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[GTN LI:2] THE NUMBER OF INCREASING NONCONSECUTIVE SUBPATHS IN A PATH

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Let $G = (V, E)$ be a connected simple graph of order n. Suppose that $f\colon V \to \{1, 2, \ldots, n\}$ is a vertex labeling of G; that is, f is a bijection. Let $d_f(G)$ denote the number of *increasing nonconsecutive paths* in G (that is, the labels of each subpath of G are in increasing order and no two labels in the sequence are consecutive). Let

$$d(G) = \max_f d_f(G),$$

where f runs through all the vertex labelings of G. This concept was defined by Gargano *et al.* **[1]**. These authors asked for the value of $d(P_n)$, where P_n is a path graph of order n.

Let $P_n = x_1x_2\ldots x_n$ be a path graph of order n. Suppose θ is a labeling of P_n such that $\theta(x_i) = a_i$, $1 \le i \le n$. Then we use the sequence $\theta = (a_1 \ldots, a_n)$ to denote the labeled path P_n under θ, where $\{a_1, \ldots, a_n\} = \{1, \ldots, n\}$. For simplicity, we use $d(n)$ instead of $d(P_n)$. In this paper, we provide an answer to the question in **[1]**.

Let $\theta = (a_1, \ldots, a_n)$ be a labeling of P_n. A *section* of θ is a subsequence $(a_i, a_{i+1}, \ldots, a_j)$ for some pair of integers i and j, with $1 \le i \le j \le n$. A *monotonic nonconsecutive section* of θ is a section of θ that is a monotonic sequence, no two terms of which are consecutive. Thus, a labeled subpath of P_n corresponds to a section of θ; an increasing nonconsecutive subpath corresponds to a monotonic nonconsecutive section of θ.

We use $d(\theta)$ to denote the number of monotonic nonconsecutive sections of θ. Thus, $d(n) = \max_\theta d(\theta)$, where θ runs through all labelling of P_n.

A *maximal monotonic nonconsecutive section* of θ is a monotonic nonconsecutive section $(a_i, a_{i+1}, \ldots, a_j)$ of θ such that neither $(a_{i-1}, a_i, a_{i+1}, \ldots, a_j)$, if $i > 1$, nor $(a_i, a_{i+1}, \ldots, a_j, a_{j+1})$, if $j < n$, are monotonic nonconsecutive sections. A subpath corresponding to a maximal monotonic nonconsecutive section of θ is called a *maximal monotonic nonconsecutive path* in θ.

> **Lemma 1:** Any increasing nonconsecutive subpath in a labeled graph is of order at most $\lfloor (n+1)/2 \rfloor$. Moreover, suppose that n is odd, then there is at most one increasing nonconsecutive path of order $(n+1)/2$, and if there is an increasing nonconsecutive path of order $(n+1)/2$, then the second longest increasing nonconsecutive path has order at most $(n-1)/2$.

Proof: Let θ be a labeling of P_n. Let $\sigma = (b_1, \ldots, b_s)$ be a maximal monotonic nonconsecutive section of θ. Without loss of generality, assume that σ is an increasing sequence. Then $2(s-1) \le b_s - b_1 \le n-1$. Thus, $s \le (n+1)/2$, that is, $s \le \lfloor (n+1)/2 \rfloor$.

Suppose that n is odd and there is a monotonic nonconsecutive section of θ with order $s = (n+1)/2$. Without loss of generality, assume that this sequence is increasing. Then, from the previous inequality, we have $b_s = n$ and $b_1 = 1$. Hence, this sequence must be $(1, 3, \ldots, n)$. Clearly it is unique.

