Lorentz and Poincaré Group

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Chapter 1

Introduction

We are starting here with Eistein's postulates of special relativity. The first postulate, Principle of relativity, tells that all physical laws are the same for all inertial frames of reference, regardless of their relative state of motion. The second postulate, The universal speed of light, tells that the speed of light in free space is the same in all inertial frames of reference, regardless of their relative state of motion. Lorentz transformation can be considered as the direct consequence of second postulate.

In physics, the Lorentz transformations are a six-parameter family of linear transformations from a coordinate frame in space-time to another frame that moves at a constant velocity relative to the former.The most common form of Lorentz transformation equation by considering the constant velocity along x direction is

$$t' = \gamma(t - vx/c^2) \tag{1.1}$$

$$x' = \gamma(x - vt) \tag{1.2}$$

$$y' = y \tag{1.3}$$

$$z' = z \tag{1.4}$$

Here, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

1.1 Derivation of Lorentz transformation

Consider two reference frame,S and S'. S' is moving with constant velocity, v. Considering an event(something that happens at certain point of space-time), a spherical pulse that start at t=0, when both frames are at rest and at origin.

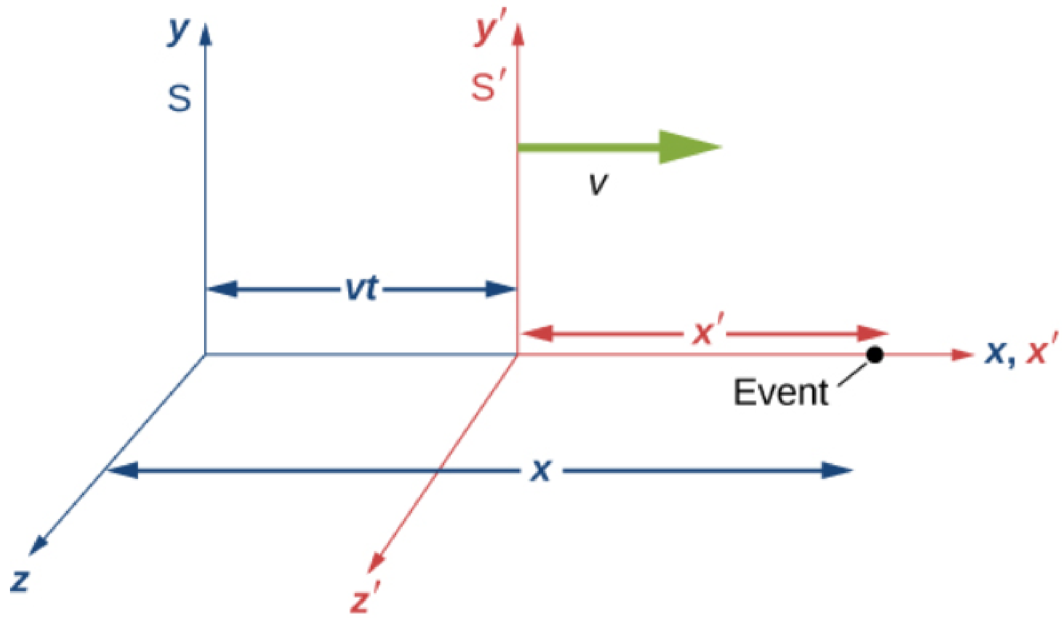

Figure 1.1: An event occurs at (x,0,0,t) in S and at (x',0,0,t') in S

At time, t in S frame,radius of spherical pulse,r = ct
At time, t' in S' frame,radius of spherical pulse, r' = ct'
Expressing these in cartesian co-ordinate gives

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{1.5}$$

$$x'^2 + y'^2 + z'^2 - c^2t^2 = 0 \tag{1.6}$$

here $y' = y$ and $z' = z$

$$x^2 - c^2t^2 = x'^2 - c^2t^2 \tag{1.7}$$

eq(1.5) and eq(1.6) are called Space- time invariant/interval, which is a constant regardless of the reference frame we are considering. At time t, an observer in S finds the origin of S' to be at $x = vt$. And distance from the origin of S' to the event $x'\sqrt{1-v^2/c^2}$. The position of event from origin of S at time t becomes

$$x = vt + x'\sqrt{1-v^2/c^2} \tag{1.8}$$

and

$$x' = \frac{x - vt}{x'\sqrt{1-v^2/c^2}} \tag{1.9}$$

and

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \tag{1.10}$$

The equations relating the time and position of the events as seen in S are then

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \tag{1.11}$$

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \tag{1.12}$$

$$y' = y \tag{1.13}$$

$$z' = z \tag{1.14}$$

1.2 Some definitions

In this section I am going to generalize some terms used and introducing some concepts related to space time diagram.

- Event in space-time

  In physics, and in particular relativity, an event is the instantaneous physical situation or occurrence associated with a point in space-time.

  An event is characterized by $x^{\mu}, \mu = 0, 1, 2, 3$

  $$x^0 = ct, x^i = \mathbf{x} \tag{1.15}$$

- Coordinate Four-vector, Length of Vectors

  Let $x^{\mu}{}_1$ and $x^{\mu}{}_2$ represents two events. The difference between two events defines a coordinate four- vector $x^{\mu} = x_1^{\mu} - x_2^{\mu}$.

