MA110: Lecture 18

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Real Quadratic Forms

Let $n \in \mathbb{N}$. A **real n-ary quadratic form** Q is a homogeneous polynomial of degree 2 in n variables with coefficients in $\mathbb{R}$. Thus

$$\begin{aligned} Q(x_1, \ldots, x_n) &:= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} x_j x_k \\ &= \sum_{j=1}^{n} \alpha_{jj} x_j^2 + \sum_{1 \le j < k \le n} (\alpha_{jk} + \alpha_{kj}) x_j x_k, \end{aligned}$$

where $\alpha_{jk} \in \mathbb{R}$ for $j, k = 1, \ldots, n$.

Examples Let $a, b, c, a', b', c' \in \mathbb{R}$.

$n = 1 : Q(x) := a\,x^2$ (unary quadratic form)

$n = 2 : Q(x, y) := a\,x^2 + b\,y^2 + a'xy$ (binary quadratic form)

$n = 3 : Q(x, y, z) := a\,x^2 + b\,y^2 + c\,z^2 + a'xy + b'yz + c'zx$ (ternary quadratic form)

For $n \in \mathbb{N}$, consider an $n \times n$ real matrix $\mathbf{A} := [a_{jk}]$.
Then for $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}}$,

$$\mathbf{x}^{\mathsf{T}} \mathbf{A}\, \mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{n} a_{1k} x_k \\ \vdots \\ \sum_{k=1}^{n} a_{nk} x_k \end{bmatrix} = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{jk} x_k \right) x_j$$

$$= \sum_{j=1}^{n} a_{jj} x_j^2 + \sum_{1 \le j < k \le n} (a_{jk} + a_{kj}) x_j x_k,$$

which is an n-ary quadratic form.
In fact, $Q(x_1, \ldots, x_n) = \mathbf{x}^{\mathsf{T}} \mathbf{A}\, \mathbf{x}$ for all $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$ in $\mathbb{R}^{n \times 1}$ if and only if

$$\alpha_{jk} + \alpha_{kj} = a_{jk} + a_{kj} \quad \text{for all } j, k = 1, \ldots, n.$$

In general, many $n \times n$ matrices induce the same quadratic form. For example, the matrices $\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 11 & 2 \end{bmatrix}$ induce the same binary quadratic form.

But if we require the matrix $\mathbf{A} := [a_{jk}]$ inducing the quadratic form Q to be symmetric, that is, $a_{jk} = a_{kj}$ for all j, k, then

$$a_{jk} = \frac{1}{2}(\alpha_{jk} + \alpha_{kj}) \quad \text{for all } j, k = 1, \ldots, n.$$

Thus given an n-ary quadratic form Q, there is a unique $n \times n$ real symmetric matrix $\mathbf{A}$ such that $Q(x_1 \ldots, x_n) = \mathbf{x}^{\mathsf{T}} \mathbf{A}\, \mathbf{x}$ for all $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \in \mathbb{R}^{n \times 1}$; in fact

$$\mathbf{A} := [a_{jk}], \quad \text{where } a_{jk} := \frac{1}{2}(\alpha_{jk} + \alpha_{kj}),\ j, k = 1, \ldots, n.$$

This real symmetric matrix $\mathbf{A}$ is called the **matrix associated with** the quadratic form Q, and we write $Q = Q_{\mathbf{A}}$.

A real n-ary quadratic form Q is said to be a **diagonal quadratic form** if there are $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that

$$Q(x_1, \dots, x_n) = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2.$$

It is clear that a quadratic form Q is diagonal if and only if Q is associated with a diagonal matrix $\mathbf{D}$, that is, $Q = Q_{\mathbf{D}}$. Using the spectral theorem for real symmetric matrices, we show that every quadratic form can be orthogonally transformed to a diagonal quadratic form.

Theorem (Principle Axis Theorem)

Let Q be a real quadratic form and $\mathbf{A} \in \mathbb{R}^{n\times n}$ be the symmetric matrix associated with Q. If $\mathbf{C}$ is an orthogonal matrix such that the matrix $\mathbf{D} := \mathbf{C}^{\mathsf{T}}\mathbf{A}\,\mathbf{C}$ is diagonal, then $Q(\mathbf{x}) = Q_{\mathbf{D}}(\mathbf{y})$, where $\mathbf{y} := \mathbf{C}^{\mathsf{T}}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^{n\times 1}$.

