1. A function $f : (0, 1) \to \mathbb{R}$ which is concave and differentiable on $(0, 1)$ is

   (A) $f(x) = e^x$ (B) $f(x) = \log(x^2)$ (C) $f(x) = (\log x)^2$ (D) $f(x) = -|x-1/2|$

   SOLUTION. Option (D) is not differentiable. Checking the second derivatives gives that (B) is the answer for the given interval.

2. Let $P_1 = \left\{0 < \frac{1}{3} < \frac{1}{2} < 1\right\}$ and $P_2 = \left\{0 < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < 1\right\}$ be partitions of $[0, 1]$. For a bounded function $f : [0, 1] \to \mathbb{R}$, let $L_1 = L(P_1, f)$, $U_1 = U(P_1, f)$, $L_2 = L(P_2, f)$, $U_2 = U(P_2, f)$. Then

   (A) $L_1 \leq L_2$ and $U_1 \leq U_2$ (B) $L_1 \leq L_2$ and $U_1 \geq U_2$
   (C) $L_1 \geq L_2$ and $U_1 \leq U_2$ (D) $L_1 \geq L_2$ and $U_1 \geq U_2$

   SOLUTION. Let $P = \{x_1, x_2, \ldots, x_n\}$. Let P' be the partition obtained from P by adding a point x' between x_i and x_{i+1}. Observe that

$$\inf_{[x_i, x_{i+1}]} f(x) \leq \min\left\{\inf_{[x_i, x']} f(x), \inf_{[x', x_{i+1}]} f(x)\right\}.$$

   Thus $L(P, f) \leq L(P', f)$. Analogously, one can show that $U(P, f) \geq U(P', f)$. Applying this to $P = P_1$ and $P' = P_2$ yields $L_1 \leq L_2$ and $U_1 \geq U_2$.

3. Let $f : [0, 1] \to \mathbb{R}$ be a function such that $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = f(x)^2$ is Riemann integrable. Which of the following statements is **FALSE**?

   (A) f is bounded. (B) f can have infinitely many points of discontinuity.
   (C) f is Riemann integrable. (D) g can have infinitely many points of discontinuity.

   SOLUTION. f must be bounded since g is. Take f to be the function in (5.1) in notes. Then f is not Riemann integrable, but g is identically 1 on $[0, 1]$ and hence, is Riemann integrable. Further, f has infinitely many points of discontinuity. Finally, take g to be function in (5.2) in notes. It has infinitely many points of discontinuity.

4. The area enclosed between $f(x) = \sin x$ and $g(x) = \sin 2x$ for $0 \leq x \leq \pi$ is

   (A) 2 (B) $\frac{5}{2}$ (C) $\sqrt{3}$ (D) 3

   SOLUTION. Again, this reduces to computing the integral

$$\int_0^{\pi/3} (\sin 2x - \sin x)\, dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x)\, dx = \frac{1}{4} + \frac{9}{4} = \frac{5}{2}.$$

5. The length of the curve $C(t) = \left(t, \int_0^t \sqrt{\frac{1-x}{1+x}}\, dx\right)$ for $t \in [0, 1]$ is

   (A) $\sqrt{2}$ (B) $\sqrt{2} - 1$ (C) $\sqrt{2}(\sqrt{2} - 1)$ (D) $2\sqrt{2}(\sqrt{2} - 1)$

   SOLUTION. The length of the curve is given by

$$\int_0^1 \sqrt{1 + C'(t)^2}\, dt = \int_0^1 \sqrt{1 + \frac{1-t}{1+t}}\, dt = \int_0^1 \frac{\sqrt{2}}{\sqrt{1+t}}\, dt = 2\sqrt{2}(\sqrt{2} - 1),$$

   where the first equality follows from FTC.