  The length of 4 vector can be defined as

  $$|x|^2 = \mathbf{x}^2 - (x^0)^2 = \mathbf{x}^2 - c^2t^2 \tag{1.16}$$

  In terms of metric tensor,$g^{\mu\nu}$, the length vector can be defined as

  $$|x|^2 = x_{\mu}x^{\nu} \tag{1.17}$$

where x_μ is the co-vector of x^μ, a vector is dual space to the space of x^μ and is defined as

$$x_\mu = g^{\mu\nu} x^\nu \tag{1.18}$$

eq(1.18) in eq(1.17)

$$|x|^2 = g^{\mu\nu} x^\mu x^\nu \tag{1.19}$$

The metric is known as Minkowski metric and is defined in Minkowski space, which has as a signature (-1,1,1,1) . More explicitly

$$g^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.20}$$

As $g^{\mu\nu}$ are independent of coordinate (i.e. matrix elements are constant) and the matrix $g^{\mu\nu}$ is diagonal hence the space-time is flat and the coordinate system associated to it are orthogonal.

Chapter 2

Lorentz Group

2.1 Lorentz Group

2.1.1 Homogeneous Lorentz Transformation

Homogeneous Lorentz transformation are continuous linear transformation on the unit coordinate vector and the component vector given by :

$$\hat{e}_\mu \longrightarrow \hat{e}'_\mu = \hat{e}_\nu \Lambda^\nu_\mu \tag{2.1}$$

$$\hat{x}^\mu \longrightarrow \hat{x}'^\mu = \Lambda^\mu_\nu x^\nu \tag{2.2}$$

which preserves the length of the 4-vector.

$$|x|^2 = |x'|^2 \tag{2.3}$$

Combining above equations one can reformulate the condition on Lorentz transformations Λ without referring to any specific 4-vector as either,

$$g_{\mu\nu}\Lambda^\mu_\lambda \Lambda^\nu_\sigma = g_{\lambda\sigma} \tag{2.4}$$

Or,

$$\Lambda^\mu_\lambda \Lambda^\nu_\sigma g^{\mu\nu} = g^{\mu\nu} \tag{2.5}$$

Where $g_{\mu\nu} = g^{\mu\nu}$ .This result is an apparent generalization of rotations in 3-dimensional Euclidean space. If we suppress the indices in equations [2.4] and equation [2.5] it can be written as

$$\Lambda^{-1} = g\Lambda^T g^{-1} \tag{2.6}$$

So Λ^{-1} and Λ^T exists and also satisfy the proper Lorentz transformation properties. The above result can be compared with the orthogonal property of $S0(3)$

group. $RR^T = I$. The only difference with above condition is instead of I (identity matrix) there is g-matrix which is the metric tensor of the Minkowski space.for Euclidean space metric tensor is identity matrix. So equation [2.6] condition is called the pseudo orthogonal condition.

Taking the determinant on both sides of Eq. (2.6), we obtain $(\det\Lambda)^2 = 1$, hence $\det(\Lambda) = \pm 1$.Here, we only consider those transformations which are continuously connected to the identity transformation. So from the well known determinant identity of tensor form , we must have ,

$$\det\Lambda = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu}\Lambda^{\nu}_{\lambda}\Lambda^{\delta}_{\sigma}\varepsilon^{\alpha\beta\nu\delta} \tag{2.7}$$

where $\varepsilon^{\alpha\beta\nu\delta}$ is the 4-dimensional totally anti-symmetric unit tensor. Hence we observe that the tensor $\varepsilon^{\alpha\beta\nu\delta}$ remains invariant under Lorentz transformation. From eq.(2.4), eq.(2.5) and eq.(2.7) one can conclude that under Lorentz transformation the tensor $\varepsilon^{\alpha\beta\nu\delta}$ and $g^{\mu\nu}$ remain invariant.

We also note that, setting $\lambda = \sigma = 0$ in equation (2.4), we obtain the condition

$$(\Lambda^0_0)^2 - \sum_i (\Lambda^i_0)^2 = 1 \tag{2.8}$$

This implies $(\Lambda^0_0)^2 \geq 1$. Hence $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$. Here we only consider the case $\Lambda^0_0 \geq 1$. The condition $\Lambda^0_0 \leq -1$ is for time reversal. Since $\Lambda^0_0 = 1$ for the identity transformation , continuity requires that all proper Lorentz transformations have

$$(\Lambda^0_0)^2 \geq 1 \tag{2.9}$$

From the above discussion we can conclude:

Theorem : Characterization of Homogeneous Lorentz transformation:

Homogeneous Lorentz transformations are linear transformations of 4 x 4 matrices with $(\Lambda^0_0) \geq 1$ that leave two special tensors $g^{\mu\nu}$ and $\varepsilon^{\alpha\beta\nu\delta}$ invariant.

