MA 110:
Lecture 04

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Recall: A square matrix $\mathbf{A} \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$ is said to be invertible if there is $\mathbf{B} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{AB} = \mathbf{I} = \mathbf{BA},$$

and in this case, $\mathbf{B}$ is called an inverse of $\mathbf{A}$.

We have seen examples of square matrices that are invertible and also those that are not invertible. Further we noted that:

- If a square matrix $\mathbf{A} \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$ is invertible, then it has a unique inverse, and it is denoted by $\mathbf{A}^{-1}$
- If a square matrix $\mathbf{A} \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$ is invertible, then so is its transpose $\mathbf{A}^{\mathsf{T}}$ and in this case,

$$(\mathbf{A}^{\mathsf{T}})^{-\mathbf{1}} = (\mathbf{A}^{-\mathbf{1}})^{\mathsf{T}}.$$

We now relate the invertibility of a square matrix $\mathbf{A}$ to the solutions of the homogeneous system $\mathbf{Ax} = \mathbf{0}$.

Proposition

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Then $\mathbf{A}$ is invertible if and only if the linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution.

Proof. Suppose $\mathbf{A}$ is invertible. Then by definition, there is $\mathbf{B} \in \mathbb{R}^{n \times n}$ such that $\mathbf{BA} = \mathbf{I}$. If $\mathbf{x} \in \mathbb{R}^{n \times 1}$ satisfies $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x} = \mathbf{BAx} = \mathbf{B}(\mathbf{Ax}) = \mathbf{B0} = \mathbf{0}$. Thus the linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution.

Conversely, suppose the linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution. Let $\mathbf{y} = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{n \times 1}$. We transform the augmented matrix $[\mathbf{A}|\mathbf{y}]$ to a matrix $[\mathbf{A}'|\mathbf{y}']$, where $\mathbf{A}'$ is in REF. By our previous result, $\mathbf{A}'$ has n nonzero rows, and so back substitution gives a unique $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{n \times 1}$ such that $\mathbf{A}'\mathbf{x} = \mathbf{y}'$. Hence $\mathbf{Ax} = \mathbf{y}$.

Further, the process of the back substitution shows that the entries $x_1, \dots, x_n$ of $\mathbf{x}$ are given as follows:

$$\begin{array}{rcl} x_n & = & c'_{nn} y'_n \\ x_{n-1} & = & c'_{(n-1)(n-1)} y'_{n-1} + c'_{(n-1)n} y'_n \\ \vdots & \vdots & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ x_2 & = & c'_{22} y'_2 + \cdots + \cdots + \cdots + c'_{2n} y'_n \\ x_1 & = & c'_{11} y'_1 + c'_{12} y'_2 + \cdots + \cdots + \cdots + c'_{1n} y'_n, \end{array}$$

where $\mathbf{y}' = \begin{bmatrix} y'_1 & \cdots & y'_n \end{bmatrix}^{\mathsf{T}}$ and $c'_{jk} \in \mathbb{R}$ for $j, k = 1, \dots, n$.

Also, since $\mathbf{y}'$ is obtained from $\mathbf{y}$ by performing EROs (which are of the type $R_i \longleftrightarrow R_j$, $R_i + \alpha R_j$ and αR_j) on $[\mathbf{A}|\mathbf{y}]$, we see that each $y'_1, \dots, y'_n$ is a linear combination of the entries $y_1, \dots, y_n$ of $\mathbf{y}$. As a result, each $x_1, \dots, x_n$ is a linear combination of $y_1, \dots, y_n$.

Thus there is $c_{jk} \in \mathbb{R}$ for $j, k = 1, \ldots, n$ (not depending on $y_1, \ldots, y_n$) such that

$$\begin{array}{rcl} x_1 & = & c_{11}y_1 + c_{12}y_2 + \cdots + c_{1n}y_n \\ x_2 & = & c_{21}y_1 + c_{22}y_2 + \cdots + c_{2n}y_n \\ \vdots & \vdots & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ x_n & = & c_{n1}y_1 + c_{n2}y_2 + \cdots + c_{nn}y_n. \end{array}$$

