Module 16

Properties of Laplace Transform

Objective: To understand the properties of Laplace Transform and associating the knowledge of properties of ROC in response to different operations on signals.

Introduction :

We are aware that the Laplace transform of a continuous signal *x(t)* is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

And inverse Laplace transform is given by

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

The Properties of Laplace transform simplifies the work of finding the s-domain equivalent of a time domain function when different operations are performed on signal like time shifting, time scaling, time reversal etc. These properties also signify the change in ROC because of these operations.

These properties are also used in applying Laplace transform to the analysis and characterization of LTI systems.

Description :

1. **Linearity of the Laplace Transform**
   Statement:
   If $x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)$ with a region of convergence denoted as R_1
   and$x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s)$ with a region of convergence denoted as R_2
   then $ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\leftrightarrow} aX_1(s) + bX_2(s)$, with ROC containing $R_1 \cap R_2$
   Proof:
   Consider the linear combination of two signals $x_1(t)$ and $x_2(t)$ as $z(t)=ax_1(t)+bx_2(t)$. Now, take the Laplace transform of *z(t)* as

$$\mathcal{L}\{z(t)\} = \mathcal{L}\{ax_1(t) + bx_2(t)\} = \int_{-\infty}^{\infty} \{ax_1(t) + bx_2(t)\}e^{-st}dt$$

$$= a\int_{-\infty}^{\infty} x_1(t)e^{-st}dt + b\int_{-\infty}^{\infty} x_2(t)e^{-st}dt$$

$$= aX_1(s) + bX_2(s)$$

The resulting ROC is as large as the region in common between the independent ROCs. However, there may be pole-zero cancellation in the linear combination, which results in extending the ROC beyond the common region.

Illustration:

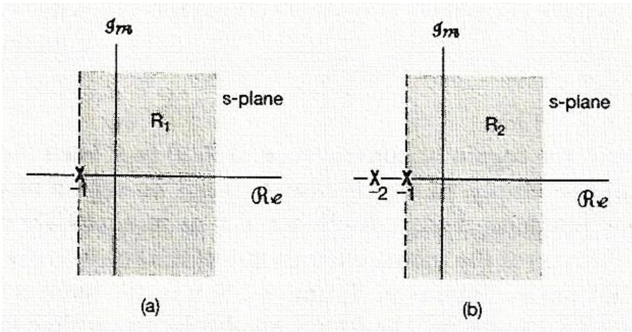
In this example, we illustrate the fact that the ROC for the Laplace transform of a linear combination of signals can sometimes extend beyond the intersection of the ROCs for the individual terms. Consider

$$x(t) = x_1(t) - x_2(t)$$

where the Laplace transforms of $x_1(t)$ and $x_2(t)$ are, respectively

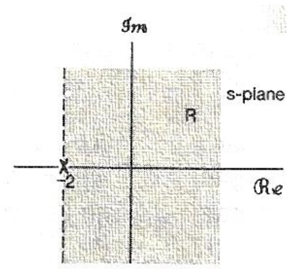
$X_1(s) = \frac{1}{s+1}$, Re{s}>-1 and $X_2(s) = \frac{1}{(s+1)(s+2)}$, Re{s}>-1

The pole-zero plot, including the ROCs for $X_1(s)$ and $X_2(s)$, is shown below respectively in figure (a) and (b)



$$X(s) = X_1(s) - X_2(s) = \frac{1}{(s+1)} - \frac{1}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}$$

Thus, in the linear combination of $x_1(t)$ and $x_2(t)$, the pole t s=-1 is cancelled by a zero at s=-1. The pole-zero plot for X(s) is shown below



The intersection of ROCs for $X_1(s)$ and $X_2(s)$ is Re{s}>-1. However, since the ROC is always bounded by a pole or infinity, for this example the ROC for X(s) can be extended to the left to be bounded by the pole at s=-2, because of the pole-zero cancellation at s=-1.