A general Homogeneous Lorentz transformation have 6 real parameter. This can be seen as follows : the 4 x 4 real matrix Λ has 16 arbitrary elements; Eq. (2.5) contains 10 independent constraints : as both sides of the equation are symmetric in the $(\mu\nu)$ indices. Since Eq. (2.7) follows from (2.5), it does not lead to an additional relational constraint. (It only imposes restrictions on the range of solutions to Eq. (2.5), such as the exclusion of spatial and time inversions and related transformations.)

2.1.2 Example of Lorentz Transformations

Example:1

Rotations in the 3 spatial dimensions are examples of Lorentz transformations in this generalized sense. They are of the form,

$$(R)^{\mu}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R^{i}_{j} & \\ 0 & & & \end{pmatrix} \tag{2.10}$$

where $(R)^{i}_{j}$ denotes ordinary 3 x 3 rotation matrices. It is straightforward to verify that this form satisfies Eqs. (2.5) and (2.7)

Example 2: Loentz Boosts

The Lorentz transformation which mix spatial coordinate with the time coordinate are called the Lorentz Boosts The simplest of these is a Lorentz boost along a given coordinate axis, say the x-axis given in the matrix form :

$$(\mathbf{L}_1)^{\mu}_{\nu} = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.11}$$

This corresponds physically to the transformation between two coordinate frames moving with respect to each other along the x-direction at the speed $v = c\tanh\zeta$.

The parameter ζ and the physical variable c, v are related by the following equations:

$$\beta = \frac{v}{c} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \text{and} \quad \sinh\zeta = \beta\gamma \quad \cosh\zeta = \gamma \tag{2.12}$$

2.1.3 Definition:Proper Lorentz Group :

The set of all proper Lorentz trans- formations $\{\Lambda\}$ satisfying the conditions of Eqs. (2.5)-(2.7) forms the Proper Lorentz Group or, in short, the Lorentz Group. It will be denoted by the symbol $\mathbf{L}_{+}$ and it is wriiten as $SO(3,1)$.

The group consists of all special “orthogonal” 4 x 4 matrices (pseudo orthogonal). (3,1)the quotation marks here signifies the non-Euclidean signature of the invariant metric $g_{\mu\nu}$ (-1, l, l, 1). Thus, Λ matrices for Lorentz boosts (such as given by Eq. (2.11)) are not unitary like the rotation matrices. As the Lorentz group is non compact the finite dimensional irreducible representation of the Lorentz group is non unitary. This can be shown in next to next unit, we will see that the generator of the Lorentz boost are anti-Hermitian that is why the representation are non-unitary.However the infinite dimensional representations of Lorentz group are unitary (we can construct through constructing the Casmiri operator) and its eigen value represent the infinite dimensional representation which is unitary.

We observe that in example-2 we introduce the hyperbolic angle variable.The motivation to include the hyperbolic angle variable is to relate it with rotation in hyperbolic plane. The range of the ζ varies $(-\infty \leq \zeta \leq \infty)$. So Lorentz group is non compact as the paramatrization space is unbounded.

More about the Lorentz boost :
In first unit of our report we have an example of Lorentz transformation. That is nothing but the Lorentz boost along the x-axis. Example-1 is about the rotation. Rotations forms a subgroup of Lorentz group i.e. $S0(3)$ group. whereas boosts do not form a subgroup. That means multiplication of two boost is not close. However Boosts along with rotations forms a subgroup. That is why we can always decompose a Lorentz transformation to product of rotation and boosts. Moreover In Einsteins special theory of relativity the variable ζ plays an important rule.Unlike Newtonian and Galilean relativity, In Einstein's STR, the velocity do not add up but rather the $\tanh\zeta$ add up as velocity added up in Newtonian relativity.

2.1.4 Lorentz - Vector :

Any 4-component object v^{μ}, transforming under Lorentz transformations as the coordinate component x^{μ} transform, we call it Lorentz Vector.

Definition: Scalar product

The Scalar product of two object u^{μ} and v^{ν} is defind as

$$u.v = g_{\mu\nu}u^{\mu}v^{\nu} = -u^0v^0 + \mathbf{u}.\mathbf{v} \tag{2.13}$$

Contravarient and covarient Component :

The 4-ordinary of represented by the contavarient representation. The covarient representation of the vector v^{μ} is given by :

$$v_{\mu} = g_{\mu\nu}g^{\nu} \tag{2.14}$$

The scalar product is given by in the contracting index form :

$$u.v = u_{\mu}v^{\nu} = u^{\mu}v_{\nu} \tag{2.15}$$

Theorem : transformation of covarient components :

The covariant components of a 4-vector v transform under proper Lorentz transformations as

$$v_{\mu} \longrightarrow v'_{\mu} = v_{\mu}(\Lambda^{-1})^{\nu}_{\mu} \tag{2.16}$$

The above theorem explicitly indicates that the quantity $(v^{\mu}u_{\mu})$ is Lorentz invariant.

