D \setminus B \subseteq U$ and V surrounds U in D.*

We now combine these results on the homology of surrounding pairs with information about both $\mathbb{X}$ as a metric space and our function. Let $(\mathbb{X}, \mathbf{d})$ be a metric space and $D \subseteq \mathbb{X}$ be a compact subspace. Let $f : D \to \mathbb{R}$ be a c-Lipschitz function and $B_\alpha = f^{-1}((-\infty, a])$ denote the α-sublevel set of f for $\alpha \in \mathbb{R}$. We introduce a constant ω as a threshold that defines our "boundary" as a sub-levelset of the function f. Let P be a finite subset of D and $Q_\alpha := P \cap B_\alpha$ for $\alpha \in \mathbb{R}$. Let $\zeta \geq \delta > 0$ and $\omega \in \mathbb{R}$ be constants such that $P^\delta \subseteq D$. Here, δ will serve as our communication radius where ζ is reserved for use in Section 4. [2]

▶ **Lemma 5.** *Let $i : \mathrm{H}_0(\overline{Q^\delta_{\omega+c\delta}}, \overline{P^\delta}) \to \mathrm{H}_0(\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta})$.*
If B_ω surrounds D in $\mathbb{X}$ then $\mathbf{dim}\ \mathrm{H}_0(\overline{B_\omega}, \overline{D}) \geq \mathbf{rk}\ i$.

Proof. Choose a basis for $\mathbf{im}\ i$ such that each basis element is represented by a point in $P^\delta \setminus Q^\delta_{\omega+c\delta}$. Let $x \in P^\delta \setminus Q^\delta_{\omega+c\delta}$ be such that $i[x] \neq 0$. So there exits some $p \in P$ such that $\mathbf{d}(p, x) < \delta$ and $p \notin Q_{\omega+c\delta}$, otherwise $x \in Q^\delta_{\omega+c\delta}$. Therefore, because f is c-Lipschitz,

$$f(x) \geq f(p) - c\mathbf{d}(x, p) > \omega + c\delta - c\delta = \omega.$$

So $x \in \overline{B_\omega}$ and, because $x \in P^\delta \subseteq D$, $x \in D \setminus B_\omega$. Because i and $\ell : \mathrm{H}_0(\overline{B_\omega}, \overline{D}) \to \mathrm{H}_0(\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta})$ are induced by inclusion $\ell[x] = i[x] \neq 0$ in $\mathrm{H}_0(\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta})$. That is, every element of $\mathbf{im}\ i$ has a preimage in $\mathrm{H}_0(\overline{B_\omega}, \overline{D})$, so we may conclude that $\mathbf{dim}\ \mathrm{H}_0(\overline{B_\omega}, \overline{D}) \geq \mathbf{rk}\ i$. ◀

[2] We will set $\zeta = 2\delta$ in the proof of our interleaving with Rips complexes but the TCC holds for all $\zeta \geq \delta$.

Note that, while there is a surjective map from $\mathrm{H}_0(\overline{B_\omega}, \overline{D})$ to $\mathbf{im}\ i$ this map is not necessarily induced by inclusion, as $Q^\delta_{\omega+c\delta} \not\subseteq B_\omega$. We therefore must introduce a larger space $B_{\omega+c(\delta+\zeta)}$ that contains $Q^\delta_{\omega+c\delta}$ in order to provide a criteria for the injectivity of $\ell : \mathrm{H}_0(\overline{B_\omega}, \overline{D}) \to \mathrm{H}_0(\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta})$ in terms of $\mathbf{rk}\ i$.

$$\begin{tikzcd}
(P^\delta, Q^\delta_{\omega-c\zeta}) \arrow[r, hook] \arrow[d, hook] & (P^\delta, Q^\delta_{\omega+c\delta}) \arrow[d, hook] \\
(D, B_\omega) \arrow[r, hook] & (D, B_{\omega+c(\delta+\zeta)}),
\end{tikzcd}
\quad
\begin{tikzcd}
(\overline{B_{\omega+c(\delta+\zeta)}}, \overline{D}) \arrow[r, hook] \arrow[d, hook] & (\overline{B_\omega}, \overline{D}) \arrow[d, hook] \\
(\overline{Q^\delta_{\omega+c\delta}}, \overline{P^\delta}) \arrow[r, hook] & (\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta}).
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{H}_0(\overline{B_{\omega+c(\delta+\zeta)}}, \overline{D}) \arrow[r, "j"] \arrow[d, "m"] & \mathrm{H}_0(\overline{B_\omega}, \overline{D}) \arrow[d, "\ell"] \\
\mathrm{H}_0(\overline{Q^\delta_{\omega+c\delta}}, \overline{P^\delta}) \arrow[r, "i"] & \mathrm{H}_0(\overline{Q^\delta_{\omega-c\zeta}}, \overline{P^\delta}).
\end{tikzcd} \tag{1}$$

3.0.0.1 Assumptions

We will first require the map $\mathrm{H}_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ to be *surjective*—as we approach ω from *above* no components *appear*. This ensures that the rank of the map j is equal to the dimension of $\mathbf{dim}\ \mathrm{H}_0(\overline{B_\omega}, \overline{D})$ so our map ℓ induced by inclusion depends only on $\mathrm{H}_0(\overline{B_\omega}, \overline{D})$ and $\mathbf{im}\ i$.

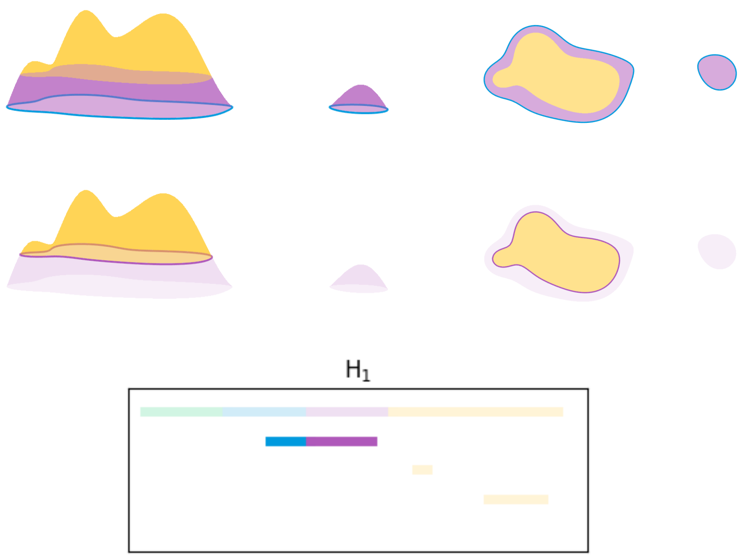


Figure 1 (Assumption 1) The blue levelset does not satisfy Assumption 1 as the smaller component is “pinched out” in the orange region.

We also assume that $\mathrm{H}_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is *injective*—as we move away from ω moving *down* no components *disappear*. Lemma 6 uses Assumption 2 to provide a computable upper bound on $\mathbf{rk}\ j$, its proof can be found in the full version of this paper.

▶ **Lemma 6.** *If $\mathrm{H}_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega+c(\delta+\zeta)})$ is injective and each component of $D \setminus B_\omega$ contains a point in P then* $\mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \mathbf{dim}\ \mathrm{H}_0(D \setminus B_\omega)$.

