Knight's Tour on Hexagonal Nets*

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Abstract

The Knight's Tour Problem is that: can a knight visit each block of a chessboard exactly once, by a sequence of knight's moves, and returns to the initial hexagonal cell. This paper introduces a chessboard in the shape of a hexagonal net and adapted moves comparable to that of the standard chess pieces. J. J. Watkins [1] shows that there exists a general solution of the knight's tours on triangular honeycombs of size 8 and larger. In this paper we shall show that in general, knight's tours on hexagonal nets of size 4 or larger exist. This is accomplished by breaking down the hexagonal net into six triangular honeycombs plus one single cell.

Key words and phrases : knight's tour, knight's move, hexagonal net, triangular honeycomb.

AMS 1991 subject classification : 05C45

1. Introduction

Research in the knight's tour problem goes back to De Moivre and Euler [2]. Two excellent sources for the history of this problem can be found in W. W. Rouse Ball [2] and Martin Gardner [3]. There were some results of knight's tour problem on various rectangular chessboards [4-9]. In 1991, A. J. Schwenk [10] answered completely the question of which rectangular chessboards have a knight's tour. More recently, J. J. Watkins [1] and R. Hoenigman [11] considered the knight's tour problem for surfaces other than the plane, such as torus.

The hexagonal net of order n is a graph formed by a hexagon in the center, surrounded by $(n-1)$ rings of hexagonal cells [12]. On hexagonal net the knight moves two hexagons in a single direction and then one hexagon either to the left or to the right

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60 degrees, as illustrated in Figure 1. Throughout this paper, a *k-tour* means a knight's tour.

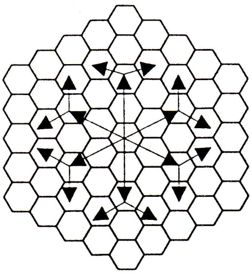

Figure 1: Knight's Move of the Center Hexagon on Hexagonal Net of Order 5

For hexagonal nets of order 1 and 2, it is obvious that there is no k-tour. For hexagonal net of order 3, the hexagonal cell in the center has no knight's move to any other cells. Therefore, the smallest hexagonal net that has a k-tour is a board of order 4.

2. Idea for Finding Knight's Tour of Hexagonal Nets

To find a k-tour on hexagonal nets of order 9 or above, we can break down a hexagonal net into six triangular honeycombs plus one hexagonal cell in the center. For example, a hexagonal net of order 9 can be broken down into six triangular honeycombs of order 8 (Figure 2). Since each triangular honeycomb has a k-tour, a k-tour on the hexagonal net can be formed by connecting the k-tours on triangular honeycombs. But this method fails for the hexagonal nets of order 4 to 8, because no k-tour is possible on triangular honeycombs of order smaller than 8.

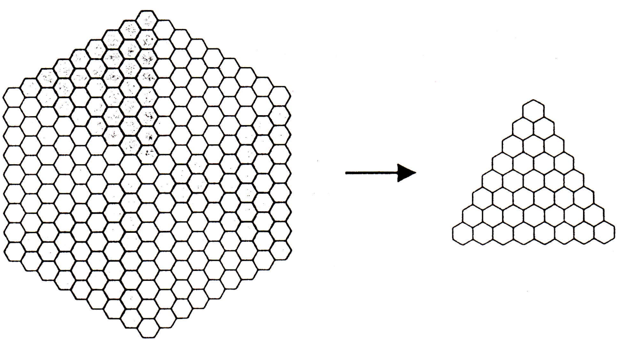

Figure 2: Breaking Down the Hexagonal Net of Order 9

For hexagonal net of order 4 to 8, the knight's tours are showed in Figure 3.