2. **Time Shifting**

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$with ROC= R

then $x(t-\tau) \overset{\mathcal{L}}{\leftrightarrow} e^{-s\tau}X(s)$with ROC= R

Proof:

$$\mathcal{L}\{x(t-\tau)\} = \int_{-\infty}^{\infty} x(t-\tau)e^{-st}dt$$

Let t-τ=p

$$= \int_{-\infty}^{\infty} x(p)e^{-s(p+\tau)}dt$$

$$= e^{-s\tau} \int_{-\infty}^{\infty} x(p)e^{-sp}dt$$

$$= e^{-s\tau}X(s)$$

Illustration:

As productof X(s) with $e^{-s\tau}$ will not effect the poles of X(s), ROC remains unaltered

3. **Shifting in s-Domain**

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $e^{s_o t}x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s-s_o)$ with ROC= R+Re{s_o}

Proof:

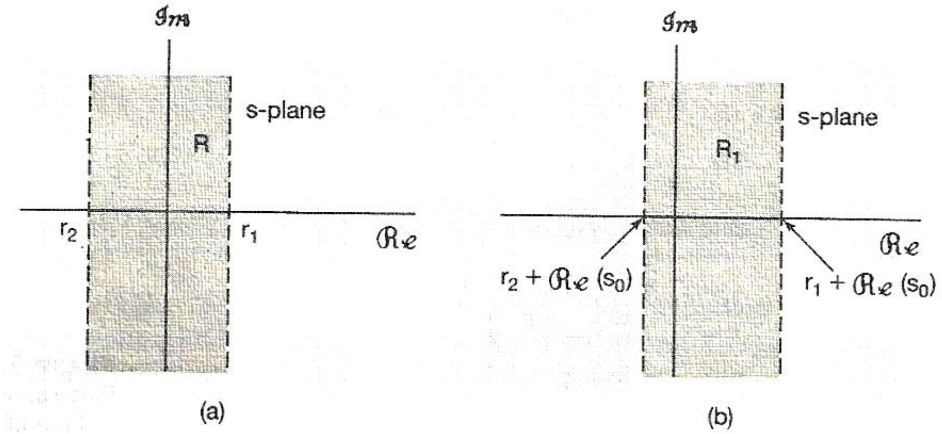
$$\mathcal{L}\{e^{s_o t}x(t)\} = \int_{-\infty}^{\infty} e^{s_o t}x(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-(s-s_o)t}dt$$

$$= X(s-s_o)$$

Illustration:

That is, the ROC associated with X(s-s_o) is that of X(s), shifted by Re{s_o}.Thus, for any value s that is in R, the value s+Re{s_o} will be in R_1.This is illustrated in figure below. Figure (a) and (b) represents ROC of X(s) and X(s-s_o) respectively.



Note that if X(s) has a pole or zero at $s=a$, then X(s-s_o) has a pole or zero at s-s_o=a, i.e.,$s=a+s_o$.

A special case is observed when $s_o = j\omega_o$, i.e., when a signal is used to modulate a periodic complex exponential $e^{j\omega_o t}$.

In this case $e^{j\omega_o t}x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s - j\omega_o)$ with ROC= R .

This is true because ROC depends on real part of 's' not the imaginary part.

4. Time Scaling

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $x(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right)$ with ROC= $R_1 = a$R

Proof:

To prove this we have to consider two cases: *a (real)* is positive and *a* is negative.

Case 1: For *a*>0:

$$\mathcal{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-st}dt$$

Using the substitution of $\lambda = at;\ dt = ad\lambda$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-\left(\frac{s}{a}\right)\lambda}d\lambda$$

$$= \frac{1}{a}X\left(\frac{s}{a}\right)$$

Case 2: For *a*< 0 :

$$\mathcal{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-st}dt$$

Using the substitution of $\lambda = at;\ dt = ad\lambda$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-\left(\frac{s}{a}\right)\lambda}d\lambda$$

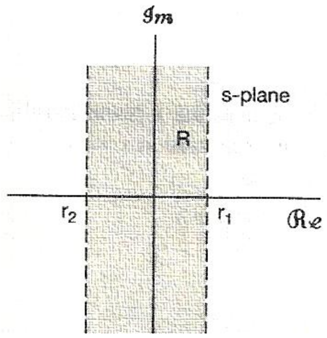
$$= -\frac{1}{a}X\left(\frac{s}{a}\right)$$

