CS105 2024 Midsem: DIC on Discrete Structures

45 marks, 120min

Answer key and grading scheme

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1. (2+3=5 marks) Let X and Y be finite sets with $|X| = 10, |Y| = 2$. Then
   (a) What is the number of functions from X to Y?
   (b) What is the number of surjections from X to Y?

   Give a short justification for your answers.

   (a) 1 mark for the answer + 1 mark for explanation , In any function possible from X to Y each of i in X there are possible values for f(i) , so total number of functions $2^{(|X|)} = 2^{10}$

   (b) 1.5 mark for the answer + 1.5 mark for explanation, Off the total $2^k i10$ functions , the functions f,g , f(x) = y1 for all x , and g(x) = y2 for all x are the only functions which are not surjective ,hence total surjective functions $=2^{10} - 2$

2. (2+3+4=9 marks) Prove or disprove:
   (a) There exists a natural number $n \geq 1$ such that $n^2 + 3n + 2$ is prime.
   (b) For any positive integer $n \geq 2$ if there is no prime number p, such that $1 \leq p \leq \sqrt{n}$ which divides n, then n must be prime.
   (c) For every positive integer n, there exists a consecutive sequence of n positive integers which are all composite.

   (a) False (1 mark)
   1 mark for any valid justification
   Following are some common correct justifications:
      i. Factorizing the polynomial into $(n+2)(n+1)$. Now, this number must be divisible by both $n+1$ and $n+2$. Since $n \geq 1$, $n+1 \geq 2$ and $n+1 < n^2+3n+2$. Thus, $n+1$ is a factor of the term which is neither 1 nor the term. Thus, the term is composite always. (0.5 marks deducted if factors not being 1 isnt argued)
      ii. Disproof by contradiction. Assume that there exists a natural number $n \geq 1$ satisfying the given property. Then, either n is even or n is odd. In the first case, $n^2 + 3n + 2$ is divisible by 2. In the second case, let n be $2k + 1$ for some natural number k. $n^2 + 3n + 2 = 4k^2 + 4k + 1 + 6k + 3 + 2 = 4k^2 + 10k + 4$, which is also divisible by 2. Thus, $n^2 + 3n + 2$ is always divisible by 2. The only way it is prime is if it is 2 itself, but if $n \geq 1$, $n^2 + 3n + 2 \geq 3 > 2$, so this is never possible.
      iii. Factorizing the polynomial into $(n+2)(n+1)$. Let $n+1$ be m, either m or m+1 is divisible by 2. Then proceed with the earlier argument.

(b) True (1 marks)

2 marks for complete justification.

Approach 1

We prove the contrapositive of this statement. If the number n is not prime, there is a prime factor less than or equal to $\sqrt{n}$.
If the number is not prime, $\exists a, b$ such that $n = a\dot{b}$ and $a \neq 1$, $a < n$, $b \neq 1$, $b < n$.

Now, if both $a > \sqrt{n}$ and $b > \sqrt{n}$, $ab = n > n$ which is a contradiction. So, either $a < \sqrt{n}$ or $b < \sqrt{n}$. WLOG, let $a < \sqrt{n}$.

Now, by FTA, $a \neq 1$ must have a prime factor $p \leq a \leq \sqrt{n}$

Hence, n must have a prime factor $p \leq \sqrt{n}$.

(max 1.5 if FTA not mentioned.)

Alternate Solution:

We prove the contrapositive of this statement. If the number n is not prime, there is a prime factor less than or equal to $\sqrt{n}$.
If n has a prime factor p, p divides n and hence we can construct a set $\{a | a \text{ is prime and there exists } b \text{ such that } 1 \leq b < n \text{ and } ab = n\}$.
Take the minimum of this set, which exists by WOP.
Obviously, the corresponding b is greater than or equal to a for a to be the minimum element, hence $n \geq a^2$.
Thus, $a \leq \sqrt{n}$, is the prime that proves our contrapositive.