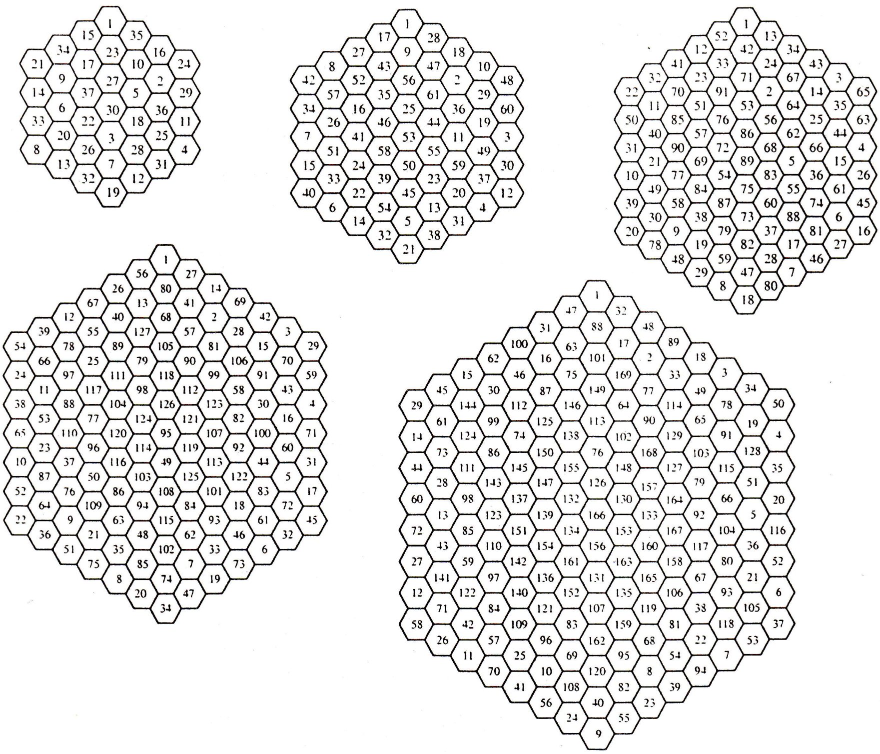

Figure 3: Tours of Hexagonal Nets of Orders 4 to 8

3. Algorithm for Hexagonal Nets

To produce a k-tour for the hexagonal net of order 9, we first break down to six triangular honeycombs of order 8. A k-tour on triangular honeycomb of order 8 is shown in Figure 4.

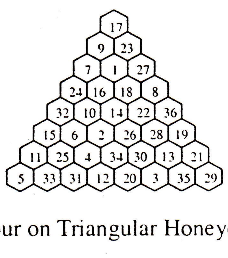

Figure 4: Knight's Tour on Triangular Honeycomb of Order 8.

We duplicate the k-tours from figure 4 for the six pieces on the hexagonal net of order 9, as shown in figure 5. In particular, we have highlighted the key cells where splicing will take place. We begin at cell 1 at the upper left side triangular honeycomb and finish the k-tour on the triangular honeycomb at cell 36. From there we jump to the cell 1 on the left side triangular honeycomb and end up at cell 36, and this continuous until we get to cell 36 on the right side triangular honeycomb. Then we jump to cell 2 on the upper right side triangular honeycomb and finish at cell 1, (i.e. from cell 2 to cell 3, through cell 36, and then cell 1). Finally, we move to the cell in the center and return to cell 1 on the upper left side triangular honeycomb. This is a k-tour on the 9-hexagonal net.

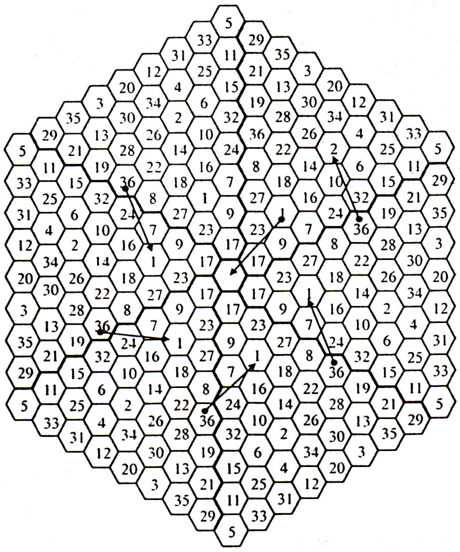

Figure 5: Subdivision of Hexagonal Net of Order 9.

