Linear Algebra
Lecture 17

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Recall: In the last lecture, we proved

Proposition (Spectral Theorem for Normal Matrices)

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$. Then $\mathbf{A}$ is normal if and only if $\mathbf{A}$ is unitarily diagonalizable.

We then turned to self-adjoint matrices and proved:

Lemma

A matrix $\mathbf{A} \in \mathbb{K}^{n\times n}$ is diagonal with all diagonal entries real if and only if it is upper triangular and self-adjoint.

Proof. Let $\mathbf{A} := \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Then clearly $\mathbf{A}$ is upper triangular. Also, it is self-adjoint since $\mathbf{A}^* = \mathrm{diag}(\overline{\lambda}_1, \dots, \overline{\lambda}_n) = \mathrm{diag}(\lambda_1, \dots, \lambda_n) = \mathbf{A}$.

Conversely, suppose $\mathbf{A}$ is upper triangular and self-adjoint. Then $\mathbf{A}^*$ is lower triangular. Since $\mathbf{A}^* = \mathbf{A}$, we see that $\mathbf{A}$ is in fact diagonal with all diagonal entries real. $\square$

Proposition (Spectral Theorem for Self-Adjoint Matrices)

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$. Then $\mathbf{A}$ is self-adjoint if and only if $\mathbf{A}$ is unitarily diagonalizable and all eigenvalues of $\mathbf{A}$ are real.

Proof. Suppose $\mathbf{A}$ is unitarily diagonalizable and all eigenvalues of $\mathbf{A}$ are real. Then $\mathbf{D} = \mathbf{U}^*\mathbf{A}\mathbf{U}$ for some unitary matrix $\mathbf{U}$ and diagonal matrix $\mathbf{D}$. The diagonal entries entries of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$, and so they are real. Hence $\mathbf{D}^* = \mathbf{D}$. Consequently, $\mathbf{A}^* = (\mathbf{U}\mathbf{D}\mathbf{U}^*)^* = \mathbf{U}\mathbf{D}^*\mathbf{U}^* = \mathbf{U}\mathbf{D}\mathbf{U}^* = \mathbf{A}$. Thus $\mathbf{A}$ is self-adjoint.

Conversely, suppose $\mathbf{A}$ is self-adjoint. By Schur's theorem, $\mathbf{B} = \mathbf{U}^*\mathbf{A}\mathbf{U}$ for some unitary matrix $\mathbf{U}$, and upper triangular matrix $\mathbf{B}$. Now $\mathbf{B}^* = (\mathbf{U}^*\mathbf{A}\mathbf{U})^* = \mathbf{U}^*\mathbf{A}^*\mathbf{U} = \mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{B}$. So $\mathbf{B}$ is self-adjoint. Hence by the Lemma above, $\mathbf{B}$ is diagonal with all diagonal entries real. This proves that $\mathbf{A}$ is unitarily diagonalizable and all eigenvalues of $\mathbf{A}$ are real.

Our short proof of the spectral theorem for self-adjoint matrices is based on Schur's theorem. This result can also be deduced from part (ii) of the spectral theorem for normal matrices since every self-adjoint matrix in normal, provided we independently show that every eigenvalue of a self-adjoint matrix is real. The latter statement can be easily proved.

Proposition

If $\mathbf{A} \in \mathbb{C}^{n \times n}$ is self-adjoint, then every eigenvalue of $\mathbf{A}$ is real.

Proof. Let λ be an eigenvalue of a self-adjoint matrix $\mathbf{A}$, and let $\mathbf{x}$ be a corresponding unit eigenvector. Then

$$\lambda = \lambda\, \mathbf{x}^*\mathbf{x} = \mathbf{x}^*\mathbf{A}\,\mathbf{x} = \mathbf{x}^*\mathbf{A}^*\mathbf{x} = (\mathbf{A}\,\mathbf{x})^*\mathbf{x} = (\lambda\,\mathbf{x})^*\mathbf{x} = \overline{\lambda}\,\mathbf{x}^*\mathbf{x} = \overline{\lambda}.$$

Hence λ is real. $\square$

Finally, let us consider a real symmetric matrix $\mathbf{A}$, that is, $\mathbf{A} \in \mathbb{R}^{n \times n}$ such that $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$. We shall prove a spectral theorem for $\mathbf{A}$ which involves only real scalars.

A unitary matrix with real entries is also called an **orthogonal matrix**. Thus $\mathbf{C} \in \mathbb{R}^{n \times n}$ is orthogonal if its columns form an orthonormal subset of $\mathbb{R}^{n \times 1}$. Clearly, an orthogonal matrix is invertible and its inverse is the same as its transpose.