(c) True (1 mark if started proof assuming true)

Given $n \geq 2$ (the case for $n = 1$ is easy to see), consider the sequence of consecutive numbers $(n+1)!+2, (n+1)!+3, \ldots, (n+1)!+(n+1)$. This is a sequence of n consecutive numbers, with each number $> n+1$. Also observe that the i^{th} number here has $i+1 \leq (n+1)$ as a factor (this cannot be the only factor other than 1 as the i^{th} number is greater than $(n+1)$). Hence each number in this sequence is composite.

Hence, for any n, there exists a sequence of n consecutive composite numbers.

2 marks for constructing valid sequence, 1 for arguing all numbers are composite.

Alternate solution by Aditya Neeraje

First, we will prove a 2-number variant of the Chinese Remainder Theorem. Given p, q which are coprime, and numbers a, b such that $1 \leq a < p$, $1 \leq b < q$, we can always find a natural number less than pq whose remainder mod p is a and whose remainder mod q is b. For this, consider the pq pairs of numbers $\{(x \mod p, x \mod q) | x \in \{0, \cdots, pq-1\}\}$. No two values of x in $\{0, \cdots, pq-1\}$ can map to the same tuple, otherwise there would be two elements less than pq whose absolute difference is divisible by pq, which is not possible. Given that the domain has pq elements and the range also has pq elements, and the function is injective, the function is also surjective (If you delete all elements which are mapped to, and you are left with some elements, we have deleted pq elements but still have some elements left over in a set with pq elements).
Now, consider the first k primes. Let these be $\{p_1, \cdots, p_k\}$ We will construct a number such that n is divisible by p_1, n+1 by p_2 up to n+k-1 by p_k, and n is greater than or equal to $2 \cdot p_1 \cdot \cdots \cdot p_k$.
This is provable by induction. For the base case, if k is 1, we can simply choose 4. Now, given a number n_{k-1} satisfying all these properties, let a_{k-1} be the remainder when n_{k-1} is divided by $p_1 \cdot \cdots \cdot p_{k-1}$. Let b be the remainder when $p_k - k + 1$ is divided by p_k. Now, by the two-element variant of the Chinese Remainder Theorem proven above, we can find a new element b_k such that b_k is $a_{k-1} \mod p_1 \cdot \cdots \cdot p_{k-1}$ and $b \mod p_k$. Let n_k be $b_k + 2p_1 \cdot \cdots \cdot p_k$. This is our required value of n, the starting point for our sequence of k elements.

Approach 3

Any argument involving non-boundedness of difference between primes, density of primes or any such thing has been awarded a maximum of total 2 marks assuming correctness of argument.

3. (2.5+2.5=5 marks) Write True or False. If true, provide a proof. Else give a counter-example.

   (a) If X is a finite set and $f : X \to X$ is an injection, then f is a bijection.

   (b) If X is a countably infinite set and $f : X \to X$ is an injection, then f is a bijection.

a.

True. (1 mark)
Proof: (1.5 marks)

- Let X be a finite set with $|X| = n$ (i.e., X contains n elements).
- An injection (one-to-one function) maps distinct elements of X to distinct elements of X. This means $f(x_1) \neq f(x_2)$ for all $x_1 \neq x_2$.
- Since the domain and the codomain of f are the same set X, and f is injective, every element in the codomain must be mapped to by some element in the domain (since $|X| = n$, there are n distinct images for n elements of the domain).
- Therefore, f is also surjective (onto), which means f is a bijection.

b.

False. (1 mark)
Counterexample: (1.5 marks)

- Let $X = \mathbb{N}$ (the set of natural numbers), which is countably infinite.
- Define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = n + 1$. This function is injective because if $f(n_1) = f(n_2)$, then $n_1 + 1 = n_2 + 1$, which implies $n_1 = n_2$.
- However, f is not surjective, because there is no $n \in \mathbb{N}$ such that $f(n) = 1$ (or more generally $f(n) = k$ for any $k \in \mathbb{N}$).
- Hence, f is injective but not bijective.

4. (4 marks) What is the number of equivalence relations on a set of 4 elements? Explain your reasoning.

