# MA 110
# Lecture 01

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# Course Information

Course Code: MA 110

Course Name: Linear Algebra and Differential Equations

Instructors for the first part (Linear Algebra) are Professors Sudhir Ghorpade and Saurav Bhaumik.

Instructors for the second part (Differential Equations) are Professors Ronnie M. Sebastian and Mallesham Kummari.

# Aim of the course: linear algebra

The first part of this core course aims at an introduction to the fundamental concepts of linear algebra, including systems of linear equations, matrices, linear transformations, vector spaces, eigenvalues, and eigenvectors.

# Description/Syllabus of the Linear Algebra part

Vectors in $\mathbb{R}^n$, linear independence and dependence, linear span of a set of vectors, vector subspaces of $\mathbb{R}^n$, basis of a vector subspace. Systems of linear equations, matrices and Gauss elimination, row space, null space, and column space, rank of a matrix. Determinants and rank of a matrix in terms of determinants. Abstract vector spaces, linear transformations, matrix of a linear transformation, change of basis and similarity, rank-nullity theorem. Inner product spaces, Gram-Schmidt process, orthonormal bases, projections and least squares approximation. Eigenvalues and eigenvectors, characteristic polynomials, eigenvalues of special matrices (orthogonal, unitary, hermitian, symmetric, skew-symmetric, normal), algebraic and geometric multiplicity, diagonalization by similarity transformations, spectral theorem for real symmetric matrices, application to quadratic forms.

# Basic Information (contd..)

### Grading Policy

There will be two common quizzes (scheduled on 22 January and 12 February 2025) and one final exam.

The allotted 50 marks will be split as follows:

| | | |
|---|---|---|
| Common Quiz 1 | : | 10 marks |
| Common Quiz 2 | : | 10 marks |
| Final exam | : | 30 marks |

### Attendance Policy

**Attendance in lectures and tutorials is COMPULSORY. Students who do not meet 80% attendance will be awarded the DX grade.**

# Introduction to linear Algebra

Linear algebra deals with vectors and matrices and, more generally, with vector spaces and linear transformations.

Linear algebra is central to almost all areas of mathematics. For example, in geometry, basic objects such as lines, planes and rotations are represented in terms of linear algebra. Mathematical analysis applies linear algebra to function spaces.

Linear algebra is also used in most sciences and fields of engineering, because it allows modelling many natural phenomena, and computing efficiently with such models.

# Introduction to linear algebra contd.

Linear algebra provides a vital arena where the interaction of Mathematics and machine computation is seen.
Many of the problems studied in Linear Algebra are amenable to systematic and even algorithmic solutions, and this makes them implementable on computers.
Numerous Applications within and outside Mathematics. For example, Google page rank algorithm is based on notions and results from Linear Algebra.

# Notation

- $\mathbb{N} := \{1, 2, 3, \ldots\}$
- $\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{R} :=$ the set of all real numbers

For $n \in \mathbb{N}$, let us consider the **Euclidean space**

$$\mathbb{R}^n := \{(x_1, \ldots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \ldots, n\}.$$

We let $\mathbf{0} := (0, \ldots, 0)$. Also, for $\boldsymbol{x} := (x_1, \ldots, x_n)$ and $\boldsymbol{y} := (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, and for $\alpha \in \mathbb{R}$, we define

$$\begin{array}{rrcl} \text{(sum)} & \boldsymbol{x} + \boldsymbol{y} & := & (x_1 + y_1, \ldots, x_n + y_n) \in \mathbb{R}^n, \\ \text{(scalar multiple)} & \alpha\, \boldsymbol{x} & := & (\alpha\, x_1, \ldots, \alpha\, x_n) \in \mathbb{R}^n, \\ \text{(scalar product)} & \boldsymbol{x} \cdot \boldsymbol{y} & := & x_1 y_1 + \cdots + x_n y_n \in \mathbb{R}. \end{array}$$

# Matrices

Let $m, n \in \mathbb{N}$. An $m \times n$ **matrix** $\mathbf{A}$ with real entries is a rectangular array of real numbers arranged in $m$ rows and $n$ columns, written as follows:

$$\mathbf{A} := \begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{j1} & \cdots & \boxed{a_{jk}} & \cdots & a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{bmatrix} = [a_{jk}],$$

where $a_{jk} \in \mathbb{R}$ is called the $(j, k)$th **entry** of $\mathbf{A}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$.

Let $\boxed{\mathbb{R}^{m \times n}}$ denote the set of all $m \times n$ matrices with real entries. If $\mathbf{A} := [a_{jk}]$ and $\mathbf{B} := [b_{jk}]$ are in $\mathbb{R}^{m \times n}$, then we say $\mathbf{A} = \mathbf{B} \iff a_{jk} = b_{jk}$ for all $j = 1, \ldots, m$ and $k = 1, \ldots, n$.