Suppose that n is odd and θ contains a maximal nonconsecutive section of order $(n+1)/2$. Without loss of generality, assume this maximal nonconsecutive section is $(1, 3, \ldots, n)$. Suppose that τ is another monotonic nonconsecutive path. Then τ contains at most one odd integer. If τ contains no odd integers, then the order of τ is less than $(n-1)/2$. Suppose that τ contains an odd integer. Then it is either 1 or n. If it contains 1, then $\tau = (c_1, \ldots, c_t, 1)$. Thus, τ is decreasing, the c_js are even, and $c_t \ge 4$. Hence, the order of τ is less than or equal to $(n+1)/2$. The argument for n contained in τ is similar. ■

Lemma 2: There are $k(k-1)/2$ monotonic nonconsecutive subpaths of order at least 2 in every monotonic nonconsecutive path graph of order k.

Let τ be a sequence of positive integers. We use $e(\tau)$ to denote the number of monotonic nonconsecutive sections (increasing paths) of order greater than 1. Let $\theta = (a_1 \leq 1, \ldots, a_n)$ be a labeled path graph. Suppose that θ contains a total number s maximal monotonic nonconsecutive paths, say $\theta_1, \ldots, \theta_s$, of orders $n_1, \ldots, n_s$, respectively, where $1 \leq n_i \leq \lfloor (n+1)/2 \rfloor$. It is easy to see that

$$d(\theta) = f(s;n_1, \ldots, n_s) = n + \sum_{i=1}^{s} e(\theta_i) = n + \sum_{i=1}^{s} \frac{1}{2} n_i(n_i - 1).$$

Let $e(n) = \max_\theta e(\theta)$. Then our aim is to determine $e(n)$, that is, to maximize $\sum_{i=1}^{s} \frac{1}{2} n_i(n_i-1)$ among all labelling.

Example 1: Let $\theta = (1, 4, 3, 5, 2)$. Then θ contains three maximal monotonic nonconsecutive sections; namely, (1, 4), (3, 5), and (5, 2). Hence, $s = 3$, $n_1 = n_2 = n_3 = 2$. There are, altogether, eight monotonic nonconsecutive subsequences: (1, 4), (3, 5), (5, 2), (1), (2), (3), (4), and (5).

Example 2: Let $\theta = (1, 4, 3, 2, 5)$. Then θ contains three maximal monotonic nonconsecutive sections; namely, (1, 4), (3), and (2, 5). Thus, $s = 3$, $n_1 = 2$, $n_2 = 1$, and $n_3 = 2$. There is a total of seven monotonic nonconsecutive subsequences: (1, 4), (2, 5), (1), (2), (3), (4), and (5).

Example 3: Suppose $k \geq 2$. Let $\theta = (1, 3, 5, \ldots, 2k-1, 2, 4, \ldots, 2k)$. Then θ contains three maximal monotonic nonconsecutive sections: $(1, 3, 5, \ldots, 2k-1)$, $(2k-1, 2)$, and $(2, 4, \ldots, 2k)$. Thus,

$$d(\theta) = 2k + \frac{k(k-1)}{2} + 1 + \frac{k(k-1)}{2} = k^2 + k + 1.$$

Hence, $d(2k) \geq k^2 + k + 1$.

Let $\theta = (1, 3, 5, \ldots, 2k-1, 2k+1, 2k-2, \ldots, 4, 2, 2k)$. Then, θ contains three maximal monotonic nonconsecutive sections: $(1, 3, 5, \ldots, 2k-1, 2k+1)$, $(2k+1, 2k-2, \ldots, 4, 2)$, and $(2, 2k)$. Thus,

$$d(\theta) = 2k + 1 + \frac{(k+1)k}{2} + \frac{k(k-1)}{2} + 1 = k^2 + 2k + 2.$$

Hence, $d(2k+1) \geq k^2 + 2k + 2$.

Lemma 3: Suppose a and b are two integers such that $a, b \geq 1$. Then $a(a-1) + b(b-1) \leq (a+b-1)(a+b-2)$.

Proof: It is clear that

$$a(a-1) + b(b-1) = (a+b-1)(a+b-2) - 2(a-1)(b-1) \leq (a+b-1)(a+b-2).$$

■

Corollary 4: Suppose θ_1 and θ_2 are two maximal monotonic nonconsecutive sections with orders a and b, respectively. Then $e(\theta_1) + e(\theta_2) \leq \frac{1}{2}(a+b-1)(a+b-2)$. (Note that the total number of integers involved in θ_1 or θ_2 is either $a+b$ or $a+b-1$.) ■

Lemma 5: For any labeling θ of P_{2k} with $k \geq 2$, $e(\theta) \leq k^2 - k + 1$.

