

1 Some Useful Theorems

Lemma 1 (Splitting Lemma (Hatcher p. 147)). *For a short exact sequence*

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$$

of abelian groups the following statements are equivalent

1. *There is a homomorphism $p : B \rightarrow A$ such that $p \circ i = \text{Id}_A$.*
2. *There is a homomorphism $s : C \rightarrow B$ such that $j \circ s = \text{Id}_C$.*
3. *There is an isomorphism $B \cong A \oplus C$ making the commutative diagram below, where the maps in the lower row are the obvious ones $a \mapsto (a, 0)$ and $(a, c) \mapsto c$.*

$$\begin{array}{ccccccc} & & & B & & & \\ & & i \nearrow & \downarrow \cong & \nwarrow j & & \\ 0 \rightarrow & A & & & & C & \rightarrow 0 \\ & \searrow & & \downarrow & & \nearrow & \\ & & & A \oplus C & & & \end{array}$$

Lemma 2 (The Five-Lemma (Hatcher p. 129)). *In a commutative diagram of abelian groups as below, if the two rows are exact and α, β, δ , and ε are isomorphisms then γ is an isomorphism.*

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \varepsilon \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E' \end{array}$$

Theorem 1 (Alexander Duality). *If D is a compact, locally contractible, nonempty, proper subspace of S^d then for all k there is an isomorphism*

$$\Gamma_D^k : \tilde{H}_k(D) \rightarrow \tilde{H}^{d-k-1}(S^d \setminus D).$$

If (D, B) is a pair of such subspaces of S^d then for all k there is an isomorphism

$$\Gamma_{(D,B)}^k : \tilde{H}_k(D, B) \rightarrow \tilde{H}^{d-k}(S^d \setminus B, S^d \setminus D).$$

Lemma 3 (Lemma 3.2 from [?]). *Given a sequence $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F$ of homomorphisms between finite-dimensional vector spaces, if $\text{rk}(A \rightarrow F) = \text{rk}(C \rightarrow D)$ then this quantity also equals the rank of $B \rightarrow E$. Similarly, if $A \rightarrow B \rightarrow C \rightarrow E \rightarrow F$ is a sequence of homomorphisms such that $\text{rk}(A \rightarrow F) = \dim C$ then $\text{rk}(B \rightarrow E) = \dim C$.*

TODO

- **Excision**

2 Separation

Definition 1 (Separated). *Two subsets U, V of a topological space X are **separated in X** if each is disjoint from the other's closure: $\text{cl}(U) \cap V = \emptyset$ and $U \cap \text{cl}(V) = \emptyset$.*

Definition 2 (Separation). *We say that a subset B **separates** a topological space X with the pair (U, V) if B, U, V partitions X , U, V are separated in X , and any path from U to V in X intersects B .*

Lemma 4. *If B separates X with the pair (U, V) then for all k the short exact sequence*

$$0 \rightarrow H_k(V) \xrightarrow{i_*} H_k(X \setminus B) \xrightarrow{j_*} H_k(U) \rightarrow 0$$

splits.

Proof. Because $X \setminus B$ is the disjoint union of U and V we know that $i_* : H_k(V) \rightarrow H_k(X \setminus B)$ is the map induced by inclusion and $p_* : H_k(X \setminus B) \rightarrow H_k(V)$ is induced by the restriction of the identity on $X \setminus B$ to V . Thus $p_* \circ i_* = \text{Id}_{H_k(V)}$ and therefore, by Lemma 1 the sequence splits. \square

Corollary 1. *If B separates X with the pair (U, V) then for all k*

$$H_k(X \setminus B) \cong H_k(U) \oplus H_k(V).$$

Lemma 5. *If B separates X with the pair (U, V) then for all k*

$$H_k(U) \cong H_k(X \setminus B, V).$$

Proof. First note that the short exact sequence

$$0 \rightarrow H_k(V) \rightarrow H_k(U) \oplus H_k(V) \rightarrow H_k(U) \rightarrow 0$$

extends to a long exact sequence with the zero map $\partial_*^k : H_k(U) \rightarrow H_k(V)$ as $\text{im } j_*^k = H_k(U) = \ker \partial_*^k$ and $\text{im } \partial_*^k = \ker i_*^{k-1} = \mathbf{0}_{H_{k-1}(V)}$. Consider the following commutative diagram where the bottom row is the long exact sequence of the pair $(X \setminus B, V)$

$$\begin{array}{ccccccccccc} \dots & \rightarrow & H_k(V) & \xrightarrow{i_*^k} & H_k(U) \oplus H_k(V) & \xrightarrow{j_*^k} & H_k(U) & \xrightarrow{\partial_*^k} & H_{k-1}(V) & \xrightarrow{i_*^{k-1}} & H_{k-1}(U) \oplus H_{k-1}(V) & \rightarrow \dots \\ & & \downarrow f_*^k & & \downarrow g_*^k & & \downarrow h_*^k & & \downarrow f_*^{k-1} & & \downarrow g_*^{k-1} & \\ \dots & \rightarrow & H_k(V) & \xrightarrow{\widehat{i_*^k}} & H_k(X \setminus B) & \xrightarrow{\widehat{j_*^k}} & H_k(U) & \xrightarrow{\widehat{\partial_*^k}} & H_{k-1}(V) & \xrightarrow{\widehat{i_*^{k-1}}} & H_{k-1}(X \setminus B) & \longrightarrow \dots \end{array}$$

