

From Coverage Testing to Topological Scalar Field Analysis

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1 Abstract

The topological coverage criterion (TCC) can be used to test whether an underlying space is sufficiently well covered by a given data set. Given a sufficiently dense sample, topological scalar field analysis (SFA) can give a summary of the shape of a real-valued function on its domain. The goal of this paper is to put these theories together so that one can test coverage with the TCC while computing a summary with SFA. The challenge is that the TCC requires a well-defined boundary that is not generally available in the setting of SFA. To overcome this, we show how the scalar field itself can be used to define a boundary that can be used to confirm coverage. This requires an interpretation of the TCC that resolves one of the major barriers to wider use. It also extends SFA methods to the setting in which coverage is only confirmed in a subset of the domain.

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11 1 Introduction

In the topological analysis of scalar fields (SFA), one computes a topological summary capturing qualitative and quantitative shape information from a set of points endowed with a metric and a real-valued function. That is, we have points with distances and a real number assigned to each point. More generally, it suffices to have a neighborhood graph identifying the pairs of points within some distance. The topological computation often takes the form of persistent homology and integrates local information about the function into global information about its *behavior* as whole. In prior work, Chazal et al. [3] showed that for sufficiently dense samples on sufficiently smooth spaces, the persistence diagram can be computed with some guarantees. In followup work, Buchet et al. [1] extended this result to show how to work with noisy inputs. A fundamental assumption required to have strong guarantees is that the underlying space is sufficiently well-sampled. In this paper, we show how to combine scalar field analysis with the theory of topological coverage testing to simultaneously compute the persistence diagram while testing that the underlying space is sufficiently well-sampled.

Initiated by De Silva and Ghrist [10, 6, 7], the theory of homological sensor networks addresses the problem of testing coverage of a bounded domain by a collection of sensors without coordinates. The main result is the topological coverage criterion, which, in its most general form, states that under reasonable geometric assumptions, the d -dimensional homology of a pair of simplicial complexes built on the neighborhood graph will be nontrivial if and only if there is sufficient coverage (see Section 3 for the precise statements). This relative persistent homology test is called the Topological Coverage Criterion (TCC).



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33 Superficially, the methods of SFA and TCC are very similar. Both construct similar
34 complexes and compute the persistent homology of the homological image of a complex on
35 one scale into that of a larger scale. They even overlap on some common techniques in their
36 analysis such as the use of the Nerve theorem and the Rips-Čech interleaving. However,
37 they differ in some fundamental way that makes it difficult to combine them into a single
38 technique. The main difference is that the TCC requires a clearly defined boundary. Not
39 only must the underlying space be a bounded subset of \mathbb{R}^d , the data must also be labeled to
40 indicate which input points are close to the boundary. This requirement is perhaps the main
41 reason why the TCC can so rarely be applied in practice.

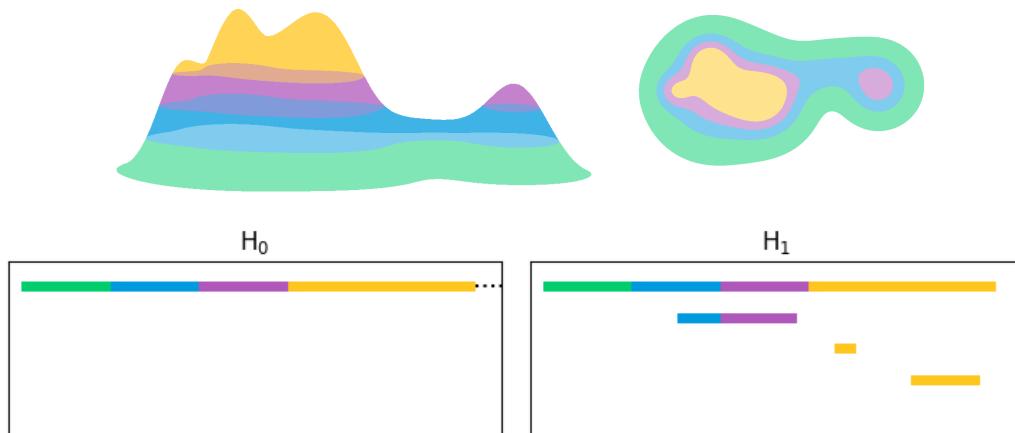
42 In applications to data analysis it is more natural to assume that the data measures
43 some unknown function. We can then replace this requirement with assumptions about the
44 function itself. Indeed, these assumptions could relate the behavior of the function to the
45 topological boundary of the space. However, the generalized approach by Cavanna et al. [2]
46 allows much more freedom in how the boundary is defined.

47 We consider the case in which we have incomplete data from a particular sublevel set
48 of our function. Our goal is to isolate this data so we can analyze the function in only the
49 verified region. From this perspective, the TCC confirms that we not only have coverage,
50 but that the sample we have is topologically representative of the region near, and above
51 this sublevel set. We can then re-use the same machinery to analyze a *part* of the function
52 in a specific way.

53 Contribution

54 We will re-cast the TCC as a way to verify that the persistent homology of a scalar field
55 can be *partially* approximated by a given sample. Specifically, we will relate the persistent
56 homology of a function relative to a *static* sublevel set to a *truncation* of the full diagram.
57 That is, beyond a certain point the full diagram remains unchanged, allowing for possible
58 reconstruction. This is in comparison with the *restricted* diagram obtained by simply ignoring
59 part of the domain. We therefore present relative persistent homology as an alternative to
60 restriction in a way that extends the TCC to the analysis of scalar fields.

61 Section 2 establishes notation and provides an overview of our main results in Sections 3
62 and 4. In Section 5 we introduce an interpretation of the relative diagram as a truncation of
63 the full diagram that is motivated by a number of experiments in Section 6.



64 2 Summary

65 Let \mathbb{X} denote an orientable d -manifold and $D \subset \mathbb{X}$ a compact subspace. For a c -Lipschitz
 66 function $f : D \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$ let $B_\alpha := f^{-1}((-\infty, \alpha])$ denote the α -sublevel set of f . Our
 67 sample will be denoted P , and the subset of points sampling B_α will be denoted $Q_\alpha := P \cap B_\alpha$.
 68 For ease of exposition let

69 $D_{\lfloor \alpha \rfloor w} := B_\alpha \cup B_w$

70 denote the *truncated* α sublevel set and

71 $P_{\lfloor \alpha \rfloor w} := Q_\alpha \cup Q_w$

72 denote its sampled counterpart for all $\alpha, w \in \mathbb{R}$.

73 We will select a sublevel set B_ω to serve as our boundary. Specifically, we require that
 74 B_ω surrounds D , where the notion of a surrounding set is defined formally in Section 3. This
 75 distinction allows us to generalize the standard proof of the geometric TCC as properties of
 76 surrounding pairs.

77 Results

78 Suppose B_ω surrounds D in \mathbb{X} and $\delta < \varrho_D/4$, where ϱ_D denotes the *strong convexity radius*
 79 of D (see Chazal et al. [3]). As a minimal assumption we require that every component of
 80 $D \setminus B_\omega$ contains a point in P . We also make additional technical assumptions on P and δ
 81 with respect to the pair (D, B_ω) (see Section 3 and Lemma 25 of the Appendix).

82 Theorem 6 If

83 I. $H_0(D \setminus B_{\omega+5c\delta} \hookrightarrow D \setminus B_\omega)$ is *surjective*,

84 II. $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-3c\delta})$ is *injective*,

85 and

86 $\text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$

89 then $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D . ¹

90 This formulation of the TCC states that our approximation by a nested pair of Rips
 91 complexes captures the homology of the pair (D, B_ω) in a specific way. We use this fact
 92 to interleave our sample with the relative diagram of the filtration $\{(D_{\lfloor \alpha \rfloor w}, B_\omega)\}_{\alpha \in \mathbb{R}}$. This
 93 is done by generalizing our regularity assumptions near $D \setminus B_\omega$ in a way that allows us to
 94 interleave persistence modules relative to static sublevels.

95 **Theorem 17** Suppose $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D . If

96 I. $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is *surjective* and

97 II. $H_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an *isomorphism*

98 for all k then the persistent homology modules of

99 $\{\mathcal{R}^{2\delta}(P_{\lfloor \alpha \rfloor w-2c\delta}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor \alpha \rfloor w+c\delta}, Q_{\omega+c\delta})\}_{\alpha \in \mathbb{R}}$

100 and $\{(D_{\lfloor \alpha \rfloor w}, B_\omega)\}_{\alpha \in \mathbb{R}}$ are $4c\delta$ interleaved.

87 ¹ We state this result using constants that will be used to prove the interleaving. The statement of
 88 Theorem 6 parameterizes the region around ω in terms of $\zeta \geq \delta$ as $[\omega - c(\delta + \zeta), \omega + c(\delta + \zeta)]$.

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101 The main challenges we face come from the fact that the sublevel set B_ω and our
 102 approximation by the inclusion $\mathcal{R}^{2\delta}(Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(Q_{\omega+c\delta})$ remain *static* throughout.
 103 Using the fact that $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D we define an *extension* $(D, \mathcal{E}Q_{\omega-2c\delta}^\delta)$ of the
 104 pair $(P^\delta, Q_{\omega-2c\delta}^\delta)$ that has isomorphic relative homology by excision. These extensions give
 105 us a sequence of inclusion maps

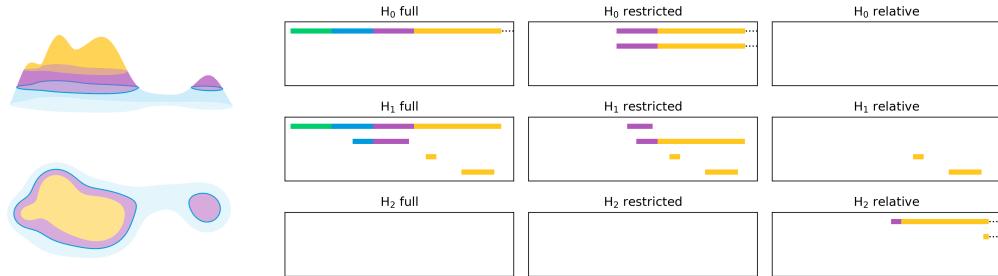
$$106 \quad B_{\omega-3c\delta} \hookrightarrow \mathcal{E}Q_{\omega-2c\delta}^{2\delta} \hookrightarrow B_\omega \hookrightarrow \mathcal{E}Q_{\omega+c\delta}^{4\delta} \hookrightarrow B_{\omega+5c\delta}$$

107 that can be used along with our regularity assumptions to prove the interleaving.

108 Relative, Truncated, and Restricted Persistence Diagrams

109 For fixed $\omega \in \mathbb{R}$ we will refer to the persistence diagram associated with the filtration
 110 $\{(D_{[\alpha]_\omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ as the **relative diagram** of f . In Section 5 we relate the relative diagram
 111 to the *full* diagram of the sublevel set filtration $\{B_\alpha\}_{\alpha \in \mathbb{R}}$. Specifically, we define the
 112 **truncated diagram** to be the subdiagram consisting of features born *after* ω in the full.
 113 In Section 6 we compare the relative and truncated diagrams to the **restricted diagram**,
 114 defined to be that of the sublevel set filtration of $f|_{D \setminus B_\omega}$.

115 Note that the truncated sublevel sets $D_{[\alpha]_\omega}$ are equal to the union of B_ω and the restricted
 116 sublevel sets. It is in this sense that B_ω is *static* throughout—it is contained in every sublevel
 117 set of the relative filtration. As we will not have verified coverage in B_ω we cannot analyze
 118 the function in this region directly. We therefore have two alternatives: *restrict* the domain
 119 of the function to $D \setminus B_\omega$, or use relative homology to analyze the function *relative* to this
 120 region using excision.



121 **Figure 1** Full, restricted, and relative barcodes of the function (left).

122 Outline of Sections 3 and 4

123 We will begin with our statement of the TCC in Section 3. This requires the introduction
 124 of surrounding pairs before proving our reformulation of the TCC (Theorem 6). Section 4
 125 formally introduces extensions and partial interleavings of image modules which will be used
 126 to interleave our approximation with the relative diagram (Theorem 17).

