MA 110:
Lecture 07

Saurav Bhaumik
Department of Mathematics
IIT Bombay

Spring 2025

Review of last lecture

We have discussed the following important notions.

- Rank of a matrix
- Vector subspaces (of $\mathbb{R}^{n\times 1}$).
- Basis of a subspace
- Dimension of a subspace
- Span of a subset of a vector subspace.
- Null space and the column space of a matrix.

And we proved several important results such as:

- Characterizations of a basis of a vector subspace
- Rank-Nullity Theorem
- Fundamental Theorem for Linear Systems:

Remark: The notion of a vector subspace of $\mathbb{R}^{1\times m}$ is defined similarly. The corresponding notions of basis, dimension, span, etc. are defined in an identical manner.

Let us restate two earlier results which are in conformity with the Rank-Nullity Theorem. Let $\mathbf{A}$ be an $n \times n$ matrix. Then

$$\boxed{\mathbf{A} \text{ is invertible} \iff \text{nullity}\,\mathbf{A} = 0 \iff \text{rank}\,\mathbf{A} = n}.$$

Further, $\text{rank}\,\mathbf{A} = n \iff \mathcal{C}(\mathbf{A}) = \mathbb{R}^{n\times 1}$.

We are now in a position to state and prove a comprehensive result regarding solutions of a system of m linear equations in n unknowns that we started with, namely

$$\begin{array}{rcll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 & \quad (1) \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 & \quad (2) \\
\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots & \vdots & & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m & \quad (m)
\end{array}$$

As usual, we write this as $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} := [a_{jk}] \in \mathbb{R}^{m\times n}$, $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}}$ and $\mathbf{b} := \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}^{\mathsf{T}}$.

Theorem (Fundamental Theorem for Linear Systems: FTLS)

Let $m, n \in \mathbb{N}$ and $\mathbf{A}$ be an $m \times n$ matrix with real entries. Suppose $\boxed{\operatorname{rank} \mathbf{A} = r}$.

(i) **Homogeneous Linear System**: $\mathbf{A}\mathbf{x} = \mathbf{0}$ **(H)**

The solution space $\{\mathbf{x} \in \mathbb{R}^{n\times 1} : \mathbf{A}\mathbf{x} = \mathbf{0}\}$ of (H) is a subspace of $\mathbb{R}^{n\times 1}$ of dimension $\boxed{n-r}$.

In particular, $r = n$ if and only if $\mathbf{0}$ is the only solution of (H).

If $r < n$, then there are linearly independent solutions $\mathbf{x}_1, \ldots, \mathbf{x}_{n-r}$ of (H) and every solution of (H) is a unique linear combination of these $\mathbf{x}_1, \ldots, \mathbf{x}_{n-r}$.

(ii) **General Linear System**: $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{b} \in \mathbb{R}^{m\times 1}$ **(G)**

(G) has a solution if and only if $\operatorname{rank}[\mathbf{A}|\mathbf{b}] = r$. In this case, let $\mathbf{x}_0$ be a particular solution of (G). If $\mathbf{x}$ is a solution of (G), then $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of (H) above.

Proof. (i) The solution space of the homogeneous linear system (H) is just the nullspace $\mathcal{N}(\mathbf{A})$ of $\mathbf{A}$, and we have seen that its dimension, that is, the nullity of $\mathbf{A}$, is equal to $n - r$.

We note that $r = n \iff n - r = 0$, that is, the dimension of $\mathcal{N}(\mathbf{A})$ is zero; in other words, $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$. This means that $\mathbf{0}$ is the only solution of (H).

Let now $r < n$. Then $\mathcal{N}(\mathbf{A})$ has a basis consisting of $n - r$ elements, say $\mathbf{x}_1, \ldots, \mathbf{x}_{n-r}$. Hence every element of the solution space is a unique linear combination of the elements in this basis.

