Midsem Practice Questions (Optional)

September 2024

Note

The following questions may be challenging and are intended solely for practice purpose. Dont be discouraged if you cant solve even a single one in 2 hrs (as TAs who were in your positions some years back, believe us, we suffered through the same).
They are optional and meant to allow you to take a feel of what all you can do with the few techniques you learnt in this course by now. Feel free to attempt them as you see fit.

A. Proof Techniques

A.1 Prove that no right angled triangle with integer sides can have an area which is a perfect square.

A.2 Show that n straight lines divide a 2-D plane into:

$$\frac{n^2+n+2}{2}$$

regions, provided that no two such lines are parallel and no three meet at a single point.

A.3 Prove that we can cover a $2^n \times 2^n$ chessboard, $n \geq 2$, with L-shaped tiles such that no two tiles overlap and all but one of the four center squares are covered.

A.4 Prove that f_n and f_{n+1} are relatively prime for all $n \geq 1$. That is, they share no common factor other than 1, where $f_1 = 0$, $f_2 = 1$, and f_n is the n-th Fibonacci number.

A.5 Suppose S is a set of $n+1$ integers. Prove that there exists a, b distinct in S such that $a - b$ is a multiple of n.

A.6 Prove or disprove: For every two positive integers a, b, if an integer linear combination of a and b^2 equals 1, then so does an integer linear combination of a^2 and b.

A.7 Consider a tournament consisting of n players. Each player plays against every other player exactly once, and there are no draws. Show that you can order the players as $P_1, P_2, \ldots, P_n$ such that $\forall i = 1, 2, \ldots, n-1$, P_i defeated P_{i+1} in the tournament.
(Hint: Try using induction on n. When you have an ordering of $n-1$ players, show that there would always exist a place for a new player P in the sequence such that he is defeated by the player in front of him and defeats a player behind him OR he is to be placed at the extreme ends.)

A.8 Let $a_1, a_2, \cdots a_{2n+1}$ be a set of 2n+1 non-negative integers such that if any one of them is removed, the remaining set can be partitioned into sets of n integers each both of which have the same sum. Prove that $a_1 = a_2 \cdots a_{2n+1}$.

B. Sets, Functions and Countability

B.1 Suppose U is the power set of the set $S = \{1, 2, 3, 4, 5, 6\}$. For any $T \in U$, let $|T|$ denote the number of elements in T and T' denote the complement of T. For any $T, R \in U$ let $T \setminus R$ be the set of all elements in T which are not in R. Which one of the following is true?

A. $\forall X \in U, (|X| = |X'|)$

B. $\exists X \in U, \exists Y \in U, (|X| = 5, |Y| = 5 \text{ and } X \cap Y = \varnothing)$

C. $\forall X \in U, \forall Y \in U, (|X| = 2, |Y| = 3 \text{ and } X \cap Y = \varnothing)$

D. $\forall X \in U, \forall Y \in U, (X \setminus Y = Y' \setminus X')$

B.2 For any two sets A, B, is it the case that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$? If yes, prove your statement; if no, give a counterexample. (Note: A, B are arbitrary sets, so you cannot assume them to be finite or even countable!)

B.3 Let $A_1, A_2, \ldots$ be an infinite sequence of sets such that the intersection of any finite number of sets in the sequence is non-empty. Is it true that there exists an element x such that $x \in A_i$ for all i? If so, prove it; else give an example where it is false.
Suppose $a_1, a_2, \ldots$ is an infinite sequence of numbers such that the *gcd* of any finite set of numbers in the sequence is greater than 1. Prove that there exists a prime p such that p divides a_i for all i.

B.4 Prove that the countable union of countable sets is countable.

B.5 Show that the following sets are countable:

1. The set of all strings of finite length over a finite alphabet.
2. The set of real numbers that are solutions of quadratic equations $ax^2 + bx + c = 0$, where a, b, c are integers
3. More generally, the set of all numbers which are the solution of a polynomial equation with integer coefficients.

