Quiz 1: DIC on Discrete Structures

Answer Key and Grading Scheme

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1. (9 marks) True or false. If true, give a short proof, if false give a counter-example and a short justification why its a counter-example.
   (a) For all $x, y \in \mathbb{N}$, if $(x \cdot y^2)$ is odd, then $(5x + 3y)$ is odd.
   (b) Let A, B be two infinite sets. If A is countable and $B \subseteq A$ then B must be countable.
   (c) For sets $A, B \subseteq U$ if there is a bijection from A to B then there is a bijection from complement of A, i.e., $U \setminus A$ to complement of B, i.e., $U \setminus B$.

Solution.

a) It is false.
Take $x = 1$, $y = 1$.
$5x + 3y = 5 \cdot 1 + 3 \cdot 1 = 8$ which is even
[1 mark for false. 1.5 marks for counter-example, 0.5 marks for justification why it is a counter-example.]

b) It is true.
As B is an infinite set, there exists a surjection from B to $\mathbb{N}$.
As A is countable, there exists a injection from A to $\mathbb{N}$ say g.
Claim: function h from B to $\mathbb{N}$ defined as $h(x) = g(x)$ for all $x \in B$ is also an injection. As B is a subset of A and g is well defined, h is also well defined. Assume h is not an injection. Then there exists $x, y \in B, x \neq y$ for which $h(x) = h(y)$. However, as B is subset of A, both $x, y \in A, x \neq y$ and, construction of h, $g(x) = g(y)$ which is a contradiction to the fact g is an injection. Thus, h is an injection, and thus B has both, a surjection and an injection to $\mathbb{N}$ and thus B is countable.

[1 mark for true, 1 Mark for choosing claim, 1 mark for proving injection.]

c) It is false.
Take $A = \mathbb{N}, B = \mathbb{N} \setminus \{0\}$ and $U = \mathbb{N}$. Thus, there exists a bijection from A to B which is, say f, $f(x) = x + 1$ but A's complement is the emptyset, ϕ, while the complement of B is $\{0\}$. As they are finite sets of different size there can be no bijection between them. Thus the statement is false.
[1 mark for false. 1.5 marks for counter example, 0.5 marks for justification why it is a counter-example]

2. (5 marks) Consider the statement:
   Every integer greater than 7 is the sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.
   Write this as a proposition using quantifiers and prove it formally by using (strong) induction or WOP.

Solution.
Proposition: (1 mark)

$$\forall n \geq 8,\ \exists k \geq 0 \text{ and } \exists l \geq 0 \text{ such that } n = 3k + 5l.$$

We will prove this proposition using the principle of mathematical induction.
Base Case: (1 mark) We begin by proving the proposition for the initial case $n = 8$.
For $n = 8$, consider the values $k = 1$ and $l = 1$. Then:

$$n = 3 \cdot 1 + 5 \cdot 1 = 3 + 5 = 8.$$

Thus, $P(8)$ is true, demonstrating that the base case holds.
Inductive Hypothesis: (1 mark) Assume that for some arbitrary integer $n \geq 8$, the proposition $P(n)$ holds. This means there exist nonnegative integers $k \geq 0$ and $l \geq 0$ such that:

$$n = 3k + 5l.$$

Inductive Step: (2 mark) We need to show that $P(n + 1)$ is also true; that is, we need to demonstrate that there exist nonnegative integers $m \geq 0$ and $p \geq 0$ such that:

$$n + 1 = 3m + 5p.$$

To achieve this, we analyze two cases based on the value of l.
Case 1: $l = 0$
If $l = 0$, then $n = 3k$. Given that $n \geq 8$, it follows that $3k \geq 8$, implying $k \geq 3$. In this scenario, we consider $n + 1 = 3k + 1$. To express $n + 1$ in the form $3m + 5p$, we can rewrite it as:

$$n + 1 = 3k + 1 = 3(k - 3) + 9 + 1 = 3(k - 3) + 2 \cdot 5.$$

Since $k \geq 3$, $k - 3 \geq 0$. Thus, setting $m = k - 3$ and $p = 2$, we have nonnegative integers $m \geq 0$ and $p \geq 0$ such that:

