Indian Institute of Technology Bombay

MA 110: Linear Algebra and Differential Equations

Spring 2025

Code A SRG/SB

Mid-semester Examination

Day: Sunday Date : 23.02.2025
Time : 8.30 AM – 10.30 AM Max Marks : 30

NOTE: 1. Attempt all the questions. Answers to the questions should be written in the answerbook provided separately. Begin the answer to a question on a new page. Answers to all subparts of a question should appear together.

2. Please make sure that your roll number and division and tutorial batch is written on the answerbook as well as every supplement you may take. Failure to do so may result in a penalty of 2 marks.

3. In the following, m, n denote positive integers, $\mathbb{R}$ the set of all real numbers, $\mathbb{C}$ the set of all complex numbers, and $\mathbb{K}$ can be $\mathbb{R}$ or $\mathbb{C}$. Also $\mathbf{I}_n$ denotes the $n \times n$ identity matrix

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Q. 1 (i) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices with entries in $\mathbb{R}$ such that $\mathbf{AB} = \mathbf{A}^2 + \mathbf{A} + \mathbf{I}_n$. Show that $\mathbf{A}$ and $\mathbf{B}$ commute, that is, show that $\mathbf{AB} = \mathbf{BA}$. [2 marks]

(ii) Consider an $m \times n$ matrix $\mathbf{A}$ and an $n \times m$ matrix $\mathbf{B}$ with entries in $\mathbb{R}$. Show that the nullity of $(\mathbf{AB} - \mathbf{I}_m)$ is the same as the nullity of $(\mathbf{BA} - \mathbf{I}_n)$ Deduce that $(\mathbf{AB} - \mathbf{I}_m)$ is invertible if and only $(\mathbf{BA} - \mathbf{I}_n)$ is invertible. [3 marks]

Q. 2 Let V be a vector space over $\mathbb{K}$ and S be a subset of V. (i) Define when S is said to be linearly dependent. Also define what is meant by the span of S. [2 marks]

(ii) Prove the following crucial result: If S contains s elements and $S \subset \text{span}(R)$ for some set R with r elements and if $s > r$, then S is linearly dependent. [3 marks]

Q. 3 (i) Give an example of a square matrix $\mathbf{A}$ such that $\mathbf{A}^2 + \mathbf{I} = \mathbf{0}$, where $\mathbf{I}$ denotes the identity matrix and $\mathbf{0}$ the zero matrix (of the same size as $\mathbf{A}$). [1 mark]

(ii) Show that if n is odd, then there does not exist any $n \times n$ matrix $\mathbf{A}$ with entries in $\mathbb{R}$ such that $\mathbf{A}^2 + \mathbf{I} = \mathbf{0}$. [4 marks]

Q. 4 Let $\mathbf{A}$ be an $n \times n$ matrix with entries in $\mathbb{C}$.

(i) Define what is meant by the adjoint $\mathbf{A}^*$ of $\mathbf{A}$. Show that $\text{rank}(\mathbf{A}^*) = \text{rank}(\mathbf{A}^*\mathbf{A})$.

[2 marks]

(ii) Assume that all the entries of $\mathbf{A}$ are in $\mathbb{R}$ and $\mathbf{A}^* = \mathbf{A}$. If all the eigenvalues of $\mathbf{A}$ are positive real numbers, then show that $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for every nonzero column vector $\mathbf{x} \in \mathbb{R}^{n\times 1}$. [3 marks]

Q. 5 (i) Let $V = \mathbb{R}^{2\times 2}$ be the space of all 2×2 matrices with entries in $\mathbb{R}$ and consider the ordered basis $E = \{\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{21}, \mathbf{E}_{22}\}$ of V, where $\mathbf{E}_{jk}$ denotes the 2×2 matrix whose (j,k)th entry is 1 and all other entries are 0. Let $T : V \to V$ be the linear map defined by $T(\mathbf{A}) = \mathbf{A}^T$, the transpose of $\mathbf{A}$. Find the matrix $\mathbf{M}_E^E(T)$ of T with respect to E. [2 marks]

(ii) Consider the quadric surface in $\mathbb{R}^3$ given by

$$7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy - 36 = 0,$$

and let $Q(x, y, z)$ be the ternary quadratic form associated to this surface. Find an orthogonal set of coordinates u, v, w such that $Q(x, y, z)$ becomes a diagonal form in u, v, w. Write explicitly the expressions for u, v, w in terms of x, y, z. [3 marks]

Q. 6 (i) Find the best approximation for a solution of the following system of linear equations:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

[2 marks]

(ii) Let $C[-3, 3]$ be the vector space of all continuous real-valued functions defined on the interval $[-3, 3]$. Consider the usual inner product on $C[-3, 3]$ defined by

$$\langle f, g \rangle = \int_{-3}^{3} f(t)g(t)dt$$

Let W be the subspace of $C[-3, 3]$ spanned by the polynomial functions 1 and t. Find the orthogonal projection of the function $t^2 + t$ onto W. [3 marks]