CS105 (DIC on Discrete Structures)
Problem set 6

- Attempt *all* questions.
- Apart from things proved in lecture, you cannot assume anything as "obvious". Either quote previously proved results or provide clear justification for each statement.

Basic

Posets and Lattices

1. Give an example of a poset with five elements that is a lattice and an example of another poset with five elements which is not a lattice.
2. Prove or disprove: Every totally ordered set is a lattice.

Counting

3. How many functions are there from a set of size 15 to a set of size 12?
4. Prove the following identities by counting the size of a suitably designed set in two different ways.
   (a) $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$
   (b) $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

Advanced

Posets and Lattices

5. Let L be a lattice. For any two elements $x, y \in L$ we use $x \vee y$ to denote the least upper bound of $\{x, y\}$ and $x \wedge y$ to denote the greatest lower bound of $\{x, y\}$ (note that both these elements exist by the definition of a lattice).

(a) Show the following properties for all $x, y, z \in L$,

   i. (commutative laws) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$
   ii. (associative laws) $((x \vee y) \vee z) = (x \vee (y \vee z))$ and $((x \wedge y) \wedge z) = (x \wedge (y \wedge z))$.
   iii. (absorption laws) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$
   iv. (idempotency laws) $x \vee x = x$ and $x \wedge x = x$.

(b) Use the above to prove that every finite nonempty subset of a lattice must have a greatest lower bound and a least upper bound.

6. For all $t > 0$, prove formally that any poset with n elements must have either a chain of length greater than t or an antichain with at least $\frac{n}{t}$ elements.

7. *Consider a permutation of the numbers from 1 to n arranged as a sequence from left to right on a line. Using Mirsky's theorem done in class, prove that there exists a $\sqrt{n}$-length subsequence of these numbers that is completely increasing or completely decreasing as you move from right to left.

   For example, the sequence 2, 3, 4, 7, 9, 5, 6, 1, 8 has an increasing subsequence of length 3, for example: 2, 3, 4, and a decreasing subsequence of length 3, for example: 9, 6, 1. (Hint: Use the previous question!)

Counting

8. Two hundred students participated in a math contest. The had six problems to solve. Each problem was correctly solved by at least 120 participants. Prove, using double counting, that there must be two participants such that every problem was solved by at least one of these students.