$$n + 1 = 3m + 5p.$$

Case 2: $l > 0$
If $l > 0$, then $l - 1 \geq 0$. Given that $n = 3k + 5l$, we can express $n + 1$ as:

$$n + 1 = 3k + 5l + 1 = 3(k + 2) - 6 + 5(l - 1) + 5 + 1 = 3(k + 2) + 5(l - 1).$$

Here, $k + 2 \geq 0$ and $l - 1 \geq 0$ by the assumption. Hence, by setting $m = k + 2$ and $p = l - 1$, both m and p are nonnegative integers. Thus, we have:

$$n + 1 = 3m + 5p.$$

Conclusion:
In both cases, we have shown that if the proposition $P(n)$ is true for some $n \geq 8$, then $P(n + 1)$ is also true. By the principle of mathematical induction, the proposition holds for all $n \geq 8$.

Proof using the Well-Ordering Principle (WOP)

Statement: (1 mark)

$$\forall n\,(n > 7 \to \exists a \geq 0\,\exists b \geq 0\,(n = 3a + 5b))$$

Proof:

Assumption for Contradiction (0.5 mark):

Assume, for contradiction, that there exists an integer $n > 7$ that cannot be expressed as $n = 3a + 5b$ where a and b are nonnegative integers.

Let S be the set of all such integers greater than 7 that cannot be expressed in this form. By our assumption, S is non-empty.

Applying the Well-Ordering Principle (0.5 mark):

By the Well-Ordering Principle, any non-empty set of positive integers has a least element. Let m be the smallest element in S.

Thus, $m > 7$ and m cannot be expressed as $m = 3a + 5b$.

Checking Smaller Values for Contradiction(2 mark):

Since $m > 7$, consider the numbers $m - 3$ and $m - 5$:

- **Case 1:** $m - 3 \geq 8$
  Since m is the smallest element in S, $m-3$ must not be in S (otherwise, it would contradict the minimality of m).
  Therefore, $m - 3$ can be expressed as $m - 3 = 3a' + 5b'$ for some nonnegative integers a' and b'.
  Then, $m = 3(a' + 1) + 5b'$, which implies m can be expressed as $3a + 5b$, a contradiction.

- **Case 2:** $m - 5 \geq 8$
  Similarly, if $m - 5 \geq 8$, $m - 5$ must not be in S.
  Thus, $m - 5 = 3a'' + 5b''$ for some nonnegative integers a'' and b''.
  Then, $m = 3a'' + 5(b'' + 1)$, which again implies m can be expressed as $3a + 5b$, leading to a contradiction.

In both cases, we have shown that m can be expressed as a sum of a multiple of 3 and a multiple of 5, contradicting the assumption that $m \in S$.

Verification for Base Cases $8, 9, 10, 11, 12$ **(1 mark):**

We verify that the smallest integers greater than 7 can all be expressed as $3a + 5b$:

- $8 = 3 \times 1 + 5 \times 1$
- $9 = 3 \times 3 + 5 \times 0$
- $10 = 3 \times 0 + 5 \times 2$
- $11 = 3 \times 2 + 5 \times 1$
- $12 = 3 \times 4 + 5 \times 0$

Since these base cases are covered, and every integer greater than 12 can be reached by repeatedly adding 3 or 5, all integers greater than 7 can be expressed as $3a + 5b$.

Conclusion:

Since assuming the set S is non-empty leads to a contradiction, the set S must be empty. Therefore, every integer greater than 7 is indeed the sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5. **Proposition:**

$$\forall n \in \mathbb{Z},\ (n > 7) \implies (\exists x, y \in \mathbb{Z}_{\geq 0} \text{ such that } n = 3x + 5y)$$

Proof by Strong Induction

$$\forall n \geq 8,\ \exists k \geq 0 \text{ and } \exists l \geq 0 \text{ such that } n = 3k + 5l.\color{red}{(1 mark)}$$

Base Cases (1 mark)

We first verify the proposition for all integers n such that $8 \leq n \leq 12$.