Let $0 \leq r < m$, $0 \leq s < n$. By deleting $r$ rows and $s$ columns from $\mathbf{A}$, we obtain an $(m - r) \times (n - s)$ **submatrix** of $\mathbf{A}$.

An $n \times n$ matrix, that is, an element of $\mathbb{R}^{n \times n}$, is called a **square matrix** of size $n$.

- A square matrix $\mathbf{A} = [a_{jk}]$ is called **symmetric** if $a_{jk} = a_{kj}$ for all $j, k$.
- A square matrix $\mathbf{A} = [a_{jk}]$ is called **skew-symmetric** if $a_{jk} = -a_{kj}$ for all $j, k$.
- A square matrix $\mathbf{A} = [a_{jk}]$ is called a **diagonal matrix** if $a_{jk} = 0$ for all $j \neq k$.
- A diagonal matrix $\mathbf{A} = [a_{jk}]$ is called a **scalar matrix** if all diagonal entries of $\mathbf{A}$ are equal.

Two important scalar matrices are the **identity matrix I** in which all diagonal elements are equal to 1, and the **zero matrix O** in which all diagonal elements are equal to 0.

A square matrix $\mathbf{A} = [a_{jk}]$ is called **upper triangular** if $a_{jk} = 0$ for all $j > k$, and **lower triangular** if $a_{jk} = 0$ for all $j < k$.

Note: A matrix $\mathbf{A}$ is upper triangular as well as lower triangular if and only if $\mathbf{A}$ is a diagonal matrix.

Examples

The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is symmetric, while the matrix

$\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$ is skew-symmetric.

Note: Every diagonal entry of a skew-symmetric matrix is 0 since $a_{jj} = -a_{jj} \implies a_{jj} = 0$ for $j = 1, \ldots, n$.

## Examples

The matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is diagonal, while $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a scalar matrix.

The matrix $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ is upper triangular,

while the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{bmatrix}$ is lower triangular.

A **row vector a** of length $n$ is a matrix with only one row consisting of $n$ real numbers; it is written as follows:

$$\mathbf{a} = \begin{bmatrix} a_1 & \cdots & a_k & \cdots & a_n \end{bmatrix},$$

where $a_k \in \mathbb{R}$ for $k = 1, \ldots, n$. Here $\mathbf{a} \in \mathbb{R}^{1\times n}$.

A **column vector b** of length $n$ is a matrix with only one column consisting of $n$ real numbers; it is written as follows:

$$\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ \vdots \\ b_n \end{bmatrix}, \text{ where } b_k \in \mathbb{R} \text{ for } k = 1, \ldots, n. \text{ Here } \mathbf{b} \in \mathbb{R}^{n\times 1}.$$

When $n = 1$, we may identify $[\alpha] \in \mathbb{R}^{1\times 1}$ with $\alpha \in \mathbb{R}$.

# Operations on Matrices

Let $m, n \in \mathbb{N}$, and let $\mathbf{A} := [a_{jk}]$ and $\mathbf{B} := [b_{jk}]$ be $m \times n$ matrices. Then the $m \times n$ matrix $\mathbf{A} + \mathbf{B} := [a_{jk} + b_{jk}]$ is called the **sum** of $\mathbf{A}$ and $\mathbf{B}$. Also, if $\alpha \in \mathbb{R}$, then the $m \times n$ matrix $\alpha\mathbf{A} := [\alpha a_{jk}]$ is called the **scalar multiple** of $\mathbf{A}$ by $\alpha$.

These operations follow the usual rules: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$, which we write as $\mathbf{A} + \mathbf{B} + \mathbf{C}$, $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$, $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$ and $\alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A}$, which we write as $\alpha\beta\mathbf{A}$. Also, we write $(-1)\mathbf{A}$ as $-\mathbf{A}$, and $\mathbf{A} + (-\mathbf{B})$ as $\mathbf{A} - \mathbf{B}$.

The **transpose** of an $m \times n$ matrix $\mathbf{A} := [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T} := [a_{kj}]$ (in which the rows and the columns of $\mathbf{A}$ are interchanged).

Clearly, $(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}$, $(\mathbf{A} + \mathbf{B})^\mathsf{T} = \mathbf{A}^\mathsf{T} + \mathbf{B}^\mathsf{T}$ and $(\alpha\mathbf{A})^\mathsf{T} = \alpha\mathbf{A}^\mathsf{T}$.