There is a natural covariant four-vector, the 4-gradient ∂_{μ} and it is easy to see that ,

$$\partial_{\mu} \longrightarrow \partial'_{\mu} = \frac{\partial}{\partial x'_{\mu}} = \frac{\partial x^{\lambda}}{\partial x'_{\mu}}\frac{\partial}{\partial x^{\lambda}} = (\Lambda^{-1})^{\lambda}_{\mu}\frac{\partial}{\partial x^{\lambda}} \tag{2.17}$$

2.1.5 Decomposition of Lorentz Transformation

A general element of the proper Lorentz group $\mathbf{L}_{+}$ can be uniquely written in the factorized form

$$\Lambda = \mathrm{R}(\alpha, \beta, 0)\mathbf{L}_3(\zeta).\mathrm{R}(\theta, \phi, \psi)^{-1} \tag{2.18}$$

Where $\mathbf{L_3}(\zeta)$ is s Lorentz boosts along positive z-direction by the velocity $v = c\tanh\zeta$ and Euler angles for rotation have the usual meaning. I have not gone through the proof of the theorem , it can be easily proved by the properties of proper Lorentz transformation, and the fact that rotations form a subgroup.

It is not hard to see that, in this parametrization, a pure rotation corresponds to $\zeta = 0$ and a pure Lorentz boost along the direction $\hat{n}(\phi, \theta)$ corresponds to $\psi = 0$,, $\alpha = \phi$ and $\beta = \theta$.

Chapter 3

Relation of SO(3,1) and SL(2) & the Poincare Group

3.1 Relation of the Proper Lorentz Group to SL(2)

In analogy to the connection between the rotation group SO(3) to the special unitary group SU(2) there is the natural correspondence between the Lorentz group SO(3,1) and special linear group SL(2).

We can associate each space time point x^μ with a 2x2 Hermitian matrix :-

$$x^\mu \to X = \sigma^\mu . x^\mu \tag{3.1}$$

σ^μ, μ = 0,1,2,3 are the Pauli matrices. Hence,

$$X = \sigma^0 . x^0 + \sigma^1 . x^1 + \sigma^2 . x^2 + \sigma^3 . x^3 \tag{3.2}$$

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x^0 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x^1 + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} x^2 + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x^3 \tag{3.3}$$

$$X = \begin{bmatrix} x^0 & 0 \\ 0 & x^0 \end{bmatrix} + \begin{bmatrix} 0 & x^1 \\ x^1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -ix^2 \\ ix^2 & 0 \end{bmatrix} + \begin{bmatrix} x^3 & 0 \\ 0 & -x^3 \end{bmatrix} \tag{3.4}$$

Thus the explicit form of X is given by :-

$$X = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix} \tag{3.5}$$

Length of the 4-vector corresponds to the negative of the determinant of X

$$-det(X) = |\mathbf{x}|^2 - |x^0|^2 \tag{3.6}$$

where,

$$|\mathbf{x}|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \quad |x^0|^2 = c^2t^2 \tag{3.7}$$

3.1.1 Lorentz Transformation

A Lorentz transformation Λ on the 4 vector x^μ induces a linear transformation on the matrix X , which preserves its hermicity and the value of the determinant.

$$X \xrightarrow{\Lambda} X' = AXA^\dagger \tag{3.8}$$

here, A is a 2x2 invertible matrix, dependent on Λ which satisfies :-

$$det(A).det(A^\dagger) = |detA|^2 = 1 \tag{3.9}$$

The value of the determinant X does not change under linear transformation:-
Proof:-

$$\begin{aligned} det(X') &= det(AXA^\dagger) \\ &= det(A)det(X)det(A^\dagger) \\ &= det(A) \end{aligned} \tag{3.10}$$

The herimicity of X is preserved under linear transformation:-
Proof:-

$$X' = AXA^\dagger \tag{3.11}$$

$$\begin{aligned} (X')^\dagger &= (AXA^\dagger)^\dagger \\ &= (A^\dagger)^\dagger X^\dagger A^\dagger \\ &= AX^\dagger A^\dagger \\ &= AXA^\dagger \end{aligned} \tag{3.12}$$

From 3.11 and 3.12, $X' = (X')^\dagger$
The matrix A is determined up to an overall phase factor:-

$$det(A).det(A^\dagger) = 1 \tag{3.13}$$

$$det(A).(det(A))^* = 1$$

$$det(A) = e^{i\alpha} \tag{3.14}$$

The phase is fixed by choosing $det(A) = 1$ Thus, there is a natural correspondence between elements of the Lorentz group SO(3,1) and the special linear group SL(2) (2x2 invertible matrices with unit determinant). On replacing A with -A, the RHS of equation 3.8 and det(A) will not change.Thus to each $\lambda \in [SO(3,1)]$ is associated two SL(2) matrices $A(\Lambda)$.

3.2 Poincare Group

The set of transformations in Minkowski space consisting of all translations and proper Lorentz transformations and their products form a group P, called the Poincaré Group, or the inhomogeneous Lorentz group.
A general element of the Poincare group is denoted by g(b,Λ), it induces the coordinate transformation :-

$$x^{\mu} \xrightarrow{g} x^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + b^{\mu} \tag{3.15}$$

A transformation $g(b,\Lambda)$ followed by another, $g(b',\Lambda')$, is equivalent to a single transformation given by the group multiplication rule.
This can be seen by applying (3.15) a second time,

$$x' \xrightarrow{g'} x'' = \Lambda' x' + b' = (\Lambda'\Lambda)x + (\Lambda' b + b) \tag{3.16}$$