127 3 The Topological Coverage Criterion (TCC)

128 A positive result from the TCC requires that we have a subset of our cover to serve as the
 129 boundary. That is, the condition not only checks that we have coverage, but also that
 130 we have a pair of spaces that reflects the pair (D, B) topologically. We call such a pair a
 131 *surrounding pair* defined in terms of separating sets. It has been shown that the TCC can

be stated in terms of these surrounding pairs [2]. Moreover, this work made assumptions directly in terms of the *zero dimensional* persistent homology of the domain close to the boundary. This allows us enough flexibility to define our surrounding set as a sublevel of a c -Lipschitz function f and state our assumptions in terms of its persistent homology.

► **Definition 1** (Surrounding Pair). *Let X be a topological space and (D, B) a pair in a topological space X . The set B surrounds D in X if B separates X with the pair $(D \setminus B, X \setminus D)$. We will refer to such a pair as a surrounding pair in X .*

The following lemma generalizes the proof of the TCC as a property of surrounding sets. We will then combine these results on the homology of surrounding pairs with information about both \mathbb{X} as a metric space and our function.

► **Lemma 2.** *Let (D, B) be a surrounding pair in X and $U \subseteq D, V \subseteq U \cap B$ be subsets. Let $\ell : H_0(X \setminus B, X \setminus D) \rightarrow H_0(X \setminus V, X \setminus U)$ be induced by inclusion.*

If ℓ is injective then $D \setminus B \subseteq U$ and V surrounds U in D .

Let (\mathbb{X}, \mathbf{d}) be a metric space and $D \subseteq \mathbb{X}$ be a compact subspace. For a c -Lipschitz function $f : D \rightarrow \mathbb{R}$ we introduce a constant ω as a threshold that defines our “boundary” as a sublevel set B_ω of the function f . Let P be a finite subset of D and $\zeta \geq \delta > 0$ and be constants such that $P^\delta \subseteq \text{int}_{\mathbb{X}}(D)$. Here, δ will serve as our communication radius where ζ is reserved for use in Section 4.²

► **Lemma 3.** *Let $i : H_0(\overline{Q_{\omega+c\delta}^\delta}, \overline{P^\delta}) \rightarrow H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$.*

If B_ω surrounds D in \mathbb{X} then $\dim H_0(\overline{B_\omega}, \overline{D}) \geq \text{rk } i$.

Proof. Choose a basis for $\text{im } i$ such that each basis element is represented by a point in $P^\delta \setminus Q_{\omega+c\delta}^\delta$. Let $x \in P^\delta \setminus Q_{\omega+c\delta}^\delta$ be such that $i[x] \neq 0$. So there exists some $p \in P$ such that $\mathbf{d}(p, x) < \delta$ and $p \notin Q_{\omega+c\delta}$, otherwise $x \in Q_{\omega+c\delta}^\delta$. Therefore, because f is c -Lipschitz,

$$f(x) \geq f(p) - c\mathbf{d}(x, p) > \omega + c\delta - c\delta = \omega.$$

So $x \in \overline{B_\omega}$ and, because $x \in P^\delta \subseteq D$, $x \in D \setminus B_\omega$. Because i and $\ell : H_0(\overline{B_\omega}, \overline{D}) \rightarrow H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$ are induced by inclusion $\ell[x] = i[x] \neq 0$ in $H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$. That is, every element of $\text{im } i$ has a preimage in $H_0(\overline{B_\omega}, \overline{D})$, so we may conclude that $\dim H_0(\overline{B_\omega}, \overline{D}) \geq \text{rk } i$. ◀

Note that, while there is a surjective map from $H_0(\overline{B_\omega}, \overline{D})$ to $\text{im } i$ this map is not necessarily induced by inclusion. We therefore must introduce a larger space $B_{\omega+c(\delta+\zeta)}$ that contains $Q_{\omega+c\delta}^\delta$ in order to provide a criteria for the injectivity of $\ell : H_0(\overline{B_\omega}, \overline{D}) \rightarrow H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta})$ in terms of $\text{rk } i$. We have the following commutative diagrams of inclusion maps the induced maps between complements in \mathbb{X} .

$$\begin{array}{ccc} (P^\delta, Q_{\omega-c\zeta}^\delta) & \xhookrightarrow{\quad} & (P^\delta, Q_{\omega+c\delta}^\delta) \\ \downarrow & & \downarrow \\ (D, B_\omega) & \xhookrightarrow{\quad} & (D, B_{\omega+c(\delta+\zeta)}), \quad H_0(\overline{Q_{\omega+c\delta}^\delta}, \overline{P^\delta}) \xrightarrow{i} H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta}). \end{array} \quad \begin{matrix} H_0(\overline{B_{\omega+c(\delta+\zeta)}}, \overline{D}) \xrightarrow{j} H_0(\overline{B_\omega}, \overline{D}) \\ \downarrow m \\ H_0(\overline{Q_{\omega-c\zeta}^\delta}, \overline{P^\delta}) \end{matrix} \quad (1)$$

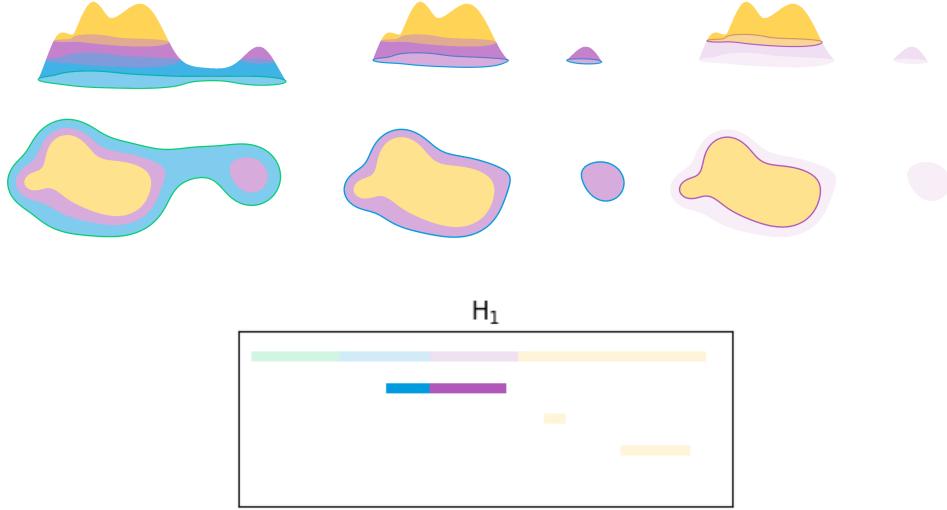
² We will set $\zeta = 2\delta$ in the proof of our interleaving with Rips complexes but the TCC holds for all $\zeta \geq \delta$.

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167 Assumptions

168 We will first require the map $H_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ to be *surjective*—as we approach
 169 ω from *above* no components *appear*. This ensures that the rank of the map j is equal to the
 170 dimension of $\dim H_0(\overline{B_\omega}, \overline{D})$ so ℓ depends only on $H_0(\overline{B_\omega}, \overline{D})$ and $\text{im } i$.

171 We also assume that $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is *injective*—as we move away from ω
 172 moving *down* no components *disappear*. Lemma 4 uses Assumption 2 to provide a computable
 173 upper bound on $\text{rk } j$.



174 **Figure 2** The blue level set does not satisfy either assumption as the smaller component is not in
 175 the inclusion from blue to green and it “pinched out” in the yellow region. This can be seen in the
 176 barcode shown as a feature that is born in the blue region and dies in the purple region.

177 ► **Lemma 4.** If $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega+c(\delta+\zeta)})$ is injective and each component of $D \setminus B_\omega$
 178 contains a point in P then $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$.

179 Nerves and Duality

182 Recall that the Nerve Theorem states that for a good open cover \mathcal{U} of a space X the inclusion
 183 map from the *Nerve* of the cover to the space $\mathcal{N}(\mathcal{U}) \hookrightarrow X$ is a homotopy equivalence.³ The
 184 Persistent Nerve Lemma [4] states that this homotopy equivalence commutes with inclusion
 185 on the level of homology. We note that the standard proof of the Nerve Theorem [9], and
 186 therefore the Persistent Nerve Lemma [4], extends directly to pairs of good open covers $(\mathcal{U}, \mathcal{V})$
 187 of pairs (X, Y) such that \mathcal{V} is a subcover of \mathcal{U} .⁴

188 Recalling the definition of the strong convexity radius ϱ_D (see Chazal et al. [3]) \mathcal{U} is a
 189 good open cover whenever $\varrho_D > \varepsilon$. As the Čech complex is the Nerve of a cover by a union
 190 of balls we will let $\mathcal{N}_w^\varepsilon : H_k(\check{\mathcal{C}}^\varepsilon(P, Q_w)) \rightarrow H_k(P^\varepsilon, Q_w^\varepsilon)$ denote the isomorphism on homology
 191 provided by the Nerve Theorem for all $k, w \in \mathbb{R}$ and $\varepsilon < \varrho_D$.

193 Under certain conditions Alexander Duality provides an isomorphism between the k
 194 relative cohomology of a compact pair in an orientable d -manifold \mathbb{X} with the $d-k$ dimensional

180 ³ In a good open cover every nonempty intersection of sets in the cover is contractible.

181 ⁴ $\{V_i\}_{i \in I}$ is a subcover of $\{U_i\}_{i \in I}$ if $V_i \subseteq U_i$ for all $i \in I$.

homology of their complements in \mathbb{X} (see Spanier [11]). For finitely generated (co)homology over a field the Universal Coefficient Theorem can be used with Alexander Duality to give a natural isomorphism $\xi_w^\varepsilon : H_d(P^\varepsilon, Q_w^\varepsilon) \rightarrow H_0(D \setminus Q_w^\varepsilon, D \setminus P^\varepsilon)$.⁵ This isomorphism holds in the specific case when $P^\varepsilon \subseteq \text{int}_{\mathbb{X}}(D)$ and $D \setminus P^\varepsilon, D \setminus Q_w^\varepsilon$ are locally contractible. We therefore provide the following definition for ease of exposition.

► **Definition 5** ((δ, ζ, ω)-Sublevel Sample). *For $\zeta \geq \delta > 0$, $\omega \in \mathbb{R}$, and a c -Lipschitz function $f : D \rightarrow \mathbb{R}$ a finite point set $P \subset D$ is said to be a (δ, ζ, ω) -sublevel sample of f if every component of $D \setminus B_\omega$ contains a point in P , $P^\delta \subset \text{int}_{\mathbb{X}}(D)$, and $D \setminus P^\delta, D \setminus Q_{\omega-c\zeta}^\delta$, and $D \setminus Q_{\omega+c\delta}^\delta$ are locally path connected in \mathbb{X} .*

Because this isomorphism is natural and the isomorphism provided by the Nerve Theorem commutes with maps induced by inclusion the composition $\xi \mathcal{N}_w^\varepsilon := \xi_w^\varepsilon \circ \mathcal{N}_w^\varepsilon$ gives an isomorphism that commutes with maps induced by inclusion for all $w \in \mathbb{R}$ and $\varepsilon < \varrho_D$.

► **Theorem 6** (Algorithmic TCC). *Let \mathbb{X} be an orientable d -manifold and let D be a compact subset of \mathbb{X} . Let $f : D \rightarrow \mathbb{R}$ be c -Lipschitz function and $\omega \in \mathbb{R}$ and $\delta \leq \zeta < \varrho_D$ be constants such that $P \subset D$ is a (δ, ζ, ω) -sublevel sample of f and $B_{\omega-c(\zeta+\delta)}$ surrounds D in \mathbb{X} .*

If $H_0(D \setminus B_{\omega+c(\delta+\zeta)} \hookrightarrow D \setminus B_\omega)$ is surjective, $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is injective, and $\text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-c\zeta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta}))$ then $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-c\zeta}^\delta$ surrounds P^δ in D .