The number of equivalence relations on a set is equivalent to the number of **partitions** of the set, where each partition corresponds to one equivalence relation.
If S is a set with 4 elements, say $S = \{a, b, c, d\}$, then we are looking for the number of ways to partition S into non-empty, disjoint subsets (i.e., equivalence classes).
We will categorize the partitions based on the number of equivalence classes:

(a) **All elements in separate sets (4 partitions):**

$$\{\{a\}, \{b\}, \{c\}, \{d\}\}$$

Number of partitions: 1

(b) **One set of 2 elements, and two sets of single elements (3 partitions):** There are $\binom{4}{2} = 6$ ways to choose which pair will be grouped:

$$\{\{a, b\}, \{c\}, \{d\}\}, \{\{a, c\}, \{b\}, \{d\}\}, \{\{a, d\}, \{b\}, \{c\}\}, \ldots$$

Number of partitions: 6

(c) **Two sets of 2 elements each (2 partitions):** There are $\frac{\binom{4}{2}}{2} = 3$ ways to form these pairs:

$$\{\{a, b\}, \{c, d\}\}, \{\{a, c\}, \{b, d\}\}, \{\{a, d\}, \{b, c\}\}$$

Number of partitions: 3

(d) **One set of 3 elements, and one set of 1 element (2 partitions):** There are $\binom{4}{3} = 4$ ways to choose the single element:

$$\{\{a, b, c\}, \{d\}\}, \{\{a, b, d\}, \{c\}\}, \{\{a, c, d\}, \{b\}\}, \{\{b, c, d\}, \{a\}\}$$

Number of partitions: 4

(e) **All elements in the same set (1 partition):**

$$\{\{a, b, c, d\}\}$$

Number of partitions: 1

Thus, the total number of partitions is:

$$1 + 6 + 3 + 4 + 1 = 15$$

Therefore, the number of equivalence relations on a set of 4 elements is $\boxed{15}$.
(1 marks for the answer, if correct;1 marks for mentioning Partition: additional 2 marks for a correct justification (1 marks if one of case is missing.))

5. (3+2+5=10 marks) Let $f : A \to B$ be a function. Let us define the relation R_f by xR_fy iff $f(x) = f(y)$.

   (a) Prove that R_f is an equivalence relation.

   (b) Do there exist two distinct functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $R_f = R_g$? If yes provide these functions with a justification, else prove that it is impossible to have such functions.

   (c) The fiber of the function f, is defined by $fib(f) = \{f^{-1}(b) \mid b \in B\} = \{\{a \in A \mid f(a) = b\} \mid b \in B\}$. Let f be a surjective function. Then what is the relation between equivalence classes of R_f and $fib(f)$? Does this hold for any arbitrary function f? Why or why not?

a) (1 mark)Reflexive: As f is a well defined function, $\forall_x f(x) = f(x) \to xR_f x \to R_f$ is reflexive. (1 mark)Symmetric: $\forall_{xy} xR_f y \to f(x) = f(y) \to f(y) = f(x) \to yR_f x \to R_f$ is symmetric. (1 mark)Transitive: $\forall_{xyz} xR_f y \;\&\; yR_f z \to f(x) = f(y) \;\&\; f(y) = f(z) \to f(x) = f(z) \to xR_f z \to R_f$ is transitive. As R_f is reflexive, symmetric and transitive, it is an equivalence relation.

b) (Any Valid Example 1 mark) $\forall_x f(x) = x^2 \;\&\; g(x) = |x|$. For these 2 functions $R_f = R_g$.

(Any valid justification 1 mark) $\forall_{x,y} xR_f y \to x^2 = y^2 \to |x| = |y| \to xR_g y \to R_f \subseteq R_g$. Similarly, $\forall_{x,y} xR_g y \to |x| = |y| \to x^2 = y^2 \to xR_f y \to R_g \subseteq R_f$. Thus, $R_f = R_g$ and hence they are the same relation.

Note: There are many other trivial combinations for functions f and g.

c) (1 mark relation) $fib(f)$ is the set of all equivalence classes of R_f.

