CS105 (Discrete Structures)
Exercise Problem Set 3

August 17, 2024

Instructions:

- Attempt *all* questions.
- *Some* of the answers will be discussed during the help sessions, but again you are expected to have attempted *all* the questions.
- If you have any doubts or you find any typos in the questions, post them on piazza at once!
- In the following, "disprove" means you have to give a counterexample.

Part 1

1. Give an example of infinite sets A, B and functions f, g, h all from A to B such that

   - f is an injection but not a surjection,
   - g is a surjection but not an injection, and
   - h is an injection and surjection.

   Is h a bijection?

2. Prove or disprove the following (with complete justifications): Let A, B, C be non-empty sets.

   (a) If there is an injection from A to B then there is a surjection from B to A.

   (b) If there is an injection from A to B then there is a surjection from A to B.

   (c) If there is an bijection from A to B and an injection from B to C, then there is an injection from A to C.

   (d) If $A \subseteq B$ but $A \neq B$, then there must exist an injection from A to B but there can exist no surjection from A to B.

3. Show that the function $f(x) = |x|$ from set of real numbers to set of nonnegative real numbers has no inverse, but if the domain is restricted to the set of nonnegative real numbers, the resulting function is invertible.

4. Prove or disprove: There is a bijection from the set of all integers $\mathbb{Z}$ to the set of all natural numbers $\mathbb{N}$.

Part 2

5. Let A be any infinite set. Prove carefully that there is a surjection from A to $\mathbb{N}$. We said in class that this implies that the natural numbers are the "smallest" infinite set! Do you agree? What about the set of even numbers or set of all primes? Discuss.

6. (Schröder Bernstein Theorem) For any two sets A, B, show that if there exist injective functions $f : A \to B$ and $g : A \to B$ between the sets A, B, then there exists a bijection between A and B. *Note: If you cannot prove it yourself, read the proof!*