(ii) Let $\mathbf{b} \in \mathbb{R}^{n \times 1}$. Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ be the n columns of $\mathbf{A}$. Then

$$\mathbf{Ax} = x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n \quad \text{for } \mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{n \times 1}.$$

Hence $\mathbf{Ax} = \mathbf{b}$ for some $\mathbf{x} \in \mathbb{R}^{n \times 1}$ if and only if $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$, that is, $\mathbf{b} \in \mathcal{C}(\mathbf{A})$.

Since every column of $\mathbf{A}$ is also a column of the augmented matrix $[\mathbf{A}|\mathbf{b}]$, the column space $\mathcal{C}(\mathbf{A})$ of $\mathbf{A}$ is contained in the column space $\mathcal{C}([\mathbf{A}|\mathbf{b}])$ of $[\mathbf{A}|\mathbf{b}]$. It follows that $\mathbf{b} \in \mathcal{C}(\mathbf{A})$ if and only if $\mathcal{C}([\mathbf{A}|\mathbf{b}]) = \mathcal{C}(\mathbf{A})$, that is, the column rank of $[\mathbf{A}|\mathbf{b}]$ is equal to the column rank of $\mathbf{A}$. So $\operatorname{rank}[\mathbf{A}|\mathbf{b}] = \operatorname{rank}\mathbf{A} = r$.

Let $\mathbf{x}_0$ be a particular solution of (G), that is, let $\mathbf{x}_0 \in \mathbb{R}^{n\times 1}$ satisfy $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$. Then for any $\mathbf{x} \in \mathbb{R}^{n\times 1}$, we see that $\mathbf{A}\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$, that is, $\mathbf{x}$ is a solution of (G) if and only if $\mathbf{x}_h := \mathbf{x} - \mathbf{x}_0$ is a solution of (H). The proof is complete. $\square$

Remark

The above theorem is of immense theoretical importance. It tells us precisely when solutions exist, and also describes the nature of solutions of a linear system of equations.
For example, it says that when there is a nonzero solution of a homogeneous linear system, there are infinitely many solutions.

Further, when a homogeneous system has infinitely many solutions, it says that they can be described by a one parameter family, or a two parameter family etc.

It may seem that to implement the results of the above theorem, we must first find the rank of the coefficient matrix $\mathbf{A}$ of the linear system. This is not necessary.

We have already seen that we may directly proceed to find the solutions of the linear system by considering the augmented matrix $[\mathbf{A}|\mathbf{b}]$ and transform the coefficient matrix $\mathbf{A}$ to a row echelon form by the Gauss Elimination Method and then use Back Substitution. This process itself reveals all possibilities.

In particular, when the rank r of $\mathbf{A}$ is less than the number n of variables, we have shown how to construct a set S of basic solutions of an homogeneous linear system. This set S is in fact a basis of the solution space of the system. That is the reason for using the terminology 'basic solutions'.

Row Space and Column Space

Definition

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. The row space of A, denoted $\mathcal{R}(\mathbf{A})$, is defined as the subspace of $\mathbb{R}^{1 \times n}$ spanned by the row vectors of $\mathbf{A}$.

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Here are some important observations.

- The row-rank of $\mathbf{A}$ is precisely the dimension of $\mathcal{R}(\mathbf{A})$.
- If $\mathbf{A}' \in \mathbb{R}^{m \times n}$ is obtained from $\mathbf{A}$ by an elementary row operation, then $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}')$.
- If $\mathbf{A}' \in \mathbb{R}^{m \times n}$ is in REF, then the nonzero rows of $\mathbf{A}'$ form a basis of $\mathcal{R}(\mathbf{A}')$.
- A basis of $\mathcal{R}(\mathbf{A})$ is given by the nonzero rows of its REF.
- If $\mathbf{A}'$ is obtained from $\mathbf{A}$ by an elementary row operation, then $\mathcal{C}(\mathbf{A})$ need not be equal to $\mathcal{C}(\mathbf{A}')$.

Lemma

The columns of $\mathbf{A}$ *corresponding to the pivotal columns of its REF form a basis of* $\mathcal{C}(\mathbf{A})$.

