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PRODUCT-CORDIAL INDEX SET FOR CARTESIAN PRODUCT OF A GRAPH WITH A PATH*

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Abstract

Let $G=(V,E)$ be a connected simple graph. A labeling $f:V\to\mathbb{Z}_2$ induces two edge labelings $f^+,f^*:E\to\mathbb{Z}_2$ defined by $f^+(xy)=f(x)+f(y)$ and $f^*(xy)=f(x)f(y)$ for each $xy\in E$. For $i\in\mathbb{Z}_2$, let $v_f(i)=|f^{-1}(i)|$, $e_{f^+}(i)=|(f^+)^{-1}(i)|$ and $e_{f^*}(i)=|(f^*)^{-1}(i)|$. A labeling f is called friendly if $|v_f(1)-v_f(0)|\le 1$. For a friendly labeling f of a graph G, the friendly index of G under f is defined by $i^+_f(G)=e_{f^+}(1)-e_{f^+}(0)$. Also the product-cordial index of G under f is defined by $i^*_f(G)=e_{f^*}(1)-e_{f^*}(0)$. In this paper, we show a relation between these two indices. Moreover, the product-cordial index sets of grids are determined.

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1. Introduction

In this paper, all graphs are simple and connected. All undefined symbols and concepts may be looked up from [1]. Let $G=(V,E)$ be a connected simple graph. A labeling $f:V\to\mathbb{Z}_2$ induces two edge labelings $f^+,f^*:E\to\mathbb{Z}_2$ defined by $f^+(xy)=f(x)+f(y)$ and $f^*(xy)=f(x)f(y)$ for each $xy\in E$. For $i\in\mathbb{Z}_2$, let $v_f(i)=|f^{-1}(i)|$, $e_{f^+}(i)=|(f^+)^{-1}(i)|$ and $e_{f^*}(i)=|(f^*)^{-1}(i)|$. A labeling f is called *friendly* if $|v_f(1)-v_f(0)|\le 1$. For a friendly labeling f of a graph G, the *friendly index* of G under f is defined by $i^+_f(G)=e_{f^+}(1)-e_{f^+}(0)$. The set

$$\mathrm{FFI}(G)=\{i^+_f(G)\mid f \text{ is a friendly labeling of } G\}$$

is called the *full friendly index set* of G which was first introduced in [5]. Also the *product-cordial index* of G under f is defined by $i^*_f(G)=e_{f^*}(1)-e_{f^*}(0)$. The set

$$\mathrm{FPCI}(G)=\{i^*_f(G)\mid f \text{ is a friendly labeling of } G\}$$

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is called the *full product-cordial index set* of G, which was first introduced in [7]. The full product-cordial index sets of torus and twisted cylinders were found in [7] and [8], respectively. The set of absolute values of product-cordial indices of a graph is called the *product-cordial index set* of that graph. The product-cordial index sets of paths, cycles, wheels, complete graphs, complete bipartite graphs, double stars, cylinders, generalized wheels were found in [2–4]. For a brief history of product-cordial labeling, interested readers are referred to [2]. Throughout this paper, we use the term 'labeling' to mean a vertex labeling whose values are taken in $\mathbb{Z}_2$. Note that $i_f^+(G)$ and $i_f^*(G)$ can be extended to any labeling.

For a fixed labeling f, a vertex v is called *k-vertex* if $f(v)=k$ and an edge is called an *(i,j)-edge* if it is incident with an i-vertex and a j-vertex. An edge e is called *k-edge* if $f^*(e)=k$. We define the number of (i,j)-edges by $E_f(i,j)$. Then

$$\begin{aligned} e_{f^+}(1) &= E_f(1,0) = E_f(0,1), & e_{f^+}(0) &= E_f(1,1) + E_f(0,0); \\ e_{f^*}(1) &= E_f(1,1), & e_{f^*}(0) &= E_f(0,0) + E_f(1,0). \end{aligned}$$

Since $e_{f^+}(1) + e_{f^+}(0) = e_{f^*}(1) + e_{f^*}(0) = q$ the size of the graph G,

$$i_f^+(G) = 2e_{f^+}(1) - q = 2E_f(1,0) - q = q - 2e_{f^+}(0); \tag{1.1}$$

$$i_f^*(G) = 2e_{f^*}(1) - q = 2E_f(1,1) - q = q - 2e_{f^*}(0). \tag{1.2}$$

Lemma 1.1 ([6]). *Let f be any labeling of a graph G with q edges. If the degree sum of 1-vertices is s, then $i_f^+(G) = 2s - 4E_f(1,1) - q$.*

Combining (1.2) and Lemma 1.1 we have

Corollary 1.1. *Let f be any labeling of a graph G with q edges. If the degree sum of 1-vertices is s, then $2i_f^*(G) = 2s - 3q - i_f^+(G)$.*

Lemma 1.2. *Let $f : V(C_n) \to \mathbb{Z}_2$ be any labeling. Suppose $v_f(1) = z+k$ and $v_f(0) = z$ for some $k \geq 0$. Note that $2z+k = n$. Then $e_{f^*}(1) \geq k$.*

Proof. We provide two proofs here.

It is obvious when $z = 0$. So we assume $z \geq 1$. We view a labeling of C_n as a circular binary sequence. A all 1 sequence between two consecutive 0's is called a section (it may be empty sequence). So there are z sections. Suppose there are a empty sections and $z-a$ nonempty sections. To generate these nonempty sections, we have to put $z+k$ 1's into $z-a$ sections such that each section contains at least one 1. So we put $z-a$ 1's into each of these $z-a$ sections first. Now it remains $k+a$ 1's to put into some nonempty sections. When we put one 1 into a nonempty section, it creates one 1-edge. So after putting all the remaining 1's, they create $k+a$ 1-edge. Hence, $e_{f^*}(1) = k+a \geq k$.

Alternative proof:

Applying Corollary 1.1 on a cycle and by (1.1) and (1.2) we have $e_{f^*}(1) = (s - e_{f^+}(1))/2 = k + z - e_{f^+}(1)/2$. Since each 0-vertex induced at most two $(1,0)$-edges, $e_{f^*}(1) \geq k$. □

By the first proof of Lemma 1.2 we have the following corollary.