Combining the two cases, we get $x(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right)$ with ROC= $R_1 = a$R

Illustration:

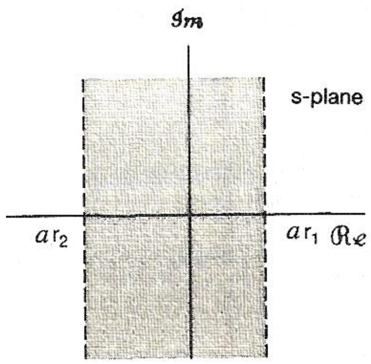
For example Laplace transform of $x(t)=e^{bt}u(t)$ is $X(s) = \frac{1}{s+b}$ with ROC:Re{s}=$\sigma > -b$ representing right-sided signal. Then $\frac{1}{|a|}X\left(\frac{s}{a}\right) = \frac{1}{|a|}\frac{1}{\left(\frac{s}{a}+b\right)}$ with ROC: $Re\left\{\frac{s}{a}\right\} > -b \Rightarrow Re\{s\} > a(-b)$ representing ROC= $R_1 = a$R

Let the ROC of X(s) is given as shown below

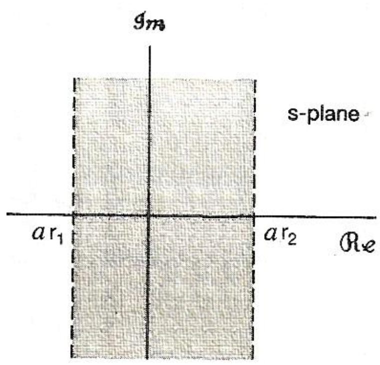


Change in ROC is also explained with different ranges of *a*

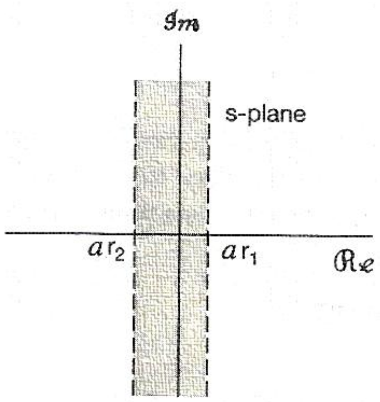
(i) *If a >1 then the resultant ROC is expanded*



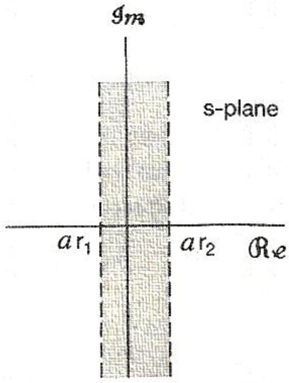
(ii) *If a <-1 then the resultant ROC expands and the bounds get reversed*



(iii) *If 0 < a < 1 then the resultant ROC is compressed*

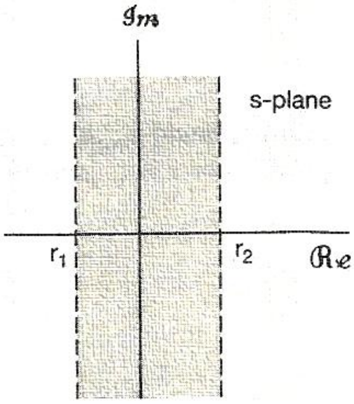


(iv) *If -1 < a < 0 then the resultant ROC is compressed and the bounds get reversed*



(v) *If a = -1, then it gives rise to* ***Time Reversal*** *operation with the statement*

$x(-t) \overset{\mathcal{L}}{\leftrightarrow} X(-s)$ with ROC= R_1=-R



5. **Conjugation**

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $x^*(t) \overset{\mathcal{L}}{\leftrightarrow} X^*(s^*)$ with ROC= R

