Reinforced Concrete

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Part I

Materials for Reinforced Concrete

Introduction

Concrete, the most widely used human-made material, is an artificial building material, made of *aggregates* bonded together with a fluid *cement* that hardens to form a hard matrix that binds the materials together into a durable stone-like material.

The essential components of concrete are **cement**, **aggregates** and **water**.

Roman used concrete made with natural (pozzolana) and easy to produce (quicklime) cements. Concrete technology was mostly forgotten formore than 1000 years until the invention of Portland cement.

Today, concrete is typically made with Portland cement, that reacts with water to form the hard matrix.

Different cements can be used to produce concrete, e.g., also asphalt is a concrete in which the role of cement is taken by bitumen.

Introduction — Steel

Concrete has a very good strength with respect to compressive stresses, but not so much w/r to tensile stresses.

Romans used concrete in their construction, but used it to build walls, arches and domes in which the resistance to loadings was possible by developing mostly compressive stresses.

The reinforced concrete, where concrete works in compression and reinforcement steel works in tension, was developed starting mid XIX century, initially in France and later in every industrial country that could produce steel and cement as it made possible economical constructions.

The reinforcement steel currently used in Italy is a low-carbon alloy that is not particularly strong but exhibits a good ductility, useful in earthquake resistant designs.

Cement

- Cement, a *hydraulic binder* (that needs water to react), is a fine powder that, mixed with water, produces a slurry that in a few hours hardens due to a chemical reaction of *hydration*.

  The hydration process continues as long as there is available water in concrete, leading to an increase in the strength of it.

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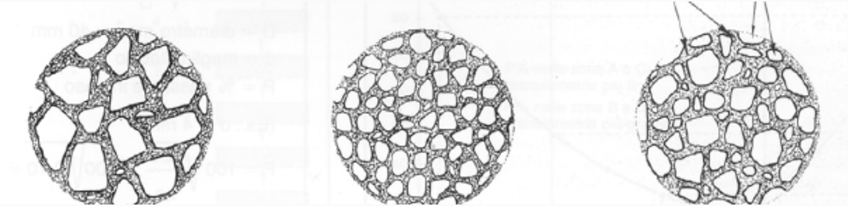
- The most common cement is Portland cement, obtained finely grinding *clinker*, that in turn is produced by sintering (fusing together without melting) limestone and clay into a special oven (the cement kiln). Cement today contains also different components to improve its characteristics.

Aggregates are the solid skeleton of concrete and constitute the main part, in weight, of concrete. Their quality is important for a good concrete.

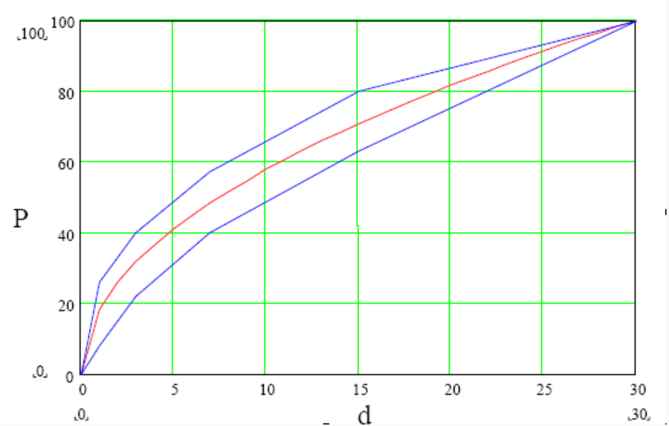
Aggregates are made by gravel and sand, mixed in different sizes to get a suitable *granulometry* — the percentage of voids in the aggregate should be as low as possible to have a good concrete using the least possible amount of cement.



Large, large voids — small, small voids but small strength — optimal.

Granulometric curve

The weight percent p of aggregates that are finer than a given diameter d, $p(d)$ is usually described by the Fuller curve,



$$p(d) = 100\sqrt{\frac{d}{d_{\max}}}$$

Another formulation has $p(d) = 100\left(\frac{d}{d_{\max}}\right)^{0.45}$

Aggregate characteristics

Large grains should be from high strength rock, small grains should be from siliceous sands rather than calcareous ones.

Aggregates should be clean, without clay or organic materials that influence bonding between aggregates and cement.

Aggregates influence

- concrete cost,
- strength,
- elastic modulus.

Water

Water in concrete serves two purposes

- reacts with cement, turning the slurry in a solid mass and
- acts as a fluidifying, permitting to cast the slurry in forms.

One one side, any excess of water w/r to the amount required to react with cement leads to a weaker concrete, on the other not enough water leads to problems when casting the concrete — in general more water than the chemically required amount is used.

It is recommended to use best quality water, free of residuals of organic or chemical nature — impurities interfere with the hydration reaction, leading to weaker concrete.

In particular the presence of sulfates leads to undesirable phenomena of concrete degradation.

	mass [kg]	volume [dm^3]
air	—	20
water	180	180
cement	300	100
sand	580	223
gravel	1240	477

Typical composition of a cubic metre of concrete

Concrete strength is measured using compression tests on cylinders or cubes of standard dimensions, computing the stress corresponding to the specimen collapse.

Strength classification

- low strength, $f_c \leq 20\,\mathrm{MPa}$,
- medium strength, $20\,\mathrm{MPa} \leq f_c \leq 40\,\mathrm{MPa}$,
- low strength, $40\,\mathrm{MPa} \leq f_c$.

Density classification

- Regular concrete, approx. $2400\,\mathrm{kg\,m^{-3}}$,
- light concrete, lighter than $1800\,\mathrm{kg\,m^{-3}}$,
- Regular concrete, heavier than $3200\,\mathrm{kg\,m^{-3}}$.

Strength depends on...

Cement

Strength increases almost linearly with cement quantity up to values of 500 kg per cubic metre.

Aggregates

Granulometry and quality of aggregates are both important.

Water/Cement ratio

The chemically required amount of water is 30 kg every 100 kg of cement, but such a cement paste is impossible to work with, w/o additives at least 60/100 is required.

Additives

Commonly used additives are plasticizers that permit to use lower water/cement ratio, increasing the workability of concrete.

Other additives

Other than plasticizers (or fluidifiers)

Accelerators speed up the hydration (hardening) of the concrete, useful to mature concrete in cold weather.

Retarders slow the hydration of concrete, used in large pours where partial setting is undesirable.

Air entraining agents add tiny air bubbles, which reduces damage during freeze-thaw cycles, increasing durability (at the price of a strength decrement)

Pigments can be used for aesthetics.

Corrosion inhibitors to minimize the corrosion of steel.

Environmental conditions

Hardening speed increases with temperature, but a dry hot climate and direct sunlight are nefarious, because lead to evaporation of water near the surface, lower strength and cracking.

In Summer concrete casting must be kept wet and covered to reduce evaporation.

On the other hand, a cold climate slows down the hydration process and hardening. Further, if water freezes, hydration stops and bonds between cement and cement and aggregates are ruined by water-ice expansion.

Cover can be attained using mats, towels, sand or waterproof film.

Hardening

In presence of free water hardening can last for years, with always increasing strengths.

Humidity influences the final strength of concrete.

A fast hardening can be obtained using hot steam.

If a concrete is workable, the pourings are compact, free of cavities and other defects.

Workability is measured using Abrams' cone.

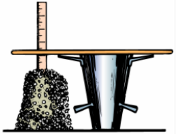
1. Stand on the two foot pieces of cone to hold in firmly in the place during Steps 1 though 4. Fill cone mold 1/3 full by volume [2-5/8" (67 mm) high] with the concrete sample and rod it with 25 strokes using a round, straight steel rod of 5/8" (16 mm) diameter x 24" (600 mm) long with a hemispherical tip end. Uniformly distribute strokes over the cross section of each layer. For the botton layer. this will necessitate inclining the rod slightly and making approximately half the strokes near the perimeter (out edge), then progressing with vertical strokes spirally toward the center.

2. Fill cone 2/3 full by volume (half the height) and again rod 25 times with rod just penetrating into, but not througth, the first layer. Distribute strokes evenly as described in Step 1.

3 Fill cone to overflowing and again rod 25 times with rod just penetrating into, but not through, the second layer. Again distribute strokes evenly.

4 Strikes off excess concrete form top of cone with the steel rod so the cone is exactly level full. Clean the overflow away from the base of the cone mold.

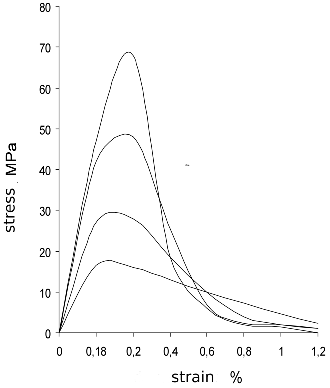
5 Immediately after completion of Step 4, the operation of raising the mold shall be performed in 5±2 sec. by a steady upward lift with no lateral or torsional motion being imparted to the concrete. The entire operation from the start of the filling through removal of the mold shall be carried out without interruption and shall be completed within an elasped time of 2-1/2 minutes



6 Place the steel rod horizontally across the inverted mold so the rod extends over the slumped concrete. Immediately measure the distance from botton of the steel rod to the displaced original center of the specimen. This distance, to the nearest 1/4 inch (6 mm), is the slump of the concrete. If a decided falling away or shearing off concrete from one side or portion of the mass occurs, disregard the test and make a new test on another portion of the sample.

Mechanical Characteristics

Compression tests are performed on 15x15x15 cm cubes or on cylinders 15 cm wide and 30 cm high,



Strain-stress curves are different for different concrete types.

We have hence two different rupture stress measures and two different characteristic values of the strength, the characteristic stress f_c and the cubic characteristic stress, $f_{c,\text{cubic}}$, where $f_c < f_{c,\text{cubic}}$, in any case the strength of 95% of the specimens is greater than the characteristic value.

Strength class

In design, we specify a class for the concrete, implying that the manufacturer has to demonstrate that the characteristic strength of the cast concrete is no less than a design value.

Specifying a concrete class we specify not only a characteristic value of the compressive strength but also a number of other design parameters.

Strength classes for concrete															Analytical relation / Explanation
f_{ck} (MPa)	12	16	20	25	30	35	40	45	50	55	60	70	80	90	
$f_{ck,cube}$ (MPa)	15	20	25	30	37	45	50	55	60	67	75	85	95	105	
f_{cm} (MPa)	20	24	28	33	38	43	48	53	58	63	68	78	88	98	$f_{cm} = f_{ck}+8$(MPa)
f_{ctm} (MPa)	1,6	1,9	2,2	2,6	2,9	3,2	3,5	3,8	4,1	4,2	4,4	4,6	4,8	5,0	$f_{ctm}=0{,}30\times f_{ck}^{(2/3)}$ ≤C50/60 $f_{ctm}=2{,}12\cdot\ln(1+(f_{cm}/10))$ > C50/60
$f_{ctk,0,05}$ (MPa)	1,1	1,3	1,5	1,8	2,0	2,2	2,5	2,7	2,9	3,0	3,1	3,2	3,4	3,5	$f_{ctk,0,05} = 0{,}7\times f_{ctm}$ 5% fractile
$f_{ctk,0,95}$ (MPa)	2,0	2,5	2,9	3,3	3,8	4,2	4,6	4,9	5,3	5,5	5,7	6,0	6,3	6,6	$f_{ctk,0,95} = 1{,}3\times f_{ctm}$ 95% fractile
E_{cm} (GPa)	27	29	30	31	32	34	35	36	37	38	39	41	42	44	$E_{cm} = 22[(f_{cm})/10]^{0,3}$ (f_{cm} in MPa)
ε_{c1} (‰)	1,8	1,9	2,0	-2,1	2,2	2,25	2,3	2,4	2,45	2,5	2,6	2,7	2,8	2,8	see Figure 3.2 ε_{c1} (‰) $= 0{,}7 f_{cm}^{0,31} < 2.8$
ε_{cu1} (‰)	3,5									3,2	3,0	2,8	2,8	2,8	see Figure 3.2 for $f_{ck} \ge 50$ Mpa ε_{cu1}(‰)$=2{,}8+27[(98-f_{cm})/100]^4$
ε_{c2} (‰)	2,0									2,2	2,3	2,4	2, 5	2,6	see Figure 3.3 for $f_{ck} \ge 50$ Mpa ε_{c2}(‰)$=2{,}0+0{,}085(f_{ck}-50)^{0,53}$
ε_{cu2} (‰)	3,5									3,1	2,9	2,7	2,6	2,6	see Figure 3.3 for $f_{ck} \ge 50$ Mpa ε_{cu2}(‰)$=2{,}6+35[(90-f_{ck})/100]^4$
n	2,0									1,75	1,6	1,45	1,4	1,4	for $f_{ck} \ge 50$ Mpa $n=1{,}4+23{,}4[(90-f_{ck})/100]^4$
ε_{c3} (‰)	1,75									1,8	1,9	2,0	2,2	2,3	see Figure 3.4 for $f_{ck} \ge 50$ Mpa ε_{c3}(‰)$=1{,}75+0{,}55[(f_{ck}-50)/40]$
ε_{cu3} (‰)	3,5									3,1	2,9	2,7	2,6	2,6	see Figure 3.4 for $f_{ck} \ge 50$ Mpa ε_{cu3}(‰)$=2{,}6+35[(90-f_{ck})/100]^4$

	f_{ck}	R_{ck}	E_c	f_{ctm}	f_{cfm}	f_{ctk}	f_{cfk}	$\min(f_{bk})$	$\max(f_{bk})$
C16/20	16	20	28600	1.90	2.29	1.33	1.60	2.10	3.00
C20/25	20	25	29900	2.21	2.65	1.55	1.86	2.44	3.48
C25/30	25	30	31400	2.56	3.08	1.80	2.15	2.83	4.04
C28/35	28	35	32300	2.77	3.32	1.94	2.32	3.05	4.36
C32/40	32	40	33300	3.02	3.63	2.12	2.54	3.33	4.76
C35/45	35	45	34000	3.21	3.85	2.25	2.70	3.54	5.06
C40/50	40	50	35200	3.51	4.21	2.46	2.95	3.87	5.53
C45/55	45	55	36200	3.80	4.55	2.66	3.19	4.18	5.98
C50/60	50	60	37200	4.07	4.89	2.85	3.42	4.49	6.41

Design strength is deduced from f_{ck} using a reduction factor $\alpha_{cc} = 0.85$ to take into account all long-term effects (α_{cc} is not applied when you take into account instantaneous effects) and a material safety factor, specifically for concrete, $\gamma_c = 1.50$

$$f_{cd} = \alpha_{cc} \frac{f_{ck}}{\gamma_c}$$

$$f_{cd} = 0.5667 f_{ck} = \frac{17}{30} f_{ck} = \left(\frac{1}{2} + \frac{1}{15}\right) f_{ck}$$

Young modulus

Concrete has not a clearly defined linear elastic behavior, conventionally for a tested specimen the elastic modulus is the secant modulus for a stress $f = 0.4 f_c$. At any rate E is correlated to the strength of concrete, the relationship postulated by the Italian code being

$$E_c = 22\,000\,\text{MPa}\left(\frac{f_{ck} + 8\,\text{MPa}}{10\,\text{MPa}}\right)^{0.3}$$

For medium strength concrete, 30 to 35 GPa.