- **Case $n = 8$:**
$$8 = 3 \cdot 1 + 5 \cdot 1$$
Thus, $x = 1$ and $y = 1$.
- **Case $n = 9$:**
$$9 = 3 \cdot 3 + 5 \cdot 0$$
Thus, $x = 3$ and $y = 0$.
- **Case $n = 10$:**
$$10 = 3 \cdot 0 + 5 \cdot 2$$
Thus, $x = 0$ and $y = 2$.
- **Case $n = 11$:**
$$11 = 3 \cdot 2 + 5 \cdot 1$$
Thus, $x = 2$ and $y = 1$.
- **Case $n = 12$:**
$$12 = 3 \cdot 4 + 5 \cdot 0$$
Thus, $x = 4$ and $y = 0$.

All these base cases satisfy the proposition. Therefore, the proposition holds for $n = 8, 9, 10, 11,$ and 12.

Inductive Step(1+2 mark)

For the **inductive step**, assume that the proposition holds for all integers k such that $8 \leq k \leq n$, where $n \geq 12$. In other words, assume that:

$$\forall k \in \mathbb{Z},\ (8 \leq k \leq n) \implies (\exists x, y \in \mathbb{Z}_{\geq 0} \text{ such that } k = 3x + 5y)$$

We need to show that the proposition also holds for $n + 1$. That is, we must prove that there exist nonnegative integers x' and y' such that:

$$n + 1 = 3x' + 5y'$$

By the **inductive hypothesis**, since $n - 2 \geq 10$ for $n \geq 12$, there exist nonnegative integers x and y such that:

$$n - 2 = 3x + 5y$$

To find $n + 1$, add 3 to both sides of the equation:

$$n + 1 = (n - 2) + 3 = 3x + 5y + 3 = 3(x + 1) + 5y$$

Here, we define $x' = x + 1$ and $y' = y$. Both x' and y' are nonnegative integers, as $x \geq 0$ and $y \geq 0$.
Therefore, we have shown that $n + 1 = 3x' + 5y'$, satisfying the proposition for $n + 1$.

Conclusion

By the principle of strong induction, the proposition holds for all integers $n > 7$. Hence, every integer greater than 7 can be expressed as the sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.

3. (6 marks) Which of the following sets are countable? Justify your answer. If uncountable, use Cantor's diagonalization to show that this is the case.

   (a) The set of all functions from $\{0,1\}$ to $\mathbb{Z}$.

   (b) The set of all functions from $\mathbb{Z}$ to $\{0,1\}$.

Solution.

a) Yes, it is countable. [0.5 for answer(countable)]
We can create a bijection from set of all function $f : \mathbb{Z} \to \{0,1\}$ to the set $\mathbb{Z} \times \mathbb{Z}$.
Claim: g defined as:

$$g(f) = (f(0), f(1)) \quad [\text{1 mark for denoting as pair}]$$

is a bijection from set of all functions from $\{0,1\}$ to $\mathbb{Z}$ to the set $\mathbb{Z} \times \mathbb{Z}$.
As all functions in domain have $\{0,1\}$ as their domain, the function g satisfies the first property of being well defined.
As all fs are well defined functions, g satisfies second property of being well defined.

$$\begin{aligned}
g(f_1) &= g(f_2) && (1)\\
f_1(0) &= f_2(0) \text{ from } (1) && (2)\\
f_1(1) &= f_2(1) \text{ from } (1) && (3)\\
f_1 &= f_2 \text{ from } (2) \text{ and } (3) && (4)
\end{aligned}$$