As f_*^k is the identity map and, by Corollary 1, g_*^k is an isomorphism for all k it follows that h_*^k is an isomorphism for all k by Lemma 2. \square

Definition 3 (Surrounding). *We say that $B \subset D$ **surrounds** $D \subset X$ in X if B separates X with the pair $(D \setminus B, X \setminus D)$. We will refer to such a pair (D, B) as a **surrounding pair** in X .*

We define the *extension* of a surrounding pair (D, B) in X to be the pair

$$(X, \hat{B}) = (D \cup (X \setminus D), B \cup (X \setminus D)).$$

Note that, for $X \subset \mathcal{X}$ the extension of a surrounding pair in X is a surrounding pair in \mathcal{X} .

Lemma 6. *If (D, B) is a surrounding pair of open sets in a topological space X then for all k*

$$H_k(D, B) \cong H_k(X, \hat{B}).$$

Proof. Note,

$$\begin{aligned} X \setminus (\text{int}(D) \cup \text{int}(\hat{B})) &= X \cap \overline{(\text{int}(D) \cup \text{int}(\hat{B}))} \\ &= X \cap \text{cl}(\overline{D}) \cap \text{cl}(\overline{\hat{B}}) \\ &= X \cap \overline{D} \cap \text{cl}(\overline{\hat{B}}), \end{aligned}$$

because D is open, thus \overline{D} is closed, and therefore equal to its closure.

Now,

$$\begin{aligned} \text{cl}(\overline{\hat{B}}) &= \text{cl}(\overline{B \cup (X \setminus D)}) \\ &= \text{cl}(\overline{B \cup (X \cap \overline{D})}) \\ &= \text{cl}(\overline{(B \cup X) \cap (B \cup \overline{D})}) \\ &= \text{cl}(\overline{X \cup (D \cap \overline{B})}) \\ &= \text{cl}(D \setminus B). \end{aligned}$$

Therefore,

$$X \setminus (\text{int}(D) \cup \text{int}(\hat{B})) = (X \setminus D) \cap \text{cl}(D \setminus B) = \emptyset$$

for a surrounding pair (D, B) in X which requires that $(X \setminus D)$ and $D \setminus B$ are separated in X .

So $\text{int}(D) \cup \text{int}(\hat{B}) = X$ thus $H_k(D, B) \cong H_k(X, \hat{B})$ by excision. \square

The following is a corollary of Theorem 1 (Alexander Duality).

Corollary 2. *If (D, B) is a surrounding pair of locally contractible, nonempty, proper subspaces in S^d then for all k*

$$\tilde{H}_k(D, B) \cong \tilde{H}^{d-k}(D \setminus B).$$

In the following we will assume that (D, B) is a surrounding pair of locally contractible, nonempty, proper subspaces in S^d . Let $(\overline{B}, \overline{D}) = (S^d \setminus B, S^d \setminus D)$ denote the complement of the pair (D, B) in S^d .

Lemma 7. *If (D, B) is a surrounding pair of locally contractible, nonempty, proper subspaces of S^d then*

$$i_*^k : \tilde{H}_{k+1}(D, B) \rightarrow \tilde{H}_k(B)$$

is injective and

$$j_*^k : \tilde{H}_k(B) \rightarrow \tilde{H}_k(D)$$

is surjective for all k .

Proof. We have the following commutative diagram of long exact sequences of the pairs (D, B) and $(\overline{B}, \overline{D})$.

$$\begin{array}{ccccc}
\tilde{H}_{k+1}(D, B) & \xrightarrow{\partial_*^{k+1}} & \tilde{H}_k(B) & \xrightarrow{i_*^k} & \tilde{H}_k(D) \\
\downarrow \Gamma_{(D,B)}^{k+1} & & \downarrow \Gamma_B^k & & \downarrow \Gamma_D^k \\
\tilde{H}^{d-k-1}(\overline{B}, \overline{D}) & \xrightarrow{\overline{j_*^{d-k-1}}} & \tilde{H}^{d-k-1}(\overline{B}) & \xrightarrow{\overline{i_*^{d-k-1}}} & \tilde{H}^{d-k-1}(\overline{D})
\end{array} \tag{1}$$

Because B surrounds D we have that

$$\tilde{H}^{d-k-1}(\overline{B}) \cong \tilde{H}^{d-k-1}(D \setminus B) \oplus H_k(\overline{D})$$

by Corollary 1¹, where $\tilde{H}^{d-k-1}(D \setminus B) \cong \tilde{H}^{d-k-1}(\overline{B}, \overline{D})$ by Lemma 5. It follows that $\overline{j_*^{d-k-1}}$ is injective and $\overline{i_*^{d-k-1}}$ is surjective.