Poisson modulus

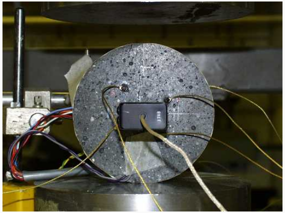
The Poisson modulus is $\nu = 0.20$ for not cracked concrete and $\nu = 0$ for cracked concrete.

Thermal expansion coefficient

It is $\alpha_T = 10^{-5}\,\text{K}^{-1}$

Concrete has a brittle failure in tension.

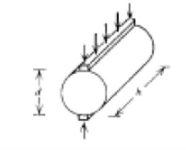
The tensile strength, that is correlated to compressive strength, is often measured using indirect tests.

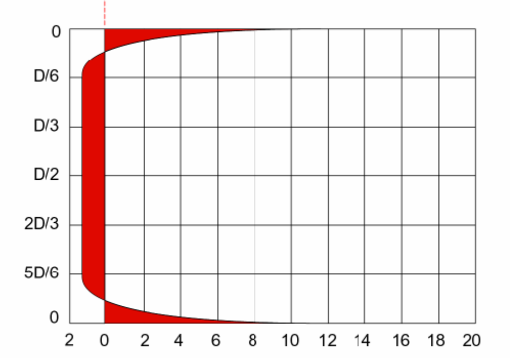


Mean tensile strength can be expressed in terms of f_{ck}, $f_{ctm} = 0.30 f_{ck}^{0.67}$ (beware that 0.30 holds for f_{ck} in MPa), characteristics values are obtained multiplying f_{ctm} by 0.70 or 1.30 respectively.

Tensile strength for bending is larger than for a centered axial force,

Brazilian test

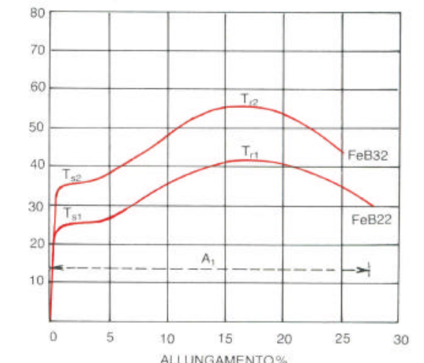


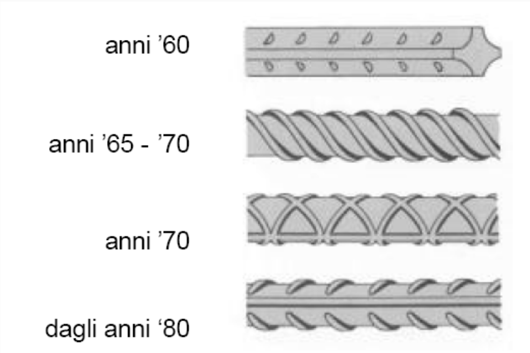


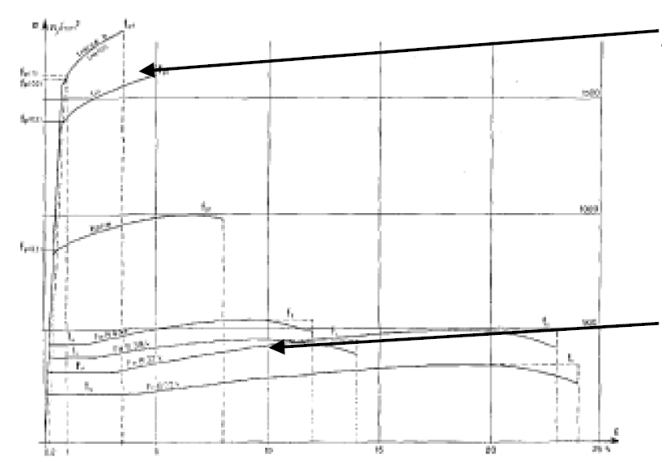
The reinforcement is of two different types of steel, **B450C** and **B450A**. For both classes the characteristic yield and tear strengths are the same

$$f_{yk} = 450\,\text{MPa}, \quad f_{tk} = 540\,\text{MPa}.$$

Steel **B450C** ensures a tear strain not less than 7.5%, **B450A** not less than 2.5%.







Acciai ad alta resistenza (cap)

Acciai da Cemento Armato

The steel bars are bond to the surrounding concrete by means of a chemical bond and a mechanical bond (ribbed bars).

Bond is necessary to transfer stresses between the two materials.

The stresses are exchanged in terms of tangential stresses, leading to a diagonal tension field and rupture of concrete in tension, the bonding strength is hence associate to concrete's tensile strength.

$$f_{bk} = 2.25\,\eta_1\,\eta_2\,f_{tck}$$

$\eta_1 = (1.0,\ 0.7)$ describes *bonding conditions*,
$\eta_2 = (1,\ (132\,\text{mm} - \Phi)/100\,\text{mm})$ depends on bar diameter Φ.

The development length is the minimum length of anchorage between steel and concrete that permits yielding of a reinforcement bar.

$$N_d = \frac{\pi\Phi^2}{4} f_{yd} = l_{b,\mathrm{rqrd}} \pi\Phi f_{Bud} \quad \longrightarrow \quad l_{b,\mathrm{rqrd}} = \frac{\Phi}{4}\frac{f_{yd}}{f_{cd}}$$

The development length can be reduced anchoring the bar mechanically using curved ends, hooks, welded plates.

Durability

Durability

Durability is the ability of a material to resist to environmental actions, chemical attacks, abrasion, any process leading to a loss of performances.

Concrete degradation

mechanical abrasion, erosion, collision, explosion

physical frost-thaw cycles, fire

structural overloads, fatigue, cyclic loads, adaptation

chemical alkaline aggregates, acid attack, sulfate/sulfide attack, clear water attack.

Reinforcement degradation

corrosion *carbonation*, chlorides, stray electric currents.

Durability

- Expected environmental conditions
- Use of the structure
- Required performances
- Material properties
- Members shape and construction details
- Quality assurance for the whole process, workmanship
- Maintenance

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1. Determine the level of environmental aggression.
2. Choice of materials, design of members and details to resist environmental hazards during the working life of the construction.

Concrete, on its own, protects the reinforcement from corrosion.

Concrete is alkaline, iron oxide is formed, iron is *passivated*.

When the alkalinity of concrete decreases, iron hydroxide is formed and (a) strength is reduced to zero in the corroded part and (b) there is a strong volume increase of the corroded part, that swells and expels the concrete cover.

Carbonation calcium reacts with **CO2** in air, pH diminishes and re-bars are exposed to corrosion.
The process is slow, faster if concrete has more pores hence faster for high values of w/c ratio.
The carbonated depth in mm is $d_c = k\sqrt{y}$, where k depends on w/c ratio and y is time in years. E.g, $k(0.6) = 10.1$, $k(0.5) = 7.0$ and $k(0.4) = 3.8$.

Chlorides when ion `Cl` is present in significant quantities the protective action of concrete weakens because of increased porosity, water and oxygen can reach and corrode re bars.

- exogenous: sea water or anti thaw salts: the `Cl` ion enters the concrete by (a) suction in dry concrete and (b) diffusion in pore water in wet concrete,
- endogenous: contaminated aggregates, sea sand!

Degradation depends on

- saturation levels of pores, if pore is almost full of water the increase of volume of ice exerts an internal pressure that damages concrete
- wetness of concrete, external layers are saturated and subject to ice expansion, leading to the opening of cracks,
- presence of cracks, that drives water to inner layers of concrete.

The problem is mitigated using low w/c ratios and/or entraining air into concrete (using specific additives) forming microscopic bubbles that act as chambers of compensation when ice forms (note that these bubbles reduce concrete strength).

Prescriptions in favor of Durability

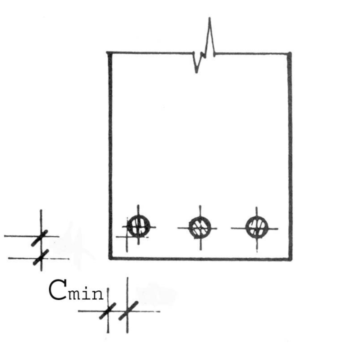
- Structural conception
- Choice of materials
- Details of constructions
- Execution
- Quality controls
- Surveillance and maintenance
- Special measures (stainless steel, coatings, cathodic protection, etc)

The single most important action to favor durability is the specification of an appropriate concrete cover c over the reinforcement.

$$c_{\text{nom}} = c_{\text{min}} + \Delta c_{\text{dev}}$$

$$c_{\text{min}} = \max\{c_{\text{min},b},\, c_{\text{min,dur}} + \Delta c_{\text{dur},\gamma} - \Delta c_{\text{dur,st}} - \Delta c_{\text{dur,add}},\, 10\,\text{mm}\}$$

$c_{\text{min},b}$	minimum cover for bonding
$c_{\text{min,dur}}$	m.c. for environmental hazards
$\Delta c_{\text{dur},\gamma}$	increment as safety factor
$\Delta c_{\text{dur,st}}$	reduction for use of stainless steel
$\Delta c_{\text{dur,add}}$	reduction for use of protective additives.



Single bar $c_{\min,b} = \Phi$

Bar bundles $c_{\min,b} = \Phi\sqrt{n_b} \leq 55\,\text{mm}$
where $n_b \leq (3, 4)$ is the nominal number of bars, 4 for vertical, compressed bars and 3 in all other cases.

If aggregate max diameter is larger than $32\,\text{mm}$ $c_{\min,b}$ must be incremented by $5\,\text{mm}$.

Class	Cause of degradation	# of sub-classes
X0	No hazard	1
XC	Carbonation	4
XD	Chlorides (no sea water)	3
XS	Chlorides (in sea salt)	3
XF	Frost thaw cycles	4
XA	Different chemical hazard	3

	Environment desc.	Examples
XC1	Dry or permanently wet	Inside building with low moisture content or permanently immersed in water
XC2	Wet, rarely dry	Surfaces exposed to water, foundations
XC3	Moderately wet	Inside, moderate or high moisture, exterior protected from rain
XC4	Cyclically wet and dry	Surfaces exposed to water not in XC2

	Environment desc.	Examples
XD1	Moderate moisture	Surfaces in contact with saline atmosphere
XD2	Wet, rarely dry	Pools, concrete exposed to industrial waters containing chlorides
XD3	Cyclically wet and dry	Bridge parts exposed to splashes, parking floors

	Environment desc.	Examples
XF1	Moderate water saturation, no anti-frost agents	Vertical surfaces exposed to rain and frost
XF2	… with anti-frost agents	Vertical surfaces of road structures
XF3	Elevate saturation, no agents	… as XF1
XF4	… with a/f agents or sea water	Roads, bridge decks, marine structures…

Minimum strength class

Carbonation	XC1, XC2	C25/30
	XC3, XC4	C30/37
Chlorides	XD1, XD2, XS1	C30/37
	XD3, XS2, XS3	C35/45
Frost+Thaw	XF1, XF2, XF3	C30/37
Chemical hazard	XA1, XA2	C30/37
	XA3	C35/45

The reference *structural class* for a working life of 50 years is 4, this class has to be corrected according to the following table

Structural Class							
Criterion	**Exposure Class according to Table 4.1**						
	X0	XC1	XC2 / XC3	XC4	XD1	XD2 / XS1	XD3 / XS2 / XS3
Service Life of 100 years	increase class by 2	increase class by 2	increase class by 2	increase class by 2	increase class by 2	increase class by 2	increase class by 2
Strength Class 1) 2)	≥ C30/37 reduce class by 1	≥ C30/37 reduce class by 1	≥ C35/45 reduce class by 1	≥ C40/50 reduce class by 1	≥ C40/50 reduce class by 1	≥ C40/50 reduce class by 1	≥ C45/55 reduce class by 1
Member with slab geometry (position of reinforcement not affected by construction process)	reduce class by 1	reduce class by 1	reduce class by 1	reduce class by 1	reduce class by 1	reduce class by 1	reduce class by 1
Special Quality Control ensured	reduce class by 1	reduce class by 1	reduce class by 1	reduce class by 1	reduce class by 1	reduce class by 1	reduce class by 1

Environmental Requirement for c_{min} (mm)							
Structural Class	Exposure Class according to Table 4.1						
	X0	XC1	XC2 / XC3	XC4	XD1 / XS1	XD2 / XS2	XD3 / XS3
1	10	10	10	15	20	25	30
2	10	10	15	20	25	30	35
3	10	10	20	25	30	35	40
4	10	15	25	30	35	40	45
5	15	20	30	35	40	45	50
6	20	25	35	40	45	50	55

In lack of prescription from the national annexes (no Italian prescriptions) the recommended values are:

$\Delta c_{\text{dur},\gamma} = 0\,\text{mm}$,

$\Delta c_{\text{dur,st}} = 0\,\text{mm}$,

$\Delta c_{\text{dur,add}} = 0\,\text{mm}$.

Recommended value: $\Delta c_{\text{dev}} = 10\,\text{mm}$.

Production subjected to quality control: $5\,\text{mm} \leq \Delta c_{\text{dev}} \leq 10\,\text{mm}$.

Production subjected to quality control and rejection of non conformal items: $0\,\text{mm} \leq \Delta c_{\text{dev}} \leq 10\,\text{mm}$.

In case of pours on irregular surfaces the recommended values, the minimum values are 40 mm (regularized surface) or 75 mm (direct pour on soil).

Example

A building with a planned working life of 50 years has external parts in concrete, indirectly exposed to rain, in class XC3 for the carbonation hazard.