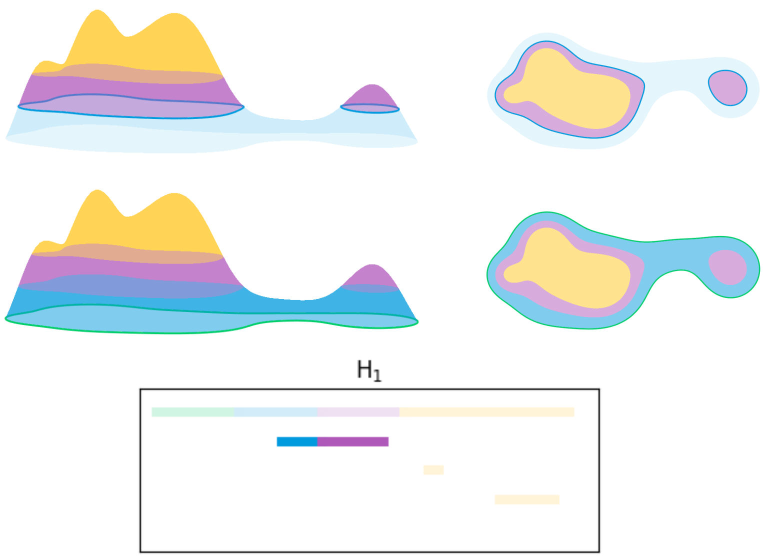


Figure 2 (Assumption 2) The blue levelset does not satisfy Assumption 2 as the smaller component is not in the inclusion from blue to green.

The full version of this paper details how to construct the following isomorphism using the Nerve Theorem along with Alexander Duality and the Universal Coefficient Theorem.

$$\xi\mathcal{N}_w^{\varepsilon,k} : \mathrm{H}_d(\check{\mathcal{C}}^\varepsilon(P, Q_w)) \to \mathrm{H}_0(D \setminus Q_w^\varepsilon, D \setminus P^\varepsilon).$$

This isomorphism holds in the specific case when $P^\varepsilon \subseteq \mathbf{int}_{\mathbb{X}}(D)$ and $D \setminus P^\varepsilon$, $D \setminus Q_w^\varepsilon$ are locally contractible. We therefore provide the following definition for ease of exposition

▶ **Definition 7** ((δ, ζ, ω)-Sublevel Sample). *For $\zeta \geq \delta > 0$, $\omega \in \mathbb{R}$, and a c-Lipschitz function $f : D \to \mathbb{R}$ a finite point set $P \subset D$ is said to be a (δ, ζ, ω)**-sublevel sample** of f if every component of $D \setminus B_\omega$ contains a point in P, $P^\delta \subset \mathbf{int}_{\mathbb{X}}(D)$, and $D \setminus P^\delta$, $D \setminus Q^\delta_{\omega - c\zeta}$, and $D \setminus Q^\delta_{\omega + c\delta}$ are locally path connected in $\mathbb{X}$.*

▶ **Theorem 8** (Algorithmic TCC). *Let $\mathbb{X}$ be an orientable d-manifold and let D be a compact subset of $\mathbb{X}$. Let $f : D \to \mathbb{R}$ be c-Lipschitz function and $\omega \in \mathbb{R}$ and $\delta \leq \zeta < \varrho_D$ be constants such that $P \subset D$ is a (δ, ζ, ω)-sublevel sample of f and $B_{\omega - c(\zeta+\delta)}$ surrounds D in $\mathbb{X}$.*

If $\mathrm{H}_0(D \setminus B_{\omega + c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ is surjective, $\mathrm{H}_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega + c(\delta+\zeta)})$ is injective, and

$$\mathbf{rk}\ \mathrm{H}_d(\mathcal{R}^\delta(P, Q_{\omega - c\zeta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega + c\delta})) \geq \mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega - c\zeta}))$$

then $D \setminus B_\omega \subseteq P^\delta$ and $Q^\delta_{\omega - c\zeta}$ surrounds P^δ in D.

Proof. Because P is a (δ, ζ, ω)-sublevel sample we have isomorphisms $\xi\mathcal{N}^\delta_{\omega - c\zeta}$ and $\xi\mathcal{N}^\delta_{\omega + c\delta}$ that commute with $q_{\check{\mathcal{C}}}$ and $i : \mathrm{H}_0(D \setminus Q^\delta_{\omega + c\delta}, D \setminus P^\delta) \to \mathrm{H}_0(D \setminus Q^\delta_{\omega - c\zeta}, D \setminus P^\delta)$. Let $q_{\mathcal{R}} : \mathrm{H}_d(\mathcal{R}^\delta(P, Q_{\omega - c\zeta})) \to \mathrm{H}_d(\mathcal{R}^{2\delta}(P, Q_{\omega + c\delta}))$ be induced by inclusion. Then $\mathbf{rk}\ q_{\check{\mathcal{C}}} \geq \mathbf{rk}\ q_{\mathcal{R}}$ as $q_{\mathcal{R}}$ factors through $q_{\check{\mathcal{C}}}$. As we have assumed $\mathrm{H}_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega - c(\delta+\zeta)})$ Lemma 6 implies $\mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega - c\zeta})) \geq \mathbf{dim}\ \mathrm{H}_0(D \setminus B_\omega)$. It follows that, whenever $\mathbf{rk}\ q_{\mathcal{R}} \geq \mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega - c\zeta}))$, we have

$$\mathbf{rk}\ i = \mathbf{rk}\ q_{\check{\mathcal{C}}} \geq \mathbf{rk}\ q_{\mathcal{R}} \geq \mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega - c\zeta})) \geq \mathbf{dim}\ \mathrm{H}_0(D \setminus B_\omega).$$

Because j is surjective by hypothesis $\mathbf{rk}\ j = \mathbf{dim}\ \mathrm{H}_0(\overline{B_\omega}, \overline{D}) = \mathbf{dim}\ \mathrm{H}_0(D \setminus B_\omega)$ so $\mathbf{rk}\ j \geq \mathbf{rk}\ i$ by Lemma 5. Therefore, $\mathbf{rk}\ j = \mathbf{rk}\ i$ as we have shown $\mathbf{rk}\ i \geq \mathbf{dim}\ \mathrm{H}_0(D \setminus B_\omega)$. Because P is a finite point set we know that $\mathbf{im}\ i$ is finite-dimensional and, because $\mathbf{rk}\ i = \mathbf{rk}\ j$, $\mathbf{im}\ j = \mathrm{H}_0(\overline{B_\omega}, \overline{D})$ is finite dimensional as well. So $\mathbf{im}\ j$ is isomorphic to $\mathbf{im}\ i$ as a subspace of $\mathrm{H}_0(\overline{Q^\delta_{\omega - c\zeta}}, \overline{P^\delta})$ which, because j is surjective, requires the map ℓ induced by inclusion to be injective. Therefore $D \setminus B_\omega \subseteq P^\delta$ and $Q^\delta_{\omega - c\zeta}$ surrounds P^δ in D by Lemma 4. ◀

4 From Coverage Testing to the Analysis of Scalar Fields

Because the TCC only confirms coverage of a *superlevel* set $D \setminus B_\omega$, we cannot guarantee coverage of the entire domain. Indeed, we could compute the persistent homology of the *restriction* of f to the superlevel set we cover in the standard way [3]. Instead, we will approximate the persistent homology of the sublevel set filtration *relative to* the sublevel set B_ω.

■ **Figure 3** Full, restricted, and relative barcodes of the function (left).

We will first introduce the notion of an extension which will provide us with maps on relative homology induced by inclusion via excision. However, even then, a map that factors through our pair (D, B_ω) is not enough to prove an interleaving of persistence modules by inclusion directly. To address this we impose conditions on sublevel sets near B_ω which generalize the assumptions made in the TCC on maps induced by the inclusions

$$D \setminus B_{\omega + c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega \hookrightarrow D \setminus B_{\omega - c(\delta+\zeta)}$$

on 0-dimensional homology, to assumptions on maps induced by the corresponding inclusions

$$B_{\omega - c(\delta+\zeta)} \hookrightarrow B_\omega \hookrightarrow B_{\omega + c(\delta+\zeta)}$$

on homology in all dimensions k.