By commutativity of Diagram 1 and because $\Gamma_{(D,B)}^{k+1}$, Γ_B^k and Γ_D^k are isomorphisms we have that

$$\partial_*^{k+1} = (\Gamma_B^k)^{-1} \circ \overline{j_*^{d-k-1}} \circ \Gamma_{(D,B)}^{k+1}$$

is injective and

$$i_*^k = (\Gamma_D^k)^{-1} \circ \overline{i_*^{d-k-1}} \circ \Gamma_B^{k+1}$$

is surjective. □

We note that this implies the following for non-reduced homology² for subsets of \mathbb{R}^d .³

Corollary 3. *If (D, B) is a surrounding pair of locally contractible, nonempty, proper subspaces of \mathbb{R}^d then*

$$i_*^k : H_{k+1}(D, B) \rightarrow H_k(B)$$

is injective and

$$j_*^k : H_k(B) \rightarrow H_k(D)$$

is surjective for all k .

¹**TODO** show/state this holds for cohomology as well.

²**TODO** reasoning:

- consider $H_1(D, B) \rightarrow H_0(B)$.
- $\tilde{H}_0(B) \rightarrow \tilde{H}_0(D)$ surjective implies $H_0(B) \rightarrow H_0(D)$ surjective (right?).

³**TODO** reasoning:

- $S^d \cong \mathbb{R}^d \cup \{\infty\}$.
- Only requires spaces *and* complements remain compact?

3 Separating Covers

In the following let $\mathbf{d}(x, y) = \|x - y\|$ denote the euclidean distance between points $x, y \in \mathbb{R}^d$. For $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ let

$$\mathbf{d}_A(x) = \min_{a \in A} \mathbf{d}(x, a)$$

denote the distance from x to the set A . In the following, we will use open metric balls

$$\mathbf{ball}_\varepsilon(x) = \{y \in \mathbb{R}^d \mid \mathbf{d}(x, y) < \varepsilon\}$$

and offsets

$$A^\varepsilon = \mathbf{d}_A^{-1}[0, \varepsilon) = \{x \in \mathbb{R}^d \mid \mathbf{d}_A(x) < \varepsilon\}.$$

Definition 4 (Separating Cover). *For $\delta > 0$, $\gamma > \delta$, and finite subsets $P \subset D$, $Q \subset P \cap B$ we say that (P, Q) is an **(open) separating (δ, γ) -cover** of a surrounding pair (D, B) if*

- (a) $D \setminus B \subseteq P^\delta$,
- (b) Q^δ surrounds P^δ in D , and
- (c) $\hat{Q}^\delta \subseteq B \subseteq \hat{Q}^\gamma$.

4 Chasing

Theorem 2. *Let $(\mathcal{D}, \mathcal{B})$ be a surrounding pair of open subsets of \mathbb{R}^d and let $P \subset \mathcal{D}$ be a finite subset of \mathcal{D} . Let (P, Q) be an (open) separating (δ, γ) -cover of $(\mathcal{D}, \mathcal{B})$ for $\gamma > \delta > 0$. Let (D_0, B_0) and (D_1, B_1) be surrounding pairs of nonempty, compact subsets of \mathbb{R}^d such that $B_0 \subseteq \hat{Q}^\delta$, $\hat{Q}^\gamma \subseteq B_1$, and*

$$(D_0, B_0) \subset (\mathcal{D}, \mathcal{B}) \subset (D_1, B_1).$$

If

$$\mathbf{im} \, H_k((D_0, B_0) \hookrightarrow (D_1, B_1)) \cong H_k(\mathcal{D}, \mathcal{B})$$

then

$$\mathbf{im} \, H_k((P^\delta, Q^\delta) \hookrightarrow (P^\gamma, Q^\gamma)) \cong H_k(\mathcal{D}, \mathcal{B}).$$

Proof. In the following let

$$\begin{aligned} \eta_B^k &: H_k(B_0) \rightarrow H_k(B_1), \\ \eta_D^k &: H_k(D_0) \rightarrow H_k(D_1), \text{ and} \\ \eta^k &: H_k(D_0, B_0) \rightarrow H_k(D_1, B_1). \end{aligned}$$

Consider the commutative diagram of long exact sequences of the pairs (D_0, B_0) , $(\mathcal{D}, \mathcal{B})$ and (D_1, B_1) .

$$\begin{array}{ccccccccc}
H_k(B_0) & \xrightarrow{i_0^k} & H_k(D_0) & \xrightarrow{j_0^k} & H_k(D_0, B_0) & \xrightarrow{\partial_0^k} & H_{k-1}(B_0) & \xrightarrow{i_0^{k-1}} & H_{k-1}(D_0) \\
\downarrow a^k & & \downarrow b^k & & \downarrow d^k & & \downarrow a^{k-1} & & \downarrow b^{k-1} \\
H_k(\mathcal{B}) & \xrightarrow{i_*^k} & H_k(\mathcal{D}) & \xrightarrow{j_*^k} & H_k(\mathcal{D}, \mathcal{B}) & \xrightarrow{\partial_*^k} & H_{k-1}(\mathcal{B}) & \xrightarrow{i_*^{k-1}} & H_{k-1}(\mathcal{D}) \\
\downarrow f^k & & \downarrow g^k & & \downarrow h^k & & \downarrow f^{k-1} & & \downarrow g^{k-1} \\
H_k(B_1) & \xrightarrow{i_1^k} & H_k(D_1) & \xrightarrow{j_1^k} & H_k(D_1, B_1) & \xrightarrow{\partial_1^k} & H_{k-1}(B_1) & \xrightarrow{i_1^{k-1}} & H_{k-1}(D_1)
\end{array} \tag{2}$$