Proof. Let $\mathbf{A}'$ be an REF of $\mathbf{A}$. Suppose the columns $\mathbf{c}'_{\mathbf{i_1}}, \cdots, \mathbf{c}'_{\mathbf{i_r}}$ of $\mathbf{A}'$ are the pivotal columns. Let $\mathbf{c}_{\mathbf{i_1}}, \cdots, \mathbf{c}_{\mathbf{i_r}}$ be the corresponding columns of $\mathbf{A}$. It is enough to prove that $\mathbf{c}_{\mathbf{i_1}}, \cdots, \mathbf{c}_{\mathbf{i_r}}$ are linearly independent. If they were not so, then there would be constants $\alpha_{i_1}, \cdots, \alpha_{i_r}$, not all zero such that

$$\alpha_{i_1}\mathbf{c}_{\mathbf{i_1}} + \cdots + \alpha_{i_r}\mathbf{c}_{\mathbf{i_r}} = \mathbf{0}.$$

Consider the vector $\mathbf{x} = (x_1, \ldots, x_n)^T$ such that $x_j = \alpha_j$ if j is one of $i_1, \ldots, i_r$, and $x_j = 0$ other wise. Then $\mathbf{Ax} = \mathbf{0}$, thus $\mathbf{A}'\mathbf{x} = \mathbf{0}$ as well. But this means,

$$\alpha_{i_1}\mathbf{c}'_{\mathbf{i_1}} + \cdots + \alpha_{i_r}\mathbf{c}'_{\mathbf{i_r}} = \mathbf{0},$$

which is a contradiction because $\mathbf{c}'_{\mathbf{i_1}}, \cdots, \mathbf{c}'_{\mathbf{i_r}}$ being the pivotal columns of a matrix in REF, are linearly independent. $\square$

Example: Consider the 5×6 matrix $\mathbf{A}$ and its REF $\mathbf{A}'$ given by

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} \boxed{1} & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & \boxed{-1} & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\operatorname{rank} \mathbf{A} = 3$. A basis of the row space $\mathcal{R}(\mathbf{A})$ is given by

$$\left\{ \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -2 & 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \right\}$$

whereas a basis for the column space $\mathcal{C}(\mathbf{A})$ is given by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 15 \\ 18 \end{bmatrix} \right\}.$$

Determinants

You already know formulas for determinants of 1×1, 2×2 and 3×3 matrices. Let us recall them.

$$\det \begin{bmatrix} a_1 \end{bmatrix} = a_1, \quad \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1 \quad \text{and}$$

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2),$$

which is also equal to
$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$.

We shall presently give formulas for the determinant of an $n \times n$ matrix, that is, of a matrix of size n, where $n \in \mathbb{N}$, and we shall explore their use in matrix theory.

Let $n \in \mathbb{N}$ and $\mathbf{A} := \begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{j1} & \cdots & \boxed{a_{jk}} & \cdots & a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}$.

The determinant of $\mathbf{A}$ is a real number defined inductively as follows. For $n := 1$, define $\det \mathbf{A} := a_{11}$. Let $n \geq 2$, and suppose we have defined the determinant of any $(n-1)\times(n-1)$ matrix. For $j, k = 1, \ldots, n$, let $\mathbf{A}_{jk}$ denote the submatrix of $\mathbf{A}$ obtained by deleting the jth row and the kth column of $\mathbf{A}$, and let $M_{jk} := \det \mathbf{A}_{jk}$, called the (j, k)th **minor** of $\mathbf{A}$. Define

$$\det \mathbf{A} := a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{1+k}a_{1k}M_{1k} + \cdots + (-1)^{1+n}a_{1n}M_{1n}.$$

This is also known as the **expansion for the determinant of A in terms of the first row of A**.

An immediate consequence of our definition is the following.

Proposition

If an $n \times n$ matrix $\mathbf{A}$ is lower triangular, then the determinant of $\mathbf{A}$ is the product of its diagonal entries.