Corollary 1.2. *Let $f : V(C_n) \to \mathbb{Z}_2$ be any labeling. Suppose $v_f(0) = z$ and $n \geq 2z$. Then $e_{f^*}(1) = n - 2z + a$, where $a = E_f(0,0)$.*

Corollary 1.3. *Let $f : V(P_n) \to \mathbb{Z}_2$ be any labeling. Suppose $v_f(0) = z$, $n \geq 2z$ and $E_f(0,0) = a$. Then*

$$e_{f^*}(1) = \begin{cases} n-2z+a+1 & \text{if two pendents are labeled by } 0, \\ n-2z+a-1 & \text{if two pendents are labeled by } 1, \\ n-2z+a & \text{otherwise.} \end{cases}$$

Proof. Add an extra edge to P_n to form the cycle C_n. By considering the labels of two pendents of P_n we will get the corollary. □

Consider the graph $G \times H$, the Cartesian product of G and H. For $x \in V(G)$, the *vertical graph* H_x is the graph induced by $\{(x,y) \mid y \in V(H)\}$; and for $y \in V(H)$, the *horizontal graph* G_y is the graph induced by $\{(x,y) \mid x \in V(G)\}$. Clearly $H_x \cong H$ and $G_y \cong G$. Sometimes when G is a cycle (resp. a path), we will call G_y a horizontal cycle (resp. horizontal path). It is similar for vertical graph.

2. Application to Cylinders

In the paper of Kwong *et al.* [2], they provided two friendly labelings of $C_m \times P_n$ and tried to illustrate the possible maximum value of $e(1)$, the number of 1-edges under a friendly labeling. But they did not make any justification on the maximum value of $e(1)$ not excess the proposed values. There is also some confusion on presenting the bound. For example, when m and n are odd, $e(1) \leq n(m-1) - (m-1)/2$ shows in [2, page 142], and $e(1) \leq n(m-1)$ shows in [2, page 143]. In this section, we make a supplement on that.

Theorem 2.1. *Let f be a friendly labeling of $C_m \times P_n$. Then*

$$e_{f^*}(1) \geq \begin{cases} 0 & \text{if } m \text{ is even,} \\ \lfloor \frac{n}{2} \rfloor & \text{if } m \text{ is odd.} \end{cases}$$

Proof. It is obviously that $e_{f^*}(1) \geq 0$. So we only consider odd m. Suppose there are a horizontal cycles containing more 0-vertices than 1-vertices. Thus there are at least $a-1$ (should be a if mn is even) 1-vertices more than 0-vertices lying in the remaining horizontal cycles totally. Applying Lemma 1.2 on each remaining horizontal cycle, we have $e_{f^*}(1) \geq a-1$. Since m is odd and there are $n-a$ horizontal cycles containing more 1-vertices than 0-vertices, by Lemma 1.2 again we have $e_{f^*}(1) \geq n-a$. Hence $e_{f^*}(1) \geq \max\{a-1, n-a\} \geq \lfloor n/2 \rfloor$. □

Let f be any labeling of a graph G. For a fixed $i \in \mathbb{Z}_2$, a subgraph H of G is called *i-pure* (under f) if $f(u) = i$ for every vertex $u \in V(H)$. H is called *mixed*, if it is not pure.

Lemma 2.1. *Suppose C is a mixed cycle under a labeling f. If $v_f(1) = b$ with $b \geq 1$, then $e_{f^*}(1) \leq b-1$. The equality holds if and only if C contains a 1-pure path of length $b-1$.*

Proof. Since the number of 1-edges is the size of the subgraph H induced by all the 1-vertices. Since C is a mixed cycle, $H \neq C$. Hence H is a forest and then the size of H is at most $b-1$. Moreover, the size of H is $b-1$ if and only if H is a tree which is a path of length $b-1$ in C.

By using the similar proof as above, we have the following corollary.

Corollary 2.1. *Let f be a labeling of a path P. If $e_f(1) = b$ with $b \geq 1$, then $e_{f^*}(1) \leq b-1$. The equality holds if and only if P is a 1-pure path of length $b-1$.*

Lemma 2.2. *For $m,n \in \mathbb{N}$, let $g(y,s) = y+s$ be defined on the hyperbola $(y-m)(s-n) = \lceil mn/2 \rceil$, where $m - \lceil mn/2 \rceil / \lceil n/2 \rceil \leq y \leq \lfloor m/2 \rfloor$. Then the maximum value of g is*

$$m+n-2\sqrt{\left\lceil \frac{mn}{2} \right\rceil} \text{ when } \left\lceil \frac{m}{2} \right\rceil^2 < \left\lceil \frac{mn}{2} \right\rceil \text{ and } \left\lceil \frac{n}{2} \right\rceil^2 < \left\lceil \frac{mn}{2} \right\rceil;$$
$$m+\left\lfloor \frac{n}{2} \right\rfloor - \frac{\lceil \frac{mn}{2} \rceil}{\lceil \frac{n}{2} \rceil} \text{ when } \left\lceil \frac{n}{2} \right\rceil^2 \geq \left\lceil \frac{mn}{2} \right\rceil;$$
$$n+\left\lfloor \frac{m}{2} \right\rfloor - \frac{\lceil \frac{mn}{2} \rceil}{\lceil \frac{m}{2} \rceil} \text{ when } \left\lceil \frac{m}{2} \right\rceil^2 \geq \left\lceil \frac{mn}{2} \right\rceil.$$

Proof. By using simple calculus we can show that the local maximum point occurs at $(m-\sqrt{\lceil mn/2 \rceil}, n-\sqrt{\lceil mn/2 \rceil})$. If the maximum point lies in the interior of the hyperbola, then $m-\sqrt{\lceil mn/2 \rceil} > m - \lceil mn/2 \rceil / \lceil n/2 \rceil$ and $n-\sqrt{\lceil mn/2 \rceil} > n - \lceil mn/2 \rceil / \lceil m/2 \rceil$.