Definition

A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ *is called* **orthogonally diagonalizable** *if there is an orthogonal matrix* $\mathbf{C} \in \mathbb{R}^{n \times n}$ *and a diagonal matrix* $\mathbf{D} \in \mathbb{R}^{n \times n}$ *such that* $\mathbf{D} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}$.

Proposition (Spectral Theorem for Real Symmetric Matrices)

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Then $\mathbf{A}$ is symmetric if and only if $\mathbf{A}$ is orthogonally diagonalizable. In this case, $\mathbf{A}$ has n real eigenvalues counted according to their algebraic multiplicities.

Proof of the Spectral Theorem for Real Symmetric Matrices:
Suppose $\mathbf{A}$ is orthogonally diagonalizable. Then $\mathbf{D} = \mathbf{C}^\mathsf{T}\mathbf{A}\mathbf{C}$ for some orthogonal matrix $\mathbf{C}$ and diagonal matrix $\mathbf{D}$. in $\mathbb{R}^{n\times n}$. Now $\mathbf{D}^\mathsf{T} = \mathbf{D} \implies \mathbf{A}^\mathsf{T} = (\mathbf{C}\mathbf{D}\mathbf{C}^\mathsf{T})^\mathsf{T} = \mathbf{C}\mathbf{D}^\mathsf{T}\mathbf{C}^\mathsf{T} = \mathbf{C}\mathbf{D}\mathbf{C}^\mathsf{T} = \mathbf{A}$. Thus $\mathbf{A}$ is symmetric.

Conversely, suppose $\mathbf{A}$ is symmetric. By the spectral theorem for self-adjoint matrices, there is a unitary matrix $\mathbf{U} \in \mathbb{C}^{n\times n}$ and a diagonal matrix $\mathbf{D} \in \mathbb{C}^{n\times n}$ such that $\mathbf{D} = \mathbf{U}^{-\mathbf{1}}\mathbf{A}\mathbf{U}$, that is, $\mathbf{A}\mathbf{U} = \mathbf{U}\mathbf{D}$. Since the diagonal entries of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$, we see that $\mathbf{D} \in \mathbb{R}^{n\times n}$.

Let λ be an eigenvalue of $\mathbf{A}$. Then $\lambda \in \mathbb{R}$. Using the Gauss Elimination Method, we may find the basic solutions of the homogeneous linear system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. Their entries are real since all entries of $\mathbf{A}$ are real and $\lambda \in \mathbb{R}$. These solutions form a basis for the eigenspace of $\mathbf{A}$ corresponding to λ.

Further, we can use the Gram-Schmidt Orthogonalization Process for these basic solutions to obtain an orthonormal basis for the eigenspace of **A** corresonding to λ. In this process, the entries of the basis vectors remain real. Putting together the orthonormal bases for the eigenspaces of **A** corresponding to distinct eigenvalues, we get an orthonormal set (since eigenvectors corresponding to distinct eigenvalues of a normal matrix are orthogonal); moreover, this set contains exactly n vectors (since the sum of geometric multiplicities of eigenvalues of **A** is n because **A** is diagonalisable). So the matrix $\mathbf{C} \in \mathbb{R}^{n \times n}$ formed by these n vectors as its columns is orthogonal and moreover $\mathbf{C}^{-1}\mathbf{AC}$ is a diagonal matrix. This proves that **A** is orthogonally diagonalizable. □

Remark: The above proof suggests a constructive method to orthogonally diagonalize a real symmetric matrix (and similarly, to unitary diagonalize a complex self-adjoint matrix), provided we know its eigenvalues. We illustrate this with an example.

Example: Let $\mathbf{A} := \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$. Clearly $\mathbf{A}$ is real symmetric.

$\boxed{1.}$ $p_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I}) = (3-t)^2(3+t)$. So the eigenvalues of $\mathbf{A}$ are $\mu_1 = 3$ with $m_1 = g_1 = 2$, and $\mu_2 = -3$ with $m_2 = g_2 = 1$.

$\boxed{2.}$ (i) $(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} -2 & 2 & -2 \\ 2 & -2 & 2 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} -2 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\iff x_1 - x_2 + x_3 = 0$ by the Gauss Elimination Method.

Hence $\boxed{\mathbf{x}_{11} := \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{\mathsf{T}}}$ and $\boxed{\mathbf{x}_{12} := \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^{\mathsf{T}}}$ form a basis for the null space $\mathcal{N}(\mathbf{A} - 3\mathbf{I})$.

