Department of Mathematics
Indian Institute of Technology Bombay
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MA 110 : Linear Algebra and Differential Equations

Spring 2025

Instructors

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Name	:	
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Syllabus and Course Outline of Part 1 (Linear Algebra) of MA 110

Instructors and their Coordinates

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Syllabus (Up to Mid-Sem) [Full syllabus at: https://portal.iitb.ac.in/asc/Courses/RunningCourses.jsp]

Vectors in $\mathbb{R}^n$, notion of linear independence and dependence, linear span of a set of vectors, vector subspaces of $\mathbb{R}^n$, basis of a vector subspace.

Systems of linear equations, matrices and Gauss elimination, row space, null space, and column space, rank of a matrix. Determinants and rank of a matrix in terms of determinants.

Abstract vector spaces, linear transformations, matrix of a linear trans-formation, change of basis and similarity, rank-nullity theorem.

Inner product spaces, Gram-Schmidt process, orthonormal bases, projections and least squares approximation.

Eigenvalues and eigenvectors, characteristic polynomials, eigenvalues of special matrices (orthogonal, unitary, hermitian, symmetric, skew-symmetric, normal), algebraic and geometric multiplicity, diagonalization by similarity transformations, spectral theorem for real symmetric matrices, application to quadratic forms.

Texts/References:

1. H. Anton, Elementary linear algebra with applications (8th Edition),John Wiley (1995).
2. G. Strang, Linear algebra and its applications (4th Edition), Thomson (2006).
3. S. Kumaresan, Linear algebra - A Geometric approach, Prentice Hall of India (2000).
4. E. Kreyszig, Advanced engineering mathematics (8th Edition), John Wiley (1999)

Evaluation

Quiz 1	22 January 2025	8.30 A.M.-9.25 A.M.	10 marks
Quiz 2	12 February 2025	8.30 A.M.-9.25 A.M.	10 marks
Mid-Sem Exam	During 22 Feb 2025 – 2 Mar 2025	2 hours	30 marks

Course Outline

1. **Solutions of Linear Systems (Lec 01 to Lec 09)**
   (a) Operations on Matrices, Linear systems, Row Echelon Form. (Lecs 01-02)
   (b) Gauss Elimination Method, Inverse of a Matrix, Row Canonical Form, Linear Dependence. (Lecs 03-04)
   (c) Row Rank and Column Rank of a Matrix, Basis and Dimension of a Subspace. (Lec 05)
   (d) Span of Vectors, Rank-Nullity Theorem for a Matrix, Fundamental Theorem for Linear Systems. (Lec 06)
   (e) Properties of Determinants, Rank of a Matrix in terms of Determinants, Cramer's Rule, Cofactor Matrix. (Lecs 07-08)

(f) Linear Transformations, Matrix of a Linear Transformation w.r.t. Given Bases, Complex Numbers. (Lecs 08-09)

2. **Matrix Eigenvalue Problems (Lec 10 to Lec 18)**
   (a) Geometric Multiplicity of an Eigenvalue, Similarity of Matrices, Diagonaizable Matrix, Characteristic Polynomial, Algebraic Multiplicity of an Eigenvalue, Fundamental Theorem of Algebra. (Lecs 10-11)
   (b) Relation between Geometric and Algebraic Multiplicities, Diagonalizability, Inner Product and Norm. (Lec 12)
   (c) Orthogonality, Gram-Schmidt Orthogonalization, Orthonormal Basis. (Lec 13)
   (d) Unitary Matrix, Unitarily Similar Matrices, Schur's Theorem and its Consequences. (Lec 14)
   (e) Unitarily Diagonalizable Matrices, Spectral Theorem for Normal Matrices. (Lec 15)
   (f) Spectral Theorem for Self-Adjoint and Real Symmetric Matrices. (Lec 16)
   (g) Real Quadratic Forms, Conic Sections, Quadric Surfaces. (Lec 17)
   (h) Orthogonal Projection, Projection Theorem in $\mathbb{K}^{n\times 1}$, Least Squares Approximation. (Lec 18)

3. **Abstract Vector Spaces (Lec 19 to Lec 21)**
   (a) Span of Vectors, Linear Dependence, Examples of Finite Dimensional Vector Spaces, Basis. (Lec 19)
   (b) Linear Transformations, Rank-Nullity Theorem for a Linear map, Matrix of a Linear Map w.r.t. Given Bases. (Lec 20)
   (c) Space of All Linear Transformations from a Vector Space to a Vector Space, Change of Bases, Solutions of Operator Equations, Eigenvalue Problems for Linear Operators. (Lec 20)
   (d) Inner Product Spaces, Orthogonal Projection Theorem, Linear Maps on Inner Product Spaces, Hermitian Operators. (Lec 21)