B.6 Given an infinite set A, show that there exists a bijection from A to $A \times A$ if and only if there exists a bijection from A to $A \times A \times A$. *(Hint: Try constructing a bijection for $A \to A \times A \times A$ given a bijection from $A \to A \times A$. For the other direction, try constructing both an injection and a surjection from $A \to A \times A$, and use the Schröder-Bernstein Theorem to finish your argument.)*

B.7 Let f be a function from a set A to itself. Suppose there exists a number k such that f^k is the identity function. Prove that f is a bijection. Prove that the converse is true if A is finite, but is not true if A is infinite. For a finite set A with n elements, what is the smallest number k such that for every bijection f from A to A, $f^k = I$? Prove your answer.

B.8 The function $f : \mathbb{N} \to \mathbb{N}$ is defined as follows:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 3n + 1, & \text{if } n \equiv 1 \pmod 4 \\ 3n - 1, & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Prove that for any n, there exists a k such that $f^k(n) = 1$, where f^k is the function f composed with itself k times.

B.9 Show that there is no injection from the set of reals R to the set of functions $f : \mathrm{R} \to \mathrm{R}$. In fact, the set of functions on reals is of the same cardinality as the power set of R[1].

[1] There is no need to prove the last sentence. It is just a note from the instructor's side. However, to prove that for an infinite set A of cardinality k, the cardinality of 2^A equals the cardinality of A^A, observe that $|A| = |A \times A|$ and that 2^k is the cardinality of the power set of A, while k^k is the cardinality of the power set of A$\times$ A.

C. Relations, Equivalence Relations and Posets

C.1 A relation R is called a *Circular Relation* if

$$[(a,b) \in R \wedge (b,c) \in R] \rightarrow [(c,a) \in R].$$

Then which of the following is True ? Also Prove your Answer.

1. If S is reflexive and symmetric, then S is an equivalence relation.
2. If S is circular and symmetric, then S is an equivalence relation.
3. If S is reflexive and circular, then S is an equivalence relation.
4. If S is transitive and circular, then S is an equivalence relation.

C.2 Suppose there are n students and m companies. A relation R from students to companies is defined by $(s,c) \in R$ iff the student s is to be interviewed by company c. The interviews are conducted in 15-minute slots. Each student can be interviewed by at most one company per slot, and each company can interview at most one student per slot. Suppose that every student is to be interviewed by at most D companies and every company has to interview at most D students. Prove that it is always possible to schedule the interviews so that at most D slots are required. Is it always possible to find such a schedule such that the slots for every company are consecutive, though they may start at different times? If so, prove it; otherwise, find a counterexample.

C.3 Let R be any arbitrary relation defined on a set A. Let I denote the identity relation, and R^{-1} the converse of R. Let R^k be the relation defined by $R^0 = I$ and $R^k = R * R^{k-1}$ for $k > 0$. Write down an expression for the smallest equivalence relation E containing R, using these and other standard set operations. E is smallest in the sense that any equivalence relation that contains R must also contain E. Suppose E is an equivalence relation on a finite set A with n elements and k equivalence classes. What is the minimum number of elements in a relation R on A such that the smallest equivalence relation containing R is the given relation E? Prove your answer using induction.

C.4 In a partially ordered set, a chain is a totally ordered subset. For example, in the set $\{1,2,3,4,5,6\}$, the divisibility relation is a partial order, and $\{1,2,4\}$ and $\{1,3,6\}$ are chains.
(a) What is the longest chain on the set $\{1,2,\ldots,n\}$ using the divisibility relation? How many distinct chains have this length? Consider all positive values of n.
(b) What is the longest chain in the powerset of a set A with $|A| = n$ under the $\subseteq$ relation? How many distinct chains have this length?

C.5 Prove or disprove that every finite poset with a maximum and a minimum element is a lattice.

C.6 Let $(X, \leq)$, $(Y, \preceq)$ be two posets. We say they are isomorphic if there exists a bijection $f : X \rightarrow Y$ such that for every $x, y \in X$, we have $x \leq y$ iff $f(x) \preceq f(y)$. They are said to be non-isomorphic if they are not isomorphic.
(a) Draw Hasse diagrams of all non-isomorphic 3-element posets.
(b) Prove that any two finite total orders $(X, \leq)$, $(Y, \preceq)$, such that $|X| = |Y|$, are isomorphic. What if the sets were infinite?
(c) Find two non-isomorphic total ordering relations on the set of all natural numbers $\mathbb{N}$. Prove that they are non-isomorphic.
(d) Let S be the set of all pairwise non-isomorphic total orderings on $\mathbb{N}$. Is S finite or infinite? If finite, give an upper bound on $|S|$. If infinite, state whether it is countable or uncountable. Prove your claim.

