MA 110:
Lecture 03

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Suppose an $m \times n$ matrix $\mathbf{A}$ is in a REF, and there are r nonzero rows. Let the r pivots be in the columns $k_1, \ldots, k_r$ with $k_1 < \cdots < k_r$, and let the columns $\ell_1, \ldots, \ell_{n-r}$ be nonpivotal. Then $\boxed{x_{k_1}, \ldots, x_{k_r}}$ are called the **pivotal variables** and $\boxed{x_{\ell_1}, \ldots, x_{\ell_{n-r}}}$ are called the **free variables**.

Let $\mathbf{b} := [b_1 \cdots b_r \; b_{r+1} \cdots b_m]^\mathsf{T}$, and consider the linear system

$$\mathbf{Ax} = \mathbf{b}.$$

Important Observations

1. The linear system has a solution $\iff b_{r+1} = \cdots = b_m = 0$. This is known as the **consistency condition**.
2. Let the consistency condition $b_{r+1} = \cdots = b_m = 0$ be satisfied. Then we obtain a **particular solution** $\mathbf{x}_0 := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T}$ of the linear system by letting $x_k := 0$ if $k \in \{\ell_1, \ldots, \ell_{n-r}\}$, and then by determining the pivotal variables $x_{k_1}, \ldots, x_{k_r}$ by back substitution.

3. We obtain $n - r$ **basic solutions** of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ as follows. Fix $\ell \in \{\ell_1, \ldots, \ell_{n-r}\}$. Define $\mathbf{s}_\ell := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T}$ by $x_k := 1$ if $k = \ell$, while $x_k := 0$ if $k \in \{\ell_1, \ldots, \ell_{n-r}\}$ but $k \neq \ell$. Then determine the pivotal variables $x_{k_1}, \ldots, x_{k_r}$ by back substitution.

4. Let $\mathbf{s} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{n \times 1}$ be any solution of the homogeneous system, that is, $\mathbf{As} = \mathbf{0}$. Then $\mathbf{s}$ is a linear combination of the $n - r$ basic solutions $\mathbf{s}_{\ell_1}, \ldots, \mathbf{s}_{\ell_{n-r}}$. To see this, let $\mathbf{y} := \mathbf{s} - x_{\ell_1}\mathbf{s}_{\ell_1} - \cdots - x_{\ell_{n-r}}\mathbf{s}_{\ell_{n-r}}$. Then $\mathbf{Ay} = \mathbf{As} - x_{\ell_1}\mathbf{As}_{\ell_1} - \cdots - x_{\ell_{n-r}}\mathbf{As}_{\ell_{n-r}} = \mathbf{0}$, and moreover, the kth entry of $\mathbf{y}$ is 0 for each $k \in \{\ell_1, \ldots, \ell_{n-r}\}$. It then follows that $\mathbf{y} = \mathbf{0}$, that is, $\mathbf{s} = x_{\ell_1}\mathbf{s}_{\ell_1} + \cdots + x_{\ell_{n-r}}\mathbf{s}_{\ell_{n-r}}$. Thus we find that the general solution of the homogeneous system is given by

$$\boxed{\mathbf{s} = \alpha_1\mathbf{s}_{\ell_1} + \cdots + \alpha_{n-r}\mathbf{s}_{\ell_{n-r}}, \text{ where } \alpha_1, \ldots, \alpha_{n-r} \in \mathbb{R}}.$$

5. The general solution of $\mathbf{Ax} = \mathbf{b}$ is given by

$$\boxed{\mathbf{x} = \mathbf{x}_0 + \alpha_1 \mathbf{s}_{\ell_1} + \cdots + \alpha_{n-r} \mathbf{s}_{\ell_{n-r}}, \text{ where } \alpha_1, \ldots, \alpha_{n-r} \in \mathbb{R},}$$

provided the consistency condition is satisfied.

Example

$$\text{Let} \quad \mathbf{A} := \begin{bmatrix} 0 & \boxed{2} & 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & \boxed{3} & 5 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} := \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

As we have seen earlier, here $m = 4$, $n = 6$, $r = 3$, pivotal columns: $\boxed{2}$, $\boxed{4}$ and $\boxed{5}$, and nonpivotal columns: $\boxed{1}$, $\boxed{3}$, $\boxed{6}$.