(1 for showing each element is an equivalence class and 1 for showing all equivalence classes are covered) Each element of $fib(f)$ is a subset of elements of our domain having the same value for f. Logically: $\forall_{c \in fib(f)}(\exists_{b \in B} \forall_{a \in c} f(a) = b) \to (\forall_{a_1,a_2 \in c} a_1 R_f a_2) \to c$ is an equivalence class of R_f. Also all equivalence classes are covered. This is because all equivalence classes have a representative element of the range which is then used to create the corresponding set in $fib(f)$. Logically: $\forall_{e=equivalenceclass} \exists_{b \in B}(\forall_{a \in A} f(a) = b \to a \in e)$. $b \in B \to \exists_{c \in fib(f)} \forall_{d \in A} f(d) = b \to d \in c$. Thus, we can see $e = c \to$ all equivalence classes are covered. Therefore, $fib(f)$ is the set of all equivalence classes of R_f.

(1 mark) No, it doesn't hold for any arbitrary function.

(1 mark justification). For a function, $f : A \to B$, which is not surjective, $\exists_{b \in B} f^{-1}(b) = \phi$. By definition of an equivalence class, it should have an element. Thus, in that scenario, $fib(f)$ is not the set of all equivalence classes but it's the set of all equivalence classes $\bigcup \phi$

6. (5+5+2=12 marks) Let $S = \{I_1, \ldots I_n\}$ be a set of n open intervals on the real line with rational endpoints. That is, for all $1 \leq j \leq n$, $I_j = (a_j, b_j)$ for some $a_j, b_j \in \mathbb{Q}$.

(a) Suppose we know that any two of the intervals have a non-empty intersection, i.e., $I_i \cap I_j \neq \emptyset$ for all $1 \leq i, j \leq n$. Prove by induction that the intersection of all intervals in S is non-empty, i.e., $\bigcap_{j=1}^{n} I_j \neq \emptyset$.

(b) Suppose instead that we only know that for some positive integer k, among any $k+1$ of the intervals from S, there are two with a non-empty intersection. Then, prove that there exists a set of k points on the real line such that each interval in S contains at least one of them.

(c) Suppose set S is a countably infinite set of open intervals. Then does the statement in (b) above hold? If yes, prove it, else give a counter-example.

i. We show a stronger claim by induction: Given the finite set S of open intervals (a_i, b_i), where any two have a non-zero intersection, their intersection is non-empty **and** this intersection is an interval (α, β) with $\alpha = \max_i a_i$ and $\beta = \min_i b_i$

1 for claim, or any other correct proposition to be proved by induction. Marks to be awarded only if the claim is also proved correctly

We induct on the cardinality of S, $n = |S|$

Base Case: $n = 1$: This holds true since the intersection is the interval (a_1, b_1) itself.

0.5 mark for the base case, with marks awarded only if the proof is correct

Induction hypothesis and step:

We now assume it to be true for $n = k$ for some $k \geq 1$, and show it for $n = k + 1$.

The intersection of all the intervals will be $(I_1 \cap I_2 \cdots \cap I_k) \cap I_{k+1}$. By our induction hypothesis, the intersection of first k intervals is the non-empty open interval (α, β). Hence the required intersection is $(\alpha, \beta) \cap I_{k+1}$. Consider the cases on their intersection:

- They have a non-zero intersection, the intersection will be $(\max(\alpha, a_{k+1}), \min(\beta, b_{k+1}))$. It is easy to see then that both the parts of our claim hold true for $k + 1$.
- Their intersection is empty. In this case, we either have the interval I_{k+1} lying completely to the left of the interval (α, β) or completely to the right. Let us analyse the former case (the latter has the same argument).
  In case of former, we have $b_{k+1} < \alpha \implies b_{k+1} < \max_i a_i$. Consider the index $i^* = argmax_i(a_i)$, we then have $b_{k+1} < a_{i^*}$. Hence the interval I_{k+1} would lie to the left of the interval I_{i^*}, which implies is has a zero intersection with the interval I_{i^*}. This is a contrdiction because no two intervals in S have an empty intersection. Hence this case is not possible.