4. Sufficient Condition

The important thing to notice about this example is how the position of the highlighted cells 1 and 36 in the triangular honeycomb relative to the other triangular honeycombs in anti-clockwise direction. Actually, except for the upper right triangular honeycomb, position of the highlighted cells are in exactly the same relative position.

Thus, a sufficient condition of a k-tour of triangular honeycombs that allows the splicing is that

The k- tour includes two moves:

- *A move between the 2^{nd} hexagon in the 2^{nd} row (the hexagon 1 in Figure 4, for example) and the 5^{th} hexagon in the 1^{st} row (the hexagon 36 in Figure 4, for example).*
- *A move between the 2^{nd} hexagon in the 2^{nd} row and the 3^{rd} hexagon in the 4^{th} row (the hexagon 2 in Figure 4, for example).*

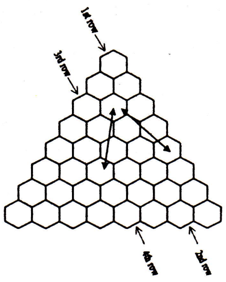

Figure 6: Sufficient Condition for Finding the Tour on Hexagonal Net.

If we can find a k-tour on all triangular honeycombs of order larger than 8 with the sufficient condition described above, then we can show that there exists a k-tour on each hexagonal net of order 4 or above. We now give an inductive argument to show that a k-tour with the sufficient condition exists for any triangular honeycomb of order at least 8. The idea of the proof comes from J. J. Watkins [1].

5. Algorithm for Triangular Honeycombs

Figure 7 shows a k-tours on triangular honeycomb of order 8 and 9. Notice that they both have the sufficient condition described above.

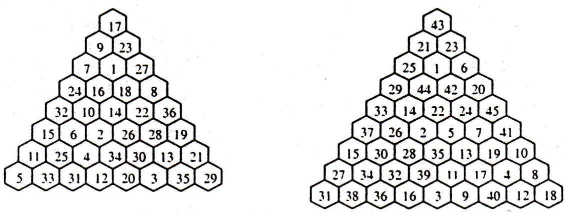

Figure 7: Knight's Tours of Triangular Honeycombs of order 8 & 9.

To produce a k-tour for triangular honeycomb of order 17, we can subdivide it into smaller pieces and then splice together k-tours of the individual pieces to form a k-tour of the whole board. We illustrate the technique by using the two k-tours from Figure 7 to produce a k-tour for the board of order 17.

In Figure 8, we subdivide the board of order 17 into four smaller boards, three of order 8 and one of order 9, and duplicate the k-tours from Figure 7 for the four pieces. In particular, we have highlighted the key cells where the splicing will take place. To produce a k-tour for the board of order 17, we begin at cell 1 in the lower left and finish at cell 36. From there we jump to the highlighted cell 19 in the center and proceed along the k-tour, that is from 19 to 20 and 21 and so on, until the highlighted cell 31 is reached. Then we jump to the highlighted cell 13 of the top triangular honeycomb and complete the k-tour of that board at the highlighted cell 12. Next, we jump back to the center board at the highlighted cell 32 and continue the k-tour up to the highlighted cell 43. From there we jump to the highlighted cell 25 of the right lower board and complete the k-tour of that board at the highlighted cell 24. Then, we jump back to the center board at the highlighted cell 44 and complete the k-tour in that board, that is from 44 to 45 to 1 to 2, and so on, finishing at the highlighted cell 18. This gives a k-tour of triangular honeycomb of order 17.