Define $\mathbf{C} := \left[c_{jk}\right] \in \mathbb{R}^{n \times n}$. Then $\mathbf{x} = \mathbf{Cy}$, and so $\mathbf{ACy} = \mathbf{A}(\mathbf{Cy}) = \mathbf{Ax} = \mathbf{y}$. Letting $\mathbf{y} := \mathbf{e}_k \in \mathbb{R}^{n \times 1}$, we see that $(\mathbf{AC})\mathbf{e_k} = \mathbf{e_k}$ for $k = 1, \ldots, n$. Hence $\mathbf{AC} = \mathbf{I}$. We still need to show that $\mathbf{CA} = \mathbf{I}$. For this, consider the linear system $\mathbf{Cx} = \mathbf{0}$. Note that $\mathbf{Cx} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{ACx} = \mathbf{A}(\mathbf{Cx}) = \mathbf{A0} = \mathbf{0}$. Thus the linear system $\mathbf{Cx} = \mathbf{0}$ has only the zero solution.

Hence by what we have proved above, there is $\mathbf{D} \in \mathbb{R}^{n \times n}$ such that $\mathbf{CD} = \mathbf{I}$. Now, $\mathbf{D} = \mathbf{ID} = (\mathbf{AC})\mathbf{D} = \mathbf{A}(\mathbf{CD}) = \mathbf{AI} = \mathbf{A}$. Thus $\mathbf{AC} = \mathbf{I} = \mathbf{CA}$, and so $\mathbf{A}$ is invertible. $\square$

Corollary

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. If there is $\mathbf{B} \in \mathbb{R}^{n \times n}$ such that either $\mathbf{BA} = \mathbf{I}$ or $\mathbf{AB} = \mathbf{I}$, then $\mathbf{A}$ is invertible, and $\mathbf{A}^{-1} = \mathbf{B}$.

Proof. Let $\mathbf{B} \in \mathbb{R}^{n \times n}$ be such that $\mathbf{BA} = \mathbf{I}$. If $\mathbf{x} \in \mathbb{R}^{n \times 1}$ satisfies $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x} = \mathbf{BAx} = \mathbf{B}(\mathbf{Ax}) = \mathbf{B0} = \mathbf{0}$. Thus the linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution. By the previous proposition, $\mathbf{A}$ is invertible. Then there is $\mathbf{C} \in \mathbb{R}^{\mathbf{n} \times \mathbf{n}}$ such that $\mathbf{AC} = \mathbf{I}$, and $\mathbf{B} = \mathbf{C}$. Hence $\mathbf{A}^{-1} = \mathbf{B}$.

Next, let $\mathbf{B} \in \mathbb{R}^{n \times n}$ be such that $\mathbf{AB} = \mathbf{I}$. Then $\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} = \mathbf{I}^{\mathsf{T}} = \mathbf{I}$. By what we have just proved, $\mathbf{A}^{\mathsf{T}}$ is invertible, and $(\mathbf{A}^{\mathsf{T}})^{-1} = \mathbf{B}^{\mathsf{T}}$. Hence $\mathbf{A} = (\mathbf{A}^{\mathsf{T}})^{\mathsf{T}}$ is invertible, and $\mathbf{A}^{-1} = (\mathbf{B}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{B}$. $\square$

Note: The above result is a definite improvement over requiring the existence of a matrix $\mathbf{B}$ satisfying both $\mathbf{BA} = \mathbf{I}$ and $\mathbf{AB} = \mathbf{I}$ for the invertibility of a square matrix $\mathbf{A}$.

Proposition

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices. Then $\mathbf{AB}$ is invertible if and only if $\mathbf{A}$ and $\mathbf{B}$ are invertible, and then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be invertible. Using the associativity of matrix multiplication, we easily see that

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}.$$

Hence $\mathbf{AB}$ is invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ by the previous corollary.

Conversely, let $\mathbf{AB}$ be invertible. Then there is $\mathbf{C}$ such that $(\mathbf{AB})\mathbf{C} = \mathbf{I} = \mathbf{C}(\mathbf{AB})$. Since $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} = \mathbf{I}$, we see that $\mathbf{A}$ is invertible, and $\mathbf{A}^{-1} = \mathbf{BC}$. Also, since $(\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{I}$, we see that $\mathbf{B}$ is invertible and $\mathbf{B}^{-1} = \mathbf{CA}$, again by the previous corollary. $\square$

Proposition

Let A be an $n \times m$ matrix.
(i) If there is an $m \times n$ matrix B such that $BA = I_{m\times m}$ then $n \leq m$.
(ii) If there is an $n \times m$ matrix C such that $AC = I_{n\times n}$ then $m \leq n$.
(iii) If there are matrices B and C such that $BA = I$, $AC = I$ then $m = n$ and $B = C$.