Because (D_0, B_0) and (D_1, B_1) are nonempty, compact surrounding pairs of \mathbb{R}^d we can embed them in $S^d \cong \mathbb{R}^d \cup \{\infty\}$ in order to show that i_0^k and i_1^k are surjective by Lemma 7. So, by exactness, $\mathbf{im} \ i_0^k = \mathbf{ker} \ j_0^k = H_k(D_0)$ and $\mathbf{im} \ i_1^k = \mathbf{ker} \ j_1^k = H_k(D_1)$. It follows that for any $[y''] \in \mathbf{im} \ \eta^k$ with preimage $[y] \in H_k(D_0, B_0)$ we must have that $[y''] \in \mathbf{cok} \ j_1^k$ and $[y] \in \mathbf{cok} \ j_0^k$. That is, there must exist nonzero $[z] = \partial_0^k[y]$ in $H_{k-1}(B_0)$ and $[z''] = \partial_1^k[y'']$ in $H_{k-1}(B_1)$ such that $\eta_B^{k-1}[z] = [z'']$. Moreover, because η^k factors through $H_k(\mathcal{D}, \mathcal{B})$ and η_B^{k-1} factors through $H_{k-1}(\mathcal{B})$ there must exist nonzero $[y'] = d^k[y]$ in $H_k(\mathcal{D}, \mathcal{B})$ and $[z'] = \partial_*^k[y'] = a^{k-1}[z]$ in $H_{k-1}(\mathcal{B})$ such that $h^k[y'] = [y'']$ and $f^{k-1}[z'] = [z'']$.

Consider the long exact sequence of the pair $(\mathcal{D}, \hat{Q}^\delta)$

$$\dots \rightarrow H_k(\mathcal{D}, \hat{Q}^\delta) \xrightarrow{\partial_\delta^k} H_{k-1}(\hat{Q}^\delta) \xrightarrow{p_\delta^{k-1}} H_{k-1}(\mathcal{D}) \rightarrow \dots,$$

and the following commutative diagrams taken from the long exact sequences of the pairs $(\mathcal{D}, \hat{Q}^\delta)$ and $(\mathcal{D}, \mathcal{B})$.

$$\begin{array}{ccc}
H_{k-1}(\hat{Q}^\delta) & \xrightarrow{p_\delta^{k-1}} & H_{k-1}(\mathcal{D}) \\
& \searrow \psi_\delta^{k-1} & \nearrow i_*^{k-1} \\
& H_{k-1}(\mathcal{B}) &
\end{array} \tag{3}$$

where $\psi_\delta^{k-1} : H_{k-1}(\hat{Q}^\delta) \rightarrow H_{k-1}(\mathcal{B})$ is induced by inclusion.

Letting $\phi_0^{k-1} : H_{k-1}(B_0) \rightarrow H_{k-1}(\hat{Q}^\delta)$ be the homomorphism induced by inclusion we have that $a^{k-1} = \psi_\delta^{k-1} \circ \phi_0^{k-1}$ so, because

$$\eta_B^{k-1} = f^{k-1} \circ a^{k-1} = f^{k-1} \circ \psi_\delta^{k-1} \circ \phi_0^{k-1}$$

and $\partial_1^k \circ \eta^k = \partial_0^k \circ \eta_B^{k-1}$ it follows that $\phi_0^{k-1} \circ \partial_0^k[y]$ is nonzero in $H_{k-1}(\hat{Q}^\delta)$. Moreover, because $a^{k-1} \circ \partial_0^k[y] \in \mathbf{im} \ \partial_*^k$ we have that $a^{k-1} \circ \partial_0^k[y] \in \mathbf{ker} \ i_*^{k-1}$ by exactness, thus $\phi_0^{k-1} \circ \partial_0^k[y] \in \mathbf{ker} \ p_\delta^{k-1}$ as $p_\delta^{k-1} = i_*^{k-1} \circ \psi_\delta^{k-1}$. That is, $\phi_0^{k-1} \circ \partial_0^k[y] \in \mathbf{im} \ \partial_\delta^k$ by exactness. We can therefore construct a homomorphism $\mu^k : H_k(D_0, B_0) \rightarrow H_k(P^\delta, Q^\delta)$ for $[y] \in H_k(D_0, B_0)$ as the preimage of $\phi_0^{k-1} \partial_0^k[y]$ in $H_k(P^\delta, Q^\delta)$ for nonzero $[y''] = \eta^k[y]$, 0 otherwise.

