MA110 Tutorial Problems

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Contents

1 Tutorial 1

1.1 Question 2 (a)

Question: Find the general solution for the following equations.

(a) $y' + 3\,y = \cos 10x$.

We have the differential equation

$$y' + 3y = \cos 10x\,. \tag{1}$$

It is easy to see that the solution to the ODE $y' + 3y = 0$ is e^{-3x} . So on substituting $y = u(x)e^{-3x}$ into (1), we get

$$\begin{aligned} u'(x)e^{-3x} + u(x)(-3e^{-3x}) + 3u(x)e^{-3x} &= \cos 10x \\ u'(x)e^{-3x} &= \cos 10x \\ u'(x) &= e^{3x}\cos 10x \\ u(x) &= \int e^{3x}\cos 10x\,dx\,. \end{aligned}$$

Using integration by parts, it is easy to see that the above integral is

$$u(x) = \frac{e^{3x}(10\sin 10x + 3\cos 10x)}{109} + C\,.$$

Putting this back into $y = u(x)e^{-3x}$, we get the general solution of the differential equation, which is

$$\begin{aligned} y(x) &= \frac{e^{3x}(10\sin 10x + 3\cos 10x)}{109}e^{-3x} + Ce^{-3x} \\ &= \frac{10\sin 10x + 3\cos 10x}{109} + Ce^{-3x}\,. \end{aligned}$$

1.2 Question 2 (b)

Question: Find the general solution for the following equations.

(b) $y' + 2\,y = x^2\,.$

Step 1: Homogeneous Solution

The associated homogeneous equation is:

$$y' + 2y = 0.$$

Its solution is:

$$y_h = Ce^{-2x},$$

where C is an arbitrary constant.

Step 2: Particular Solution

We use variation of parameters by assuming a solution of the form:

$$y_p = u(x)e^{-2x},$$

with $u(x)$ to be determined.
Differentiate:

$$y_p' = u'(x)e^{-2x} - 2u(x)e^{-2x}.$$

Substitute y_p and y_p' into the original equation:

$$\left(u'(x)e^{-2x} - 2u(x)e^{-2x}\right) + 2\left(u(x)e^{-2x}\right) = u'(x)e^{-2x} = x^2.$$

Multiplying by e^{2x} gives:

$$u'(x) = x^2e^{2x}.$$

Integrate both sides:

$$u(x) = \int x^2e^{2x}\,dx.$$

Integration by parts: Let

$$A = x^2, \qquad dB = e^{2x}dx,$$

$$dA = 2x\,dx, \qquad B = \frac{e^{2x}}{2}.$$

Then,

$$\int x^2e^{2x}dx = \frac{x^2e^{2x}}{2} - \int xe^{2x}dx.$$

Next, integrate $\int xe^{2x}dx$ by parts:

$$C = x, \qquad dD = e^{2x}dx,$$
$$dC = dx, \qquad D = \frac{e^{2x}}{2}.$$

Thus,

$$\int xe^{2x}dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}.$$

Substitute back:

$$u(x) = \frac{x^2e^{2x}}{2} - \left(\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right) = e^{2x}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right).$$

Therefore, the particular solution is:

$$y_p = u(x)e^{-2x} = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}.$$

Step 3: General Solution

The general solution is the sum of the homogeneous and particular solutions:

$$y = y_h + y_p = Ce^{-2x} + \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}.$$

1.3 Question 4 (a)

Question: Find the general solution for the following equations.

(a) $xy' + 2y = 8x^2$.

Given the differential equation:

$$xy'(x) + 2y(x) = 8x^2 \,.$$

We can reduce it to standard form by dividing both sides by x

$$y'(x) + \frac{2}{x}y(x) = 8x \,.$$

Let us first solve the homogeneous equation:

$$\begin{aligned} y'(x) + \frac{2}{x}y(x) &= 0 \\ \frac{y'(x)}{y(x)} &= -\frac{2}{x} \\ d(\ln|y(x)|) &= d(-2\ln|x|) \,. \end{aligned}$$

Integrating both sides and using $\ln C$ as the constant of integration:

$$\ln|y(x)| = \ln\frac{1}{x^2} + \ln C \,.$$

Using the logarithm property and exponentiating both sides:

$$|y(x)| = \frac{C}{x^2} \,.$$

The modulus on $y(x)$ can be absorbed into the constant C:

$$y(x) = \frac{C}{x^2} \,.$$

Now solving the original equation by assuming $y(x) = \frac{U(x)}{x^2}$ and substituting:

$$\begin{aligned} x\left(\frac{U'(x)}{x^2} - 2\frac{U(x)}{x^3}\right) + 2\frac{U(x)}{x^2} &= 8x^2 \\ \frac{U'(x)}{x} &= 8x^2 \\ U'(x) &= 8x^3 \,. \end{aligned}$$

Integrating both sides:

$$U(x) = 2x^4 + C' \,.$$

Thus, the final solution is:

$$y(x) = \frac{2x^4 + C'}{x^2} \,.$$

1.4 Question 5 (a)

Question: Solve the following non-linear differential equation: $y' = 2y - 10y^2$. Rewriting the equation, we get $y' - 2y = -10y^2$. The equation is of the type of a Bernoulli equation. To solve it, we find a nonzero solution to $y' - 2y = 0$. By observation e^{2x} is such a solution. In the original equation, postulate $y = u(x)e^{2x}$. This gives:

$$\begin{aligned} y' - 2y &= (ue^{2x})' - 2ue^{2x} \\ &= u'e^{2x}\,. \end{aligned}$$

Thus, the original equation reduces to

$$u'e^{2x} = -10u^2e^{4x}.$$

Simplifying we obtain $u' = -10u^2e^{2x}$. Clearly $u \equiv 0$ is a solution. Otherwise dividing by u gives

$$-\frac{u'}{u^2} = 10e^{2x}$$

which has solutions $u = \dfrac{1}{5e^{2x} + C}$. Putting this all together, we get either $y \equiv 0$ or $y = \dfrac{e^{2x}}{5e^{2x} + C}$ for some $C \in \mathbb{R}$.

1.5 Question 5 (b)

Question: Solve the following non-linear differential equations: $5x^2y' - 3xy + e^xy^6 = 0$. The equation we have here is

$$5x^2\frac{dy}{dx} - 3xy + e^xy^6 = 0.$$

Rewriting our equation:

$$\frac{dy}{dx} - \frac{3}{5x}y = -\frac{e^x}{5x^2}y^6.$$

Note that the expression above matches the form of a Bernoulli equation given by

$$\frac{dy}{dx} + p(x)y = f(x)y^r.$$

Consider the differential equation

$$\begin{aligned}
\frac{dy}{dx} - \frac{3}{5x}y &= 0 \\
\frac{y'}{y} &= \frac{3}{5x} \\
\frac{d}{dx}ln(y) &= \frac{3}{5x} \\
ln(y) &= \int \frac{3}{5x}dx = \frac{3}{5}ln(x) + c \\
y &= e^{3ln(x)/5+c} = e^cx^{3/5}.
\end{aligned}$$

We look for solutions of type $y = x^{3/5}u(x)$.
Substituting into the original and simplifying we get

$$\begin{aligned}
\frac{du}{dx} &= -\frac{e^xx^{18/5}u^6(x)}{5x^{13/5}} \\
\frac{du}{dx} &= -\frac{e^x}{5}xu^6(x) \\
\int u^{-6}du &= -\frac{1}{5}\int e^xxdx \\
u^{-5} &= \int e^xxdx.
\end{aligned}$$

Using integration by parts for $\int e^xxdx$

$$\int xe^xdx = x\int e^x - \int \frac{d}{dx}(x)\int e^xdx = xe^x - e^x.$$

Thus,

$$u^{-5} = xe^x - e^x + c.$$
$$u = (xe^x - e^x + c)^{-1/5}.$$

Recall that $y = x^{3/5}u(x)$:

$$y = x^{3/5}(xe^x - e^x + c)^{-1/5}.$$

1.6 Question 7 (a)

Question: Following may not be separable but can be made separable by substitution.

(a) $y' = \dfrac{-6x + y - 3}{2x - y - 1}$

We solve it using the variable separable method with a substitution.

Step 1: Shifting Variables

We introduce the transformations:

$$x = X + \alpha, \quad y = Y + \beta$$

Substituting these into the given equation:

$$y' = \frac{-6(X + \alpha) + (Y + \beta) - 3}{2(X + \alpha) - (Y + \beta) - 1}$$

Expanding:

$$y' = \frac{(-6X + Y) + (-6\alpha + \beta - 3)}{(2X - Y) + (2\alpha - \beta - 1)}$$

We choose α and β such that unnecessary terms vanish:

$$-6\alpha + \beta - 3 = 0$$

$$2\alpha - \beta - 1 = 0$$

Solving these equations:

$$\alpha = -1 \quad \beta = -3$$

Thus, the transformed variables are:

$$x = X - 1, \quad y = Y - 3.$$

Substituting these into the equation:

$$Y' = \frac{-6X + Y}{2X - Y}.$$

In the above, note that y' has suddenly become Y'. We will not justify this step here, however, see the next page for a slightly more "conceptual" way of understanding this solution.

Step 2: Substituting $v = \frac{Y}{X}$

Setting:

$$v = \frac{Y}{X}, \quad \text{so} \quad Y = vX$$

Differentiating:

$$y' = v + X\frac{dv}{dX}.$$

Using this substitution in our equation:

$$v + X\frac{dv}{dX} = \frac{-6 + v}{2 - v}$$

Rearranging:

$$X\frac{dv}{dX} = \frac{v^2 - v - 6}{2 - v}$$

Rewriting:

$$\frac{2 - v}{(v-3)(v+2)}dv = \frac{dX}{X}$$

Step 3: Partial Fraction Decomposition

We express:

$$\frac{2 - v}{(v-3)(v+2)} = \frac{A}{v-3} + \frac{B}{v+2}$$

Multiplying both sides by $(v-3)(v+2)$:

$$A(v+2) + B(v-3) = 2 - v$$

Setting $v = 3$:

$$5A = -1 \Rightarrow A = -\frac{1}{5}$$

Setting $v = -2$:

$$-5B = 4 \Rightarrow B = -\frac{4}{5}$$

Thus:

$$\int \left(-\frac{1}{5}\frac{dv}{v-3} - \frac{4}{5}\frac{dv}{v+2}\right) = \int \frac{dX}{X}$$

Step 4: Integration

Integrating both sides:

$$-\frac{1}{5}\ln|v-3| - \frac{4}{5}\ln|v+2| = \ln|X| - C$$

where $C \in \mathbb{R}$. Exponentiating:

$$|X| = |v-3|^{-1/5}|v+2|^{-4/5}e^C$$

Letting $C' = e^C \in \mathbb{R}^+$

$$X|v-3|^{1/5}|v+2|^{4/5} = C'$$

Substituting $v = \frac{Y}{X} = \frac{y+3}{x+1}$ (again, conceptually this is not very clear, although it works):

$$(x+1)|(y+3) - 3(x+1)|^{1/5}|(y+3) + 2(x+1)|^{4/5} = C'$$

Simplifying:

$$(x+1)|y-3x|^{1/5}|y+2x+5|^{4/5} = C'$$

Raising both sides to the power of 5:

$$(y-3x)(y+2x+5)^4 = C'^5$$

is the solution.