C.7 Find a finite poset P with the following property or show that none exists - the longest chain in P is of length m. P can be written as the union of two chains C_1 and C_2, but this is not possible if we add the condition that the size of C_1 is m.

C.8 Consider the poset $(\mathcal{P}(S), \subseteq)$ for a finite set S. A chain $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m$ is defined to be a symmetric chain if $|A_{i+1}| = |A_i| + 1$ and $|A_1| + |A_m| = |S|$. Prove that the set $\mathcal{P}(S)$ can be partitioned into symmetric chains.

C.9 For a finite set S, let $P(S)$ be the set of partitions of S. Define the relation $\preceq$ on $P(S)$ as:

$$A \preceq B \iff \forall x, y \in S,\ x \sim_A y \implies x \sim_B y$$

where $\sim_A, \sim_B$ are the equivalence relations defined by the partitions. This is equivalent to saying that all the equivalence classes of A are contained in or are a refinement of those of B.

Show that $\preceq$ is a partial order. Show that $(P(S), \preceq)$ has a minimum and maximum element, and that it is a lattice.

Let $\vee, \wedge$ stand for the lub (least upper bound) and glb (greatest lower bound) operations. Show that these do not follow the distributive laws, i.e.

$$a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c), \quad a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$$

using a partition lattice.

C.10 Prove or disprove that the lub ($\vee$) and gub ($\wedge$) operators satisfy the following properties in every lattice:

1. (Distributivity) $\forall x \forall y \forall z (x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z))$
2. $\forall x \forall y \forall z (x \leq z \rightarrow (x \vee (y \wedge z) = (x \vee y) \wedge z))$

D. Counting

D.1 You have a bag with 2013 black balls and 2024 white balls. Without looking, you draw two balls from the bag and apply the following rule:

- If both balls are of the same color, you throw them both away.
- Otherwise, you throw away the black ball and return the white ball to the bag.

You keep repeating this process. If at some stage there is exactly one ball left in the bag, what can you say about the color of the ball?
(Hint: Parity checking.)

D.2 Suppose S be a set with n elements. Let $A \subseteq S$ and $B \subseteq S$. How many ordered pairs (A, B) are possible such that $A \subseteq B$?

D.3 Using combinatorial arguments, prove that for all a, b, and prime p, p^2 divides:

$$\binom{pa}{pb} - \binom{a}{b}.$$

(You may use the fact that p divides $\binom{p}{k}$ when $0 < k < p$.)

(Hint: Use Double Counting on pairs (j, k) for $d(j) = k$.)

D.4 Given below is an encoding format for an arbitrary sequence of integers of finite length. For the given encoding format:

1. Firstly, determine if we can find an encoding for every sequence. That is, determine if this even gives us a function from the set of finite sequences to $\mathbb{N}^3$.
2. Determine if this function is injective and if it is surjective, viewing it as a function to $\mathbb{N}^3$.
3. Also construct an injection from $\mathbb{N}^3$ to the set of sequences of finite length.
4. Using the above results or otherwise, prove that the set of finite sequences of integers is countable.

Encoding Algorithm: Given a sequence $a_1, a_2, \cdots a_l$, we encode it as a tuple (l, m, k). l is the number of elements in the sequence, and m is the largest element of the sequence. Let d be the smallest number greater than m which is divisible by every number in $\{1, 2, \cdots l\}$. Let k be the lowest number such that $k \equiv a_i \mod i \cdot d + 1 \; \forall i \leq l$.
Then (l,m,k) is the required encoded format.
Given an encoded sequence (l,m,k), we propose the following decoding algorithm: knowing l and m, we can find the value of d above. Then $a_i = rem(k, i \cdot d + 1)$ for each i from 1 to l (both inclusive).
Note that we have not proven that either of these algorithms works. It is possible that no suitable k exists in the encoding algorithm. It is also possible that more than one sequence maps to the same 3-tuple, in which case the decoding algorithm is faulty.
Hint: Chinese Remainder Theorem.