Note: A square matrix $\mathbf{A}$ is symmetric $\iff \mathbf{A}^\mathsf{T} = \mathbf{A}$.

In particular, the preceding operations can be performed on row vectors, and also on column vectors since they are particular types of matrices.

Notice that the sum $\mathbf{a}_1 + \mathbf{a}_2$ of two row vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ of the same length follows the parallelogram law, and so does the sum $\mathbf{b}_1 + \mathbf{b}_2$ of two column vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ of the same length. (Note: All vectors 'originate' from the zero vector.)

Also, note that the transpose of a row vector is a column vector, and vice versa.

We shall often write a column vector $\mathbf{b} := \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^{n \times 1}$, as

$\begin{bmatrix} b_1 & \cdots & b_k & \cdots & b_n \end{bmatrix}^\mathsf{T}$ in order to save space.

Let $m, n \in \mathbb{N}$. Let $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$.

If $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^{1 \times n}$, then

$$\alpha_1 \mathbf{a}_1 + \cdots + \alpha_m \mathbf{a}_m \in \mathbb{R}^{1 \times n}$$

is called a **(finite) linear combination** of $\mathbf{a}_1, \ldots, \mathbf{a}_m$.

Similarly, if $\mathbf{b}_1, \ldots, \mathbf{b}_m \in \mathbb{R}^{n \times 1}$, then

$$\alpha_1 \mathbf{b}_1 + \cdots + \alpha_m \mathbf{b}_m \in \mathbb{R}^{n \times 1}$$

is a **(finite) linear combination** of $\mathbf{b}_1, \ldots, \mathbf{b}_m$.

In particular, for $k = 1, \ldots, n$, consider the column vector $\mathbf{e}_k := \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{n \times 1}$, where the $k$th entry is 1 and all other entries are 0.

If $\mathbf{b} = \begin{bmatrix} b_1 & \cdots & b_k & \cdots & b_n \end{bmatrix}^{\mathsf{T}}$ is any column vector of length $n$, then it follows that $\mathbf{b} = b_1\mathbf{e}_1 + \cdots + b_k\mathbf{e}_k + \cdots + b_n\mathbf{e}_n$, which is a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are known as the **basic column vectors** in $\mathbb{R}^{n \times 1}$.

Let $\mathbf{A} := [a_{jk}] \in \mathbb{R}^{m \times n}$.

Then $\mathbf{a}_j := \begin{bmatrix} a_{j1} & \cdots & a_{jn} \end{bmatrix} \in \mathbb{R}^{1 \times n}$ is called the $j$th **row vector of A** for $j = 1, \ldots, m$, and we write $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{bmatrix}$.

Also, $\mathbf{c}_k := \begin{bmatrix} a_{1k} & \cdots & a_{mk} \end{bmatrix}^{\mathsf{T}}$ is called the $k$th **column vector of A** for $k = 1, \ldots, n$, and we write $\mathbf{A} = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix}$.

## Examples

Let $\mathbf{A} := \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} 1 & 0 & 2 \\ -1 & 4 & 1 \end{bmatrix}$.

Then $\mathbf{A} + \mathbf{B} := \begin{bmatrix} 3 & 1 & 1 \\ -1 & 7 & 2 \end{bmatrix}$ and $5\mathbf{A} = \begin{bmatrix} 10 & 5 & -5 \\ 0 & 15 & 5 \end{bmatrix}$.

The row vectors of $\mathbf{A}$ are $\begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 3 & 1 \end{bmatrix}$.

The column vectors of $\mathbf{A}$ are $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Also, $\mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \\ -1 & 1 \end{bmatrix}$.

# Matrix multiplication

So if $m, n, p \in \mathbb{N}$, $\mathbf{A} := [a_{jk}] \in \mathbb{R}^{m \times n}$ and $\mathbf{B} := [b_{jk}] \in \mathbb{R}^{n \times p}$, then $\mathbf{AB} \in \mathbb{R}^{m \times p}$, and for $j = 1, \ldots, m$; $k = 1 \ldots, p$,

$$\mathbf{AB} = [c_{jk}], \quad \text{where } c_{jk} := \mathbf{a}_j \mathbf{b}_k = \sum_{\ell=1}^{n} a_{j\ell} b_{\ell k}.$$

Note that the $(j, k)$th entry of $\mathbf{AB}$ is a product of the $j$th row vector of $\mathbf{A}$ with the $k$th column vector of $\mathbf{B}$ as shown below:

$$\begin{bmatrix} & & & & \\ a_{j1} & \cdots & a_{j\ell} & \cdots & a_{jn} \\ & & & & \end{bmatrix} \begin{bmatrix} b_{1k} \\ \vdots \\ b_{\ell k} \\ \vdots \\ b_{nk} \end{bmatrix}$$

Clearly, the product $\mathbf{AB}$ is defined only when the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

Note that $\mathbf{AI} = \mathbf{A}, \mathbf{IA} = \mathbf{A}, \mathbf{AO} = \mathbf{O}$ and $\mathbf{OA} = \mathbf{O}$ whenever these products are defined.