3.2.1 Theorem I

A general element of the Poincare group can be written in the factorised form:

$$g(b,\Lambda) = T(b)\Lambda \tag{3.17}$$

where $T(b) = g(b,E)$ (E is the unit matrix) is a translation, and $\Lambda = g(0,\Lambda)$ is a proper Lorentz transformation. From equation(3.16) we have:-

$$g(b',\Lambda')g(b,\Lambda) = g(\Lambda' b + b', \Lambda'\Lambda) \tag{3.18}$$

Set,

$$g(b',\Lambda') = g(0,\Lambda) \tag{3.19}$$

$$g(b,\Lambda) = g(b,E) \tag{3.20}$$

Substituting this in the above equation:-

$$g(0,\Lambda)g(b,E) = g(\Lambda b + 0, \Lambda) \tag{3.21}$$

$$\begin{aligned} \Lambda T(b) &= g(\Lambda b, \Lambda) \\ &= T(\Lambda b).\Lambda \end{aligned} \tag{3.22}$$

Equation 3.22 is obtained using 3.17. Multiplying Λ^{-1} on both sides of (3.22) gives :-

$$\Lambda T(b)\Lambda^{-1} = T(\Lambda b).\Lambda\Lambda^{-1} \tag{3.23}$$

$$\Lambda T(b)\Lambda^{-1} = T(\Lambda b) \tag{3.24}$$

3.2.2 Theorem II

Let Λ be an arbitrary proper Lorentz transformation and $T(b)$ a 4-dimensional translation. Then
(i) the Lorentz transformed translation is another translation, i.e.

$$\Lambda T(b)\Lambda^{-1} = T(\Lambda b) \tag{3.25}$$

(ii) the group of translations forms an invariant subgroup of the Poincare group
Using equations (3.17) and (3.25)

$$\begin{aligned} g(b,\Lambda)T(a)g(b,\Lambda)^{-1} &= T(b)\Lambda T(a)\Lambda^{-1}T(b)^{-1} \\ &= T(b)T(\Lambda a)T(b)^{-1} \\ &= T(\Lambda a) \end{aligned} \tag{3.26}$$

Hence, the set of all translations is invariant under operations of the full group $g(b,\Lambda)$

Chapter 4

Generators of the Lorentz and Poincare Group

4.1 Theorem: Lorentz Transformation Property of translation generators P

Under the Lorentz group, the Poincare group translation generators $\{P^{\mu}\}$ transform as 4 - coordinate vectors,

$$\Lambda P_{\mu}\Lambda^{-1} = P_{\nu}\Lambda^{\nu}_{\mu} \qquad \text{for all} \quad \Lambda \in \tilde{L}_{+} \tag{4.1}$$

Correspondingly, the contravariant generators $\{P^{\mu}\}$ transform as

$$\Lambda P_{\mu}\Lambda^{-1} = (\Lambda^{-1})^{\mu}_{\nu}P^{\nu} \tag{4.2}$$

4.2 Generators of Poincare Group

The Poincare Group has 10 generators one for each of its independent one parameter subgroup.s The covariant generators of translations P_{μ} are defined by the following expression for the infinitesimal translation (db):

$$T(db) = E - \mathrm{i}db^{\mu}P_{\mu} \tag{4.3}$$

where E is the unit matrix and (db) is the arbitrarily small 4-dimensional displacement vector.

$$P^{\mu} = g^{\mu\nu}P_{\nu} \tag{4.4}$$

The corresponding contravariant generators $\{P_{\mu}\}$ are defined by:

$$P^{\mu} = g^{\mu\nu}P_{\nu} \tag{4.5}$$

Hence

$$P^0 = -P_0 \qquad \text{and} \qquad P^i = P_i \qquad (i = 1, 2, 3). \tag{4.6}$$

The generators for spatial translations P_i are realized as momentum operators in physical applications of the translation group. Correspondingly, the generator for time translations P_0 are shown to be related to the energy operator—or Hamiltonian-—in physics. In this context, $P = \{P^\mu\}$ collectively as the *four-momentum operator*.

4.3 Generators of Lorentz Transformations $J_{\mu\nu}$

There are six distinct pairings of the indices $(\mu,\ \nu)$ - three involving 3- dimensional rotations $(1,\quad 2)$, $(2,\quad 3)$, $(3,\quad 1)$ and 3 corresponding to Lorentz boosts $(0,\quad 1)$, $(0,\quad 2)$, $(0,\quad 3)$.

The covariant geneartors $J_{\mu\nu}$ are anti-symmetric tensors defined by the following for infinitesimal "rotations" in Minkowski space:

$$\Lambda(\delta\omega) = E - \frac{1}{2}\delta\omega^{\mu\nu}J_{\mu\nu} \tag{4.7}$$

where

$$\delta\omega^{\mu\nu} = -\delta\omega_{\mu\nu} \tag{4.8}$$

are anti-symmetrical infinitesimal parameters.