Time Table for Lectures & Tutorials

Course No.	Div	Departments	Lecture		Tutorial	
			Slot	Venue	Slot	Venue
MA 110	D1	ME, EN, ES, MA (310)	1A, 1B, 1C	LA 201	X2	LT 001,002,003,004,005,006,101,102, 103
	D2	AE, CE, EP, IE (324)	8A, 8B	LA 201	X1	LT 001,002,003,004,005,006,101,102, 103
	D3	CS, EC, MM (352)	1A, 1B, 1C	LA 202	X1 2B	LT 201,202,203,204,205 LT 201,202,203,204 (for CS)
	D4	EE, CL (330)	8A, 8B	LA 202	X3	LT 001,002,003,004,005,006,101,102, 103

Policy for Attendance

Attendance in both lecture and tutorial classes is compulsory. Students who fail to attend 80% of the lecture and tutorial classes may be awarded DX grade.

Linear Algebra: Tutorials

Tutorial 1 (on Lectures 1 and 2)

1.1 Let $\mathbf{A}$ be a square matrix. Show that there is a symmetric matrix $\mathbf{B}$ and there is a skew-symmetric matrix $\mathbf{C}$ such that $\mathbf{A} = \mathbf{B} + \mathbf{C}$. Are $\mathbf{B}$ and $\mathbf{C}$ unique?

1.2 Let $\mathbf{A} := \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Write (i) the second row of $\mathbf{AB}$ as a linear combination of the rows of $\mathbf{B}$ and (ii) the second column of $\mathbf{AB}$ as a linear combination of the columns of $\mathbf{A}$.

1.3 Let $\mathbf{A} := \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -24 & 8 & -5 \\ 0 & -7 & 2 & 2 \end{bmatrix}$. Assuming that $\mathbf{A}$ is invertible, find the last column and the last row of $\mathbf{A}^{-1}$.

1.4 Show that the product of two upper triangular matrices is upper triangular. Is this true for lower triangular matrices?

1.5 The **trace** of a square matrix is the sum of its diagonal entries. Show that for any $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$,

$$\text{trace}\,(\mathbf{A}+\mathbf{B}) = \text{trace}\,(\mathbf{A}) + \text{trace}\,(\mathbf{B}) \quad \text{and} \quad \text{trace}\,(\mathbf{AB}) = \text{trace}\,(\mathbf{BA}.$$

1.6 Find all solutions of the linear system $\mathbf{Ax} = \mathbf{b}$, where

(i) $\mathbf{A} := \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 0 & -1 & 6 & 6 \end{bmatrix}^{\mathsf{T}}$,

(ii) $\mathbf{A} := \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 5 & -2 & 9 \end{bmatrix}^{\mathsf{T}}$,

(iii) $\mathbf{A} := \begin{bmatrix} 0 & 2 & -2 & 1 \\ 2 & -8 & 14 & -5 \\ 1 & 3 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 2 & 2 & 8 \end{bmatrix}^{\mathsf{T}}$

by reducing $\mathbf{A}$ to a row echelon form.

Tutorial 2 (on Lectures 3, 4 and 5)

2.1 Find the Row Canonical Form of $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$.

2.2 Let $\mathbf{A} := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Find $\mathbf{A}^{-1}$ by Gauss-Jordan method.

2.3 An $m \times m$ matrix $\mathbf{E}$ is called an **elementary matrix** if it is obtained from the identity matrix $\mathbf{I}$ by an elementary row operation. Write down all elementary matrices.

(i) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. If an elementary row operation transforms $\mathbf{A}$ to $\mathbf{A}'$, then show that $\mathbf{A}' = \mathbf{E}\mathbf{A}$, where $\mathbf{E}$ is the corresponding elementary matrix.

(ii) Show that every elementary matrix is invertible, and find its inverse.

(iii) Show that a square matrix $\mathbf{A}$ is invertible if and only if it is a product of finitely many elementary matrices.

2.4 Let S and T be subsets of $\mathbb{R}^{n \times 1}$ such that $S \subset T$. Show that if S is linearly dependent then so is T, and if T is linearly independent then so is S. Does the converse hold?