Examples

(i) Let $\mathbf{A} := \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix}_{2\times 3}$ and $\mathbf{B} := \begin{bmatrix} 1 & 6 & 0 & 2 \\ 2 & -1 & 1 & -2 \\ 2 & 0 & -1 & 1 \end{bmatrix}_{3\times 4}$.

Then $\mathbf{AB} = \begin{bmatrix} 2 & 11 & 2 & 1 \\ 8 & -3 & 2 & -5 \end{bmatrix}_{2\times 4}$.

(ii) In general, $\mathbf{AB} \neq \mathbf{BA}$. For example, if $\mathbf{A} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $\mathbf{AB} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, while $\mathbf{BA} := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Note that $\mathbf{BA} = \mathbf{O}$, while $\mathbf{AB} = \mathbf{B} \neq \mathbf{O}$. Since $\mathbf{A} \neq \mathbf{I}$,

we see that the so-called cancellation law does not hold.

Let $\mathbf{A} := [a_{jk}] \in \mathbb{R}^{m\times n}$, and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the basic column vectors in $\mathbb{R}^{n\times 1}$. Then for $k = 1, \ldots, n$,

$$\mathbf{A}\,\mathbf{e}_k = \begin{bmatrix} a_{1k} \\ \vdots \\ a_{jk} \\ \vdots \\ a_{mk} \end{bmatrix},$$ which is the $k$th column of $\mathbf{A}$.

# Properties of Matrix Multiplication

Consider matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\alpha \in \mathbb{R}$. Then it is easy to see that $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$, $\mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB}$ and $(\alpha\mathbf{A})\mathbf{B} = \alpha\mathbf{AB} = \mathbf{A}(\alpha\mathbf{B})$, if sums & products are well-defined.

Matrix multiplication also satisfies the **associative law**:

**Proposition**

Let $m, n, p, q \in \mathbb{N}$. If $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$ and $\mathbf{C} \in \mathbb{R}^{p\times q}$, then $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (which we shall write as $\mathbf{ABC}$).

Proof. Let $\mathbf{A} := [a_{jk}]$, $\mathbf{B} := [b_{jk}]$ and $\mathbf{C} := [c_{jk}]$. Also, let $(\mathbf{AB})\mathbf{C} := [\alpha_{jk}]$ and $\mathbf{A}(\mathbf{BC}) := [\beta_{jk}]$. Then

$$\alpha_{jk} = \sum_{i=1}^{p}\left(\sum_{\ell=1}^{n} a_{j\ell}b_{\ell i}\right)c_{ik} = \sum_{\ell=1}^{n} a_{j\ell}\left(\sum_{i=1}^{p} b_{\ell i}c_{ik}\right) = \beta_{jk}$$

for $j = 1, \ldots, m$ and $k = 1, \ldots, q$. Hence the result. $\square$

Also, the transpose of a product is the product of the transposes in the reverse order:

**Proposition**

Let $m, n, p \in \mathbb{N}$. If $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$, then $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$.

*Proof.* Let $\mathbf{A} := [a_{jk}]$, $\mathbf{B} := [b_{jk}]$ and $\mathbf{AB} := [c_{jk}]$.

Also, let $\mathbf{A}^{\mathsf{T}} := [a'_{jk}]$, $\mathbf{B}^{\mathsf{T}} := [b'_{jk}]$ and $(\mathbf{AB})^{\mathsf{T}} := [c'_{jk}]$. Then

$$c_{jk} = \sum_{\ell=1}^{n} a_{j\ell} b_{\ell k} \quad \text{and so} \quad c'_{jk} = c_{kj} = \sum_{\ell=1}^{n} a_{k\ell} b_{\ell j}$$

for $j = 1, \ldots, m$; $k = 1, \ldots, p$. Suppose $\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} := [d_{jk}]$. Then

$$d_{jk} = \sum_{\ell=1}^{n} b'_{j\ell} a'_{\ell k} = \sum_{\ell=1}^{n} b_{\ell j} a_{k\ell} = c'_{jk}$$

for $j = 1, \ldots, m$; $k = 1, \ldots, p$. Hence the result. $\square$

# Matrix Multiplication Revisited

Let $m, n, p \in \mathbb{N}$, $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$. For $j = 1, \ldots, m$, let $\mathbf{a}_j := \begin{bmatrix} a_{j1} & \cdots & a_{jn} \end{bmatrix}$ be the $j$th row of $\mathbf{A}$, and for $k = 1, \ldots, p$, let $\mathbf{b}_k := \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix}$ be the $k$th column of $\mathbf{B}$. Then $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{bmatrix}$, in terms of its rows. Also, $\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_p \end{bmatrix}$, in terms of its columns. Note: $\mathbf{AB}$ has $m$ rows and $p$ columns.