1.7 Question 7 (a) (slightly more conceptual deduction)

Question: Following may not be separable but can be made separable by substitution.

(a) $y' = \dfrac{-6x + y - 3}{2x - y - 1}$

We solve it using the variable separable method with a substitution.

Step 1: Shifting Variables

We introduce two functions X and Y of x by:

$$X = x - \alpha, \quad Y(x) = y(x + \alpha) - \beta$$

Substituting these into the given equation:

$$\frac{dy}{dx}(x) = \frac{dY}{dx}(x - \alpha) = \frac{-6(X + \alpha) + (Y(x - \alpha) + \beta) - 3}{2(X + \alpha) - (Y(x - \alpha) + \beta) - 1}$$

Expanding:

$$\frac{dy}{dx}(x) = \frac{dY}{dx}(x - \alpha) = \frac{(-6X + Y(X)) + (-6\alpha + \beta - 3)}{(2X - Y(X)) + (2\alpha - \beta - 1)}$$

We choose α and β such that unnecessary terms vanish:

$$-6\alpha + \beta - 3 = 0$$

$$2\alpha - \beta - 1 = 0$$

Solving these equations:

$$\alpha = -1 \quad \beta = -3$$

Thus, the necessary transformations are:

$$X = x + 1, \quad Y(x) = y(x - 1) + 3$$

Substituting these into the equation:

$$\frac{dy}{dx}(x) = \frac{dY}{dx}(x + 1) = \frac{-6X + Y}{2X - Y}.$$

An important point to note here is that if we "change coordinates" from x to $X = x + 1$, then $\frac{dY}{dx}(x+1) = \frac{dY}{dX}(X)$. We will not explain or justify this statement. However, this statement enables us to say that the above equation is equivalent to:

$$\frac{dY}{dX}(X) = Y'(X) = \frac{-6X + Y}{2X - Y}.$$

Step 2: Substituting $v = \frac{Y}{X}$

Setting:

$$v = \frac{Y}{X}, \quad \text{so} \quad Y = vX$$

Differentiating:

$$y' = v + X\frac{dv}{dX}.$$

Using this substitution in our equation:

$$v + X\frac{dv}{dX} = \frac{-6+v}{2-v}$$

Rearranging:

$$X\frac{dv}{dX} = \frac{v^2-v-6}{2-v}$$

Rewriting:

$$\frac{2-v}{(v-3)(v+2)}dv = \frac{dX}{X}$$

Step 3: Partial Fraction Decomposition

We express:

$$\frac{2-v}{(v-3)(v+2)} = \frac{A}{v-3} + \frac{B}{v+2}$$

Multiplying both sides by $(v-3)(v+2)$:

$$A(v+2) + B(v-3) = 2-v$$

Setting $v = 3$:

$$5A = -1 \Rightarrow A = -\frac{1}{5}$$

Setting $v = -2$:

$$-5B = 4 \Rightarrow B = -\frac{4}{5}$$

Thus:

$$\int \left(-\frac{1}{5}\frac{dv}{v-3} - \frac{4}{5}\frac{dv}{v+2} \right) = \int \frac{dX}{X}$$

Step 4: Integration

Integrating both sides:

$$-\frac{1}{5}\ln|v-3|-\frac{4}{5}\ln|v+2|=\ln|X|-C$$

where $C \in \mathbb{R}$. Exponentiating:

$$|X|=|v-3|^{-1/5}|v+2|^{-4/5}e^{C}$$

Letting $C' = e^C \in \mathbb{R}^+$

$$X|v-3|^{1/5}|v+2|^{4/5}=C'$$

Raising both sides to the fifth power we get

$$(Y(X)-3X)(Y(X)+2X)^4=C\,.$$

But note that $Y(X)=Y(x+1)=y(x)+3$. Putting this into the above we get

$$(y(x)+3-3x-3)(y(x)+3+2x+2)^4=C\,,$$

that is,

$$(y(x)-3x)(y(x)+2x+5)^4=C'^5$$

is the solution. To be completely sure, we can different this and check that the derivative satisfies the differential equation we started with.

1.8 Question 13 (a)

Question: In each of following problems solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

(a) $y' + y^3 = 0,\ y(0) = y_0$

Given the initial value problem:

$$\frac{dy}{dx} + y^3 = 0, \quad y(0) = y_0,$$

we can rewrite it as:

$$\frac{dy}{dx} = -y^3.$$

If $y_0 = 0$, then $y = 0$ satisfies $\frac{dy}{dx} = -y^3$ since both sides equal zero. Thus, $y(x) = 0$ is a valid solution for all $x \in \mathbb{R}$ (trivial solution).

For $y \neq 0$, we separate variables:

$$\frac{dy}{-y^3} = dx.$$

$$-\int y^{-3} dy = \int dx.$$

$$\int y^{-3} dy = \frac{y^{-2}}{-2} = -\frac{1}{2y^2}.$$

$$-\left(-\frac{1}{2y^2}\right) = x + C.$$

$$\frac{1}{2y^2} = x + C.$$

Solving for y:

$$y^2 = \frac{1}{2(x+C)}.$$

$$y = \pm\frac{1}{\sqrt{2(x+C)}}.$$

Using $y(0) = y_0$:

$$y_0 = \pm\frac{1}{\sqrt{2C}}.$$

$$C = \frac{1}{2y_0^2}.$$

Final Solution

$$y(x) = \pm\frac{1}{\sqrt{2(x+\frac{1}{2y_0^2})}}.$$

For the solution to be real:

$$\frac{1}{y_0^2} + 2x > 0.$$

Thus, the solution is valid for:

$$x > -\frac{1}{2y_0^2}.$$

Hence, the solution of the differential equation is:

- If $y_0 = 0$, then the solution is $y(x) = 0$ for all $x \in \mathbb{R}$.
- If $y_0 \neq 0$, the solution is:

  $$y(x) = \pm\frac{1}{\sqrt{\frac{1}{y_0^2} + 2x}}.$$

  for all $x \in \left(-\frac{1}{2y_0^2}, \infty\right)$.

2 Tutorial 2

2.1 Question 1 (b)

Question: Determine if the following equations are exact and solve them.

(b) $\left(\dfrac{1}{x} + 2x\right) + \left(\dfrac{1}{y} + 2y\right)\frac{dy}{dx} = 0.$

Check for Exactness

The given equation is of the form:

$$M(x,y)dx + N(x,y)dy = 0$$

where

$$M(x,y) = \frac{1}{x} + 2x, \quad N(x,y) = \frac{1}{y} + 2y.$$

To check if the equation is exact, compute the partial derivatives:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\text{M(x,y)}\right) = 0,$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\text{N(x,y)}\right) = 0.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Find the Function $G(x,y)$

Since the equation is exact, there exists a function $G(x,y)$ such that:

$$\frac{\partial G}{\partial x} = M = \frac{1}{x} + 2x,$$

$$\frac{\partial G}{\partial y} = N = \frac{1}{y} + 2y.$$

Integrating $M(x,y)$ with Respect to x

$$G(x,y) = \int M(x,y)dx = \log|x| + x^2 + g(y),$$

where $g(y)$ is a function of y.

Differentiate $G(x, y)$ with Respect to y

$$\frac{\partial G}{\partial y} = g'(y).$$

Setting this equal to $N = \frac{1}{y} + 2y$, we get:

$$g'(y) = \frac{1}{y} + 2y$$

Solving for $g(y)$:

$$g(y) = \log|y| + y^2 + C.$$

Final Solution

Thus, the function $G(x, y)$ is:

$$G(x, y) = \log|x| + x^2 + \log|y| + y^2 + C.$$

And hence the solution to the differential equation is $e^{x^2+y^2}xy = C'$.

2.2 Question 2 (c)

Question: Solve the following IVP.

(c) $(9x^2 + y - 1) - (4y - x)\frac{dy}{dx} = 0,\ y(1) = 0.$

The given equation is of the form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

where

$$M(x, y) = 9x^2 + y - 1, \quad N(x, y) = -(4y - x).$$

To check if the equation is exact, compute the partial derivatives:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(9x^2 + y - 1) = 1,$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-4y + x) = 1.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Find the Function $G(x, y)$

Since the equation is exact, there exists a function $G(x, y)$ such that:

$$\frac{\partial G}{\partial x} = M = 9x^2 + y - 1,$$

$$\frac{\partial G}{\partial y} = N = -(4y - x).$$

Integrating $M(x, y)$ with Respect to x

$$G(x, y) = \int (9x^2 + y - 1)dx = 3x^3 + xy - x + g(y).$$

where $g(y)$ is a function of y.

Differentiate $G(x, y)$ with Respect to y

$$\frac{\partial G}{\partial y} = x + g'(y).$$

Setting this equal to $N = -(4y - x)$, we get:

$$x + g'(y) = -4y + x.$$

$$g'(y) = -4y.$$

Integrating for $g(y)$

$$g(y) = -2y^2 + C.$$

Thus, the function $G(x, y)$ is:

$$G(x, y) = 3x^3 + xy - x - 2y^2 + C.$$

Solve for the Initial Condition

Since the differential equation is exact, the implicit solution is:

$$3x^3 + xy - x - 2y^2 = C.$$

Using the initial condition $y(1) = 0$:

$$3(1)^3 + (1)(0) - 1 - 2(0)^2 = C.$$

$$3 - 1 = C \Rightarrow C = 2.$$

Thus, the solution is:

$$3x^3 + xy - x - 2y^2 = 2.$$

2.3 Question 5

Question: Suppose M and N are continuous and have continuous partial derivatives M_y and N_x that satisfy the exactness condition $M_y = N_x$ on an open rectangle R around (x_0, y_0). Show that if (x, y) is in R and

$$F(x,y) = \int_{x_0}^{x} M(s, y_0)ds + \int_{y_0}^{y} N(x,t)dt,$$

then $F_x = M$ and $F_y = N$.