For the external parts a concrete class C30/37 is required, while for the internal parts a class 25/30 is permitted.

A sensible design choice is to choose the same class, C30/37, for both the internal and the external parts (the cost difference is minimal).

Example

What about the concrete cover? The starting class is S4, no reductions are applicable for the external parts and for exposure class XC3 the minimum cover is 25 mm.

For the internal parts, because the class used is better than the class strictly necessary we are in class S3, hence the minimum cover is of 10 mm.

Again a minimum cover of 25 mm should be chosen. A deeper cover is requested in case of $\Phi > 25$ mm to fulfill the requirements in term of bond.

The cover must be measured from the external parts of stirrups.

What about slabs?

No external parts, so XC1 — the structural class is S2 because there is slab behavior and better concrete: the minimum cover for protection is hence 10 mm, less to the cover required for bonding.

Hypothesizing $\Phi \leq 14\,\mathrm{mm}$ with $5\,\mathrm{mm}$ for safety we have $c_{\mathrm{min}} = 19\,\mathrm{mm} = 20\,\mathrm{mm}$ for the cover of slabs re bars.

Part II

R.C. Beams — Axial Force, Bending Moment

Axial Force

- Linear Model
    - tension
    - compression
- Non Linear Model
    - tension
    - compression
- Design & Requirements

Due to bonding between concrete and steel the strains in concrete and in steel are the same, $\boldsymbol{\epsilon}_c = \boldsymbol{\epsilon}_s = \boldsymbol{\epsilon}$ and the stresses are $\boldsymbol{\sigma}_c = E_c\boldsymbol{\epsilon},\ \boldsymbol{\sigma}_s = E_s\boldsymbol{\epsilon}$.

From the first eq. it is $\boldsymbol{\epsilon} = \boldsymbol{\sigma}_c/E_c$ and substituting in the second one the steel stress is $\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_c \cdot E_s/Ec$.

The resultant of constant stresses is $N = A_c\boldsymbol{\sigma}_c + A_s\boldsymbol{\sigma}_s$, with the positions $n = E_s/E_c$, $\boldsymbol{\sigma}_s = n\boldsymbol{\sigma}_c$ it is

$$N = A_c\boldsymbol{\sigma}_c + n \cdot A_s\boldsymbol{\sigma}_c = \boldsymbol{\sigma}_c \cdot (A_c + n \cdot A_s) \quad \Rightarrow$$

$$\Rightarrow \quad \boldsymbol{\sigma}_c = N/(A_c + nA_s),\ \boldsymbol{\sigma}_s = n\boldsymbol{\sigma}_c$$

The stress in concrete can be computed with reference to an homogenized area $A_{\text{hom}} = A_c + n \cdot A_s$.

> *The coefficient n is the* homogenization coefficient. *— For instantaneous loadings n is the ratio of the Young modules, for long duration loadings we have to take into account the long term deformation of concrete (viscosity) and, conventionally, we use $n = 15$.*

We have two possibilities,

not cracked concrete the stress in concrete is below the tension strength, the formulas seen for compression are still valid,

cracked concrete when the loading exceeds the concrete strength concrete cracks and no stresses are transmitted,

$$\boldsymbol{\sigma}_c = 0, \quad N = 0 \cdot A_c + \boldsymbol{\sigma}_s \cdot A_s \Rightarrow \boldsymbol{\sigma}_s = N/A_s$$

It is possible that the centroids of the not cracked and the cracked section are not coincident and that a (small) bending moment is introduced when the section cracks.

In this case we are interested in the strength of the section at the ultimate limit state.

Tension — Concrete is cracked and gives no contribution, the design strength is

$$N_{Rd} = f_{yd}A_s.$$

Compression — Materials' stresses are both equal to the respective design values and the design strength is

$$N_{Rd} = f_{cd}A_c + f_{yd}A_s.$$

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No columns are exempt from bending moment, NTC (Italian Technical Norm for Constructions) prescribes that columns are verified for N and a bending moment $M_e = N \cdot e$ *due to an* accidental eccentricity $e = \max(20\,\text{mm}, h/20)$ *where h is the column's height.*

- Strength: $N_{Ed} \leq N_{Rd}$.
- Minimum steel: $A_s \geq 0.10 N_{Ed}/f_{yd}$.
- Steel concrete ratio: $0.003 \leq A_s/A_c \leq 0.040$,
  in seismic zones: $0.010 \leq A_s/A_c \leq 0.040$.
- Minimum diameter of longitudinal re bar: $\Phi \geq 12\,\text{mm}$.
- Maximum distance of l.r.: $d \leq 300\,\text{mm}$.
- Min diameter of ties (stirrups): $\Phi \geq \max(6\,\text{mm}, \Phi_{\text{max}}/4)$.
- Max tie distance: $s \leq \min(250\,\text{mm}, 12\Phi_{\text{min}})$.

EC2 requires that the distance between stirrups is reduced corresponding to a beam-column node to $0.6 s_{\text{standard}}$

Ties?

Ties in columns are the same as stirrups in beams.

In beams stirrups are required to provide shear strength and in columns we have no shear... so why there is a requirement for ties/stirrups in columns?

Ties are required for

- limiting the lateral instability of compressed re bars, that are very slender beams subjected to compressive forces, when the concrete cover is damaged and
- limiting the transversal expansion of compressed concrete (Poisson effect), this leads to a marked increase in ductility when the concrete is plasticized in compression — this effect is particularly important in seismic areas.

From a preliminary structural analysis we know only N_{Ed}, but we have to verify the column for a combination of N and M...

A convenient approach is to dimension the column so that its design strength is consistently larger than the design action.

For *normal* storey heights it's possible to see that a 15% stronger section is appropriate, so we will design our columns conservatively with a 20% larger strength, placing the most of the over-strength on the side of reinforcing steel that contributes most of the bending strength, using

$$A_c \geq N_{Ed}/f_{cd}, \quad A_s \geq 0.20 N_{Ed}/f_{yd}.$$

Tension, not cracked

- gross section 300 mm × 500 mm of C25/30 concrete
- 6 Φ20 re bars, B450C steel
- $N_{Ed} = 175\,\text{kN}$, tension

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$$E_c = 31\,500\,\text{MPa} \Rightarrow n = 200000/31500 = 6.35,$$
$$A_{\text{hom}} = 300 \cdot 500 + 6.35 \cdot 6 \cdot 10^2 \cdot \pi = 161\,970\,\text{mm}^2,$$
$$\sigma_c = N/A = 175\,000\,\text{N}/161\,970\,\text{mm}^2 = 1.08\,\text{MPa},$$
$$\sigma_s = n\sigma_c = 6.35 \cdot 1.08\,\text{MPa} = 6.90\,\text{MPa},$$
$$g_{ctk} = 1.8\,\text{MPa},\ f_{ctd} = f_{ctk}/1.5 = 1.2\,\text{MPa}.$$

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&g_{ctk} = 1.8\,\mathrm{MPa},\ f_{ctd} = f_{ctk}/1.5 = 1.2\,\mathrm{MPa}.
\end{aligned}
$$

Our section can resist its design action.

Which is the action that leads to cracked concrete and that we use to verify the steel?

$$N_{\text{crack}} = A_{\text{hom}} f_{ctk} = 291.5\,\text{kN}$$

The steel stress is, before cracking, $n\,f_{ctk} = 11.43\,\text{MPa}$ and after cracking we have

$$\sigma_s = 291\,500\,\text{N}/A_s = 154.7\,\text{MPa}.$$

Same section as before,
$N_{Ed} = 1000\,\text{kN}$ in compression.

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$N_{Ed} = 1000\,\text{kN}$ in compression.

In this case (compression) we deal with a long duration loading, hence $n = 15$,

$A_{\text{hom}} = 150\,000\,\text{mm}^2 + 15 \cdot 1884\,\text{mm}^2 = 178\,260\,\text{mm}^2$,
$\sigma_c = N_{Ed}/A_{\text{hom}} = 1 \times 10^6\,\text{N}/178\,260\,\text{mm}^2 = 5.61\,\text{MPa}$,
$\sigma_s = n \cdot \sigma_c = 15 \cdot 5.61\,\text{MPa} = 84.2\,\text{MPa}$.

C25/30 concrete, B450C steel, $N_{Ed} = 1750\,\text{kN}$ in compression.
$f_{cd} = 14.17\,\text{MPa}$, $fyd = 392.3\,\text{MPa}$.

The minimum areas of concrete and steel, using our design formulas, are

$$A_c \geq 1750\,\text{MPa}/14.17\,\text{MPa} = 123\,500\,\text{mm}^2$$

$$A_s \geq 0.2 \cdot 1750\,\text{MPa}/391.3\,\text{MPa} = 894\,\text{mm}^2$$

Respecting *exactly* these constraints, having an additional, self-imposed constraint in the column width $b = 300\,\text{mm}$ we can choose $h = 500\,\text{mm}$, while for steel we can use 6 Φ14, with $A_s = 914\,\text{mm}^2$

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As we are actually over-sizing our column to provide bending strength, we should investigate also the possibility of using $h = 400\,\mathrm{mm}$, the concrete area is *almost* OK but the steel area is larger than the area strictly required.

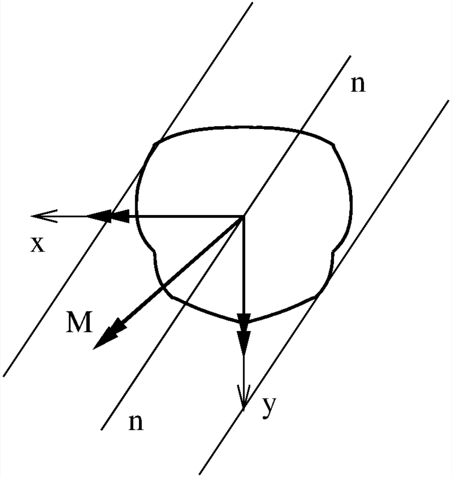
Bending Moment

Bending Moment

- Linear, not cracked.
- Linear. cracked.
- Non Linear, not cracked.
- Non Linear, cracked.

Our assumptions:

- homogenized section ($A_{\text{hom}} = n \cdot A_s$),
- tensile stress in concrete less than f_{ctd}, so that we have the contribution of all the concrete section.



Example

C25/30 concrete, B450C steel,
$b = 300\,\mathrm{mm}$, $h = 500\,\mathrm{mm}$, $4\Phi 20 + 2\Phi 14$, $n = 6.35$, $c = 40\,\mathrm{mm}$
$M_{Ed} = 50\,\mathrm{kNm}$.

Example

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$b = 300\,\text{mm}$, $h = 500\,\text{mm}$, $4\Phi20 + 2\Phi14$, $n = 6.35$, $c = 40\,\text{mm}$
$M_{Ed} = 50\,\text{kNm}$.

$A_s = 1256\,\text{mm}^2$, $A'_s = 308\,\text{mm}^2$, $A_c = 150\,000\,\text{mm}^2$,
$A_{\text{hom}} = 159\,930\,\text{mm}^2$
$S = 1500000 \cdot 250 + 6.35 \cdot (308 \cdot 40 + 1256 \cdot 460) = 41\,247\,000\,\text{mm}^3$
$y_G = S/A = 257.9\,\text{mm}$
$J = \frac{300 \cdot 500^3}{12} + 150000 \cdot (250 - 257.9)^2 + 308 \cdot (40 - 257.9)^2 + 1256 \cdot (460 - 257.9)^2$
$\Rightarrow J = 3\,552\,980\,000\,\text{mm}^4$

Example, continued

$$\sigma_{\mathrm{c,top}} = 50 \times 10^6\,\mathrm{Nmm}(-257.9\,\mathrm{mm})/3553 \times 10^6\,\mathrm{mm}^4 = -3.63\,\mathrm{MPa},$$
$$\sigma_{\mathrm{c,bot}} = 50 \times 10^6\,\mathrm{Nmm}(+242.1\,\mathrm{mm})/3553 \times 10^6\,\mathrm{mm}^4 = +3.41\,\mathrm{MPa},$$
$$\sigma_{\mathrm{s,top}} = 6.3550 \times 10^6\,\mathrm{Nmm}(-217.9\,\mathrm{mm})/3553 \times 10^6\,\mathrm{mm}^4 = -19.5\,\mathrm{MPa},$$
$$\sigma_{\mathrm{s,bot}} = 6.3550 \times 10^6\,\mathrm{Nmm}(+202.1\,\mathrm{mm})/3553 \times 10^6\,\mathrm{mm}^4 = +18.1\,\mathrm{MPa}.$$

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For C25/30 the characteristic tension strength in bending is $f_{cfk} = 2.16\,\text{MPa}$ hence the section is, in effect, cracked.

The moment that cause cracking is

$$\begin{aligned} M_{\text{crack}} &= J \cdot f_{cfk}/y_{\text{bot}} \\ &= \frac{3553 \times 10^6\,\text{mm}^4 \cdot 2.16\,\text{MPa}}{242.1\,\text{mm}} \\ &= 242.1\,\text{mm}. \end{aligned}$$

In this case we don't know the position of the neutral axis, we want to determine it writing an equation of equilibrium.

Glossary

b base of rectangular section

h height of the section

c concrete cover (center to edge)

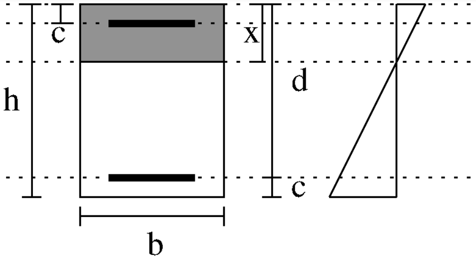
d depth of tensile steel

x (unknown) distance of the neutral axis from the compressed edge

z arm of the internal couple

A_s area of tensile steel

A_s' area of compressed steel



In general, the neutral axis must be barycentric or, equivalently, the static moment w.r.t the neutral axis is equal to zero. For a rectangular section, using the homogenization coefficient n

$$S_x = -\frac{bx^2}{2} + (nA_s')(c - x) + (nA_s)(d - x) = 0,$$

$$\left(\frac{b}{2}\right)x^2 + n(A_s + A_s')x - n(A_s d + A_s' c) = 0,$$

$$x^2 + 2n\frac{A_s + A_s'}{b}x - 2n\frac{A_s d + A_s' c}{b} = 0.$$

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$$x^2 + 2n\frac{A_s + A'_s}{b}x - 2n\frac{A_s d + A'_s c}{b} = 0.$$

The position of the neutral axis does not depend on strength but only on geometry and n, and is determined solving a 2nd degree equation.