Proof. Because P is a (δ, ζ, ω) -sublevel sample we have isomorphisms $\xi \mathcal{N}_{\omega-c\zeta}^\delta$ and $\xi \mathcal{N}_{\omega+c\delta}^\delta$ that commute with $q_{\check{\mathcal{C}}} : H_d(\check{\mathcal{C}}^\delta(P, Q_{\omega-c\zeta})) \rightarrow H_d(\check{\mathcal{C}}^{2\delta}(P, Q_{\omega+c\delta}))$ and $i : H_0(D \setminus Q_{\omega+c\delta}^\delta, D \setminus P^\delta) \rightarrow H_0(D \setminus Q_{\omega-c\zeta}^\delta, D \setminus P^\delta)$. Let $q_{\mathcal{R}} : H_d(\mathcal{R}^\delta(P, Q_{\omega-c\zeta})) \rightarrow H_d(\mathcal{R}^{2\delta}(P, Q_{\omega+c\delta}))$ be induced by inclusion. Then $\text{rk } q_{\check{\mathcal{C}}} \geq \text{rk } q_{\mathcal{R}}$ as $q_{\mathcal{R}}$ factors through $q_{\check{\mathcal{C}}}$. As we have assumed $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ Lemma 4 implies $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$. It follows that, whenever $\text{rk } q_{\mathcal{R}} \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta}))$, we have

$$\text{rk } i = \text{rk } q_{\check{\mathcal{C}}} \geq \text{rk } q_{\mathcal{R}} \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega).$$

Because j is surjective by hypothesis $\text{rk } j = \dim H_0(\overline{B_\omega}, \overline{D}) = \dim H_0(D \setminus B_\omega)$ so $\text{rk } j \geq \text{rk } i$ by Lemma 3. As we have shown $\text{rk } i \geq \dim H_0(D \setminus B_\omega)$ it follows that $\text{rk } j = \text{rk } i$. Because P is a finite point set we know that $\text{im } i$ is finite-dimensional and, because $\text{rk } i = \text{rk } j$, $\text{im } j = H_0(\overline{B_\omega}, \overline{D})$ is finite dimensional as well. So $\text{im } j$ is isomorphic to $\text{im } i$ as a subspace of $H_0(Q_{\omega-c\zeta}^\delta, P^\delta)$ which, because j is surjective, requires the map ℓ to be injective. Therefore $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-c\zeta}^\delta$ surrounds P^δ in D by Lemma 2. ◀

4 From Coverage Testing to the Analysis of Scalar Fields

Because the TCC only confirms coverage of a *superlevel* set $D \setminus B_\omega$, we cannot guarantee coverage of the entire domain. Indeed, we could compute the persistent homology of the *restriction* of f to the superlevel set we cover in the standard way [3]. Instead, we will approximate the persistent homology of the sublevel set filtration *relative to* the sublevel set B_ω . In the next section we will discuss an interpretation of the relative diagram that is motivated by examples in Section 6.

We will first introduce the notion of an extension which will provide us with maps on relative homology induced by inclusion via excision. However, even then, a map that factors

⁵ For the construction of this isomorphism see the Appendix.

235 through our pair (D, B_ω) is not enough to prove an interleaving of persistence modules by
 236 inclusion directly. To address this we impose conditions on sublevel sets near B_ω which
 237 generalize the assumptions made in the TCC.

238 **4.1 Extensions and Image Persistence Modules**

239 Suppose D is a subspace of X . We define the extension of a surrounding pair in D to a
 240 surrounding pair in X with isomorphic relative homology.

241 ► **Definition 7 (Extension).** If V surrounds U in a subspace D of X let $\mathcal{EV} := V \sqcup (D \setminus U)$
 242 denote the (disjoint) union of the separating set V with the complement of U in D . The
 243 **extension of** (U, V) **in** D is the pair $(D, \mathcal{EV}) = (U \sqcup (D \setminus U), V \sqcup (D \setminus U))$.

244 Lemma 8 states that we can use these extensions to interleave a pair (U, V) with a
 245 sequence of subsets of (D, B) . Lemma 9 states that we can apply excision to the relative
 246 homology groups in order to get equivalent maps on homology that are induced by inclusions.

247 ► **Lemma 8.** Suppose V surrounds U in D and $B' \subseteq B \subset D$.
 248 If $D \setminus B \subseteq U$ and $U \cap B' \subseteq V \subseteq B'$ then $B' \subseteq \mathcal{EV} \subseteq B$.

249 ► **Lemma 9.** Let (U, V) be an open surrounding pair in a subspace D of X .
 250 Then $H_k((U \cap A, V) \hookrightarrow (A, \mathcal{EV}))$ is an isomorphism for all k and $A \subseteq D$ with $\mathcal{EV} \subset A$.

251 The TCC uses a nested pair of spaces in order to filter out noise introduced by the sample.
 252 This same technique is used to approximate the persistent homology of a scalar fields [3]. As
 253 modules, these nested pairs are the images of homomorphisms between homology groups
 254 induced by inclusion, which we refer to as image persistence modules.

255 ► **Definition 10 (Image Persistence Module).** The **image persistence module** of a homo-
 256 mophism $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$ is the family of subspaces $\{\Gamma_\alpha := \text{im } \gamma_\alpha\}$ in \mathbb{V} along with linear
 257 maps $\{\gamma_\alpha^\beta := v_\alpha^\beta|_{\text{im } \gamma_\alpha} : \Gamma_\alpha \rightarrow \Gamma_\beta\}$ and will be denoted by $\text{im } \Gamma$.

258 While we will primarily work with homomorphisms of persistence modules induced by
 259 inclusions, in general, defining homomorphisms between images simply as subspaces of the
 260 codomain is not sufficient. Instead, we require that homomorphisms between image modules
 261 commute not only with shifts in scale, but also with the functions themselves.

264 ► **Definition 11 (Image Module Homomorphism).** Given $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$ and $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$
 265 along with $(F, G) \in \text{Hom}^\delta(\mathbb{U}, \mathbb{S}) \times \text{Hom}^\delta(\mathbb{V}, \mathbb{T})$ let $\Phi(F, G) : \text{im } \Gamma \rightarrow \text{im } \Lambda$ denote the family
 266 of linear maps $\{\phi_\alpha := g_\alpha|_{\Gamma_\alpha} : \Gamma_\alpha \rightarrow \Lambda_{\alpha+\delta}\}$. $\Phi(F, G)$ is an **image module homomorphism**
 267 of degree δ if the following diagram commutes for all $\alpha \leq \beta$.⁶

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\gamma_\alpha[\beta-\alpha]} & V_\beta \\ \downarrow f_\alpha & & \downarrow g_\beta \\ S_{\alpha+\delta} & \xrightarrow{\lambda_{\alpha+\delta}[\beta-\alpha]} & T_{\beta+\delta} \end{array} \quad (2)$$

269 The space of image module homomorphisms of degree δ between $\text{im } \Gamma$ and $\text{im } \Lambda$ will be
 270 denoted $\text{Hom}^\delta(\text{im } \Gamma, \text{im } \Lambda)$.

271 The composition of image module homomorphisms are image module homomorphisms. Proof
 272 of this fact can be found in the Appendix.

262 ⁶ We use the notation $\gamma_\alpha[\beta-\alpha] = v_\alpha^\beta \circ \gamma_\alpha$, $\lambda_\alpha[\beta-\alpha] = t_\alpha^\beta \circ \lambda_\alpha$ to denote the composition of homomorphisms
 263 between persistence modules and shifts in scale.

²⁷³ **Partial Interleavings of Image Modules**

²⁷⁴ Image module homomorphisms introduce a direction to the traditional notion of interleaving.
²⁷⁵ As we will see, our interleaving via Lemma 13 involves partially interleaving an image module
²⁷⁶ to two other image modules whose composition is isomorphic to our target.

²⁷⁷ ▶ **Definition 12** (Partial Interleaving of Image Modules). *An image module homomorphism*
²⁷⁸ $\Phi(F, G)$ *is a **partial δ -interleaving of image modules**, and denoted $\Phi_M(F, G)$, if there*
²⁷⁹ *exists $M \in \text{Hom}^\delta(\mathbb{S}, \mathbb{V})$ such that $\Gamma[2\delta] = M \circ F$ and $\Lambda[2\delta] = G \circ M$.*

²⁸⁰ Lemma 13 uses partial interleavings of a map Λ with $\mathbb{U} \rightarrow \mathbb{V}$ and $\mathbb{V} \rightarrow \mathbb{W}$ along with the
²⁸¹ hypothesis that $\mathbb{U} \rightarrow \mathbb{W}$ is isomorphic to \mathbb{V} to interleave $\mathbf{im} \Lambda$ with \mathbb{V} . When applied, this
²⁸² hypothesis will be satisfied by assumptions on our sublevel set similar to those made in the
²⁸³ TCC.

²⁸⁴ ▶ **Lemma 13.** *Suppose $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$, $\Pi \in \text{Hom}(\mathbb{V}, \mathbb{W})$, and $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$.*

²⁸⁵ *If $\Phi_M(F, G) \in \text{Hom}^\delta(\mathbf{im} \Gamma, \mathbf{im} \Lambda)$ and $\Psi_G(M, N) \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbf{im} \Pi)$ are partial*
²⁸⁶ *δ -interleavings of image modules such that Γ is a epimorphism and Π is a monomorphism*
²⁸⁷ *then $\mathbf{im} \Lambda$ is δ -interleaved with \mathbb{V} .*

²⁸⁸ **4.2 Proof of the Interleaving**

²⁸⁹ For $w, \alpha \in \mathbb{R}$ let \mathbb{D}_w^k denote the k th persistent (relative) homology module of the filtration
²⁹⁰ $\{(D_{\lfloor \alpha \rfloor w}, B_w)\}_{\alpha \in \mathbb{R}}$ with respect to B_w , and let $\mathbb{P}_w^{\varepsilon, k}$ denote the k th persistent (relative) homology module of $\{(P_{\lfloor \alpha \rfloor w}^\varepsilon, Q_w^\varepsilon)\}_{\alpha \in \mathbb{R}}$. Similarly, let $\check{C}\mathbb{P}_w^{\varepsilon, k}$ and $\mathcal{R}\mathbb{P}_w^{\varepsilon, k}$ denote the corresponding Čech and Vietoris-Rips filtrations, respectively. We will omit the dimension k and write \mathbb{D}_w
²⁹² (resp. \mathbb{P}_w^ε) if a statement holds for all dimensions. If Q_w^δ surrounds P^δ in D let $\mathcal{E}\mathbb{P}_w^\varepsilon$ denote
²⁹⁴ the k th persistent homology module of the filtration of extensions $\{(\mathcal{E}P_{\lfloor \alpha \rfloor w}^\varepsilon, \mathcal{E}Q_w^\varepsilon)\}$ for any
²⁹⁵ $\varepsilon \geq \delta$, where $\mathcal{E}P_{\lfloor \alpha \rfloor w}^\varepsilon = P_{\lfloor \alpha \rfloor w}^\varepsilon \cup (D \setminus P^\delta)$.

²⁹⁶ Lemma 14 follows directly from the definition of truncated sublevel sets. This is used
²⁹⁷ to extend Lemma 8 to persistence modules in Lemma 15 in order to provide a sequence of
²⁹⁸ shifted homomorphisms $\mathbb{D}_{\omega-3c\delta} \xrightarrow{F} \mathcal{E}\mathbb{P}_{\omega-2c\delta}^\varepsilon \xrightarrow{M} \mathbb{D}_\omega \xrightarrow{G} \mathcal{E}\mathbb{P}_{\omega+c\delta}^{2\varepsilon} \xrightarrow{N} \mathbb{D}_{\omega+5c\delta}$ of varying degree.
²⁹⁹ These homomorphisms are then combined with those given by the Nerve theorem and the
³⁰⁰ Rips-Čech interleaving in Lemma 16 to obtain partial interleavings required for our proof of
³⁰¹ Theorem 17.