Proof: Suppose that θ contains s maximal monotonic nonconsecutive paths, $\theta_1, \ldots, \theta_s$, of orders $n_1, \ldots, n_s$, respectively, where $1 \leq n_i \leq k$. We may require that there is no common integer belonging to both θ_i and θ_j if $|j-i| \geq 2$. We partition the set $\{\theta_1, \ldots, \theta_s\}$ as follows: Let t be the largest index such that the total number of integers involved in $\theta_1, \ldots, \theta_t$ is less than $k+1$. Then put $\theta_1, \ldots, \theta_t$ in the first class C_1. Let t' be the next largest index such that the total number of integers involved in $\theta_{t+1}, \ldots, \theta_{t'}$ is less than $k+1$. Then put $\theta_{t+1}, \ldots, \theta_{t'}$ in the second class C_2. The remaining maximal monotonic nonconsecutive sections, if any, are put into the third class, C_3. Note that there must be less than $k+1$ integers involved in the last class of maximal monotonic nonconsecutive paths.

Let a_i be the total numbers of integers involved in some subset of the maximal monotonic nonconsecutive paths in C_i, $i = 1, 2, 3$. Then, $0 \leq a_i \leq k$, $n \leq a_1 + a_2 + a_3 \leq n+2$. By applying Lemma 3 repeatedly, we obtain

$$\sum_{i=1}^{s} \frac{1}{2} n_i(n_i - 1) \le \sum_{i=1}^{3} \frac{1}{2} a_i(a_i - 1).$$

A simple calculation shows that the function $g(x, y, z) = x(x-1) + y(y-1) + z(z-1)$, where $x, y, z \in \mathbb{R}$, $0 \le x, y, z \le k$, and $2k \le x + y + z \le 2k + 2$, has only one local extremum in the interior of the domain of g. This extremum is a local minimum. Thus, the maximum value of g is attained at the boundary of the domain. By using the method of *Lagrange multipliers* is easy to see that there is no maximum in the interior of the boundary planes or edges. So the maximum is attained at the vertices of the domain of g. The vertices of the domain (by symmetry we only list the vertices with $x \ge y \ge z$) are $(k, k, 0)$ and $(k, k, 2)$. Thus, $g(x, y, z) \le 2k(k-1) + 2$. Hence, $e(\theta) \le k^2 - k + 1$. ■

Lemma 6: For any labeling θ of P_{2k+1} with $k \ge 2$, $e(\theta) \le k^2 + 1$.

Proof: From Lemma 1, if $x = k + 1$, then $0 \le y, z \le k$ and $k \le y + z \le k + 2$. In this case, by the method of Lagrange multipliers, there are four vertices $(y, z) = (k, 0), (0, k), (k, 2), (2, k)$ that need to be considered. In this case, $g(x, y, z) \le 2k^2 + 2$.

Now restrict to the domain $0 \le x, y, z \le k$ and $2k + 1 \le x + y + z \le 2k + 3$. Then, g attains its maximum value at $(k, k, 3)$ by an argument similar to that used in the proof of Lemma 5. Thus, $g(x, y, z) \le 2k^2 - 2k + 6$. Since $k \ge 2$, $g(x, y, z) \le 2k^2 + 2$ and, hence, $e(\theta) \le k^2 + 1$. ■

Combining Lemmas 5 and 6 we have:

Theorem 7: For $k \ge 2$, $d(2k) = k^2 + k + 1$ and $d(2k + 1) = k^2 + 2k + 2$. ■

Reference

[1] M.L. Gargano, M. Lewinter, and J.F. Malerba; On the number of increasing nonconsecutive paths and cycles in labeled graphs, *Graph Theory Notes of New York*, **XLIV:1**, New York Academy of Sciences, 8–9 (2003).

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