Considering only the positive root,

$$x = n\frac{A_s + A_s'}{b}\left(-1 + \sqrt{1 + \frac{2}{n}\,\frac{A_s bd + A_s' bc}{(A_s + A_s')^2}}\right).$$

(The result was simplified w.r.t. the standard solution of a 2nd degree equation).

Having x we have

$$J = \frac{bx^3}{3} + nA_s(d-x)^2 + nA_s'(c-x)^2,$$

$$\sigma_{c,\max} = -\frac{M}{J}x, \quad \sigma_s = n\frac{M}{J}(d-x), \quad \sigma_s' = n\frac{M}{J}(c-x),$$

$$z = \frac{M}{A_s\sigma_s} = \frac{M}{A_s\, nM(d-x)/J} = \frac{J}{nA_s(d-x)}.$$

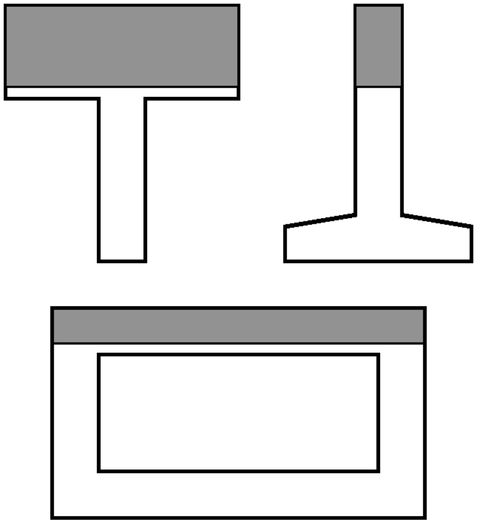
Example

Sheet1

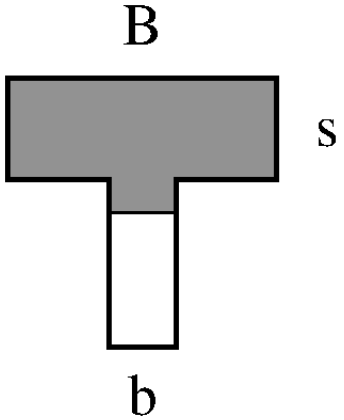
	Data			Results	
b	300	mm	x	128.376	mm2
h	500	mm	J	1,103,956,609.838	mm4
c	40	mm	f_c	-5.814	N/mm2
n	6.35		f_s	95.376	N/mm2
As	1256	mm2	f'_s	-25.417	N/mm2
A's	308	mm2	z	417.390	mm
d	460	mm			
M	50000000	N mm			

In many cases (T beams, inverted T beams, box beams) the formulas for rectangular beams are correct — as long as the neutral axis is inside the flange.

If we find otherwise, we have to take into account the variation of the section width when we determine the static moment expression.



If, using the formula for a rectangular beam, the neutral axis is inside the flange OK, otherwise we write



$$S = Bx^2/2 - (B-b)(x-s)^2/2 + nA'_s(x-c) + nA_s(x-d).$$

For circular or trapezoidal sections it's easy to write the analytical expression of S, e.g., for an isosceles trapezoid where a is the width of the compressed edge and b the width of the tensioned edge, the contribution of concrete to S is

$$S_c = \int_0^x (a + \frac{b-a}{h} y)(x-y)\mathrm{d}y = \frac{b-a}{6h} x^3 + \frac{a}{2} x^2$$

to which we have to add the steel contributions.

```
from scipy.optimize import newton

def S(x):
    Sc  = (b-a)*x**3/6/h + a*x**2/2
    Ssc, Sst = n * Asc * (x-c), n * Ast * (x-(h-c))
    return Sc + Ssc + Sst

a, b, h, c = 400, 200, 500, 40
Asc, Ast = 308, 1256
n = 6.35
M = 50E6

x = newton(S, h/3) ; print('x =', x)

J = (b*x**4 - a*x**4 + 4*(a*(h*x**3))) / (12*h)
J+= Asc * n * (x-c)**2 + Ast * n * (h-c-x)**2 ; print('J =', J)
z = J/(Ast*n*(h-c-x))
fc, fsc, fst = M/J*(0-x), n*M/J*(c-x), n*M/J*(h-c-x)
print('f_c = %.2f, f_sc = %.2f, f_st = %.2f'%(fc, fsc, fst))
```

Non Linear Model

- The section remains planar
- materials reach the limit state
  - the stresses in concrete are, with $\eta = \epsilon/\epsilon_{c2}$,

$$\sigma_c = -f_{cd} \begin{cases} \eta \cdot (2 - \eta) & \text{for } 0 \leq \eta \leq 1 \\ 1 & \text{for } 1 \leq \eta \leq \epsilon_{cu}/\epsilon_{c2} \end{cases}$$

  - the stresses in steel are

$$\sigma_s = E_s \cdot \epsilon, \qquad -f_{yd} \leq \sigma_s \leq +f_{yd}.$$

- The characteristics of stresses must be computed by direct integration,

$$N = \int \sigma \, dA, \qquad M = \int \sigma \cdot y \, dA.$$

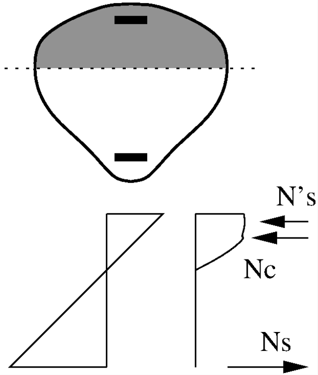
$\epsilon_{c2} = 0.0020$ *and* $\epsilon_{cu} = 0.0035$ *are **defined** by EC2.*

Operational assumptions

- the section is symmetrical w.r.t a vertical axis
- the bending moment acts in the same vertical plane
- neutral axis at distance x from the compressed edge
- at the limit state the deformation at the compressed edge is equal to $\epsilon_{cu} = 3.5‰$
- for a section in pure bending $N_c + N_s' = N_s$
- N_c is expressed in terms of coefficient $\boldsymbol{\beta}$

$$N_c = -f_{cd} A_{c,\mathrm{comp}} \boldsymbol{\beta}.$$

- The section remains planar, the diagram of deformations is linear and it is possible to compute the stresses (in concrete or in steel).
- In every strip at distance y from the neutral axis the stresses are constant and it is possible to compute the resultant force using integration.
- In the end, we must determine the position of the resultant N_c.



N_c AND COEFFICIENTS β AND κ

We start defining the area A_c, the area of concrete above the neutral axis, the area of compressed concrete.

We want to find N_c in terms of a filling coefficient β and in terms of the position of the n.a.:

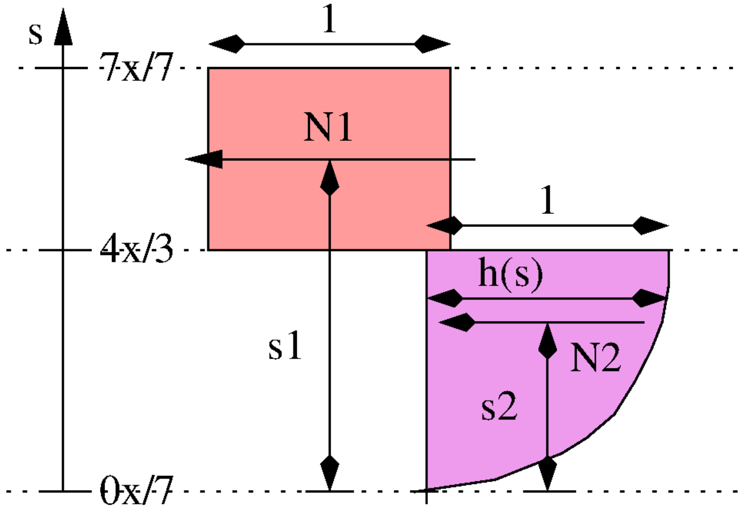
$$N_c(x) = \beta f_{cd} A_c(x)$$

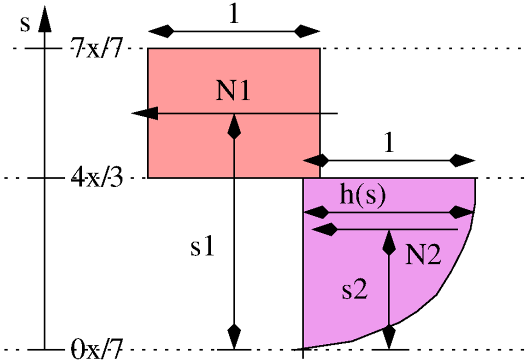
and we want to represent the moment of N_c w.r.t. the n.a. in terms of an additional coefficient κ

$$M_c(x) = N_c(x)(1-\kappa)x = \beta f_{cd} A_c(x)(1-\kappa)x.$$

The coefficient κ defines the position of the resultant measured from the compressed edge, κx.

Rectangular Section





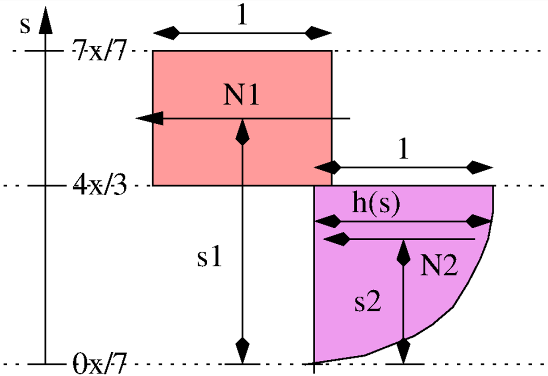
$$N_1 = \frac{3}{7}x, \quad s_1 = \frac{4}{7}x + \frac{1}{2}\frac{3}{7}x = \frac{11}{14}x$$

$$M_1 = N_1 s_1 = \frac{1}{49}\frac{33}{2}x^2$$

$$h(s) = \frac{2as - s^2}{a^2} \qquad \text{dove} \qquad a = \frac{4}{7}x$$

$$N_2 = \int_0^a h(s)\,\mathrm{d}s = \frac{2}{3}a = \frac{8}{21}x$$

$$M_2 = \int_0^a h(s)\,s\,\mathrm{d}s = \frac{5}{12}a^2 = \frac{1}{49}\frac{20}{3}x^2$$



$$N = N_1 + N_2 = \frac{17}{21}x$$

$$M = M_1 + M_2 = \frac{1}{49}\frac{139}{6}x^2$$

$$N = \beta x = \frac{17}{21}x \quad \Rightarrow \quad \beta = \frac{17}{21} = 0.809524$$

$$s_G = \frac{M}{N} = \frac{139}{238}x$$

$$s_G = (1-\kappa)x \quad \Rightarrow \quad \kappa = \frac{99}{238} = 0.415966$$

For non rectangular sections it is possible to use similar procedures, starting from the parabola-rectangle stress-strain diagram and performing the integral taking into account the variability of the width b.

For rectangular and non-rectangular sections it is possible to use the stress block diagram:

$$\sigma_c = \begin{cases} 0 & \epsilon \leq 0.2\epsilon_{cu} \\ f_{cd} & \text{otherwise} \end{cases}$$

The results obtained using the stress block are usually very good approximations to the results obtained using the parabola-rectangle diagram

$$N_s = +s\, A_s\, f_{yd}$$
$$N'_s = -s'\, A'_s\, f_{yd}$$

where s and s' are the rates of work of the reinforcement steel,

$$s = \sigma_s / f_{yd}, \quad s' = -\sigma'_s / f_{yd}.$$

Another definition is the ratio of compressed to tensile reinforcement

$$\mu = A'_s / A_s.$$

BTW the values of the stresses in steel can be deduced by the linear diagram of deformations.

For $A'_s = 0$ and assuming that $s = 1$ (this hypothesis must be verified ex-post) the equation of equilibrium, $N = 0$, is written

$$A_s f_{yd} - \beta f_{cd} bx = 0 \quad \Rightarrow \quad x = \frac{A_s f_{yd}}{\beta f_{cd} b}.$$

Having computed x the strain at the level of the reinforcement is

$$\epsilon_s = \epsilon_{cu} \frac{d - x}{x}$$

and to verify the hypothesis of yielded steel it should be

$$\epsilon_s \geq \epsilon_{yd}.$$

This is usually verified, except when an excessive percentage of reinforcement is present.

The compressed steel is yielded if $\epsilon_s' \leq -\epsilon_{yd}$, i.e.,

$$\epsilon_{cu}\frac{x-c}{x} \leq -\epsilon_{yd}.$$

For B450C steel, this translates to

$$x \geq 2.27\,c.$$

We start assuming that both reinforcements are yielded and write

$$A_s f_{yd} - A_s' f_{yd} - \beta f_{cd}\, bx = 0 \quad \Rightarrow \quad x = \frac{(A_s - A_s')f_{yd}}{\beta f_{cd}\, b}$$

and verify that $-\epsilon_{cu}\frac{d-x}{x} \geq \epsilon_{yd}$ and that $x \geq 2.27\,c$.

It is common to find that $x < 2.27\,c$ 1) when we deal with a shallow beam or a slab because c is large w.r.t. to the total height of the beam and 2) for large values of A_c' because x gets smaller.

If $\epsilon_s' > -\epsilon_{yd}$ the top reinforcement is still elastic and it is

$$s' = \frac{\epsilon_s'}{\epsilon_{yd}} = \frac{x-c}{x}\frac{\epsilon_{cu}}{\epsilon_{yd}}$$

and the new equation of equilibrium is

$$-f_{cd}\,\beta\,bx + \left(A_s - A_s'\frac{x-c}{x}\frac{\epsilon_{cu}}{\epsilon_{yd}}\right)f_{yd}$$

multiplying by x

$$(\beta b f_{cd})x^2 - (A_s - A_s'\frac{\epsilon_{cu}}{\epsilon_{yd}})f_{yd}\,x - A_s'\frac{\epsilon_{cu}}{\epsilon_{yd}}f_{yd}c = 0.$$

With

$$\omega = \frac{A_s}{bd}\frac{f_{yd}}{f_{cd}}, \ A'_s = \mu A_s, \ \mu_1 = \mu\frac{\epsilon_{cu}}{\epsilon_{yd}}$$

the equation of equilibrium can be simplified

$$\beta x^2 - (1-\mu_1)\omega d x - \omega\mu_1 c d = 0$$

whose positive solution is

$$\frac{x}{d} = \frac{\omega}{2\beta}\left((1-\mu_1) + \sqrt{(1-\mu_1)^2 + \frac{4\beta\mu_1 c}{\omega d}}\right)$$

Finally, writing the moment of the steel forces with respect to the point of application of N_c

$$M_{Rd} = f_{yd}\Big((d-\kappa x)A_s + s'(\kappa x - c)A'_s\Big).$$

For the usual section,

$$x = \frac{(1256 - 308) \cdot 391.3}{0.81 \cdot 300 \cdot 14.17} = 107.7\,\mathrm{mm}.$$

Is it the top steel yielded? It's $2.27c = 91\,\mathrm{mm}$, hence we can conclude that it is yielded.