³⁰² ▶ **Lemma 14.** *If $\delta \leq \varepsilon$ and $t, \alpha \in \mathbb{R}$ then $P^\delta \cap D_{\lfloor \alpha - c\varepsilon \rfloor t - c\varepsilon} \subseteq P_{\lfloor \alpha \rfloor t}^\varepsilon \subseteq D_{\lfloor \alpha + c\varepsilon \rfloor t + c\varepsilon}$.*

³⁰³ ▶ **Lemma 15.** *Let $s + 3c\delta \leq t + 2c\delta \leq u \leq v - c\delta \leq w - 5c\delta$ and $\varepsilon \in [\delta, 2\delta]$. If Q_t^δ surrounds
³⁰⁴ P^δ in D and $D \setminus B_u \subseteq P^\delta$ then the following homomorphisms are induced by inclusions:*

$$(305) \quad (F, G) \in \text{Hom}^{c\delta}(\mathbb{D}_s, \mathcal{E}\mathbb{P}_t^\varepsilon) \times \text{Hom}^{2c\delta}(\mathbb{D}_u, \mathcal{E}\mathbb{P}_v^{2\varepsilon}), \quad (M, N) \in \text{Hom}^{c\varepsilon}(\mathcal{E}\mathbb{P}_t^\varepsilon, \mathbb{D}_u) \times \text{Hom}^{2c\varepsilon}(\mathcal{E}\mathbb{P}_v^{2\varepsilon}, \mathbb{D}_w).$$

³⁰⁶ ▶ **Lemma 16.** *For $\delta < \varrho_D$ let $\Gamma \in \text{Hom}(\mathbb{D}_s, \mathbb{D}_u)$, $\Pi \in \text{Hom}(\mathbb{D}_u, \mathbb{D}_w)$, and $\Lambda \in \text{Hom}(\mathcal{R}\mathbb{P}_t^{2\delta}, \mathcal{R}\mathbb{P}_v^{4\delta})$
³⁰⁷ be induced by inclusions for $s + 3c\delta \leq t + 2c\delta \leq u \leq v - c\delta \leq w - 5c\delta$.*

³⁰⁸ *If Q_t^δ surrounds P^δ in D and $D \setminus B_u \subseteq P^\delta$ then there is a partial $2c\delta$ interleaving
³⁰⁹ $\Phi^* \in \text{Hom}^{2c\delta}(\mathbf{im} \Gamma, \mathbf{im} \Lambda)$ and a partial $4c\delta$ interleaving $\Psi^* \in \text{Hom}^{4c\delta}(\mathbf{im} \Lambda, \mathbf{im} \Pi)$.*

³¹⁰ **Proof.** Because the shifted homomorphisms provided by Lemma 15 are all induced by
³¹¹ inclusions the following diagram commutes for all $\alpha \leq \beta$. So we have image module

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312 homomorphisms $\Phi(F, G) \in \text{Hom}^{2c\delta}(\text{im } \Gamma, \text{im } C \circ A)$ and $\Psi(M, N) \in \text{Hom}^{4c\delta}(\text{im } E \circ C, \text{im } \Pi)$.

$$\begin{array}{ccccc} H_k(D_{\lfloor \alpha - 2c\delta \rfloor_s}, B_s) & \xrightarrow{f_{\alpha-2c\delta}} & H_k(\mathcal{E}P_{\lfloor \alpha \rfloor_t}^\delta, \mathcal{E}Q_t^\delta) & H_k(\mathcal{E}P_{\lfloor \alpha \rfloor_t}^{2\delta}, \mathcal{E}Q_t^{2\delta}) & \xrightarrow{m_\alpha} H_k(D_{\lfloor \alpha + 4c\delta \rfloor_u}, B_u) \\ \downarrow \gamma_{\alpha-2c\delta}[\beta-\alpha] & & \downarrow c_\alpha[\beta-\alpha] \circ a_\alpha & & \downarrow e_\beta \circ c_\alpha[\beta-\alpha] \\ H_k(D_{\lfloor \beta - 2c\delta \rfloor_u}, B_u) & \xrightarrow{g_{\beta-2c\delta}} & H_k(\mathcal{E}P_{\lfloor \beta \rfloor_v}^{2\delta}, \mathcal{E}Q_v^{2\delta}) & H_k(\mathcal{E}P_{\lfloor \beta \rfloor_v}^{4\delta}, \mathcal{E}Q_v^{4\delta}) & \xrightarrow{n_\beta} H_k(D_{\lfloor \beta + 4c\delta \rfloor_w}, B_w) \end{array}$$

314 Because the isomorphisms provided by Lemma 9 are given by excision they are induced
315 by inclusion, and therefore give isomorphisms $\mathcal{E}_z^\varepsilon \in \text{Hom}(\mathbb{P}_z^\varepsilon, \mathcal{E}\mathbb{P}_z^\varepsilon)$ for any $z \in \mathbb{R}$ such that Q_z^ε
316 surrounds P^δ in D . For any $\varepsilon < \varrho_D$ we have isomorphisms $\mathcal{N}_z^\varepsilon \in \text{Hom}(\check{\mathcal{C}}\mathbb{P}_z^\varepsilon, \mathbb{P}_z^\varepsilon)$ that commute
317 with maps induced by inclusions by the Persistent Nerve Lemma. So the compositions $\mathcal{E}_z^\varepsilon \circ \mathcal{N}_z^\varepsilon$
318 isomorphisms that commute with maps induced by inclusion as well. These compositions,
319 along with the Rips-Čech interleaving, provide maps $\mathcal{E}\mathbb{P}_t^\delta \xrightarrow{F'} \mathcal{R}\mathbb{P}_t^{2\delta} \xrightarrow{M'} \mathcal{E}\mathbb{P}_t^{2\delta}$ and $\mathcal{E}\mathbb{P}_v^{2\delta} \xrightarrow{G'} \mathcal{R}\mathbb{P}_v^{4\delta} \xrightarrow{N'} \mathcal{E}\mathbb{P}_v^{4\delta}$ that commute with maps induced by inclusions. So we have the following
320 commutative diagram:
321

$$\begin{array}{ccccccc} \mathcal{E}\mathbb{P}_t^\delta & \xrightarrow{A} & \mathcal{E}\mathbb{P}_t^{2\delta} & \xrightarrow{C} & \mathcal{E}\mathbb{P}_v^{2\delta} & \xrightarrow{E} & \mathcal{E}\mathbb{P}_v^{4\delta} \\ \searrow F' & & \nearrow M' & & \searrow G' & & \nearrow N' \\ \mathcal{R}\mathbb{P}_t^{2\delta} & \xrightarrow{\Lambda} & \mathcal{R}\mathbb{P}_v^{4\delta} & & & & \end{array} \quad (3)$$

322 That is, we have image module homomorphisms $\Phi'(F', G')$ and $\Psi'(M', N')$ such that $A = M' \circ F'$, $E = N' \circ G'$, and $\Lambda = G' \circ C \circ M'$. Because image module homomorphisms compose
323 we have we have $\Phi^* = \Phi' \circ \Phi \in \text{Hom}^{2c\delta}(\text{im } \Gamma, \text{im } \Lambda)$ and $\Psi^* = \Psi \circ \Psi' \in \text{Hom}^{4c\delta}(\text{im } \Lambda, \text{im } \Pi)$.
324 Because G, M, C are induced by inclusions $C[3c\delta] = G \circ M$, so $\Lambda[3c\delta] = C' \circ C[3c\delta] \circ M' =$
325 $G' \circ (G \circ M) \circ M'$ as G', M' commute with maps induced by inclusions. In the same way,
326 $\Gamma[3c\delta] = M \circ (A \circ F) = M \circ (M' \circ F') \circ F$ and $\Pi[5c\delta] = N \circ (E \circ G) = N \circ (N' \circ G') \circ G$.

327 Let $F^* := F' \circ F$, $G^* := G' \circ G$, $M^* := M' \circ M$, and $N^* := N' \circ N$. So $\Phi_{M^*}^*$ is a
328 partial $2c\delta$ interleaving as $\Gamma[3c\delta] = M^* \circ F^*$ and $\Lambda[3c\delta] = G^* \circ M^*$, and $\Psi_{G^*}^*$ is a partial $4c\delta$
329 interleaving as $\Lambda[3c\delta] = G^* \circ M^*$ and $\Pi[5c\delta] = N^* \circ G^*$. \blacktriangleleft

322 The partial interleavings given by Lemma 16, along with assumptions that imply
323 $\text{im}(\mathbb{D}_{\omega-3c\delta} \rightarrow \mathbb{D}_{\omega+5c\delta}) \cong \mathbb{D}_\omega$, provide the proof of Theorem 17 by Lemma 13.

324 ▶ **Theorem 17.** *Let \mathbb{X} be a d -manifold, $D \subset \mathbb{X}$ and $f : D \rightarrow \mathbb{R}$ be a c -Lipschitz function.
325 Let $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $B_{\omega-3c\delta}$ surrounds D in \mathbb{X} . Let $P \subset D$ be
326 a finite subset and suppose $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $H_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an
327 isomorphism for all k .*

328 If $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D then the k th persistent homology
329 module of $\{\mathcal{R}^{2\delta}(P_{\lfloor \alpha \rfloor_{\omega-2c\delta}}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor \alpha \rfloor_{\omega+c\delta}}, Q_{\omega+c\delta})\}_{\alpha \in \mathbb{R}}$ is $4c\delta$ -interleaved with that
330 of $\{(D_{\lfloor \alpha \rfloor_\omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$.

321 **Proof.** Let $\Lambda \in \text{Hom}(\mathcal{R}\mathbb{P}_{\omega-2c\delta}^{2c\delta}, \mathcal{R}\mathbb{P}_{\omega+c\delta}^{4c\delta})$, $\Gamma \in \text{Hom}(\mathbb{D}_{\omega-3c\delta}, \mathbb{D}_\omega)$, and $\Pi \in \text{Hom}(\mathbb{D}_\omega, \mathbb{D}_{\omega+5c\delta})$
322 be induced by inclusions. Because $\delta < \varrho_D/4$, $D \setminus B_\omega \subseteq P^\delta$ and $Q_{\omega-2c\delta}^\delta$ surrounds P^δ in D
323 we have a partial $2c\delta$ interleaving $\Phi^* \in \text{Hom}^{2c\delta}(\text{im } \Gamma, \text{im } \Lambda)$ and a partial $4c\delta$ interleaving
324 $\Psi^* \in \text{Hom}^{4c\delta}(\text{im } \Lambda, \text{im } \Pi)$ by Lemma 16. As we have assumed that $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$
325 is surjective and $H_k(B_\omega) \cong H_k(B_{\omega+5c\delta})$ the five-lemma implies γ_α is surjective and π_α is
326 an isomorphism (and therefore injective) for all α . So Γ is an epimorphism and Π is a
327 monomorphism, thus $\text{im } \Lambda$ is $4c\delta$ -interleaved with \mathbb{D}_ω by Lemma 13 as desired. \blacktriangleleft

348 5 Approximation of the Truncated Diagram

349 We will relate the relative persistence diagram that we have approximated in the previous
 350 section to a truncation of the full diagram. Let \mathbb{L}^k denote the k th persistent homology
 351 module of the sublevel set filtration $\{B_\alpha\}_{\alpha \in \mathbb{R}}$. As in the previous section, let \mathbb{D}_ω^k denote
 352 the k th persistent (relative) homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$. Throughout we will
 353 assume that we are taking homology in a field \mathbb{F} and that the homology groups $H_k(B_\alpha)$ and
 354 $H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega)$ are finite dimensional vector spaces for all k and $\alpha \in \mathbb{R}$. We will use the
 355 interval decomposition of \mathbb{L}^k to give a decomposition of the relative module \mathbb{D}_ω^k in terms of a
 356 *truncation* of \mathbb{L}^k . Recall, the *truncated diagram* is defined to be that of \mathbb{L}^k consisting only of
 357 those features born after ω . For fixed $\omega \in \mathbb{R}$ we will define the truncation \mathbb{T}_ω^k of \mathbb{L}^k in terms
 358 of the intervals decomposing \mathbb{L}^k that are in $[\omega, \infty)$.