The corresponding contravariant generators are:

$$J^{\mu\nu} = g^{\mu\nu}J_{\lambda\sigma}g^{\sigma\nu} \tag{4.9}$$

Hence for $m = 1,\ 2,\ 3$

$$J^{mn} = J_{mn} \qquad \text{and} \qquad J^{0m} = -J_{0m} = J_{m0}. \tag{4.10}$$

Rotations in 3 spatial dimensions are examples of Lorentz transformations in this generalized sense. They are of the form,

$$(R)^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R^i_j & \\ 0 & & & \end{pmatrix} \tag{4.11}$$

where $(R)^i_j$ denotes ordinary $3X3$ rotation matrices.

$$\Lambda(\delta\omega) = E - \frac{1}{2}\delta\omega^{\mu\nu}J_{\mu\nu} \tag{4.12}$$

By combining the above equations of Lorentz transformations, the generators can be expressed as 4x4 matrices J_{mn}. J_{mn} has non vanishing terms only at (m, n) and (n, m) positions.

A spatial rotation in the (m, n) plane can be interpreted as rotation around the k-axis where (k, m, n) is some permutation of (1, 2, 3).

$$R(\delta\theta) = E - i\delta\theta^k J_k \tag{4.13}$$

By comparing with the above notation for generators for rotations and equa. 4.7, the following identification can be made,

$$\delta\theta^1 = \delta\omega^{23} \qquad J_1 = J_{23} \qquad \text{plus cyclic permutations} \tag{4.14}$$

In a more compact notation it can be written as

$$J_k = \frac{1}{2}\varepsilon^{kmn} J_{mn} \qquad J_{mn} = \varepsilon^{kmn} J_k \tag{4.15}$$

These generators can be expressed as 4 x 4 matrices. The matrix for J_{mn} has non-vanishing elements only at the (m, n) and (n, m) positions. The association of rotations in the (m-n) plane with a unique "axis" $(\hat{k})$ perpendicular to that plane is a special property of 3-dimensions. In the 4-dimensional Minkowski space, as well as in other higher dimensional spaces, them subspace perpendicular to a plane is multi-dimensional, there is no unique "axis" associated with a set of one-parameter rotations. It is most natural to use the second-rank tensor notation J", for generators of "rotations" in the $(\mu\nu)$ plane .

The three generators of special Lorentz transformations (or Lorentz boosts) mix the time axis with one of the spatial dimensions. When focusing on this class of transformations, the following notation shall be used,

$$\delta\xi^m = \delta\omega^{m0} \qquad K_m \equiv J_{m0} \tag{4.16}$$

hence

$$\Lambda(\delta\xi) = E - \mathrm{i}\delta\xi^m K_m \tag{4.17}$$

The 4 x 4 matrices for Km can be derived in the same way as for Jm, making use of expressions of Lorentz boosts, specialized to infinitesimal transformations. For example,

$$(K_1) = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.18}$$

likewise for the other generators. Finite Lorentz boosts assume the familiar form

$$\Lambda(\xi) = exp(-\xi^m K_m) \tag{4.19}$$

Similarly, the general proper Lorentz transformations can be written as

$$\Lambda(\omega) = exp(-\frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}). \tag{4.20}$$

4.4 Transformation Law of Lorentz Generators

Let $\Lambda(\omega)$ be a proper Lorentz transformation and Ω be another arbitrary Lorentz transformation, then

$$\Omega\,\Lambda(\omega)\,\Omega^{-1} = \Lambda(\omega') \qquad \text{where} \qquad \omega'^{\mu\nu} = \Omega^{\mu}_{\lambda}\,\Omega^{\nu}_{\sigma}\,\omega^{\lambda\sigma} \tag{4.21}$$

The generators $\{J_{\mu\nu}\}$ transform under Ω as components of a second rank tensor.

$$\Omega\,J_{\mu\nu}\,\Omega^{-1} = J_{\lambda\sigma}\,\Omega^{\lambda}_{\mu}\,\Omega^{\sigma}_{\nu} \tag{4.22}$$

4.5 Lie Algebra of Lorentz and Poincare Group

Separating the spatial and time components of Lorentz transformation generators and Poincare translation generators and writing the Lie Algebra in terms of $\{P^0,\, P_m,\, J_m,\, K_m\}$

$$\begin{aligned}
[P^0,\, P_m] &= [P_n,\, P_m] = 0 \\
[P^0,\, J_n] &= 0 \\
[P_m,\, J_n] &= i\varepsilon^{mnl}P_l \\
[P_m,\, K_n] &= i\delta_{mn}P^0 \\
[P^0,\, K_n] &= iP_n
\end{aligned} \tag{4.23}$$

The second commutation relation indicates that P^0 is a scalar under 3-dimensional rotations and J_n is invariant under time translations. The commutation relation between P_m, J_n and K_n sate that translations and Lorentz boosts in different spatial directions commute but they mix if both involve the same direction in space.

$$\begin{aligned}
[J_m,\, J_n] &= i\varepsilon^{mnl}J_l \\
[K_m,\, J_n] &= i\varepsilon^{mnl}K_l \\
[K_m,\, K_n] &= -i\varepsilon^{mnl}J_l
\end{aligned} \tag{4.24}$$

These are the Lie algebra of the proper Lorentz group. The negative sign on the right-hand side of the last equation is noteworthy. It is another manifestation of the non-compactness of the group.The negative sign originates, from that of the Minkowski metric associated with the time dimension.