Which is the design bending strength?

$$\begin{aligned} M_{Rd} &= (1256 \cdot (460 - 0.416 \cdot 107.7) + 308 \cdot (0.414 \cdot 107.7 - 40)) \cdot 391.4 \\ &= 204.6 \times 10^6\,\mathrm{Nmm} = 204.6\,\mathrm{kNm} \end{aligned}$$

Let's change $A_s' = 782\,\text{mm}^2$,

$$x = \frac{(1256 - 782) \cdot 391.3}{0.81 \cdot 300 \cdot 14.17} = 53.9\,\text{mm} < 91\,\text{mm}$$

We have to follow the other procedure...

$$\mu = \frac{7.82}{12.56} = 0.623, \quad \mu_1 = 0.623 \frac{3.5}{1.96} = 1.114, \quad \omega = \frac{1256}{300 \cdot 460} \frac{391.3}{14.17} = 0.251$$

Substituting these values in the equation for the neutral axis we have $x = 72.0\,\text{mm}$ and it is

$$s' = \frac{72.0 - 40}{72.0} \frac{3.5}{1.96} = 0.796$$

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It is interesting to note that in this case $M_{Rd} = 208.9\,\text{kNM}$, our previous value is $204.6\,\text{kNm}$. We have more than doubled A_s' to have a really small increment in the design strength.

In general, critical sections must be ductile. A good ductility ensures that plastic adaption (i.e., redistribution of bending moments) is possible and that, during seismic excitation, large amount of energy can be dissipated.

- Deep beams,
- low reinforcements percentages,
- presence of compressed reinforcement.

It is accepted that $\epsilon_s \geq 1\%$ at the ultimate limit state ensures a good ductility. For our design we can start imposing $x = 0.25d$ that is equivalent to $\epsilon_s = 3\epsilon_{cu} = 1.05\%$.

Design of a rectangular beam, single reinforcement

We want to design a ductile beam section to resist a design action M_{Ed}, if we fix in advance (see durability requirement.s) the concrete class we have the need to determine

- the steel area A_s and
- either the width b or the depth d of the beam.

Let's start with $N_c = \beta f_{cd} bx$, applied at a distance κx from the edge hence the lever arm, the distance from the resultants, is

$$z = d - \kappa x = (1 - \kappa \xi) d = \zeta d$$

With $\xi = 0.25$ and $\kappa = 0.416$ it is $\zeta = 0.896$ and the bending strength is

$$M = -N_c \cdot z = \beta \xi (1 - \kappa \xi) f_{cd}\, bd^2 = \frac{1}{r^2}\, bd^2$$

The dimensional design coefficient r is

$$r = \frac{1}{\sqrt{\beta \xi (1 - \kappa \xi) f_{cd}}}.$$

We must be careful to write all terms in the same units...

$$r = \frac{1}{\sqrt{\beta\xi(1-\kappa\xi) f_{cd}}}.$$

If we use, as it is customary, kNm as the unit for bending moments we must express b and d in metres and stresses in kN/m^2 or kPa (note: 1 N/mm^2 = 1 MPa = 1000 kPa).

For $\xi = 0.25$ and, e.g., C25/30 concrete, $f_{cd} = 14.17\,\mathrm{MPa} = 14\,170\,\mathrm{kPa}$, it is

$$r = (\beta\xi(1-\kappa\xi) f_{cd})^{-0.50} = (0.81 \cdot 0.25(1-0.416 \cdot 0.25)14170)^{-0.50} = 0.0197$$

The design bending strength is

$$M_{Rd} = \frac{0.30 \cdot 0.46^2}{0.0197^2} = 163.6\,\mathrm{kNm}.$$

We can manipulate the equation of design strength

$$M_{Rd} = \frac{bd^2}{r^2}$$

using $M_{Rd} = M_{Ed}$ and solving w.r.t the other variables we have design equations for ductile sections,

$$d = \sqrt{\frac{M_{Ed}}{b}}\, r, \quad b = \frac{M_{Ed}}{d^2}\, r^2$$

and eventually

$$A_s = \frac{M_{Ed}}{z \cdot f_{yd}} \approx \frac{M_{Ed}}{0.9d \cdot f_{yd}}.$$

- $\xi = 0.25$
- $A'_s = \mu A_s$
- $c = \gamma d$
- $\epsilon'_s = -\epsilon_{cu}\frac{x-c}{c} = -\epsilon_{cu}\frac{\xi-\gamma}{\xi}$, for $\xi = 0.25 \Rightarrow \epsilon'_s = -\epsilon_{cu}(1-4\gamma)$
- top reinforcement yielded, for B450C steel, if $\gamma \leq 0.11$
- $s' = \min(1, -\epsilon'_s/\epsilon_{yd}) = \min(1, (1-4\gamma)\frac{\epsilon_{cu}}{\epsilon_{yd}})$
- $\sigma'_s = -s' f_{yd}$, the compression stress in top steel is a function of γ,
- $N'_s = -s'\,\mu\,N_s$
- the resultant of steel forces act at a distance $d + d_1$ from the compressed edge (below the tensile steel), from equilibrium

$$-N'_s \cdot (d + d_1 - c) = N_s \cdot d_1 \Rightarrow d_1 = d \cdot s' \cdot \mu \frac{1-\gamma}{1-s'\mu},$$

The bending moment can be written as the product of N_c by the distance between N_c and the resultant of steel forces,

$$M = \beta\, bx f_{cd}(d + d_1 - \kappa\xi d) = \beta\xi\, bd\, f_{cd}\left((1-\kappa\xi) + \frac{s'\mu(1-\gamma)}{1-s'\mu}\right)d$$

$$= bd^2\, \beta\xi(1-\kappa\xi) f_{cd}\left(1 + \frac{s'\mu}{1-s'\mu}\,\frac{1-\gamma}{1-\kappa\xi}\right) = bd^2\,\frac{1}{r^2}\,\frac{1}{k^2} = bd^2\,\frac{1}{r'^2}$$

Because $1-\gamma \approx 1-\kappa\xi$ (distance of top steel to bottom steel, distance of centre of concrete compression to bottom steel) we can write, with good approximation ($< 1\%$ for reasonable values of γ),

$$\frac{1}{k^2} \approx \frac{1}{1-s'\mu}$$

For the same reasons, still $z \approx 0.9\,d$ and

$$A_s = M/(0.0\,d\,f_{yd}).$$

These parameters does not depend on the concrete class

γ	0.10	0.15	0.20
s'	1.000	0.716	0.358

γ	0.10	0.15	0.20
$\mu = 0.00$	0.896	0.896	0.896
$\mu = 0.25$	0.897	0.888	0.887
$\mu = 0.50$	0.898	0.880	0.879

Table 1: Values of ζ

r AND r'

	γ	0.1000	0.1500	0.2000
• C20/25	$\mu = 0.00$	0.0221	0.0221	0.0221
	$\mu = 0.25$	0.0191	0.0201	0.0212
	$\mu = 0.50$	0.0156	0.0178	0.0202
• C25/30	$\mu = 0.00$	0.0197	0.0197	0.0197
	$\mu = 0.25$	0.0171	0.0180	0.0189
	$\mu = 0.50$	0.0139	0.0160	0.0181
• C32/40	$\mu = 0.00$	0.0174	0.0174	0.0174
	$\mu = 0.25$	0.0151	0.0159	0.0167
	$\mu = 0.50$	0.0123	0.0141	0.0160

The reinforcement percentage is

$$\rho = \frac{A_s}{bd} = \frac{M}{0.9df_{yd}}\frac{1}{bd} = \frac{bd^2}{r^2}\frac{1}{bd^2 0.9f_{yd}}$$

solving for r

$$r = (0.9\rho f_{yd})^{-0.50}$$

or, in different words, there is a relationship between ρ and the design coefficient, $r(1.0\%) = 0.017$,

For smaller r we have larger ρ — that's OK, for better concrete (i.e., decreasing values of r we have smaller sections of concrete and consequently we need more reinforcements, in absolute terms (the section depth decreases) and as a percentage (the concrete area decreases).

We always have a little top steel, at least because it's needed to support the stirrups, so we design our beam considering at least $\mu = 0.25$.

W.r.t. s', for $d/c \geq 9$ it is $s' = 1.0$, for $d/c = 7$ we have still a good contribution from compressed steel as $s' = 0.72$ but the work rate is rapidly decreasing, $s'(5) = 0.36$ or, in other words, deep beams are better from this point of view.

We can start our design choosing a reasonable value of ρ (1%, 1.5% maybe) and the corresponding value of r.

We have fixed, say, the concrete class, $\rho = 0.015$ and $b = 300\,\text{mm}$, known M_{Ed} we can easily determine the depth d and the steel area A_s (using $z = 0.9d$), we must still determine the needed amount of the compressed steel.

As a first step, one must compute the design strength using r (or r' for the μ value associated to the compressed steel already present). If $bd^2 > M_{Ed}r^2$ (or $bd^2 > M_{Ed}r'^2$) the compressed steel present in the beam is enough, otherwise we must determine the needed additional steel.

First we determine the *unbalanced moment*

$$\Delta M = M_{Ed} - bd^2/r^2,$$

next the rate of work $s' = \min(1, (25 - 100\gamma)/14)$ and finally

$$\Delta A_s' = \frac{\Delta M}{(d - c)s' f_{yd}}.$$

$M_{Ed} = 220\,\text{kNm}$, C25/30, $b = 0.3\,\text{m}$, determine d, A_s and possibly A'_s.

With $r' = 0.018$ it is $d = r'\sqrt{M/b} = 0.018\sqrt{220/0.3} = 0.49\,\text{m}$

Considering a concrete cover $c = 40\,\text{mm}$ and sticking with multiples of 10 cm we choose $h = 600\,\text{mm}$ and $d = 560\,\text{mm}$

The steel area is $A_s = 220 \times 10^6\,\text{Nmm}/(0.9 \cdot 560\,\text{mm} \cdot 391\,\text{N/mm}^2 = 1115.5\,\text{mm}^2$ corresponding to $\rho = 0.62\%$ and the design strength (neglecting the possible presence of top steel) is

$$M_{Rd} = bd^2/r^2 = 0.30 \cdot 0.56^2/0.0197^2 = 242\,\text{kNm} > M_{Ed}.$$

No additional top steel is needed.

Everything is the same but we accept the idea of more reinforcement, $\rho \approx 1\%$ and consequently $r' = 0.016$.

The depth should be $d = 0.47\,\text{m}$ but we choose $h = 500\,\text{mm}$ and $d = 460\,\text{mm}$.

The steel area is $A_s = 1360\,\text{mm}^2$ and the bending strength is $M_{Rd} = 163.6\,\text{kNm} < M_{Ed}$, hence we need to dimension the compressed steel,

$$\Delta M = 220\,\text{kNm} - 163.6\,\text{kNm} = 56.4\,\text{kNm}$$

and, because $s' = 1$ ($\gamma \approx 1/12 < 1/9$) we have

$$A_s' = \frac{56.4 \times 10^6\,\text{Nmm}}{(460\,\text{mm} - 40\,\text{mm})391.3\,\text{N/mm}^2} = 340\,\text{mm}^2$$

A shallow beam, $h = 240\,\text{mm}$, $c = 40\,\text{mm}$ and $M_{Ed} = 120\,\text{kNm}$.

Using $r' = 0/019$, with $d = 200\,\text{mm}$, $\gamma = 0.20$ and $s' = 0.358$ we have

$$b = M \cdot r'^2/d^2 = 120 \cdot 0.019^2/0.2^2 = 1100\,\text{mm}$$

. The tensile steel area is

$$A_s = 120 \times 10^6\,\text{Nmm}/(0.9\,200\,\text{mm}\,391.3\,\text{N/mm}^2) = 1700\,\text{mm}^2$$

and the no compression reinforcement design strength is

$$M_{Rd} = 113.4\,\text{kNm} < M_{Ed}$$

, the unbalanced moment is $\Delta M = 6.6\,\text{kNm}$ and eventually

$$A'_s = \frac{6.6 \times 10^6\,\text{Nmm}}{160\,\text{mm}0.358\,391.3\,\text{N/mm}^2} = 300\,\text{mm}^2.$$

Part III

R.C. Beams - Shear and Combined Actions

Shear Force, linear behavior

When a beam id subjected to a shear force V, said force must be equilibrated by tangential stresses that act on the surface of the beam.

If we have a Cartesian coordinate system with z corresponding to the beam axis, x and y corresponding to to the beam section's principal axes and $V = V_y$, for equilibrium we must have that $\int \boldsymbol{\tau}_{zy} dA = V_y$ and $\int \boldsymbol{\tau}_{zx} dA = 0$.

The stresses $\boldsymbol{\tau}_{zx}$ are needed to respect the infinitesimal equilibrium on the perimeter of the beam.

It is possible to derive an exact solution for an elastic beam subjected to shear force but it is convenient to use an approximate formula due to D. Zhuravskii (1855),

$$\bar{\tau}_{zy} = \frac{V_y}{bJ_x} S(y)/$$

that gives the *mean value* of the tangential stress over a chord of length b (i.e., $b\bar{\tau}_{zy} = \int_0^b \tau_{zy} dx$).

Real values of τ_{zy} are minimum at the center of the chord and the variation is very small when the chord b is small with respect to the total height h of the section.

For a rectangular beam with $h/b = 2$ relative error is $\approx 3\%$, for $b/h = 2$ (shallow beams) the error is $\approx 12\%$ and hence is starting to become relevant.