2.5 Are the following sets linearly independent?

(i) $\{\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 5 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\} \subset \mathbb{R}^{1 \times 3}$,

(ii) $\{\begin{bmatrix} 1 & 9 & 9 & 8 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 8 \end{bmatrix}\} \subset \mathbb{R}^{1 \times 4}$,

(iii) $\{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^\mathsf{T}, \begin{bmatrix} 3 & -5 & 2 \end{bmatrix}^\mathsf{T}, \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^\mathsf{T}\} \subset \mathbb{R}^{3 \times 1}$.

2.6 Given a set of s linearly independent row vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_i, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_s\}$ in $\mathbb{R}^{1 \times n}$ and $\alpha \in \mathbb{R}$, show that the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_i + \alpha \mathbf{a}_j, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_s\}$ is linearly independent.

2.7 Find the ranks of the following matrices.

(i) $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$, (ii) $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$.

2.8 Are the following subsets of $\mathbb{R}^{3 \times 1}$ subspaces?

(i) $\{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\mathsf{T} : x_1, x_2, x_3 \in \mathbb{R}, x_1 + x_2 + x_3 = 0\}$,

(ii) $\{\begin{bmatrix} x_1 + x_2 + x_3 & x_2 + x_3 & x_3 \end{bmatrix}^\mathsf{T} : x_1, x_2, x_3 \in \mathbb{R}\}$,

(iii) $\{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\mathsf{T} : x_1, x_2, x_3 \in \mathbb{R}, x_1 x_2 x_3 = 0\}$

(iv) $\{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\mathsf{T} : x_1, x_2, x_3 \in \mathbb{R}, |x_1|, |x_2|, |x_3| \leq 1\}$.

If so, find a basis for each, and also its dimension.

2.9 Describe all subspaces of $\mathbb{R}$, $\mathbb{R}^{2 \times 1}$, $\mathbb{R}^{3 \times 1}$ and $\mathbb{R}^{4 \times 1}$. Can you visualise them geometrically?

Tutorial 3 (on Lectures 6 and 7)

3.1 Let V be a subspace of $\mathbb{R}^{n\times 1}$ with $\dim V = r$, and let S be a finite subset of V such that $\operatorname{span} S = V$. Suppose S has s elements. Show that (i) $s \geq r$, (ii) if $s = r$, then S is a basis for V, (iii) if $s > r$, then S contains basis for V.

3.2 Let $\mathbf{A}' \in \mathbb{R}^{m\times n}$ be in a REF. Show that the pivotal columns of $\mathbf{A}'$ form a basis for the column space $\mathcal{C}(\mathbf{A}')$.

3.3 Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. The set $\mathcal{R}(\mathbf{A})$ consisting of all linear combinations of the rows of $\mathbf{A}$ is called the **row space** of $\mathbf{A}$. Show that $\mathcal{R}(\mathbf{A})$ is a subspace of $\mathbb{R}^{1\times n}$. If $\mathbf{A}'$ is obtained from $\mathbf{A}$ by EROs, then prove that $\mathcal{R}(\mathbf{A}') = \mathcal{R}(\mathbf{A})$. Further, show that the dimension of $\mathcal{R}(\mathbf{A})$ is equal to the rank of $\mathbf{A}$.

3.4 Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$. Show that $\operatorname{rank}\mathbf{AB} \leq \min\{\operatorname{rank}\mathbf{A}, \operatorname{rank}\mathbf{B}\}$.

3.5 Let $\mathbf{A} := \begin{bmatrix} 0 & 0 & 0 & -2 & 1 \\ 0 & 2 & -2 & 14 & -1 \\ 0 & 2 & 3 & 13 & 1 \end{bmatrix}$. Find the rank and the nullity of $\mathbf{A}$. What is the dimension of the solution space of the homogeneous equation $\mathbf{Ax} = \mathbf{0}$? If $\mathbf{b} := \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}^{\mathsf{T}}$, find the general solution of $\mathbf{Ax} = \mathbf{b}$.

3.6 Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$, where $a, b, c \in \mathbb{R}$. Also, prove an analogous formula for a determinant of order n, known as the **Vandermonde determinant**.

3.7 For $n \in \mathbb{N}$, prove that

$$\det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & \cdot & & & & \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} = (-1)^{n(n-1)/2}.$$

3.8 For $n \in \mathbb{N}$, prove that

$$\det \begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 2 & 3 & \dots & n-1 & n \\ 3 & 3 & 3 & \dots & n-1 & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n-1 & n-1 & \dots & n-1 & n \\ n & n & n & \dots & n & n \end{bmatrix} = (-1)^{n+1}n.$$

3.9 Find $\operatorname{rank}\mathbf{A}$ using determinants, where $\mathbf{A}$ is

$$\text{(i)} \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix}.$$

Verify by transforming $\mathbf{A}$ to a REF.

