Catalan Numbers

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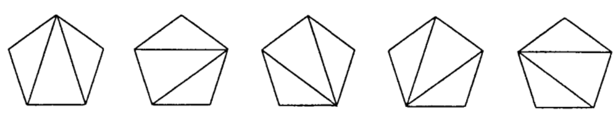
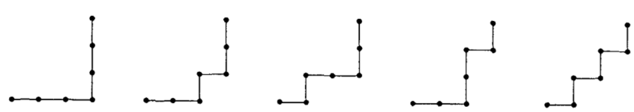
1 Catalan Numbers

It is well known that the Catalan numbers, denoted by $C_n = \frac{\binom{2n}{n}}{n+1}$, are given by the recursion $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. A couple of nice proofs for this fact can be found on Wikipedia and elsewhere.
Your exercise is to prove that the following counting problems give us the n$^{\text{th}}$ Catalan number, by somehow reducing the problem to the Catalan recursion or to a question taught in class as an example of the Catalan recursion. The first few, you have already done, and are only a recap

1. Division of a (n+2)-gon into n triangles by non-intersecting diagonals.

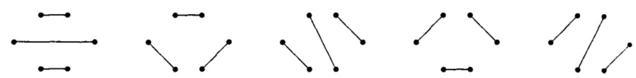


2. Lattice paths from (0,0) to (n,n) with Right or Up steps, never rising above the main diagonal.



3. Paths from (0,0) to (2n,0) with steps (1,1) and (1,-1) only (Dyck paths) never falling below the x-axis.

4. n non-intersecting chords joining 2n points on the circumference of a circle.



5. Ways of connecting 2n points on a line by n non-intersecting arcs containing two points each and lying above the points.

6. Ways of drawing in the plane n+1 points lying on a horizontal line L and n arcs connecting them such that the arcs do not pass below L, the graph thus formed is a tree, no two arcs intersect in their interior and at every vertex, all arcs exit in the same direction - either left or right.

7. Sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$:

111 112 113 122 123

8. Sequences $a_1 < a_2 < \cdots < a_{n-1}$ of integers with $1 \leq a_i \leq 2i$.

12 13 14 23 24

9. Sequences $a_1, \cdots a_n$ of integers with $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$.

000 001 010 011 012

10. Sequences of n-1 integers a_i such that $a_i \leq 1$ and all partial sums are non-negative.

$0, 0 \quad 0, 1 \quad 1, -1 \quad 1, 0 \quad 1, 1$

11. Sequences of n integers such that $a_i \geq -1$, all partial sums are non-negative, and the total sum is equal to 0.

$$0,0,0 \quad 0,1,-1 \quad 1,0,-1 \quad 1,-1,0 \quad 2,-1,-1$$

12. Permutations $a_1, a_2 \cdots a_n$ of n with longest decreasing subsequence of length at most 2. These are called 321 avoiding permutations.

123 213 132 312 231

13. Permutations of n for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (also called 312 avoiding permutations).

123 132 213 231 321

14. (*) Sequences a_i of n integers such that i $\leq a_i \leq n$ and if $i \leq j \leq a_i$, $a_j \leq a_i$.

123 133 223 323 333

15. Sequences $< a_i >$ of n integers such that $1 \leq a_i \leq i$ and such that if $a_i = j$, then $a_{i-r} \leq j - r$ for $1 \leq r < j$.

111 112 113 121 123

16. Permutations of the multiset $(1, 1, 2, 2, \cdots, n, n)$ so that the indices of the first occurences of $1, 2 \cdots n$ appear in increasing order and there is no subsequence of the form abab, where a and b are two distinct numbers from [n].

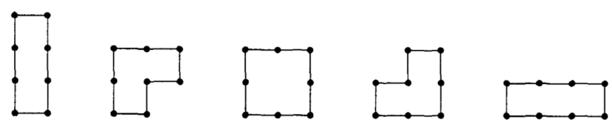
112233 112332 122331 123321 122133

17. Stack-Sortable permutations - The permutations of n which can be sorted using an algorithm which uses a single stack:
    - Initialize an empty stack
    - For each input value x, while the stack is nonempty and x is larger than the top item on the stack, pop the stack to the output, and then push x onto the stack

- While the stack is nonempty, pop it to the output

123 132 213 312 321

18. (*) Lattice paths with n+1 steps each, starting at (0,0) using steps of (0,1) or (1,0), ending at the same point, and only intersecting at the beginning or end.



19. (*) Pairs of lattice paths with n-1 steps each, starting at (0,0) and using steps of (1,0) or (0,1), such that one never rises above the other.

20. Standard Young Tableaux of shape (n,n) (top and bottom rows are both increasing. Any column is also ordered in increasing order)

123	124	125	134	135
456	356	346	256	246

21. Ways to stack coins, with n consecutive coins in the bottom row.