359 **Truncated Interval Modules**

360 For an interval $I = [s, t] \subseteq \mathbb{R}$ let $I_+ := [t, \infty)$ and $I_- := (-\infty, s]$. For $\omega \in \mathbb{R}$ let \mathbb{F}_ω^I denote the
 361 interval module consisting of vector spaces $\{F_{\lfloor \alpha \rfloor \omega}^I\}_{\alpha \in \mathbb{R}}$ and linear maps $\{f_{\lfloor \alpha, \beta \rfloor \omega}^I : F_{\lfloor \alpha \rfloor \omega}^I \rightarrow F_{\lfloor \beta \rfloor \omega}^I\}_{\alpha \leq \beta}$ where

$$363 \quad F_{\lfloor \alpha \rfloor \omega}^I := \begin{cases} F_\alpha^I & \text{if } \omega \in I_- \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{\lfloor \alpha, \beta \rfloor \omega}^I := \begin{cases} f_{\alpha, \beta}^I & \text{if } \omega \in I_- \\ 0 & \text{otherwise.} \end{cases}$$

364 For a collection \mathcal{I} of intervals let $\mathcal{I}_\omega := \{I \in \mathcal{I} \mid \omega \in I\}$.

365 ► **Lemma 18.** Suppose \mathcal{I}^k and \mathcal{I}^{k-1} are collections of intervals that decompose \mathbb{L}^k and \mathbb{L}^{k-1} ,
 366 respectively. Then for all k the k th persistent homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ is equal
 367 to

$$368 \quad \bigoplus_{I \in \mathcal{I}^k} \mathbb{F}_\omega^I \oplus \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} \mathbb{F}^{I+}.$$

369 **Main Theorem**

370 Let \mathcal{I}^k denote the decomposing intervals of \mathbb{L}^k for all k . Let

$$371 \quad \mathbb{T}_\omega^k := \bigoplus_{I \in \mathcal{I}^k} \mathbb{F}_\omega^I$$

372 denote the ω -truncated k th persistent homology module of \mathbb{L}^k and

$$373 \quad \mathbb{L}_\omega^{k-1} := \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} \mathbb{F}^{I+}.$$

374 denote the submodule of \mathbb{D}_ω^k consisting of intervals $[\beta, \infty)$ corresponding to features $[\alpha, \beta)$ in
 375 \mathbb{L}^{k-1} such that $\alpha \leq \omega < \beta$. Now, by Lemma 18 the k th persistent (relative) homology module
 376 of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$ is $\mathbb{D}_\omega^k = \mathbb{T}_\omega^k \oplus \mathbb{L}_\omega^{k-1}$. Our main theorem combines this decomposition
 377 with our coverage and interleaving results of Theorems 6 and 17.

378 ► **Theorem 19.** Let \mathbb{X} be an orientable d -manifold and let D be a compact subset of \mathbb{X} . Let
 379 $f : D \rightarrow \mathbb{R}$ be a c -Lipschitz function and $\omega \in \mathbb{R}$, $\delta < \varrho_D/4$ be constants such that $P \subset D$ is a
 380 $(\delta, 2\delta, \omega)$ -sublevel sample of f and $B_{\omega-3c\delta}$ surrounds D in \mathbb{X} .

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381 Suppose $H_k(B_{\omega-3c\delta} \hookrightarrow B_\omega)$ is surjective and $H_k(B_\omega \hookrightarrow B_{\omega+5c\delta})$ is an isomorphism for
 382 all k . If

$$383 \text{rk } H_d(\mathcal{R}^\delta(P, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{2\delta}(P, Q_{\omega+c\delta})) \geq \dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-2c\delta}))$$

384 then the k th (relative) homology module of

$$385 \{\mathcal{R}^{2\delta}(P_{\lfloor \alpha \rfloor \omega-2c\delta}, Q_{\omega-2c\delta}) \hookrightarrow \mathcal{R}^{4\delta}(P_{\lfloor \alpha \rfloor \omega+c\delta}, Q_{\omega+c\delta})\}_{\alpha \in \mathbb{R}}$$

386 is $4c\delta$ -interleaved with $T_\omega^k \oplus \mathbb{L}_\omega^{k-1}$: the k th persistent homology module of $\{(D_{\lfloor \alpha \rfloor \omega}, B_\omega)\}_{\alpha \in \mathbb{R}}$.

Proof.

387 : (

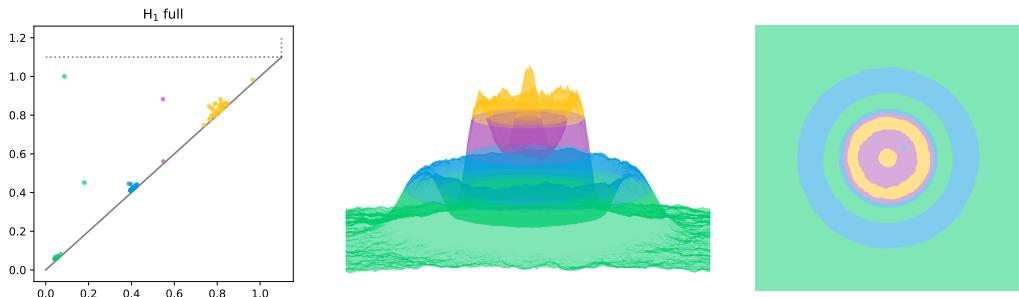
388



389 6 Experiments

390 In this section we will discuss a number of experiments which illustrate the benefit of
 391 truncated diagrams, and their approximation by relative diagrams, in comparison to their
 392 restricted counterparts. We will focus on the persistent homology of functions on a square
 393 2d grid. We chose as our function a radially symmetric damped sinusoid with random noise,
 394 depicted in Figure 3, as it has prominent persistent homology in dimension one.

395 **Experimental setup.**



396 **Figure 3** The H_1 persistence diagram of the sinusoidal function pictured to the right. Features
 397 are colored by birth time, infinite features are drawn above the dotted line.

399 Throughout, the four interlevel sets shown correspond to the ranges $[0, 0.3]$, $[0.3, 0.5]$,
 400 $[0.5, 0.7]$, and $[0.7, 1]$, respectively. Our persistent homology computations were done primarily
 401 with Dionysus augmented with custom software for computing representative cycles of
 402 infinite features.⁷ The persistent homology of our function was computed with the lower-star
 403 filtration of the Freudenthal triangulation on an $N \times N$ grid over $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. We
 404 take this filtration as $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ where P is the set of grid points and $\delta = \sqrt{2}/N$.

405 We note that the purpose of these experiments is not to demonstrate the effectiveness of our
 406 approximation by Rips complexes, but to demonstrate the relationships between restricted,

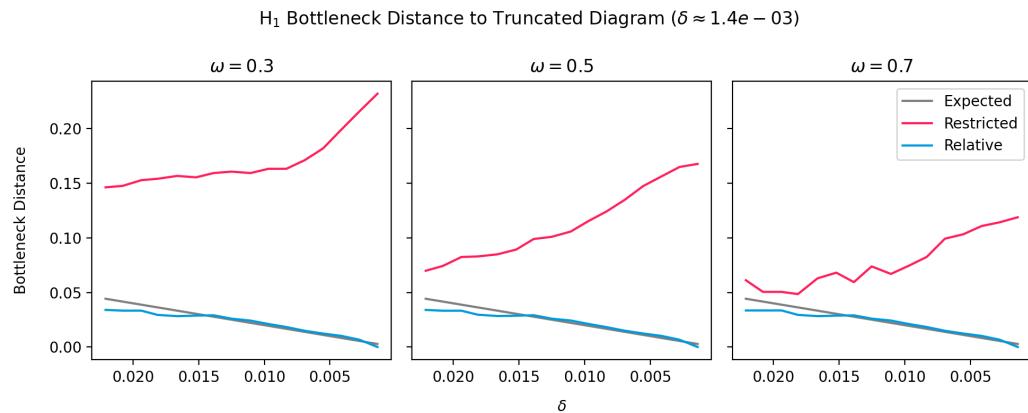
398 ⁷ 3D figures were made with Mayavi, all other figures were made with Matplotlib.

relative, and truncated diagrams. Therefore, for simplicity, we will omit the inclusion $\mathcal{R}^{2\delta}(P_\alpha) \hookrightarrow \mathcal{R}^{4\delta}(P_\alpha)$ and take the persistent homology of $\{\mathcal{R}^{2\delta}(P_\alpha)\}$ with sufficiently small δ as our ground-truth.

In the following we will take $N = 1024$, so $\delta \approx 1.4 \times 10^{-3}$, as our ground-truth. Figure 3 shows the *full diagram* of our function with features colored by birth time. Therefore, for $\omega = 0.3, 0.5, 0.7$ the *truncated diagram* is obtained by successively removing features in each interlevel set. Recall the *restricted diagram* is that of the function restricted to the ω super-levelset filtration, and computed with $\{\mathcal{R}^{2\delta}(P_\alpha \setminus Q_\omega)\}$. We will compare this restricted diagram with the *relative diagram*, computed as the relative persistent homology of the filtration of pairs $\{\mathcal{R}^{2\delta}(P_\alpha, Q_\omega)\}$.

417 The issue with restricted diagrams.

Figure ?? shows the bottleneck distance from the truncated diagram at full resolution ($N = 1024$) to both the relative and restricted diagrams with varying resolution. Specifically, the function on a 1024×1024 grid is down-sampled to grids ranging from 64×64 to 1024×1024 . We also show the expected bottleneck distance to the true truncated diagram given by the interleaving in Theorem 17 in black.



423 **Figure 4** Comparison of the bottleneck distance between the truncated diagram and those of the
424 restricted and relative diagrams with increasing resolution.

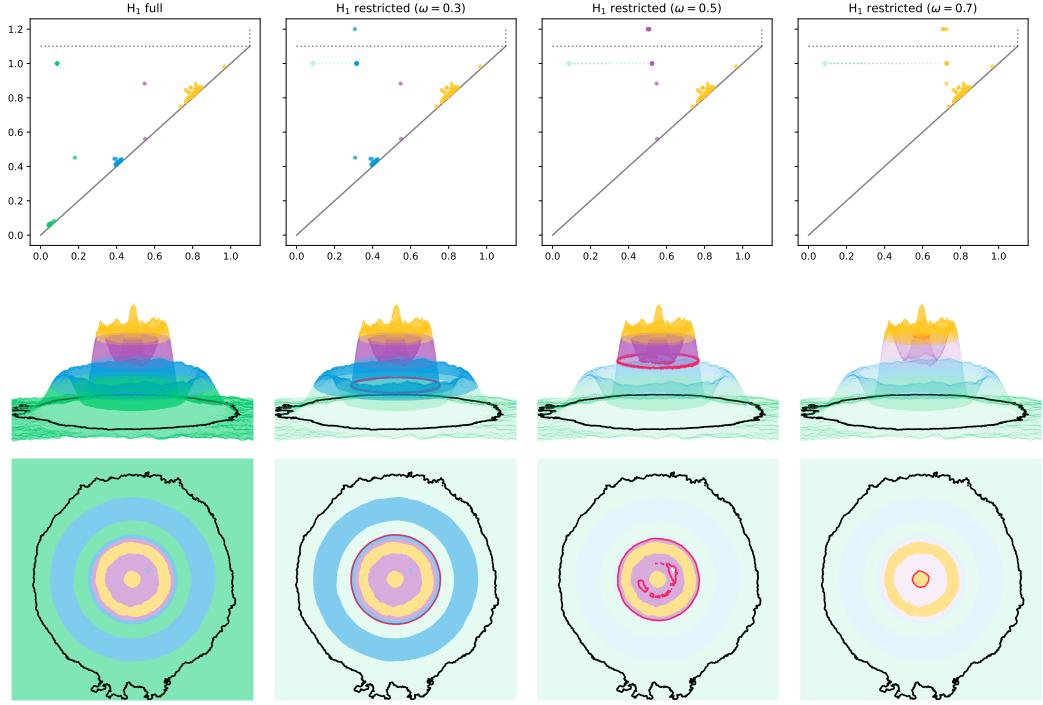
425 As we can see, the relative diagram clearly performs better than the restricted diagram,
426 which diverges with increasing resolution. Recall that 1-dimensional features that are born
427 before ω and die after ω become infinite 2-dimensional features in the relative diagram, with
428 birth time equal to the death time of the corresponding feature in the full diagram. These
429 same features remain 1-dimensional figures in the restricted diagram, but with their birth
430 times shifted to ω .

435 Figure 5 shows this distance for a feature that persists throughout the diagram. As the
436 restricted diagram in full resolution the restricted filtration is a subset of the full filtration,
437 so these features can be matched by their death simplices. For illustrative purposes we also
438 show the representative cycles associated with these features.