Tutorial 4 (on Lectures 8, 9 and 10)

4.1 Find the value(s) of α for which Cramer's rule is applicable. For the remaining value(s) of α, find the number of solutions, if any.

$$\begin{array}{ccccccc} \mathrm{x} & + & 2\mathrm{y} & + & 3\mathrm{z} & = & 20 \\ \mathrm{x} & + & 3\mathrm{y} & + & \mathrm{z} & = & 13 \\ \mathrm{x} & + & 6\mathrm{y} & + & \alpha\mathrm{z} & = & \alpha. \end{array}$$

4.2 Find the cofactor matrix $\mathbf{C}$ of the matrix $\mathbf{A}$, and verify $\mathbf{C}^{\mathsf{T}}\mathbf{A} = (\det \mathbf{A})\mathbf{I} = \mathbf{A}\mathbf{C}^{\mathsf{T}}$. If $\det \mathbf{A} \neq 0$, find $\mathbf{A}^{-1}$, where $\mathbf{A}$ is

$$\text{(i)} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ (ii)} \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \text{ (iii)} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

4.3 Find the matrix of the linear transformation $T : \mathbb{R}^{3\times 1} \to \mathbb{R}^{4\times 1}$ defined by $T\left(\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathsf{T}}\right) := \begin{bmatrix} x_1+x_2 & x_2+x_3 & x_3+x_1 & x_1+x_2+x_3 \end{bmatrix}^{\mathsf{T}}$ with respect to the ordered bases (i) $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^{3\times 1}$ and $F = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ of $\mathbb{R}^{4\times 1}$,

(ii) $E' = (\mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_3+\mathbf{e}_1)$ of $\mathbb{R}^{3\times 1}$ and $F' = (\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_1, \mathbf{e}_4+\mathbf{e}_1+\mathbf{e}_2)$ of $\mathbb{R}^{4\times 1}$, first showing that E' is a basis for $\mathbb{R}^{3\times 1}$ and F' is a basis for $\mathbb{R}^{4\times 1}$.

4.4 Let $\mathbf{A} \in \mathbb{R}^{4\times 4}$. Let $\mathbf{P} := \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Show that $\mathbf{P}$ is invertible. Find an ordered bases E of $\mathbb{R}^{4\times 1}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{M}_E^E(T_{\mathbf{A}})$.

4.5 Let $\lambda \in \mathbb{K}$. Find the geometric multiplicity of the eigenvalue λ of each of the following matrices:

$$\mathbf{A} := \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \mathbf{B} := \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \mathbf{C} := \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Also, find the eigenspace associated with λ in each case.

4.6 Let $\mathbf{A} := \begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix}$. Show that 3 is an eigenvalue of $\mathbf{A}$, and find all eigenvectors of $\mathbf{A}$ corresponding to it. Also, show that $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^{\mathsf{T}}$ is an eigenvector of $\mathbf{A}$, and find the corresponding eigenvalue of $\mathbf{A}$.

4.7 Let $\theta \in (-\pi, \pi]$, $\mathbf{A} := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\mathbb{K} = \mathbb{C}$. Show that $\cos\theta \pm i\sin\theta$ are eigenvalues of $\mathbf{A}$. Find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix, and check your answer.

4.8 Let $n \geq 2$ and $\mathbf{A} := \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$, that is, $a_{jk} = 1$ for all $j, k = 1, \ldots, n$. Find $\operatorname{rank} \mathbf{A}$ and $\operatorname{nullity} \mathbf{A}$. Find an eigenvector of $\mathbf{A}$ corresponding to a nonzero eigenvalue by inspection. Find two distinct eigenvalues of $\mathbf{A}$ along with their geometric multiplicities, and find bases for the eigenspaces. Show that $\mathbf{A}$ is diagonalizable, and find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

Tutorial 5 (on Lectures 11, 12 and 13)

5.1 Find all eigenvalues, and their geometric as well as algebraic multiplicities of the following matrices. Are they diagonalizable? If so, find invertible $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

$$\text{(i) } \mathbf{A} := \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}, \quad \text{(ii) } \mathbf{A} := \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, \quad \text{(iii) } \mathbf{A} := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

5.2 Let $\mathbf{A} := \begin{bmatrix} 2 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}$. Find a necessary and sufficient condition on a, b, c for $\mathbf{A}$ to be diagonalizable.