Leibniz Rule: Let F be a function defined by

$$F(x) = \int_{a_1(x)}^{a_2(x)} f(x,t)\,dt.$$

Assume that $a_1(x)$ and $a_2(x)$ and their derivatives are continuous for $a \leq x \leq b$. Further, $f(x,t)$ and $\frac{\partial}{\partial x} f(x,t)$ are continuous (in both t and x) in some open rectangle containing $a \leq x \leq b$ and $a_1(x) \leq t \leq a_2(x)$. Then, for $a \leq x \leq b$,

$$\frac{d}{dx}F(x) = f(x, a_2(x))a_2'(x) - f(x, a_1(x))a_1'(x) + \int_{a_1(x)}^{a_2(x)} \frac{\partial}{\partial x} f(x,t)\,dt.$$

Solution: We apply the Leibniz rule given above to solve this. Note first that for any y_1 such that $(x, y_1) \in R$,

$$\frac{d}{dx}F(x,y_1) = \frac{d}{dx}\int_{x_0}^{x} M(s,y_0)ds + \frac{d}{dx}\int_{y_0}^{y_1} N(x,t)dt. \tag{2}$$

By Leibniz rule, the first term on the right hand side of (2) leads us to

$$\frac{d}{dx}\int_{x_0}^{x} M(s,y_0)ds = M(x,y_0) - 0 + \int_{x_0}^{x} \frac{d}{dx}M(s,y_0)ds = M(x,y_0). \tag{3}$$

Since N_x is continuous, by applying the Leibniz rule and using $M_y = N_x$ on R, the second term on the right hand side of (2) leads to

$$\frac{d}{dx}\int_{y_0}^{y_1} N(x,t)dt = \int_{y_0}^{y_1} \frac{\partial}{\partial x}N(x,t)dt = \int_{y_0}^{y_1} \frac{\partial}{\partial y}M(x,t)dt = M(x,y_1) - M(x,y_0). \tag{4}$$

Here, the last equality is valid by the Fundamental theorem of calculus since M_y is continuous. Putting the expressions of (3) and (4) into (2), we get that

$$F_x(x,y_1) = \frac{d}{dx}F(x,y_1) = M(x,y_1), \quad (x,y_1) \in R,$$

and hence, $F_x = M$ on R.

Following a similar approach, we now verify the remaining part of the proof. Note that for any x_1 with $(x_1, y) \in R$,

$$\frac{d}{dy}F(x_1, y) = \frac{d}{dy}\int_{x_0}^{x_1} M(s, y_0)ds + \frac{d}{dy}\int_{y_0}^{y} N(x_1, t)dt. \tag{5}$$

Since M_y is continuous, by Leibniz rule, the first term on the right hand side of (5) reduces to

$$\frac{d}{dy}\int_{x_0}^{x_1} M(s, y_0)ds = \int_{x_0}^{x_1} \frac{d}{dy}M(s, y_0)ds = 0. \tag{6}$$

Similarly, by applying the Leibniz rule, the second term on the right hand side of (5) reduces to

$$\frac{d}{dy}\int_{y_0}^{y} N(x_1, t)dt = N(x_1, y) - 0 + \int_{y_0}^{y} \frac{\partial}{\partial y}N(x_1, t)dt = N(x_1, y), \tag{7}$$

since for any t with $(x_1, t) \in R$, the function $N(x_1, t)$ is independent of y in the first variable. Putting the expressions of (6) and (7) into (5), we get that

$$F_y(x_1, y) = \frac{d}{dy}F(x_1, y) = N(x_1, y), \quad (x_1, y) \in R,$$

and hence, $F_y = N$ on R. This completes the proof.

2.4 Question 8 (a)

Question: Solve the following after finding an integrating factor.

(a) $(27xy^2 + 8y^3) + (18x^2y + 12xy^2)\frac{dy}{dx} = 0$.

Let us define $M(x, y) := 27xy^2 + 8y^3$ and $N(x, y) := 18x^2y + 12xy^2$.

We check if the differential equation is exact, that is, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$$\frac{\partial M}{\partial y} = 54xy + 24y^2, \quad \frac{\partial N}{\partial x} = 36xy + 12y^2.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

In order to make the differential equation exact, we hope to find a function $\mu(x)$ such that

$$\mu(x)M(x, y) + \mu(x)N(x, y)\,\frac{dy}{dx} = 0$$

is exact. In order for this to be the case,

$$\mu \cdot \frac{\partial M}{\partial y} = \mu \cdot \frac{\partial N}{\partial x} + \mu' \cdot N$$

Note that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{18xy + 12y^2}{18x^2y + 12xy^2} = \frac{1}{x}.$$

Thus, we are in case 1 and we obtain $\mu(x) = x$. The exact ODE is thus

$$(27x^2y^2 + 8xy^3)\,dx + (18x^3y + 12x^2y^2)\,dy = 0.$$

Thus, there exists a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = 27x^2y^2 + 8xy^3, \quad \frac{\partial \psi}{\partial y} = 18x^3y + 12x^2y^2$$

On solving the first partial differential equation,

$$\psi(x, y) = 9x^3y^2 + 4x^2y^3 + h(y).$$

On differentiating the obtained $\psi(x, y)$ with respect to y,

$$\frac{\partial \psi}{\partial y} = 18x^3y + 12x^2y^2 + h'(y).$$

Setting this equal to $18x^3y + 12x^2y^2$ gives $h'(y) = 0$, so $h(y) = C, C \in \mathbb{R}$.
The solution to the differential equation is thus

$$\boxed{x^2y^2(9x + 4y) = C}$$

2.5 Question 8 (b)

Question: Solve the following after finding an integrating factor.

(b) $-y + (x^4 - x)\frac{dy}{dx} = 0.$

This equation can be written as:

$$(x^4 - x)y' - y = 0$$

Comparing with $M(x, y) + N(x, y)y'$
we get $M(x, y) = -y$ and $N(x, y) = (x^4 - x)$
$M_y = -1$ and $N_x = 4x^3 - 1$

$$\frac{M_y - N_x}{N} = \frac{-4x^3}{x^4 - x} = p(x)$$

This means the integrating factor (μ) is independent of y:

$$\begin{aligned} \mu &= e^{\int p(x)\,dx} \\ &= e^{\int \frac{-4x^3\,dx}{x^4 - x}} \\ &= e^{\int \frac{-4x^2\,dx}{x^3 - 1}} \end{aligned}$$

To solve the integral, substitute $v = x^3 - 1$, thus $dv = (3x^2)dx$.

$$\begin{aligned} \mu &= e^{\int \frac{-4dv}{3v}} \\ &= e^{-\frac{4}{3}\ln|v|} \\ &= |v|^{-\frac{4}{3}} \\ &= |x^3 - 1|^{-\frac{4}{3}} \end{aligned}$$

Multiplying by the integrating factor

$$\begin{aligned} |x^3 - 1|^{-\frac{4}{3}}((x^4 - x)y' - y) &= 0 \\ (x^3 - 1)^{-\frac{4}{3}}((x^4 - x)y' - y) &= 0 \\ x(x^3 - 1)^{-\frac{1}{3}}y' - (x^3 - 1)^{-\frac{4}{3}}y &= 0 \end{aligned}$$

This equation is now exact, so there exists a $\phi(x, y) = 0$ which a solution such that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= -(x^3 - 1)^{-\frac{4}{3}}y \\ \frac{\partial \phi}{\partial y} &= x(x^3 - 1)^{-\frac{1}{3}} \end{aligned}$$

Integrating the second equation,

$$\phi(x,y) = x(x^3-1)^{-\frac{1}{3}}y + c(x)$$

Differenciating w.r.t x

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= y((x^3-1)^{-\frac{1}{3}} - \frac{1}{3}3x^3(x^3-1)^{-\frac{4}{3}}) + c'(x) \\
-(x^3-1)^{-\frac{4}{3}}y &= y((x^3-1)^{-\frac{1}{3}} - x^3(x^3-1)^{-\frac{4}{3}}) + c'(x) \\
-(x^3-1)^{-\frac{4}{3}}y &= y(x^3-1)^{-\frac{4}{3}}(x^3-1-x^3) + c'(x) \\
-(x^3-1)^{-\frac{4}{3}}y &= -y(x^3-1)^{-\frac{4}{3}} + c'(x) \\
c'(x) &= 0 \\
c(x) &= k
\end{aligned}$$

Therefore the solution is

$$\phi(x,y) = x(x^3-1)^{-\frac{1}{3}}y + k = 0$$

2.6 Question 8 (c)

Question: Solve the following after finding an integrating factor.

(c) $y \sin y + x(\sin y - y \cos y)\frac{dy}{dx} = 0.$

Compare the given equation with the standard form $M(x,y)dx + N(x,y)dy = 0$:

$$M(x,y) = y \sin y,$$
$$N(x,y) = x(\sin y - y \cos y).$$

Step 1: Check Exactness

Compute partial derivatives:

$$\frac{\partial M}{\partial y} = \sin y + y \cos y,$$
$$\frac{\partial N}{\partial x} = \sin y - y \cos y.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

Step 2: Find Integrating Factor

To find the integrating factor, check if either $\frac{M_y - N_x}{N}$ or $\frac{N_x - M_y}{M}$ is a function of only x or y.
First, compute $\frac{M_y - N_x}{N}$:

$$\frac{M_y - N_x}{N} = \frac{(\sin y + y \cos y) - (\sin y - y \cos y)}{x(\sin y - y \cos y)} = \frac{2y \cos y}{x(\sin y - y \cos y)}.$$

This is not a function of x alone, as it depends on both x and y.
Next, compute $\frac{N_x - M_y}{M}$:

$$\frac{N_x - M_y}{M} = \frac{(\sin y - y \cos y) - (\sin y + y \cos y)}{y \sin y} = \frac{-2y \cos y}{y \sin y} = -2 \cot y.$$

This is a function of y alone. Therefore, the integrating factor $\mu(y)$ can be found as follows:

Derivation of $\mu(y)$

For the equation to become exact after multiplying by $\mu(y)$, the following condition must hold:

$$(\mu M)_y = (\mu N)_x.$$

Expanding this:

$$\mu_y M + \mu M_y = \mu N_x.$$

Rearranging:

$$\mu_y M = \mu N_x - \mu M_y.$$

Divide through by μM:

$$\frac{\mu_y}{\mu} = \frac{N_x - M_y}{M}.$$

From earlier, we know:

$$\frac{N_x - M_y}{M} = -2\cot y.$$

Thus:

$$\frac{\mu_y}{\mu} = -2\cot y.$$

Integrate both sides with respect to y:

$$\ln|\mu| = -2\ln|\sin y| + C.$$

Exponentiating both sides:

$$\mu(y) = \exp(-2\ln|\sin y|) = \frac{1}{\sin^2 y} = \csc^2 y.$$

Step 3: Multiply by $\mu(y)$

Multiply the original equation by $\mu(y) = \csc^2 y$:

$$\underbrace{y\csc y}_{\mu M}\,dx + \underbrace{x(\csc y - y\csc y\cot y)}_{\mu N}\,dy = 0.$$

Step 4: Verify Exactness

Compute the new partial derivatives:

$$\frac{\partial}{\partial y}(y\csc y) = \csc y - y\csc y\cot y,$$

$$\frac{\partial}{\partial x}(x(\csc y - y\csc y\cot y)) = \csc y - y\csc y\cot y.$$

Since $\frac{\partial}{\partial y}(y\csc y) = \frac{\partial}{\partial x}(x(\csc y - y\csc y\cot y))$, the equation is now exact.