Now, consider the long exact sequence of the pair $(\mathcal{D}, \hat{Q}^\gamma)$

$$\dots \rightarrow H_k(\mathcal{D}, \hat{Q}^\gamma) \xrightarrow{\partial_\gamma^k} H_{k-1}(\hat{Q}^\gamma) \xrightarrow{p_\gamma^{k-1}} H_{k-1}(\mathcal{D}) \rightarrow \dots$$

We have the following commutative diagrams

$$\begin{array}{ccc} H_{k-1}(\mathcal{B}) & \xrightarrow{i_*^{k-1}} & H_{k-1}(\mathcal{D}) \\ & \searrow \psi_\gamma^{k-1} & \nearrow p_\gamma^{k-1} \\ & H_{k-1}(\hat{Q}^\gamma) & \end{array} \quad (4a)$$

$$\begin{array}{ccc} H_k(\mathcal{D}, \hat{Q}^\gamma) & \xrightarrow{\partial_\gamma^k} & H_{k-1}(\hat{Q}^\gamma) \\ \downarrow \nu^k & & \downarrow \phi_1^{k-1} \\ H_k(D_1, B_1) & \xrightarrow{\partial_1^k} & H_{k-1}(B_1) \end{array} \quad (4b)$$

where $\psi_\gamma^{k-1}, \sigma_\gamma^{k-1}, \phi_1^{k-1}$ and ν^k are induced by inclusion.

Because $[z'] \in \mathbf{im} \partial_*^k$ we have $[z'] \in \mathbf{ker} i_*^{k-1}$ by exactness and, by commutativity of diagram 4a, $[z'] \in \mathbf{ker} p_\gamma^{k-1} \circ \psi_\gamma^{k-1}$. Noting that f^{k-1} factors through $H_{k-1}(\hat{Q}^\gamma)$ as $f^{k-1} = \phi_1^{k-1} \circ \psi_\gamma^{k-1}$ we have that $\psi_\gamma^{k-1}[z']$ is nonzero in $H_{k-1}(Q^\gamma)$. So $\psi_\gamma^{k-1}[z'] \in \mathbf{ker} p_\gamma^{k-1}$ thus $\psi_\gamma^{k-1}[z'] \in \mathbf{im} H_k(\mathcal{D}, \hat{Q}^\gamma)$ by exactness.

So we may conclude that η^k factors through $\hat{\tau}^k : H_k(\mathcal{D}, \hat{Q}^\delta) \rightarrow H_k(\mathcal{D}, \hat{Q}^\gamma)$ with the maps $\mu^k = (\partial_\delta^k)^{-1} \circ \phi_0^{k-1} \circ \partial_0^k$ and $\nu^k : H_k(\mathcal{D}, \hat{Q}^\gamma) \rightarrow H_k(D_1, B_1)$ induced by inclusion. We therefore have the following sequence of homomorphisms

$$H_k(D_0, B_0) \xrightarrow{\mu^k} H_k(\mathcal{D}, \hat{Q}^\delta) \rightarrow H_k(\mathcal{D}, \mathcal{B}) \rightarrow H_k(\mathcal{D}, \hat{Q}^\gamma) \xrightarrow{\nu^k} H_k(D_1, B_1).$$

So, by Lemma 3, $\mathbf{im} \eta^k \cong H_k(\mathcal{D}, \mathcal{B})$ implies $\mathbf{im} \hat{\tau}^k \cong H_k(\mathcal{D}, \mathcal{B})$. The result follows from Lemma 6 as

$$\mathbf{im} H_k((P^\delta, Q^\delta) \hookrightarrow (P^\gamma, Q^\gamma)) \cong \mathbf{im} \hat{\tau}^k \cong H_k(\mathcal{D}, \mathcal{B}).$$

□

5 Connection with the TCC

5.1 Assumptions

Let $P \subset \mathcal{D}$ be a finite collection of sensors p with the following capabilities.

Sensor Capabilities

- (Communication Radii)** detect the presence, but not location or distance, of sensors within distances $\delta > 0$ and $\gamma \geq 3\delta$, and discriminate between sensors within each scale,
- (Coverage Radius)** cover a radially symmetric subset of the domain with radius δ ,

We will refer to the following preliminary assumptions about pairs (D_0, B_0) and (D_1, B_1) for $\delta > 0$ and $\gamma \geq 3\delta$.

Geometric Assumptions

1. **(Domain)** (D_0, B_0) and (D_1, B_1) are surrounding pairs of nonempty, compact subsets of \mathbb{R}^d with $(D_0^{\delta+\gamma}, B_0^{\delta+\gamma}) \subset (D_1, B_1)$.
2. **(Boundary)** $H_0(D_1 \setminus B_1 \hookrightarrow D_0 \setminus B_0^{2\delta})$ is surjective.

In the following let $Q = P \cap B_0^\delta$ and $(\mathcal{D}, \mathcal{B}) = (D_0^{2\delta}, B_0^{2\delta})$.