5.3 Let $k \in \mathbb{N}$ and

$$\mathbf{A} := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & -1 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{K}^{2k \times 2k},$$

that is, $\mathbf{A}$ has all diagonal entries 0, the subdiagonal entries are $1, 0, 1, 0 \ldots, 1, 0$, and the superdiagonal entries are $-1, 0, -1, 0, \ldots, -1, 0$. Find the characteristic polynomial of $\mathbf{A}$, all eigenvalues of $\mathbf{A}$, and their algebraic as well as geometric multiplicities.

5.4 Let $\lambda \in \mathbb{K}$. Show that λ is an eigenvalue of $\mathbf{A}$ if and only if $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}^*$, but their eigenvectors can be very different.

5.5 Let $\mathbf{A} \in \mathbb{K}^{n \times n}$. Show that 0 is an eigenvalue of $\mathbf{A}$ if and only if 0 is an eigenvalue of $\mathbf{A}^*\mathbf{A}$, and its geometric multiplicity is the same. Deduce $\operatorname{rank} \mathbf{A}^*\mathbf{A} = \operatorname{rank} \mathbf{A}$.

5.6 Let $\mathbf{A} := \begin{bmatrix} 2 & i & 1+i \\ -i & 3 & 1 \\ 1-i & -1 & 8 \end{bmatrix}$. Show that no eigenvalue of $\mathbf{A}$ is away from one of the diagonal entries of $\mathbf{A}$ by more than $1 + \sqrt{2}$.

5.7 A square matrix $\mathbf{A} := [a_{jk}]$ is called **strictly diagonally dominant** if $|a_{jj}| > \sum_{k \neq j} |a_{jk}|$ for each $j = 1, \ldots, n$. If $\mathbf{A}$ strictly diagonally dominant, show that $\mathbf{A}$ is invertible.

5.8 Let $\mathbf{A} \in \mathbb{K}^{n \times n}$. Define $\alpha_2 := \max\{\|\mathbf{A}\mathbf{x}\| : \|\mathbf{x}\| = 1\}$, $\alpha_\infty := \max\left\{ \sum_{k=1}^{n} |a_{jk}| : j = 1, \ldots, n \right\}$ and $\alpha_1 := \max\left\{ \sum_{j=1}^{n} |a_{jk}| : k = 1, \ldots, n \right\}$, where $\mathbf{A} := [a_{jk}]$. Show that $|\lambda| \leq \min\{\alpha_2, \alpha_\infty, \alpha_1\}$ for every eigenvalue λ.

5.9 Let $\mathbf{x}, \mathbf{y} \in \mathbb{K}^{n \times 1}$. Prove the **parallelogram law**: $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$. In case $\mathbf{x}$ and $\mathbf{y}$ are both nonzero, prove the **cosine law**, which says that $\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$, where the angle $\theta \in [0, \pi]$ between nonzero $\mathbf{x}$ and $\mathbf{y}$ is defined to be $\cos^{-1}\big(\Re \langle \mathbf{x}, \mathbf{y} \rangle / \|\mathbf{x}\|\|\mathbf{y}\|\big)$.

Tutorial 6 (on Lectures 14, 15 and 16)

6.1 Orthonormalize the following ordered subsets of $\mathbb{K}^{4\times 1}$.

(i) $(\mathbf{e}_1,\, \mathbf{e}_1+\mathbf{e}_2,\, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)$

(ii) $\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4,\, -\mathbf{e}_1+\mathbf{e}_2,\, -\mathbf{e}_1+\mathbf{e}_3,\, -\mathbf{e}_1+\mathbf{e}_4)$.

6.2 Use the Gram-Schmidt Orthogonalization Process to orthonormalize the ordered subset

$$\left(\begin{bmatrix}1 & -1 & 2 & 0\end{bmatrix}^{\mathsf{T}}, \begin{bmatrix}1 & 1 & 2 & 0\end{bmatrix}^{\mathsf{T}}, \begin{bmatrix}3 & 0 & 0 & 1\end{bmatrix}^{\mathsf{T}}\right)$$

and obtain an ordered orthonormal set $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. Also, find $\mathbf{u}_4$ such that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ is an ordered orthonormal basis for $\mathbb{K}^{4\times 1}$. Express the vector $\begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^{\mathsf{T}}$ as a linear combination of these four basis vectors.

6.3 Show that $\mathbf{A} \in \mathbb{K}^{n\times n}$ is unitary if and only if its rows form an orthonormal subset of $\mathbb{K}^{1\times n}$.