Proof:

$$\mathcal{L}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-st}dt$$

as we know that $s=\sigma+j\omega$

$$= \int_{-\infty}^{\infty} x^*(t)e^{-\sigma t}e^{-j\omega t}\,dt$$

$$= \left(\int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{j\omega t}\,dt\right)^*$$

$$= \left(\int_{-\infty}^{\infty} x(t)e^{-(\sigma-j\omega)t}dt\right)^*$$

$$= \left(\int_{-\infty}^{\infty} x(t) e^{-(s^*)t} dt \right)^*$$

$$= \left(X(s^*)\right)^* = X^*(s^*)$$

Also$X(s) = X^*(s^*)$when x(t) is real.

Illustration:

If x(t) is real then and if X(s) has a pole or zero at s=so, then X(s) also has a pole or zero at the complex conjugate point $s = s_o^*$. As only imaginary part changes and not the real part, ROC remains unaltered.

6. **Convolution Property**

Statement:

If $x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)$ with ROC = R$_1$

and$x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s)$ with ROC = R$_2$

then $x_1(t) * x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s).X_2(s)$, with ROC containing $R_1 \cap R_2$

Proof:

$$\mathcal{L}\{z(t)\} = \mathcal{L}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} \{x_1(t) * x_2(t)\} e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right\} e^{-st} dt$$

Interchanging the order of integrations

$$\mathcal{L}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} x_1(\tau) \left\{ \int_{-\infty}^{\infty} x_2(t-\tau) e^{-st} dt \right\} d\tau$$

$= \int_{-\infty}^{\infty} x_1(\tau) \{e^{-s\tau} X_2(s)\} d\tau$(Since from Time shifting property)

$$= X_2(s) \int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} d\tau$$

$$= X_1(s).X_2(s)$$

Illustration:

In a manner, like the linearity property, the ROC of $X_1(s).X_2(s)$ includes the intersection of the ROCs of X$_1$(s) and X$_2$(s) and may be larger if pole-zero cancellation occurs in the product.

For example, if

$X_1(s) = \frac{s+1}{s+2}$, Re{s} > -2 and $X_2(s) = \frac{s+2}{s+1}$, Re{s} > -1,

then $X_1(s).X_2(s) = 1$, and its ROC is the entire s-plane.

Note: This property plays an important role in the analysis of LTI systems.

7. **Differentiation in the Time Domain**

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $\frac{dx(t)}{dt} \overset{\mathcal{L}}{\leftrightarrow} s\,X(s)$ with ROC containing R

Proof:

Inverse Laplace transform is given by

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

Differentiating above on both sides with respect to ‘t’

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \{sX(s)\}e^{st}ds$$

Comparing both equations $s\,X(s)$ is the Laplace transform of $\frac{dx(t)}{dt}$.

Illustration:

The ROC of $s\,X(s)$ includes the ROC of X(s) and may be larger if X(s) has a first-order pole at s=0 that is cancelled by the multiplication by s.

For example, if *x(t)=u(t),* then $X(s) = \frac{1}{s}$, with an ROC that is Re{s} > 0. The derivative of x(t) is an impulse with an associated Laplace transform that is unity and an ROC that is the entire s-plane.

8. **Differentiation in the s-Domain**

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $-tx(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{dX(s)}{ds}$ with ROC = R

Proof:

Laplace transform is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Differentiating above on both sides with respect to ‘s’

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} \{-tx(t)\} e^{-st} dt$$

Comparing both equations $\frac{dX(s)}{ds}$ is the Laplace transform of $-tx(t)$.

Illustration:

Let us find the Laplace transform of $x(t) = te^{-at}u(t)$

Since $e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}, \; ROC: Re\{s\} > -a$

From the property, it follows that

$$te^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}, \; ROC: Re\{s\} > -a$$

If the property is repeated, we obtain

$$t^2 e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{2!}{(s+a)^3}, \; ROC: Re\{s\} > -a$$

And, more generally

$$t^n e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{n!}{(s+a)^{n+1}}, \; ROC: Re\{s\} > -a$$

From the above result, it is observed that differentiating rational form s-domain function will result in multiple order poles. Therefore, ROC remains same as X(s).