4.1 Extensions and Image Persistence Modules

Suppose D is a subspace of X. We define the extension of a surrounding pair in D to a surrounding pair in X with isomorphic relative homology.

▶ **Definition 9** (Extension). *If V surrounds U in a subspace D of X let $\mathcal{E}V := V \sqcup (D \setminus U)$ denote the (disjoint) union of the separating set V with the complement of U in D. The* ***extension of*** *(U, V)* ***in*** *D is the pair $(D, \mathcal{E}V) = (U \sqcup (D \setminus U), V \sqcup (D \setminus U))$.*

Lemma states that we can use these extensions to interleave a pair (U, V) with a sequence of subsets of (D, B). Lemma we can apply excision to the relative homology groups in order to get equivalent maps on homology that are induced by inclusions. Proof of these facts, and ther extensions to homomorphisms of persistence modules in the next section, can be found in the full version of this paper.

▶ **Lemma 10.** *Suppose V surrounds U in D and $B' \subseteq B \subset D$.*
If $D \setminus B \subseteq U$ and $U \cap B' \subseteq V \subseteq B'$ then $B' \subseteq \mathcal{E}V \subseteq B$.

▶ **Lemma 11.** *Let (U, V) be an open surrounding pair in a subspace D of X.*
Then $\mathrm{H}_k(U \cap A, V) \hookrightarrow (A, \mathcal{E}V))$ is an isomorphism for all k and $A \subseteq D$ with $\mathcal{E}V \subset A$.

▶ **Definition 12** (Image Persistence Module)**.** *The **image persistence module** of a homomorphism $\Gamma \in \mathrm{Hom}(\mathbb{U}, \mathbb{V})$ is the family of subspaces $\{\Gamma_\alpha := \mathbf{im}\ \gamma_\alpha\}$ in $\mathbb{V}$ along with linear maps $\{\gamma_\alpha^\beta := v_\alpha^\beta|_{\mathbf{im}\ \gamma_\alpha} : \Gamma_\alpha \to \Gamma_\beta\}$ and will be denoted by $\mathbf{im}\ \Gamma$.*

▶ **Definition 13** (Image Module Homomorphism)**.** *Given $\Gamma \in \mathrm{Hom}(\mathbb{U}, \mathbb{V})$ and $\Lambda \in \mathrm{Hom}(\mathbb{S}, \mathbb{T})$ along with $(F, G) \in \mathrm{Hom}^\delta(\mathbb{U}, \mathbb{S}) \times \mathrm{Hom}^\delta(\mathbb{V}, \mathbb{T})$ let $\Phi(F, G) : \mathbf{im}\ \Gamma \to \mathbf{im}\ \Lambda$ denote the family of linear maps $\{\phi_\alpha := g_\alpha|_{\Gamma_\alpha} : \Gamma_\alpha \to \Lambda_{\alpha+\delta}\}$. $\Phi(F, G)$ is an **image module homomorphism of degree** δ if the following diagram commutes for all $\alpha \leq \beta$.*[3]

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\gamma_\alpha[\beta-\alpha]} & V_\beta \\ \downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle g_\beta} \\ S_{\alpha+\delta} & \xrightarrow{\lambda_{\alpha+\delta}[\beta-\alpha]} & T_{\beta+\delta} \end{array} \tag{2}$$

The space of image module homomorphisms of degree δ between $\mathbf{im}\ \Gamma$ and $\mathbf{im}\ \Lambda$ will be denoted $\mathrm{Hom}^\delta(\mathbf{im}\ \Gamma, \mathbf{im}\ \Lambda)$.

The composition of image module homomorphisms are image module homomorphisms. Proof of this fact can be found in the full version of this paper.

4.1.0.1 Partial Interleavings of Image Modules

Image module homomorphisms introduce a direction to the traditional notion of interleaving. As we will see, our interleaving via Lemma 15 involves partially interleaving an image module to two other image modules whose composition is isomorphic to our target.

▶ **Definition 14** (Partial Interleaving of Image Modules)**.** *An image module homomorphism $\Phi(F, G)$ is a **partial δ-interleaving of image modules**, and denoted $\Phi_M(F, G)$, if there exists $M \in \mathrm{Hom}^\delta(\mathbb{S}, \mathbb{V})$ such that $\Gamma[2\delta] = M \circ F$ and $\Lambda[2\delta] = G \circ M$.*

uses partial interleavings surrounding a module $\mathbb{V}$ to prove an interleaving of an image module with $\mathbb{V}$. Its proof is straightforward and can be found in the full version of this paper. It uses partial interleavings of a map Λ with $\mathbb{U} \to \mathbb{V}$ and $\mathbb{V} \to \mathbb{W}$ along with the hypothesis that $\mathbb{U} \to \mathbb{W}$ is isomorphic to $\mathbb{V}$ to interleave $\mathbf{im}\ \Lambda$ with $\mathbb{V}$. When applied, this hypothesis will be satisfied by assumptions on our sublevel set similar to those made in the TCC.

▶ **Lemma 15.** *Suppose $\Gamma \in \mathrm{Hom}(\mathbb{U}, \mathbb{V})$, $\Pi \in \mathrm{Hom}(\mathbb{V}, \mathbb{W})$, and $\Lambda \in \mathrm{Hom}(\mathbb{S}, \mathbb{T})$.*
If $\Phi_M(F, G) \in \mathrm{Hom}^\delta(\mathbf{im}\ \Gamma, \mathbf{im}\ \Lambda)$ and $\Psi_G(M, N) \in \mathrm{Hom}^\delta(\mathbf{im}\ \Lambda, \mathbf{im}\ \Pi)$ are partial δ-interleavings of image modules such that Γ is a epimorphism and Π is a monomorphism then $\mathbf{im}\ \Lambda$ is δ-interleaved with $\mathbb{V}$.

[3] Recall that $\gamma_\alpha[\beta - \alpha] = v_\alpha^\beta \circ \gamma_\alpha$ and $\lambda_\alpha[\beta - \alpha] = t_\alpha^\beta \circ \lambda_\alpha$.

4.1.0.2 Proof of the Interleaving

For $w, \alpha \in \mathbb{R}$ let $\mathbb{D}_w^k$ denote the kth persistent (relative) homology module of the filtration $\{(D_{\lfloor \alpha \rfloor_w}, B_w)\}_{\alpha\in\mathbb{R}}$ with respect to B_w, and let $\mathbb{P}_w^{\varepsilon;k}$ denote the kth persistent (relative) homology module of $\{(P^\varepsilon_{\lfloor \alpha \rfloor_w}, Q^\varepsilon_w)\}_{\alpha\in\mathbb{R}}$. Similarly, let $\check{\mathcal{C}}\mathbb{P}_w^{\varepsilon,k}$ and $\mathcal{R}\mathbb{P}_w^{\varepsilon,k}$ denote the corresponding Čech and Vietoris-Rips filtrations, respectively. We will omit the dimension k and write $\mathbb{D}_w$ (resp. $\mathbb{P}^\varepsilon_w$) if a statement holds for all dimensions.