6.4 Let $E := (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be the standard basis for $\mathbb{K}^{n\times 1}$, and let $F := (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be an orthonormal basis for $\mathbb{K}^{n\times 1}$. If I denotes the identity map from $\mathbb{K}^{n\times 1}$ to $\mathbb{K}^{n\times 1}$, then show that the matrix $\mathbf{M}_E^F(I)$ is unitary.

6.5 Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and let λ be an eigenvalue of $\mathbf{A}$. Show that $p(\lambda)$ is an eigenvalue of $p(\mathbf{A})$ for every polynomial $p(t)$.

6.6 Suppose $\mathbf{A} \in \mathbb{C}^{3\times 3}$ satisfies $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} = 6\mathbf{I}$.

If $5 \le \det \mathbf{A} \le 7$, determine the eigenvalues of $\mathbf{A}$.

Is $\mathbf{A}$ diagonalizable?

6.7 Let $\mathbf{A} \in \mathbb{K}^{n\times n}$, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$ with a corresponding orthonormal set of eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Show that $\mathbf{A} = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^* + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^*$. ($\mathbf{x}\mathbf{y}^*$= outer product of $\mathbf{x}, \mathbf{y}$)

6.8 Let $\mathbf{A} \in \mathbb{K}^{n\times n}$, and $\lambda \in \mathbb{K}$.

(i) Show that λ is an eigenvalue of $\mathbf{A}$ if and only $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}^*$.

(ii) Let $\mathbf{A}$ be unitary. Show that $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$. If λ is an eigenvalue of $\mathbf{A}$, then show that $|\lambda| = 1$.

(iii) Let $\mathbb{K} = \mathbb{C}$ and let $\mathbf{A}$ skew self-adjoint. If λ is an eigenvalue of $\mathbf{A}$, then show that $i\lambda \in \mathbb{R}$.

6.9 Let $\mathbf{A} := [a_{jk}] \in \mathbb{C}^{n\times n}$, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$, counting algebraic multiplicities. Show that $\mathbf{A}$ is normal $\iff \sum_{1\le j,k\le n} |a_{jk}|^2 = \sum_{j=1}^{n} |\lambda_j|^2$.

6.10 A matrix $\mathbf{A} \in \mathbb{K}^{n\times n}$ is called **nilpotent** if there is $m \in \mathbb{N}$ such that $\mathbf{A}^m = \mathbf{O}$. If $\mathbf{A}$ is upper triangular with all diagonal entries equal to 0, then show that $\mathbf{A}$ is nilpotent. Further, if $\mathbf{A} \in \mathbb{C}^{n\times n}$, then show that $\mathbf{A}$ is nilpotent if and only if 0 is the only eigenvalue of $\mathbf{A}$.

Tutorial 7 (on Lectures 17, 18 and 19)

7.1 Let $\mathbf{A} \in \mathbb{C}^{n\times n}$. Show that $\mathbf{A}$ is self-adjoint if and only if $\mathbf{A}$ is normal and all eigenvalues of $\mathbf{A}$ are real.

7.2 State and prove a spectral theorem for skew self-adjoint matrices with complex entries.

7.3 Find an orthonormal basis for $\mathbb{K}^{4\times 1}$ consisting of eigenvectors of

$$\mathbf{A} := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}.$$

Write down a spectral representation of $\mathbf{A}$, and find $\mathbf{A}^7\mathbf{x}$, where $\mathbf{x} := \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^{\mathsf{T}}$

7.4 A self adjoint matrix $\mathbf{A}$ is called **positive definite** if $\langle \mathbf{A}\,\mathbf{x}, \mathbf{x}\rangle > 0$ for all nonzero $\mathbf{x} \in \mathbb{K}^{n\times 1}$. Show that a self-adjoint matrix is positive definite if and only if all eigenvalues of $\mathbf{A}$ are positive.

7.5 Real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are placed on the 4 corners of a square in clockwise order initially. In the next step,

α_1 is replaced by $\beta_1 := (\alpha_2 + \alpha_4)/2$,

α_2 is replaced by $\beta_2 := (\alpha_3 + \alpha_1)/2$,

α_3 is replaced by $\beta_3 := (\alpha_4 + \alpha_2)/2$ and

α_4 is replaced by $\beta_4 := (\alpha_1 + \alpha_3)/2$.

Find the numbers placed on the corners of the square after k such steps. (Hint: Find a set of 4 orthonormal eigenvectors of the matrix $\mathbf{A} := \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$ and use the spectral theorem for $\mathbf{A}$.)