Since $b_4 = 0$, the linear system $\mathbf{Ax} = \mathbf{b}$ is consistent.

For a particular solution of $\mathbf{Ax} = \mathbf{b}$, let $x_1 = x_3 = x_6 = 0$.
Then $x_5 + 2x_6 = 2 \implies x_5 = 2$,
$3x_4 + 5x_5 + 0x_6 = 1 \implies x_4 = -3$,
$2x_2 + x_3 + 0x_4 + 2x_5 + 5x_6 = 0 \implies x_2 = -2$.
Thus $\mathbf{x}_0 := \begin{bmatrix} 0 & -2 & 0 & -3 & 2 & 0 \end{bmatrix}^\mathsf{T}$ is a particular solution.

Basic solutions of $\mathbf{Ax} = \mathbf{0}$:
$x_1 = 1, x_3 = x_6 = 0$ gives $\mathbf{s}_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$,
$x_3 = 1, x_1 = x_6 = 0$ gives $\mathbf{s}_3 := \begin{bmatrix} 0 & -1/2 & 1 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$,
$x_6 = 1, x_1 = x_3 = 0$ gives $\mathbf{s}_6 := \begin{bmatrix} 0 & -1/2 & 0 & 10/3 & -2 & 1 \end{bmatrix}^\mathsf{T}$.

The general solution of $\mathbf{Ax} = \mathbf{b}$ is given by
$\mathbf{x} = \mathbf{x}_0 + \alpha_1 \mathbf{s}_1 + \alpha_3 \mathbf{s}_3 + \alpha_6 \mathbf{s}_6$, that is,
$x_1 = \alpha_1,\ x_2 = -2 - (\alpha_3 + \alpha_6)/2,\ x_3 = \alpha_3,\ x_4 = -3 + 10\alpha_6/3,$
$x_5 = 2(1 - \alpha_6),\ x_6 = \alpha_6$, where $\alpha_1, \alpha_3, \alpha_6 \in \mathbb{R}$.

Conclusion

Suppose an $m \times n$ matrix $\mathbf{A}$ is in a REF, and let r be the number of nonzero rows of $\mathbf{A}$. If $\mathbf{b} \in \mathbb{R}^{m\times 1}$, then the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has

(i) no solution if one of $b_{r+1}, \ldots, b_m$ is nonzero.

(ii) a unique solution if $b_{r+1} = \cdots = b_m = 0$ and $r = n$.

(iii) infinitely many solutions if $b_{r+1} = \cdots = b_m = 0$ and $r < n$. (In this case, there are $\boxed{n - r \text{ free variables}}$ which give $\boxed{n - r \text{ degrees of freedom}}$.)

Considering the case $\mathbf{b} = \mathbf{0} \in \mathbb{R}^{m\times 1}$ and recalling that $r \leq m$, we obtain the following important results.

Proposition

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be in REF with r nonzero rows. Then the linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the zero solution if and only if $r = n$. In particular, if $m < n$, then $\mathbf{A}\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Gauss Elimination Method (GEM)

We have seen how to solve the linear system $\mathbf{Ax} = \mathbf{b}$ when the matrix $\mathbf{A}$ is in a row echelon form (REF).

We now explain the **Gauss Elimination Method** (GEM) by which we can transform any $\mathbf{A} \in \mathbb{R}^{m \times n}$ to a REF.
This involves the following two **elementary row operations** (EROs):

Type I: Interchange of two rows

Type II: Addition of a scalar multiple of a row to another row

We shall later consider the following elementary row operation:

Type III: Multiplication of a row by a nonzero scalar

Consider the matrix $T_{i,j}$, which is obtained from the identity matrix by interchanging the row i and the row j. Then

$$T_{i,j} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

Then for any matrix A, the result of interchanging the row i and row j of A is given by the matrix $T_{i,j}A$.