If Q^ε_w surrounds P^ε in D let $\mathcal{E}\mathbb{P}^\varepsilon_w$ denote the kth persistent homology module of the filtration of extensions $\{(\mathcal{E}P^\varepsilon_{\lfloor \alpha \rfloor_w}, \mathcal{E}Q^\varepsilon_w)\}$, where $\mathcal{E}P^\varepsilon_{\lfloor a \rfloor_w} = P^\varepsilon_{\lfloor a \rfloor_w} \cup (D \setminus P^\varepsilon)$. Lemma 11 can be extended to show that we have isomorphisms $\mathcal{E}^\varepsilon_w \in \mathrm{Hom}(\mathbb{P}^\varepsilon_w, \mathcal{E}\mathbb{P}^\varepsilon_w)$ of persistence modules induced by inclusions. If $\varepsilon < \varrho_D$ then we for any $\alpha \in \mathbb{R}$ the inclusion $\check{\mathcal{C}}^\varepsilon(P_{\lfloor \alpha \rfloor_w}, Q_w) \hookrightarrow (P^\varepsilon_{\lfloor \alpha \rfloor_w}, Q^\varepsilon_w)$ is a homotopy equivalence by the Nerve Theorem. As the module homomorphisms of $\check{\mathcal{C}}\mathbb{P}^\varepsilon_w$ and $\mathbb{P}^\varepsilon_w$ are induced by inclusion we have an isomorphism $\mathcal{N}^\varepsilon_w \in \mathrm{Hom}(\check{\mathcal{C}}\mathbb{P}^\varepsilon_w, \mathbb{P}^\varepsilon_w)$ of persistence modules that commutes with maps induced by inclusions by the Persistent Nerve Lemma As the isomorphisms of $\mathcal{E}^\varepsilon_w$ are given by excision they are induced by inclusions, so the composition $\mathcal{E}\mathcal{N}^\varepsilon_w := \mathcal{E}^\varepsilon_w \circ \mathcal{N}^\varepsilon_w$ is an isomorphism that commutes with maps induced by inclusion as well. The following lemma uses these isomorphisms along with inclusions $\mathcal{I}^\varepsilon_w \in \mathrm{Hom}(\check{\mathcal{C}}\mathbb{P}^\varepsilon_w, \mathcal{R}\mathbb{P}^{2\varepsilon}_w)$ and $\mathcal{J}^\varepsilon_w \in \mathrm{Hom}(\mathcal{R}\mathbb{P}^\varepsilon_w, \check{\mathcal{C}}\mathbb{P}^\varepsilon_w)$ to establish image module homomorphisms by maps $\Sigma^\varepsilon_w \in \mathrm{Hom}(\mathbb{P}^\varepsilon_w, \mathcal{R}\mathbb{P}^{2\varepsilon}_w)$ and $\Upsilon^\varepsilon_w \in \mathrm{Hom}(\mathcal{R}\mathbb{P}^\varepsilon_w, \mathbb{P}^\varepsilon_w)$. Proof of this lemma, along with the existence of the maps $\mathcal{E}\mathcal{N}^\varepsilon_w$ can be found in the full version of this paper.

▶ **Lemma 16.** *For $w \in \mathbb{R}$ and $\varepsilon \le \varrho_D/4$ let $\Lambda^\varepsilon \in \mathrm{Hom}(\mathcal{E}\mathbb{P}^\varepsilon_w, \mathcal{E}\mathbb{P}^\eta_z)$ and $\mathcal{R}\Lambda \in \mathrm{Hom}(\mathcal{R}\mathbb{P}^{2\varepsilon}_w, \mathcal{R}\mathbb{P}^{4\varepsilon}_z)$. Then $\tilde{\Phi}(\Sigma^\varepsilon_w, \Sigma^{2\varepsilon}_z) \in \mathrm{Hom}(\mathbf{im}\ \Lambda^\varepsilon, \mathbf{im}\ \mathcal{R}\Lambda)$ and $\tilde{\Psi}(\Upsilon^{2\varepsilon}_w, \Upsilon^{4\varepsilon}_z) \in \mathrm{Hom}(\mathbf{im}\ \mathcal{R}\Lambda, \mathbf{im}\ \Lambda^{2\varepsilon})$ are image module homomorphisms.*

Suppose $Q^\delta_{\omega-2c\delta}$ surrounds P^δ in D and $D \setminus B_\omega \subseteq P^\delta$.Then, because f is c-Lipschitz, $B_{\omega-3c\delta} \cap P^\delta \subseteq Q^\delta_{\omega-2c\delta}$ and $B_\omega \cap P^\delta \subseteq Q^{2\delta}_{\omega+c\delta}$. Similarly, $Q^{2\delta}_{\omega-2c\delta} \subseteq B_\omega$ and $Q^{4\delta}_{\omega+c\delta} \subseteq B_{\omega+5c\delta}$. Therefore, by Lemma 10

$$B_{\omega-3c\delta} \subseteq \mathcal{E}Q^\delta_{\omega-2c\delta} \subseteq \mathcal{E}Q^{2\delta}_{\omega-2c\delta} \subseteq B_\omega \subseteq \mathcal{E}Q^{2\delta}_{\omega+c\delta} \subseteq \mathcal{E}Q^{4\delta}_{\omega+c\delta} \subseteq B_{\omega+5c\delta}.$$

We have the following commutative diagrams of persistence modules where all maps are induced by inclusions.

$$\begin{array}{ccc} \mathbb{D}_{\omega-3c\delta} & \xrightarrow{\Gamma} & \mathbb{D}_\omega \\ \downarrow F & & \downarrow G \\ \mathcal{E}\mathbb{P}^\delta_{\omega-2c\delta} & \xrightarrow{\Lambda} & \mathcal{E}\mathbb{P}^{2\delta}_{\omega+c\delta} \end{array} \quad (3a) \qquad \begin{array}{ccc} \mathcal{E}\mathbb{P}^{2\delta}_{\omega-2c\delta} & \xrightarrow{\Lambda'} & \mathcal{E}\mathbb{P}^{4\delta}_{\omega+c\delta} \\ \downarrow M & & \downarrow N \\ \mathbb{D}_\omega & \xrightarrow{\Pi} & \mathbb{D}_{\omega+5c\delta} \end{array} \quad (3b)$$

In the following let $\mathcal{R}\Lambda \in \mathrm{Hom}(\mathcal{R}\mathbb{P}^{2\delta}_{\omega-2c\delta}, \mathcal{R}\mathbb{P}^{2\delta}_{\omega+c\delta})$ be induced by inclusion. Clearly, $\Phi(F,G)$ is an image module homomorphism of degree $2c\delta$ and $\Psi(M,N)$ is an image module homomorphism of degree $4c\delta$. By Lemma 16 we have image module homomorphisms $\tilde{\Phi}(\Sigma^\delta_{\omega-2c\delta}, \Sigma^{2\delta}_{\omega+c\delta})$ and $\tilde{\Psi}(\Upsilon^{2\delta}_{\omega-2c\delta}, \Upsilon^{4\delta}_{\omega+c\delta})$. Therefore, as the composition of image module homomorphisms are image module homomorphisms we have

$$\mathcal{R}\Phi := \tilde{\Phi} \circ \Phi \in \mathrm{Hom}^{2c\delta}(\mathbf{im}\ \Gamma, \mathbf{im}\ \mathcal{R}\Lambda) \ \text{ and } \ \mathcal{R}\Psi := \Psi \circ \tilde{\Psi} \in \mathrm{Hom}^{4c\delta}(\mathbf{im}\ \mathcal{R}\Lambda, \mathbf{im}\ \Pi)$$

given by the compositions

$$\mathcal{R}\Phi(\mathcal{R}F, \mathcal{R}G) := (\Sigma^\delta_{\omega-2c\delta} \circ F, \Sigma^{2\delta}_{\omega+c\delta} \circ G) \ \text{ and } \ \mathcal{R}\Psi(\mathcal{R}M, \mathcal{R}N) := (M \circ \Upsilon^{2\delta}_{\omega-2c\delta}, N \circ \Upsilon^{4\delta}_{\omega+c\delta}).$$

Because all maps are induced by inclusions, or commute with maps induced by inclusions it can be shown that $\mathcal{R}\Phi_{\mathcal{R}M}$ is a partial $2c\delta$-interleaving of image modules and $\mathcal{R}\Psi_{\mathcal{R}G}$ is a partial $4c\delta$-interleaving of image modules by a straightforward diagram chasing argument. Proof of these facts can be found in the full version of this paper. These maps, along with assumptions that imply $\mathbf{im}(\mathbb{D}_{\omega-3c\delta} \to \mathbb{D}_{\omega+5c\delta}) \cong \mathbb{D}_{\omega}$ provide the proof of Theorem 17 by Lemma **??**.