Chapter 5

Harmonic Oscillator

5.1 Covariant Harmonic Oscillator Differential Equations

Let us consider the differential equation for a hadron consisting of two quarks bound together by a harmonic oscillator potential of unit strength:

$$\left\{-2\left[\left(\frac{\partial}{\partial x_a^{\mu}}\right)^2+\left(\frac{\partial}{\partial x_b^{\mu}}\right)^2+\left(\frac{1}{16}\right)(x_a^{\mu}-x_b^{\mu})^2+m_0^2\right\}\phi(x_a,x_b)=0, \quad (5.1)$$

where x_a and x_b are space-time coordinates for the first and second quarks respectively. To simplify the above differential equation, we introduce new coordinate variables:

$$\begin{aligned} X &= (x_a+x_b)/2, \\ x &= (x_a-x_b)/2\sqrt{2}. \end{aligned} \quad (5.2)$$

The four-vector X specifies where the hadron is located in space-time, while the variable x measures the space-time separation between the quarks. In terms of these variables, Equation (5.1) can be written as

$$\left(\frac{\partial^2}{\partial X_{\mu}^2}-m_0^2+\frac{1}{2}\left[\frac{\partial^2}{\partial x_{\mu}^2}+x_{\mu}^2\right]\right)\phi(X,x)=0. \quad (5.3)$$

This equation is separable in the X and x variables. Thus

$$\phi(X,x)=f(X)\psi(x) \quad (5.4)$$

and f(X) and $\psi(x)$ satisfy the following differential equations, respectively:

$$\left(\frac{\partial^2}{\partial X_\mu^2} - m_0^2 - (\lambda+1)\right) f(X) = 0, \tag{5.5}$$

$$\frac{1}{2}\left(-\frac{\partial^2}{\partial x_\mu^2} + x_\mu^2\right)\psi(x) = (\lambda+1)\psi(x) \tag{5.6}$$

Equation (5.5) is a Klein-Gordon equation, and its solution takes the form

$$f(X) = exp[\pm i p_\mu X^\mu],$$

with

$$-P^2 = -P_\mu P^\mu = M^2$$

where P and M are the four-momentum and mass of the hadron respectively. The eigenvalue λ is determined from the solution of Equation (5.6)

5.2 Normalizable Solutions of the Relativistic Oscillator Equation

Now, we consider the separation of the space and time variables and write the four dimensional harmonic oscillator equation of Equation (5.6) as

$$\left(-\nabla^2 + \frac{\partial^2}{\partial t^2} + [(\mathbf{x})^2 - t^2]\right)\psi(x) = (\lambda+1)\psi(x) \tag{5.7}$$

The Lorentz transformation of the internal coordinates from the laboratory frame to the hadronic rest frame takes the form

$$\begin{aligned} x' = x, \quad y' = y,\\ z' = (z-\beta t)/(1-\beta^2)^{1/2}\\ t' = (t-\beta z)/(1-\beta^2)^{1/2} \end{aligned} \tag{5.8}$$

where β is the velocity of the hadron moving along the z direction. The primed quantities are the coordinate variables in the hadronic rest frame. In terms of the primed variables, the oscillator differential equation is

$$\left(-\nabla'^2+\frac{\partial^2}{\partial t^2}+[(\mathbf{x'})^2-t'^2]\right)\psi(x)=(\lambda+1)\psi(x) \tag{5.9}$$

This form is identical to that of Equation (5.7), due to the fact that the oscillator differential equation is Lorentz-invariant. Now, we consider the solution of the above equation of the type:

$$\psi_\beta(x)=\left(-\frac{1}{\pi}\right)\left(\frac{1}{2}\right)^{(a+b+n+k)/2}\left(\frac{1}{a!b!n!k!}\right)^{1/2}H_a(x')H_b(y')H_n(z')H_k(t')\times \\ \times exp\left[-\frac{1}{2}(x'^2+y'^2+z'^2+t'^2)\right] \tag{5.10}$$

where a, b, n, and k are integers, and $H_aCx'), H_b(y')...$ are the Hermite polynomials. This wave function is normalizable, but the eigenvalue takes the values

$$(\lambda=a+b+n-k) \tag{5.11}$$

For finite λ, we have infinite combinations of a,b,n and k, which makes it infinitely degenerate. It is difficult to give physical interpretations to infinite-component wave function. So to find a finite set, we invoke restriction that there be no time-like oscillations in the Lorentz frame in which the hadron is at rest and take $k=0$.