Proof. Indeed, if $BA = I$ and if $x \in \mathbb{R}^{n\times 1}$ is a vector such that $Ax = 0$ then $x = Ix = BAx = 0$. On the other hand if $n > m$ then there is at least one nonzero solution to $Ax = 0$. This proves (i). For (ii), note that $AC = I$ implies $I = C^T A^T$, so by (i), as A^T is of order $n \times m$, $m \leq n$. For (iii), if $BA = I = AC$ then by (i) and (ii) we know $n = m$. Again,
$B = BI = B(AC) = (BA)C = IC = C$. $\square$

Row Canonical Form (RCF)

As we have seen, a matrix **A** may not have a unique REF. However, a special REF of **A** turns out to be unique.

An $m \times n$ matrix **A** is said to be in a **row canonical form** (RCF) or a **reduced row echelon form** (RREF) if
(i) it is in a row echelon form (REF),
(ii) all pivots are equal to 1 and
(iii) in each pivotal column, all entries above the pivot are (also) equal to 0.

For example, the matrix

$$\mathbf{A} := \begin{bmatrix} 0 & \boxed{1} & * & 0 & 0 & * \\ 0 & 0 & 0 & \boxed{1} & 0 & * \\ 0 & 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in a RCF, where $*$ denotes any real number.

Note: If $\mathbf{A}$ is in REF, then in each pivotal column, all entries below the pivot are 0. If $\mathbf{A}$ is in fact in RCF and has r nonzero rows, then the $r \times r$ submatrix formed by the first r rows and the r pivotal columns is the $r \times r$ identity matrix $\mathbf{I}$.

Suppose an $m \times n$ matrix $\mathbf{A}$ is in RCF and has r nonzero rows. If $r = n$, then it has n pivotal columns, that is, all its columns are pivotal, and so $\mathbf{A} = \mathbf{I}$ if $m = n$, and $\mathbf{A} = \begin{bmatrix} \mathbf{I} \\ \\ \mathbf{O} \end{bmatrix}$ if $m > n$, where $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{O}$ is the $(m - n) \times n$ zero matrix.

To transform an $m \times n$ matrix to a RCF, we first transform it to a REF by elementary row operations of type I and II. Then we multiply a row containing a pivot p by $1/p$ (which is an elementary row operation of type III), and then we add a suitable nonzero multiple of this row to each preceding row.

Every matrix has a unique RCF. (Proof by induction on n)

Example

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 16 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} \boxed{1} & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & \boxed{-1} & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is in REF,

$$\xrightarrow{\text{EROs}} \begin{bmatrix} \boxed{1} & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & \boxed{1} & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} \boxed{1} & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is in RCF.

Recall: A square matrix $\mathbf{A}$ is invertible if and only if the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution.

Proposition

An $n \times n$ matrix is invertible if and only if it can be transformed to the $n \times n$ identity matrix by EROs.

Proof. Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ is invertible. Using EROs, transform $\mathbf{A}$ to a matrix $\mathbf{A}' \in \mathbb{R}^{n\times n}$ such that $\mathbf{A}'$ is in a RCF. Since $\mathbf{A}$ is invertible, the linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution. Hence $\mathbf{A}'$ has n nonzero rows, and so each of the n columns of $\mathbf{A}'$ is pivotal. Also, the number of rows of $\mathbf{A}$ is equal to the number of its columns, that is, $m = n$. Therefore $\mathbf{A}' = \mathbf{I}$.

Conversely, suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ can be transformed to the $n \times n$ identity matrix $\mathbf{I}$ by EROs. Since $\mathbf{Ix} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$ for $\mathbf{x} \in \mathbb{R}^{n\times 1}$, we see that the linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution. Hence $\mathbf{A}$ is invertible. $\square$