Proof. Let $\mathbf{x} \in \mathbb{R}^{n\times 1}$ and $\mathbf{y} := \mathbf{C}^{\mathsf{T}}\mathbf{x} = \mathbf{C}^{-1}\mathbf{x}$. Then $\mathbf{x} = \mathbf{C}\mathbf{y}$ and

$$Q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^{\mathsf{T}}\mathbf{A}\,\mathbf{x} = (\mathbf{C}\mathbf{y})^{\mathsf{T}}\mathbf{A}(\mathbf{C}\mathbf{y}) = \mathbf{y}^{\mathsf{T}}(\mathbf{C}^{\mathsf{T}}\mathbf{A}\mathbf{C})\mathbf{y} = \mathbf{y}^{\mathsf{T}}\mathbf{D}\mathbf{y} = Q_{\mathbf{D}}(\mathbf{y}).$$

To diagonalise a real n-ary quadratic form Q, we first write down the (real symmetric) matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ associated with Q. We then find an orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ consisting of eigenvectors of $\mathbf{A}$ corresponding to its eigenvalues $\lambda_1, \ldots, \lambda_n$ counted according to their algebraic multiplicities. If we let

$$\mathbf{C} := \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_n\end{bmatrix} \quad \text{and} \quad \mathbf{D} := \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

Then $Q(\mathbf{x}) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$, where $\mathbf{y} := \mathbf{C}^{\mathsf{T}}\mathbf{x} = \begin{bmatrix}\mathbf{u}_1^{\mathsf{T}} \\ \vdots \\ \mathbf{u}_n^{\mathsf{T}}\end{bmatrix} \mathbf{x}$.

Example

Let us transform the quadratic form $Q(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3 - 4x_3x_1$ to a diagonal form. Here $\mathbf{A} := \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$ is the associated matrix.

We have seen before that
$\mathbf{u}_1 := \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}^\mathsf{T}$, $\mathbf{u}_2 := \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}^\mathsf{T}$
and $\mathbf{u}_3 := \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}^\mathsf{T}$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues 3, 3 and -3 respectively, and they form an orthonormal basis for $\mathbb{R}^{3\times 1}$. Hence let

$$\mathbf{C} := \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \quad \text{and} \quad \mathbf{D} := \operatorname{diag}(3, 3, -3).$$

Then $\mathbf{C}^\mathsf{T}\mathbf{A}\,\mathbf{C} = \mathbf{D}$, and so $Q(\mathbf{x}) = 3(y_1^2 + y_2^2 - y_3^2)$, where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} := \mathbf{C}^\mathsf{T}\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

that is, $y_1 = (x_1 + x_2)/\sqrt{2}$, $y_2 = (-x_1 + x_2 + 2x_3)/\sqrt{6}$ and $y_3 = (x_1 - x_2 + x_3)/\sqrt{3}$.

Conic Sections

A **conic section** is the locus in $\mathbb{R}^2$ of an equation

$$a\,x^2 + b\,y^2 + c\,xy + a'x + b'y + c' = 0,$$

where $a, b, c, a', b', c' \in \mathbb{R}$ and at least one among a, b, c is nonzero. We assume WLOG that not both a and b are negative. It can be proved that the conic is one of these:
(i) the empty set
(ii) a single point
(iii) one or two straight lines
(iv) an ellipse
(v) a hyperbola
(vi) a parabola.

Terms of the second degree on the LHS of the equation give

$$Q(x, y) := a\,x^2 + b\,y^2 + c\,xy.$$

It is a binary quadratic form. It determines the type of the conic.

The (real symmetric) matrix associated with Q is

$$\mathbf{A} := \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix}.$$

Hence the equation of the given conic section becomes

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a' & b' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + c' = 0,$$

that is,

$\begin{bmatrix} x & y \end{bmatrix} \mathbf{A} \begin{bmatrix} x & y \end{bmatrix}^{\mathsf{T}} + \begin{bmatrix} a' & b' \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}^{\mathsf{T}} + c' = 0.$

Let $\mathbf{C} := [\mathbf{u}_1, \mathbf{u}_2]$ be an orthogonal matrix whose columns $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors of $\mathbf{A}$ with corresponding eigenvalues λ_1 and λ_2, and let $\mathbf{D} := \mathrm{diag}(\lambda_1, \lambda_2)$ so that $\mathbf{C}^{\mathsf{T}} \mathbf{A} \, \mathbf{C} = \mathbf{D}$.