We have the max value of $b\bar{\tau}_{zy}$ when the static moment S_y is maximized, i.e., when it is computed with respect to the barycentric axis.

We can relate the resultant of compressive stresses and $S_{y,\max}$:

$$N_c = \int_0^{y_G} \sigma_z b\,dy = \int_0^{y_G} E\chi_y y b\,dy = E\chi_y S_{y,\max} \Rightarrow S_{y,\max} = \frac{N_c}{E\chi_y};$$

because it is $\chi_y = M_x/EJ_x = N_c z/EJ_x$ we can write $J_x = N_c z/E\chi_y$ and substitute in the Zhuravskii formula

$$\max(b\bar{\tau}_{zy}) = V_y \frac{S_{y,\max}}{J_x} = \frac{V_y}{z}.$$

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Rectangular section $b \times h$ subjected to a shear force V_y.

- $J_x = bh^3/12$,
- $S_{y,\max} = (b \cdot h/2) \cdot (h/4) = bh^2/8$,
- $\bar{\tau}_{zy,\max} = \dfrac{V_y S_{y,\max}}{b\,J_x} = \dfrac{V_y bh^2/8}{b \cdot bh^3/12} = \dfrac{3}{2}\dfrac{V_y}{bh}$.

$\bar{\tau}_{zy,\max}$ is 50% larger than the mean shear stress V_y/bh.

Rectangular section $b \times h$ subjected to a shear force V_y.

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$\bar{\tau}_{zy,\max}$ is 50% larger than the mean shear stress V_y/bh.

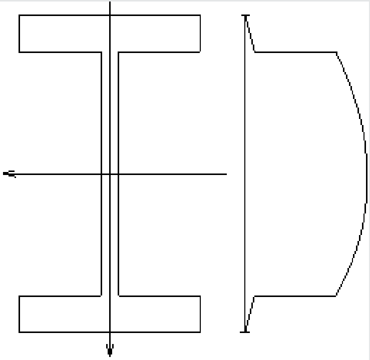
It's easy to derive that, for a rectangular section,

$$\bar{\tau}_{zy} = \left(1 - \left(\frac{2y}{h}\right)^2\right) \bar{\tau}_{zy,\max}$$

i.e., the mean stress varies according to a parabola.

I shaped beam

Aside, a schematic representation of the mean stresses in an I beam, we have a fast variation of $\bar{\tau}_{zy}$ in the flanges and a slow variation in the web, moreover we have a discontinuity where there is a sharp variation of the chord b



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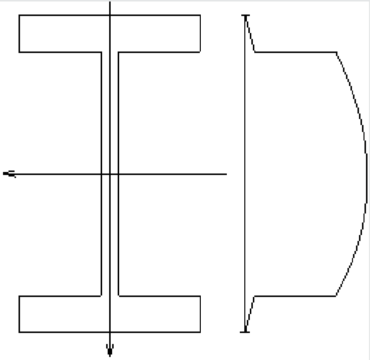
Because the variation of S_y in the web is small neglecting the flanges contribution we can write

$$\bar{\tau}_{zy} \approx \frac{V_y}{b_w h_w}.$$



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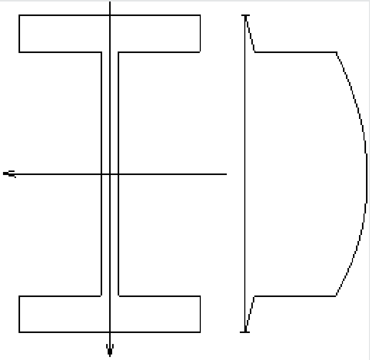
Because the variation of S_y in the web is small neglecting the flanges contribution we can write

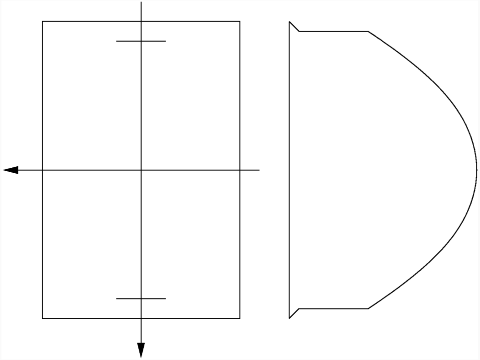
$$\bar{\tau}_{zy} \approx \frac{V_y}{b_w h_w}.$$

We have to recognize that in the flanges there is a significant value of $\bar{\tau}_{zx}$, as its apparent from the equilibrium of the flange-web node.



For a reinforced concrete section, completely reacting, we have just to take into account that the contribution of the re bars must be *homogenized*, if we start from the top and consider what happens for $y = c$ it is

$$S(c^{+}) = S(c^{-}) + n \cdot A'_s \cdot (y_G - c).$$

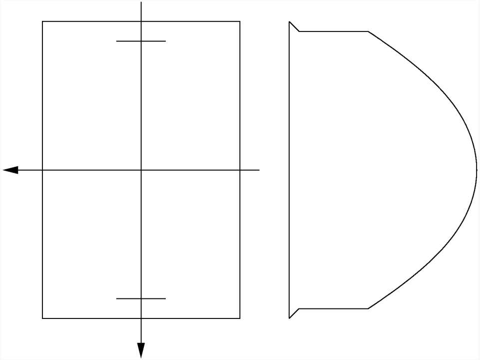


For a reinforced concrete section, completely reacting, we have just to take into account that the contribution of the re bars must be *homogenized*, if we start from the top and consider what happens for $y = c$ it is

$$S(c^+) = S(c^-) + n \cdot A_s' \cdot (y_G - c).$$

Analogously for the bottom bars

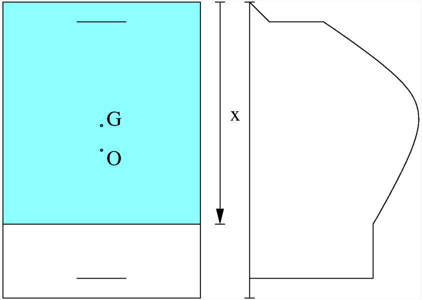
$$S(d^+) = S(d^-) + n \cdot A_s \cdot (y_G - c).$$



Now the maximum value of S_y is reached corresponding to the centroid G^* of the reacting section (compressed concrete and both re bars):

- we have a first discontinuity at $y = c$,
- S_y increases until $y = y_{G^*}$,
- S_y decreases until $y = y_x$, y_x being the position of the neutral axis,
- until $y = d$ we have a constant value of S_y because we have no variation in the stress resultant,
- for $y = d$ we have another discontinuity, $S^+ = 0$,
- for $y > d$ it is $S_y = 0$.

It is worth to note that, however, $b\bar{\tau}_{zy} = V/z$.



Pure bending

The centroid of the reacting section coincides with the neutral axis, because $N_C = N_T$. $S_{y,\max}$ is found on the neutral axis and below it remains constant until the bottom reinforcement.

For a beam under pure bending we have, approximately,

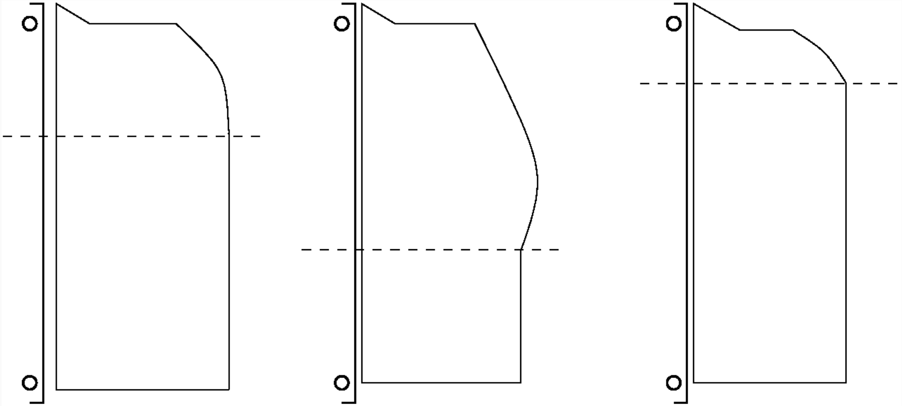
$$z = 0.9d \Rightarrow \bar{\tau}_{zy,\max} = \frac{V}{0.9bd}.$$

Bending & compression

In this case $y_x > y_G$ because $N_C > N_T$ and the maximum shear stress is found inside the compressed concrete.

Bending & tension

On the contrary, $y_x < y_G$ because $N_C < N_T$ and we don't reach a maximum tangential stress neither in compressed concrete nor in cracked concrete, because there the tangential stress equals the value reached in the neutral axis position.



Over the neutral axis the state of stress of concrete is a combination of normal compression stress and tangential stress, one of the principal stresses is tensile.

Nearing the neutral axis, the compressive stresses decrease and the tangential stresses increase, the principal stresses are rotating toward an inclination of 45^0.

In cracked concrete the normal stresses equals zero and we have a diagonal tension field, with compressive and tensile principal stresses both equal to a value that, for pure bending, is $\bar{\tau}_{zy,\max}$.

Concrete does not resist tensile stress, the cracks below the neutral axis are rotated 45^0 while approaching the bottom edge the cracks become sub vertical.

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To equilibrate the tension field that develops when the tangential stresses are high it is necessary to use *transversal reinforcement*.

Because concrete cracking is a fragile collapse and because our aim is to design a non-fragile structure, it is mandatory to use *transversal reinforcement*.

The use of transversal reinforcement is required by every building regulation, for all the principal structural members, with the exception of slabs. Every regulation requires a *minimum amount* of transversal reinforcement exactly to prevent a fragile collapse.

Cranked bars

Because the tensions are inclined at 45^0 they can be equilibrated by *cranked bars*, bent up at 45^0.

The use of cranked bars is very efficient, because

- the slope of the bar is close to the direction of the tensile principal stresses,
- using a single cranked bar it is possible to use it in the bottom where moment is positive (centre of span) and where moment is negative (supports),

but this efficient use of material is overbalanced by the higher cost associated whit manufacturing a reinforcement cage using cranked bars.

Cranked bars are interesting in countries where the cost of manufacturing is low with respect to the cost of materials.

Stirrups

Stirrups are disposed in the vertical direction, typically as closed loops surrounding the longitudinal bars and are intended to equilibrate the vertical component of the tension field.
Stirrups have multiple functions beyond the shear resistance:

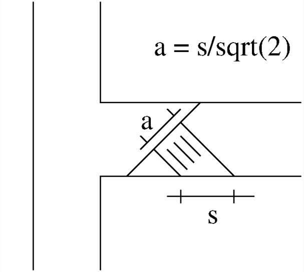
- stirrups allow the correct positioning of the longitudinal bars, both top and bottom bars, because it keeps them aligned,
- stirrups offer some amount of lateral containment of concrete, enhancing its strength and its ductility.

In view of these additional beneficial effects, EC2 requires that, at least, a fraction of the shear force is equilibrated using stirrups.

Tension stress about the neutral axis has an inclination of 45º and its value, equal to $\bar{\tau}_{zy,\max}$ is $\boldsymbol{\sigma} = V/(z \cdot b)$.

Considering an horizontal distance between cranked bars s, the corresponding distance measured across a crack is $a = s/\sqrt{2}$ and the resultant of the normal stresses, that must be equilibrated by a single bar of area A_s, is $R = a \cdot b \cdot \boldsymbol{\sigma} = Vs/(\sqrt{2}z)$.
The stress, $\boldsymbol{\sigma}_s = R/A_s$ is

$$\boldsymbol{\sigma}_s = \frac{V \cdot s}{\sqrt{2}z \cdot A_s}.$$

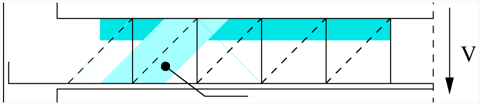


If you use closer bars (s being smaller) you have smaller values of the tension in steel.
You can find the bar area in terms of the spacing

$$\frac{V \cdot s}{\sqrt{2}z \cdot A_s} \leq \bar{\sigma} \Rightarrow A_s \geq \frac{V \cdot s}{\sqrt{2}z \cdot \bar{\sigma}}.$$

Stress in stirrups

Considering only stirrups, we have 45^0 diagonal concrete struts spaced at z (their base equal to their height).



From the node equilibrium it is apparent that the stirrup axial force is $N_{st} = V$ considering notional stirrups spaced at a distance equal to z.

Real stirrups being spaced at a distance $s < z$, the real axial force is

$$N_{st} = \frac{V \cdot s}{z}$$

and the stress in stirrups is

$$\sigma_{st} = \frac{V}{A_{st}} \frac{s}{z}.$$

Stirrups

Consider a rectangular section $30 \times 60\,\text{cm}$ with $\Phi 8$ stirrups with two legs, $n_l = 2$, distance $s = 10\,\text{cm}$, c=4 cm and $V = 120\,\text{kN}$.

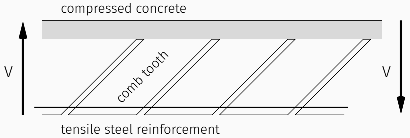
- $d = 56\,\text{cm}$, $z \approx 0.9d = 50.4\,\text{cm}$
- $A_{st} = n_l 0.5\,\text{cm}^2 = 1.0\,\text{cm}^2$,

$$\sigma_{st} = \frac{120\,000\,\text{N}}{100\,\text{mm}^2}\frac{10}{50.4} = 238.1\,\text{MPa}.$$

Shear Force, non linear behavior

- Shear strength without transversal reinforcement.
- Shear strength with transversal reinforcement.
- EC prescriptions.

At the ultimate limit state the stirrup-less beam is a comb composed of an upper part of compressed concrete, teeth of compressed concrete directed at 45^0 separated by cracks and jointed by the bottom tensile steel bars.
If the concrete teeth were hinged at the top (truss model) no equilibrium would be possible, but the teeth are fixed into the top compressed concrete...
The possible failure mechanism are



- flexural failure at the fixed end of a tooth,
- tensile failure of compressed concrete for a combination of compression and shear.

We want to find the maximum tensile stress $\boldsymbol{\sigma}_p$ at the fixed end of a tooth.