7.6 Let Q be a real quadratic form, and let $\mathbf{A}$ denote the associated real symmetric matrix. Let $g(\mathbf{x}) = \|\mathbf{x}\|^2 - 1$. If $\mathbb{Q}$ has a local extremum at a vector $\mathbf{x}_0$ subject to the constraint $g(\mathbf{x}) = 0$, then show that $\mathbf{x}_0$ is a unit eigenvector of $\mathbf{A}$, and the corresponding eigenvalue λ_0 is the corresponding Lagrange multiplier and equals $Q(\mathbf{x}_0)$.

In particular, the largest eigenvalue of $\mathbf{A}$ is the constrained maximum and the smaller eigenvalue of $\mathbf{A}$ is the constrained minimum of Q.

7.7 Which quadric surface does the equation $7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy - 36 = 0$ describe? Explain by reducing the quadratic form involved to a diagonal form. Express x, y, z in terms of the new coordinates u, v, w.

7.8 Let Y be a subspace of $\mathbb{K}^{n\times 1}$. Show that $(Y^\perp)^\perp = Y$.

7.9 Let $\mathbf{A}$ be a self-adjoint matrix. If $\langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle = 0$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then show that $\mathbf{A} = \mathbf{O}$. Deduce that if $\|\mathbf{A}^*\mathbf{x}\| = \|\mathbf{A}\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then $\mathbf{A}$ is a normal matrix, and if $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then $\mathbf{A}$ is a unitary matrix.

7.10 Let E be a nonempty subset of $\mathbb{K}^{n\times 1}$.

(i) If E is not closed, then show that there is $\mathbf{x} \in \mathbb{K}^{n\times 1}$ such that no best approximation to $\mathbf{x}$ exists from E.

(ii) If E is convex, then show that for every $\mathbf{x} \in \mathbb{K}^{n\times 1}$, there is at most one best approximation to $\mathbf{x}$ from E.

7.11 Find $\mathbf{x} := [x_1, x_2]^\mathsf{T} \in \mathbb{R}^{2\times 1}$ such that the straight line $t = x_1 + x_2 s$ fits the data points $(-1, 2)$, $(0, 0)$, $(1, -3)$ and $(2, -5)$ best in the 'least squares' sense.

7.12. Let $Q(x_1, \ldots, x_n) := \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} x_k \overline{x}_j$, where $\alpha_{jk} \in \mathbb{C}$, be a **complex quadratic form**. If the complex quadratic form $Q(x_1, \ldots, x_n)$ takes values in $\mathbb{R}$ for all $\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T} \in \mathbb{C}^{n\times 1}$, then show that there is a unique self-adjoint matrix $\mathbf{A}$ such that

$$Q(x_1, \ldots, x_n) = \mathbf{x}^* \mathbf{A}\, \mathbf{x} \quad \text{for all } \mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T} \in \mathbb{C}^{n\times 1}.$$

7.13. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be normal, and let $\mu_1, \ldots, \mu_k$ be the distinct eigenvalues of $\mathbf{A}$. Let $Y_j := \mathcal{N}(\mathbf{A} - \mu_j \mathbf{I})$ for $j = 1, \ldots, k$. Show that $\mathbb{C}^{n\times 1} = Y_1 \oplus \cdots \oplus Y_k$, which means show that every $\mathbf{x} \in \mathbb{C}^{n\times 1}$ can be written as $\mathbf{x} = \mathbf{y}_1 + \cdots + \mathbf{y}_k$ for unique $\mathbf{y}_j \in Y_j$, $1 \le j \le k$. Also, if P_j is the orthogonal projection of $\mathbb{C}^{n\times 1}$ onto Y_j (defined by $P_j(\mathbf{x}) := \mathbf{y}_j$), then show that $P_1 + \cdots + P_k = I$, $P_i P_j = O$ if $i \ne j$ and $\mathbf{A}\mathbf{x} = \mu_1 P_1(\mathbf{x}) + \cdots + \mu_k P_k(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{C}^{n\times 1}$.

Tutorial 8 (on Lectures 20 and 21)

8.1 State why the following sets are not subspaces:

(i) All $m \times n$ matrices with nonnegative entries.

(ii) All solutions of the differential equation $xy' + y = 3x^2$.

(iii) All solutions of the differential equation $y' + y^2 = 0$.

(iv) All invertible $n \times n$ matrices.