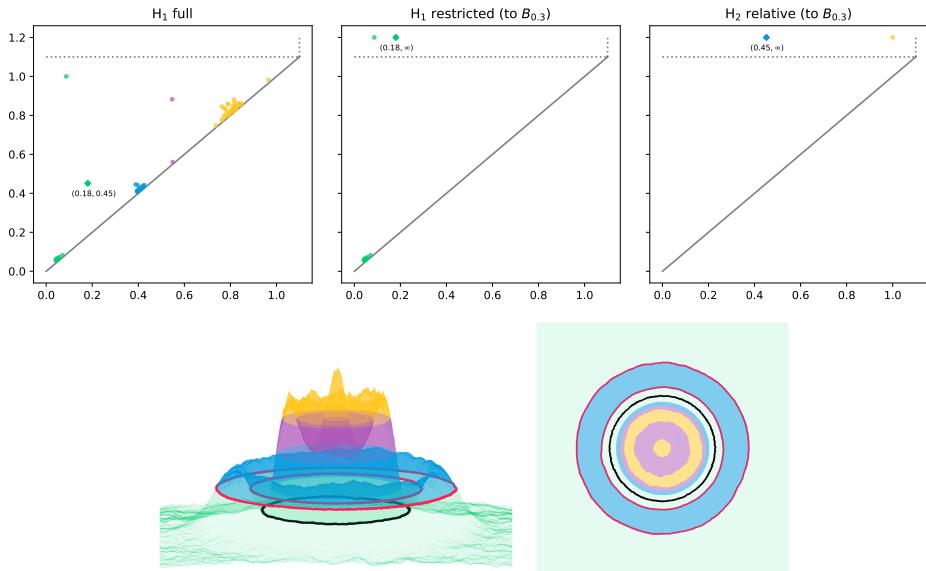
439 Relative diagrams and reconstruction.

445 Now, imagine we obtain the persistence diagram of our sub-levelset B_ω . That is, we now
446 know that we cover B_ω , or some subset, and do not want to re-compute the diagram above

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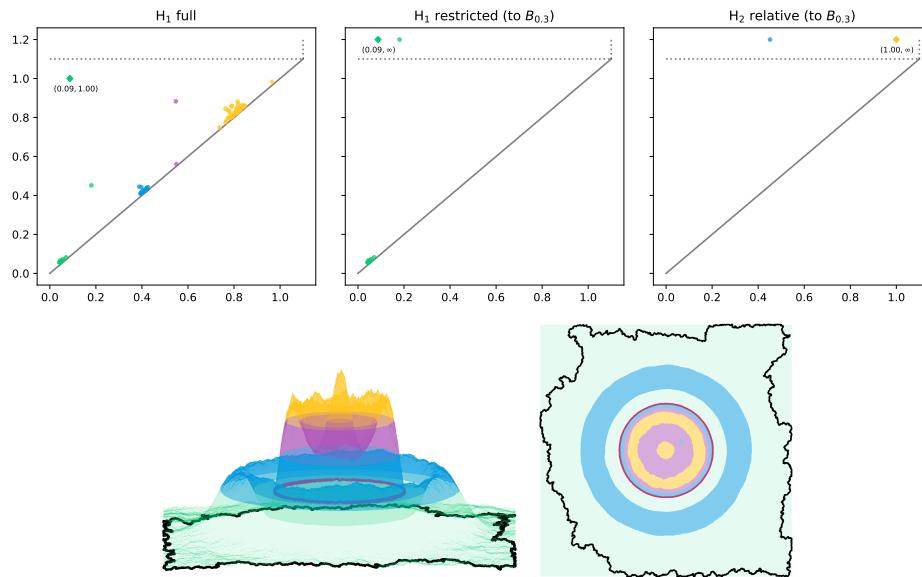
431 ■ **Figure 5** (Top) H_1 persistence diagrams of the function depicted in Figure 3 restricted to *super-*
432 levelsets at $\omega = 0.3, 0.5$, and 0.7 (on a 1024×1024 grid). The matching is shown between a feature in
433 the full diagram (marked with a diamond) with its representative cycle in black. The corresponding
434 representative cycle in the restricted diagram is pictured in red.



440 ■ **Figure 6** (Top) The indicated infinite features in the restricted and relative diagrams correspond
441 to the birth and death of the 1-feature $(0.18, 0.45)$ in the full diagram. (Bottom) In black, the
442 representative cycle of the infinite 1-feature born at 0.18 in the restricted diagram is shown in black.
443 In red, the *boundary* of the representative relative 2-cycle born at 0.45 in the relative diagram is
444 shown in red.

447 ω . If we compute the persistence diagram of the function restricted to the *sub-levelset* B_ω
 448 any 1-dimensional features born before ω that die after ω will remain infinite features in
 449 this restricted (below) diagram. Indeed, we could match these infinite 1-features with the
 450 corresponding shifted finite 1-features in the restricted (above) diagram, as shown in Figure 5.
 451 However, that would require sorting through all finite features that are born near ω and
 452 deciding if they are in fact features of the full diagram that have been shifted.

453 Recalling that these same features become infinite 2-features in the relative diagram, we
 454 can use the relative diagram instead and match infinite 1-features of the diagram restricted
 455 below to infinite 2-features in the relative diagram, as shown in Figures 6 and 7. For this
 456 example the matching is given by sorting the 1-features by ascending and the 2-features by
 457 descending birth time. How to construct this matching in general, especially in the presence
 458 of infinite features in the full diagram, is the subject of future research.



459 ■ **Figure 7** The infinite 1-features of the restricted diagram can be matched with the infinite
 460 2-features of the relative diagrams. The sequence birth times of relative 2-features in *decreasing*
 461 order correspond to the deaths of restricted 1-features in *increasing* order.

462 7 Conclusion

463 We have extended the Topological Coverage Criterion to the setting of Topological Scalar
 464 Field Analysis. By defining the boundary in terms of a sublevel set of a scalar field we
 465 provide an interpretation of the TCC that applies more naturally to data coverage. We then
 466 showed how the assumptions and machinery of the TCC can be used to approximate the
 467 persistent homology of the scalar field relative to a static sublevel set. This relative persistent
 468 homology is shown to be related to a truncation of that of the scalar field as whole, and
 469 therefore provides a way to approximate a part of its persistence diagram in the presence of
 470 un-verified data.

471 There are a number of unanswered questions and directions for future work. From the
 472 theoretical perspective, our understanding of duality limited us in providing a more elegant
 473 extension of the TCC. A better understanding of when and how duality can be applied would

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allow us to give a more rigorous statement of our assumptions. Moreover, as duality plays a central role in the TCC it is natural to investigate its role in the analysis of scalar fields. This would not only allow us to apply duality to persistent homology [8], but also allow us to provide a rigorous comparison between the relative approach and the persistent homology of the superlevel set filtration and explore connections with Extended Persistence [5].

From a computational perspective, we interested in exploring how to recover the full diagram as discussed in Section 6. Our statements in terms of sublevel sets can be generalized to disjoint unions of sub and superlevel sets, where coverage is confirmed in an *interlevel* set. This, along with a better understanding of the relationship between sub and superlevel sets could lead to an iterative approach in which the persistent homology of a scalar field is constructed as data becomes available. We are also interested in finding efficient ways to compute the image persistent (relative) homology that vary in both scalar and scale.

The problem of relaxing our assumptions on the boundary can be approached from both a theoretical and computational perspective. Ways to avoid the isomorphism we require could be investigated in theory, and the interaction of relative persistent homology and the Persistent Nerve Lemma may be used tighten our assumptions. We would also like to conduct a more rigorous investigation on the effect of these assumptions in practice.

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521 **A Omitted Proofs**

522 **Proof of Lemma 2.** This proof is in two parts.

523 **ℓ injective $\implies D \setminus B \subseteq U$** Suppose, for the sake of contradiction, that p is injective and
 524 there exists a point $x \in (D \setminus B) \setminus U$. Because B surrounds D in X the pair $(D \setminus B, \overline{D})$
 525 forms a separation of \overline{B} . Therefore, $H_0(\overline{B}) \cong H_0(D \setminus B) \oplus H_0(\overline{D})$ so

$$526 \quad H_0(\overline{B}, \overline{D}) \cong H_0(D \setminus B).$$

527 So $[x]$ is non-trivial in $H_0(\overline{B}, \overline{D}) \cong H_0(D \setminus B)$ as x is in some connected component of
 528 $D \setminus B$. So we have the following sequence of maps induced by inclusions

$$529 \quad H_0(\overline{B}, \overline{D}) \xrightarrow{f} H_0(\overline{B}, \overline{D} \cup \{x\}) \xrightarrow{g} H_0(\overline{V}, \overline{U}).$$

530 As $f[x]$ is trivial in $H_0(\overline{B}, \overline{D} \cup \{x\})$ we have that $\ell[x] = (g \circ f)[x]$ is trivial, contradicting
 531 our hypothesis that ℓ is injective.

532 **ℓ injective $\implies V$ surrounds U in D .** Suppose, for the sake of contradiction, that V does
 533 not surround U in D . Then there exists a path $\gamma : [0, 1] \rightarrow \overline{V}$ with $\gamma(0) \in U \setminus V$ and
 534 $\gamma(1) \in D \setminus U$. As we have shown, $D \setminus B \subseteq U$, so $D \setminus B \subseteq U \setminus V$.

535 Choose $x \in D \setminus B$ and $z \in \overline{D}$ such that there exist paths $\xi : [0, 1] \rightarrow U \setminus V$ with $\xi(0) = x$,
 536 $\xi(1) = \gamma(0)$ and $\zeta : [0, 1] \rightarrow \overline{D} \cup (D \setminus U)$ with $\zeta(0) = z$, $\zeta(1) = \gamma(1)$. ξ, γ and ζ all
 537 generate chains in $C_1(\overline{V}, \overline{U})$ and $\xi + \gamma + \zeta = \gamma^* \in C_1(\overline{V}, \overline{U})$ with $\partial\gamma^* = x + z$. Moreover, z
 538 generates a chain in $C_0(\overline{U})$ as $\overline{D} \subseteq \overline{U}$. So $x = \partial\gamma^* + z$ is a relative boundary in $C_0(\overline{V}, \overline{U})$,
 539 thus $\ell[x] = \ell[z]$ in $H_0(\overline{V}, \overline{L})$. However, because B surrounds D , $[x] \neq [y]$ in $H_0(\overline{B}, \overline{D})$
 540 contradicting our assumption that ℓ is injective.

541 ◀

542 **Proof of Lemma 4.** Assume there exist $p, q \in P \setminus Q_{\omega-c\zeta}$ such that p and q are connected in
 543 $\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})$ but not in $D \setminus B_\omega$. So the shortest path from p, q is a subset of $(P \setminus Q_{\omega-c\zeta})^\delta$.
 544 For any $x \in (P \setminus Q_{\omega-c\zeta})^\delta$ there exists some $p \in P$ such that $f(p) > \omega - c\zeta$ and $d(p, x) < \delta$.
 545 Because f is c -Lipschitz

$$546 \quad f(x) \geq f(p) - cd(x, p) > \omega - c(\delta + \zeta)$$

547 so there is a path from p to q in $D \setminus B_{\omega-c(\delta+\zeta)}$, thus $[p] = [q]$ in $H_0(D \setminus B_{\omega-c(\delta+\zeta)})$.