▶ **Theorem 17.** *Let $\mathbb{X}$ be a d-manifold, $D \subset \mathbb{X}$ and $f : D \to \mathbb{R}$ be a c-Lipschitz function. Let $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $B_{\omega-3c\delta}$ surrounds D in $\mathbb{X}$. Let $P \subset D$ be a finite subset and suppose $\mathrm{H}_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $\mathrm{H}_k(B_\omega) \cong \mathrm{H}_k(B_{\omega+5c\delta})$ for all k.*

If $D \setminus B_\omega \subseteq P^\delta$ and $Q^\delta_{\omega-2c\delta}$ surrounds P^δ in D then the kth persistent homology module of

$$\{\mathcal{R}^{2\delta}(P_{\lfloor\alpha\rfloor_{\omega-2c\delta}}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor\alpha\rfloor_{\omega+c\delta}}, Q_{\omega+c\delta})\}_{\alpha\in\mathbb{R}}$$

is $4c\delta$-interleaved with that of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$.

5 Approximation of the Truncated Diagram

In this section we will relate the relative persistence diagram that we have approximated in the previous section to a truncation of the full diagram. Let $\mathbb{L}^k$ denote the kth persistent homology module of the sublevel set filtration $\{B_\alpha\}_{\alpha\in\mathbb{R}}$. As in the previous section, let $\mathbb{D}^k_\omega$ denote the kth persistent (relative) homology module of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$. Throughout we will assume that we are taking homology in a field $\mathbb{F}$ and that the homology groups $\mathrm{H}_k(B_\alpha)$ and $\mathrm{H}_k(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)$ are finite dimensional vector spaces for all k and $\alpha \in \mathbb{R}$. We will use the interval decomposition of $\mathbb{L}^k$ to give a decomposition of the relative module $\mathbb{D}^k_\omega$ in terms of a *truncation* of $\mathbb{L}^k$. Recall, the *truncated diagram* is defined to be that of $\mathbb{L}^k$ consisting only of those features born after ω. For fixed $\omega \in \mathbb{R}$ we will define the truncation $\mathbb{T}^k_\omega$ of $\mathbb{L}^k$ in terms of the intervals decomposing $\mathbb{L}^k$ that are in $[\omega, \infty)$.

5.0.0.1 Truncated Interval Modules

For an interval $I = [s, t) \subseteq \mathbb{R}$ let $I_+ := [t, \infty)$ and $I_- := (-\infty, s]$. For $\omega \in \mathbb{R}$ let $\mathbb{F}^I_\omega$ denote the interval module consisting of vector spaces $\{F^I_{\lfloor\alpha\rfloor_\omega}\}_{\alpha\in\mathbb{R}}$ and linear maps $\{f^I_{\lfloor\alpha,\beta\rfloor_\omega} : F^I_{\lfloor\alpha\rfloor_\omega} \to F^I_{\lfloor\beta\rfloor_\omega}\}_{\alpha\leq\beta}$ where

$$F^I_{\lfloor\alpha\rfloor_\omega} := \begin{cases} F^I_\alpha & \text{if } \omega \in I_- \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f^I_{\lfloor\alpha,\beta\rfloor_\omega} := \begin{cases} f^I_{\alpha,\beta} & \text{if } \omega \in I_- \\ 0 & \text{otherwise.} \end{cases}$$

For a collection $\mathcal{I}$ of intervals let $\mathcal{I}_\omega := \{I \in \mathcal{I} \mid \omega \in I\}$.

▶ **Lemma 18.** *Suppose $\mathcal{I}^k$ and $\mathcal{I}^{k-1}$ are collections of intervals that decompose $\mathbb{L}^k$ and $\mathbb{L}^{k-1}$, respectively. Then the kth persistent homology module of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$ is equal to*

$$\bigoplus_{I\in\mathcal{I}^k} \mathbb{F}^I_\omega \oplus \bigoplus_{I\in\mathcal{I}^{k-1}_\omega} \mathbb{F}^{I_+}$$

for all k.

Proof. (See Appendix B) ◀

5.1 Main Theorem

Let $\mathbb{L}^k$ denote the kth persistent homology module of the sublevel set filtration $\{B_\alpha\}_{\alpha\in\mathbb{R}}$ of f and let $\mathcal{I}^k$ denote the decomposing intervals of $\mathbb{L}^k$ for all k. For a fixed $\omega \in \mathbb{R}$ let $\mathbb{D}^k_\omega$ denote the kth persistent (relative) homology module of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$. Let

$$\mathbb{T}^k_\omega := \bigoplus_{I\in\mathcal{I}^k} \mathbb{F}^I_\omega$$

denote the **ω-truncated kth persistent homology module** of $\mathbb{L}^k$. Let

$$\mathbb{L}^{k-1}_\omega := \bigoplus_{I\in\mathcal{I}^{k-1}_\omega} \mathbb{F}^{I_+}.$$

denote the submodule of $\mathbb{D}^k_\omega$ consisting of intervals $[\beta,\infty)$ corresponding to features $[\alpha,\beta)$ in $\mathbb{L}^{k-1}$ such that $\alpha \le \omega < \beta$. Now, by Lemma 18 the kth persistent (relative) homology module of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$ is

$$\mathbb{D}^k_\omega = \mathbb{T}^k_\omega \oplus \mathbb{L}^{k-1}_\omega.$$

Our main theorem combines this decomposition with our coverage and interleaving results of Theorems 8 and 17.

▶ **Theorem 19.** *Let $\mathbb{X}$ be an orientable d-manifold and let D be a compact subset of $\mathbb{X}$. Let $f : D \to \mathbb{R}$ be a c-Lipschitz function and $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $P \subset D$ is a $(\delta, 2\delta, \omega)$-sublevel sample of f and $B_{\omega-3c\delta}$ surrounds D in $\mathbb{X}$.*

Suppose $\mathrm{H}_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $\mathrm{H}_k(B_\omega) \cong \mathrm{H}_k(B_{\omega+5c\delta})$ for all k. If

$$\mathbf{rk}\ \mathrm{H}_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \ge \mathbf{dim}\ \mathrm{H}_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$$

then the kth (relative) homology module of

$$\{\mathcal{R}^{2\delta}(P_{\lfloor\alpha\rfloor_{\omega-2c\delta}}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor\alpha\rfloor_{\omega+c\delta}}, Q_{\omega+c\delta})\}_{\alpha\in\mathbb{R}}$$

is $4c\delta$-interleaved with $\mathbb{T}^k_\omega \oplus \mathbb{L}^{k-1}_\omega$: the kth persistent homology module of $\{(D_{\lfloor\alpha\rfloor_\omega}, B_\omega)\}_{\alpha\in\mathbb{R}}$.

6 Experiments

In this section we will discuss a number of experiments which illustrate the benefit of truncated diagrams, and their approximation by relative diagrams, in comparison to their restricted counterparts. We will focus on the persistent homology of functions on a square 2d grid—that is, functions with non-trivial persistent homology in dimensions zero and one. While these experiments can be conducted in dimension zero or one we will focus on H_1. We therefore chose a function with prominent persistent homology in dimension one—a radially symmetric damped sinusoid with random noise, depicted in Figure **??**.