Thus g is an injection. [1 mark for injection from set of all function to $\mathbb{Z} \times \mathbb{Z}$ (showing 2 functions cannot map to the same pair)]
$\forall (a,b) \in \mathbb{Z} \times \mathbb{Z}$, we can create a function f such that $f(0) = a$ & $f(1) = b$ and thus, $g(f) = (a,b)$.
Thus, g is a surjection.
As $\mathbb{Z}$ is countable, and cartesian product of cartesian sets is countable, $\mathbb{Z} \times \mathbb{Z}$ is countable. [0.5 for $\mathbb{Z} \times \mathbb{Z}$ is countable]
Thus, using our bijection j, we can conclude the set of all functions from $\{0,1\} \to \mathbb{Z}$ is countable.
Final Marking scheme:
[0.5 for answer(countable)
1 mark for denoting as pair
1 mark for injection from set of all function to $\mathbb{Z} \times \mathbb{Z}$
0.5 for $\mathbb{Z} \times \mathbb{Z}$ is countable]

Solution.

a) (3 marks) Yes, it is countable. [0.5 for answer (countable)]
Its enough to prove that there exists injection from the set of all $f : \{0,1\} \to \mathbb{Z}$ to $\mathbb{N}$ for us to prove it is countable.

We will define an injective mapping from the set of all $f : \{0,1\} \to \mathbb{N}$ to $\mathbb{N}$.

Consider the set of all functions $f : \{0,1\} \to \mathbb{Z}$. We can create a function g from this set to $\mathbb{N}$ which is an injection as follows:

$$g(f) = 2^{h(f(0))} \cdot 3^{h(f(1))}$$ [1 mark for finding the function.]

where $h(x)$ is bijection from $\mathbb{Z}$ to $\mathbb{N}$. (No need to prove that h is a bijection)

$$h(x) = \left\{ \begin{array}{ll} 2 \cdot x, & \text{for } x \geq 0 \\ -2 \cdot x - 1, & \text{for } x < 0 \end{array} \right\}$$

Claim: g is a well defined function.

Due to construction, this function maps each function $f : \{0,1\} \to \mathbb{Z}$ to atleast 1 natural number. Thus g satisfies 1st property of being well defined.

For $f_1 = f_2$:

$$\begin{align} f_1(0) &= f_2(0) \tag{5} \\ f_1(1) &= f_2(1) \tag{6} \\ g(f_1) &= 2^{h(f_1(0))} \cdot 3^{h(f_1(1))} \tag{7} \\ &= 2^{h(f_2(0))} \cdot 3^{h(f_2(1))}, \text{ from (1) and (2)} \tag{8} \\ &= g(f_2) \tag{9} \end{align}$$

Thus g also satisfies the 2nd property of being a well defined function. Thus g is a well defined function.

Claim: g is an injection.

For $g(f_1) = g(f_2)$,

$$2^{h(f_1(0))} \cdot 3^{h(f_1(1))} = 2^{h(f_2(0))} \cdot 3^{h(f_2(1))} \tag{10}$$

As 2 and 3 are prime numbers, both lhs and rhs are prime decompositions of the same number. From the fundamental theorem of arithmetic, they must have the same exponents for each primes:

$$\begin{align} h(f_1(0)) &= h, \text{ equating exponents of 2} \tag{11} \\ f_1(0) &= f_2(0), \text{ as h is an injection} \tag{12} \\ f_1(1) &= f_2(1), \text{ similarly equating exponents of 3} \tag{13} \\ f_1 &= f_2, \text{ from (8) and (9)} \tag{14} \end{align}$$

Therefore $g(f_1) = g(f_2) \to f_1 = f_2$, and thus g is an injection. [1.5mark for proving injection.]

Since there is an injective mapping from the set of all functions to the natural numbers, we conclude that the set of functions from $\{0,1\}$ to $\mathbb{Z}$ is countable.