For any constant α, consider the matrix $L_{i,j}(\alpha)$, which is obtained from the identity matrix by adding α in the (i, j)-th position. So, $L_{ij}(\alpha) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \alpha & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$

If A is a matrix, then the result of adding α times the row j to the row i is given by the matrix $L_{i,j}(\alpha)A$.

First we remark that if the augmented matrix $[\mathbf{A}|\mathbf{b}]$ is transformed to a matrix $[\mathbf{A}'|\mathbf{b}']$ by any of the EROs, then $\mathbf{A}\mathbf{x} = \mathbf{b} \iff \mathbf{A}'\mathbf{x} = \mathbf{b}'$ for $\mathbf{x} \in \mathbb{R}^{n\times 1}$, that is, the linear systems $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{A}'\mathbf{x} = \mathbf{b}'$ have the same solutions.

This follows by noting that an interchange of two equations does not change the solutions, neither does an addition of an equation to another, nor does a multiplication of an equation by a nonzero scalar, since these operations can be undone by similar operations, namely, interchange of the equations in the reverse order, subtraction of an equation from another, and division of an equation by a nonzero scalar.

Consequently, we are justified in performing EROs on the augmented matrix $[\mathbf{A}|\mathbf{b}]$ in order to obtain all solutions of the given linear system.

Transformation to REF

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, that is, let $\mathbf{A}$ be an $m \times n$ matrix with entries in $\mathbb{R}$. If $\mathbf{A} = \mathbf{O}$, the zero matrix, then it is already in REF.

Suppose $\mathbf{A} \neq \mathbf{O}$.

(i) Let column k_1 be the first nonzero column of $\mathbf{A}$, and let some nonzero entry p_1 in this column occur in the jth row of $\mathbf{A}$. Interchange row j and row 1. Then $\mathbf{A}$ is transformed to

$$\mathbf{A}' := \begin{bmatrix} 0 & \cdots & 0 & \boxed{p_1} & * & \cdots & * \\ 0 & \cdots & 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & * & * & \cdots & * \end{bmatrix},$$

where $*$ denotes a real number. Note: p_1 becomes the chosen pivot in row 1. (This choice may not be unique.)

(ii) Since $p_1 \neq 0$, add suitable scalar multiples of row 1 of $\mathbf{A}'$ to rows 2 to m of $\mathbf{A}'$, so that all entries in column k_1 below the pivot p_1 are equal to 0. Then $\mathbf{A}'$ is transformed to

$$\mathbf{A}'' := \begin{bmatrix} 0 & \cdots & 0 & \boxed{p_1} & * & \cdots & * \\ 0 & \cdots & 0 & \boxed{0} & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \boxed{0} & * & \cdots & * \end{bmatrix}.$$

(iii) Keep row 1 of $\mathbf{A}''$ intact, and repeat the above process for the remaining $(m-1) \times n$ submatrix of $\mathbf{A}''$ to obtain

$$\mathbf{A}''' := \begin{bmatrix} 0 & \cdots & 0 & \boxed{p_1} & * & * & * & * & \cdots & * \\ 0 & \cdots & 0 & \boxed{0} & \cdots & 0 & \boxed{p_2} & * & \cdots & * \\ 0 & \cdots & 0 & \boxed{0} & \cdots & 0 & \boxed{0} & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \boxed{0} & \cdots & 0 & \boxed{0} & * & \cdots & * \end{bmatrix},$$

where $p_2 \neq 0$ and occurs in column k_2 of $\mathbf{A}'''$, where $k_1 < k_2$.

Note: p_2 becomes the chosen pivot in row 2. (Again, this choice may not be unique.)

(iv) Keep rows 1 and 2 of $\mathbf{A}'''$ intact, and repeat the above process till the remaining submatrix has no nonzero row. The resulting $m \times n$ matrix is in REF with pivots $p_1, \ldots, p_r$ in columns $k_1, \ldots, k_r$, and the last $m - r$ rows are zero rows, where $1 \leq r \leq m$.

Notation

$R_i \longleftrightarrow R_j$ will denote the interchange of the ith row R_i and the jth row R_j for $1 \leq i, j \leq m$ with $i \neq j$.

