1. Let $\{a_n\}$ be the sequence of real numbers given by

$$a_n = \begin{cases} 1 - \frac{1}{n} & \text{if } n \text{ is odd,} \\ 1 + \frac{4}{n} & \text{if } n \text{ is even.} \end{cases}$$

The smallest positive integer n_0 such that $|a_n - 1| < \frac{1}{10}$ holds for all $n \geq n_0$ is

(A) 10 (B) 11 (C) 41 (D) 42

SOLUTION. Observe that

$$|a_n - 1| = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd,} \\ \frac{4}{n} & \text{if } n \text{ is even.} \end{cases}$$

For each positive even integer $n \leq 40$, $|a_n - 1| \geq \frac{1}{10}$, however for even integers $m \geq 42$, $|a_m - 1| < \frac{1}{10}$. Similarly, for odd integers $n \geq 11$, $|a_n - 1| < \frac{1}{10}$. Thus, 41 is the smallest possible choice for n_0.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$ be a real number such that

$$\lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h^2}$$

exists and is equal to $L \in \mathbb{R}$. Then

(A) f is continuous but not differentiable at c
(B) f is continuous and differentiable at c
(C) $L = 0$
(D) there is $\delta > 0$ such that f is constant on $(c - \delta, c + \delta) \setminus \{c\}$

SOLUTION. Let

$$g(h) = \frac{f(c+h) - f(c-h)}{2h^2}.$$

This is an odd function of h defined on $\mathbb{R} \setminus \{0\}$. Hence, if $\lim_{h \to 0} g(h)$ exists, then it must be equal to 0. A counterexample to all the other options is

$$f(x) = \begin{cases} |x - c| & \text{if } x \neq c, \\ 1 & \text{if } x = c. \end{cases}$$

3. Let $f : [0, 10] \to \mathbb{R}$ be given by $f(x) = e^{-x/5} - e^{-[x/5]}$ (where $[x]$ is the integer part of x). Then f attains its global maximum at a point in the open interval

(A) $(1, 2)$ (B) $(2, 4)$ (C) $(4, 6)$ (D) $(6, 8)$

SOLUTION. By definition of the integer part function, $x/5 \geq [x/5]$ with equality only when $x = 5n$ for some integer n. As e^{-t} is a decreasing function of t, we have $e^{-x/5} \leq e^{-[x/5]}$ and hence $f(x) \leq 0$ for all $x \in [0, 10]$. Also, f takes the value 0 iff $x/5$ is an integer. So, the global maximum of f is 0 and it is achieved at integer multiples of 5, that is, at 0, 5, 10. Thus, the correct option is $(4, 6)$.

4. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Let a and b be real numbers. The notation $a_n \not\to$ says that $\{a_n\}$ does not converge to a real number. Which one of the following statements is **FALSE**?

(A) If $a_n \to a$, then $\{a_n\}$ is bounded.
(B) If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.
(C) If $a_n \to 0$ and $\{b_n\}$ is bounded, then $a_n b_n \to 0$.
(D) If $a_n \not\to$ and $b_n \not\to$, then $a_n + b_n \not\to$.

SOLUTION. The first two statements are true and have been proved in the notes. For statement (C), let $M \in \mathbb{R}$ be such that $\max_{n\geq 1}|b_n| \leq M$. For any $\epsilon > 0$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, $|a_n - 0| \leq \epsilon/M$. It follows that for any $n \geq n_0$, $|a_n b_n - 0| = |a_n||b_n| < \epsilon/M \times M < \epsilon$, and thus $\{a_n b_n\}$ converges to zero. Statement (D) however, is false. Consider the sequences $\{a_n\}$ and $\{b_n\}$ given by $a_m = (-1)^m$ and $b_m = (-1)^{m+1}$. Neither of these sequences converge. However, $\{a_n + b_n\}$ is the identically zero sequence which converges to 0.

5. Let $\log : (0, \infty) \to \mathbb{R}$ denote the inverse function of $\exp : \mathbb{R} \to (0, \infty)$ (sending x to e^x), and $\cos^{-1} : [-1, 1] \to [0, \pi]$ denote the inverse function of $\cos : [0, \pi] \to [-1, 1]$.

   Consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \mapsto \begin{cases} e^{-x^2/2} & \text{if } x \leq 0, \\ \cos x & \text{if } x > 0. \end{cases}$$

   The largest value of δ such that $|f(x) - 1| < \frac{1}{2}$ whenever $|x| < \delta$ is

   (A) $\cos^{-1}\left(\frac{1}{2}\right) \approx 1.04$ (B) $\sqrt{\log 2} \approx 0.83$ (C) $\cos^{-1}\left(\frac{1}{4}\right) \approx 1.32$ (D) $\sqrt{\log 4} \approx 1.17$

   SOLUTION. Note: $g(x) = \cos x$ is strictly decreasing in $[0, \pi]$ and $h(x) = e^{-x^2/2}$ is strictly increasing in $(-\infty, 0]$, and both take value 1 at $x = 0$. So, the solution is $\min\{\cos^{-1}\left(\frac{1}{2}\right), |t|\}$, where t is the solution to $e^{-x^2/2} = \frac{1}{2}$. Solving for t yields $t = \sqrt{2\log 2}$. Now $\cos^{-1}\left(\frac{1}{2}\right) = \pi/3 \approx 1.04$ and $\sqrt{2\log 2} \approx 1.17$. So the smaller of the two is $\cos^{-1}\left(\frac{1}{2}\right)$.

6. A function is continuously differentiable if it is differentiable, and its derivative is continuous. Which one of the following statements is **FALSE**?

   For any continuously differentiable function $g : [0, 1] \to \mathbb{R}$ with $g(0) = 1$, $g(1) = e$ and $g'_+(0) = \frac{1}{2}$, there is $c \in (0, 1)$ such that

   (A) $g'(c) = 2cg(c)$ (B) $g'(c) = 1$ (C) $g'(c) = e - 1$ (D) $g'(c) = 2e$

   SOLUTION. Using mean value theorem on g, there is a $c \in (0, 1)$ such that $g'(c) = e - 1$. Since $g' : [0, 1] \to \mathbb{R}$ is continuous and $\frac{1}{2} < 1 < e - 1$, the intermediate value theorem gives us a $d \in (0, c)$ such that $g'(d) = 1$. Consider $f(x) = e^{-x^2} g(x)$ for all $x \in [0, 1]$. This is continuously differentiable and $f(0) = f(1) = 1$. Thus, by Rolle's theorem, there is $c \in (0, 1)$ such that $e^{-c^2}(g'(c) - 2cg(c)) = f'(c) = 0$, that is, $g'(c) = 2cg(c)$. The function

$$g(x) = \exp\left(\frac{x^2 + x}{2}\right)$$

   satisfies the hypothesis, and g' is an increasing function, with $g'(0) = \frac{1}{2}$ and $g'(1) = \frac{3}{2}e$. Thus, there is no $c \in (0, 1)$ with $g'(c) = 2e$. Alternatively, the function

$$g(x) = 1 + \frac{x}{2} + \left(e - \frac{3}{2}\right)x^2$$

   also works for the same reason. Here $g'(1) = 2e - \frac{5}{2}$.

7. Let $\{a_n\}$ be the sequence defined by

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}}.$$

   Then $\lim_{n\to\infty} a_n$ is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 3

Solution. Observe that

$$\frac{n}{\sqrt{n^2+n}} \leq a_n \leq \frac{n}{\sqrt{1+n^2}}.$$

By sandwich lemma, we get the limit to be 1.

8. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be the increment function at $x = 0$ of a differentiable function $f : \mathbb{R} \to \mathbb{R}$. Then f_1 on $\mathbb{R} \setminus \{0\}$ can be

   (A) $\sin\left(\frac{1}{x}\right)$

   (B) $x \sin\left(\frac{1}{x}\right)$

   (C) $[x]$ (where $[x]$ denotes the integer part of x)

   (D) $\begin{cases} x+1 & \text{if } x > 0, \\ x-1 & \text{if } x < 0 \end{cases}$

   Solution. The increment function of f at $x = 0$ must be continuous at 0. So the only choice that works is $x \sin\left(\frac{1}{x}\right)$. For this increment function, we can take $f(x)$ to be $x^2 \sin\left(\frac{1}{x}\right)$ on $\mathbb{R} \setminus \{0\}$, and 0 at $x = 0$.