We use the change of variables $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{C} \begin{bmatrix} u \\ v \end{bmatrix}$ to transform the quadratic form $Q(x, y)$ to a diagonal form as follows.

$$
\begin{aligned}
Q(x,y) &= \begin{bmatrix} x & y \end{bmatrix} \mathbf{A} \begin{bmatrix} x & y \end{bmatrix}^{\mathsf{T}} \\
&= \begin{bmatrix} u & v \end{bmatrix} \mathbf{C}^{\mathsf{T}} \mathbf{A}\, \mathbf{C} \begin{bmatrix} u & v \end{bmatrix}^{\mathsf{T}} \\
&= \begin{bmatrix} u & v \end{bmatrix} \mathbf{D} \begin{bmatrix} u & v \end{bmatrix}^{\mathsf{T}} \\
&= \lambda_1 u^2 + \lambda_2 v^2 = Q_{\mathbf{D}}(u,v).
\end{aligned}
$$

The ordered orthonormal basis $(\mathbf{u}_1, \mathbf{u}_2)$ determines a new set of coordinate axes, so that the locus of the original equation is given by

$$
\begin{aligned}
&\begin{bmatrix} u & v \end{bmatrix} \operatorname{diag}(\lambda_1, \lambda_2) \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} a' & b' \end{bmatrix} \mathbf{C} \begin{bmatrix} u \\ v \end{bmatrix} + c' \\
= \;& \lambda_1 u^2 + \lambda_2 v^2 + \begin{bmatrix} a' & b' \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + c' = 0.
\end{aligned}
$$

If the conic so determined is not degenerate, that is, if it does not reduce to an empty set, a point, or line(s), then the signs of λ_1 and λ_2 determine the type of the conic section as follows.

The equation represents

1. ellipse if $\lambda_1\lambda_2 > 0$, that is, both λ_1 and λ_2 are positive,

2. hyperbola if $\lambda_1\lambda_2 < 0$, that is, one of λ_1, λ_2 is positive and the other is negative,

3. parabola if $\lambda_1\lambda_2 = 0$, that is, one of λ_1, λ_2 is zero.

Note: Since $Q(x, y) := a\,x^2 + b\,y^2 + c\,xy = \lambda_1 u^2 + \lambda_2 v^2$, where not both a and b are negative, it follows that not both λ_1 and λ_2 can be negative, and since the conic is assumed to be nondegenerate, not both λ_1 and λ_2 can be equal to zero.

Examples

1. Consider the conic section given by $2x^2 + 4xy + 5y^2 + 4x + 13y - 1/4 = 0$, and the binary quadratic form $Q(x,y) := 2x^2 + 4xy + 5y^2$.

Then the associated symmetric matrix $\mathbf{A} := \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ has eigenvalues $\lambda_1 := 1$ and $\lambda_2 := 6$, and the corresponding eigenvectors $\mathbf{u}_1 := \begin{bmatrix} 2 & -1 \end{bmatrix}^\mathsf{T}/\sqrt{5}$ and $\mathbf{u}_2 := \begin{bmatrix} 1 & 2 \end{bmatrix}^\mathsf{T}/\sqrt{5}$ form an orthonormal basis for $\mathbb{R}^{2\times 1}$. Hence let

$$\mathbf{C} := \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{D} := \operatorname{diag}(1, 6).$$

Then $\mathbf{C}^\mathsf{T}\mathbf{A}\mathbf{C} = \mathbf{D}$. So $Q(x,y) = Q_\mathbf{D}(u,v) = u^2 + 6v^2$, where

$\begin{bmatrix} u \\ v \end{bmatrix} := \mathbf{C}^\mathsf{T} \begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, that is,

$u = (2x - y)/\sqrt{5}$ and $v = (x + 2y)/\sqrt{5}$.