$R_i + \alpha R_j$ will denote the addition of α times the jth row R_j to the ith row R_i for $1 \leq i, j \leq m$ with $i \neq j$.

αR_j will denote the multiplication of the jth row R_j by the nonzero scalar α for $1 \leq j \leq m$.

Remark

A matrix $\mathbf{A}$ may be transformed to different REFs by EROs.

For example, we can transform $\mathbf{A} := \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ by EROs to $\begin{bmatrix} 1 & 3 \\ 0 & -6 \end{bmatrix}$ as well as to $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, both of which are REFs.

Examples

(i) Consider the linear system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 0 \\ 6x_1 + 2x_2 + 4x_3 &= 6. \end{aligned}$$

We can check that

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} \boxed{3} & 2 & 1 & 3 \\ 0 & \boxed{-1/3} & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} \boxed{3} & 2 & 1 & 3 \\ 0 & \boxed{-1/3} & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right].$$

Here $m = 3 = n$, $r = 2$ and $b'_{r+1} = b'_3 = 12 \neq 0$. Hence the given linear system has no solution.

(ii) Consider the linear system

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80. \end{aligned}$$

As we have already seen in Lecture 2,

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 0 \\ 0 & \boxed{10} & 25 & 90 \\ 0 & 0 & \boxed{-95} & -190 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Here $m = 4$, $n = 3$, $r = 3$, pivotal columns: $\boxed{1}$, $\boxed{2}$, $\boxed{3}$.

Since $b'_{r+1} = b'_4 = 0$ and $r = n$, the linear system has a unique solution, namely $\mathbf{x}_0 := \begin{bmatrix} 2 & 4 & 2 \end{bmatrix}^{\mathsf{T}}$, which we had obtained by back substitution in Lecture 2.

(iii) Consider the linear system

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ 1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1. \end{aligned}$$

We can check that

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\xrightarrow{\text{EROs}} \left[\begin{array}{cccc|c} \boxed{3} & 2 & 2 & -5 & 8 \\ 0 & \boxed{1.1} & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] =: [\mathbf{A}'|\mathbf{b}'].$$

Here $m = 3, n = 4, r = 2$, pivotal columns: $\boxed{1}$, $\boxed{2}$, nonpivotal columns: $\boxed{3}$, $\boxed{4}$.

Since $b'_{r+1} = b'_3 = 0$, the linear system $\mathbf{A}'\mathbf{x} = \mathbf{b}'$ has a solution.

For a particular solution of $\mathbf{A}'\mathbf{x} = \mathbf{b}'$, let $x_3 = x_4 = 0$. Then
$1.1\,x_2 = 1.1 \implies x_2 = 1$,
$3x_1 + 2x_2 = 8 \implies x_1 = 2$,
Thus $\mathbf{x}_0 := \begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$ is a particular solution.

Since $r = 2 < 4 = n$, the linear system has many solutions.
For basic solutions of $\mathbf{A}'\mathbf{x} = \mathbf{0}'$, where $\mathbf{0}' = \mathbf{0}$,
let $x_3 = 1, x_4 = 0$, so that $\mathbf{s}_3 := \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}^{\mathsf{T}}$,
and $x_4 = 1, x_3 = 0$, so that $\mathbf{s}_4 := \begin{bmatrix} -1 & 4 & 0 & 1 \end{bmatrix}^{\mathsf{T}}$,

The general solution of $\mathbf{A}'\mathbf{x} = \mathbf{b}'$ is given by
$\mathbf{x} = \mathbf{x}_0 + \alpha_3\mathbf{s}_3 + \alpha_4\mathbf{s}_4$, that is,
$x_1 = 2 - \alpha_4,\ x_2 = 1 - \alpha_3 + 4\alpha_4,\ x_3 = \alpha_3,\ x_4 = \alpha_4$, where α_3, α_4 are arbitrary real numbers. These are precisely the solutions of the given linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Proposition

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then the linear system $\mathbf{Ax} = \mathbf{0}$ has only the zero solution if and only if any REF of $\mathbf{A}$ has n nonzero rows. In particular, if $m < n$, then $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution.