In both the cases, we showed that the induction hypothesis holds for $k + 1$. Hence by induction, it holds for all $k \geq 1$

3.5 marks for the proof. 0.5 at most for using proof by cases but only proving the result for the simplest cases..

ii. We present an alternative proof, based on strong induction.
Base Case: n=1, 2 and 3. The proof for 3 alone is non-trivial and will be explained here. Suppose I_1, I_2 and I_3 are intervals with all of their pairwise intersections being non-empty. Let I_1 and I_2 intersect in the interval (a, b). Let a must be the left endpoint of $I_i, i \in \{1, 2\}$ and b must be the right endpoint of $I_j, j \in \{1, 2\}$. Then, if $I_3 = (a_3, b_3)$ does not intersect with $I_1 \cap I_2$, either $b_3 \leq a$ or $a_3 \geq b$. In the first case, I_3 does not intersect I_i, and in the second case, I_3 does not intersect I_j, both of which are contradictions.
1 mark for base case here, since the base case n=3 is very vital to the remainder of the proof
Induction Hypothesis: Assume for some $k \geq 3$, for all $1 \leq n \leq k$, whenever there exists a set of n open intervals with all pairs of intervals having a non-empty intersection, the intersection of all intervals is non-empty.
1 mark. No marks if later proof is incorrect. **Induction Step:**
Consider the three intervals I_1, $\bigcap_{i=2}^{k} I_i$ and I_{k+1}. By the induction hypothesis, $\bigcap_{i=2}^{k} I_i$ is non-empty, as every pair of intervals amongst $\{I_2, \cdots, I_K\}$ has a non-empty intersection. Now, notice that the intersection of these 3 sets is $\bigcap_{i=1}^{k+1} I_i$. Also, $\bigcap_{i=1}^{k} I_i$ and $\bigcap_{i=2}^{k+1} I_i$ have non-empty intersections by the induction hypothesis. Thus, all three of the sets we have chosen pairwise have non-empty intersection. Because we have proven the claim for n=3, we have $\bigcap_{i=1}^{k+1} I_i$ is non-empty.
3 marks for the proof.

(b)

i. Construct a poset with two intervals satisfying $(a_x, a_y) < j(b_x, b_y)$ being related if $a_y < b_x$ (and equal if the intervals are equal). This is a valid poset as it is a reflexive transitive and anti-symmetric relation.

2 mark for correct poset, and some proof about it being a valid poset

A chain in this satisfies the condition that no two intervals in the chain intersect.

ii. Our condition implies that there is no chain of length $> k$.

iii. Given that, the length of the maximum chain is $\leq k$ and Mirsky's theorem (or any reasonable constructive proof involving "levels" in the Hasse diagram) allows us to construct a partition with at most k antichains.

1 marks for showing that we can partition it into k anti chains

iv. An antichain corresponds to a set of intervals such that any 2 have a non-empty pairwise intersection.

v. From Part (a) above, this also implies that the entire set of intervals has a non-empty intersection.

vi. Take any element a from the intersection corresponding to each antichain, and this set of at most k elements can be trivially extended to a set of size k, such that every interval contains at least one of these k, corresponding to the antichain in which the interval lies.

2 marks for the final argument above, that shows we can obtain k points.

At most 2 marks for partial progress towards constructing a poset, but constructing a structure with two intervals related if they intersect (this relation is not transitive)

(c) Simply set $k = 1$ and consider the intervals $(1 - \frac{1}{n}, 1)$ for positive natural numbers n. Every finite subset of these intervals has a non-empty intersection $(1 - \frac{1}{n_0}, 1)$ where n_0 is the greatest value of n corresponding to the intervals in the subset considered. But if $a < 1$ is in the infinite intersection, set $n_0 = 1 + \lceil \frac{1}{1-a} \rceil$ and note that a is not in the interval $(1 - \frac{1}{n_0}, 1)$.
Grading scheme: 2 for correct counterexample