5.2 Proof of the TCC

In the following let $\overline{X} = \mathbb{R}^d \setminus X$ for all subsets $X \subset \mathbb{R}^d$. We have the following commutative diagrams of inclusions between the pairs (P, Q) and $(\mathcal{D}, \mathcal{B})$ and their complements in \mathbb{R}^d with increasing scale.

$$\begin{array}{ccccc} (P^\delta, Q^\delta) & \hookrightarrow & (P^\gamma, Q^\gamma) & (\overline{B_1}, \overline{D_1}) & \xhookrightarrow{j} & (\overline{\mathcal{B}}, \overline{\mathcal{D}}) \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ (\mathcal{D}, \mathcal{B}) & \hookrightarrow & (D_1, B_1) & (\overline{Q^\gamma}, \overline{P^\gamma}) & \xhookrightarrow{i} & (\overline{Q^\delta}, \overline{P^\delta}). \end{array}$$

The following diagram is formed by applying the homology functor.

$$\begin{array}{ccc} H_0(\overline{B_1}, \overline{D_1}) & \xrightarrow{j_*} & H_0(\overline{\mathcal{B}}, \overline{\mathcal{D}}) \\ \downarrow & & \downarrow \\ H_0(\overline{Q^\gamma}, \overline{P^\gamma}) & \xrightarrow{i_*} & H_0(\overline{Q^\delta}, \overline{P^\delta}). \end{array} \tag{5}$$

Let $p_* : \mathbf{im} j_* \rightarrow \mathbf{im} i_*$.

Lemma 8. *Given assumptions 1 & 2, the map p_* is surjective.*

Proof. Choose a basis for $\mathbf{im} i_*$ such that each basis element is represented by a point in $P^\delta \setminus Q^\gamma$. Let $x \in P^\delta \setminus Q^\gamma$ be such that $[x]$ is non-trivial in $\mathbf{im} i_*$. Suppose $x \in \mathcal{B}$ and let $y \in B_0$ so that $\mathbf{d}(x, y) < 2\delta$.

Now, because $x \in \overline{Q^\gamma}$ by hypothesis $\mathbf{d}(x, q) \geq \gamma$ for all $q \in Q$. For any z in the shortest path between x and y we have $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) < 2\delta$, so the following inequality holds for all $q \in Q$

$$\begin{aligned} \mathbf{d}(x, q) &\geq \mathbf{d}(x, q) - \mathbf{d}(x, z) \\ &> \gamma - 2\delta \\ &\geq \delta. \end{aligned}$$

So $z \in \overline{Q^\delta}$ for all z in the shortest path from x to y . In particular, $x, y \in \overline{Q^\delta}$.

Now, suppose $y \in P^\delta$. So there exists some $p \in P$ such that $\mathbf{d}(p, y) < \delta$. So $\mathbf{d}(p, y) < \delta$ which implies $p \in Q$ thus $y \in Q^\delta$. But we have shown that $y \in \overline{Q^\delta}$, a contradiction, so we may assume that $y \in \overline{P^\delta}$.

Because $x, y \in \overline{Q^\delta}$ we have corresponding chains $x, y \in C_0(\overline{Q^\delta})$ as well as $y \in \overline{P^\delta}$ generating a chain $y \in C_0(P^\delta)$. As we have shown that $x \in \mathcal{B}$ implies that the shortest path from x to y is contained in $\overline{Q^\delta}$ there exists a path $h : [0, 1] \rightarrow \overline{Q^\delta}$ with $h(0) = x$ and $h(1) = y$ that generates a chain $h \in C_1(\overline{Q^\delta})$. So for $h \in C_1(\overline{Q^\delta}, \overline{P^\delta})$ with $\partial h = x + y$ we have that $x = \partial h + y$. Thus $[x]$ is a relative boundary and is therefore trivial in $H_0(\overline{P^\delta}, \overline{Q^\delta})$, a contradiction, as we have assumed $[x]$ is non-trivial in $\mathbf{im} i_*$. So we may conclude that $x \notin \mathcal{B}$.

So $x \in \overline{\mathcal{B}}$ and $x \in \mathcal{D} \setminus \mathcal{B}$. So $[x]$ is non-trivial in $H_0(\overline{\mathcal{B}}, \overline{\mathcal{D}})$ and, because j_* is surjective, $\mathbf{im} j_* = H_0(\overline{\mathcal{B}}, \overline{\mathcal{D}})$. So p_* is surjective as $p_*[x] = [x] \in \mathbf{im} p_*$ for all non-trivial $[x] \in \mathbf{im} i_*$. \square

Lemma 9. *Given assumptions 1 & 2, if p_* is injective then $\mathcal{D} \setminus \mathcal{B} \subseteq P^\delta$.*

Proof. Suppose, for the sake of contradiction, that p_* is injective and there exists a point $x \in (\mathcal{D} \setminus \mathcal{B}) \setminus P^\delta$. So $[x]$ is non-trivial in $H_0(\overline{\mathcal{B}}, \overline{\mathcal{D}}) = \mathbf{im} j_*$ as x is in some connected component of $\mathcal{D} \setminus \mathcal{B}$ and j_* is surjective. So we have the following sequence of maps induced by inclusions

$$H_0(\overline{\mathcal{B}}, \overline{\mathcal{D}}) \xrightarrow{f_*} H_0(\overline{\mathcal{B}}, \overline{\mathcal{D}} \cup \{x\}) \xrightarrow{g_*} H_0(\overline{Q^\delta}, \overline{P^\delta}).$$