Since $\begin{bmatrix} x \\ y \end{bmatrix} := \mathbf{C} \begin{bmatrix} u \\ v \end{bmatrix}$, substituting $x = (2u + v)/\sqrt{5}$ and $y = (-u + 2v)/\sqrt{5}$ in the given equation of the conic section, we obtain

$$u^2 + 6v^2 - \sqrt{5}u + 6\sqrt{5}v - \frac{1}{4} = 0.$$

Completing the squares, we see that

$$\left(u - \frac{\sqrt{5}}{2}\right)^2 + 6\left(v + \frac{\sqrt{5}}{2}\right)^2 = 9.$$

This is an equation of an ellipse with its centre at $(\sqrt{5}/2, -\sqrt{5}/2)$ in the uv-plane, where the u-axis and the v-axis are determined by the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

2. Consider the conic section given by $2x^2 - 4xy - y^2 - 4x + 10y - 13 = 0$, and the binary quadratic form $Q(x, y) := 2x^2 - 4xy - y^2$.

The associated symmetric matrix $\mathbf{A} := \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 3, \lambda_2 = -2$, and the corresponding eigenvectors $\mathbf{u}_1 := \begin{bmatrix} 2 & -1 \end{bmatrix}^\mathsf{T}/\sqrt{5}$ and $\mathbf{u}_2 := \begin{bmatrix} 1 & 2 \end{bmatrix}^\mathsf{T}/\sqrt{5}$ form an orthonormal basis for $\mathbb{R}^{2\times 1}$. Hence let

$$\mathbf{C} := \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{D} := \mathrm{diag}(3, -2).$$

Then $\mathbf{C}^\mathsf{T} \mathbf{A}\, \mathbf{C} = \mathbf{D}$. Hence $Q(x, y) = Q_\mathbf{D}(u, v) = 3u^2 - 2v^2$,

where $\begin{bmatrix} u \\ v \end{bmatrix} := \mathbf{C}^\mathsf{T} \begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$,

that is, $u = (2x - y)/\sqrt{5}$ and $v = (x + 2y)/\sqrt{5}$.

Thus substituting $x = (2u + v)/\sqrt{5}$ and $y = (-u + 2v)/\sqrt{5}$ in the given equation of the conic section, we obtain

$$3u^2 - 2v^2 - \frac{4}{\sqrt{5}}(2u + v) + \frac{10}{\sqrt{5}}(-u + 2v) - 13 = 0,$$

that is,

$$3u^2 - 2v^2 - \frac{1}{\sqrt{5}}(18u - 16v) - 13 = 0.$$

Completing the squares, we see that

$$\frac{(u - 3/\sqrt{5})^2}{4} - \frac{(v - 4/\sqrt{5})^2}{6} = 1.$$

This is an equation of a hyperbola with its centre $(3/\sqrt{5}, 4/\sqrt{5})$ in the uv-plane, where the u-axis and the v-axis are determined by the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

3. Consider the conic section given by $9x^2 + 24xy + 16y^2 - 20x + 15y = 0$, and the binary quadratic form $Q(x, y) := 9x^2 + 24xy + 16y^2$. Then the associated symmetric matrix $\mathbf{A} := \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ has eigenvalues $\lambda_1 := 25$ and $\lambda_2 := 0$, and the corresponding eigenvectors $\mathbf{u}_1 := \begin{bmatrix} 3 & 4 \end{bmatrix}^\mathsf{T}/5$ and $\mathbf{u}_2 := \begin{bmatrix} -4 & 3 \end{bmatrix}^\mathsf{T}/5$ form an orthonormal basis for $\mathbb{R}^{2\times 1}$. Hence let $\mathbf{C} := \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{D} := \mathrm{diag}(25, 0)$. Then $\mathbf{C}^\mathsf{T}\mathbf{A}\,\mathbf{C} = \mathbf{D}$. Thus $Q(x, y) = Q_\mathbf{D}(u, v) = 25u^2$, where

$$\begin{bmatrix} u \\ v \end{bmatrix} := \mathbf{C}^\mathsf{T}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

that is, $u = (3x + 4y)/5$ and $v = (-4x + 3y)/5$. Substituting $x = (3u - 4v)/5$ and $y = (4u + 3v)/5$ in the equation of the conic, we obtain $25u^2 + 25v = 0$, i.e., $u^2 = -v$, which is an equation of a parabola with its vertex at $(0, 0)$ in the uv-plane.

