CS105 (DIC on Discrete Structures)
Exercise Problem Sheet 4

Instructions:

- Attempt *all* questions.
- If you have any doubts or you find any typos in the questions, post them on piazza at once!

Part 1

1. True or False. You must provide a short justification for your answer.
   (a) Any subset of a countable set is countable.
   (b) An intersection of an uncountable set and a countable set must be uncountable.
   (c) The set of all irrational numbers (i.e., reals that are not rational) is countable.
   (d) The set $\{a^b \mid a, b \in \mathbb{Q}^+\}$ is countable, where $\mathbb{Q}^+$ stands for positive rationals.
2. Give an example for each of the following, if such an example exists. Else prove why it cannot exist.
   (a) A relation that is irreflexive, symmetric and not transitive.
   (b) Relations R_1 and R_2 on set S such that both are symmetric but $R_1 \cap R_2$ is not symmetric.
3. Suppose R_1 and R_2 are two equivalence relations on set S.
   (a) Is $R_1 \cap R_2$ an equivalence relation?
   (b) Is $R_1 \cup R_2$ an equivalence relation?

   For each of the above, if your answer is "yes", you must prove it, and if your answer is "no", you must provide a counterexample.
4. Prove or disprove:
   (a) There is a surjection but no injection from $\mathbb{Q} \cap [0, 1]$ to $\mathbb{N} \times \mathbb{N}$.
   (b) There is a bijection between $\mathbb{Z} \times \mathbb{N}$ and $\mathbb{Q} \times \mathbb{N} \times (\mathbb{N} \cup \{\sqrt{2}\})$.
5. Construct a bijection
   (a) from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$
   (b) from the open interval $(0, 1)$ on the real line to the closed interval $[0, 1]$.

   Prove that the function you constructed is indeed a bijection.

Part 2

6. In class we showed that there is no bijection from $\mathbb{N}$ to the set of subsets of $\mathbb{N}$. Prove that for *any* non-empty set S, there is no bijection from S to the set of all subsets of S.

7. Prove that there exists a bijection from $\mathbb{R}$ to set of all subsets of $\mathbb{N}$. Can you construct it explicitly? Also, can you conclude whether $\mathbb{R}$ is countable or uncountable from this?

8. Which of the following sets are countable? Justify with a formal proof.

   (a) Set of all functions from $\mathbb{N}$ to $\mathbb{N}$.

   (b) Set of all non-increasing functions from $\mathbb{N}$ to $\mathbb{N}$. A function $f : \mathbb{N} \to \mathbb{N}$ is said to be non-increasing if for all $x, y \in \mathbb{N}$, if $x \leq y$ then $f(x) \geq f(y)$.

9. (*) Prove that there does not exist a C-program which will always determine whether an arbitrary input-free C-program will halt.