Step 5: Find Potential Function $F(x, y)$

Integrate μM with respect to x treating y as constant:

$$F(x, y) = \int y \csc y \, dx = xy \csc y + g(y),$$

where $g(y)$ is a function of y only.
Differentiate $F(x, y)$ with respect to y and equate to μN:

$$\frac{\partial F}{\partial y} = x \csc y - xy \csc y \cot y + g'(y) = x \csc y - xy \csc y \cot y.$$

This implies $g'(y) = 0$, so $g(y) = C_1$ (constant).

Step 6: General Solution

The potential function is:

$$F(x, y) = xy \csc y + C_1 = \text{constant}.$$

Thus, the general solution is:

$$xy \csc y = C \quad \Longrightarrow \quad \boxed{xy = C \sin y}.$$

2.7 Question 10

Question: Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation $M + Ny' = 0$ has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

Consider the equation

$$M(x,y) + N(x,y)y' = 0$$

This equation becomes exact on the multiplication of an integration factor $\mu(x,y)$ iff

$$\mu(M_y - N_x) = \frac{\partial \mu}{\partial x}N - \frac{\partial \mu}{\partial y}M$$

Since we are given

$$\frac{N_x - M_y}{xM - yN} = R,$$

the condition becomes

$$\mu R(yN - xM) = \frac{\partial \mu}{\partial x}N - \frac{\partial \mu}{\partial y}M$$

$$N\left(\mu R y - \frac{\partial \mu}{\partial x}\right) + M\left(-\mu R x + \frac{\partial \mu}{\partial y}\right) = 0$$

Notice the terms being multplied by N and M. If we choose our integration factor μ such that they both individually become zero, the condition holds and the equation becomes exact.
Thus, we try to solve the two equations

$$\begin{aligned} \mu R y - \frac{\partial \mu}{\partial x} &= 0 \\ -\mu R x + \frac{\partial \mu}{\partial y} &= 0\,. \end{aligned}$$

simultaneously to get μ.
We are also given that there is a function $R(t)$ of one variable t, such that $(N_x - M_y)/(xM - yN) = R(xy)$, so we should try and use this. We may rewrite the first equation above as

$$\frac{\partial \mu}{\partial x} = \mu R y\,.$$

Let $Q(t)$ be the anti-derivative of $R(t)$. Then we observe that the partial derivative of $e^{(Q(xy))}$ with respect to x is

$$\frac{\partial e^{(Q(xy))}}{\partial x} = e^{(Q(xy))} R y\,.$$

Similarly, the partial derivative of $e^{(Q(xy))}$ with respect to y is

$$\frac{\partial e^{(Q(xy))}}{\partial y} = e^{(Q(xy))} Rx\,.$$

Thus, it follows that if we take $\mu = e^{Q(xy)}$ then μ will be an integrating factor.

2.8 Question 13 (a)

Question: Apply the Picard's iteration method to the following initial value problems and get four iterations:

(a) $y' = x + y$, $y(0) = 0$

The corresponding integral equation is

$$\phi(x) = \int_0^x (s + \phi(s))ds$$

Let $\phi_0(x) = 0$, then

$$\phi_1(x) = \int_0^x (s + \phi_0(s))ds$$

$$\phi_1(x) = \int_0^x (s + 0)ds$$

$$\phi_1(x) = \frac{x^2}{2}$$

$$\phi_2(x) = \int_0^x (s + \phi_1(s))ds$$

$$\phi_2(x) = \int_0^x (s + \frac{s^2}{2})ds$$

$$\phi_2(x) = \frac{x^2}{2} + \frac{x^3}{6}$$

$$\phi_3(x) = \int_0^x (s + \phi_2(s))ds$$

$$\phi_3(x) = \int_0^x (s + \frac{s^2}{2} + \frac{s^3}{6})ds$$

$$\phi_3(x) = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$\phi_4(x) = \int_0^x (s + \phi_3(s))ds$$

$$\phi_4(x) = \int_0^x \left(s + \frac{s^2}{2} + \frac{s^3}{6} + \frac{s^4}{24} \right) ds$$

$$\phi_4(x) = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

3 Tutorial 3

3.1 Question 1 (a)

Question: Find the general solution of $y'' - 2y' + 2y = 0$. Solve it with initial conditions

(a) $y(0) = 3, y'(0) = -2$

The given differential equation is a second-order equation with constant coefficients.

The characteristic equation is:

$$m^2 - 2m + 2 = 0$$

Solving for m:

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

The general solution is:

$$y = e^x (C_1 \cos x + C_2 \sin x)$$

Using initial conditions:

$$y(0) = e^0 (C_1 \cos 0 + C_2 \sin 0) = C_1$$

$$C_1 = 3$$

Differentiating:

$$y' = e^x (C_1 \cos x + C_2 \sin x) + e^x(-C_1 \sin x + C_2 \cos x)$$

$$y' = e^x [(C_1 + C_2) \cos x + (C_2 - C_1) \sin x]$$

At $x = 0$:

$$y'(0) = (C_1 + C_2) e^0 = C_1 + C_2$$

$$-2 = 3 + C_2$$

$$C_2 = -5$$

Thus, the final solution is:

$$y(x) = e^x (3 \cos x - 5 \sin x)$$

3.2 Question 4

Question: Find the Wronskian of a given set of solutions of $(1-x^2)y'' - 2xy' + a(a+1)y = 0$, given that $W(0) = 1$.

Rewriting the equation:

$$y'' - \frac{2x}{1-x^2}y' + \frac{a(a+1)}{1-x^2}y = 0.$$

Integrating :

$$\int_0^x \frac{2s}{1-s^2}ds.$$

Substituting $u = 1 - s^2$, so that $du = -2sds$, we get:

$$\int_0^x \frac{2s}{1-s^2}ds = \int_0^x -\frac{1}{2}d\ln(1-s^2) = -\frac{1}{2}\ln(1-x^2).$$

Thus,

$$W(f,g,x) = W(f,g,0)e^{-\frac{1}{2}\ln(1-x^2)} = e^{\ln((1-x^2)^{-1/2})}.$$

Therefore,

$$W(f,g,x) = (1-x^2)^{-1/2}.$$

3.3 Question 6 (b)

Question: Given one solution y_1, find other solution y_2 s.t. $\{y_1, y_2\}$ is linearly independent set.

(b) $x^2y'' - xy' + y = 0$; $y_1 = x$

We are given that one solution is $y_1 = x$. Using the method of variation of parameters, the second solution y_2 can be found using the formula:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx,$$

where the differential equation is written in standard form:

$$y'' + P(x)y' + Q(x)y = 0.$$

The given equation is:

$$x^2y'' - xy' + y = 0.$$

Dividing through by x^2 (for $x > 0$):

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0.$$

This gives us the standard form, with $P(x) = -\frac{1}{x}$ and $Q(x) = \frac{1}{x^2}$.

Now we simplify $e^{-\int P(x)dx}$

Here, $P(x) = -\frac{1}{x}$. Compute $-\int P(x)dx$:

$$-\int P(x)dx = -\int \left(-\frac{1}{x}\right) dx = \int \frac{1}{x} dx = \ln|x|.$$

$$e^{-\int P(x)dx} = e^{\ln|x|} = x.$$

Now substituting $e^{-\int P(x)dx}$ and $P(x)$ into the formula for y_2:

$$y_2 = x \int \frac{x}{x^2} dx = x \int \frac{1}{x} dx.$$

$$\int \frac{1}{x} dx = \ln|x| + C,$$

where C is the constant of integration. Thus:

$$y_2 = x(\ln|x| + C).$$

To ensure linear independence, we take $C = 0$ (since $y_1 = x$ is already a solution):

$$y_2 = x \ln |x|.$$

The two linearly independent solutions are:

$$y_1 = x, \quad y_2 = x \ln |x|.$$

The general solution is:

$$y(x) = C_1 x + C_2 x \ln |x|,$$

where $C_1, C_2 \in \mathbb{R}$.

3.4 Question 7

Question: Suppose p_1, p_2, q_1, q_2 are continuous on (a, b) and the equations $y''+p_1(x)y'+q_1(x)y = 0$ and $y'' + p_2(x)y' + q_2(x)y = 0$ have the same solutions on (a, b). Show that $p_1 = p_2$ and $q_1 = q_2$ on (a, b). [Hint. Use Abel's formula.]

Given are two Linear Homogeneous 2^{nd} Order ODE with the same set of solutions on (a, b)

$$y'' + p_1(x)y' + q_1(x)y = 0 \ (1) \quad y'' + p_2(x)y' + q_2(x)y = 0 \ (2)$$

Using the Dimension Theorem, there exist two linearly independent solutions $y_1(x), y_2(x), x \in (a, b)$ for both the equations.

Part 1: $p_1 = p_2$
Consider the Wronskian of y_1, y_2, $W(y_1, y_2; x)$. By Abel's theorem on (1), we have:

$$W(y_1, y_2; x) = W(y_1, y_2; x_0) \cdot \exp\left(-\int_{x_0}^{x} p_1(t)dt\right)$$

for some $x_0 \in (a, b)$.

Because the Wronskian only depends on y_1, y_2, it is the same for (1)&(2).

Because y_1, y_2 are linearly independent the Wronskian is non-zero always.

$$\frac{W(y_1, y_2; x)}{W(y_1, y_2; x_0)} = \exp\left(-\int_{x_0}^{x} p_1(t)dt\right) = \exp\left(-\int_{x_0}^{x} p_2(t)dt\right) \ \forall\, x \in (a, b)$$

Removing the exp and differentiating both sides we get,

$$p_1(x) = p_2(x) \forall x \in (a, b)$$

Part 2: $q_1 = q_2$
Proof by Contradiction: WLOG, Let $\exists\, z \in (a, b)$ for which $q_1(z) - q_2(z) > 0$. Because of the continuity of q_1, q_2 there is an open interval J, $z \in J \subseteq (a, b)$ in which $q_1 - q_2$ is greater than 0.

Consider any $\tilde{y} \in \{y_1, y_2\}$, it is a solution of $(1) - (2)$ because it is a solution of (1) and (2).

$$(1) - (2) : (q_1(x) - q_2(x))\tilde{y}(x) = 0$$

but, because $q_1 - q_2$ is non-zero in J,

$$\tilde{y} = 0 \text{ in } J$$

and as J is open, we have,

$$\tilde{y}(x) = \tilde{y}'(x) = 0 \; \forall x \in J \subseteq (a, b)$$

So, putting $\tilde{y} = y_1$ or y_2,

$$y_1 = y_1' = y_2 = y_2' = 0 \text{ in } J$$

Therefore the Wronskian $W(y_1, y_2; x)$ is 0 in J. But Wronskian is always non-zero as y_1, y_2 are linearly independent.