Proof. We prove it by induction on n. For $n = 1$, it is the definition. Suppose we know the proposition for all $(n-1) \times (n-1)$ matrices. Now $\det \mathbf{A} = a_{11} M_{11}$ since $a_{12} = \cdots = a_{1n} = 0$, etc. Now $\mathbf{A_{1,1}}$ is lower triangular and thus by the induction hypothesis, $M_{1,1}$ is the product of the diagonal entries of $\mathbf{A}$ other than a_{11}. $\square$

Next, it can be proved by induction on the size n of a matrix that $\det \mathbf{A}$ is equal to the following expansions in terms of the jth row of $\mathbf{A}$, and also in terms of the kth column of $\mathbf{A}$:

$$\det \mathbf{A} = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad \text{for each } j \in \{1, \ldots, n\}$$

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \quad \text{for each } k \in \{1, \ldots, n\}$$

(For a proof, see Kreyszig, Appendix 4, page A81.)

Proposition

Let $\mathbf{A}$ be a square matrix. Then $\det \mathbf{A}^{\mathsf{T}} = \det \mathbf{A}$.

Proof. This is obvious if $n = 1$. Let now $n \geq 2$, and assume this property for all $(n-1)\times(n-1)$ matrices. Note that $(\mathbf{A}^{\mathsf{T}})_{jk} = (\mathbf{A}_{kj})^{\mathsf{T}}$ for all $j, k = 1, \ldots, n$, that is, the submatrix obtained by deleting the jth row and the kth column of $\mathbf{A}^{\mathsf{T}}$ is the same as the transpose of the submatrix obtained by deleting the kth row and the jth column of $\mathbf{A}$. (For example,

$$\mathbf{A} := \begin{bmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \implies \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ \boxed{a_{12}} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix},$$

so that $(\mathbf{A}^{\mathsf{T}})_{21} = \begin{bmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}^{\mathsf{T}} = (\mathbf{A}_{12})^{\mathsf{T}}$.)

Let $\mathbf{A} := [a_{jk}]$ and $\mathbf{A}^{\mathsf{T}} := [a'_{jk}]$. Then $a'_{jk} = a_{kj}$ and $M'_{jk} := \det(\mathbf{A}^{\mathsf{T}})_{jk} = \det(\mathbf{A}_{kj})^{\mathsf{T}} = \det \mathbf{A}_{kj} = M_{kj}$ by the inductive hypothesis for $j, k = 1, \ldots, n$. Expanding $\det \mathbf{A}^{\mathsf{T}}$ in terms of its first row, and $\det \mathbf{A}$ in terms of its first column,

$$\begin{aligned} \det \mathbf{A}^{\mathsf{T}} &= a'_{11}M'_{11} - a'_{12}M'_{12} + \cdots + (-1)^{1+n}a'_{1n}M'_{1n} \\ &= a_{11}M_{11} - a_{21}M_{21} + \cdots + (-1)^{n+1}a_{n1}M_{n1} \\ &= \det \mathbf{A}. \quad \square \end{aligned}$$

Corollary

If $\mathbf{A}$ is upper triangular, then the determinant of $\mathbf{A}$ is the product of its diagonal entries.

Proof. Let $\mathbf{A}$ be upper triangular. Then $\mathbf{A}^{\mathsf{T}}$ is lower triangular and has the same diagonal entries as those of $\mathbf{A}$. $\square$

Let us write $\boxed{\mathbf{A} := \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_n \end{bmatrix}}$ in terms of its n columns.

Crucial Properties of the **determinant function** $\mathbf{A} \longmapsto \det \mathbf{A}$ from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$:

1. Let $k \in \{1, \ldots, n\}$ and $\boxed{\mathbf{c}_k = \alpha\, \mathbf{c}'_k + \beta\, \mathbf{c}''_k}$, where $\alpha, \beta \in \mathbb{R}$.

Then $\det \mathbf{A} = \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \alpha\, \mathbf{c}'_k + \beta\, \mathbf{c}''_k & \cdots & \mathbf{c}_n \end{bmatrix}$ is equal to $\alpha \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}'_k & \cdots & \mathbf{c}_n \end{bmatrix} + \beta \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}''_k & \cdots & \mathbf{c}_n \end{bmatrix}$.