6.0.0.1 Experimental setup.

Throughout, the inter-levelsets shown in green, blue, purple, and yellow correspond to the ranges $[0, 0.3)$, $[0.3, 0.5)$, $[0.5, 0.7)$, and $[0.7, 1)$, respectively. Our persistent homology computations were done primarily with Dionysus augmented with custom software for computing representative cycles of infinite features. [4] The persistent homology of our function was computed with the lower-star filtration of the Freudenthal triangulation on an $N \times N$ grid over $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. We take this filtration as $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ where P is the set of grid points and $\delta = \sqrt{2}/N$.

We note that the purpose of these experiments is not to demonstrate the effectiveness of our approximation by Rips complexes, but to demonstrate the relationships between restricted, relative, and truncated diagrams. Therefore, for simplicity, we will omit the inclusion $\mathcal{R}^{2\delta}(P_\alpha) \hookrightarrow \mathcal{R}^{4\delta}(P_\alpha)$ and take the persistent homology of $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ with sufficiently small δ as our ground-truth. However, in order to keep our diagrams clean we show only those features a distance at least 4δ from the diagonal. Note that these features are *not* removed from the diagram, and considered in all computations.

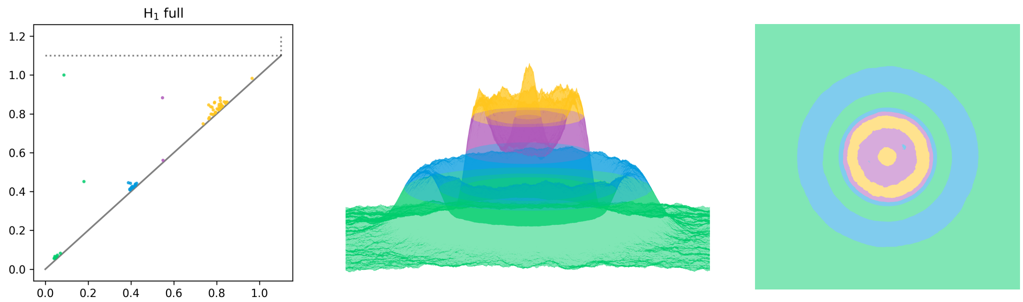


Figure 4 The H_1 persistence diagram of the sinusoidal function pictured to the right. Features are colored by birth time, infinite features are drawn above the dotted line.

In the following we will take $N = 1024$, so $\delta \approx 1.4 \times 10^{-3}$, as our ground-truth. Figure **??** shows the *full diagram* of our function with features colored by birth time. Therefore, for $\omega = 0.3, 0.5, 0.7$ the *truncated diagram* is obtained by successively removing the green, blue, and purple features. Recall the *restricted diagram* is that of the function restricted to the ω *super*-levelset filtration, and computed with $\{\mathcal{R}^{2\delta}(P_\alpha \setminus Q_\omega)\}$. We will compare this restricted diagram with the *relative diagram*, computed as the relative persistent homology of the filtration of pairs $\{\mathcal{R}^{2\delta}(P_\alpha, Q_\omega)\}$.

6.0.0.2 The issue with restricted diagrams.

In order to get an initial sense of the difference between relative and restricted diagrams we first compare the bottleneck distance of each to the truncated diagram. As we have shown the relative diagram is equal to the truncated diagram with additional infinite features we will remove all infinite features from the bottleneck computation. We therefore expect the distance between the relative and truncated diagrams to be zero for $N = 1024$.

Figure **??** shows the bottleneck distance from the truncated diagram at full resolution ($N = 1024$) to both the relative and restricted diagrams with varying resolution. Specifically, the function on a 1024×1024 grid is down-sampled to grids ranging from 64×64 to 1024×1024. We also show the expected bottleneck distance to the true truncated diagram given by the interleaving in Theorem 17 in black.

As we can see, the relative diagram clearly performs better than the restricted diagram, which diverges with increasing resolution. Recall that 1-dimensional features that are born

[4] 3D figures were made with MayAvi, all other figures were made with Matplotlib.

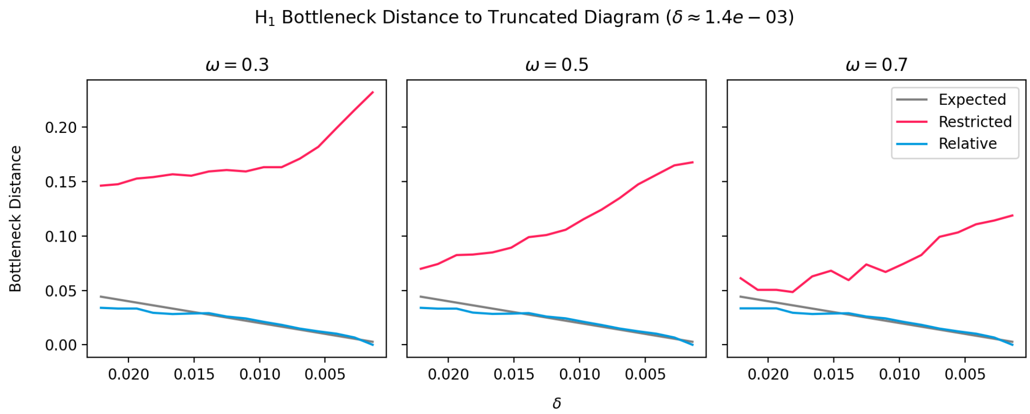


Figure 5 Comparison of the bottleneck distance between the truncated diagram of the function shown in Figure **??** approximated with $\delta \approx 1.4 \times 10^{-3}$ (1024 × 1024 grid) and those of the restricted and relative diagrams with decreasing δ (increasing grid size 64-1024).

before ω and die after ω become infinite 2-dimensional features in the relative diagram, with birth time equal to the death time of the corresponding feature in the full diagram. These same features remain 1-dimensional figures in the restricted diagram, but with their birth times shifted to ω. Indeed, the resulting restricted diagram may be closer to the full diagram for sufficiently small ω. However, the distance will be proportional to the difference between ω and the true birth time.

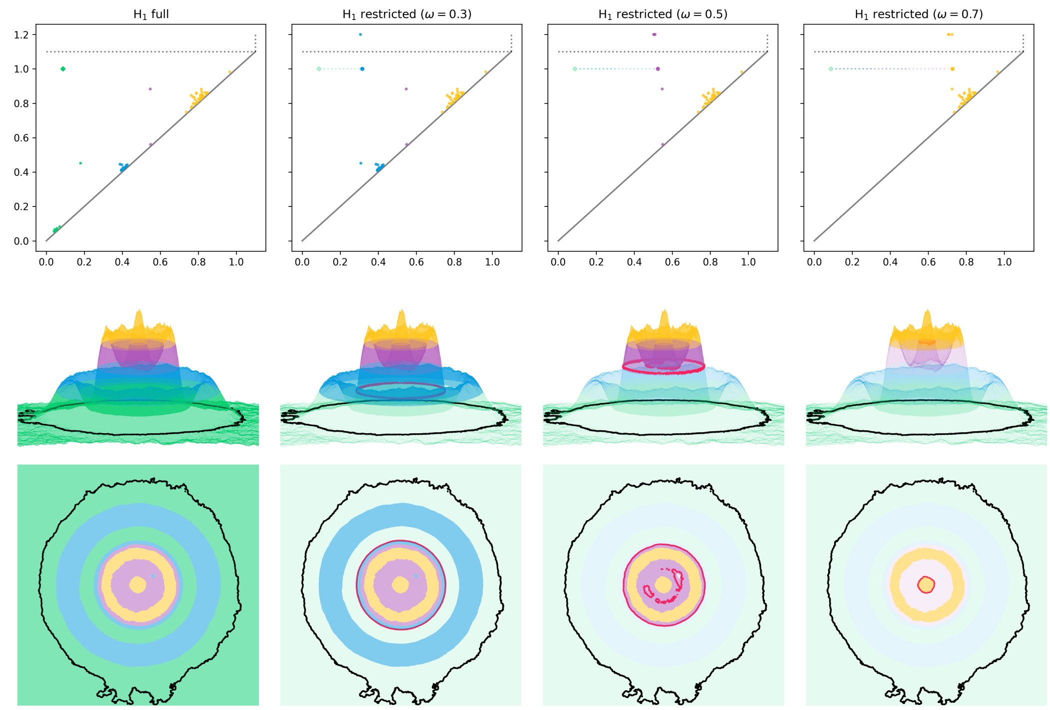
Figure 6 shows this distance for a feature that persists throughout the diagram. As the restricted diagram in full resolution the restricted filtration is a subset of the full filtration, so these features can be matched by their death simplices. For illustrative purposes we also show the representative cycles associated with these features.

