CS105 Quiz 2: DIC on Discrete Structures

25 marks, 55min

18 Oct 2024

Instructions:

- Attempt *all* questions. Write all answers and proofs carefully. You can assume results shown in class. However, if you are *making any such assumptions or using results proved in class, state them clearly.*
- All sets considered below are general sets that can be infinite. Hence, you must not assume that they are finite or countable, unless clearly specified otherwise.
- Recall that $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ denote, respectively, the set of real numbers, rational numbers, integers and natural numbers.
- In this course, one of our aims is to learn how to write good proofs, hence considerable weightage will be given to *clarity and completeness* of proofs.
- Do *not* copy or use any other unfair means. Offenders will be reported to the Disciplinary Action Committee.

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1. (2+2+2=6 marks, 15min) True or false. Give a short 2-3 line justification for your answer.
   (a) There exists no simple connected graphs that has 6 vertices whose degrees are,respectively 2, 3, 3, 4, 4, 5.
   (b) A simple complete graph on 100 vertices is Eulerian, i.e., it has a closed walk that visits every edge exactly once.
   (c) There are exactly 734 numbers between 1 and 1000 that are divisible by 2, 3 or 5.

Solution.

(a) True. The sum of degrees of all vertices in any graph must add up to an even number, however, for the degrees provided, they add up to $2+3+3+4+4+5=21$. Hence such a graph cannot exist.

(b) False. In the simple complete graph on 100 vertices (also called K_{100}), each vertex is connected to 99 other vertices, and hence has degree 99. Also, the graph is connected. As shown in class a connected graph is Eulerian if and only if each vertex has even degree. Since the latter property is violated, the graph is not Eulerian.

(c) True. Let us define sets A_i ($i \geq 2$) as set of all numbers between 1 and 1000 which are divisible by i. We are then interested in $|A_2 \cup A_3 \cup A_5|$. By PIE,

$$\begin{aligned}|A_2 \cup A_3 \cup A_5| =& |A_2| + |A_3| + |A_5| \\ & - |A_2 \cap A_3| - |A_3 \cap A_5| - |A_5 \cap A_2| \\ & + |A_2 \cap A_3 \cap A_5| \\ =& |A_2| + |A_3| + |A_5| - |A_6| - |A_{15}| - |A_{10}| + |A_{30}|\end{aligned}$$

Where the last step follows from the observation that intersection of A_i, A_j is the set of all numbers divisible by both i, j or equivalently by their LCM.

The cardinality of A_i is $\lfloor 1000/i \rfloor$ $(i, 2i, \ldots, \lfloor 1000/i \rfloor \cdot i)$. Plugging this in,

$$\begin{aligned}|A_2 \cup A_3 \cup A_5| &= 500 + 333 + 200 - 166 - 66 - 100 + 33 \\ &= 734\end{aligned}$$

(0.5 marks for correctly saying True or False, 1.5 marks for reasoning)

2. (5 marks, 15min) Messages are sent over a communications channel using two different signals. One requires 2 microseconds for transmitting, other requires 3 microseconds. Each signal is immediately followed by next and there is no lag.

   (a) Find a recurrence relation for the number of different signals that can be sent in n microseconds (for $n \geq 1$).

   (b) What are the initial conditions of this recurrence relation?

   (c) How many different messages can be sent in 13 microseconds?

Solution.

a) Let $T(n)$ be the total number messages that can be sent in n microseconds. Let's look at the first signal of every message. Either it is a 2 microsecond signal or a 3 microsecond signal. The number of messages that can be sent with the first signal as a 2 microsecond signal will be $T(n-2)$. Similarly, the number of messages that can be sent with the first signal as a 3 microsecond signal will be $T(n-3)$. Thus, the recurrence relation will be $T(n) = T(n-2) + T(n-3)$ (1.5 marks) for $n \geq 4$ (0.5 marks). (1 mark for explanation).