This is proved by expanding $\det \mathbf{A}$ in terms of its kth column. In particular,
$\det(\alpha \mathbf{A}) = \det \begin{bmatrix} \alpha \mathbf{c}_1 & \cdots & \alpha \mathbf{c}_k & \cdots & \alpha \mathbf{c}_n \end{bmatrix} = \alpha^n \det \mathbf{A}$.

2. Let $n \geq 2$. If $\boxed{\mathbf{c}_k = \mathbf{c}_\ell}$ for some $k \neq \ell$, then $\det \mathbf{A} = 0$, that is, if 2 columns of a matrix are identical, then its determinant is equal to 0. This is clear for $n = 2$, and for $n \geq 3$, this is proved by expanding $\det \mathbf{A}$ in terms of a column $\mathbf{c}_p$ of $\mathbf{A}$, where $p \neq k$ and $p \neq \ell$, and then using induction on n.

3. $\boxed{\det \mathbf{I} = 1}$, that is, the determinant of the identity matrix is equal to 1. This is obvious.

Proposition

Let $\mathbf{A}$ be a square matrix.

(i) If two columns of $\mathbf{A}$ are interchanged, then $\det \mathbf{A}$ gets multiplied by -1.

(ii) Addition of a multiple of a column to another column of $\mathbf{A}$ does not alter $\det \mathbf{A}$.

(iii) Multiplication of a column of $\mathbf{A}$ by a scalar α results in the multiplication of $\det \mathbf{A}$ by α.

Proof: Let $\mathbf{A} := \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}$, where $k \neq \ell$.

(i) Define $\alpha := \det \begin{bmatrix} \mathbf{c}_1 & \cdots & (\mathbf{c}_k + \mathbf{c}_\ell) & \cdots & (\mathbf{c}_k + \mathbf{c}_\ell) & \cdots \mathbf{c}_n \end{bmatrix}$.
Then $\alpha = 0$ since the matrix has two identical columns.

On the other hand, $\alpha = \beta + \gamma$, where
$\beta := \det \begin{bmatrix} \mathbf{c}_1 & \cdots & (\mathbf{c}_k + \mathbf{c}_\ell) & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix}$ and
$\gamma := \det \begin{bmatrix} \mathbf{c}_1 & \cdots & (\mathbf{c}_k + \mathbf{c}_\ell) & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}$.

In turn, $\beta := \beta_1 + \beta_2$ and $\gamma = \gamma_1 + \gamma_2$, where

$$\begin{aligned}
\beta_1 &= \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix}, \\
\beta_2 &= \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_\ell & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix}, \\
\gamma_1 &= \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}, \\
\gamma_2 &= \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_\ell & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}.
\end{aligned}$$

But $\beta_1 = 0 = \gamma_2$ since two columns are identical. Since $0 = \alpha = \beta + \gamma = \beta_2 + \gamma_1$, we see that $\gamma_1 = -\beta_2$, that is, $\det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}$ is equal to $-\det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_\ell & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix}$, as desired.

(ii) Suppose α times the ℓth column of $\mathbf{A}$ is added to the kth column of $\mathbf{A}$. Then $\det \begin{bmatrix} \mathbf{c}_1 & \cdots & (\mathbf{c}_k + \alpha\,\mathbf{c}_\ell) & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix}$ is equal to $\det \mathbf{A} + \alpha \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_\ell & \cdots & \mathbf{c}_\ell & \cdots \mathbf{c}_n \end{bmatrix} = \det \mathbf{A}$.

(iii) Suppose the kth column of $\mathbf{A}$ is multiplied by α. Then $\det \begin{bmatrix} \mathbf{c}_1 & \cdots & \alpha\,\mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix} = \alpha \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k & \cdots \mathbf{c}_n \end{bmatrix} = \alpha \det \mathbf{A}$.