Contradiction

$$\implies q_1 = q_2 \text{ in } (a, b)$$

3.5 Question 11 (a)

Question: Find the general solution of

(a) $x^2y'' - 3xy' + 3y = x$

Step 1: We first find two solutions to the homogeneous equation:

$$x^2y'' - 3xy' + 3y = 0\,.$$

Recall that this is the Cauchy-Euler equation. Note that the corresponding characteristic equation is

$$m^2 - 4m + 3 = (m-1)(m-3) = 0\,.$$

It follows that the two solutions to the homogeneous part are $y_1 = x$ and $y_2 = x^3$.

Step 2: Next we compute the Wronskian of the two solutions. The Wronskian $W(y_1, y_2)$ is:

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} \\ &= 2x^3 \end{aligned}$$

According to the variation of parameters method, the particular solution is: (**WARNING!! Make sure you convert the equation to standard form, and take $r(x)$ from the standard form, or else, you may get a wrong answer. In particular, you may check in this example, that if you take $r(x) = x$ then you will get the wrong answer.**)

$$y_p = -y_1 \int \frac{y_2 \cdot \frac{1}{x}}{W} dx + y_2 \int \frac{y_1 \cdot \frac{1}{x}}{W} dx$$

Computing the integrals:

$$\int \frac{y_2 \cdot \frac{1}{x}}{W} dt = \int \frac{x^3 \cdot \frac{1}{x}}{2x^3} dx = \frac{\ln x}{2}$$

$$\int \frac{y_1 \cdot \frac{1}{x}}{W} dt = \int \frac{x \cdot \frac{1}{x}}{2x^3} dt = \frac{-x^{-2}}{4}$$

Therefore, our particular solution is:

$$y_p(t) = -\frac{x\ln x}{2} - \frac{x}{4}$$

The general solution is the sum of the homogeneous and particular solutions:

$$y(x) = C_1 x + C_2 x^3 - \frac{1}{2} x \ln x$$

where C_1 and C_2 are arbitrary constants.

3.6 Question 11 (b)

Question: Find the general solution of

(b) $y'' - 3y' + 2y = 1/(1+e^{-x})$

Step 1: Solve the Homogeneous Equation
Consider the homogeneous part:

$$y'' - 3y' + 2y = 0$$

The characteristic equation is:

$$r^2 - 3r + 2 = 0$$

Factoring gives:

$$(r-2)(r-1) = 0 \quad \Rightarrow \quad r = 2, 1$$

Hence, the general solution to the homogeneous equation is:

$$y_h(x) = c_1 e^{2x} + c_2 e^{x}$$

Step 2: Compute the Wronskian
Let $y_1 = e^{2x}$ and $y_2 = e^x$, then the Wronskian is:

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \\ &= e^{2x} \cdot e^x - e^x \cdot 2e^{2x} \\ &= e^{3x} - 2e^{3x} \\ &= -e^{3x} \end{aligned}$$

Step 3: Variation of Parameters
The particular solution is:

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W} dx + y_2(x) \int \frac{y_1(x) f(x)}{W} dx$$

where $f(x) = \frac{1}{1+e^{-x}}$.
Substituting:

$$\begin{aligned} y_p(x) &= -e^{2x} \int \frac{e^x \cdot \frac{1}{1+e^{-x}}}{-e^{3x}} dx + e^x \int \frac{e^{2x} \cdot \frac{1}{1+e^{-x}}}{-e^{3x}} dx \\ &= e^{2x} \int \frac{1}{e^{2x}(1+e^{-x})} dx - e^x \int \frac{1}{e^x(1+e^{-x})} dx \end{aligned}$$

Make substitution $u = e^x$, $du = e^x dx$.

Integral 1:

$$\int \frac{1}{u^2(1+\frac{1}{u})} \cdot \frac{du}{u} = \int \frac{1}{u^3} \cdot \frac{u}{u+1} du = \int \frac{1}{u^2(u+1)} du$$

Integral 2:

$$\int \frac{1}{u(1+\frac{1}{u})} \cdot \frac{du}{u} = \int \frac{1}{u^2} \cdot \frac{u}{u+1} du = \int \frac{1}{u(u+1)} du$$

Solve Integral 2 using Partial Fractions:

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

Solving gives $A = 1$, $B = -1$, so:

$$\int \frac{1}{u(u+1)} du = \ln|u| - \ln|u+1| + C = \ln\left|\frac{u}{u+1}\right| + C$$

Substitute back $u = e^x$:

$$\ln\left|\frac{e^x}{e^x+1}\right| = \ln\left|\frac{1}{1+e^{-x}}\right|$$

Solve Integral 1: Using partial fractions:

$$\int \frac{1}{u^2(u+1)} du = -\frac{1}{u} - \ln|u| + \ln|u+1| + C$$

Substitute back $u = e^x$:

$$-\frac{1}{e^x} - x + \ln|1+e^x| + C = -e^{-x} - x + \ln|1+e^x| + C$$

Step 4: Particular Solution
Now write the particular solution:

$$\begin{aligned} y_p(x) &= e^{2x}\left(-e^{-x} - x + \ln|1+e^x|\right) - e^x \ln\left|\frac{1}{1+e^{-x}}\right| \\ &= -e^x - xe^{2x} + e^{2x}\ln|1+e^x| + e^x \ln|1+e^{-x}| \end{aligned}$$

Simplify using $\ln|1+e^{-x}| = -x + \ln|1+e^x|$:

$$\begin{aligned} y_p(x) &= -e^x - xe^{2x} + e^{2x}\ln|1+e^x| + e^x(-x + \ln|1+e^x|) \\ &= -e^x - xe^{2x} - xe^x + e^{2x}\ln|1+e^x| + e^x \ln|1+e^x| \end{aligned}$$

Step 5: General Solution

Combining homogeneous and particular solutions:

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1e^{2x} + c_2e^x - e^x - xe^{2x} - xe^x + e^{2x}\ln|1+e^x| + e^x\ln|1+e^x| \end{aligned}$$

Group terms:

$$y(x) = (c_1 - x + \ln|1+e^x|)e^{2x} + (c_2 - 1 - x + \ln|1+e^x|)e^x$$

Final Answer:

$$\boxed{y(x) = (c_1 - x + \ln|1+e^x|)e^{2x} + (c_2 - 1 - x + \ln|1+e^x|)e^x}$$

4 Tutorial 4

4.1 Question 1 (e)

Question: Solve the following differential equations

(e) $y''' - 6y'' + 12y' - 8y = 0,\ y(0) = 1,\ y'(0) = -1,\ y''(0) = -4$

Step 1: Characteristic Equation

The given differential equation is a linear homogeneous equation:

$$y''' - 6y'' + 12y' - 8y = 0.$$

We assume a solution of the form $y = e^{rx}$, leading to the characteristic equation:

$$r^3 - 6r^2 + 12r - 8 = 0.$$

Factoring,

$$(r-2)^3 = 0.$$

Thus, the characteristic roots are $r = 2, 2, 2$ (a repeated root of multiplicity 3).

Step 2: General Solution

Since $r = 2$ is a triple root, the general solution is:

$$y(x) = (c_1 + c_2 x + c_3 x^2)e^{2x}.$$

Step 3: Applying Initial Conditions

We compute the derivatives:

$$\begin{aligned} y'(x) &= (c_1 + c_2 x + c_3 x^2)' e^{2x} + (c_1 + c_2 x + c_3 x^2)(2e^{2x}) \quad \text{(Product Rule)} \\ &= (c_2 + 2c_3 x)e^{2x} + 2(c_1 + c_2 x + c_3 x^2)e^{2x} \\ &= (2c_1 + c_2 + (2c_2 + 2c_3)x + 2c_3 x^2)e^{2x}. \end{aligned}$$

For the second derivative:

$$\begin{aligned} y''(x) &= \Big((2c_1 + c_2) + (2c_2 + 2c_3)x + 2c_3 x^2\Big)' e^{2x} + \Big((2c_1 + c_2) + (2c_2 + 2c_3)x + 2c_3 x^2\Big)(2e^{2x}) \\ &= (2c_2 + 2c_3)e^{2x} + 2(2c_1 + c_2 + (2c_2 + 2c_3)x + 2c_3 x^2)e^{2x} \\ &= (4c_1 + 2c_2 + 2c_2 + 2c_3 + (4c_2 + 4c_3)x + 4c_3 x^2)e^{2x} \\ &= (4c_1 + 4c_2 + 2c_3 + 4c_3 x + 4c_3 x^2)e^{2x}. \end{aligned}$$

For the third derivative:

$$
\begin{aligned}
y'''(x) &= \left(4c_1 + 4c_2 + 2c_3 + 4c_3x + 4c_3x^2\right)' e^{2x} + \left(4c_1 + 4c_2 + 2c_3 + 4c_3x + 4c_3x^2\right)(2e^{2x}) \\
&= (4c_3 + 4c_3x)e^{2x} + 2(4c_1 + 4c_2 + 2c_3 + 4c_3x + 4c_3x^2)e^{2x} \\
&= (8c_1 + 8c_2 + 4c_3 + 8c_3x + 8c_3x^2)e^{2x}.
\end{aligned}
$$

Now, applying the initial conditions:

$$
\begin{aligned}
y(0) = 1 &\Rightarrow c_1e^0 = 1 \Rightarrow c_1 = 1. \\
y'(0) = -1 &\Rightarrow (2c_1 + c_2)e^0 = -1 \Rightarrow 2(1) + c_2 = -1 \Rightarrow c_2 = -3. \\
y''(0) = -4 &\Rightarrow (4c_1 + 4c_2 + 2c_3)e^0 = -4 \\
&\Rightarrow 4(1) + 4(-3) + 2c_3 = -4 \Rightarrow 4 - 12 + 2c_3 = -4 \Rightarrow 2c_3 = 4 \Rightarrow c_3 = 2.
\end{aligned}
$$

Step 4: Final Solution

Substituting $c_1 = 1$, $c_2 = -3$, and $c_3 = 2$, we get:

$$y(x) = (1 - 3x + 2x^2)e^{2x}.$$

Final Answer:

$$\boxed{y(x) = (1 - 3x + 2x^2)e^{2x}}$$

4.2 Question 1 (f)

Question: Solve the following differential equations

(f) $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = 0,\ y(0) = 5,\ y'(0) = -2,\ y''(0) = 6,\ y'''(0) = 8.$

Given the differential equation:

$$y^{(4)} + 2y''' - 2y'' - 8y' - 8y = 0$$

Characteristic Polynomial

$$m^4 + 2m^3 - 2m^2 - 8m - 8 = 0$$

Since $m = 2$ is a root:

$$(m-2)(m^3 + 4m^2 + 6m + 4) = 0$$

Factoring further:

$$(m-2)(m+2)(m^2 + 2m + 2) = 0$$

The roots are: $m = 2$, $m = -2$, and $m = -1 \pm i$

General Solution

$$y = C_1 e^{2x} + C_2 e^{-2x} + e^{-x}(C_3 \cos x + C_4 \sin x)$$

Using the IVP Given

Given conditions:

$$y(0) = 2C_1 + 2C_2 + C_3 = 5$$

$$y'(0) = C_1 - 2C_2 - C_3 + C_4 = -2$$

$$y''(0) = 4C_1 + 4C_2 + 2C_4 = 6$$

$$y'''(0) = 8C_1 - 8C_2 - 2C_3 + 2C_4 = 8$$

Solving for constants:

$$C_1 = 2, \quad C_2 = 4, \quad C_3 = -6, \quad C_4 = -14$$

4.3 Question 2 (a)

Question: Find the fundamental set of solutions for the following equations.