$\Delta N = V \cdot \Delta s/z$

$N_p = -\Delta N/\sqrt{2} = -V\Delta s/\sqrt{2}/z$

$M_p = \Delta N(d - x - \Delta s/4) =$

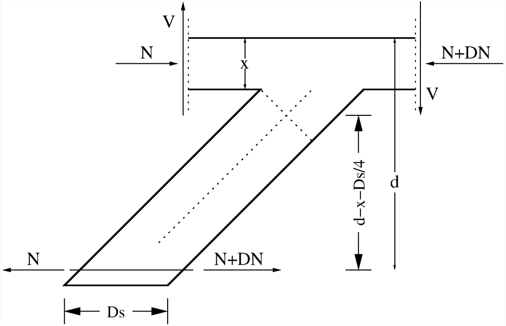
$V\Delta s(d - x - \Delta s/4)/z$

$A_p = b\Delta s/\sqrt{2}$

$W_p = b(\Delta s)^2/12$

ultimately

$$\boldsymbol{\sigma}_p = \frac{V}{bz}\,\frac{12d - 12x - 4\Delta s}{\Delta s}$$



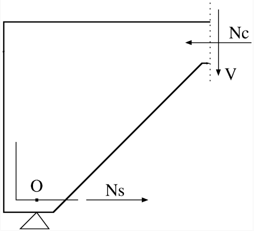
From experiments, $\Delta s = d$ and $x = d/5$m and substituting it is

$\boldsymbol{\sigma}_p = 5.6V/(bz) \approx 6.2V/(bd) \le f_{cfd} = 1.6 f_{ctd}$ and finally

$$V_{Rd} = 0.25 \cdot b \cdot d \cdot f_{ctd}$$

From rotation equilibrium with respect to O it is $N_c = V$ and using an uniform distribution of normal and tangential stresses we have

$$\sigma_z = \tau_{zy} = V/bx, \ \sigma_y = 0.$$



The principal tension stress is

$$\sigma_1 = -\frac{\sigma_z}{2} + \sqrt{\left(\frac{\sigma_z}{2}\right)^2 + \sigma_z^2} = \frac{\sqrt{5}-1}{2}\sigma_z$$

with $\sigma_z = \frac{V}{bx} \Rightarrow$

$$\sigma_1 = 0.618\frac{V}{bx}.$$

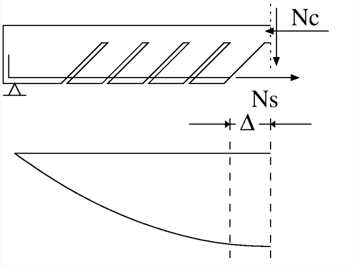
Using the tensile design strength of concrete, f_{ctd} as the tensile stress σ_1 we have

$$V_{Rd} = 1.618\, b \cdot x \cdot f_{cdt}.$$

Aggregate interlocking The faces of the crack are not smooth, when two adjacent teeth are deformed the relative displacement is partially contrasted and there is a shear transfer between them.

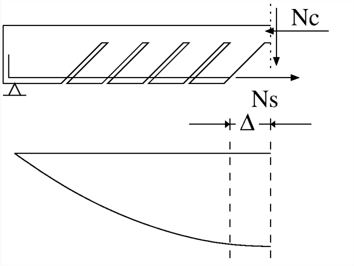
Dowel effect The bottom bars are deformed following the deformation of the teeth and because of their stiffness there is a shear transfer between adjacent teeth.

Axial force A (moderate) net compressive force lower the neutral axis, reducing both the tensile stress in the compressed band and the length of the bent teeth.



The couple of forces that is equilibrating the external moment are applied in two different positions: the compressive force is applied at a distance x from the support and the tensile force is applied at a distance $x - \Delta$ and for a 45^0 tooth $\Delta \simeq z$.

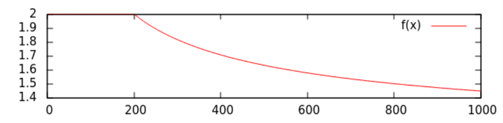
The tensile force at distance $x - z$ from the support is hence $N_t(x - z) = M(x)/z$.



The couple of forces that is equilibrating the external moment are applied in two different positions: the compressive force is applied at a distance x from the support and the tensile force is applied at a distance $x - \Delta$ and for a 45⁰ tooth $\Delta \simeq z$.
The tensile force at distance $x - z$ from the support is hence $N_t(x - z) = M(x)/z$.

We can generalize this reasoning and conclude that in general, due to the presence of diagonal concrete struts, the action in the tensile bars must be computed with reference not to the position x but $x \pm z$ (the sign is chosen so that the effect on N_t is maximized).

The design strength depends on the main failure mechanism described, plus correction to account for the beneficial effects.



Interlocking it is described by an non dimensional coefficient k, depending on d (in mm): $k = \min(2, 1 + \sqrt{200/d})$.

Dowel effect it is described in a simplified manner by the geometrical reinforcement ratio $\rho_l = A_{s,l}/A_c$.

Axial force effect it is described by $\sigma_{cp} = \min(N/A_c, 0.2f_{cd})$.

The design strength is then

$$\frac{V_{Rd,c}}{b_w d} = 0.15\sigma_{cp} + \max\left(\frac{0.18}{\gamma_c}\sqrt[3]{100\rho_l f_{ck}}, 0.035\sqrt{k^3 f_c k}\right).$$

Ribbed floor

Concrete C25/30, $h = 24\,\text{cm}$, two joists with $b_w = 10\,\text{cm}$ in each metre, $A_s = 2 \cdot (0.79 + 1.54)\text{cm}^2$ from $1\Phi10 + 1\Phi14$ in each joist, $c = 2\,\text{cm}$ and $V_{Ed} = 28\,\text{kN}$.

Ribbed floor

Concrete C25/30, $h = 24\,\text{cm}$, two joists with $b_w = 10\,\text{cm}$ in each metre,
$A_s = 2 \cdot (0.79 + 1.54)\text{cm}^2$ from $1\Phi10 + 1\Phi14$ in each joist, $c = 2\,\text{cm}$ and $V_{Ed} = 28\,\text{kN}$.
With $d = 220\,\text{mm}$ it is $k = 1 + \sqrt{200/220} = 1.953$.
The geometric ratio is $\rho_l = 4.66/20/22 = 0.0106$.
With $N = 0$ it is

$$V_{Rd,c} = 200\,\text{mm} \cdot 220\,\text{mm} \cdot 0.12 \cdot 1.953 \sqrt[3]{100 \cdot 0.0106 \cdot 25\,\text{N/mm}^2} = 30\,700\,\text{N}.$$

Ribbed floor

Concrete C25/30, $h = 24\,\text{cm}$, two joists with $b_w = 10\,\text{cm}$ in each metre,
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Because $V_{Rd,c} > V_{Ed}$ we have that a specific reinforcement is not required to resist the design shear.

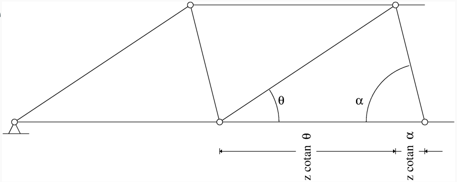
The ultimate resistance mechanism is similar to what we have seen previously, a truss whose transversal members are concrete struts and steel tendons, the difference being the inclination of the concrete struts.

At limit state, due primarily to aggregate interlocking, the principal stresses of compression rotate and are no more at an angle of 45^{0} with the horizontal but at a smaller angle.

The fictional truss struts are more inclined and, hence, more efficient.

The axial load in the strut is $N_p = V/\sin\theta$, we estimate its area as $A_p = \sin\theta \cdot (\cot\theta + \cot\alpha) z \cdot b_w$.

NTC uses a reduction factor $\nu_1 = 0.50$ for the strut, if we equate the strut stress for the design strength to the reduced strength it is

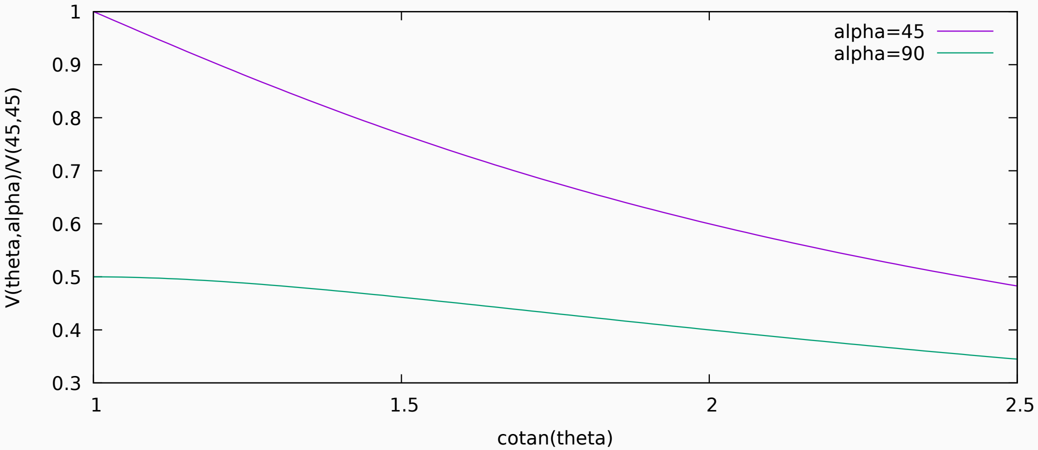


$$\frac{V_{Rd}/\sin\theta}{\sin\theta \cdot (\cot\theta + \cot\alpha) z \cdot b_w} = \nu_1 f_{cd},$$

solving with respect to V_{Rd} we have the design shear strength *for the concrete*,

$$V_{Rd,p} = \nu_1 f_{cd} b_w z \cdot (\cot\theta + \cot\alpha)\sin^2\theta = \nu_1 f_{cd} b_w z \frac{(\cot\theta + \cot\alpha)}{1 + \cot^2\theta}$$

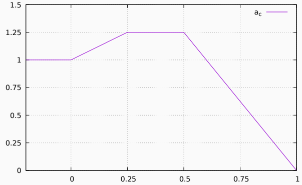
EC2 has the following restriction on the strut inclination: $1.0 \leq \cot\alpha \leq 2.5$.



alpha=45
alpha=90
V(theta,alpha)/V(45,45)
cotan(theta)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
1
1.5
2
2.5

The design strength is affected by a $N \neq 0$, using a multiplicative coefficient α_c — the effects are beneficial for small values of N and detrimental for large values of the compression force.

$$\alpha_c = \begin{cases} 1 & \text{no compressione} \\ 1 + \sigma_{cp}/f_{cd} & 0.00 \leq \sigma_{cp}/f_{cd} \leq 0.25 \\ 1.25 & 0.25 \leq \sigma_{cp}/f_{cd} \leq 0.50 \\ 2.5(1 - \sigma_{cp}/f_{cd}) & 0.50 \leq \sigma_{cp}/f_{cd} \leq 1.00 \end{cases}$$



The equivalent steel area is $A = A_w \frac{z\cdot(\cot\theta+\cot\alpha)}{z}$ and considering an equivalent tendon, with a slope α, the axial load is $N = V/\sin\alpha$.

Setting $f_{yd}A = N$ it is possible to solve for the design strength corresponding to the stirrups' yielding

$$V_{Rd,s} = A_w f_{yd} \frac{z\cdot(\cot\theta+\cot\alpha)}{s}\sin\alpha.$$

For stirrups, $\alpha = 90^0$ and $V_{Rd,s} = A_w f_{yd}\dfrac{z\cot\theta}{s}$.

Also in this case there is a tension shift,

$$\Delta = \frac{z}{2}(\cot\theta - \cot\alpha).$$

For stirrups, $0.50 \leq \Delta/z \leq 1.25$, for cranked bars ($\alpha = 45^0$) $0.00 \leq \Delta/z \leq 0.75$.

For stirrups

$$V_{Rd,p} = \alpha_c \nu_1 f_{cd} b_w z \cdot \frac{\cot\theta}{1+\cot^2\theta}, \qquad V_{Rd,s} = A_w f_{yd} \frac{z\cot\theta}{s}.$$

It is possible that the stirrup strength is always lower than the struts strength, otherwise if

$$0 \leq \cot\theta_0 = \sqrt{\frac{\alpha_c \nu_1 f_{cd} b_w s}{A_{sw} f_{yd}} - 1} \leq 2.5$$

for $\theta \leq \theta_0$ the failure happens in stirrups, otherwise the limit state is reached in struts.

NTC prescribes a minimum amount of stirrups, hence a reasonable approach is to compute $V_{Rd,s}$ for $\theta = 45^0$ and the minimum amount of steel, so we have a guaranteed value of the shear strength.

Typically the design shear action on supports is higher than the guaranteed strength, so that we can either increase the steel area (reducing the value of s) or the value of $\cot \alpha$.

A design strategy is to compute the strength for a fixed value of α and different values of s and varying the distance between stirrups as the shear action is increasing.

A last consideration, to limit the tension shift to reasonable values it may be convenient to limit the maximum possible value of $\cot \alpha$ to 2.

- minimum area of stirrups, $A_w \geq 0.0015 \cdot s \cdot b_w$,
- maximum distance of stirrups, $s \leq 0.8 \cdot d$,
- at least 50% of the steel strength must be provided by stirrups.

Deep beam verification

$V_{Ed} = 250\,\text{kN}$, C25/30, 30x50 cm, stirrups $\Phi 8$ @ $s = 15\,\text{cm}$, $n_l = 2$, $c = 4\,\text{cm}$.

Deep beam verification

$V_{Ed} = 250\,\text{kN}$, C25/30, 30x50 cm, stirrups $\Phi 8$ @ $s = 15\,\text{cm}$, $n_l = 2$, $c = 4\,\text{cm}$.

$A = 1\,\text{cm}^2/0.15\,\text{m} = 6.67\,\text{cm}^2/\text{m}$

For $\cot\alpha = 2.5$

$V_c = 0.5 \cdot 14.17 \cdot 300 \cdot (0.9 \cdot 460) \cdot 2.5/(1 + 2.5^2) = 303\,400\,\text{N}$

$V_s = 0.667 \cdot (0.9 \cdot 460) \cdot 391.3 \cdot 2.5 = 270\,000\,\text{N}.$

Using the minimum value as V_{Rd} we have

$$270\,\text{kN} = V_{Rd} \geq V_{Ed} = 250\,\text{kN}.$$

The shear strength of the beam is satisfactory.

Deep beam design

$V_{Ed} = 180\,\text{kN}$, 30x60, $c = 4$.

Deep beam design

$V_{Ed} = 180\,\text{kN}$, 30x60, $c = 4$.