[1 mark for finding the function.
1.5mark for proving injection.
0.5 for answer (countable)]

Solution.

b) Let S be the set of all functions from $\mathbb{Z}$ to $\{0, 1\}$. We claim that S is uncountable. To prove that, assume to the contrary that S is countable. Thus there exists a bijective function $f : \mathbb{Z} \longrightarrow S$(Note: as Z is countable, we can create this function same as enumeration) [0.5]. Now, define a function $g : \mathbb{Z} \longrightarrow \{0, 1\}$ as follows:

$$\forall n \in \mathbb{Z}, g(n) = \begin{cases} 0, & \text{if } (f(n))(n) = 1 \\ 1, & \text{if } (f(n))(n) = 0 \end{cases} \quad [1] \tag{15}$$

since $g \in S$ [0.5], $\exists n_0 \in \mathbb{Z}$, s.t. $f(n_0) = g$. Now, either $g(n_0) = 1$ or $g(n_0) = 0$. However, note that

$$g(n_0) = 1 \implies (f(n_0))(n_0) = 1 \implies g(n_0) = 0$$

and

$$g(n_0) = 0 \implies (f(n_0))(n_0) = 0 \implies g(n_0) = 1 \ [0.5]$$

Thus, for both $g(n_0) = 1$ or $g(n_0) = 0$, we arrive at a contradiction. This contradicts our assumption that S was countable. Hence S must be uncountable. [0.5]

[0.5 marks for conclusion (uncountable)
0.5 mark for assuming the contrary, and stating f can be enumerated
1 mark for counter example
0.5 marks for arguing that this function belongs in domain
0.5 mark for arriving at contradicton (proving that this function cannot belong to the enumeration)]

b) Alternate solution:

Let S be the set of all such functions.

Claim: There exists bijection from S to $P(N)$.

Take $g : S \to P(N)$ defined as $g(f) = \{2x | x \geq 0(x) = 0\} \bigcup \{-2x - 1 | x < 0 \ \& \ f(x) = 0\}$

By construction, g satisfies the property of being a well defined function. It has an image for all elements and there is only a single way of creating that image thus it doesn't have 2 images for the same element.

If $g(f_1) = g(f_2)$ then for all i, $f_1(i) = 0_2(i) = 0$ by construction of g. And as f_1 and f_2 have only $\{0, 1\}$ in their ranges, $f_1(i)! = 0 \iff f_2(i)! = 0$ is the same as $f_1(i) = 1_2(i) = 1$. Thus, for all i, $f_1(i) = f_2(i)$ and thus $f_1 = f_2$. Therefore g is an injection.

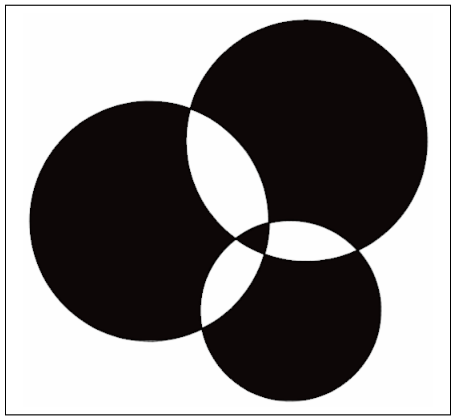
For any set $A \in P(N)$, we can construct f such that $f(i) = 0 \iff (i \geq 0 \ \& \ 2 * i \in A) \vee (i < 0 \ \& \ -2 * i - 1 \in A)$ and $f(i) = 1$ otherwise. By construction of g, you get $g(f) = A$. Thus, g is a surjection. And thus, g is a bijection. Hence, our claim is true. [1.5]

Claim: $P(N)$ is uncountable.

Assume it is countable, then we can enumerate all sets in $P(N)$. Create a set B such that $i \in B \iff i \notin C_i$ where $P(N) = \{C_0, C_1, C_2, ...\}$. As $P(N)$ is countable, there exists j such that $C_j = B$. However, if $j \in C_j \to j \notin B \to j \notin C_j$ which is contradiction. Similarly, if $j \notin C_j \to j \in B \to j \in C_j$ which is again contradiction. Therefore, our assumption is wrong. Thus, $P(N)$ is uncountable [1] and as there exists a bijection from S to $P(N)$, set S is also uncountable [0.5]. Hence proved

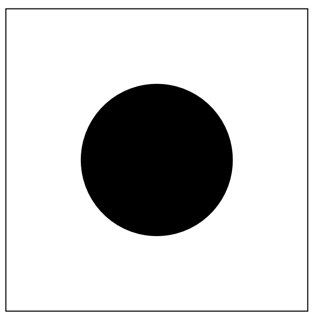
[0.5 marks for uncountable
1.5 mark for showing bijection to $P(N)$
1 mark for using cantors diagnolisation]

4. (5 marks) n circles (where $n \geq 1$) are given on the plane, dividing it into several regions. Show, by induction, that one can color the plane with two colors, such that no two regions with a common boundary line have the same color. See Figure below for an example.