Quadric Surfaces

A **quadric surface** is the locus in $\mathbb{R}^3$ of an equation

$$a\,x^2 + b\,y^2 + c\,z^2 + a'xy + b'yz + c'zx + a''x + b''y + c''z + d = 0,$$

where $a, b, c, a', b', c', a'', b'', c'', d \in \mathbb{R}$. We assume WLOG that not all three a, b and c are negative.

Terms of the second degree on the LHS of the equation give

$$Q(x, y, z) := a\,x^2 + b\,y^2 + c\,z^2 + a'xy + b'yz + c'zx.$$

It is a ternary quadratic form. It determines the type of the quadric surface. The real symmetric matrix associated with the quadratic form Q is

$$\mathbf{A} := \begin{bmatrix} a & a'/2 & c'/2 \\ a'/2 & b & b'/2 \\ c'/2 & b'/2 & c \end{bmatrix}.$$

Hence the equation of the given quadric surface becomes

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & a'/2 & c'/2 \\ a'/2 & b & b'/2 \\ c'/2 & b'/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a'' & b'' & c'' \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + d = 0,$$

that is,

$$\begin{bmatrix} x & y & z \end{bmatrix} \mathbf{A} \begin{bmatrix} x & y & z \end{bmatrix}^{\mathsf{T}} + \begin{bmatrix} a'' & b'' & c'' \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix}^{\mathsf{T}} + d = 0.$$

Let $\mathbf{C} := \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$ be an orthogonal matrix whose columns $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are eigenvectors of $\mathbf{A}$ with corresponding eigenvalues λ_1, λ_2, λ_3, and let $\mathbf{D} := \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$.

We use the change of variables $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{C} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ to transform the quadratic form $Q(x, y, z)$ to a diagonal form as follows.

$$
\begin{aligned}
Q(x,y,z) &= \begin{bmatrix} x & y & z \end{bmatrix} \mathbf{A} \begin{bmatrix} x & y & z \end{bmatrix}^{\mathsf{T}} \\
&= \begin{bmatrix} u & v & w \end{bmatrix} \mathbf{C}^{\mathsf{T}} \mathbf{A}\, \mathbf{C} \begin{bmatrix} u & v & w \end{bmatrix}^{\mathsf{T}} \\
&= \begin{bmatrix} u & v & w \end{bmatrix} \mathbf{D} \begin{bmatrix} u & v & w \end{bmatrix}^{\mathsf{T}} \\
&= \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3^2 w^2 = Q_{\mathbf{D}}(u,v,w).
\end{aligned}
$$

The ordered orthonormal basis $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ determines a new set of coordinate axes, so that the locus of the original equation is given by

$$
\begin{bmatrix} u & v & w \end{bmatrix} \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} a'' & b'' & c'' \end{bmatrix} \mathbf{C} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + d = 0.
$$

Leaving aside the degenerate cases, the primary cases are:

Equation	Surface	Eigenvalues of A
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	ellipsoid	all three positive
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}=0$	elliptic paraboloid	two positive, one zero
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	elliptic cone	two positive, one negative
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	1-sheeted hyperboloid	two positive, one negative
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	2-sheeted hyperboloid	one positive, two negative
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z}{c}=0$	hyperbolic paraboloid	one positive, one negative, one zero.

Pictures of these surfaces can be found on the Internet by searching with their names.

Example Consider the quadric surface given by

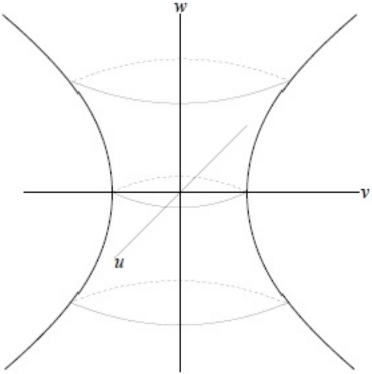
$$x^2 + y^2 + z^2 + 4xy + 4yz - 4zx - 27 = 0,$$

and the associated ternary quadratic form
$Q(x, y, z) := x^2 + y^2 + z^2 + 4xy + 4yz - 4zx$.

