CS105 (DIC on Discrete Structures)
Problem set 8

- Apart from things proved in lecture, you cannot assume anything as "obvious". Either quote previously proved results or provide clear justification for each statement.

Basic

1. Show that if there are 30 students in a class, at least two have last names beginning with the same letter.

2. In CS105 this semester there are 205 registered students. They have to be given grades, which must be of course one of following: AA, AB, BB, BC, CC, CD, DD, FR.

   (a) Can we be sure that at least 25 students have got the same grade? Why or why not?

   (b) What is the minimum number of students required in a course to be sure that at least 30 must have the same grade?

3. In any group of 7 people, show that there must be a group of 3 people who are mutual friends (that is, each pair of them are friends), or a group of 3 people who are mutually unknown to each other. You may assume that between any two people in the group, they are either friends or don't know each other.

4. Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

5. Consider the number of strings x_n of length n over the set $\{0, 1, 2\}$ such that any two consecutive numbers in the string differ by at most 1. Thus 01221 is such a string of length 5, but 01200 is not.

   (a) Write a recurrence relation for x_n.

   (b) What are x_1, x_2 and x_7?

   (c) Solve the recurrence to get a closed form solution for x_n in terms of n.

Advanced

6. How many integral solutions does $x_1 + x_2 + x_3 = 11$ have where $0 \leq x_1 \leq 3, 0 \leq x_2 \leq 4, 0 \leq x_3 \leq 6$?

7. How many ways can a convex n-sided polygon be cut into triangles by adding non-intersecting diagonals (i.e., connecting vertices with non-crossing lines)? Write a recurrence and solve it!

8. Write the formal proof showing that $R(3,4) = 9$. Note that we already saw the proof for $R(3,4) \leq 9$ in class, but your exercise is to formally rewrite it in your own words. And in addition to show that this number is indeed optimal (which we did not do in class).

9. Recall that a derangement is a permutation of objects that leaves no object in its original position. Let D_n denote the number of derangements of $\{1, \ldots, n\}$. Thus, D_n is the number of ways in which letters numbered 1 to n can be put into boxes numbered from 1 to n such that for any i, the letter i is not in box i. We wrote a recurrence for this in the earlier problem sheet. Here is a question on derangements via Principle of Inclusion Exclusion (PIE):

   (a) Obtain a closed form solution for D_n in terms of n, using PIE.

   (b) What is $\lim_{n\to\infty} \frac{D_n}{n!}$?

10. A bijection $f : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ is said to be *cool* if there exists $i \in \{1, \ldots, 2n-1\}$ such that $|f(i) - f(i+1)| = n$. Prove that, for each n, more than half the bijections from $\{1, \ldots, 2n\}$ to $\{1, \ldots, 2n\}$ are cool.