8.2 Let V denote the vector space of all polynomial functions on $\mathbb{R}$ of degree at most n. Are the following subsets of V in fact subspaces of V? (i) $W_1 := \{p \in V : p(0) = 0\}$,

(ii) $W_2 := \{p \in V : p'(0) = 0 = p''(0)\}$,

(iii) $W_3 := \{p \in V : p \text{ is an odd function}\}$.

If so, find a spanning set for each.

8.3 Let $V := C([-\pi, \pi])$. Show that $S_1 := \{1, \cos, \sin\}$ is a linearly independent subset of V, while $S_2 := \{1, \cos^2, \sin^2\}$ is a linearly dependent subset of V.

8.4 Let $V := \mathbb{R}^{1\times 2}$, and let $v_1 := [1\ 0]$, $v_2 := [1\ 1]$, $v_3 := [1\ -1]$. Explain why $(24, 12)$ can be written as a linear combination of v_1, v_2, v_3 in two different ways, namely, $4v_1 + 16v_2 + 4v_3$ and $6v_1 + 15v_2 + 3v_3$.

8.5 Let $n \in \mathbb{N}$. Let W_1, W_2, W_3, W_4 denote the subspaces of $n \times n$ real matrices which are diagonal, upper triangular, symmetric and skew-symmetric. Find their dimensions.

8.6 Let V and W be vector spaces over $\mathbb{K}$. Show that $V \times W := \{(v, w) : v \in V \text{ and } w \in W\}$ is a vector space over $\mathbb{K}$ with componentwise addition and scalar multiplication. If $\dim V = n$ and $\dim W = m$, find $\dim V \times W$.

8.7 Let $\mathbf{A} := [a_{jk}] \in \mathbb{K}^{4\times 4}$. Define $T : \mathbb{K}^{2\times 2} \to \mathbb{K}^{2\times 2}$ by

$$T\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix},$$

where $\begin{bmatrix} y_{11} & y_{12} & y_{21} & y_{22} \end{bmatrix}^{\mathsf{T}} := \mathbf{A} \begin{bmatrix} x_{11} & x_{12} & x_{21} & x_{22} \end{bmatrix}^{\mathsf{T}}$. Show that T is linear, and find the matrix of T with respect to the ordered basis $\big(\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{21}, \mathbf{E}_{22}\big)$ of $\mathbb{K}^{2\times 2}$.

8.8 Define $T : \mathcal{P}_2 \to \mathbb{K}^{2\times 1}$ by

$$T(\alpha_0 + \alpha_1 t + \alpha_2 t^2) := \begin{bmatrix} \alpha_0 + \alpha_1 & \alpha_1 + \alpha_2 \end{bmatrix}^{\mathsf{T}}$$

for $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$. If $E := (1, t, t^2)$ and $F := (\mathbf{e}_1, \mathbf{e}_2)$, then find $\mathbf{M}_F^E(T)$. Also, if $E' := \big(1, 1+t, (1+t)^2\big)$ and $F' := (\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2)$, then find $\mathbf{M}_{F'}^{E'}(T)$.

8.9 (**Parallelogram law**) Let V be an inner product space. Prove that the norm on V induced by the inner product satisfies $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$ for all $v, w \in V$.

(Conversely, if there is a norm $\|\cdot\|$ on a vector space V which satisfies the parallelogram law, then it can be shown that there is an inner product $\langle \cdot\,, \cdot \rangle$ on V such that $\langle v,\, v\rangle = \|v\|^2$ for all $v \in V$.)

8.10 For $\mathbf{A}, \mathbf{B} \in \mathbb{K}^{m\times n}$, define $\langle \mathbf{A},\, \mathbf{B} \rangle := \operatorname{trace} \mathbf{A}^*\mathbf{B}$. Show that $\langle \cdot\,, \cdot \rangle$ is an inner product on $\mathbb{K}^{m\times n}$.

8.11 Show that

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \dots\right\}$$

is an orthonormal subset of $C([-\pi, \pi])$.

(This is the beginning of the theory of Fourier Series.)

8.12 Let T be a Hermitian operator on a finite dimensional inner product space V over $\mathbb{K}$. Prove the following.

(i) $\langle T(v),\, v\rangle \in \mathbb{R}$ for every $v \in V$.

(ii) Every eigenvalue of T is real.

(iii) If $\lambda \neq \mu$ are eigenvalues of T with v and w corresponding eigenvectors of T, then $v \perp w$.

(iv) Let W be a subspace of V such that $T(W) \subset W$. Then $T(W^{\perp}) \subset W^{\perp}$.