Solutions to MA 110 Mid-Sem Exam (23 Feb 2025)

Q 1. (i) $AB = A^2 + A + I$, hence $A(B - A - I) = I$. This means, A is invertible and $A^{-1} = B-A-I$. But then $I = A^{-1}A = (B-A-I)A$, which means $BA = A^2+A+I = AB$. [2 marks]

(ii) Let $x \in N(AB - I_m)$. Then $(AB - I_m)x = 0$. Multiplying by B on the left, $BABx - Bx = 0$ as well, or $(BA - I_n)(Bx) = 0$. Thus, $T : N(AB - I_m) \to N(BA - I_n)$ sending x to $T(x) := Bx$ is a linear transformation. If $Bx = 0$ for some $x \in N(AB - I_m)$, then $(AB - I_m)x = 0$ and so $x = ABx = 0$. This proves that T is injective. Hence $\text{nullity}(AB - I_m) \le \text{nullity}(BA - I_n)$. Similarly, $\text{nullity}(BA - I_n) \le \text{nullity}(AB - I_m)$. [2 marks]

Now $(AB-I_m)$ is invertible $\iff \text{nullity}(AB-I_m) = 0 \iff \text{nullity}(BA-I_n) = 0 \iff (BA - I_n)$ is invertible. [1 mark]

Q2. (i) Let S be a subset of V. Then S is said to be *linearly dependent* if there are distinct $v_1, \ldots, v_n \in S$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$, not all zero, such that $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$.

Also, the *span* of S is the set of all (finite) linear combinations of elements of S, i.e.,

$$\text{span}(S) = \left\{ \sum_{j=1}^{n} \alpha_j v_j : n \in \mathbb{N},\ \alpha_j \in \mathbb{K},\ \text{and } v_j \in S \text{ for } j = 1, \ldots, n \right\}. \qquad [1 \text{ mark}]$$

(ii) Let $S = \{v_1, \ldots, v_s\}$ and let $R = \{w_1, \ldots, w_r\}$. Since $S \subset \text{span}(R)$, there are $\alpha_{jk} \in \mathbb{K}$ such that $v_k = \sum_{j=1}^{r} \alpha_{jk} w_j$ for $k = 1, \ldots, s$. The matrix $A = (\alpha_{jk})$ has s columns and r rows, where $s > r$. Hence there is $\mathbf{0} \neq \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix} \in \mathbb{K}^{s\times 1}$ such that $A\mathbf{x} = 0$. Thus, $\sum_{k=1}^{s} \alpha_{jk} x_k = 0$ for each $j = 1, \ldots r$. So we obtain $x_1, \ldots, x_s \in \mathbb{K}$, not all zero, such that

$$\sum_{k=1}^{s} x_k v_k = \sum_{j=1}^{r} \left(\sum_{k=1}^{s} \alpha_{jk} x_k \right) w_j = \sum_{k=1}^{s} \left(\sum_{j=1}^{r} \alpha_{jk} w_j \right) x_k = 0.$$

This proves that S is linearly dependent. [3 marks]

[Note: Some students have written that S has more elements than the spanning set R, therefore S is linearly dependent. This is just rephrasing the question. No marks have been awarded in these cases.]

Q3. (i) $A = iI_n$. [1 mark]

(ii) As A satisfies $A^2 + I = 0$, the eigenvalues satisfy $\lambda^2 + 1 = 0$. Hence the only possible eigenvalues of A are $\pm i$. [2 marks]

Since the matrix is real, its characteristic polynomial $p(t)$ is a real polynomial. Since the matrix A is of odd order, $p(t)$ has odd degree, and hence it must have a real root, which is a contradiction. [2 marks]

Q. 4. (i) The adjoint A^* of A is the conjugate transpose of A, i.e., $A^* = \overline{A}^t$. [$\frac{1}{2}$ mark]

To show that $\text{rank}(A^*) = \text{rank}(A^*A)$, we consider the images of A^* and A^*A, i.e., the images of the corresponding linear transformations. It is clear that $\text{im}(A^*A) \subset \text{im}(A^*)$. Hence $\text{rank}(A^*A) \leq \text{rank}(A^*) = \text{rank}(A)$. On the other hand, if $x \in N(A^*A)$ then $A^*Ax = 0$, and so $0 = x^*A^*Ax = (Ax)^*Ax = \langle Ax, Ax\rangle = ||Ax||^2$. Thus $x \in N(A)$. This implies $N(A^*A) \subset N(A)$; so $\text{nullity}(A^*A) \leq \text{nullity}(A)$. By Rank-Nullity Theorem, $\text{rank}(A) \leq \text{rank}(A^*A)$. Thus, $\text{rank}(A^*) = \text{rank}(A) = \text{rank}(A^*A)$. [$1\frac{1}{2}$ mark]