For $\cot\alpha = 2.5$ it is $V_c = 369\,\text{kN}$, the concrete strength is always sufficient, we can determine A_w using $\cot\alpha = 2.5$.
$A_{min} = V/(f_{yd}z2.5) = 180000/391.4/0.9/560/2.5 = 0.365\text{mm}^2/\text{mm}$.
This can be satisfied using 2Φ8@25 cm, that is $4.0\,\text{cm}^2/\text{m}$.
From moment shift we have a requirement of additional bars
$\Delta A_s = (V_{Ed}\cot\theta)/(2f_{yd}) = 580\text{mm}^2$.
As this seems excessive, it may be convenient to design with more stirrups and a smaller $\cot\alpha$.

Combined loads, Axial and Bending

If the axial load is applied corresponding to the centroid of the homogenized section, the section is subjected to an uniform field of normal stresses.

If the load is applied with an eccentricity (e_x, e_y) we can find the stress field as a superposition of the stresses due to N, $M_x = N \cdot e_y$ and $M_y = -N \cdot e_x$.

The core (or kernel) of the section is the set of points e_x, e_y such that the neutral axis is external to the section or, in other words, the section is subjected either to only compressive normal stresses or only tensile stresses.

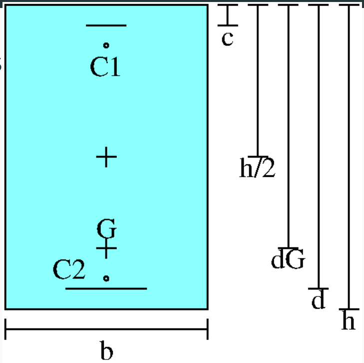
No concrete contribution, $e_x = 0$ (we'll always use this hypothesis). With A_s, A'_s the centroid position, from the top edge, is

$$d_G = \frac{dA_s + cA'_s}{A_s + A'_s}$$

The moment of inertia is

$$J_x = \frac{A_s A'_s}{A_s + A'_s}(d - c)^2$$

and the pole-centroid distances are



$$\overline{C_1 G} = \frac{A_s A'_s}{A_s + A'_s}\,\frac{(d - c)^2}{A_s c + A'_s d} \qquad \overline{C_2 G} = \frac{A_s A'_s}{A_s + A'_s}\,\frac{(d - c)^2}{A_s d + A'_s c}$$

All distances are measured, with signs, from the top edge, except the pole-centroid distances that are given in terms of absolute values.

Nominal and Effective Moment

The axial force and the bending moment are referred to the geometrical centroid of the concrete section, we must transport the axial load in G and add the moment of transport to M

$$e^* = M/N + h/2 - d_G$$

$$M^* = N \cdot e^*$$

$$\sigma_s = \frac{N}{A_s + A_s'} + \frac{M^*}{J_x}(d - d_G)$$

$$\sigma_s' = \frac{N}{A_s + A_s'} + \frac{M^*}{J_x}(c - d_G)$$

Now we have to take into account the homogenized section

$$A = bh + n(A_S + A_S'),$$

$$S = \frac{bh^2}{2} + n(A_S d + A_S' c),$$

$$d_G = S/A, \quad d_G' = h - d_G = (\frac{bh^2}{2} + n(A_S c + A_S' d))/A,$$

$$J_X = \frac{bh^3}{12} + (d_G - h/2)^2 bh + nA_S(d_G' - c)^2 + nA_S'(d_G - c)^2,$$

$$e_1 = \frac{J_X}{A \cdot d_G'}, \quad e_2 = \frac{J_X}{A \cdot d_G}.$$

We always talk about of a section compressed at the top.

Besides the dimensions already defined, we will need

- d_C distance of C, the center of N, from the top edge, $d_C > 0$ if C lies below the top edge.
- e_n distance of C from the neutral axis, $e_n > 0$ if C lies below the neutral axis.
- x distance of the neutral axis from the top edge, it is always $x > 0$.

These dimensions are not independent, it is $e_n = d_C - x$ and $d_C = e + h/2$.

For a section under eccentric load, the inertia moment with respect to the neutral axis is equal to the product between the static moment with respect to the neutral axis and e_n:

$$J_n = S_n \cdot e_n = S_n \cdot (d_C - x) = S_n \cdot (e + h/2 - x)$$

and it is

$$S_n = -\frac{bx^2}{2} - nA_s'(x - c) + nA_s(d - x),$$

$$J_n = +\frac{bx^3}{3} + nA_s'(x - c)^2 + nA_s(d - x)^2.$$

From the useful relationships of the previous slide we derive, implicitly, an equation of equilibrium,

$$\left(-\frac{bx^2}{2} - nA_s'(x-c) + nA_s(d-x)\right)\cdot(e+h/2-x) = \frac{bx^3}{3} + nA_s'(x-c)^2 + nA_s(d-x)^2.$$

Expanding, simplifying and collecting the different powers of x, we have a cubic equation

$$x^3 - 3d_C x^2 + 6\frac{n}{b}(A_s(d-d_C) + A_s'(c-d_C))x - 6\frac{n}{b}(A_s d(d-d_C) + A_s' c(c-d_C)) = 0$$

Solving for x we can compute S_n and the stresses using the so called monomial formula, where s is the downward distance from the neutral axis:

$$\boldsymbol{\sigma}_c = \frac{N}{S_n}s, \qquad \boldsymbol{\sigma}_s = n\frac{N}{S_n}s.$$

```
from scipy.optimize import newton

# dati in Newton e millimetri
M =  180E6  ; N = -450E3
h =  600.0  ; b =  300.0
n  =   15
A0 = 1000.0 ; A1 = 600.0
c  = 40.0

d  = h-c
ex = M/N
dc = ex+h/2

# coefficienti della cubica
c1 = -3*dc
c2 =  6*n*(A0*1*(d-dc)+A1*1*(c-dc))/b
c3 = -6*n*(A0*d*(d-dc)+A1*c*(c-dc))/b

x = newton(lambda x: x**3 + x**2*c1 + x*c2 + c3, h/2)
Sn = - b*x*x/2 - n*A1*(x-c) + n*A0*(d-x)

print """Posizione n-n        %10.3f mm
Momento statico      %10.3f cm^3
Sigma calcestruzzo   %10.3f N/mm^2
Sigma acciaio        %10.3f N/mm^2"""%(x,Sn/1000., -N*x/Sn,
n*N*(d-x)/Sn)
=====================================================================
Posizione n-n           286.188 mm
Momento statico      -10394.039 cm^3
Sigma calcestruzzo      -12.390 N/mm^2
Sigma acciaio           177.816 N/mm^2
```

1. Determine centroid and section nucleus.
2. If C is inside the nucleus, use formulas for completely reacting section *n.b., $M^* \neq M$).
3. You determine $S_n(x)$ and $J_n(x)$ for your section and solve the equation of equilibrium to find x, then compute $S_n(x)$ and use the monomial formulas (n.b., possibly you have to use a numerical procedure to compute/approximate S and J and solve for x).

Esempio

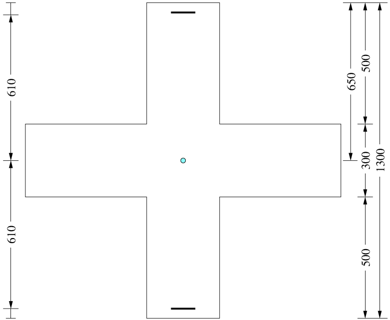
$A_s = A'_s = 1000$ mm²,

c = 40 mm

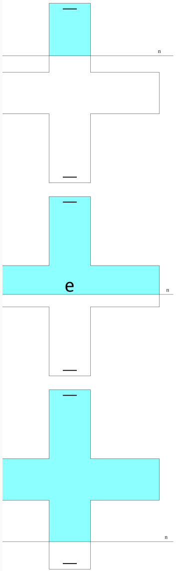
$x_G = 650$ mm

N = - 2E6 N

M = 800E6 N mm



Esempio



- $S_{n1} = -300\, x^2/2 + n\, A_s\, (40-x) + n\, A_s\, (1260-x),$

  $J_{n1} = 300\, x^3/3 + n\, A_s\, (40-x)^2 + n\, A_s\, (1260-x)^2.$

- $S_{n2} = S_{n1} - 1000\, (500-x)^2/2,$

  $J_{n2} = J_{n1} + 1000\, (x-500)^3/3.$

- $S_{n3} = S_{n1} + (1000 \times 300)\ (650-x)$

  $J_{n3} = J_{n1} + (1000 \times 300^3/12) + (1000 \times 300)\, (650-x)^2.$

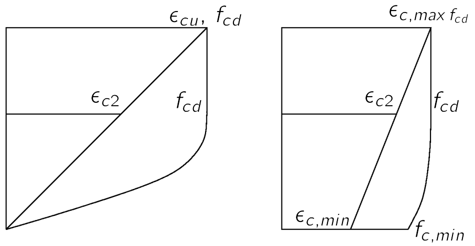
Non Linear Behavior

It is customary to determine $M_{Rd,N}$ for an assigned value of N_{Ed} and finally check that $M_{Rd,N} \geq M_{Ed}$; the steps are as follows:

1. assert that $-(A_c f_{cd} + (A_s + A_s')f_{yd}) \leq N_{Ed} \leq (A_s + A_s')f_{yd})$
2. check if the neutral axis is inside the section, $N_{Ed} \geq \beta A_c f_{cd} + \Sigma A_s f_s$,
3. if the neutral axis is inside the section, use the stress block to find x so that the resultant of stresses equals N_{Ed} and compute the moment of the stress field with respect to the geometric centre of the section, this moment is our $M_{Rd,N}$.

On the other hand if the neutral axis is external (low eccentricity) we have to take into account a parameterized conventional strain field and the derived stress field, find the stress field whose resultant equals N_{Ed}, compute the moment, as in the previous case.

On the left, the strains and the stresses when the neutral axis coincides with the lower edge of the section,
on the right the conventional strain diagram (the position of the chords on which $\epsilon = \epsilon_{c2}$ is kept fixed, the strain diagram is rotated with respect to the fixed point) and the corresponding stress diagram.



We can express the conventional strain and stresses in terms of a single parameter, it is useful to choose as our parameter η,with $\epsilon_{c,min} = \eta\epsilon_{c2}$.

Given a specific section, it is easy to compute analytically or numerically the coefficients $\beta(\eta)$ and $\kappa(\eta)$.

Alternatively we can approximate the distribution associated with η with a stress block $\sigma_c = -f_{cd}$ of equivalent height

$$h' = (1 - \frac{(1-\eta)^2}{5})\, h$$

for a rectangular section it is $\kappa = 0.5\, h'$, for different sections κ is to be computed.

For a rectangular section, we define N_0 as the smallest load for which the section is fully compressed or, i.o.w. the resultant of a limit state stress distribution that has the neutral axis on the bottom edge,

$$
\begin{aligned}
N_s &= -A_s' \min(f_{yd}, E_s \epsilon_{cu} \frac{h-c}{h}) - A_s E_s \epsilon_{cu} \frac{c}{h} \\
N_c &= -\beta b h \alpha f_{cd} \\
N_0 &= N_c + N_s
\end{aligned}
$$

If N_{Ed} is smaller, in modulus, of N_0 then the section is crossed by the neutral axis.

It is $N_{Ed} > N_0$, the neutral axis is inside the section and

$$
\begin{aligned}
N'_s &= -A'_s \min(f_{yd}, E_s \frac{x-c}{x} \epsilon_{cu}),\\
N_s &= +A_s \min(f_{yd}, E_s \frac{d-x}{x} \epsilon_{cu}),\\
N_c &= -\beta\, b\, x\, \alpha\, f_{cd}.
\end{aligned}
$$

The equation $N_{Ed} = N_c(x) + N'_s(x) + N_s(x)$ gives the value of x, the design moment strength is the moment, with respect to the centre of the concrete sections, of N'_s, N_s and N_c.

Example

Rectangular section, not fully compressed

C25/30, 30x60 cm, $A_s = 10\,\text{cm}^2$, $A'_s = 6\,\text{cm}^2$,
$N_{Ed} = -675\,\text{kN}$, $M_{Ed} = 270\,\text{kN m}$

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$N_0 = (-2066 - 235 - 47)\text{kN} = -2348\text{kN} < N_{Ed}$, the n.a. crosses the section,
by trial and error, $x = 24.15\,\text{cm}$,
$N_s = 391.3\,\text{kN}$, $N'_s = -234.0\,\text{kN}$ and $N_c = -831.5\,\text{kN}$
$M_{Rd,N} = 391.3 \cdot (0.56 - 0.30) + (-234.0) \cdot (0.04 - 0.30) + (-831.5) \cdot (0.41 \cdot 0.24 - 0.30) = 328.7\,\text{kN m} > M_{Ed}$

We have to determine the value of η — using the *signed* value of ϵ_{c2}, i.e., $\epsilon_{c2} = -0.002$ we have

$$
\begin{aligned}
\epsilon_s &= \left(\frac{c}{4h/7}(1-\eta)+\eta\right)\epsilon_{c2} & \epsilon'_s &= \left(\frac{d}{4h/7}(1-\eta)+\eta\right)\epsilon_{c2} \\
N_s &= A_s \max(-f_{yd}, E_s\epsilon_s) & N'_s &= A'_s \max(-f_{yd}, E_s\epsilon'_s) \\
\beta &= 1 - 0.19(1-\eta)^2 & N_c &= -\beta b h \alpha f_{cd} \\
N_{Ed} - N_c - N_s - N'_s &= 0
\end{aligned}
$$

The value of η for which we have equilibrium can be derived from the last equation.

When η is known, it is possible to compute the resultant in the different materials and eventually their total moment with respect to the geometrical centre of the section.

Rectangular section fully compressed

Same section,

$N_{Ed} = -2500\,\mathrm{kN}$, $M_{Ed} = 120\,\mathrm{kN\,m}$

Rectangular section fully compressed

Same section,

$N_{Ed} = -2500\,\mathrm{kN}$, $M_{Ed} = 120\,\mathrm{kN\,m}$

$N < N_0$, the section is fully compressed,

by trial and error $\eta = 0.1215$

$N_s = -90\,\mathrm{kN}$, $N'_s = -235\,\mathrm{kN}$ and $N_c = -2175\,\mathrm{kN}$

writing the moment of the forces with respect to the geometric centre

$M_{Rd} = 118.1\mathrm{kN\,m}$

our section is not OK.