1. (i) The additive inverse is not available for all entries.
(ii) If y_1, y_2 are two solutions, then for $y = y_1 + y_2$, $xy' + y = 6x^2 \neq 3x^2$.
(iii) The sum of two solutions is not a solution.
(iv) The 0 matrix is not there.

2. (i) It is a subspace spanned by $x, x^2, \ldots, x^n$.
(ii) It is a subspace spanned by $1, x^3, \ldots, x^n$.
(iii) It is a subspace spanned by $\{x^{2j+1} : 0 < 2j+1 \leq n\}$.

3. Suppose $a + b\cos t + c\sin t \equiv 0$ on $[-\pi, \pi]$, where $a, b, c \in \mathbb{R}$. Putting $t = 0$, $a + b = 0$. Putting $t = -\pi$, $a - b = 0$. Thus, $a = b = 0$. Consequently, $c\sin t \equiv 0$, which means $c = 0$ as well. This shows S_1 is linearly independent.

On the other hand, $\cos^2 t + \sin^2 t \equiv 1$, which shows S_2 is linearly dependent.

4. Note that v_1, v_2, v_3 are linearly dependent.

5. $\dim(W_1) = n$, $\dim(W_2) = n(n+1)/2 = \dim(W_3)$, $\dim(W_4) = n(n-1)/2$. To see these, note, for example, that to specify a symmetric matrix it is enough to specify the diagonal entries and entries above the diagonal, which are $1+2+\cdots+n = n(n+1)/2$ in number. Similarly for upper triangular matrices. A skew-symmetric matrix is specified by entries above diagonal, because the diagonal entries are all zero.

6. That $V \times W$ is a vector space is straightforward. If $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W, then $\{(v_i, 0), (0, w_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis of $V \times W$.

7. That T is linear is clear, the matrix of T is just $\mathbf{A}$.

8. Straight calculation.

9. In an inner product space the norm is defined as $||v||^2 = \langle v, v\rangle$. Thus, $||v+w||^2 + ||v-w||^2 = \langle v+w, v+w\rangle + \langle v-w, v-w\rangle = ||v||^2 + \langle v, w\rangle + \langle w, v\rangle + ||w||^2 + ||v||^2 - \langle v, w\rangle - \langle w, v\rangle + ||w||^2 = 2(||v||^2 + ||w||^2)$.

10. As matrix multiplication is linear in each component and as the trace is linear, $\langle A, B\rangle = \mathrm{trace}(A^*B)$ is linear in both A and B. Note that for any matrix C, $\mathrm{trace}(C) = \mathrm{trace}(C^t) = \overline{\mathrm{trace}(C^*)}$. Thus, $\langle A, B\rangle = \overline{\mathrm{trace}(A^*B)^*} = \overline{\mathrm{trace}(B^*A)} = \overline{\langle B, A\rangle}$. Finally, the (j,k)-th entry of A^*A is $\langle a_j, a_k\rangle$ where a_j is the j-th column of A. Thus $\langle A, A\rangle = \sum_j ||a_j||^2$. This is zero if and only if all a_j are zero.

11. Let $V = C[-\pi, \pi]$. Define $\langle -, -\rangle : V \times V \to \mathbb{R}$ by $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\, dt$. It is clear that $\langle -, -\rangle$ is an inner product. It is an easy exercise in calculus that $\langle \sin nt, \cos mt\rangle = 0$ for all n, m, while $\langle \sin mt, \sin nt\rangle = 0$ unless $m = n \neq 0$ and $\langle \sin nt, \sin nt\rangle = \pi$ for $n \neq 0$. Similarly, $\langle \cos nt, \cos mt\rangle = 0$ if $m \neq n$, while $\langle \cos nt, \cos nt\rangle = \pi$ if $n \neq 0$ and 2π if $n = 0$.

12. As T is Hermitian $\langle Tv, w\rangle = \langle v, Tw\rangle$.

(i) $\langle Tv, v\rangle = \langle v, Tv\rangle$, which, by the properties of the inner product, is equal to $\overline{\langle Tv, v\rangle}$.

(ii) If $Tv = \lambda v$, where $v \neq 0$, then $\lambda\langle v, v\rangle = \langle v, \lambda v\rangle = \langle v, Tv\rangle = \langle Tv, v\rangle = \langle \lambda v, v\rangle = \overline{\lambda}\langle v, v\rangle$. Thus, $\lambda = \overline{\lambda}$.

(iii) As λ, μ are real, $\lambda\langle v, w\rangle = \langle \lambda v, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \langle v, \mu w\rangle = \mu\langle v, w\rangle$. As $\lambda \neq \mu$, it follows that $\langle v, w\rangle = 0$.

(iv) Let $v \in W^{\perp}$. Let $w \in W$. Then $\langle Tv, w\rangle = \langle v, Tw\rangle = 0$ as $Tw \in W$.