Proof. We saw that these results hold if $\mathbf{A}$ itself is in REF. Since every $\mathbf{A} \in \mathbb{R}^{m \times n}$ can be transformed to a REF $\mathbf{A}'$ by EROs, and since the solutions of the linear system $\mathbf{Ax} = \mathbf{0}$ and the transformed system $\mathbf{A}'\mathbf{x} = \mathbf{0}'$, where $\mathbf{0}' = \mathbf{0}$, are the same, the desired results hold. $\square$

Note: Suppose an $m \times n$ matrix $\mathbf{A}$ is transformed by EROs to different REFs $\mathbf{A}'$ and $\mathbf{A}''$. Suppose $\mathbf{A}'$ has r' nonzero rows and $\mathbf{A}''$ has r'' nonzero rows. Then $0 \leq r', r'' \leq \min\{m, n\}$. The above result implies that $r' = n \iff r'' = n$. We shall later see that $r' = r''$ always.

A Challenge Problem

Let $\mathbf{A} \in \mathbb{R}^{9\times 4}$ and $\mathbf{B} \in \mathbb{R}^{7\times 3}$. Is there $\mathbf{X} \in \mathbb{R}^{4\times 7}$ such that $\mathbf{X} \neq \mathbf{O}$ but $\mathbf{AXB} = \mathbf{O}$?

Inverse of a Square Matrix

We now introduce a special kind of square matrices.

Let $\mathbf{A}$ be a square matrix of size $n \in \mathbb{N}$, that is, let $\mathbf{A} \in \mathbb{R}^{n \times n}$. We say that $\mathbf{A}$ is **invertible** if there is $\mathbf{B} \in \mathbb{R}^{n \times n}$ such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, and in this case, $\mathbf{B}$ is called an **inverse** of $\mathbf{A}$.

Examples

The matrix $\mathbf{A} := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is invertible. To see this, let $\mathbf{B} := \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$, and check $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$. On the other hand, the nonzero matrix $\mathbf{A} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible. To see this, let $\mathbf{B} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{\mathbf{2} \times \mathbf{2}}$ and note that $\mathbf{AB} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \neq \mathbf{I}$.

If $\mathbf{A}$ is invertible, then it has a unique inverse. In fact, if $\mathbf{AC} = \mathbf{I} = \mathbf{BA}$, then $\mathbf{C} = \mathbf{IC} = (\mathbf{BA})\mathbf{C} = \mathbf{B}(\mathbf{AC}) = \mathbf{BI} = \mathbf{B}$ by the associativity of the matrix multiplication.

If $\mathbf{A}$ is invertible, its inverse will be denoted by $\mathbf{A}^{-1}$, and so $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{AA}^{-1}$. Clearly, $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$. If $\mathbf{A}$ is invertible and if one can guess its inverse, then it is easy to verify that it is in fact the inverse of $\mathbf{A}$. Here is a case in point.

Proposition

Let $\mathbf{A}$ be a square matrix. Then $\mathbf{A}$ is invertible if and only if $\mathbf{A}^\mathsf{T}$ is invertible. In this case, $(\mathbf{A}^\mathsf{T})^{-\mathbf{1}} = (\mathbf{A}^{-\mathbf{1}})^\mathsf{T}$.

Proof. Suppose $\mathbf{A}$ is invertible and $\mathbf{B}$ is its inverse. Then $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, and so $\mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T} = \mathbf{I}^\mathsf{T} = \mathbf{A}^\mathsf{T}\mathbf{B}^\mathsf{T}$. Since $\mathbf{I}^\mathsf{T} = \mathbf{I}$, we see that $\mathbf{A}^\mathsf{T}$ is invertible and $(\mathbf{A}^\mathsf{T})^{-\mathbf{1}} = \mathbf{B}^\mathsf{T} = (\mathbf{A}^{-\mathbf{1}})^\mathsf{T}$.

Next, if $\mathbf{A}^\mathsf{T}$ is invertible, then $\mathbf{A} = (\mathbf{A}^\mathsf{T})^\mathsf{T}$ is invertible. $\square$