CS105 2024 Discrete Structures
Basics, Strong Induction
Exercise Problem Set 2

Part 1

1. Prove (by induction) or disprove: For every positive integer n,
   (a) $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}$.
   (b) if $h > -1$, then $1 + nh \leq (1+h)^n$.
   (c) 12 divides $n^4 - n^2$.

2. Consider the proposition: for all natural numbers $n \geq 5, n^2 < 2^n$.
   (a) Give a proof by induction. Do you need Strong Induction for it? Can you use Strong Induction?
   (b) Also give a proof by contradiction, using Well-Ordering-Principle.

3. Given $n \in \mathbb{N}$, consider a sequence of numbers where the n^{th} term of the sequence is defined by a "recurrence". That is,
   $$u_0 = 1, u_1 = 3, u_n = 2u_{n-1} - u_{n-2}, \text{ for } n \geq 2$$
   Prove using Strong Induction that for all $n \geq 0$, $u_n = 2n+1$. Also highlight why you need the Strong Induction in your proof.

4. Recall the statement of the fundamental theorem of arithmetic: Any integer can be written as a unique product of one or more primes. We proved the existence part in class. Show the uniqueness, i.e., there is indeed a *unique* decomposition.

Part 2

5. In a cricket tournament, every two teams played against each other exactly once. After all games were over, each team wrote down the names of the other teams they defeated, and the names of those teams defeated by some team they defeated. Give 2 proofs, one using induction and one using contradiction, that at least one team listed the names of every other team! Note: Assume no ties.

6. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show, using induction, that there is a car which can complete a lap by collecting gas from the other cars on its way around.

7. Consider the following game:
   - There are two piles of matches.
   - Two players take turns removing any positive (i.e., non-zero) number of matches they want from one of the two piles.

- The player who removes the last match wins.

Show (by Strong induction!) that, if the two piles contain the same number of matches initially, then the second player can always win the game.

8. Use the Well Ordering-Principle to show the following:

   (a) Define g as the GCD (greatest common divisor) of natural numbers a and b if $g|a$ and $g|b$, and for all natural numbers d, if $d|a$ and $d|b$ then $d|g$.

   Show that any two positive integers a, b have a unique greatest common divisor (hint: consider the set of numbers of the form $ax + by$).

   (b) The equation $4a^3 + 2b^3 = c^3$ does not have any solutions over $\mathbb{N} - \{0\}$. What about the equation $a^4 + b^4 + c^4 = d^4$ over $\mathbb{Z}$?