We have already transformed the associated symmetric matrix
$\mathbf{A} := \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$ to a diagonal form, and have obtained
$Q(x, y, z) = Q_{\mathbf{D}}(u, v, w) = 3(u^2 + v^2 - w^2)$ (with x_1, x_2, x_3 and y_1, y_2, y_3 in place of x, y, z and u, v, w),
where $\mathbf{D} := \operatorname{diag}(3, 3, -3)$ and
$u = (x+y)/\sqrt{2}$, $v = (-x+y+2z)/\sqrt{6}$, $w = (x-y+z)/\sqrt{3}$.

Under this change of coordinates, the quadric surface reduces to $u^2 + v^2 - w^2 = 9$.

This is an equation of a one-sheeted hyperboloid in the *uvw*-space, as shown in the following figure, where the *u*-axis, the *v*-axis and the *w*-axis are determined by the eigenvectors $\mathbf{u}_1 := \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}^{\mathsf{T}}$, $\mathbf{u}_2 := \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}^{\mathsf{T}}$ and $\mathbf{u}_3 := \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}^{\mathsf{T}}$. (See Lecture 16.)



Orthogonal Projection

Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Recall that in Lecture 13, we have defined the (perpendicular) projection of $\mathbf{x} \in \mathbb{K}^{n\times 1}$ in the direction of nonzero $\mathbf{y} \in \mathbb{K}^{n\times 1}$ as follows:

$$P_{\mathbf{y}}(\mathbf{x}) := \frac{\langle \mathbf{y}, \mathbf{x}\rangle}{\langle \mathbf{y}, \mathbf{y}\rangle}\,\mathbf{y}.$$

In particular, if $\mathbf{y}$ is a unit vector, then $P_{\mathbf{y}}(\mathbf{x}) = \langle \mathbf{y}, \mathbf{x}\rangle \mathbf{y}$.

We noted that the vector $P_{\mathbf{y}}(\mathbf{x})$ is a scalar multiple of the vector $\mathbf{y}$, and proved the important relation

$$\big(\mathbf{x} - P_{\mathbf{y}}(\mathbf{x})\big) \perp \mathbf{y}.$$

As a consequence,

$$\begin{aligned} \|\mathbf{x} - P_{\mathbf{y}}(\mathbf{x})\|^2 &= \langle \mathbf{x} - P_{\mathbf{y}}(\mathbf{x}), \mathbf{x} - P_{\mathbf{y}}(\mathbf{x})\rangle \\ &= \langle \mathbf{x}, \mathbf{x} - P_{\mathbf{y}}(\mathbf{x})\rangle \\ &= \|\mathbf{x}\|^2 - \langle \mathbf{x}, P_{\mathbf{y}}(\mathbf{x})\rangle. \end{aligned}$$

More generally, let Y be a nonzero subspace of $\mathbb{K}^{n\times 1}$. We would like to find a (perpendicular) projection of $\mathbf{x} \in \mathbb{K}^{n\times 1}$ into Y, that is, we want to find $\mathbf{y} \in Y$ such that $(\mathbf{x} - \mathbf{y}) \in Y^{\perp}$. (This $\mathbf{y}$ is 'the foot of the perpendicular' from $\mathbf{x}$ into Y.)

If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is an orthonormal basis for the subspace Y, then a vector belongs to $Y^{\perp}$ if and only if it is orthogonal to each $\mathbf{u}_j$ for $j = 1, \ldots, k$. As we saw while studying G-S OP, the vector

$$\tilde{\mathbf{y}} := \mathbf{x} - P_{\mathbf{u}_1}(\mathbf{x}) - \cdots - P_{\mathbf{u}_k}(\mathbf{x}) = \mathbf{x} - \langle \mathbf{u}_1,\, \mathbf{x}\rangle \mathbf{u}_1 - \cdots - \langle \mathbf{u}_k,\, \mathbf{x}\rangle \mathbf{u}_k$$

is orthogonal to each $\mathbf{u}_j$ for $j = 1, \ldots, k$, and so $\tilde{\mathbf{y}} \in Y^{\perp}$.

Since the vector $\mathbf{y} := \langle \mathbf{u}_1,\, \mathbf{x}\rangle \mathbf{u}_1 + \cdots + \langle \mathbf{u}_k,\, \mathbf{x}\rangle \mathbf{u}_k$ belongs to Y, it is a (perpendicular) projection of $\mathbf{x}$ in Y.

The following result shows that this is the only vector in Y that works!