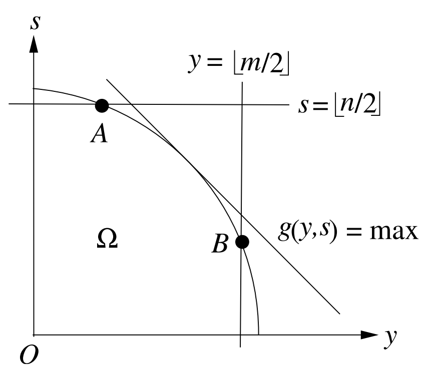

Figure 1. The feasible region.

The last conditions are equivalent to $\lceil m/2 \rceil^2 < \lceil mn/2 \rceil$ and $\lceil n/2 \rceil^2 < \lceil mn/2 \rceil$, respectively. It is easy to get the maximum value when the maximum point is A or B (referred to Fig. 1). Hence we have the lemma. □

Corollary 2.2. *For $m,n \in \mathbb{N}$, let $g(y,s) = y+s$ be defined in the region Ω bounded by the hyperbola $(y-m)(s-n) = \lceil mn/2 \rceil$, $0 \leq y \leq \lfloor m/2 \rfloor$ and $0 \leq s \leq \lfloor n/2 \rfloor$. If y and s are only taken integral values, then the maximum value of g does not excess*

1. *$m+n-\left\lceil 2\sqrt{\lceil mn/2 \rceil} \right\rceil$ when $\lceil m/2 \rceil^2 < \lceil mn/2 \rceil$ and $\lceil n/2 \rceil^2 < \lceil mn/2 \rceil$,*

2. *$m+\lfloor n/2 \rfloor - \lceil \lceil mn/2 \rceil / \lceil n/2 \rceil \rceil$ when $\lceil n/2 \rceil^2 \geq \lceil mn/2 \rceil$,*

3. $n+\lfloor m/2\rfloor-\lceil\lceil mn/2\rceil/\lceil m/2\rceil\rceil$ *when* $\lceil m/2\rceil^2\geq\lceil mn/2\rceil$.

Theorem 2.2. *Let f be a friendly labeling of $C_m\times P_n$, where $m\geq 3$ and $n\geq 2$. Then*

(1) for even m and n,
$$e_{f^*}(1)\leq\begin{cases} mn-n-m/2 & \textit{if } m\geq 2n,\\ mn-m & \textit{otherwise};\end{cases}$$

(2) for even m and odd n,
$$e_{f^*}(1)\leq\begin{cases} mn-n-m/2 & \textit{if } m\geq 2n,\\ mn-m-1 & \textit{otherwise};\end{cases}$$

(3) for odd m and even n,
$$e_{f^*}(1)\leq\begin{cases} mn-n-(m+1)/2 & \textit{if } m\geq 2n-1,\\ mn-m & \textit{otherwise};\end{cases}$$

(4) for odd m and n,
$$e_{f^*}(1)\leq\begin{cases} mn-n-(m-1)/2 & \textit{if } m\geq 2n-1\\ mn-m & \textit{otherwise}.\end{cases}$$

Proof. In order to create as many as 1-edges we may assume the number of 1-vertices is not less than the number of 0-vertices. Let r be the number of 1-pure horizontal cycles and let s be the number of 0-pure horizontal cycles. Note that $0\leq r,s\leq\lfloor n/2\rfloor$. So there are $n-r-s$ mixed horizontal cycles containing $\lceil mn/2\rceil-mr$ 1-vertices totally. Moreover, each vertical path contains at least r 1-vertices. If $r\geq 1$, then by Lemma 2.1 and Corollary 2.1 we have

$$\begin{aligned} e_{f^*}(1) &\leq (mr)+\left[\lceil\tfrac{mn}{2}\rceil-mr-(n-r-s)\right]+(\lceil\tfrac{mn}{2}\rceil-m)=2\lceil\tfrac{mn}{2}\rceil-m-n+r+s\\ &\leq 2\lceil\tfrac{mn}{2}\rceil-m-n+2\lfloor\tfrac{n}{2}\rfloor=\begin{cases} mn-m & \text{if } n \text{ is even},\\ mn-m & \text{if } m \text{ and } n \text{ are odd},\\ mn-m-1 & \text{if } m \text{ is even and } n \text{ is odd}.\end{cases} \end{aligned}\tag{2.1}$$

Suppose $r=0$. Let y be the number of 0-pure vertical paths. Note that $0\leq y\leq\lfloor m/2\rfloor$. Then by Lemma 2.1 and Corollary 2.1 we have

$$e_{f^*}(1)\leq\left[\lceil\tfrac{mn}{2}\rceil-(n-s)\right]+\left[\lceil\tfrac{mn}{2}\rceil-(m-y)\right]=2\lceil\tfrac{mn}{2}\rceil-m-n+y+s.\tag{2.2}$$

Since f is friendly, the number of 0-vertices contained in the 0-pure cycles and 0-pure paths must not greater than $\lfloor mn/2\rfloor$. That is $ny+ms-ys\leq\lfloor mn/2\rfloor$ or equivalent to $(y-m)(s-n)\geq\lceil mn/2\rceil$. We want to maximize $g(y,s)=y+s$ under the above conditions. By the convexity of the feasible region, the maximum point must lie on the hyperbola, where $m-\lceil mn/2\rceil/\lceil n/2\rceil\leq y\leq\lceil m/2\rfloor$.

Case 1. Suppose the maximum point lies in the interior of the hyperbola. By Lemma 2.2 the maximum value of $e_{f^*}(1)$ may be $2\lceil mn/2\rceil-2\sqrt{\lceil mn/2\rceil}$ under the conditions $\lceil m/2\rceil^2<\lceil mn/2\rceil$ and $\lceil n/2\rceil^2<\lceil mn/2\rceil$.

Case 1-1. Suppose n is even. By the above discussion we have $e_{f^*}(1)\leq mn-2\sqrt{mn/2}$ under the condition $\lceil m/2\rceil^2<mn/2$ and $n^2/4<mn/2$. Since $m/2\leq\lceil m/2\rceil$, these conditions imply that $n/2<m<2n$. This implies $m<2\sqrt{mn/2}$. We have $e_{f^*}(1)<mn-m$.

Case 1-2. Suppose m is even and n is odd. We still obtain that $m^2 < 2mn$ and $e_{f^*}(1) < mn-m$.