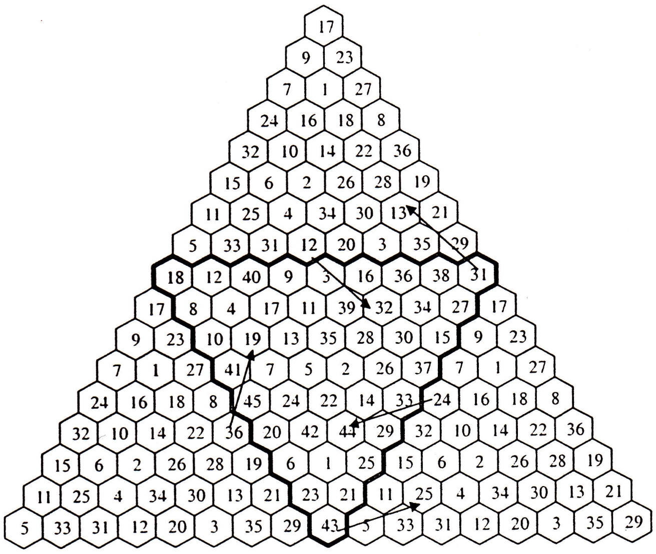

Figure 8: Subdivision of Triangular Honeycomb of Order 17.

6. Crucial Property for Odd Order Triangular Honeycombs

The important thing to notice is how the position of the highlighted cells 1 and

36 in the order 8 in the lower left relative to the highlighted cells 18 and 19 in the order 9 board in the center allows us to splice together these two k-tours. Moreover, highlighted cells 24 and 25 and highlighted cell 12 and 13 are in exactly the same position.

The crucial property of the tour that allowed the splicing was that:

for each of the three corners, the knight's tour includes a move between the 5th hexagon in the 1st row, and the 2nd hexagon in the 2nd row.

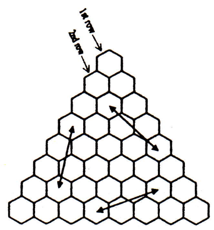

Figure 9: Crucial Property for Odd Order.

The highlighted cells in the center board are automatically in position because the corner cells always include these moves.

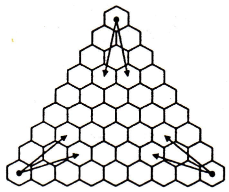

Figure 10: Knight's Move on the Corner Hexagons.

The tour produced for the order 17 board also has the crucial property described above and so, in turn, could itself be used inductively to produce tours of larger boards. This tour for the order 17 board also has the sufficient condition mentioned in section 4.

The above technique works for all boards of odd order; that is, a k-tour for a board of order $2k+1 (k \geq 8)$ can be produced by splicing together k-tours for three boards of order k and one board of order $k+1$ if a k-tours for the smaller boards have the crucial property described above.

7. The Crucial Property for Even Order Triangular Honeycombs

For boards of even order, the same basic idea works, but the position of the key cells are different because the geometry of the subdivision is different. We will illustrate this by producing a k-tour for a board of order 18, again using the k-tours of order 8 and order 9 in figure 7. This time we have three boards of order 9 and one board of order 8 in the center, as shown in Figure 11. We have suppressed the number except for the key cells where splicing takes place.

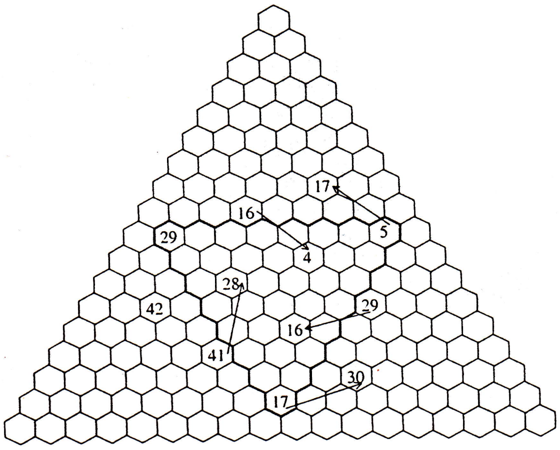

Figure 11: Subdivision of Triangular Honeycomb of Order 18

The crucial property of the tour that allowed the splicing was that:

for each of the three corners, the knight's tour includes a move between the 6th hexagon in the 1st row and the 3rd hexagon in the 2nd row.

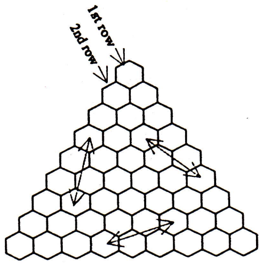

Figure 12: Crucial Property for Even Order

The k-tour produced in triangular honeycomb of order 18 also has the crucial property described above and the sufficient condition described in section 4. Thus, a k-tour for a board of order $2k(k \geq 9)$ can be produced by splicing together tours for three boards of order k and one board of order $(k\text{-}1)$.