We imagine a setting where we would like to classify a function using a sample that cannot be verified below some known ω. That is, we can only check for coverage of the super-levelset $D \setminus B_\omega$ using the variation of the TCC we have introduced in the previous sections. We would then like to classify the function with the bottleneck distance to a set of known functions based on the region we cover. However, as we have shown, the restricted diagram may contain artifacts of features born before ω which will skew our measurement. Instead, as ω is known, we can compare the *relative* diagram the collection of *truncated* diagrams of known functions to get a better classification.

6.0.0.3 Relative diagrams and reconstruction.

Now, imagine we obtain the persistence diagram of our sub-levelset B_ω. That is, we now know that we cover B_ω, or some subset, and do not want to re-compute the diagram above ω. If we compute the persistence diagram of the function restricted to the *sub*-levelset B_ω any 1-dimensional features born before ω that die after ω will remain infinite features in this restricted (below) diagram. Indeed, we could match these infinite 1-features with the corresponding shifted finite 1-features in the restricted (above) diagram, as shown in Figure 6. However, that would require sorting through all finite features that are born near ω and deciding if they are in fact features of the full diagram that have been shifted.

Recalling that these same features become infinite 2-features in the relative diagram, we can use the relative diagram instead and match infinite 1-features of the diagram restricted below to infinite 2-features in the relative diagram, as shown in Figures **??** and **??**. For this



456 **Figure 6** (Top) H_1 persistence diagrams of the function depicted in Figure **??** restricted to
457 *super*-levelsets at $\omega = 0.3, 0.5$, and 0.7 (on a 1024×1024 grid). The matching is shown between a
458 feature in the full diagram (marked with a diamond) with its representative cycle in black. The
459 corresponding representative cycle in the restricted diagram is pictured in red.

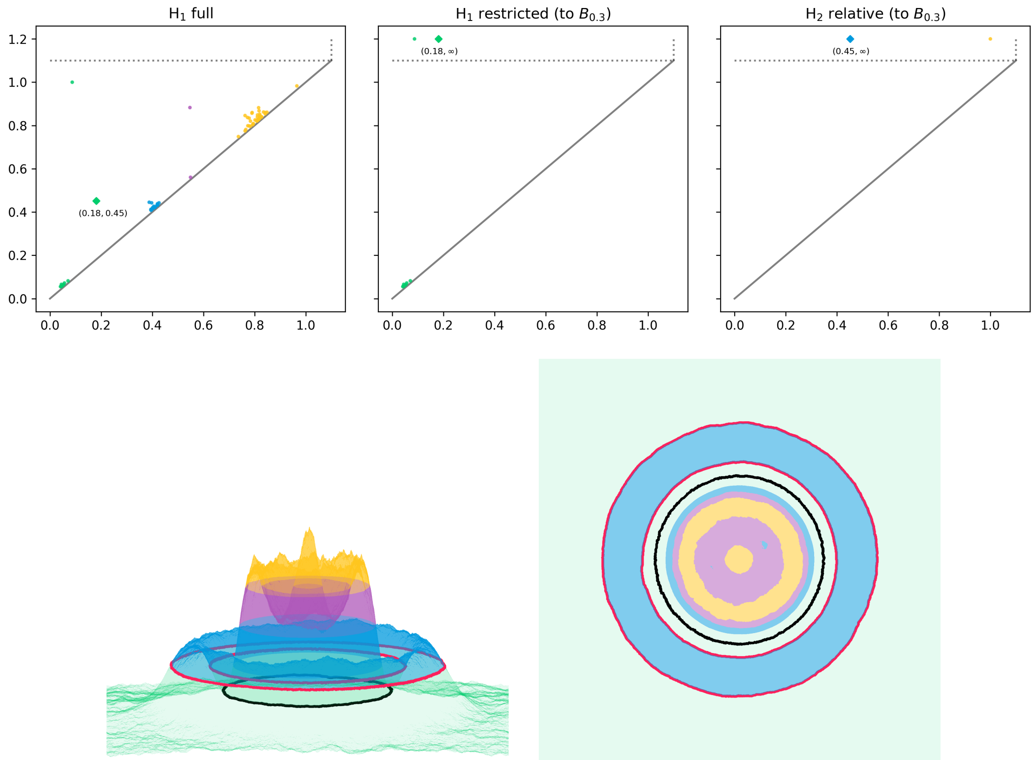


Figure 7 (Left) Full H_1 persistence diagram, (middle) H_1 persistence diagram of the function restricted to the *sub*-levelset $B_{0.3}$, (right) H_2 persistence diagram of the the function realtive to the sub-levelset $B_{0.3}$. (Bottom) In black, the representative cycle of the infinite 1-feature born at 0.18 in the restricted diagram is shown in black. In red, the *boundary* of the representative *relative* 2-cycle born at 0.45 in the relative diagram is shown in red. The indicated infinite features in the restricted and relative diagrams correspond to the birth and death of the 1-feature $(0.18, 0.45)$ in the full diagram.

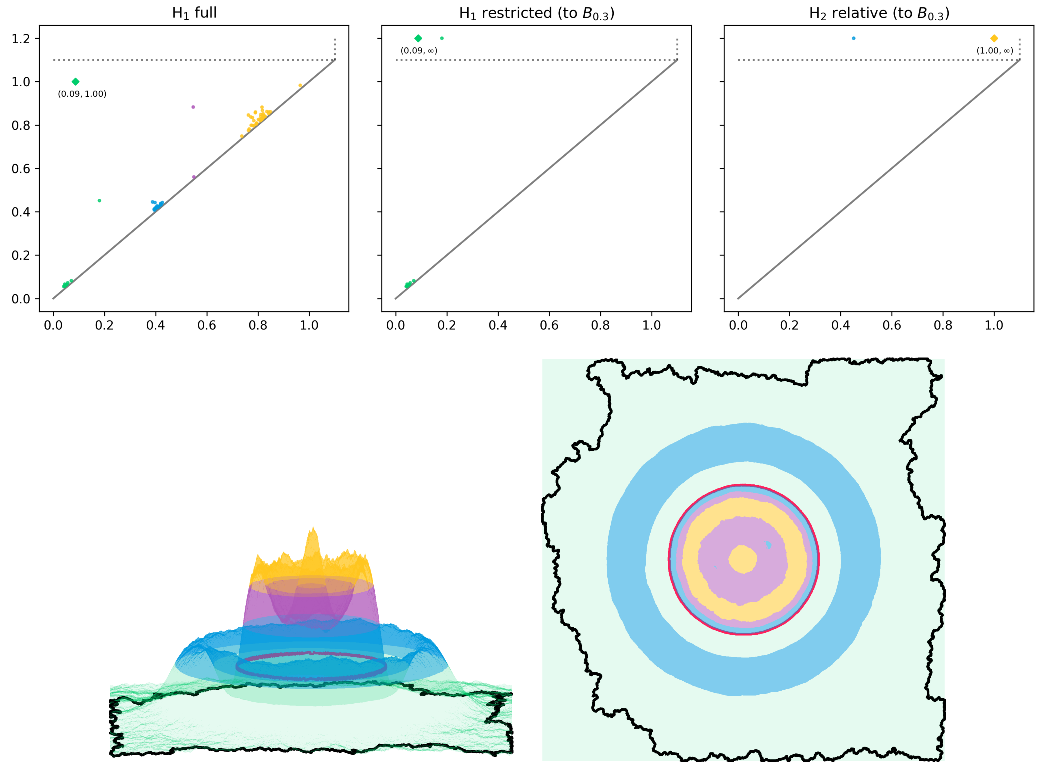


Figure 8 The infinite 1-features of the restricted diagram can be matched with the infinite 2-features of the relative diagrams. The sequence birth times of relative 2-features in *decreasing* order correspond to the deaths of restricted 1-features in *increasing* order.

example the matching is given by sorting the 1-features by ascending and the 2-features by descending birth time. How to construct this matching in general, especially in the presence of infinite features in the full diagram, is the subject of future research.