548 But we have assumed that $[p] \neq [q]$ in $H_0(D \setminus B_\omega)$, contradicting our assumption that
 549 $H_0(D \setminus B_\omega \hookrightarrow D \setminus B_{\omega-c(\delta+\zeta)})$ is injective, so any p, q connected in $\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})$ are
 550 connected in $D \setminus B_\omega$. That is, $\dim H_0(\mathcal{R}^\delta(P \setminus Q_{\omega-c\zeta})) \geq \dim H_0(D \setminus B_\omega)$. ◀

551 **A.1 Extensions**

552 **Proof of Lemma 8.** Note that $B' \setminus (D \setminus U) = B' \cap U \subseteq V$ implies $B' \subseteq V \sqcup (D \setminus U) = \mathcal{E}V$.
 553 Moreover, because $V \subseteq B$ and $D \setminus B \subseteq U$ implies $D \setminus U \subset D \setminus (D \setminus B) = B$, we have

$$554 \quad \mathcal{E}V = V \sqcup (D \setminus U) \subseteq B \cup (D \setminus U) = B.$$

555 So $B' \subseteq \mathcal{E}V \subseteq B$ as desired. ◀

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556 **Proof of Lemma 9.** Because V surrounds U in D , $(U \setminus V, D \setminus U)$ is a separation of $D \setminus V$, a
557 subspace of D . So $\text{cl}_D(U \setminus V) \setminus U = \text{cl}_D(U \setminus V) \cap (D \setminus U) = \emptyset$ which implies $\text{cl}_D(U \setminus V) \subseteq$
558 $U = \text{int}_D(U)$ as U is open in D . Therefore,

$$\begin{aligned} \text{cl}_D(D \setminus U) &= D \setminus \text{int}_D(U) \\ &\subseteq D \setminus \text{cl}_D(U \setminus V) \\ &= \text{int}_D(D \setminus (U \setminus V)) \\ &= \text{int}_D(\mathcal{E}V). \end{aligned}$$

563 SO,

$$\begin{aligned} \text{H}_k(U \cap A, V) &= \text{H}_k(A \setminus (D \setminus U), \mathcal{E}V \setminus (D \setminus U)) \\ &\cong \text{H}_k(A, \mathcal{E}V) \end{aligned}$$

566 for all k and any $A \subseteq D$ such that $\mathcal{E}V \subset A$ by Excision. \blacktriangleleft

567 A.2 Image Modules

568 **► Lemma 20.** Suppose $\Gamma \in \text{Hom}(\mathbb{U}, \mathbb{V})$, $\Lambda \in \text{Hom}(\mathbb{S}, \mathbb{T})$, and $\Lambda' \in \text{Hom}(\mathbb{S}', \mathbb{T}')$. If $\Phi(F, G) \in$
569 $\text{Hom}^\delta(\mathbf{im} \Gamma, \mathbf{im} \Lambda)$ and $\Phi'(F', G') \in \text{Hom}^{\delta'}(\mathbf{im} \Lambda, \mathbf{im} \Lambda')$ then $\Phi''(F' \circ F, G' \circ G) := \Phi' \circ \Phi \in$
570 $\text{Hom}^{\delta+\delta'}(\mathbf{im} \Gamma, \mathbf{im} \Lambda')$.

571 **Proof.** Because $\Phi(F, G)$ is an image module homomorphism of degree δ we have $g_{\beta-\delta} \circ$
572 $\gamma_{\alpha-\delta}[\beta - \alpha] = \lambda_\alpha[\beta - \alpha] \circ f_{\alpha-\delta}$. Similarly, $g'_\beta \circ \lambda_\alpha[\beta - \alpha] = \lambda'_{\alpha+\delta'}[\beta - \alpha] \circ f'_\alpha$. So $\Phi''(F' \circ$
573 $F, G' \circ G) \in \text{Hom}^{\delta+\delta'}(\mathbf{im} \Gamma, \mathbf{im} \Lambda')$ as

$$\text{g}'_\beta \circ (g_{\beta-\delta} \circ \gamma_{\alpha-\delta}[\beta - \alpha]) = (g'_\beta \circ \lambda_\alpha[\beta - \alpha]) \circ f_{\alpha-\delta} = \lambda_{\alpha+\delta'}[\beta - \alpha] \circ f'_\alpha \circ f_{\alpha-\delta}$$

575 for all $\alpha \leq \beta$. \blacktriangleleft

576 **Proof of Lemma 13.** For ease of notation let Φ denote $\Phi_M(F, G)$ and Ψ denote $\Psi_G(M, N)$.

577 If Γ is an epimorphism γ_α is surjective so $\Gamma_\alpha = V_\alpha$ and $\phi_\alpha = g_\alpha|_{\Gamma_\alpha} = g_\alpha$ for all α . So
578 $\mathbf{im} \Gamma = \mathbb{V}$ and $\Phi \in \text{Hom}^\delta(\mathbb{V}, \mathbf{im} \Lambda)$.

579 If Π is a monomorphism then π_α is injective so we can define a natural isomorphism
580 $\pi_\alpha^{-1} : \Pi_\alpha \rightarrow V_\alpha$ for all α . Let Ψ^* be defined as the family of linear maps $\{\psi_\alpha^* := \pi_\alpha^{-1} \circ \psi_\alpha : \mathbb{A}_\alpha \rightarrow V_{\alpha+\delta}\}$. Because Ψ is a partial δ -interleaving of image modules, $n_\alpha \circ \lambda_\alpha = \pi_{\alpha+\delta} \circ m_\alpha$.
582 So, because $\psi_\alpha = n_\alpha|_{\Lambda_\alpha}$ for all α ,

$$\begin{aligned} \mathbf{im} \psi_\alpha^* &= \mathbf{im} \pi_{\alpha+\delta}^{-1} \circ \psi_\alpha \\ &= \mathbf{im} \pi^{-1} \circ (n_\alpha \circ \lambda_\alpha) \\ &= \mathbf{im} \pi^{-1} \circ (\pi_{\alpha+\delta} \circ m_\alpha) \\ &= \mathbf{im} m_\alpha. \end{aligned}$$

587 It follows that $\mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ \psi_\alpha^* = \mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha$

588 Similarly, because Ψ is a δ -interleaving of image modules $n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha = w_{\alpha+\delta}^{\beta+\delta} \circ \pi_{\alpha+\delta} \circ m_\alpha$.
589 Moreover, because Π is a homomorphism of persistence modules, $w_{\alpha+\delta}^{\beta+\delta} \circ \pi_{\alpha+\delta} = \pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta}$,
590 SO

$$\text{n}_\beta \circ t_\alpha^\beta \circ \lambda_\alpha = \pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha.$$

592 As $\psi_\beta \circ \lambda_\alpha^\beta = n_\beta \circ \lambda_\alpha^\beta = n_\beta \circ t_\alpha^\beta|_{\Lambda_\alpha}$ it follows

$$\begin{aligned} 593 \quad \mathbf{im} \psi_\beta^* \circ \lambda_\alpha^\beta &= \mathbf{im} \pi_{\beta+\delta}^{-1} \circ (n_\beta \circ t_\alpha^\beta \circ \lambda_\alpha) \\ 594 &= \mathbf{im} \pi_{\beta+\delta}^{-1} \circ (\pi_{\beta+\delta} \circ v_{\alpha+\delta}^{\beta+\delta}) \circ m_\alpha \\ 595 &= \mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ m_\alpha \\ 596 &= \mathbf{im} v_{\alpha+\delta}^{\beta+\delta} \circ \psi_\alpha^*. \end{aligned}$$

597 So we may conclude that $\Psi^* \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbb{V})$.

598 So $\Phi \in \text{Hom}^\delta(\mathbb{V}, \mathbf{im} \Lambda)$ and $\Psi_G^* \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbb{V})$. As we have shown, $\mathbf{im} \psi_{\alpha-\delta}^* = \mathbf{im} m_{\alpha-\delta}$ so $\mathbf{im} \phi_\alpha \circ \psi_{\alpha-\delta}^* = \mathbf{im} \phi_\alpha \circ m_{\alpha-\delta}$. Moreover, because γ_α is surjective $\phi_\alpha = g_\alpha$ and, because Φ is a partial δ -interleaving of image modules, $g_\alpha \circ m_{\alpha-\delta} = t_{\alpha-\delta}^{\alpha+\delta} \circ \lambda_{\alpha-\delta}$. As $\lambda_{\alpha-\delta}^{\alpha+\delta} = t_{\alpha-\delta}^{\alpha+\delta}|_{\mathbf{im} \lambda_{\alpha-\delta}}$ it follows that $\mathbf{im} \phi_\alpha \circ \psi_{\alpha-\delta}^* = \mathbf{im} \lambda_{\alpha-\delta}^{\alpha+\delta}$.

600 Finally, $\psi_\alpha^* \circ \phi_\alpha = \pi_{\alpha+\delta}^{-1} \circ n_\alpha \circ g_{\alpha-\delta}$ where, because Ψ is a partial δ -interleaving of image 601 modules, $n_\alpha \circ g_{\alpha-\delta} = w_{\alpha-\delta}^{\alpha+\delta} \circ \pi_{\alpha-\delta}$. Because Π is a homomorphism of persistence modules 602 $w_{\alpha-\delta}^{\alpha+\delta} \circ \pi_{\alpha-\delta} = \pi_{\alpha+\delta} \circ v_{\alpha-\delta}^{\alpha+\delta}$. Therefore,

$$\begin{aligned} 603 \quad \psi_\alpha^* \circ \phi_\alpha &= \pi_{\alpha+\delta}^{-1} \circ n_\alpha \circ g_{\alpha-\delta} \\ 604 &= \pi_{\alpha+\delta}^{-1} \circ (\pi_{\alpha+\delta} \circ v_{\alpha-\delta}^{\alpha+\delta}) \\ 605 &= v_{\alpha-\delta}^{\alpha+\delta} \end{aligned}$$

606 which, along with $\phi_\alpha \circ \mathbf{im} \psi_{\alpha-\delta}^* = \lambda_{\alpha-\delta}^{\alpha+\delta}$ implies Diagrams ?? and ?? commute with 607 $\Phi \in \text{Hom}^\delta(\mathbb{V}, \mathbf{im} \Lambda)$ and $\Psi^* \in \text{Hom}^\delta(\mathbf{im} \Lambda, \mathbb{V})$. We may therefore conclude that $\mathbf{im} \Lambda$ and 608 \mathbb{V} are δ -interleaved. \blacktriangleleft

611 A.3 Partial Interleavings

612 **Proof of Lemma 14.** Suppose $x \in P^\delta \cap D_{[\alpha-c\varepsilon]_t-c\varepsilon}$. Because x in P^δ there exists some 613 $p \in P$ such that $d(x, p) < \delta$. Because f is c -Lipschitz $f(p) \leq f(x) + c\delta < f(x) + c\delta$. If $\alpha \leq t$ then 614 $x \in B_{t-c\varepsilon}$ implies $f(p) < t - c\varepsilon + c\delta \leq t$ so $x \in Q_t^\varepsilon$ as $\delta \leq \varepsilon$. If $\alpha \geq t$ then 615 $x \in B_{\alpha-c\varepsilon}$ which implies $f(p) \leq \alpha$ $x \in Q_\alpha^\varepsilon$. So $P^\delta \cap D_{[\alpha-c\varepsilon]_t-c\varepsilon} \subseteq P_{[\alpha]_t}^\varepsilon$ as $P_{[\alpha]_t} = Q_t^\varepsilon \cup Q_\alpha^\varepsilon$.

616 Now, suppose $x \in P_{[\alpha]_t}^\varepsilon$. If $\alpha \leq t$ then $x \in Q_t^\varepsilon \subseteq B_{t+c\varepsilon}$ because f is c -Lipschitz. Similarly, 617 $\alpha > t$ implies $x \in Q_\alpha^\varepsilon \subseteq B_{\alpha+c\varepsilon}$, so $P_{[\alpha]_t}^\varepsilon \subseteq D_{[\alpha+c\varepsilon]_t+c\varepsilon}$ as $D_{[\alpha+c\varepsilon]_t+c\varepsilon} = B_{t+c\varepsilon} \cup B_{\alpha+c\varepsilon}$. \blacktriangleleft

618 **Proof of Lemma 15.** Because Q_t^δ surrounds P^δ in D and $\delta \leq \varepsilon$, $t < v$ we know Q_t^ε and Q_v^ε 619 surround P^δ in D . As $P^\delta \cap B_s \subseteq Q_t^\varepsilon$ and $P^\delta \cap B_u \subseteq Q_v^{2\varepsilon}$ for all $\varepsilon \in [\delta, 2\delta]$ Lemma 8 implies 620 that we have a sequence of inclusions $B_s \subseteq \mathcal{E}Q_t^\varepsilon \subseteq B_u \subseteq \mathcal{E}Q_v^{2\varepsilon} \subseteq B_w$.