(a) $(D^2+9)^3D^2y = 0$.

We are given the characteristic polynomial corresponding to a second-order differential equation with constant coefficients. Solving for D, we have:

$$D^2 = 0 \Rightarrow D = 0 \text{ (with multiplicity 2).}$$

$$(D^2+9)^3 = 0 \Rightarrow D^2 = -9 \Rightarrow D = \pm 3i \text{ (with multiplicity 3).}$$

For a real repeated root $D = 0$ (multiplicity 2), the solution is given by:

$$y_1(x) = e^{0x}(C_1 + C_2x) = C_1 + C_2x.$$

For the complex repeated roots $D = \pm 3i$ (multiplicity 3), the solution takes the form:

$$y_2(x) = e^{3ix}(c_3 + c_4x + c_5x^2) + e^{-3ix}(c_6 + c_7x + c_8x^2).$$

Since we need real roots, using Euler's formula, we rewrite it in terms of sine and cosine:

$$y_2(x) = C_3 \sin 3x + C_4x \sin 3x + C_5x^2 \sin 3x + C_6 \cos 3x + C_7x \cos 3x + C_8x^2 \cos 3x.$$

Thus, the general solution to the given differential equation is:

$$y(x) = C_1 + C_2x + C_3 \sin 3x + C_4x \sin 3x + C_5x^2 \sin 3x + C_6 \cos 3x + C_7x \cos 3x + C_8x^2 \cos 3x.$$

$\therefore$ The fundamental set of solutions is given by:

$$\{1, x, \sin 3x, x \sin 3x, x^2 \sin 3x, \cos 3x, x \cos 3x, x^2 \cos 3x\}.$$

4.4 Question 3 (a)

Question: Find a particular solution using Anhilator method. Write down the Anhilator explicitly. Do not evaluate the coefficients.

(a) $y''' - 2y'' + y' = t^3 + 2e^t$

Here we have $Ly = y''' - 2y'' + y'$. Let $z(t)$ and $w(t)$ be such that $Lz = t^3$ and $Lw = 2e^t$. Then $L(z + w) = t^3 + 2e^t$.
Let us first solve $Lz = t^3$. We know that t^3 is a solution of $My = D^4 y = 0$.

Consider $MLz = D^4(D^3 - 2D^2 + D)y = D^5(D - 1)^2 y = 0$. We know that the solution in this case will be of the form

$$z(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4 + c_6 e^t + c_7 t e^t$$

But note here that $c_2 + c_6 e^t + c_7 t e^t$ is a solution to $Ly = 0$. Hence particular solution in this case is $z(t) = c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4$.

Similarly, consider $Lz = 2e^t$. We know that $2e^t$ is a solution of $My = (D - 1)y = 0$. Consider $MLz = (D - 1)(D^3 - 2D^2 + D)y = (D - 1)^3 y = 0$. We know that the solution in this case will be of the form

$$w(t) = c_1 + c_2 e^t + c_3 t e^t + c_4 t^2 e^t.$$

And in this case, the particular solution is $w(t) = c_4 t^2 e^t$.
Thus,

$$y_p = z(t) + w(t) = c_2 t + c_3 t^2 + c_4 t^3 + c_5 t^4 + c_4' t^2 e^t$$

. (We are not asked to evaluate the coefficients here.)

4.5 Question 3 (b)

Question: Find a particular solution using Anhilator method. Write down the Anhilator explicitly. Do not evaluate the coefficients.

(b) $y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t$.

Here $L = D^4 - D^3 - D^2 + D = D(D-1)^2(D+1)$ and $r(t) = t^2 + 4 + t\sin t$.

It is easy to see that, D^3 annihilates $t^2 + 4$. We know that $t^{m-1}e^{ax}\sin bx \in \operatorname{Ker}((D-a)^2 + b^2)^m$, therefore $(D^2+1)^2$ annihilates $t\sin t$.

Combining both, the annihilator of $(t^2 + 4 + t\sin t)$ is determined to be

$$D^3(D^2+1)^2.$$

Let $A = D^3(D^2+1)^2$.

A solution y of $Ly = 0$ is also a solution of $(AL)y = 0$, i.e.

$$D^4(D^2+1)^2(D-1)^2(D+1)y = 0.$$

Now, $AL = D^4(D^2+1)^2(D-1)^2(D+1)$ has characteristic equation

$$x^4(x^2+1)^2(x-1)^2(x+1) = 0.$$

The roots of this equation are 0 (with multiplicity 4), 1 (with multiplicity 2), $\pm i$ (each with multiplicity 2), -1 (with multiplicity 1).

A general solution of $(AL)y = 0$ is of the form

$$c_1 + c_2x + c_3x^2 + c_4x^3 + (c_5 + c_6x)e^x + c_7e^{-x} + (c_8 + c_9x)\cos x + (c_{10} + c_{11}x)\sin x,$$

where $c_1, \ldots, c_{11} \in \mathbb{R}$.

Here $c_1 + (c_5 + c_6x)e^x + c_7e^{-x}$ is a solution of the homogeneous part $Ly = 0$.

Therefore a particular solution is given by

$$x(c_2 + c_3x + c_4x^2) + (c_8 + c_9x)\cos x + (c_{10} + c_{11}x)\sin x,$$

where $c_2, c_3, c_4, c_8, c_9, c_{10}, c_{11} \in \mathbb{R}$.

4.6 Question 4 (e)

Question: Find the general solution using the annihilator method (method of undetermined coefficients).

(e) $y''' - y'' - y' + y = 2e^{-t} + 3$

Note here $Ly = (D^3 - D^2 - D + 1)y = 2e^{-t} + 3$. Additionally we can take the annihilator $A = D^2 + D$ such that $A(2e^{-t} + 3) = 0$ ($D + 1$ annihilates the e^{-t} term, while D annihilates the constant).
We know that any solution for

$$(D^3 - D^2 - D + 1)y = 2e^{-t} + 3$$

will also be a solution for

$$(D^2 + D)(D^3 - D^2 - D + 1)y = 0$$

The characteristic equation for $AL = (D^2 + D)(D^3 - D^2 - D + 1)$ is

$$(x^2 + x)(x^3 - x^2 - x + 1) = x(x + 1)^2(x - 1)^2$$

Thus the general solution for $(AL)(y) = 0$ is

$$c_1 + c_2e^{-t} + c_3te^{-t} + c_4e^t + c_5te^t$$

where we already know that $c_2e^{-t} + c_4e^t + c_5te^t$ is a solution for the homogeneous part $(D^3 - D^2 - D + 1)y = (D - 1)^2(D + 1)y = 0$
We now need to determine the particular solution $y_p = c_1 + c_3te^{-t}$, which we do by plugging this in the original equation and solving $y_p''' - y_p'' - y_p' + y_p = 2e^{-t} + 3$. This gives us

$$\begin{aligned}
y_p &= c_1 + c_3te^{-t} \\
y_p' &= c_3e^{-t} - c_3te^{-t} \\
y_p'' &= -c_3e^{-t} - c_3e^{-t} + c_3te^{-t} = -2c_3e^{-t} + c_3te^{-t} \\
y_p''' &= 2c_3e^{-t} + c_3e^{-t} - c_3te^{-t} \\
y_p''' - y_p'' - y_p' + y_p &= 4c_3e^{-t} + c_1 = 2e^{-t} + 3
\end{aligned}$$

Comparing coefficients, we get $c_3 = \frac{1}{2}$ and $c_1 = 3$. Thus the general solution is

$$c_1e^{-x} + c_2e^t + c_3te^t + \frac{1}{2}te^{-t} + 3$$

4.7 Question 4 (f)

Question: Find the general solution using the annihilator method (method of undetermined coefficients).

(f) $y^{(4)} - 4y'' = 3t + \cos t$.

Solution.

Step 1: Solve the homogeneous equation.
Consider the homogeneous ODE

$$y^{(4)} - 4y'' = 0.$$

Let $D = \frac{d}{dt}$. Then the equation becomes

$$(D^4 - 4D^2)y = 0.$$

Assume a solution of the form $y = e^{rt}$. Substituting gives the characteristic equation

$$r^4 - 4r^2 = 0 \quad \Longrightarrow \quad r^2(r^2 - 4) = 0.$$

Thus, $r^2 = 0$ (double root) and $r^2 = 4$ yielding $r = 2$ and $r = -2$. Therefore, the general homogeneous solution is

$$y_h(t) = C_1 + C_2\,t + C_3\,e^{2t} + C_4\,e^{-2t},$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

Step 2: Solve the non-homogeneous parts separately.
We now split the forcing term into two parts and solve the corresponding ODEs systematically.

