Endsem Practice Questions (Optional)

Nov 2024

Note

The following questions may be challenging and are intended solely for practice purpose. Dont be discouraged if you cant solve even a single one in 2 hrs (as TAs who were in your positions some years back, believe us, we suffered through the same).
They are optional and meant to allow you to take a feel of what all you can do with the few techniques you learnt in this course by now. Feel free to attempt them as you see fit.

1 Proof Techniques

Pf.1 Consider a tournament consisting of n players. Each player plays against every other player exactly once, and there are no draws. Show that you can order the players as $P_1, P_2, \ldots, P_n$ such that $\forall i = 1, 2, \ldots, n-1$, P_i defeated P_{i+1} in the tournament. (Hint: Try using induction on n. When you have an ordering of $n-1$ players, show that there would always exist a place for a new player P in the sequence such that he is defeated by the player in front of him and defeats a player behind him OR he is to be placed at the extreme ends).

Pf.2 Prove that f_n and f_{n+1} are 'relatively prime' for all $n \geq 1$. That is, they share no factor in common other than the number 1 where f_n is the nth fibonacci number. $f_1 = 0, f2 = 1$.

Pf.3 Suppose S is a set of $n+1$ integers. Prove that there exists a, b distinct in S such that $a-b$ is a multiple of n.

Pf.4 Prove or disprove that for every two positive integers a, b, if an integer linear combination of a and b^2 equals 1, then so does an integer linear combination of a^2 and b.

2 Sets, Functions, Countability

S.1 Given an infinite set A, show that there exists a bijection from A to $A \times A$ if and only if there exists a bijection from A to $A \times A \times A$. (Hint: Try constructing a bijection for A to $A \times A \times A$ given a bijection from A to $A \times A$. For the other direction, try constructing an injection and a surjection both from A to $A \times A$ and use Schröder Bernstein Theorem to finish your argument).

S.2 An interval is a set $I \subseteq \mathbb{R}$ with the property that for all real numbers x, y, and z, if $x \in I$, $z \in I$, and $x < y < z$, then $y \in I$. An interval is nondegenerate if it contains at least two different real numbers. Suppose $\mathcal{F}$ is a set of nondegenerate intervals and $\mathcal{F}$ is pairwise disjoint. Prove that $\mathcal{F}$ is countable.

3 Posets, relations

P.1 Let $(X, \leq)$, $(Y, \preceq)$ be two posets. We say that they are isomorphic if there exists a bijection $f : X \to Y$ such that for every $x, y \in X$, we have $x \leq y$ iff $f(x) \preceq f(y)$. They are said be non-isomorphic if they are not isomorphic.

(a) Draw Hasse diagrams of all non-isomorphic 3-element posets.

(b) Prove that any two finite total orders $(X, \leq), (Y, \preceq)$, such that $|X| = |Y|$, are isomorphic. What if the sets were infinite?

(c) Find two non-isomorphic total ordering relations on the set of all natural numbers $\mathbb{N}$. Prove that they are indeed non-isomorphic.

(d) Let S be the set of all pairwise non-isomorphic total orderings on $\mathbb{N}$. Note that, part (c) above shows that S has at least 2 elements. Is S finite or infinite? If it is finite, give an upper bound on $|S|$. If it is infinite, write whether it is countable or uncountable. Prove your claim.

P.2 Suppose there are n students and m companies. A relation R from students to companies is defined by $(s, c) \in R$ iff the student s is to be interviewed by company c. The interviews are conducted in slots of 15 minutes. A student can be interviewed by at most one company in a slot and a company can interview at most one student in a slot. Suppose that every student is to be interviewed by at most D companies and every company has to interview at most D students. Prove that it is always possible to schedule the interviews so that at most D slots are required. Is it always possible to find such a schedule such that the slots for every company are consecutive, though they may start at different times? If so prove it, else find a counterexample.

P.3 Let R be any arbitrary relation defined on a set A. Let I denote the identity relation, and R^{-1} the converse of R. Let R^k be the relation defined by $R^0 =$ I and $R^k = R * R^{k-1}$ for $k > 0$. Write down an expression for the smallest equivalence relation E containing R using these and other standard set operations. E is smallest in the sense that any equivalence relation that contains R must also contain E. Suppose E an equivalence relation on a finite set A with n elements and k equivalence classes. What is the minimum number of elements in a relation R on A such that the smallest equivalence relation containing R is the given relation E? Prove your answer using induction.