Case 1-3. Suppose both m and n are odd. By the above discussion we have $e_{f^*}(1) \le mn+1-2\sqrt{(mn+1)/2}$. The condition $\lceil m/2\rceil^2 < \lceil mn/2\rceil$ becomes $(m+1)^2 < 2(mn+1)$. It is equivalent to $m+1 < 2\sqrt{(mn+1)/2}$. We have $e_{f^*}(1) < mn-m$.

Case 2. Suppose the maximum point is $A = (m - \lceil mn/2\rceil/\lceil n/2\rceil, \lfloor n/2\rfloor)$ or $B = (\lfloor m/2\rfloor, n-\lceil mn/2\rceil/\lceil m/2\rceil)$. In this case, if the maximum point is A, then $\lceil n/2\rceil^2 \ge \lceil mn/2\rceil$; if the maximum point is B, then $\lceil m/2\rceil^2 \ge \lceil mn/2\rceil$.

Case 2-1. Suppose m and n are even. Then $B = (m/2, 0)$ and $A = (0, n/2)$ are lattice points. Thus, $e_{f^*}(1) \le mn-n-m/2$ if $m \ge 2n$ and $e_{f^*}(1) \le mn-m-n/2$ if $2m \le n$.

Case 2-2. Suppose m is even and n is odd. If $m \ge 2n$, then the maximum point is B and we have $e_{f^*}(1) \le mn-n-m/2$. If $(n+1)^2 \ge 2mn$, then the maximum point is A. Hence $e_{f^*}(1) \le mn-(n+1)/2+\lfloor -mn/(n+1)\rfloor = mn-\lceil (n+1)/2+mn/(n+1)\rceil = mn-\lceil\big((n+1)^2+2mn\big)/(2(n+1))\rceil \le mn-\lceil 2mn/(n+1)\rceil < mn-m$ since $n > 1$.

Case 2-3. Suppose m is odd and n is even. If $(m+1)^2 \ge 2mn$, then the maximum point is B. So $e_{f^*}(1) \le mn-(m+1)/2-mn/(m+1) = mn-(m+1)/2-n+n/(m+1)$. Hence $e_{f^*}(1) \le mn-(m+1)/2-n$. Note that $(m+1)^2 \ge 2mn$ implies $m+2+1/m \ge 2n$. Since m is odd, the last inequality is equivalent to $m \ge 2n-1$. If $n \ge 2m$, then the maximum point is A. Hence $e_{f^*}(1) \le mn-m-n/2$.

Case 2-4. Suppose m and n are odd. Suppose $(m+1)^2 \ge 2(mn+1)$. It implies that $m+2-1/m \ge 2n$. So it is equivalent to $m+1 \ge 2n$. Then $e_{f^*}(1) \le mn+1-(m+1)/2-(mn+1)/(m+1) = mn-(m-1)/2-n+(n-1)/(m+1)$. Since $(m+1)^2 \ge 2(mn+1)$, $(n-1)/(m+1) \ge (n-1)(m+1)/(2(mn+1)) = 1/2+(n-m)/(2(mn+1)) < 1$. Hence $e_{f^*}(1) \le mn-(m-1)/2-n$.
Suppose $(n+1)^2 \ge 2(mn+1)$. Similarly it implies that $1+n \ge 2m$. Then $e_{f^*}(1) \le mn+1-\big((n+1)^2+2(mn+1)\big)/(2(n+1)) \le mn+1-(2(mn+1))/(n+1) = mn-(2mn-n+1)/(n+1)$. Since $n \ge 2$, the right hand side of the inequality is less then or equal to $mn-(mn+m)/(n+1) = mn-m$.

Comparing the above results with (2.1), we have the theorem. □

Note that all bounds are attainable (see [2]).

3. Application to Grids

For $m \ge 2$ and $n \ge 2$, the Cartesian product $P_m \times P_n$ is a graph with vertex set consisting of mn vertices labeled $u_{i,j}$ (or u_{ij}), where $1 \le i \le m$ and $1 \le j \le n$. Two vertices u_{ij} and u_{hk} are adjacent in $P_m \times P_n$ if either $i = h$ and $|j-k| = 1$, or $j = k$ and $|i-h| = 1$. Note that $P_m \times P_n$ is a graph of order mn and size $2mn-m-n$. It is also called a *grid*. Without loss of generality, we always assume $m \ge n$.

Theorem 3.1. *Let f be a friendly labeling of $P_m \times P_n$, where $m \geq 2$ and $n \geq 2$. Then*

1. *for even m,*
$$e_{f^*}(1) \leq \begin{cases} mn - n - m/2 & \text{if } m \geq 2n, \\ mn - \left\lceil 2\sqrt{mn/2} \right\rceil & \text{if } n \leq m < 2n. \end{cases}$$

2. *for odd m and even n,*
$$e_{f^*}(1) \leq \begin{cases} mn - n - (m+1)/2 & \text{if } m \geq 2n+1, \\ mn - \left\lceil 2\sqrt{mn/2} \right\rceil & \text{if } n \leq m \leq 2n-1; \end{cases}$$

3. *for odd m and n,*
$$e_{f^*}(1) \leq \begin{cases} mn - n - (m-1)/2 & \text{if } m \geq 2n+1 \\ mn + 1 - \left\lceil 2\sqrt{(mn+1)/2} \right\rceil & \text{if } n \leq m \leq 2n-1. \end{cases}$$

Proof. Similar to the proof of Theorem 2.2 we may assume the number of 1-vertices is not less than the number of 0-vertices. Let r be the number of 1-pure horizontal paths and let s be the number of 0-pure horizontal paths. If $r \geq 1$, then by a similar argument as the proof of Theorem 2.2, we have

$$\begin{aligned} e_{f^*}(1) &\leq (m-1)r + [\lceil \tfrac{mn}{2} \rceil - mr - (n-r-s)] + (\lceil \tfrac{mn}{2} \rceil - m) = 2\lceil \tfrac{mn}{2} \rceil - m - n + s \\ &\leq 2\lceil \tfrac{mn}{2} \rceil - m - n + \lfloor \tfrac{n}{2} \rfloor = \begin{cases} mn - m - \frac{n}{2} & \text{if } n \text{ is even,} \\ mn - m - \frac{n+1}{2} & \text{if } m \text{ is even and } n \text{ is odd,} \\ mn - m - \frac{n-1}{2} & \text{if } m \text{ and } n \text{ are odd.} \end{cases} \end{aligned} \tag{3.1}$$

If $r = 0$, then we still obtain (2.2). So we have to maximize $g(y,s) = y + s$ when (y,s) is a lattice point and lies in Ω (see Fig. 1), which is defined in Corollary 2.2.