(ii) Similarly, $\boxed{\mathbf{x}_{21} := \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^{\mathsf{T}}}$ forms a basis for the null space $\mathcal{N}(\mathbf{A} + 3\mathbf{I})$.

3. Gram-Schmidt Orthogonalization Process gives
$\mathbf{u}_{11} := \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}^{\mathsf{T}}$ and
$\mathbf{u}_{12} := \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}^{\mathsf{T}}$, which form an orthonormal basis for $\mathcal{N}(\mathbf{A} - 3\mathbf{I})$.

Also, $\mathbf{u}_{21} := \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}^{\mathsf{T}}$ forms an orthonormal basis for $\mathcal{N}(\mathbf{A} + 3\mathbf{I})$.

4. Let $\mathbf{U} := \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

and $\mathbf{D} := \operatorname{diag}(3, 3, -3)$. Then $\mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{D}$. Note that since $\mathbf{U}$ is real (and unitary), we can also say that $\mathbf{U}$ is orthogonal and $\mathbf{U}^{\mathsf{T}}\mathbf{A}\mathbf{U} = \mathbf{D}$.

We now show that the unitary matrix $\mathbf{U}$ and the diagonal matrix $\mathbf{D}$ satisfying $\mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{D}$ we have found are not unique.

For example, let us interchange the order of the columns of $\mathbf{U}$ and make a corresponding interchange in the diagonal entries of $\mathbf{D}$.

Thus $\mathbf{U} := \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$

and $\mathbf{D} := \mathrm{diag}(3, -3, 3)$ would also satisfy $\mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{D}$.

Further, we could have chosen $\mathbf{x}_{11} := \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^{\mathsf{T}}$ and $\mathbf{x}_{12} := \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}^{\mathsf{T}}$ as basis vectors for the null space $\mathcal{N}(\mathbf{A} - 3\mathbf{I})$, and orthonormalized them to obtain $\mathbf{u}_{11} := \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{\mathsf{T}}$ & $\mathbf{u}_{12} := \begin{bmatrix} -2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}^{\mathsf{T}}$.

Then $\mathbf{U} := \begin{bmatrix} 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

and $\mathbf{D} := \mathrm{diag}(3, 3, -3)$ would also satisfy $\mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{D}$.

Final Comments

The spectral theorems proved in this lecture say the following.

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be normal. Then there is an orthonormal set of eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of $\mathbf{A}$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.

If $\mathbf{x} \in \mathbb{C}^{n \times 1}$, then $\mathbf{x} = \langle \mathbf{u}_1, \mathbf{x} \rangle \mathbf{u}_1 + \cdots + \langle \mathbf{u}_n, \mathbf{x} \rangle \mathbf{u}_n$ since $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is an orthonormal basis for $\mathbb{C}^{n \times 1}$. Hence

$$\mathbf{A}\,\mathbf{x} = \lambda_1 \langle \mathbf{u}_1, \mathbf{x} \rangle \mathbf{u}_1 + \cdots + \lambda_n \langle \mathbf{u}_n, \mathbf{x} \rangle \mathbf{u}_n \quad \text{for all } \mathbf{x} \in \mathbb{C}^{n \times 1}.$$

In particular, if $\mathbf{A}$ is self-adjoint, then $\lambda_1, \ldots, \lambda_n$ are real. This also holds if $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric (with $\mathbf{u}_1, \ldots, \mathbf{u}_n$ having all real entries).

The above formula gives a spectral representation of $\mathbf{A}$. This is useful in evaluating the action of powers of $\mathbf{A}$ or more generally, polynomials in $\mathbf{A}$, on column vectors.

Let $k \in \mathbb{N}$. Then $\mathbf{A}^k(\mathbf{u}_j) = \lambda_j^k \mathbf{u}_j$ for each $j = 1, \ldots, n$, and so

$$\mathbf{A}^k \mathbf{x} = \sum_{j=1}^{n} \lambda_j^k \langle \mathbf{u}_j, \mathbf{x} \rangle \mathbf{u}_j \quad \text{for all } \mathbf{x} \in \mathbb{C}^{n \times 1}.$$

More generally, if $p(t)$ is any polynomial, then

$$p(\mathbf{A})\, \mathbf{x} = \sum_{j=1}^{n} p(\lambda_j) \langle \mathbf{u}_j, \mathbf{x} \rangle \mathbf{u}_j \quad \text{for all } \mathbf{x} \in \mathbb{C}^{n \times 1}.$$