6. Let $f : X \longrightarrow X$ be a continuous function defined on $X \subseteq \mathbb{R}$. Then $c \in X$ is said to be a *fixed point* of f if $f(c) = c$. The function f always has a fixed point for X equal to

   (A) $(0, 1)$ (B) $[0, 1)$ (C) $[0, 1]$ (D) $(0, \infty)$

   SOLUTION. For options (a) and (d), the continuous function $x \mapsto x/2$ has no fixed points. For option (b), the function $x \mapsto (x/2 + 1/2)$ does not have a fixed point. We shall now show that every continuous function from $[0, 1]$ to $[0, 1]$ must have a fixed point. If either $f(0) = 0$ or $f(1) = 1$ hold, we are done. So suppose that $f(0) > 0$ and $f(1) < 1$. The function $g : [0, 1] \longrightarrow [0, 1]$ such that $g(x) = x - f(x)$ is continuous with $g(0) < 0 < g(1)$. The intermediate value property tells us that there exists $r \in [0, 1]$ such that $g(r) = 0$, and this r is a fixed point for f.

7. For any positive integer k, let $a_k = \log(k + 2)$. For any positive integer n, let

$$P_n = \left\{0 < \frac{a_1}{a_n} < \cdots < \frac{a_i}{a_n} < \cdots < \frac{a_n}{a_n} = 1\right\}$$

   be a partition of $[0, 1]$. Then $\lim_{n\to\infty} \|P_n\|$ is

   (A) 0 (B) $\frac{1}{2}$ (C) e^{-2} (D) $\log 2$

   SOLUTION. We have

$$\begin{aligned}
\|P_n\| &= \max\left\{\max_{2\le i\le n} \frac{a_i - a_{i-1}}{a_n}, \frac{a_1}{a_n}\right\} \\
&= \max\left\{\max_{2\le i\le n} \frac{\log\left(\frac{i+2}{i+1}\right)}{\log(n+2)}, \frac{\log 3}{\log(n+2)}\right\} \\
&\le \max\left\{\max_{2\le i\le n} \frac{1}{i+1}\frac{1}{\log(n+2)}, \frac{\log 3}{\log(n+2)}\right\} \\
&= \max\left\{\frac{1}{3}\frac{1}{\log(n+2)}, \frac{\log 3}{\log(n+2)}\right\},
\end{aligned}$$

   where we have used the inequality $\log(1 + x) \le x$ for all $x \ge 0$. Since both terms in the last equality go to 0 as $n \to \infty$, we conclude that $\|P_n\| \to 0$ as $n \to \infty$.

8. Let $f(x) = \begin{cases} 1 + x^2 & \text{if } x \le 0 \\ 1 + x & \text{if } x > 0 \end{cases}$ and $F(x) = \int_{x^2}^{x} f(t)\, dt$ for $x \in \mathbb{R}$. Then $F'(0)$

   (A) is -1 (B) is 0 (C) is 1 (D) does not exist

   SOLUTION. Observe that f is continuous. For $-1 \le x \le 1$, we have

$$F(x) = \int_{-10}^{x} f(t)\ dt - \int_{-10}^{x^2} f(t)\ dt.$$

   Using FTC, F is differentiable for $-1 < x < 1$ and

$$F'(x) = f(x) - 2xf(x^2), \quad \text{for all } -1 < x < 1.$$

   Setting $x = 0$, we have $F'(0) = f(0) = 1$.

9. The volume of the solid obtained by revolving the region between $y = x$ and $y = x^2$ for $0 \le x \le 1$, about the line $x = -1$, is

   (A) $\frac{\pi}{2}$ (B) π (C) $\frac{3\pi}{2}$ (D) 2π

   SOLUTION. The required volume is $\int_0^1 \pi((1 + \sqrt{x})^2 - (1 + x)^2)\, dx = \frac{\pi}{2}$.