Note that having a single point of contact between two regions does not count as a common boundary line.

Solution. We shall prove by induction, that proposition: $P(n)$: "The regions of any figure with n circles can be colored using two colors, such that no two regions sharing a common boundary have the same color" holds for all $n \geq 1$.
Base case: $n = 1$: It can be seen a figure with 1 circle can be colored using two colors (black and white in the figure below) which satisfies the constraints. [1 mark, 0 if base case is not 1]

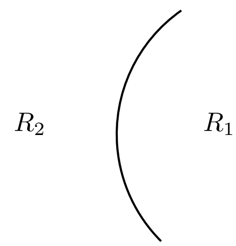


Induction hypothesis: Assume now that for some $k \geq 1$, the proposition $P(k)$ holds true. We shall now show that with this assumption, $P(k+1)$ also holds true. [1 mark]
Induction step: Consider any figure F_1 with $n = k+1$ circles. Since $n \geq 1$, we can choose a circle $\mathcal{C}$ from the figure. Consider the figure F_2 obtained from removing this circle $\mathcal{C}$. F_2 is a figure with k circles, and by our induction hypothesis, F_2 can be colored using two colors. We now have two cases:
(Note: We analyse this case only if we do not assume that the circles are distinct)
Case 1: The circle $\mathcal{C}$ coincides with some circle in F_2: In this case, the original figure F_1 is exactly the same as F_2, and the coloring of F_2 is also a valid coloring for the F_1.

Case 2: The circle $\mathcal{C}$ does not coincide with any of the k circles in F_2:
Claim: The coloring obtained by flipping the colors (black goes to white and vice versa) of regions inside $\mathcal{C}$ in the coloring for F_2 gives a valid coloring for F_1

Proof: Assume for the sake of contradiction, that after the above mentioned operation, we have two regions sharing a boundary, and having the same color. We have:

- Both the regions are outside the circle $\mathcal{C}$: In this case, the boundary they share cannot be from $\mathcal{C}$, and hence must be from one of the k circles. Also, since they are both outside, they have the same coloring as in F_2. But then, we have two regions in F_2 sharing a common boundary and having the same colors in the original coloring, violating the assumption that the original coloring was a valid coloring in F_2.
- Both the regions are inside the circle $\mathcal{C}$: We have a similar situation as above, with the only change being both the regions would have their colors flipped from the original coloring. Hence, we still arrive at the same contradiction of the original coloring being an invalid coloring in F_2.
- One region (say R_1) is inside $\mathcal{C}$ and the other (say R_2) is outside:



In this case, a common boundary for the two regions is a part of circle $\mathcal{C}$ (and this part of boundary is exclusively a part of $\mathcal{C}$, since if two circles have the same arc, they must be identical), and hence in F_2 (where this boundary is absent), they must be part of the same region, say R (as can be seen by the figure above). R would have had one color in F_2, and the operation would have flipped the color of only R_1, and thus the two regions would have had different colors after the operation, making this case impossible.

Hence, by contradiction, we conclude that the above claim holds true.
In both the cases, we are able to obtain a valid coloring for F_1. Thus $P(k+1)$ holds true assuming $P(k)$ holds true.
Hence by induction, $P(n)$ holds for all $n \geq 1$.
[3 marks for correct proof in induction step. If student uses the above proof: 1 marks for the correct operation + 2 mark for correctness of the operation. Otherwise, marks to be awarded based on progress made in the proof.]
[If proved without induction, marks awarded out of a max of 3 based on progress made]

5. (0 marks) In your opinion, who was more awesome: Cantor or Russell? Why?