CS105 (DIC on Discrete Structures)
Problem set 5

- Attempt *all* questions.
- Apart from things proved in lecture, you cannot assume anything as "obvious". Either quote previously proved results or provide clear justification for each statement.

Basic

1. Consider the set of all bit-strings, i.e., finite sequences over $\{0, 1\}$. E.g., $0101, 000, 11111110$ are all bit-strings.

   For any two bit-strings u, v, let us define a relation by uRv iff u and v contain the same number of 1's.

   Is R an equivalence relation? Why or why not?

2. Consider the set $S = \{1, 2, 3, 4\}$ with the subseteq relation $\subseteq$. We know that $(\mathcal{P}(S), \subseteq)$ is a poset.

   (a) Draw its Hasse diagram.

   (b) What is the length of the longest chain in this poset?

   (c) What is the size of the largest anti-chain in this poset?

3. Consider the poset $(\mathbb{Z}^+, |)$, i.e., positive integers with divisbility ordering.

   (a) Give an example of a chain of length 5 in this poset.

   (b) Give an example of an anti-chain of length 5 in this poset.

   (c) Does this poset have:

      i. a minimal element
      ii. a maximal element
      iii. a minimum or least element
      iv. a maximum or greatest element
      v. an infinite chain
      vi. an infinite anti-chain

   For each of the above, if you claim there exists one, give an example, otherwise explain why there can't be any.

Advanced

4. A maximal chain is a chain that is not a subset of a larger chain. Prove or disprove: every maximal chain in a finite poset $(S, \preceq)$ contains a minimal element of S.

5. Consider a necklace made of 3 beads, each of which can be either red, white or blue. Let S be the set of all such necklaces. Define the following relation R on S as: $N_1 \; R \; N_2$ iff necklace N_2 can be obtained from necklace N_1 by rotating it (and *not* allowing to flip the necklace).

   (a) Show that R is an equivalence relation.

   (b) What are the equivalence classes of R?

   (c) Is the number of elements in each equivalence class the same? Is there a relationship between the number of elements in an equivalence class of R and the total number of elements in S?

   (d) If in the definition of the relation, we allow flipping of the necklace as well: that is, $N_1 \; R' \; N_2$ iff necklace N_2 can be obtained from necklace N_1 by rotating or flipping it. Is R' an equivalence relation? Why or why not?