9. Integration in the Time Domain

Statement:

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}X(s)$ with ROC containing $R \cap \{Re\{s\} > 0\}$

Proof:

This can be derived using convolution property as

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

$$\mathcal{L}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \mathcal{L}\{x(t) * u(t)\} = X(s).\mathcal{L}\{u(t)\} = X(s)\frac{1}{s}$$

Illustration:

As $\mathcal{L}\{u(t)\} = \frac{1}{s}, \; ROC: Re\{s\} > 0$ and ROC of X(s) is R, then the resultant ROC of $\frac{1}{s}X(s)$ contains $R \cap \{Re\{s\} > 0\}$

10. The Initial and Final Value Theorems

Statement:

If x(t) and $\frac{dx(t)}{dt}$ are Laplace transformable, and under the specific constraints that x(t)=0 for t<0 containing no impulses at the origin,one can directly calculate, from the

Laplace transform, the initial value *x(0⁺)*, i.e., x(t) as t approaches zero from positive values of t. Specifically the ***initial -value theorem*** states that

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Also, if x(t)=0 for t<0 and, in addition, x(t) has a finite limit as t→∞, then the ***final-value theorem*** says that

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Proof:

To prove these theorems, we need to consider the Unilateral Laplace transform of $\frac{dx(t)}{dt}$

Unilateral Laplace transform of {x(t)}= $\int_{0+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$

$$= [x(t)e^{-st}]_{0^+}^{\infty} + s\int_{0^+}^{\infty} x(t)e^{-st} dt$$

$$= x(\infty)e^{-\infty} - x(0^+) + sX(s)$$

$$= sX(s) - x(0^+)$$

Initial value theorem

From the above discussion, we know that

$$\int_{0+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = sX(s) - x(0^+)$$

Applying the $\lim_{s\to\infty}$ *on both sides*

$$0 = \lim_{s\to\infty} sX(s) - x(0^+)$$

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Final value theorem

we know that

$$\int_{0+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = sX(s) - x(0^+)$$

Applying the $\lim_{s\to 0}$ *on both sides*

$$[x(t)]_{0^+}^{\infty} = \lim_{s\to 0} sX(s) - x(0^+)$$

$$\lim_{t\to\infty} x(t) - x(0^+) = \lim_{s\to\infty} sX(s) - x(0^+)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Note: The initial and final-value theorems can be useful in checking the correctness of the Laplace transform calculations for a signal.

Illustration:

For example, consider the signal $x(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t)$

We see that $x(0^+)=2$

Now finding the Laplace transform of x(t)

$$e^{-2t}u(t) + e^{-t}\cos(3t)u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)};\ \text{ROC: Re\{s\}} > -1$$

Applying the initial value theorem

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\left\{\frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}\right\} = \lim_{s\to\infty}\left\{\frac{2s^3+5s^2+12s}{s^3+4s^2+14s+20}\right\} = 2$$

Summary:

Property	Signal	Laplace Transform	ROC
	$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
Linearity	$ax_1(t)+bx_2(t)$	$aX_1(s)+bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0t}x(t)$	$X(s-s_0)$	Shifted version of R (i.e., s is in the ROC if $s-s_0$ is in R)
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final-Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Examples:

Solved Problems:

Problem 1: Find the Laplace transform and ROC of $x(t) = e^{-4t}u(t-2)$

Solution:

We know that $$e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}, \ \ ROC: Re\{s\} > -a$$

$$e^{-4t}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+4}, \ \ ROC: Re\{s\} > -4$$

Signal $e^{-4t}u(t)$is now delayed by 2 units to get $e^{-4(t-2)}u(t-2)$

Therefore, applying time shifting property

$$e^{-4(t-2)}u(t-2) = e^{8}e^{-4t}u(t-2) \overset{\mathcal{L}}{\leftrightarrow} \frac{e^{-2s}}{s+4}, \ \ ROC: Re\{s\} > -4$$

$$e^{-4t}u(t-2) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{e^{8}}\frac{e^{-2s}}{s+4}, \ \ ROC: Re\{s\} > -4$$