P.4 In a partially ordered set, a chain is a totally ordered subset. For example, in the set 1, 2, 3, 4, 5, 6, the divisibility relation is a partial order and 1, 2, 4 and 1, 3, 6 are chains.

(a) What is the longest chain on the set 1, 2, . . . , n using the divisibility relation? How many distinct chains have this length? For the second part, make sure to consider all positive values of n.

(b) What is the longest chain on the powerset of a set A with $|A| =$ n with the $\subseteq$ relation? How many distinct chains have this length?

4 Counting

C.1 You have a bag with 2013 black balls and 2024 white balls. Without looking, you draws two balls from the bag and apply the following rule. If both balls are of the same colour, you throw them both away. Otherwise, you throw away the black ball and return the white ball to the bag. You keep repeating this process. If at some stage there is exactly one ball left in the bag. What do you say about the color of the ball? (Hint: Parity Checking)

C.2 Let any $2n$ points be chosen in the plane so that no 3 are collinear,and let any n of them be colored red and the other n blue. Prove that it is always possible to pair up the n red points with the n blue ones in $1-1$ fashion (r, b) so that no 2 of the n segments rb, which connect the members of a matched pair, intersect. (Hint: Since n is finite, there exists altogether only a finite number of ways of pairing up the sets of red and blue points. For each of pairing the sum of n segment is also finite, albeit different for different set of matching. But any arrangement with minimum total length must be free of intersections. Why?)

C.3 There are 6 red balls and 8 green balls in a bag. Five balls are drawn out at random and placed in a red box; the remaining balls are put in a green box. What is the probability that the number of red balls in the green box plus the number of green balls in the red box is not a prime number? Probability is the branch of mathematics concerning numerical descriptions of how likely an event is to occur, or how likely it is that a proposition is true. For calculating probability, we simply divide the number of cases favorable for the event (in this case, "the number of red balls in the green box plus the number of green balls in the red box is not a prime number") over the the number of total outcomes possible (total number of draws possible).

C.4 Recall that $R(k, \ell)$ denotes the minimum number of nodes such that any 2-coloring of a (complete) graph on $R(k, \ell)$ nodes has either a complete graph on k nodes with all edges colored Red or a complete graph on ℓ nodes whose edges are all colored Blue. For instance, in class we saw that $R(3, 3) = 6$.

(a) Give an upper bound for $R(3, 5)$? You may state clearly and use any of the results proved in class or tutorials.

(b) Can you show a tight lower bound for $R(3, 5)$?

C.5 Consider a set T of size n. What is the number of ordered pairs $< A, B >$ of subsets of T such that $A \cap B = \phi$. Note that we count $< A, B >$ and $< B, A >$ as different, and either or both of them can be the ϕ.

C.6 Let $x_1, x_2 \cdots x_m$ be m distinct natural numbers less than or equal to n. Prove that if $m > \frac{n+1}{2}$, there exist indices $1 \leq i \leq j < k \leq m$ such that $x_i + x_j = x_k$. What if $m \geq \frac{n+1}{2}$?

C.7 Ramsey's theorem has been proven in class for two colors (say c_1 and c_2). In a generalized form, one could claim that given any n-tuple of colors $c_1, \cdots, c_n$, we can find a number $R(s_1, \cdots, s_n)$ such that amongst any graph with $R(s_1, \cdots, s_n)$ or more vertices, there must be a set of s_i vertices connected by edges of color c_i for at least one value of i. Prove or disprove this generalized statement.

5 Graph Theory

GT.1 Given a graph where the degree of each vertex is at least d, show that there exists a path of length d in the graph. (Hint: Consider any maximal path and show that it must have length at least d).

GT.2 A complete graph on n vertices is an undirected graph in which every pair of distinct vertices is connected by an edge. A simple path in a graph is one in which no vertex is repeated. Let G be a complete graph on 10 vertices. Let u, v, w be three distinct vertices in G. How many simple paths are there from u to v going through w.

GT.3 In the recent quiz, we saw that two graphs with same number of vertices and edges may not be isomorphic. But what if the two graphs also have same degrees for their vertices. Can we now say that they are isomorphic?

GT.4 If two graphs are isomorphic, prove that they contain the same number of triangles.