For the case $m \geq 2n$, by using the third case of Corollary 2.2 we will obtain similar results as Case 2 of the proof of Theorem 2.2. It is easy to see that each upper bound on $e^*_f(1)$ obtained from above is greater than the right hand side of (3.1).

Following we only consider the case when $2 \leq n \leq m < 2n$. By using Corollary 2.2 and a similar argument as Case 1 of the proof of Theorem 2.2, we have

$$e_{f^*}(1) \leq \begin{cases} mn - \lceil 2\sqrt{\frac{mn}{2}} \rceil & \text{if } mn \text{ is even,} \\ mn + 1 - \left\lceil 2\sqrt{\frac{mn+1}{2}} \right\rceil & \text{if } mn \text{ is odd.} \end{cases} \tag{3.2}$$

1. When n are even. Comparing with (3.1) and (3.2) we consider

$$\begin{aligned} X &= mn - \left\lceil 2\sqrt{\frac{mn}{2}} \right\rceil - (mn - m - \frac{n}{2}) \\ &= m + \frac{n}{2} - \left\lceil 2\sqrt{\frac{mn}{2}} \right\rceil \\ &> m + \frac{n}{2} - \left(2\sqrt{\frac{mn}{2}} + 1 \right) \\ &= m + \frac{n}{2} - 1 - \sqrt{2mn} \end{aligned} \tag{3.3}$$

Let $h(m) = m - \sqrt{2mn}$. By using simple calculus, we can see that $h(m)$ is an increasing function on m when $m \geq n$. From (3.3), we have

$$X > (1.5 - \sqrt{2})n - 1 > -1.$$

Since X is an integer, $X \geq 0$.

When n is odd. Since $m \geq n+1 \geq 4$, by a similar proof as above, we have

$$\begin{aligned} X &= m + \frac{n+1}{2} - 2\left\lceil \sqrt{\frac{mn}{2}} \right\rceil > m + \frac{n+1}{2} - 2\left(\sqrt{\frac{mn}{2}} - 1\right) \\ &> 1.5n + 0.5 - \sqrt{2(n^2+n)} > 1.5n + 0.5 - \sqrt{2}(n+0.5) \\ &= (1.5 - \sqrt{2})n + 0.5 - 0.5\sqrt{2} \geq 5 - (3.5)\sqrt{2} > 0. \end{aligned}$$

Thus we have the first part of the theorem.

2. When m is odd and n is even. We still have (3.3). Since $m \geq n+1 \geq 3$, by a similar proof as the case 2,

$$X > (1.5 - \sqrt{2})n - (0.5)\sqrt{2} > -1.$$

So we have the second part of the theorem.

3. When m and n are odd. Similar to the above, we have

$$\begin{aligned} X &= 1 - \left\lceil 2\sqrt{\frac{mn+1}{2}} \right\rceil + m + \frac{n-1}{2} \\ &> 1 - \left(2\sqrt{\frac{mn+1}{2}} + 1\right) + m + \frac{n-1}{2} \\ &= \frac{n-1}{2} + m - \sqrt{2(mn+1)} \end{aligned} \tag{3.4}$$

Let $h(m) = m - \sqrt{2(mn+1)}$. By using simple calculus, we can see that $h(m)$ is an increasing function on m when $m \geq n$. From (3.4) we have

$$\begin{aligned} X &> \frac{n-1}{2} + n - \sqrt{2(n^2+1)} > \frac{n-1}{2} + n - \sqrt{2}(n+0.5) \\ &= (1.5 - \sqrt{2})n - 0.5(1+\sqrt{2}) \geq 4 - (3.5)\sqrt{2} > -1. \end{aligned}$$

Hence $X \geq 0$. Hence we have the third part of the theorem. □

Suppose f is a labeling of a graph $P_m \times P_n$. We shall use an $n \times m$ array whose (j,i)-th entry is $f(u_{ij})$ to represent the labeling f (note that the numbering of columns is from left to right and that of rows is from bottom to top). It is obvious that $e_{f^*}(1) \geq 0$ and this lower bound is attainable by the labeling f_0, where $f_0(u_{ij}) \equiv i + j \pmod 2$. We keep this notation throughout the following of this paper.

Following we show that all upper bounds are attainable. Let f be the friendly labeling attaining the maximum value of the number of 1-edges.

When m is even:

When $m \geq 2n$. The upper bound on $e_{f^*}(1)$ is attained by the labeling

$$\begin{pmatrix} O_{n,m/2} & J_{n,m/2} \end{pmatrix},$$

where $O_{r,s}$ is the $r \times s$ zero matrix and $J_{r,s}$ is the $r \times s$ matrix whose entries are 1.

When $n \leq m < 2n$. Let $p = \left\lceil \sqrt{mn/2} \right\rceil$.

If $mn/2$ is a perfect square, then $mn/2 = p^2$. The labeling

$$\begin{pmatrix} O_{n-p,m-p} & O_{n-p,p} \\ O_{p,m-p} & J_{p,p} \end{pmatrix},$$

attains the upper bound on $e_{f^*}(1)$.

For example, $m = 18$ and $n = 16$. Then $p = 12$ and the labeling f is

$$\begin{pmatrix} O_{4,6} & O_{4,12} \\ O_{12,6} & J_{12,12} \end{pmatrix}.$$

It is easy to see that $e_{f^*}(1) = 264$.