To complete the proof, we show k-tours for triangular honeycombs of order 10 to 16 satisfy both of the crucial properties and the sufficient condition in the following.

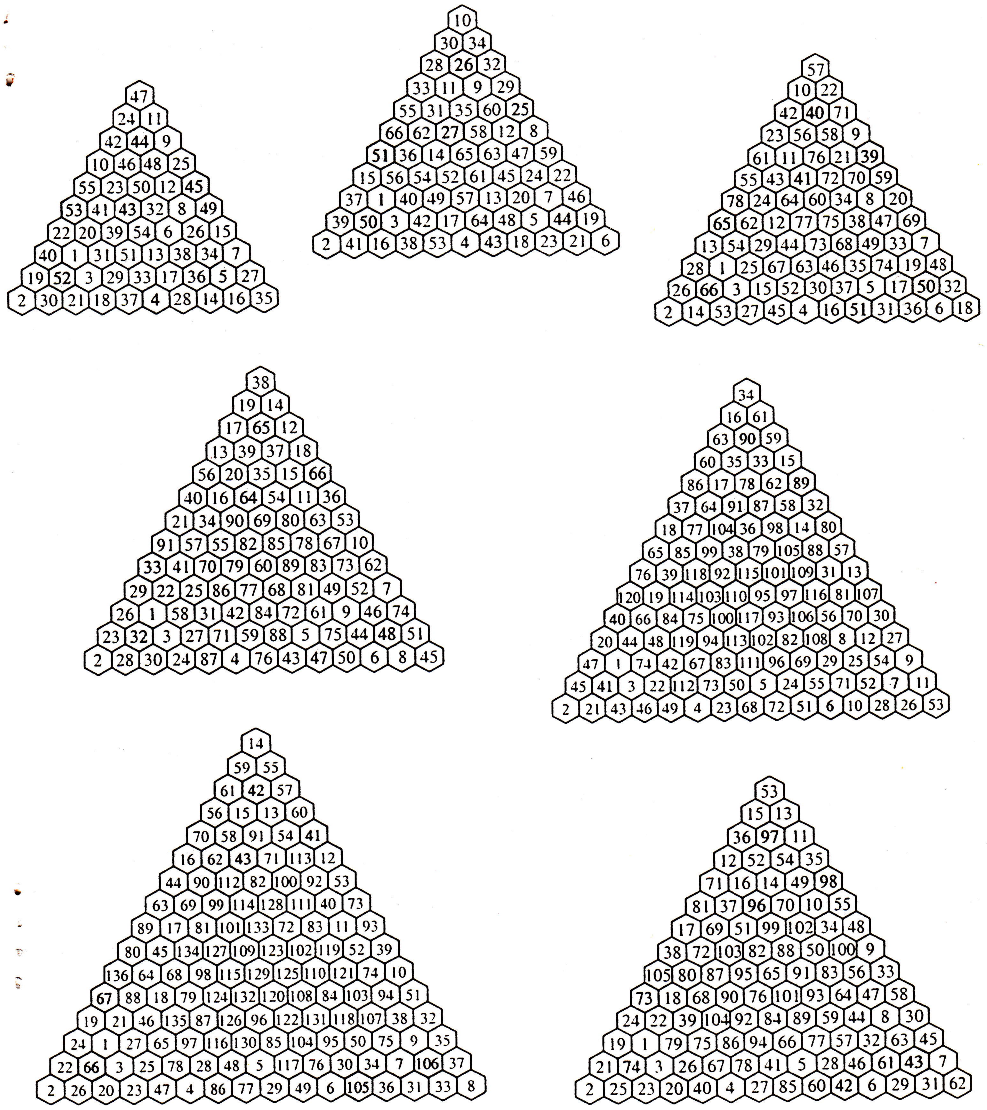

Figure 12: Knight's Tours of Triangular Honeycombs of Order 10 to 16

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