(ii) Let $\lambda_1, \ldots, \lambda_n$ be all the eigenvalues of A and let $(x_1, \ldots, x_n)$ be an orthonormal basis of $\mathbb{R}^{n\times 1}$ such that $Ax_j = \lambda_j x_j$ for all j. Then for every $x \in \mathbb{R}^{n\times 1}$, we can write $x = \sum_{j=1}^n \alpha_j x_j$ for some $\alpha_j \in \mathbb{R}$. Also if $x \neq 0$, then $\alpha_j \neq 0$ for some j. Thus for every nonzero $x \in \mathbb{R}^{n\times 1}$, we find $x^tAx = x^t(\sum_j \alpha_j\lambda_j x_j) = \sum_j \alpha_j^2\lambda_j > 0$ if all $\lambda_j > 0$. [3 marks]

Q. 5 (i) $T(E_{11}) = E_{11}$, $T(E_{22}) = E_{22}$, $T(E_{12}) = E_{21}$, $T(E_{21}) = E_{12}$. [1 mark]
Thus the matrix of T with respect to the ordered basis $(E_{11}, E_{12}, E_{21}, E_{22})$ is

$$M_E^E(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad [1 \text{ mark}]$$

(ii) The given equation can be rewritten as $q(x, y, z) = 36$, where

$$q(x, y, z) = X^tAX, \quad \text{where} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } A = \begin{pmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{pmatrix}.$$

The characteristic polynomial of A is $p(A) = \det(A - \lambda I_3) = -(\lambda - 18)(\lambda - 6)(\lambda + 12)$. Thus, the eigenvalues are $18, 6, -12$. [1 mark]

Next, consider

$$A - 18I_3 = \begin{pmatrix} -11 & -1 & -10 \\ -1 & -11 & 10 \\ -10 & 10 & -20 \end{pmatrix}.$$

Solving $(A - 18I_3)x = 0$ using Gaussian elimination, we get a solution $x_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. So x_1 is an eigenvector for $\lambda = 18$. Similarly, we get

$$\text{an eigenvector } x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for } \lambda = 6 \quad \text{and} \quad \text{an eigenvector } x_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \text{ for } \lambda = -12.$$

[1 mark]

[Note: If two out of three eigenvectors are correct, $\frac{1}{2}$ mark has been awarded.]

Since A is symmetric, the eigenvectors x_1, x_2, x_3 with distinct eigenvalues form an orthogonal set. Hence if we let $y_1 = x_1/||x_1||$, $y_2 = x_2/||x_2||$ and $y_3 = x_3/||x_3||$, then

(y_1, y_2, y_3) is an ordered orthonormal basis of $\mathbb{R}^{3\times 1}$. Consequenntly, the matrix

$$P = [y_1\, y_2\, y_3] = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix} \text{ is unitary and satisfies } P^t A P = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -12 \end{pmatrix}.$$

Now if we let

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = P^t \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

then the quadratic form $q(x, y, z)$ is transformed to the diagonal form $18u^2 + 6v^2 - 12w^2$. Thus the desired orthogonal change of coordinates ig given by

$$\begin{aligned} u &= (-x + y + z)/\sqrt{3} \\ v &= (x + y)/\sqrt{2} \\ w &= (x - y + 2z)/\sqrt{6}x \end{aligned}$$

[1 mark]

[Note: If two of u, v, w are correct, $\frac{1}{2}$ mark has been awarded.]

Q. 6 (i) An orthonormal basis for the column space $\mathcal{C}(A)$ of the matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ is given by $(u_1 u_2)$, where $u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. [1 mark]

Here $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The orthogonal projection of b on $\mathcal{C}(A)$ is

$$v = \langle u_1, b\rangle u_1 + \langle u_2, b\rangle u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} + 0 = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}.$$

Therefore, the best approximation for a solution is given by $x = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$, as it solves $Ax = v$. [1 mark]

(ii) Consider the functions $w_1 = 1$ and $w_2 = t$ which form a basis of W. Observe that $\langle w_1, w_2\rangle = \int_{-3}^{3} t\,dt = 0$. Thus w_1, w_2 are orthogonal,. Also note that $||1||^2 = \int_{-3}^{3} dt = 6$ and $||t||^2 = [t^3/3]_{t=-3}^{t=3} = 18$. Thus if we let $u_1 = 1/\sqrt{6}$ and $u_2 = t/\sqrt{18}$, then we see that (u_1, u_2) is an ordered orthonormal basis of W. [1 mark]

Note that $\langle t^2, u_1\rangle = 18\sqrt{6}$, $\langle t^2, u_2\rangle = 0, \langle t, u_1\rangle = 0$ and $\langle t, u_2\rangle = \sqrt{18}$. [1 mark]
Hence the orthogonal projection of $t^2 + t$ on W is

$$\langle t^2 + t, u_1\rangle u_1 + \langle t^2 + t, u_2\rangle u_2 = 3 + t.$$ [1 mark]