## Rows of AB

Fix $j \in \{1, \ldots, m\}$, and consider the $j$th row of $\mathbf{AB}$, namely, $\mathbf{a}_j\mathbf{B} = [\mathbf{a}_j\mathbf{b}_1 \cdots \mathbf{a}_j\mathbf{b}_p]$. For $k = 1, \ldots, p$, the $k$th entry of the $j$th row of $\mathbf{AB}$ is $\mathbf{a}_j\mathbf{b}_k = a_{j1}b_{1k} + \cdots + a_{jn}b_{nk}$, where $b_{1k}, \ldots, b_{nk}$ are the $k$th entries of the $n$ row vectors of $\mathbf{B}$.

Thus we see that for $j = 1, \ldots, m$, the $j$th row of $\mathbf{AB}$ is a linear combination of the $n$ row vectors of $\mathbf{B}$ with coefficients $a_{j1}, \ldots, a_{jn}$ provided by the $j$th row of $\mathbf{A}$.

Columns of $\mathbf{AB}$

Fix $k \in \{1, \ldots, p\}$, and consider the $k$th column of $\mathbf{AB}$, namely, $\mathbf{Ab}_k = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_k \\ \vdots \\ \mathbf{a}_m\mathbf{b}_k \end{bmatrix}$. For $j = 1, \ldots, m$, the $j$th entry of the $k$th column of $\mathbf{AB}$ is $\mathbf{a}_j\mathbf{b}_k = a_{j1}b_{1k} + \cdots + a_{jn}b_{nk}$, that is, $b_{1k}a_{j1} + \cdots + b_{nk}a_{jn}$, where $a_{j1}, \ldots, a_{jn}$ are the $j$th entries of the $n$ columns of $\mathbf{A}$.
Thus we see that for $k = 1, \ldots, n$, the $k$th column of $\mathbf{AB}$ is a linear combination of the $n$ column vectors of $\mathbf{A}$ with coefficients $b_{1k}, \ldots, b_{nk}$ provided by the $k$th column of $\mathbf{B}$.

These descriptions of the rows and columns of $\mathbf{AB}$ are useful.

## Example

As we have seen,

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 0 & 2 \\ 2 & -1 & 1 & -2 \\ 2 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 11 & 2 & 1 \\ 8 & -3 & 2 & -5 \end{bmatrix}, \quad \text{where}$$

$$\begin{aligned}
\begin{bmatrix} 2 & 11 & 2 & 1 \end{bmatrix} &= 2\begin{bmatrix} 1 & 6 & 0 & 2 \end{bmatrix} + 1\begin{bmatrix} 2 & -1 & 1 & -2 \end{bmatrix} \\
&\quad -1\begin{bmatrix} 2 & 0 & -1 & 1 \end{bmatrix}, \\
\begin{bmatrix} 8 & -3 & 2 & -5 \end{bmatrix} &= 0\begin{bmatrix} 1 & 6 & 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 & 1 & -2 \end{bmatrix} \\
&\quad +1\begin{bmatrix} 2 & 0 & -1 & 1 \end{bmatrix}, \\
\begin{bmatrix} 2 \\ 8 \end{bmatrix} &= 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \\
\begin{bmatrix} 11 \\ -3 \end{bmatrix} &= 6\begin{bmatrix} 2 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{etc.}
\end{aligned}$$

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# Linear System

Let $m, n \in \mathbb{N}$. A **linear system** of $m$ equations in the $n$ unknowns $x_1, ..., x_n$ is given by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 && (1)\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 && (2)\\
\vdots \qquad \vdots \qquad \vdots \qquad & \vdots \quad \vdots && \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, && (m)
\end{aligned}$$

where $a_{jk} \in \mathbb{R}$ for $j = 1, \ldots, m$; $k = 1, \ldots, n$ and also $b_j \in \mathbb{R}$ for $j = 1, \ldots, m$ are given.

Let $\mathbf{A} := [a_{jk}] \in \mathbb{R}^{m \times n}$, $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{n \times 1}$ and $\mathbf{b} := \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{m \times 1}$. Using matrix multiplication, we write the linear system as

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$