Suppose $mn/2$ is not a perfect square. Then $(p-1)^2 + 1 \leq mn/2 \leq p^2 - 1$

If $p^2 - p < mn/2$, then $1 \leq q = mn/2 - p^2 + p \leq p - 1$. Then $p^2 - p + 1/4 < p^2 - p + q < p^2$. Hence $p - 1/2 < \sqrt{p^2 - p + q} < p$. Then we have,

$$mn - \left\lceil 2\sqrt{mn/2} \right\rceil = 2q + 2p^2 - 2p - \left\lceil 2\sqrt{p^2 - p + q} \right\rceil = 2q + 2p^2 - 2p - 2p = mn - 2p$$

which is attained by the labeling

$$\begin{pmatrix} O_{n-p+1,m-p-1} & O_{n-p+1,1} & O_{n-p+1,p} \\ O_{p-1,m-p-1} & \alpha_q & J_{p-1,p} \end{pmatrix},$$

where $\alpha_q = \begin{pmatrix} O_{p-1-q,1} \\ J_{q,1} \end{pmatrix}$.

If $p^2 - p \geq mn/2$, then $1 \leq q = mn/2 - (p-1)^2 \leq p - 1$. Then $(p-1)^2 < (p-1)^2 + q \leq (p-1)^2 + (p-1) < (p-1/2)^2$. Hence $p - 1 < \sqrt{(p-1)^2 + q} < p - 1/2$. Then we have,

$$mn - \left\lceil 2\sqrt{mn/2} \right\rceil = 2q + 2p^2 - 4p + 2 - \left\lceil 2\sqrt{(p-1)^2 + q} \right\rceil = 2p^2 - 6p + 2q + 3 = mn - 2p + 1$$

which is attained by the labeling

$$\begin{pmatrix} O_{n-p+1,m-p} & O_{n-p+1,1} & O_{n-p+1,p-1} \\ O_{p-1,m-p} & \alpha_q & J_{p-1,p-1} \end{pmatrix}.$$

When m is odd and n is even:

The upper bound on $e_{f^*}(1)$ is attainable when $m \geq 2n + 1$. The labeling is

$$\begin{pmatrix} O_{n/2,(m-1)/2} & J_{n/2,1} & J_{n/2,(m-1)/2} \\ O_{n/2,(m-1)/2} & O_{n/2,1} & J_{n/2,(m-1)/2} \end{pmatrix}.$$

When $n \le m < 2n$. It is the same as the case when m is even.

When both m and n are odd:
The upper bound on $e_{f^*}(1)$ is attainable when $m \ge 2n+1$. The labeling is

$$\begin{pmatrix} O_{(n+1)/2,(m-1)/2} & J_{(n+1)/2,1} & J_{(n+1)/2,(m-1)/2} \\ O_{(n-1)/2,(m-1)/2} & O_{(n-1)/2,1} & J_{(n-1)/2,(m-1)/2} \end{pmatrix}.$$

When $n \le m < 2n$. It is the same as the case when m is even, but $p = \lceil (mn+1)/2 \rceil$ and q is redefined as $(mn+1)/2 - p^2 + p$ and $(mn+1)/2 - (p-1)^2$, respectively.

4. Full PC-Index Set of Grids

For convenience, we use $[a,b]$ to denote the set of integers between a and b inclusively, where $a, b \in \mathbb{Z}$.

4.1. Even m

In this subsection, we assume $m = 2h$. Let f_0 be the initial labeling of $P_m \times P_n$ for the following procedure. Then $e_{f_0^*}(1) = 0$.

Procedure A. Let $j = 1$ and $\alpha_{1,0} = f_0$.

Step 1: If $j = n$, then stop. If j is odd, then let $i = 1$, define $\alpha_{-1,j} = \alpha_{1,j-1}$ and do Step O-1 (the odd subroute). If j is even, then let $i = 2h-1$, define $\alpha_{2h+1,j} = \alpha_{2h-1,j-1}$ and do Step E-1 (the even subroute).

Step O-1: If $i > 2h$, then go to Step 2. If not, then based on $\alpha_{i-2,j}$ swap the labels of $u_{i,l}$ and $u_{i+1,l}$, for $l = 1, \dots, j$. Denote the new labeling by $\alpha_{i,j}$.

Step O-2: Increase i by 2 and repeat Step O-1.

Step E-1: If $i < 1$, then go to Step 2. If not, then based on $\alpha_{i+2,j}$ swap the labels of $u_{i,l}$ and $u_{i+1,l}$, for $l = 1, \dots, j$. Denote the labeling by $\alpha_{i,j}$.

Step E-2: Decrease i by 2 and repeat Step E-1.

Step 2: Increase j by 1 and repeat Step 1.

One can see that after performing one of the subroutes once, the number of 1-edge increases by 1. So after performing Procedure A, we show that for each $i \in [0, (n-1)h]$ there is a friendly labeling g such that $e_{g^*}(1) = i$.

Let the last labeling be β after performing Procedure A. Then

$$\beta = \begin{cases} \begin{pmatrix} O_{n,1} & J_{n,1} & \cdots & O_{n,1} & J_{n,1} \end{pmatrix} & \text{if } n \text{ is odd,} \\ \begin{pmatrix} J_{n,1} & O_{n,1} & \cdots & J_{n,1} & O_{n,1} \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

After taking vertical reflection on the array β when n is even, we may always assume that $\beta = \begin{pmatrix} O_{n,1} & J_{n,1} & \cdots & O_{n,1} & J_{n,1} \end{pmatrix}$. Note that $e_{\beta^*}(1) = (n-1)h$.

Procedure B. Let $i = h-1$.

Step 1: If $i < 1$, then stop. For $j = 1$ to n, swap the labels of $u_{2i,j}$ and $u_{h+i,j}$ based on the last labeling and let the new labeling be $\beta_{i,j}$.

Step 2: Decrease i by 1. Go to Step 1.

So we have $e_{\beta^*_{h-1,1}}(1) = (n-1)h$, $e_{\beta^*_{h-1,2}}(1) = (n-1)h+1, \ldots,$ $e_{\beta^*_{h-1,n-1}}(1) = (n-1)h+(n-2)$, $e_{\beta^*_{h-1,n}}(1) = (n-1)h+n$, $e_{\beta^*_{h-2,1}}(1) = (n-1)h+n$, etc.