(a) Solve $y^{(4)} - 4y'' = 3t$.
(i) Apply an annihilator: The forcing term $3t$ is a polynomial of degree 1. Its annihilator is D^2. Applying D^2 to both sides gives:

$$D^2\big(y^{(4)} - 4y''\big) = D^2(3t) = 0.$$

This yields the 6th-order homogeneous ODE:

$$y^{(6)} - 4y^{(4)} = 0.$$

The characteristic equation is

$$r^6 - 4r^4 = r^4(r^2 - 4) = 0,$$

so the roots are $r = 0$ (multiplicity 4) and $r = \pm 2$. Thus, the general solution of this 6th-order equation is

$$y(t) = A + Bt + Ct^2 + Dt^3 + Ee^{2t} + Fe^{-2t}.$$

(ii) Identify the particular solution: Notice that the homogeneous solution of the original 4th-order ODE is

$$y_h(t) = C_1 + C_2 t + C_3 e^{2t} + C_4 e^{-2t}.$$

Thus, the extra terms $Ct^2 + Dt^3$ in the 6th-order solution provide a particular solution for $y^{(4)} - 4y'' = 3t$. Assume

$$y_{p1}(t) = Ct^2 + Dt^3.$$

Compute the derivatives:

$$y''_{p1}(t) = 2C + 6Dt, \quad y^{(4)}_{p1}(t) = 0.$$

Then,

$$y^{(4)}_{p1} - 4y''_{p1} = -4(2C + 6Dt) = -8C - 24Dt.$$

Setting this equal to $3t$ gives:

$$-8C - 24Dt = 3t.$$

Equate coefficients:

$$-24D = 3 \quad \Longrightarrow \quad D = -\frac{1}{8}, \qquad -8C = 0 \quad \Longrightarrow \quad C = 0.$$

Thus, the particular solution for the $3t$ part is

$$y_{p1}(t) = -\frac{1}{8}\,t^3.$$

(b) Solve $y^{(4)} - 4y'' = \cos t.$

(i) Apply an annihilator: For the forcing term $\cos t$, an appropriate annihilator is $D^2 + 1$. Applying $(D^2 + 1)$ to both sides:

$$(D^2 + 1)\big(y^{(4)} - 4y''\big) = (D^2 + 1)(\cos t) = 0.$$

This produces a 6th-order homogeneous ODE whose characteristic equation yields additional roots $r = \pm i$. Thus, the general solution of the 6th-order ODE includes extra terms $E\cos t + F\sin t$.
(ii) Identify the particular solution: Since the homogeneous solution of the original 4th-order ODE does not include $\cos t$ or $\sin t$, we directly set

$$y_{p2}(t) = E\cos t + F\sin t.$$

Compute the necessary derivatives:

$$y''_{p2}(t) = -E\cos t - F\sin t, \quad y^{(4)}_{p2}(t) = E\cos t + F\sin t.$$

Thus,

$$y_{p2}^{(4)} - 4y_{p2}'' = \big(E\cos t + F\sin t\big) - 4\big(-E\cos t - F\sin t\big) = 5E\cos t + 5F\sin t.$$

Setting this equal to $\cos t$ gives:

$$5E\cos t + 5F\sin t = \cos t.$$

Hence,

$$5E = 1 \quad \Longrightarrow \quad E = \frac{1}{5}, \qquad 5F = 0 \quad \Longrightarrow \quad F = 0.$$

Thus, the particular solution for the $\cos t$ part is

$$y_{p2}(t) = \frac{1}{5}\cos t.$$

Step 3: Combine the solutions.
The particular solution for the full non-homogeneous ODE is the sum of the two parts:

$$y_p(t) = y_{p1}(t) + y_{p2}(t) = -\frac{1}{8}\,t^3 + \frac{1}{5}\cos t.$$

Thus, the final general solution is

$$y(t) = y_h(t) + y_p(t) = C_1 + C_2 t + C_3 e^{2t} + C_4 e^{-2t} - \frac{1}{8}\,t^3 + \frac{1}{5}\cos t.$$

4.8 Question 5

Question: Let $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$. Let y_1 be a solution to the corresponding homogeneous equation. Then making the substitution uy_1 in the differential gives a second order equation of the form $Q_0(x)u'' + Q_1(x)u' = F$. This is really a first order equation in variable $z = u'$ and can be solved using the variation of parameters method. This is called the method of **reduction of order**. Use the method of reduction of order to solve $(2-x)y''' + (2x-3)y'' - xy' + y = 0$ given that $y_1(x) = e^x$ is a solution.
The differential equation given is

$$(2-x)y^{'''} + (2x-3)y^{''} - xy^{'} + y = 0 \tag{8}$$

We have been given that $y_1(x) = e^x$ is a solution for the given differential equation. If we consider a second solution of the type $y(x) = u(x)y_1(x) = u(x)e^x$. Then,

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\left(e^x u(x)\right) \\
&= e^x u'(x) + e^x u(x) \\
&= e^x \big(u(x) + u'(x)\big)
\end{aligned}$$

$$\begin{aligned}
y''(x) &= \frac{d}{dx}\big(y'(x)\big) \\
&= \frac{d}{dx}\Big(e^x \big(u(x) + u'(x)\big)\Big) \\
&= e^x \big(u(x) + u'(x)\big) + e^x \big(u'(x) + u''(x)\big) \\
&= e^x \big(u''(x) + 2u'(x) + u(x)\big)
\end{aligned}$$

$$\begin{aligned}
y'''(x) &= \frac{d}{dx}\big(y''(x)\big) \\
&= \frac{d}{dx}\Big(e^x \big(u''(x) + 2u'(x) + u(x)\big)\Big) \\
&= e^x \big(u''(x) + 2u'(x) + u(x)\big) + e^x \big(u'''(x) + 2u''(x) + u'(x)\big) \\
&= e^x \big(u'''(x) + 3u''(x) + 3u'(x) + u(x)\big)
\end{aligned}$$

On substituting the above calculated values of y, y' and y''' into (8) and factoring out the common factor $e^x \neq 0$, we get

$$e^x\Big[(2-x)\big(u'''(x)+3u''(x)+3u'(x)+u(x)\big) + (2x-3)\big(u''(x)+2u'(x)+u(x)\big) - x\big(u'(x)+u(x)\big) + u(x)\Big] \; = \; 0$$

Thus, $u(x)$ must satisfy

$$(2-x)\big(u'''(x)+3u''(x)+3u'(x)+u(x)\big) + (2x-3)\big(u''(x)+2u'(x)+u(x)\big) - x\big(u'(x)+u(x)\big) + u(x) \; = \; 0$$

On collecting the terms corresponding to $u'(x), u''(x)$ and $u'''(x)$, the equation simplifies to

$$(2-x)\,u'''(x) \; + \; (3-x)\,u''(x) \; = \; 0.$$

If we denote

$$v(x) \; = \; u''(x).$$

Then $v'(x) = u'''(x)$ and the equation becomes,

$$(2-x)\,v'(x) + (3-x)\,v(x) = 0$$

$$v'(x) = -\frac{3-x}{2-x}\,v(x)$$

This is a separable equation. On separating we get,

$$\frac{dv}{v} = -\frac{3-x}{2-x}\,dx$$

$$\frac{dv}{v} = -\left(1 + \frac{1}{2-x}\right)\,dx$$

Integrating both sides:

$$\int \frac{dv}{v} = -\int \left(1 + \frac{1}{2-x}\right)\,dx$$

$$\ln|v(x)| = -x + \ln|2-x| + C$$

Thus,

$$v(x) = u''(x) = A\,e^{-x}(2-x)$$

On integrating once,

$$u'(x) \; = \; \int u''(x)\,dx \; = \; A \; \int (2-x)\,e^{-x}\,dx$$

The integral $\int (2-x)e^{-x}dx$ can be evaluated using integration by parts as follows:

$$\begin{aligned}\int (2-x)\,e^{-x}\,dx &= (2-x)(-e^{-x}) - \int (-1)(-e^{-x})dx \\ &= -(2-x)\,e^{-x} - \int e^{-x}\,dx \\ &= -(2-x)\,e^{-x} + e^{-x} + C \\ &= (x-1)\,e^{-x} + C\end{aligned}$$

so

$$u'(x) \;=\; A\,e^{-x}(x-1) + K_1.$$

Integrating one more time to get $u(x)$,

$$u(x) \;=\; \int \Big[A\,e^{-x}(x-1) + K_1\Big]\,dx \;=\; A\int e^{-x}(x-1)\,dx \;+\; K_1\,x$$

Evaluating $\int (x-1)e^{-x}dx$ using integral by parts,

$$\begin{aligned}\int (x-1)\,e^{-x}\,dx &= (x-1)(-e^{-x}) - \int (1)(-e^{-x})dx \\ &= (1-x)\,e^{-x} - e^{-x} + D \\ &= -xe^{-x} + D\end{aligned}$$

Thus,

$$u(x) \;=\; -\,A\,x\,e^{-x} \;+\; K_1\,x \;+\; K_2,$$

where A, K_1, K_2 are constants of integration.
As $y = e^x\,u(x)$. Hence,

$$y(x) \;=\; e^x\big[-\,A\,x\,e^{-x} + K_1\,x + K_2\big] \;=\; -\,A\,x \;+\; K_1\,x\,e^x \;+\; K_2\,e^x.$$

Hence, the solution for this differential equation will be of the form

$$y(x) = -Ax + K_1xe^x + K_2e^x + Be^x = (-A)x + K_1xe^x + (K_2+B)e^x$$

Renaming the constants as $C_1 = (-A), C_2 = K_1$ and $C_3 = (K_2+B)$.

Hence, the solution is

$$\boxed{y(x) \;=\; C_1\,x \;+\; C_2\,xe^x \;+\; C_3\,e^x.}$$

5 Tutorial 5

5.1 Question 2 (e)

Question: Find the Laplace transform of following functions.

(e) $f(t) = \begin{cases} e^{-t}, & 0 \leq t < 1 \\ e^{-2t}, & t \geq 1 \end{cases}$

The Laplace transform of a function $f(t)$ is given by:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

Given the piecewise function:

$$f(t) = \begin{cases} e^{-t}, & 0 \leq t < 1 \\ e^{-2t}, & t \geq 1 \end{cases}$$

We compute the Laplace transform in two parts:

$$F(s) = \int_0^1 e^{-t}e^{-st}dt + \int_1^\infty e^{-2t}e^{-st}dt$$

First Integral:

$$I_1 = \int_0^1 e^{-(s+1)t}dt$$

Evaluating:

$$I_1 = \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^1$$

$$I_1 = \frac{1 - e^{-(s+1)}}{s+1}$$

Second Integral:

$$I_2 = \int_1^\infty e^{-(s+2)t}dt$$

Evaluating:

$$I_2 = \left[\frac{e^{-(s+2)t}}{-(s+2)}\right]_1^\infty$$

$$I_2 = \frac{e^{-(s+2)}}{s+2}$$

Final Expression:

$$F(s) = \frac{1 - e^{-(s+1)}}{s+1} + \frac{e^{-(s+2)}}{s+2}$$

5.2 Question 3 (a)

Question: (a) Prove that if $L(f(t)) = F(s)$, then $L(t^k f(t)) = (-1)^k F^{(k)}(s)$.

[**Hint:** Assume that we can differentiate the integral $\int_0^\infty e^{-st} f(t)dt$ with respect to s under the integral sign.]

Solution: Using Leibniz's rule for differentiating under the integral sign, we get:

$$\frac{d^k}{ds^k} \left(\int_0^\infty e^{-st} f(t)\, dt \right) = \int_0^\infty \frac{\partial^k}{\partial s^k} \left(e^{-st} f(t) \right) dt$$

Now, differentiate $e^{-st} f(t)$ with respect to s:

$$\frac{\partial^k}{\partial s^k} \left(e^{-st} f(t) \right) = (-1)^k t^k e^{-st} f(t)$$

Thus, we have:

$$\frac{d^k}{ds^k} F(s) = \int_0^\infty (-1)^k t^k e^{-st} f(t)\, dt$$

Therefore,

$$\mathcal{L}(t^k f(t)) = (-1)^k F^{(k)}(s)$$

5.3 Question 8 (a)

Question: Find the Laplace transform of the following functions.