b) $T(1) = 0$ because no message is possible.
$T(2) = 1$ because only 1 message is possible with exactly 1 2 microsecond signal.
$T(3) = 1$ because only 1 message is possible with exactly 1 3 microsecond signal.
(1 marks if all 3, 0.5 marks if 2, 0 marks if 1)

c) $T(4) = T(2) + T(1) = 1$
$T(5) = T(3) + T(2) = 2$
$T(6) = T(4) + T(3) = 2$
$T(7) = T(5) + T(4) = 3$
$T(8) = T(6) + T(5) = 4$
$T(9) = T(7) + T(6) = 5$
$T(10) = T(8) + T(7) = 7$
$T(11) = T(9) + T(8) = 9$
$T(12) = T(10) + T(9) = 12$
$T(13) = T(11) + T(10) = 16$
(0.5 marks for correct answer, 0.5 marks for showing some calculation)

3. (4 marks) Prove that among any $n+1$ positive integers one can find 2 distinct numbers so that their difference is divisible by n.

Solution.
Consider any $n+1$ positive integers $x_1, x_2, \ldots, x_{n+1}$. Consider buckets $0, 1, \ldots, n-1$ and place x_i in bucket j if x_i has remainder j when divided by n (this is well defined since the remainder can only be one of $0, 1, \ldots, (n-1)$).(1.5 marks for buckets)
We have $n+1$ numbers in n buckets, and hence by PHP(1 marks for php), one of the two buckets will have (at-least) two numbers x_i, x_j with the remainder r, such that $i \neq j$. Then $x_i = q_i n + r$ and $x_j = q_j n + r$ and hence $x_i - x_j = (q_i - q_j)n \implies n|(x_i - x_j)$ (1.5 marks for reasoning).
Hence, there exist two numbers whose difference is divisible by n.
0.5 marks cut in reasoning if not mentioned $i \neq j$.

4. (5 marks, 15min) The class of CS105 is going for a monsoon trek and the CRs decide that each student must carry, in his/her backpack, the following items:

- at least two bottles of water,
- at most one towel,
- an even (non-zero) number of socks,
- at most 3 bananas,
- at least 1 energy bar: but the energy bar comes in packs of 4.

Let $Bag(n)$ be the number of ways to pack the bag with n items, such that all the above conditions are satisfied. What is $Bag(5)$? Find a closed-form expression for $Bag(n)$ in terms of n. (hint: you may use generating functions).

Solution. Let's look at the generating function for each individual case.

- Atleast two bottles of water: $g_1(x) = x^2 + x^3 + x^4 + \cdots = \frac{x^2}{1-x}$
- Atmost 1 towel: $g_2(x) = 1 + x = \frac{1-x^2}{1-x}$
- An even non-zero number of socks: $g_3(x) = x^2 + x^4 + x^6 + \cdots = \frac{x^2}{1-x^2}$
- Atmost 3 bananas: $g_4(x) = 1 + x + x^2 + x^3 = \frac{1-x^4}{1-x}$
- Atleast 1 energy bar, but the energy bar comes in packs of 4: $g_5(x) = x^4 + x^8 + x^{12} + \cdots = \frac{x^4}{1-x^4}$

(0.5 marks * 5 = 2.5)

Thus, the generating function $g(x)$ for total number of items is $g(x) = g_1(x) \times g_2(x) \times g_3(x) \times g_4(x) \times g_5(x) = x^8 \times (1-x)^{-3}$(0.5 marks).

Thus $bag(5)$ is the coefficient of x^5 in $g(x)$ which is 0.(0.5 marks)

Also, closed form expression for $Bag(n)$ is the coefficient of x^n in $g(x)$. For $n < 8$, $Bag(n) = 0$(0.5 marks). For $n >= 8$, $Bag(n)$ =coefficient of x^{n-8} in $(1-x)^{-3}$(1 mark for correct piecewise function) which is $\binom{(n-8)+3-1}{3-1} = \binom{n-6}{2}$ (1 marks). Therefore,

$$Bag(n) = \begin{cases} 0 & \text{if n} < 8, \\ \binom{n-6}{2} & \text{otherwise.} \end{cases}$$

5. (0 marks) Write the name of any two TAs of the course. Bonus multiplier marks will be given if you name all TAs!