As $f_*[x]$ is trivial in $H_0(\overline{\mathcal{B}}, \overline{\mathcal{D}} \cup \{x\})$ we have that $p_*[x] = (g_* \circ f_*)[x]$ is trivial, contradicting our hypothesis that p_* is injective. \square

Lemma 10. *Given assumptions 1 & 2, if the map p_* is injective then Q^δ surrounds P^δ in \mathcal{D} .*

Proof. Suppose, for the sake of contradiction, that Q^δ does not surround P^δ in \mathcal{D} . Then there exists a path $\pi : [0, 1] \rightarrow \overline{Q^\delta}$ with $\pi(0) \in P^\delta \setminus Q^\delta$ and $\pi(1) \in \mathcal{D} \setminus P^\delta$. By Lemma 9 we know that $\mathcal{D} \setminus \mathcal{B} \subset P^\delta$ and, because $Q^\delta \subset \mathcal{B}$ it follows that $\mathcal{D} \setminus \mathcal{B} \subset P^\delta \setminus Q^\delta$.

Choose $x \in \mathcal{D} \setminus \mathcal{B}$ and $y \in \overline{\mathcal{D}}$ such that there exist paths $\pi_x : [0, 1] \rightarrow P^\delta \setminus Q^\delta$ with $\pi_x(0) = x$, $\pi_x(1) = \pi(0)$ and $\pi_y : [0, 1] \rightarrow \overline{\mathcal{D}} \cup (\mathcal{D} \setminus P^\delta)$ with $\pi_y(0) = y$, $\pi_y(1) = \pi(1)$. π_x, π_y and π all generate chains in $C_1(\overline{Q^\delta}, \overline{P^\delta})$ and $\pi_x + \pi + \pi_y = \pi^* \in C_1(\overline{Q^\delta}, \overline{P^\delta})$ with $\partial \pi^* = x + y$. Moreover, y generates a chain in $C_0(\overline{P^\delta})$ as $\overline{\mathcal{D}^{2\delta}} \subseteq \overline{P^\delta}$. So $x = \partial \pi^* + y$ is a relative boundary in $C_0(\overline{Q^\delta}, \overline{P^\delta})$ thus $[x] = 0 = [y]$ in $H_0(\overline{Q^\delta}, \overline{P^\delta})$ and therefore $[x] = [y]$ in $\mathbf{im} i_*$. However, because \mathcal{B} surrounds \mathcal{D} we know that $[x] \neq [y]$ in $H_0(\overline{\mathcal{B}}, \overline{\mathcal{D}}) \cong \mathbf{im} j_*$, contradicting our assumption that p_* is injective. \square

Recall that Q^δ surrounding P^δ in \mathcal{D} implies that \hat{Q}^δ , and therefore \hat{Q}^γ , surrounds \mathcal{D} in \mathbb{R}^d .

Lemma 11. *Given assumptions 1 & 2, if p_* is injective then $B_0 \subseteq \hat{Q}^\delta$.*

Proof. Given assumptions 1 & 2 and p_* injective we have that $\mathcal{D} \setminus \mathcal{B} \subseteq P^\delta$ and Q^δ surrounds P^δ in \mathcal{D} by Lemmas 9 and 10. Recalling that $\mathcal{B} = B_0^{2\delta}$, $P \subset D_0$ and $Q = P \cap B_0^\delta$ we first note that $B_0 \cap P^\delta \subseteq Q^\delta$ as $x \in B_0 \cap P^\delta$ implies there exists some $p \in P$ such that $\mathbf{d}(x, p) < \delta$ which, for $x \in B_0$ implies that $p \in Q = P \cap B_0^\delta$, and therefore that $x \in Q^\delta$. It follows that

$$B_0 \cap (P^\delta \setminus Q^\delta) = B_0 \cap P^\delta \cap \overline{Q^\delta} \subseteq Q^\delta \cap \overline{Q^\delta} = \emptyset.$$

As Q^δ surrounds P^δ in \mathcal{D} we have

$$B_0 \subset \mathcal{D} = (P^\delta \setminus Q^\delta) \sqcup Q^\delta \sqcup (\mathcal{D} \setminus P^\delta)$$

where $B_0 \cap (P^\delta \setminus Q^\delta) = \emptyset$. It therefore follows that $B_0 \subseteq Q^\delta \sqcup (\mathcal{D} \setminus P^\delta) = \hat{Q}^\delta$. \square

Lemma 12. *Given assumptions 1 & 2, if p_* is injective then $\mathcal{B} \subseteq \hat{Q}^\gamma$.*

Proof. As $\mathcal{B} = B_0^{2\delta}$ we know that for all $x \in \mathcal{B}$ there exists some $y \in B_0$ such that $\mathbf{d}(x, y) < 2\delta$. By Lemma 11 we know that $B_0 \subseteq \hat{Q}^\delta = Q^\delta \sqcup (\mathcal{D} \setminus P^\delta)$ so either $y \in \mathcal{D} \setminus P^\delta$ or $y \in Q^\delta$.