In the example which we have just worked out, we obtain

$$\mathbf{A}^k \mathbf{x} = 3^k \langle \mathbf{u}_{11}, \mathbf{x} \rangle \mathbf{u}_{11} + 3^k \langle \mathbf{u}_{12}, \mathbf{x} \rangle \mathbf{u}_{12} + (-3)^k \langle \mathbf{u}_{21}, \mathbf{x} \rangle \mathbf{u}_{21}$$

for all $\mathbf{x} \in \mathbb{K}^{3 \times 1}$ and all $k \in \mathbb{N}$, where

$$
\begin{aligned}
\langle \mathbf{u}_{11}, \mathbf{x} \rangle \mathbf{u}_{11} &= \frac{1}{2}(x_1 + x_2) \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{\mathsf{T}} \\
\langle \mathbf{u}_{12}, \mathbf{x} \rangle \mathbf{u}_{12} &= \frac{1}{6}(-x_1 + x_2 + 2x_3) \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}^{\mathsf{T}} \\
\langle \mathbf{u}_{21}, \mathbf{x} \rangle \mathbf{u}_{21} &= \frac{1}{3}(x_1 - x_2 + x_3) \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^{\mathsf{T}}.
\end{aligned}
$$

For instance, if $\mathbf{x} := \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^{\mathsf{T}}$, then

$$
\mathbf{A}^k \mathbf{x} = 3^k \frac{3}{2} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{\mathsf{T}} + 3^k \frac{7}{6} \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}^{\mathsf{T}} + (-3)^k \frac{2}{3} \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^{\mathsf{T}}
$$

for each $k \in \mathbb{N}$.

Real Quadratic Forms

Let $n \in \mathbb{N}$. A **real n-ary quadratic form** Q is a homogeneous polynomial of degree 2 in n variables with coefficients in $\mathbb{R}$. Thus

$$\begin{aligned} Q(x_1, \ldots, x_n) &:= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} x_j x_k \\ &= \sum_{j=1}^{n} \alpha_{jj} x_j^2 + \sum_{1 \le j < k \le n} (\alpha_{jk} + \alpha_{kj}) x_j x_k, \end{aligned}$$

where $\alpha_{jk} \in \mathbb{R}$ for $j, k = 1, \ldots, n$.

Examples Let $a, b, c, a', b', c' \in \mathbb{R}$.

$n = 1 : Q(x) := a\,x^2$ (unary quadratic form)

$n = 2 : Q(x, y) := a\,x^2 + b\,y^2 + a'xy$ (binary quadratic form)

$n = 3 : Q(x, y, z) := a\,x^2 + b\,y^2 + c\,z^2 + a'xy + b'yz + c'zx$
(ternary quadratic form)

For $n \in \mathbb{N}$, consider an $n \times n$ real matrix $\mathbf{A} := [a_{jk}]$.
Then for $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}}$,

$$\mathbf{x}^{\mathsf{T}} \mathbf{A}\, \mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{n} a_{1k} x_k \\ \vdots \\ \sum_{k=1}^{n} a_{nk} x_k \end{bmatrix} = \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} a_{jk} x_k \Big) x_j$$

$$= \sum_{j=1}^{n} a_{jj} x_j^2 + \sum_{1 \le j < k \le n} (a_{jk} + a_{kj}) x_j x_k,$$

which is an n-ary quadratic form.
In fact, $Q(x_1, \ldots, x_n) = \mathbf{x}^{\mathsf{T}} \mathbf{A}\, \mathbf{x}$ for all $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$ in $\mathbb{R}^{n \times 1}$ if and only if

$$\alpha_{jk} + \alpha_{kj} = a_{jk} + a_{kj} \quad \text{for all } j, k = 1, \ldots, n.$$

In general, many $n \times n$ matrices induce the same quadratic form. For example, the matrices $\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 11 & 2 \end{bmatrix}$
induce the same binary quadratic form.

But if we require the matrix $\mathbf{A} := [a_{jk}]$ inducing the quadratic form Q to be symmetric, that is, $a_{jk} = a_{kj}$ for all j, k, then

$$a_{jk} = \frac{1}{2}(\alpha_{jk} + \alpha_{kj}) \quad \text{for all } j, k = 1, \ldots, n.$$

Thus given an n-ary quadratic form Q, there is a unique $n \times n$ real symmetric matrix $\mathbf{A}$ such that $Q(x_1 \ldots, x_n) = \mathbf{x}^\mathsf{T} \mathbf{A}\, \mathbf{x}$ for all $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \in \mathbb{R}^{n\times 1}$; in fact

$$\mathbf{A} := [a_{jk}], \quad \text{where } a_{jk} := \frac{1}{2}(\alpha_{jk} + \alpha_{kj}),\ j, k = 1, \ldots, n.$$

This real symmetric matrix $\mathbf{A}$ is called the **matrix associated with** the quadratic form Q, and we write $Q = Q_{\mathbf{A}}$.