GT.5 Prove that at any party there must be two people who have shaken hands with the same number of others present.

GT.6 A mouse wants to eat a 3*3*3 cube of cheese, in which there is a cherry in the exact center of the cube. It begins at a corner and, at each step, eats a whole 1*1*1 cube, before going on to an adjacent 1*1*1 cube (i.e., which shares a face with it). Can the mouse eat the 1*1*1 cube containing the cherry last (for dessert), i.e., can it end in the center of the 3*3*3 cube? Model the problem as a bipartite graph and solve it using graph-theoretical techniques/properties that you have learnt in this course.

GT.7 Let G be a bipartite graph with bipartition (A, B) and assume that every $X \subseteq A$ satisfies $|N(X)| \geq |X|$. Define a set $X \subseteq A$ to be optimal if $|N(X)| = |X|$. Prove that whenever $X_1, X_2 \subseteq A$ are optimal. then $X_1 \cap X_2$ is also optimal. Use this to give a new proof of Hall's Marriage Theorem by induction on $|A|$. (hint: let $x \subseteq A$ and try to find a good vertex to pair with x).

GT.8 Let A be a finite set with subsets $A_1, A_2, \ldots, A_n$ and let $d_1, d_2, \ldots, d_n \in \mathbb{N}$. Show that there are disjoint subsets $D_k \subseteq A_k$ with $|D_k| = d_k$ for all $k \leq n$ if and only if the following holds: $| \cup_{i \in I} A_i| \geq \sum_{i \in I} d_i$ for all possible subsets $I \subseteq \{1, 2, \ldots n\}$. (Hint: Try constructing a bipartite graph where a perfect matching would correspond the disjoint sets we require. Then use Hall's Theorem on the bipartite graph)

GT.9 Show that a graph G having atleast 3 vertices is 2 connected iff for each pair of vertices u,v , there exist internally disjoint u-v paths in G.

GT.10 Consider a bipartite graph $G(X \cup Y, E)$. $S \subseteq Y$ is called a partial transversal if there is some matching M in the graph saturating S. Show that all the maximal partial transversals of Y are of the same size. [Hint: Assume 2 maximal subsets of unequal sizes with matchings M_1, M_2 and consider $M_1 \Delta M_2 = (M_1 \cup M_2) - (M_1 \cap M_2)$]

GT.11 Prove that the following condition is necessary and sufficient for a sequence of "degrees" $d_1 \geq d_2 \cdots d_n$ to be the degrees of the vertices of a simple graph (no loops, unweighted bidirectional edges): $(\sum_{i=1}^{n} d_i) \equiv 0 \mod 2$ and for all $k \leq n$, $\sum_{i=1}^{k} d_i \leq k(k+1) + \sum_{i=k+1}^{n} min(d_i, k)$.

GT.12 Prove that for a tree with n vertices, the average distance between two vertices is at most $\frac{n+1}{3}$.

GT.13 Prove that every n-vertex triangle-free graph has at most $\frac{n^2}{4}$ edges.

GT.14 Prove that every n-vertex nonbipartite triangle-free graph has at most $\frac{(n-1)^2}{4}$ edges.

GT.15 In a graph with n vertices and m edges, prove that the number of triangles is at least $\frac{m}{3n}(4m - n^2)$.

GT.16 Let a graph have f triangles and h edges. Show that $f^2 \leq \frac{2}{9}h^3$.

GT.17 (hard) Let a graph have g tetrahedra (complete graph on 4 vertices) and f triangles. Show that $g^3 \leq \frac{3}{32}f^4$.

GT.18 Let $\Delta(G) = max_{v \in V} d(v)$ and let $\chi(G)$ be the minimum number of colours required to colour vertices of G such that no two adjacent vertices have same colour. Prove that $\chi(G) \leq \Delta(G) + 1$.

GT.19 Given an integer n, construct a sequence of natural numbers $a_1, a_2 \cdots a_n$ satisfying the condition that for all $i \neq j, a_i \times a_{i+1} \neq a_j \times a_{j+1}$, and the number of distinct integers in the sequence should be minimum over all such sequences. Note that you only need to give a sketch of how you would construct such a sequence (without brute force checking every possible tuple of natural numbers), and it is not expected that you be able to easily code up the solution.

GT.20 Given a graph G, prove that you can always find a set of at most half of the edges so that removing these edges makes the resulting graph bipartite.