Problem 2: Given $F(s) = \frac{s+8}{s^2+6s+13}$, find *f(0)*

Solution:

Consider $$sF(s) = \frac{s(s+8)}{s^2+6s+13} = 1 + \frac{2s-13}{s^2+6s+13}$$

From the initial value theorem, we know that

$$initial\ value\ of\ f(t) = f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left(1 + \frac{2s-13}{s^2+6s+13}\right) = 1$$

Problem 3: Consider a signal y(t) which is related to two signals $x_1(t)$ and $x_2(t)$ by

$$y(t) = x_1(t-2) * x_2(-t+3)$$

Where $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-3t}u(t)$

Use properties of the Laplace transform to determine the Laplace transform Y(s) of y(t).

Solution:

We know that $$e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}, \ \ ROC: Re\{s\} > -a$$

Now we have $x_1(t) = e^{-2t}u(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s) = \frac{1}{s+2}, \ \ ROC: Re\{s\} > -2$

and $$x_2(t) = e^{-3t}u(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s) = \frac{1}{s+3}, \ \ ROC: Re\{s\} > -3$$

Using the time shifting, time scaling properties, we obtain

$$x_1(t-2) \overset{\mathcal{L}}{\leftrightarrow} e^{-2s}X_1(s) = \frac{e^{-2s}}{s+2}, \ \ ROC: Re\{s\} > -2$$

and

$$x_2(-t+3) \overset{\mathcal{L}}{\leftrightarrow} e^{-3s}X_2(-s) = \frac{e^{-3s}}{3-s}, \ \ ROC: Re\{s\} < 3$$

Therefore, using the convolution property we obtain

$$y(t) = x_1(t-2) * x_2(-t+3) \overset{\mathcal{L}}{\leftrightarrow} Y(s) = \left(\frac{e^{-2s}}{s+2}\right)\left(\frac{e^{-3s}}{3-s}\right)$$

Problem 4: Find the Laplace transform of $x(t)=te^{-4t}\,u(t)$

Solution:

We know that $$e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a},\ \ ROC{:}\,Re\{s\} > -a$$

$$e^{-4t}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+4},\ \ ROC{:}\,Re\{s\} > -4$$

Applyingdifferentiation in s-domain property $tx(t) \overset{\mathcal{L}}{\leftrightarrow} -\frac{dX(s)}{ds}$

$$tx(t) = te^{-4t}u(t) \overset{\mathcal{L}}{\leftrightarrow} -\frac{dX(s)}{ds} = -\frac{d}{ds}\left\{\frac{1}{s+4}\right\} = \frac{1}{(s+4)^2}$$

$$ROC{:}\,Re\{s\} > -4$$

Problem 5: Find the steady state response of the following system to unit step excitation

$$H(s) = \frac{1}{s+2}$$

Solution:

We know that for a linear system output$y(t) = h(t)*x(t)$

As the input $x(t)=u(t)$

As we know that $e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a},\ \ ROC{:}\,Re\{s\} > -a$

$$h(t) = e^{-2t}u(t) \overset{\mathcal{L}}{\leftrightarrow} H(s) = \frac{1}{s+2},\ \ ROC{:}\,Re\{s\} > -2$$

Output $y(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)d\tau$ using Time integration property.

Therefore, $y(t) = h(t) * u(t) = \int_{0}^{t} e^{-2\tau}d\tau = \frac{1-e^{-2t}}{2}u(t)$

Problem 6: Find the Laplace transform of impulse function using differentiation property

Solution: The differentiation of unit step function gives unit impulse function. i.e.,

$$\delta(t) = \frac{du(t)}{dt}$$

Taking Laplace transform of both sides,

$$\mathcal{L}[\delta(t)] = \mathcal{L}\left[\frac{du(t)}{dt}\right]$$

We know that $u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}$. Using differentiation property of Laplace transform above equation can be written as

$$\mathcal{L}[\delta(t)] = \mathcal{L}\left[\frac{du(t)}{dt}\right] = s\frac{1}{s} = 1, \text{ ROC:entire s-plane}$$