Now getting this condition and by simplifying, we get

$$\psi_\beta^n(z,t)=[(1/\pi 2^2 n!)]^{1/2}H_n(z')exp[-(1/2)(z'^2+t'^2)], \tag{5.12}$$

with $\lambda=n$

5.3 Transformation Properties of Harmonic Oscillator Wave Functions

Now, we are interested in obtaining the wave function for a moving hadron as a linear combination of the wave functions for the rest frame. So we apply the boost operator to the wave function for the hadron at rest,

$$\psi_{\beta\lambda}^{lm}(x)=[e^{i\eta K_3}]\psi_{0\lambda}^{lm}(x), \tag{5.13}$$

where K_3 is the boost generator along the z axis, its form is

$$K_3 = -i\left(z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}\right), \tag{5.14}$$

and η is related to velocity parameter β as

$$sinh\eta = \beta/(1-\beta^2)^{1/2} \tag{5.15}$$

If the hadron moves along the z direction, the x and y variables remain invariant. Therefore, we use the wave function of Equation (5.12) with $\beta = 0$

$$\psi_0^{n,0}(z,t) = \left[\frac{1}{\pi 2_n n!}\right]^{1/2} H_n(z)exp[-(1/2)](z^2+t^2)]. \tag{5.16}$$

The subscript 0 indicates that there are no time-like excitations: $k = O$. We are now led to consider the transformation

$$\begin{aligned}\psi_\beta^{n,0}(z,t) &= [exp(-i\eta K_3)]\,\psi_0^{n,0}(z,t)\\ &= \psi_0^{n,0}(z',t')\end{aligned} \tag{5.17}$$

This boost operator changes z and t to z' and t' respectively. However, we are interested in whether the transformation can take the linear form

$$\psi_\beta^{n,0}(z,t) = \sum_{n',k'} A_{n',k'}^{n',0}(\beta)\,\psi_0^{n',k'}(z,t) \tag{5.18}$$

Because the oscillator differential equation is Lorentz invariant, the eigenvalue λ remains invariant, and only the terms which satisfy the condition

$$n = (n'-k') \tag{5.19}$$

make non-zero contributions in the sum. Thus the above expression can be simplified to

$$\psi_\beta^{n,0}(z,t) = \sum_{k=0'}^{\infty} A_k^n(\beta)\,\psi_0^{n+k,k}(z,t) \tag{5.20}$$

This is indeed a linear unitary representation of the Lorentz group.

Using the orthogonality relation, we can write

$$\begin{aligned} A_k^n(\beta) &= \int dz\, dt\ \psi_0^{n+k,k}(z,t)\psi_\beta^{n,0}(z,t) \\ &= \frac{1}{\pi}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{1/2}\left(\frac{1}{n!(n+k)!}\right)^{1/2} \\ &\text{x}\int dz\, dt\ H_{n+k}(z)H_k(t)H_n(z') \\ &\text{x}\ \exp\left(-\frac{1}{2}(z^2+z'^2+t^2+t'^2)\right) \end{aligned} \tag{5.21}$$

If we use the generating function for the Hermite polynomial, the evaluation of the integral is straightforward, and the result is

$$A_k^n(\beta) = (1-\beta^2)^{(1+n)/2}\beta^k\frac{(n+k)!}{n!k!}^{1/2} \tag{5.22}$$

Thus the linear expansion given in Equation (5.20) can be written as

$$\begin{aligned} \psi_\beta^{n,0}(z,t) = \left[\frac{1}{\pi 2_n}\right]^{1/2}&(1-\beta^2)^{(1+n)/2}\exp\left(-\frac{1}{2}(z^2+t^2)\right) \\ \text{x}\sum_{k=0'}^{\infty}&\left(\frac{1}{2}\right)^k\left(\frac{1}{\beta}\right)^k H_{n+k}(z)H_k(t) \end{aligned} \tag{5.23}$$

To check if the transformation is unitary, we calculate the sum

$$S = \sum_{k=0'}^{\infty} A_k^n(\beta)^2 \tag{5.24}$$

According to the Equation (5.22), this sum is

$$S = (1-\beta^2)^{(1+n)/2}\sum_{k=0'}^{\infty}\frac{(n+k)!}{n!k!}(\beta^2) \tag{5.25}$$

The Binomial expansion of $[1-\beta^2]^{-(n+1)}$

$$(1-\beta^2)^{-(1+n)} = \sum_{k=0'}^{\infty}\frac{(n+k)!}{n!k!}(\beta^{2k}) \tag{5.26}$$

Therefore the sum 5 is equal to one. The linear transformation of Equation (5.20) is indeed a unitary transformation.

We have discussed above the transformation of the physical wave function with no time-like excitations. In general, for a given value of λ the infinite by infinite transformation matrix can be written as

$$\begin{bmatrix} \psi_{\beta}^{n,0} \\ \psi_{\beta}^{n+1,1} \\ \psi_{\beta}^{n+2,2} \\ \ldots \\ \ldots \\ \ldots \end{bmatrix} = \begin{bmatrix} b_{00} & b_{01} & b_{02} & \ldots & \ldots & \ldots \\ b_{00} & b_{01} & b_{02} & \ldots & \ldots & \ldots \\ b_{10} & b_{11} & b_{12} & \ldots & \ldots & \ldots \\ b_{20} & b_{21} & b_{22} & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \begin{bmatrix} \psi_{0}^{n,0} \\ \psi_{0}^{n+1,1} \\ \psi_{0}^{n+2,2} \\ \ldots \\ \ldots \\ \ldots \end{bmatrix} \tag{5.27}$$

This is the most general form of the unitary irreducible representation of the Lorentz group applicable to the covariant harmonic oscillator wave functions. We can see that it has infinite dimensions. Hence, it is proved here that the unitary irreducible representation of the Lorentz group always has infinite dimensions.

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