(a) $\dfrac{\sin \omega t}{t}$, $\omega > 0$,

We can use the property of Laplace transforms that states:

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s')ds'$$

Let $f(t) = \sin(wt)$
We know:

$$\mathcal{L}(f(t)) = F(s') = \frac{w}{w^2 + s'^2}, s' > 0$$

Now solving for $\mathcal{L}\left(\frac{\sin(wt)}{t}\right)$:

$$\mathcal{L}\left(\frac{\sin(wt)}{t}\right) = \int_s^\infty \frac{w}{w^2 + s'^2} ds', s > 0$$

Solving the Integral:

Let $s' = w\tan(y)$ and since $w > 0; s' > 0$

$$y = \tan^{-1}\left(\frac{s'}{w}\right); ds' = w\sec^2 y dy$$

$$\mathcal{L}\left(\frac{\sin(wt)}{t}\right) = \int_\theta^{\pi/2} \frac{w\sec^2 y}{w^2 + w^2.\tan^2 y} dy$$

Where $\theta = \tan^{-1}(s/w)$

since $1 + \tan^2 y = \sec^2 y$

$$\mathcal{L}\left(\frac{\sin(wt)}{t}\right) = \int_\theta^{\pi/2} dy = \frac{\pi}{2} - \theta$$

Thus,

$$\mathcal{L}\left(\frac{\sin(wt)}{t}\right) = \boxed{\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{w}\right) = \tan^{-1}\left(\frac{w}{s}\right)}$$

5.4 Question 10 (a)

Question: Find the Laplace transform of the following periodic functions.

(a) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \end{cases}, \ f(t+2) = f(t), \ t \ge 0.$

The given function is periodic with period $T = 2$, so we use the Laplace transform formula for periodic functions:

$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

For the given piecewise function:

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2. \end{cases}$$

We compute the integral:

$$I = \int_0^1 e^{-st} t\, dt + \int_1^2 e^{-st}(2 - t)\, dt.$$

First integral:

$$I_1 = \int_0^1 t e^{-st} dt.$$

Using integration by parts, let $u = t$ and $dv = e^{-st}dt$, then $du = dt$ and $v = -\frac{1}{s}e^{-st}$, giving:

$$I_1 = \left[-\frac{t}{s} e^{-st} \right]_0^1 + \int_0^1 \frac{1}{s} e^{-st} dt.$$

Evaluating:

$$I_1 = -\frac{1}{s} e^{-s} + \frac{1}{s^2}(1 - e^{-s}).$$

$$I_1 = \frac{1 - (1 + s)e^{-s}}{s^2}.$$

Second integral:

$$I_2 = \int_1^2 e^{-st}(2 - t) dt.$$

Using substitution $u = 2 - t$, then $du = -dt$, we get:

$$I_2 = \int_0^1 e^{-s(2-u)} u(-du).$$

$$= \int_0^1 ue^{-2s} e^{su} du.$$

Using integration by parts again,

$$I_2 = e^{-2s} \frac{1-(1+s)e^{-s}}{s^2}.$$

Thus, the total integral:

$$I = \frac{1-(1+s)e^{-s}}{s^2} + e^{-2s} \frac{1-(1+s)e^{-s}}{s^2}.$$

$$I = \frac{(1-(1+s)e^{-s})(1+e^{-2s})}{s^2}.$$

Finally, the Laplace transform is:

$$F(s) = \frac{1}{1-e^{-2s}} \cdot \frac{(1-(1+s)e^{-s})(1+e^{-2s})}{s^2}.$$

5.5 Question 11 (a)

Question: Find the inverse Laplace transform of the following functions.

(a) $\dfrac{3}{(s-7)^4}$

Using the First Shifting Theorem

The **First Shifting Theorem** states:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

which implies:

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$$

Identifying the Standard Form

We recognize that $\frac{3}{(s-7)^4}$ resembles a shifted version of the standard form $\frac{n!}{s^{n+1}}$.

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

For $n = 3$:

$$\mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}.$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = t^3.$$

Since our given function has a numerator of 3 instead of 6:

$$\frac{3}{(s-7)^4} = \frac{1}{2}\cdot\frac{6}{(s-7)^4},$$

we use linearity property of Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{3}{(s-7)^4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{6}{(s-7)^4}\right\}.$$

Applying the Shifting Theorem

Using $a = 7$:

$$\mathcal{L}^{-1}\left\{\frac{6}{(s-7)^4}\right\} = e^{7t}t^3.$$

So ,

$$\mathcal{L}^{-1}\left\{\frac{3}{(s-7)^4}\right\} = \frac{1}{2}e^{7t}t^3.$$

Final Answer

$$\boxed{\frac{1}{2}t^3e^{7t}}$$

5.6 Question 11 (i)

Question: Find the inverse Laplace transform of the following functions.

(i) $\dfrac{3s+2}{(s^2+4)(s^2+9)}$

We express the function as:

$$\frac{3s+2}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

Now solving for this:

$$\frac{3s+2}{(s^2+4)(s^2+9)} = \frac{As^3+9As+Bs^2+9B+Cs^3+Ds^2+4Cs+4D}{(s^2+4)(s^2+9)}$$

Grouping terms together:

$$\frac{3s+2}{(s^2+4)(s^2+9)} = \frac{(A+C)s^3+(9A+4C)s+(B+D)s^2+9B+4D}{(s^2+4)(s^2+9)}$$

Removing common factors in denominator:

$$3s+2 = (A+C)s^3+(B+D)s^2+(9A+4C)s+(9B+4D)$$

Equating coefficients of powers of s:

$$(A+C)=0, \quad B+D=0$$

$$9A+4C=3$$

$$9B+4D=2$$

Solving for A and B:

$$9A-4A=3 \Rightarrow A=\frac{3}{5}$$

$$9B-4B=2 \Rightarrow B=\frac{2}{5}$$

$$\frac{3s/5+2/5}{s^2+4} + \frac{-(3s/5+2/5)}{s^2+9}$$

Solving for C and D:

$$A = -C \Rightarrow C = \frac{-3}{5}$$

$$B = -D \Rightarrow D = \frac{-2}{5}$$

As we know:

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}, \quad s > 0, \quad \omega \in \mathbb{R}$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}, \quad s > 0, \quad \omega \in \mathbb{R}$$

Taking the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{3s/5 + 2/5}{s^2 + 4}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{3}{5}\frac{s}{s^2 + 4}\right) + \mathcal{L}^{-1}\left(\frac{1}{5}\frac{2}{s^2 + 4}\right)$$

$$= \frac{3}{5}\cos 2t + \frac{1}{5}\sin 2t$$

$$\mathcal{L}^{-1}\left(-\frac{3s/5 + 2/5}{s^2 + 9}\right)$$

$$= -\mathcal{L}^{-1}\left(\frac{3}{5}\frac{s}{s^2 + 9}\right) + \left(-\frac{2}{15}\right)\mathcal{L}^{-1}\left(\frac{3}{s^2 + 9}\right)$$

$$= -\frac{3}{5}\cos 3t - \frac{2}{15}\sin 3t$$

Final answer:

$$\frac{3}{5}\cos 2t + \frac{1}{5}\sin 2t - \frac{3}{5}\cos 3t - \frac{2}{15}\sin 3t$$

5.7 Question 12 (a)

Question: Solve the following IVP's using Laplace transforms.

(a) $y'' + 3y' + 2y = e^t,\ y(0) = 1,\ y'(0) = -6$

Taking the Laplace transform on both sides:

$$(s^2Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s-1}$$

Substituting the initial conditions $y(0) = 1$ and $y'(0) = -6$:

$$(s^2Y(s) - s + 6) + 3(sY(s) - 1) + 2Y(s) = \frac{1}{s-1}$$

$$(s^2 + 3s + 2)Y(s) = s - 3 + \frac{1}{s-1}$$

$$(s^2 + 3s + 2)Y(s) = \frac{s^2 - 4s + 4}{s-1}$$

Solving for $Y(s)$

$$Y(s) = \frac{s^2 - 4s + 4}{(s-1)(s^2 + 3s + 2)}$$

$$Y(s) = \frac{s^2 - 4s + 4}{(s-1)(s+1)(s+2)}$$

Using partial fraction decomposition:

$$\frac{s^2 - 4s + 4}{(s-1)(s+1)(s+2)} = \frac{1/6}{s-1} + \frac{-9/2}{s+1} + \frac{16/3}{s+2}$$

Thus,

$$Y(s) = \frac{-9}{2(s+1)} + \frac{16}{3(s+2)} + \frac{1}{6(s-1)}$$

Taking the inverse Laplace transform:

$$\boxed{y(t) = \frac{1}{6}e^t - \frac{9}{2}e^{-t} + \frac{16}{3}e^{-2t}}$$

5.8 Question 12 (d)

Question: Solve the following IVP's using Laplace transforms.

(d) $y'' + 4y = 3\sin t,\ y(0) = 1,\ y'(0) = -1.$

We solve the initial value problem using Laplace transform:

$$y'' + 4y = 3\sin t, \quad y(0) = 1, \quad y'(0) = -1.$$

Taking the Laplace transform of both sides:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{3\sin t\}.$$

Let the Laplace Transform of y be denoted as the following,

$$\mathcal{L}\{y\} = Y(s),$$

Then by using the Laplace transform properties, we have:

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0),$$

Also we know,

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}.$$

Substituting given initial conditions $y(0) = 1$, $y'(0) = -1$:

$$(s^2Y(s) - s(1) - (-1)) + 4Y(s) = 3 \cdot \frac{1}{s^2+1}.$$

$$(s^2+4)Y(s) - s + 1 = \frac{3}{s^2+1}.$$

$$(s^2+4)Y(s) = \frac{3}{s^2+1} + s - 1.$$

$$Y(s) = \frac{s-1}{s^2+4} + \frac{3}{(s^2+1)(s^2+4)}.$$

$$Y(s) = \frac{s-1}{s^2+4} + \frac{(s^2+4)-(s^2+1)}{(s^2+1)(s^2+4)}.$$

Using partial fraction decomposition:

$$\frac{3}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

Multiplying both sides by $(s^2+1)(s^2+4)$:

$$3 = (As+B)(s^2+4) + (Cs+D)(s^2+1).$$

Expanding and equating coefficients, we find:

$$A = 0, B = 1, C = 0, D = -1.$$

Thus,

$$Y(s) = \frac{s-1}{s^2+4} + \frac{1}{s^2+1} - \frac{1}{s^2+4}.$$

$$Y(s) = \frac{s}{s^2+4} + \frac{1}{s^2+1} - \frac{2}{s^2+4}.$$

Using linearity of Laplace transforms and inverse transforms:

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t,$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t.$$

Thus, the final solution is:

$$y(t) = \cos 2t - \sin 2t + \sin t.$$