In general, $e_{\beta^*_{i,j}}(1) = (n-1)h+(h-1-i)n+j-1$ if $j \neq n$ and $e_{\beta^*_{i,n}}(1) = (n-1)h+(h-i)n$.

So after performing Procedure B, we show that for each

$$i \in [(n-1)h, 2hn-h-n] \setminus \{2hn-h-in-1 \mid 1 \le i \le h-1\}$$

there is a friendly labeling g such that $e_{g^*}(1) = i$.

Procedure S. For each i ($1 \le i \le h-1$), based on $\beta_{i,n}$, we swap the labels of $u_{h+i,n}$ and $u_{h+i-1,n-1}$. Let the new labeling denote by σ_i. Then $e_{\sigma^*_i}(1) = 2hn-h-in-1$.

Combining the discussion above, we have the following result:

Lemma 4.1. $[0, 2hn-n-h] \subseteq \mathrm{FPCI}(P_{2h} \times P_n)$.

Combining with Theorem 3.1, we have

Theorem 4.1. *Suppose $h \ge n \ge 2$. Then* $\mathrm{FPCI}(P_{2h} \times P_n) = [0, 2hn-n-h]$.

Now we consider the case $2 \le n \le m < 2n$, i.e., $h < n$.

Suppose $hn = p^2$ for some $p \in \mathbb{N}$. We start with the last labeling $\beta_{1,n}$ after performing Procedure B. Note that $\beta_{1,n} = \begin{pmatrix} O_{n,h} & J_{n,h} \end{pmatrix}$. Let γ_1 be the array obtained from of $\beta_{1,n}$ by deleting the first p rows and the first h columns. Hence $\gamma_1 = J_{n-p,h}$. Also we let γ_0 be the array obtained from of $\beta_{1,n}$ by deleting the last $n-p$ rows, the last h columns and the first $2h-p$ columns. Hence $\gamma_0 = O_{p,p-h}$. Since $hn = p^2$, γ_1 and γ_0 have the same number of entries. We shall swap all 1's from γ_1 with all 0's from γ_0 one by one following the order defined below.

We use $A(j,i)$ to denote the (j,i)-th entry of an array A. Recall you again, the first indices indicate the columns and the second indicate the rows.

Firstly we define the order $\prec$ for the entries of γ_1 by $\gamma_1(j_1,i_1) \prec \gamma_1(j_2,i_2)$ if '$i_1 < i_2$', or '$i_1 = i_2$ and $j_1 > j_2$'. Secondly we define the order $\prec$ for the entries of γ_0 by $\gamma_0(j_1,i_1) \prec \gamma_0(j_2,i_2)$ if '$j_2 > j_1$', or '$j_1 = j_2$ and $i_2 < i_1$'.

Procedure C. We swap the 1's of γ_1 with the 0's of γ_0 according to their order starting from the first to the last one by one. When we swap a 1 with a 0 which lies in the first row of γ_0, the number of 1-edges will decrease by 1. So when we fill up the first row of γ_0 by 1, the number of 1-edges decreases from $2p^2-n-h$ to $2p^2-n-h-(p-h) = 2p^2-n-p$. After that, when we fill the 1's row by row, the number of 1-edges does not change until the last $(n-p)$ 1's which lie in the last column of γ_1. Finally, when we swap each of the last $(n-p)$ 1's, the number of 1-edges increases by 1. So the number of 1-edges increases from $2p^2-n-p$ to $2p^2-n-p+(n-p) = 2p^2-2p$. Hence we show that $[2p^2-n-p, 2p^2-2p] \subset \mathrm{FPCI}(P_{2h} \times P_n)$.

Example 4.1. Let $m = 18$ and $n = 16$. Then $mn = 12^2 \times 2$. That is, $h = 9$ and $p = 12$.

$$\left(\begin{array}{c|ccc|ccccccccc}
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
O_{4,6} & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\ \cline{2-13}
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
O_{12,6} & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1
\end{array}\right)$$

$e(1) = 263$

$$\left(\begin{array}{c|ccc|ccccccccc}
 & 0&0&0 & 0&1&1&1&1&1&1&1&1\\
O_{4,6} & 0&0&0 & 0&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 0&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\ \cline{2-13}
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
O_{12,6} & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1
\end{array}\right)$$

$e(1) = 260$

$$\left(\begin{array}{c|ccc|ccccccccc}
 & 0&0&0 & 0&0&0&0&0&0&0&0&1\\
O_{4,6} & 0&0&0 & 0&0&0&0&0&0&0&0&1\\
 & 0&0&0 & 0&0&0&0&0&0&0&0&1\\
 & 0&0&0 & 0&0&0&0&0&0&0&0&1\\ \cline{2-13}
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
O_{12,6} & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1
\end{array}\right)$$

$e(1) = 260$

$$\left(\begin{array}{c|ccc|ccccccccc}
 & 0&0&0 & 0&0&0&0&0&0&0&0&0\\
O_{4,6} & 0&0&0 & 0&0&0&0&0&0&0&0&1\\
 & 0&0&0 & 0&0&0&0&0&0&0&0&1\\
 & 0&0&0 & 0&0&0&0&0&0&0&0&1\\ \cline{2-13}
 & 0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
O_{12,6} & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1
\end{array}\right)$$

$e(1) = 261$

$$\left(\begin{array}{c|ccc|ccccccccc}
 & 0&0&0 & 0&0&0&0&0&0&0&0&0\\
O_{4,6} & 0&0&0 & 0&0&0&0&0&0&0&0&0\\
 & 0&0&0 & 0&0&0&0&0&0&0&0&1\\
 & 0&0&0 & 0&0&0&0&0&0&0&0&1\\ \cline{2-13}
 & 0&0&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
O_{12,6} & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1
\end{array}\right)$$

$e(1) = 262$

$$\left(\begin{array}{c|ccc|ccccccccc}
 & 0&0&0 & 0&0&0&0&0&0&0&0&0\\
O_{4,6} & 0&0&0 & 0&0&0&0&0&0&0&0&0\\
 & 0&0&0 & 0&0&0&0&0&0&0&0&0\\
 & 0&0&0 & 0&0&0&0&0&0&0&0&0\\ \cline{2-13}
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
O_{12,6} & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1 & 1&1&1&1&1&1&1&1&1
\end{array}\right)$$

$e(1) = 264$ $\square$

Now we assume hn is not a prefect square. Then $(p-1)^2 + 1 \le mn/2 = hn \le p^2 - 1$, where $p = \lceil \sqrt{hn} \rceil$.