If $y \in Q^\delta$ then there exists some $q \in Q$ such that $\mathbf{d}(y, q) < \delta$. Then

$$\mathbf{d}(x, q) \leq \mathbf{d}(x, y) + \mathbf{d}(y, q) < 2\delta + \delta \leq \gamma$$

which would imply $x \in Q^\gamma \subset \hat{Q}^\gamma$.

Now, suppose $y \in B_0 \cap (\mathcal{D} \setminus P^\delta)$. Because Q^δ surrounds P^δ in \mathcal{D} there is no path from $x \in \mathcal{B} \cap (P^\delta \setminus Q^\gamma) \subset P^\delta \setminus Q^\delta$ to $y \in B_0 \cap (\mathcal{D} \setminus P^\delta) \subset \mathcal{D} \setminus P^\delta$ that does not cross Q^δ . So there must be some point $z \in Q^\delta$ in the shortest path from x to y . That is, there exists some $q \in Q$ such that $\mathbf{d}(q, z) < \delta$ and $\mathbf{d}(z, x) < \mathbf{d}(x, y) < 2\delta$ so

$$\mathbf{d}(q, x) \leq \mathbf{d}(q, z) + \mathbf{d}(z, x) < \delta + 2\delta \leq \gamma.$$

So $y \in B_0 \cap (\mathcal{D} \setminus P^\delta)$ implies $x \in Q^\gamma$. \square

Theorem 3 (Geometric TCC). *Let (D_0, B_0) and (D_1, B_1) be surrounding pairs of nonempty, compact subsets of \mathbb{R}^d satisfying assumptions 1 & 2 for $\delta > 0$, and $\gamma > 3\delta$. Let $P \subset D_0$ be a finite collection of sensors and $Q = P \cap B_0^\delta$. Let $(\mathcal{D}, \mathcal{B}) = (D_0^{2\delta}, B_0^{2\delta})$ and $p_* : \mathbf{im} j_* \rightarrow \mathbf{im} i_*$ for j_*, i_* as defined in Diagram 5.*

If $\mathbf{rk} i_ \geq \mathbf{rk} j_*$ and*

$$\mathbf{im} H_k((D_0, B_0) \hookrightarrow (D_1, B_1)) \cong H_k(\mathcal{D}, \mathcal{B})$$

for all k then

$$\mathbf{im} H_k((P^\delta, Q^\delta) \hookrightarrow (P^\gamma, Q^\gamma)) \cong H_k(\mathcal{D}, \mathcal{B})$$

for all k .

Proof. Because P is a finite point set we know that $\mathbf{im} i_*$ is finite-dimensional. Because $\mathbf{rk} i_* \geq \mathbf{rk} j_*$ j_* is finite dimensional as well so p_* is injective. Therefore $\mathcal{D} \setminus \mathcal{B} \subseteq P^\delta$ by Lemma 9 and Q^δ surrounds P^δ in \mathcal{D} by Lemma 10. We can extend (P^δ, Q^δ) and (P^γ, Q^γ) to pairs $(\mathcal{D}, \hat{Q}^\delta)$ and $(\mathcal{D}, \hat{Q}^\gamma)$ surrounding \mathcal{D} in \mathbb{R}^d .

As $Q = P \cap B_0^\delta$ we have that $Q^\delta \setminus P^\delta \subset B_0^{2\delta} \setminus \mathcal{B}$. Moreover, $\mathcal{D} \setminus \mathcal{B} \subset P^\delta$ so $\mathcal{D} \setminus P^\delta \subset \mathcal{D} \setminus (\mathcal{D} \setminus \mathcal{B}) = \mathcal{B}$. So $\hat{Q}^\delta = Q^\delta \cup (\mathcal{D} \setminus P^\delta) \subseteq \mathcal{B}$. Moreover, as p_* is injective $\mathcal{B} \subseteq \hat{Q}^\gamma$ by Lemma 12, so $\hat{Q}^\delta \subseteq \mathcal{B} \subseteq \hat{Q}^\gamma$.

As $\mathcal{D} \setminus \mathcal{B} \subseteq P^\delta$, Q^δ surrounds P^δ in \mathcal{D} , and $\hat{Q}^\delta \subseteq \mathcal{B} \subseteq \hat{Q}^\gamma$ we may conclude that (P, Q) is an (open) separating (δ, γ) -cover of $(\mathcal{D}, \mathcal{B})$. Therefore, as $B_0 \subset \hat{Q}^\delta$ by Lemma 11 and $(D_0^{\delta+\gamma}, B_0^{\delta+\gamma}) \subset (D_1, B_1)$ by assumption 1 we have that $\hat{Q}^\gamma \subset B_1$. The result therefore follows from Theorem 2. \square