10. The length of the curve $C(t) = (\sin^3 t, \cos^3 t)$ for $t \in [0, \frac{\pi}{2}]$ is

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) 2

SOLUTION. The length of the curve is given by the integral

$$\int_0^{\pi/2} \sqrt{9\cos^2 t \sin^4 t + 9\sin^2 t \cos^4 t}\, dt = 3\int_0^{\pi/2} \sin t \cos t = \frac{3}{2}.$$

11. A function $f : (0, \infty) \to \mathbb{R}$ with finitely many points of inflection is $f(x)$ equal to

(A) $3x - 2$ (B) $\sin\left(\frac{1}{x}\right)$ (C) $\max\{-1 + x^2, 1 - x^2\}$ (D) $\sin x + \cos x$

SOLUTION. Note: Every point of $3x - 2$ is an inflection point. Also, $\max\{-1 + x^2, 1 - x^2\}$ has one point of inflection on $(0, \infty)$ at $x = 1$.

12. A function which is **NOT** Riemann integrable on $[-1, 1]$ is

(A) $f(x) = \sqrt{|x|}$ (B) $f(x) = [x]$ (C) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise} \end{cases}$ (D) $f(x) = e^{x^4}$

SOLUTION. (A) and (D) are monotonically increasing in $[0, 1]$ and decreasing in $[-1, 0]$, and thus they are Riemann integrable. (B) is also Riemann integrable as it has only finitely many discontinuities. However, for (C), for any partition, the difference between the upper and lower sums is at least $1/2$. To see this, use density of $\mathbb{Q}$ in $\mathbb{R}$, and estimate the difference between the two sums in $[-1, -\frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

13. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = \int_0^x e^{[t]^2}\, dt$, where $[t]$ is the integer part of t. Then

(A) F is bounded
(B) F is not differentiable at x iff x is an integer
(C) F is not continuous
(D) F is not differentiable at x iff x is a nonzero integer

SOLUTION. The integrand is a step function on $\mathbb{R}$. If x is not an integer, then the integrand is locally constant around x and hence, is differentiable due to FTC. On the other hand, if x is an integer, then FTC gives:

$$\lim_{h \to 0^-} \frac{F(x+h) - F(x)}{h} = \exp\left((x-1)^2\right) \quad \text{and} \quad \lim_{h \to 0+} \frac{F(x+h) - F(x)}{h} = \exp\left(x^2\right),$$

which are unequal and hence, F is not differentiable at x.

14. The volume of the solid in $\mathbb{R}^3$ enclosed by the cylinders $x^2 + y^2 = 9$ and $x^2 + z^2 = 9$ is

(A) $16/3$ (B) 16 (C) 48 (D) 144

SOLUTION. This is solved just like Example 5.24 in the notes.

15. The surface area of the object obtained by revolving the curve $y = \sqrt{x}$ for $\frac{3}{4} \leq x \leq 2$, about the x-axis, is

(A) 2π (B) $\frac{13\pi}{6}$ (C) $\frac{17\pi}{6}$ (D) $\frac{19\pi}{6}$

SOLUTION. The surface area is given by

$$\int_{\frac{3}{4}}^{2} 2\pi\sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx = \int_{\frac{3}{4}}^{2} 2\pi\sqrt{x + \frac{1}{4}}\, dx = 2\pi\int_1^{\frac{9}{4}} \sqrt{z}\, dz = \frac{19}{6}\pi.$$

16. For each positive integer n, let

$$G_n = \prod_{i=0}^{n} \left(2 - \frac{i}{n+1}\right)^{1-\frac{2i}{n+1}}.$$

Then $\lim_{n\to\infty} (G_n)^{\frac{1}{n+1}}$ equals

(A) 1 (B) $\frac{1}{4}\sqrt{e}$ (C) $\frac{1}{4}e^{3/2}$ (D) $\frac{1}{2}e^{5/2}$

Solution. Taking logarithm,

$$\log G_n = \sum_{i=0}^{n} \left(1 - \frac{2i}{n+1}\right) \log\left(2 - \frac{i}{n+1}\right).$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n+1} \log G_n = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \left(1 - \frac{2i}{n+1}\right) \log\left(2 - \frac{i}{n+1}\right),$$

which is precisely the integral

$$\int_0^1 (1-2x)\log(2-x)\ dx = \int_1^2 (2y-3)\log y\ dy = \frac{3}{2} - 2\log 2.$$

The answer follows by exponentiating.