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A Duality

For a pair (A, B) in a topological space X and any R module G let $\mathrm{H}^k(A, B; G)$ denote the **singular cohomology** of (A, B) (with coefficients in G) as a vector space. Let $\mathrm{H}_c^k(A, B; G)$ denote the corresponding **singular cohomology with compact support**, where $\mathrm{H}_c^k(A, B; G) \cong \mathrm{H}^k(A, B; G)$ for any compact pair (A, B).

The following corollary follows from the Universal Coefficient Theorem for singular homology (and cohomology) as vector spaces over a field $\mathbb{F}$, as the dual vector space $\mathrm{Hom}(\mathrm{H}_k(A, B), \mathbb{F})$ is isomorphic to $\mathrm{H}_k(A, B; \mathbb{F})$ for any finitely generated $\mathrm{H}_k(A, B)$.[5]

▶ **Corollary 20.** *For a topological pair (A, B) and a field $\mathbb{F}$ such that $\mathrm{H}_0(A, B)$ is finitely generated there is a natural isomorphism*

$$\nu : \mathrm{H}^0(A, B; \mathbb{F}) \to \mathrm{H}_0(A, B; \mathbb{F}).$$

Let $\overline{\mathrm{H}}^k(A, B; G)$ be the **Alexander-Spanier cohomology** of the pair (A, B), defined as the limit of the direct system of neighborhoods (U, V) of the pair (A, B). Let $\overline{\mathrm{H}}_c^k(A, B; G)$ denote the corresponding **Alexander-Spanier cohomology with compact support** where $\overline{\mathrm{H}}_c^k(A, B; G) \cong \overline{\mathrm{H}}^k(A, B; G)$ for any compact pair (A, B).

▶ **Theorem 21** (**Alexander-Poincaré-Lefschetz Duality** (Spanier [7], Theorem 6.2.17))**.** *Let X be an orientable d-manifold and (A, B) be a compact pair in X. Then for all k and R modules G there is a (natural) isomorphism*

$$\lambda : \mathrm{H}_k(X \setminus B, X \setminus A; G) \to \overline{\mathrm{H}}^{d-k}(A, B; G).$$

[5] Reference/verify.

A space X is said to be **homologically locally connected in dimension** n if for every $x \in X$ and neighborhood U of x there exists a neighborhood V of x in U such that $\tilde{\mathrm{H}}_n(V) \to \tilde{\mathrm{H}}_n(U)$ is trivial for $k \leq n$.

▶ **Lemma 22** (Spanier p. 341, Corollary 6.9.6)**.** *Let A be a closed subset, homologically locally connected in dimension n, of a Hausdorff space X, homologically locally connected in dimension n. If X has the property that every open subset is paracompact, $\mu : \overline{\mathrm{H}}_c^k(X, A; G) \to \mathrm{H}_c^k(X, A; G)$ is an isomorphism for $k \leq n$ and a monomorphism for $q = n + 1$.*

In the following we will assume homology (and cohomology) over a field $\mathbb{F}$.

▶ **Lemma 23.** *Let X be an orientable d-manifold and (A, B) a compact pair of locally path connected subspaces in X. Then*

$$\xi : \mathrm{H}_d(X \setminus B, X \setminus A) \to \mathrm{H}_0(A, B)$$

is a natural isomorphism.

Proof. Because X is orientable and (A, B) are compact $\lambda : \mathrm{H}_d(X \setminus B, X \setminus A) \to \overline{\mathrm{H}}^0(A, B)$ is an isomorphism by Theorem 21. Note that Moreover, because every subset of X is (hereditarily) paracompact every open set in A, with the subspace topology, is paracompact. For any neighborhood U of a point x in a locally path connected space there must exist some neighborhood $V \subset U$ of x that is path connected in the subspace topology. As $\tilde{\mathrm{H}}_0(V) = 0$ for any nonempty, path connected topological space V (see Spanier p. 175, Lemma 4.4.7) it follows that A (resp. B) are homologically locally connected in dimension 0. Because (A, B) is a compact pair the singular and Alexander-spanier cohomology modules of (A, B) with compact support are isomorphic to those without, thus $\mu : \overline{\mathrm{H}}^0(A, B) \to \mathrm{H}^0(A, B)$ is an isomorphism. By Corollary 20 we have a natural isomorphism $\nu : \mathrm{H}^0(A, B) \to \mathrm{H}_0(A, B)$ thus the composition $\xi := \nu \circ \mu \circ \lambda : \mathrm{H}_d(X \setminus B, X \setminus A) \to \mathrm{H}_0(A, B)$ is a natural isomorphism. ◀

▶ **Lemma 24.** *Let $\mathbb{X}$ be an orientable d-manifold let D be a compact subset of $\mathbb{X}$ with strong convexity radius $\varrho_D > \varepsilon$. Let P be a finite subset of D such that $P^\varepsilon \subset \mathbf{int}_{\mathbb{X}}(D)$ and $Q \subseteq P$.*

If $D \setminus Q^\varepsilon$ and $D \setminus P^\varepsilon$ are locally path connected then there is an isomorphism

$$\xi\mathcal{N} : \mathrm{H}_d(\check{\mathcal{C}}^\varepsilon(P, Q)) \to \mathrm{H}_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon)$$

that commutes with maps induced by inclusions.

Proof. Because Q^ε and P^ε are open in D and D is compact in $\mathbb{X}$ the complement $D \setminus Q^\varepsilon$ is closed in D, and therefore compact in $\mathbb{X}$. Moreover, because $P^\varepsilon \subset \mathbf{int}_{\mathbb{X}}(D)$, $\mathrm{H}_d(\mathbb{X} \setminus (D \setminus P^\varepsilon), \mathbb{X} \setminus (D \setminus Q^\varepsilon)) = \mathrm{H}_d(P^\varepsilon, Q^\varepsilon)$. As we have assumed these complements are locally path connected by assumption we have a natural isomorphism $\xi : \mathrm{H}_d(P^\varepsilon, Q^\varepsilon) \to \mathrm{H}_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon)$ by Lemma 23.

Because $\varepsilon > \varrho_D$ the covers by metric balls associated with P^ε and Q^ε are good, so we have isomorphisms $\mathcal{N} : \mathrm{H}_d(\check{\mathcal{C}}^\varepsilon(P, Q)) \to \mathrm{H}_d(P^\varepsilon, Q^\varepsilon)$ for all $Q \subseteq P$ by the Nerve Theorem. So the composition $\xi\mathcal{N} := \xi \circ \mathcal{N}$ is an isomorphism. Moreover, because ξ is natural and $\mathcal{N}$ commutes with maps induced by inclusions by the persistent nerve lemma the composition $\xi\mathcal{N}$ does as well. ◀

B Omitted Proofs