A real n-ary quadratic form Q is said to be a **diagonal quadratic form** if there are $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$Q(x_1, \ldots, x_n) = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2.$$

It is clear that a quadratic form Q is diagonal if and only if Q is associated with a diagonal matrix $\mathbf{D}$, that is, $Q = Q_{\mathbf{D}}$. Using the spectral theorem for real symmetric matrices, we show that every quadratic form can be orthogonally transformed to a diagonal quadratic form.

Theorem (Principle Axis Theorem)

Let Q be a real quadratic form and $\mathbf{A} \in \mathbb{R}^{n \times n}$ be the symmetric matrix associated with Q. If $\mathbf{C}$ is an orthogonal matrix such that the matrix $\mathbf{D} := \mathbf{C}^\mathsf{T} \mathbf{A}\, \mathbf{C}$ is diagonal, then $Q(\mathbf{x}) = Q_{\mathbf{D}}(\mathbf{y})$, where $\mathbf{y} := \mathbf{C}^\mathsf{T} \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^{n \times 1}$.

Proof. Let $\mathbf{x} \in \mathbb{R}^{n \times 1}$ and $\mathbf{y} := \mathbf{C}^\mathsf{T} \mathbf{x} = \mathbf{C}^{-1} \mathbf{x}$. Then $\mathbf{x} = \mathbf{C}\mathbf{y}$ and

$$Q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^\mathsf{T} \mathbf{A}\, \mathbf{x} = (\mathbf{C}\mathbf{y})^\mathsf{T} \mathbf{A} (\mathbf{C}\mathbf{y}) = \mathbf{y}^\mathsf{T} (\mathbf{C}^\mathsf{T} \mathbf{A} \mathbf{C}) \mathbf{y} = \mathbf{y}^\mathsf{T} \mathbf{D} \mathbf{y} = Q_{\mathbf{D}}(\mathbf{y}).$$

To diagonalise a real n-ary quadratic form Q, we first write down the (real symmetric) matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ associated with Q. We then find an orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ consisting of eigenvectors of $\mathbf{A}$ corresponding to its eigenvalues $\lambda_1, \ldots, \lambda_n$ counted according to their algebraic multiplicities. If we let

$$\mathbf{C} := \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix} \quad \text{and} \quad \mathbf{D} := \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

Then $Q(\mathbf{x}) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$, where $\mathbf{y} := \mathbf{C}^\mathsf{T}\mathbf{x} = \begin{bmatrix} \mathbf{u}_1^\mathsf{T} \\ \vdots \\ \mathbf{u}_n^\mathsf{T} \end{bmatrix} \mathbf{x}$.

Example

Let us transform the quadratic form $Q(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3 - 4x_3x_1$ to a diagonal form. Here $\mathbf{A} := \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$ is the associated matrix.

We have seen before that $\mathbf{u}_1 := \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\end{bmatrix}^{\mathsf{T}}$, $\mathbf{u}_2 := \begin{bmatrix}-1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6}\end{bmatrix}^{\mathsf{T}}$ and $\mathbf{u}_3 := \begin{bmatrix}1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3}\end{bmatrix}^{\mathsf{T}}$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues 3, 3 and -3 respectively, and they form an orthonormal basis for $\mathbb{R}^{3\times 1}$. Hence let

$$\mathbf{C} := \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix} \quad \text{and} \quad \mathbf{D} := \operatorname{diag}(3, 3, -3).$$

Then $\mathbf{C}^{\mathsf{T}}\mathbf{A}\,\mathbf{C} = \mathbf{D}$, and so $Q(\mathbf{x}) = 3(y_1^2 + y_2^2 - y_3^2)$, where

$$\mathbf{y} = \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} := \mathbf{C}^{\mathsf{T}}\mathbf{x} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},$$

that is, $y_1 = (x_1 + x_2)/\sqrt{2}$, $y_2 = (-x_1 + x_2 + 2x_3)/\sqrt{6}$ and $y_3 = (x_1 - x_2 + x_3)/\sqrt{3}$.