621 For any $\alpha \in \mathbb{R}$ we know that $D \setminus P^\delta \subseteq \mathcal{E}P_{[\alpha]_t}^\varepsilon$ by the definition of $\mathcal{E}P_{[\alpha]_t}^\varepsilon$. Moreover, 622 $D \setminus P^\delta \subseteq D_{[\alpha]_u}$ because $D \setminus B_u \subseteq P^\delta$. Lemma 14 therefore implies $D_{[\alpha-c\delta]_s} \subseteq \mathcal{E}P_{[\alpha]_t}^\varepsilon \subseteq$ 623 $D_{[\alpha+c\varepsilon]_u}$ as $s + c\delta \leq t \leq u - c\varepsilon$. So the inclusions $(D_{[\alpha-c\delta]_s}, B_s) \subseteq (\mathcal{E}P_{[\alpha]_t}^\varepsilon, \mathcal{E}Q_t^\varepsilon)$ induce 624 $F \in \text{Hom}^{c\delta}(\mathbb{D}_s, \mathcal{EP}_t^\varepsilon)$ and $(\mathcal{EP}_{[\alpha]_t}^\varepsilon, \mathcal{E}Q_t^\varepsilon) \subseteq (D_{[\alpha+c\varepsilon]_u}, B_u)$ induce $M \in \text{Hom}^{c\varepsilon}(\mathcal{EP}_t^\varepsilon, \mathbb{D}_u)$.

625 By an identical argument Lemma 14 implies $D_{[\alpha-2c\delta]_u} \subseteq \mathcal{E}P_{[\alpha]_v}^\varepsilon \subseteq D_{[\alpha+2c\varepsilon]_w}$ as $u + c\delta \leq$ 626 $v \leq w - 4c\delta$. So $(D_{[\alpha-2c\delta]_u}, B_u) \subseteq (\mathcal{EP}_{[\alpha]_v}^\varepsilon, \mathcal{E}Q_v^{2\varepsilon})$ induce $G \in \text{Hom}^{2c\delta}(\mathbb{D}_u, \mathcal{EP}_v^{2\varepsilon})$ and 627 $(\mathcal{EP}_{[\alpha]_v}^\varepsilon, \mathcal{E}Q_v^{2\varepsilon}) \subseteq (D_{[\alpha+2c\varepsilon]_w}, B_w)$ induce $N \in \text{Hom}^{2c\varepsilon}(\mathcal{EP}_v^{2\varepsilon}, \mathbb{D}_w)$. \blacktriangleleft

628 A.4 Truncated Interval Modules

629 **Proof of Lemma 18.** Suppose $\alpha \leq \omega$. So $H_k(D_{[\alpha]_\omega}, B_\omega) = 0$ as $D_{[\alpha]_\omega} = B_\omega \cup B_\alpha$ and 630 $\mathbb{T}_\omega^k = 0$ as $F_\alpha^I = 0$ for any $I \in \mathcal{I}^k$ such that $\omega \in I_-$. Moreover, $\omega \in I$ for all $I \in \mathcal{I}_\omega^{k-1}$, thus 631 $F_\alpha^{I+} = 0$ for all $\alpha \leq \omega$. So it suffices to assume $\omega < \alpha$.

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632 Consider the long exact sequence of the pair $H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega) = H_k(B_\alpha, B_\omega)$

$$633 \dots \rightarrow H_k(B_\omega) \xrightarrow{p_\alpha^k} H_k(B_\alpha) \xrightarrow{q_\alpha^k} H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega) \xrightarrow{r_\alpha^k} H_{k-1}(B_\omega) \xrightarrow{p_\alpha^{k-1}} H_{k-1}(B_\alpha) \rightarrow \dots$$

634 where $H_k(B_\alpha) = \bigoplus_{I \in \mathcal{I}^k} F_\alpha^I$, $H_k(B_\omega) = \bigoplus_{I \in \mathcal{I}^k} F_\omega^I$, and $p_\alpha^k = \bigoplus_{I \in \mathcal{I}^k} f_{\omega, \alpha}^I$.

635 Noting that $\text{im } q_\alpha^k \cong H_k(B_\alpha)/\ker q_\alpha^k$ where $\ker q_\alpha^k = \text{im } p_\alpha^k$ by exactness we have
636 $\ker r_\alpha^k \cong H_k(B_\alpha)/\text{im } p_\alpha^k$. By the definition of F_α^I and $f_{\omega, \alpha}^I$ we know $\text{im } f_{\omega, \alpha}^I$ is F_α^I if $\omega \in I$
637 and 0 otherwise. As $\text{im } p_\alpha^k$ is equal to the direct sum of images $\text{im } f_{\omega, \alpha}^I$ over $I \in \mathcal{I}^k$ it follows
638 that $\text{im } p_\alpha^k$ is the direct sum of those F_α^I over those $I \in \mathcal{I}^k$ such that $\omega \in I$. Now, because
639 $H_k(B_\alpha) = \bigoplus_{I \in \mathcal{I}^k} F_\alpha^I$ and each F_α^I is either 0 or \mathbb{F} the quotient $H_k(B_\alpha)/\text{im } p_\alpha^k$ is the direct
640 sum of those F_α^I such that $\omega \notin I$. Therefore, by the definition of $F_{\lfloor \alpha \rfloor \omega}^I$ we have

$$641 \quad \ker r_\alpha^k = \bigoplus_{I \in \mathcal{I}_\omega^k} F_{\lfloor \alpha \rfloor \omega}^I.$$

642 Similarly, $\text{im } r_\alpha^k = \ker p_\alpha^{k-1}$ by exactness where $\ker p_\alpha^{k-1}$ is the direct sum of kernels
643 $\ker f_{\omega, \alpha}^I$ over $I \in \mathcal{I}^{k-1}$. By the definition of F_α^I and $f_{\omega, \alpha}^I$ we know that $\ker f_{\omega, \alpha}^I$ is F_α^I if
644 $\omega \notin I$ and 0 otherwise. Noting that $\ker f_{\omega, \alpha}^I = 0$ for any $I \in \mathcal{I}^{k-1}$ such that $\omega \notin I$ it suffices
645 to consider only those $I \in \mathcal{I}_\omega^{k-1}$. It follows that $\ker f_{\omega, \alpha}^I = F_\alpha^{I+}$ for any I containing ω as
646 $\omega < \alpha$. Therefore,

$$647 \quad \text{im } r_\alpha^k = \bigoplus_{I \in \mathcal{I}^{k-1}} F_\alpha^{I+}.$$

648 We have the following split exact sequence associated with r_α^k

$$649 \quad 0 \rightarrow \ker r_\alpha^k \rightarrow H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega) \rightarrow \text{im } r_\alpha^k \rightarrow 0.$$

650 The desired result follows from the fact that for all $\alpha \in \mathbb{R}$

$$651 \quad H_k(D_{\lfloor \alpha \rfloor \omega}, B_\omega) \cong \ker r_\alpha^k \oplus \text{im } r_\alpha^k \\ 652 \quad = \bigoplus_{I \in \mathcal{I}^k} F_{\lfloor \alpha \rfloor \omega}^I \oplus \bigoplus_{I \in \mathcal{I}_\omega^{k-1}} F_\alpha^{I+}.$$

653

654 B Duality

655 For a pair (A, B) in a topological space X and any R module G let $H^k(A, B; G)$ denote
656 the **singular cohomology** of (A, B) (with coefficients in G). Let $H_c^k(A, B; G)$ denote
657 the corresponding **singular cohomology with compact support**, where $H_c^k(A, B; G) \cong$
658 $H^k(A, B; G)$ for any compact pair (A, B) .

659 The following corollary follows from the Universal Coefficient Theorem for singular
660 homology (and cohomology) as vector spaces over a field \mathbb{F} , as the dual vector space
661 $\text{Hom}(H_k(A, B), \mathbb{F})$ is isomorphic to $H_k(A, B; \mathbb{F})$ for any finitely generated $H_k(A, B)$.

662 ▶ **Corollary 21.** *For a topological pair (A, B) and a field \mathbb{F} such that $H_0(A, B)$ is finitely
663 generated there is a natural isomorphism*

$$664 \quad \nu : H^0(A, B; \mathbb{F}) \rightarrow H_0(A, B; \mathbb{F}).$$

Let $\overline{H}^k(A, B; G)$ be the **Alexander-Spanier cohomology** of the pair (A, B) , defined as the limit of the direct system of neighborhoods (U, V) of the pair (A, B) . Let $\overline{H}_c^k(A, B; G)$ denote the corresponding **Alexander-Spanier cohomology with compact support** where $\overline{H}_c^k(A, B; G) \cong \overline{H}^k(A, B; G)$ for any compact pair (A, B) .

► **Theorem 22 (Alexander-Poincaré-Lefschetz Duality)** (Spanier [11], Theorem 6.2.17)). *Let X be an orientable d -manifold and (A, B) be a compact pair in X . Then for all k and R modules G there is a (natural) isomorphism*

$$\lambda : H_k(X \setminus B, X \setminus A; G) \rightarrow \overline{H}^{d-k}(A, B; G).$$

A space X is said to be **homologically locally connected in dimension n** if for every $x \in X$ and neighborhood U of x there exists a neighborhood V of x in U such that $\tilde{H}_n(V) \rightarrow \tilde{H}_n(U)$ is trivial for $k \leq n$.

► **Lemma 23** (Spanier p. 341, Corollary 6.9.6). *Let A be a closed subset, homologically locally connected in dimension n , of a Hausdorff space X , homologically locally connected in dimension n . If X has the property that every open subset is paracompact, $\mu : \overline{H}_c^k(X, A; G) \rightarrow H_c^k(X, A; G)$ is an isomorphism for $k \leq n$ and a monomorphism for $k = n + 1$.*

In the following we will assume homology (and cohomology) over a field \mathbb{F} .

► **Lemma 24.** *Let X be an orientable d -manifold and (A, B) a compact pair of locally path connected subspaces in X . Then*

$$\xi : H_d(X \setminus B, X \setminus A) \rightarrow H_0(A, B)$$

is a natural isomorphism.

Proof. Because X is orientable and (A, B) are compact $\lambda : H_d(X \setminus B, X \setminus A) \rightarrow \overline{H}^0(A, B)$ is an isomorphism by Theorem 22. Note that Moreover, because every subset of X is (hereditarily) paracompact every open set in A , with the subspace topology, is paracompact. For any neighborhood U of a point x in a locally path connected space there must exist some neighborhood $V \subset U$ of x that is path connected in the subspace topology. As $\tilde{H}_0(V) = 0$ for any nonempty, path connected topological space V (see Spanier p. 175, Lemma 4.4.7) it follows that A (resp. B) are homologically locally connected in dimension 0. Because (A, B) is a compact pair the singular and Alexander-Spanier cohomology modules of (A, B) with compact support are isomorphic to those without, thus $\mu : \overline{H}^0(A, B) \rightarrow H^0(A, B)$ is an isomorphism. By Corollary 21 we have a natural isomorphism $\nu : H^0(A, B) \rightarrow H_0(A, B)$ thus the composition $\xi := \nu \circ \mu \circ \lambda : H_d(X \setminus B, X \setminus A) \rightarrow H_0(A, B)$ is a natural isomorphism. ◀

► **Lemma 25.** *Let \mathbb{X} be an orientable d -manifold let D be a compact subset of \mathbb{X} . Let P be a finite subset of D such that $P^\varepsilon \subset \text{int}_{\mathbb{X}}(D)$ and $Q \subseteq P$.*

If $D \setminus Q^\varepsilon$ and $D \setminus P^\varepsilon$ are locally path connected then there is a natural isomorphism

$$\xi : H_d(P^\varepsilon, Q^\varepsilon) \rightarrow H_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon).$$

Proof. Because Q^ε and P^ε are open in D and D is compact in \mathbb{X} the complement $D \setminus Q^\varepsilon$ is closed in D , and therefore compact in \mathbb{X} . Moreover, because $P^\varepsilon \subset \text{int}_{\mathbb{X}}(D)$, $H_d(\mathbb{X} \setminus (D \setminus P^\varepsilon), \mathbb{X} \setminus (D \setminus Q^\varepsilon)) = H_d(P^\varepsilon, Q^\varepsilon)$. As we have assumed these complements are locally path connected by assumption we have a natural isomorphism $\xi : H_d(P^\varepsilon, Q^\varepsilon) \rightarrow H_0(D \setminus Q^\varepsilon, D \setminus P^\varepsilon)$ by Lemma 24. ◀