Problem 7: Find the inverse Laplace transform of $\frac{3}{s} + \frac{1}{s^2+9}$

Solution:

It follows from the Linearity property for the inverse

$\mathcal{L}^{-1}\left(\frac{3}{s} + \frac{1}{s^2+9}\right) = 3\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) = 3 + \frac{1}{3}\sin(3t)$; t>0

Problem 8: Two continuous time systems with impulse responses $h_1(t) = \delta(t-1)$ and $h_2(t) = \delta(t-2)$ are connected in cascade. Find the overall impulse response of the cascaded system.

Solution: As the systems are connected in cascade, the overall impulse response is given by

$$h(t) = h_1(t) * h_1(t)$$

From convolution property

$$H(s) = H_1(s)H_2(s) = e^{-s}.e^{-2s} = e^{-3s}$$

Therefore,$h(t) = \delta(t-3)$ using time shifting property

Problem 9: If$\delta(t)$ is a unit impulse function find the Laplace transform of $\frac{d^2}{dt^2}[\delta(t)]$

Solution:

We know that $\mathcal{L}[\delta(t)] = 1$

Applying differentiation in time domain property

$$\mathcal{L}\left\{\frac{d^2}{dt^2}[\delta(t)]\right\} = s^2$$

Problem 10: Find the Laplace transform ofu(-t) and mention change in ROC

Solution:

We know that $\mathcal{L}\{u(t)\} = \frac{1}{s}$ with ROC:Re{s}>0

Applying Time-reversal property $\mathcal{L}\{u(-t)\} = -\frac{1}{s}$ with ROC:Re{s}<0

Assignment:

Problem 1:a)Show that, if x(t) is an even function, so that x(t)=x(-t), then X(s)=X(-s)
b)Show that, if x(t) is an odd function, so that x(t)=-x(-t), then X(s)=-X(-s)

Problem 2: The signal x(t)=sin(2t) oscillates between +1 and -1 as t→∞. So it does not have a final value. Show that application of a final value theorem gives an incorrect result for the signal

Problem 3: Find the steady state response of the following system to unit step excitation

$$H(s) = \frac{s+1}{s^2+3s+2}$$

Problem 4: Find the value of *y'(0)* if, $Y(s) = \frac{s^2+1}{s^2(s^2-4s+9)}$

Problem 5: The unilateral Laplace transform of*f(t)* is $\frac{1}{s^2+s+1}$. Find the unilateral Laplace transform of *g*(*t*)=*tf(t)*

Problem 6 : The impulse response of a system *is h(t)=tu(t)*.For an input *u(t−1)*, find the output.

Problem 7: If the unit step response of a network is $(1-e^{-\alpha t})$, then find its unit impulse response.

Problem 8: If $F(S) = L[f(t)] = \frac{2(s+1)}{s^2+4s+7}$ then find the initial and final values of f(t).

Problem 9: Determine the Laplace transform of the ramp function using differentiation in s-domain property

Problem 10: Find the Laplace transform of $x(t)=e^{\pi t}u(t)$and ROC

Simulation:

Example 1: Finding the Laplace transform of u(t) and u(t-1)

Program:

```
symsst ;
x=heaviside(t);
X= laplace(x);
display(X);
y=heaviside(t-1);%delayed by one unit
Y= laplace(y);
display(Y);
```

Output:

```
X =

1/s


Y =

exp(-s)/s
```

Example 2: Finding the Laplace transform of $f(t)=-1.25+3.5te^{-2t}+1.25e^{-2t}$

```
>>syms t s
f=-1.25+3.5*t*exp(-2*t)+1.25*exp(-2*t);
F=laplace(f,t,s)

F =

5/(4*(s + 2)) + 7/(2*(s + 2)^2) - 5/(4*s)

>>  simplify(F)

ans =

(s - 5)/(s*(s + 2)^2)

>>  pretty(ans)

     s - 5
  ----------
           2
  s (s + 2)
```

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