Suppose $p^2 - p < mn/2 = hn$. Similar to the discussion above, based on $\beta_{1,n}$ we let γ_1 be the array obtained from of $\beta_{1,n}$ by deleting the first $p-1$ rows and the first h columns.

Hence $\gamma_1 = J_{n-p+1,h}$. Also we let γ_0 be the array obtained from of $\beta_{1,n}$ by deleting the last $n-p+1$ rows, the last h columns and the first $2h-p-1$ columns. Hence $\gamma_0 = O_{p-1,p-h+1}$.

Note that the numbers of entries of γ_1 and γ_0 are $hn-ph+h$ and $p^2-ph+h-1$, respectively. Since $hn \le p^2-1$, the number of entries of γ_0 is greater than that of γ_1. Keeping the same order on the entries of γ_1 and γ_0 defined above, we shall perform a procedure similar to Procedure C, called Procedure C1.

When we swap a 1 with a 0 which lies in the first row of γ_0, the number of 1-edges will decrease by 1. So when we fill up the first row of γ_0 by 1, the number of 1-edges decreases from $2hn-n-h$ to $2hn-n-h-(p-h+1) = 2hn-n-p-1$. After that, when we fill the 1's row by row, the number of 1-edges does not change until the last $(n-p+1)$ 1's which lie in the last column of γ_1. Finally, when we swap each of the last $(n-p+1)$ 1's, the number of 1-edges increases by 1. So the number of 1-edges increases from $2hn-n-p-1$ to $2hn-n-p-1+(n-p+1) = 2hn-2p$. Hence we show that $[2hn-n-p-1, 2hn-2p] \subset \mathrm{FPCI}(P_{2h} \times P_n)$.

Example 4.2. Let $m = 18$ and $n = 15$. Then $mn = 135 \times 2$. That is, $h = 9$ and $p = 12$.

$$\left(\begin{array}{c|cccc|ccccccccc}
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
O_{4,5} & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\ \cline{2-14}
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
O_{11,5} & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1
\end{array}\right)$$
$e(1) = 246$

$$\left(\begin{array}{c|cccc|ccccccccc}
 & 0&0&0&0 & 0&1&1&1&1&1&1&1&1\\
O_{4,5} & 0&0&0&0 & 0&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 0&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 0&1&1&1&1&1&1&1&1\\ \cline{2-14}
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
O_{11,5} & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1
\end{array}\right)$$
$e(1) = 242$

$$\left(\begin{array}{c|cccc|ccccccccc}
 & 0&0&0&0 & 0&0&0&0&0&0&0&0&1\\
O_{4,5} & 0&0&0&0 & 0&0&0&0&0&0&0&0&1\\
 & 0&0&0&0 & 0&0&0&0&0&0&0&0&1\\
 & 0&0&0&0 & 0&0&0&0&0&0&0&0&1\\ \cline{2-14}
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
O_{11,5} & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1
\end{array}\right)$$
$e(1) = 242$

$$\left(\begin{array}{c|cccc|ccccccccc}
 & 0&0&0&0 & 0&0&0&0&0&0&0&0&0\\
O_{4,5} & 0&0&0&0 & 0&0&0&0&0&0&0&0&0\\
 & 0&0&0&0 & 0&0&0&0&0&0&0&0&0\\
 & 0&0&0&0 & 0&0&0&0&0&0&0&0&0\\ \cline{2-14}
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 0&0&0&0 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
O_{11,5} & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1\\
 & 1&1&1&1 & 1&1&1&1&1&1&1&1&1
\end{array}\right)$$
$e(1) = 246$ □

Suppose $p^2 - p \ge mn/2 = hn$. Based on $\beta_{1,n}$ we let γ_1 be the array obtained from of $\beta_{1,n}$ by deleting the first $p-1$ rows and the first h columns. Hence $\gamma_1 = J_{n-p+1,h}$. Also we let γ_0 be the array obtained from of $\beta_{1,n}$ by deleting the last $n-p+1$ rows, the last h columns and the first $2h-p$ columns. Hence $\gamma_0 = O_{p-1,p-h}$.

Note that the numbers of entries of γ_1 and γ_0 are $hn-ph+h$ and $p^2-ph+h-p$, respectively. Since $hn \le p^2-p$, the number of entries of γ_0 is greater than that of γ_1. Keeping the same order on the entries of γ_1 and γ_0 defined above, we shall perform a procedure similar to Procedue C, called Procedure C2.

When we swap a 1 with a 0 which lies in the first row of γ_0, the number of 1-edges will decrease by 1. So when we fill up the first row of γ_0 by 1, the number of 1-edges decreases from $2hn-n-h$ to $2hn-n-h-(p-h)=2hn-n-p$. After that, when we fill the 1's row by row, the number of 1-edges does not change until the last $(n-p+1)$ 1's which lie in the last column of γ_1. Finally, when we swap each of the last $(n-p+1)$ 1's, the number of 1-edges increases by 1. So the number of 1-edges increases from $2hn-n-p$ to $2hn-n-p+(n-p+1)=2hn-2p+1$. Hence we show that $[2hn-n-p, 2hn-2p+1] \subset \mathrm{FPCI}(P_{2h}\times P_n)$.

Combining the discussion above and Lemma 4.1, we have

Theorem 4.2. *Suppose $n \le 2h < 2n$. Then* $\mathrm{FPCI}(P_{2h}\times P_n) = [0, 2hn-\lceil 2\sqrt{hn}\rceil]$.

4.2. Odd m and Even n

In this subsection, we assume $m=2h+1$ and $n=2k$. For convenience we consider $P_{2k}\times P_m$ instead of $P_m\times P_{2k}$ with $m \ge 2k$ (i.e., $h \ge k$).

We also start from f_0 which is an $2k\times m$ array. We apply Procedures A, B and S (substitute n by m). By Lemma 4.1 we have $[0, 2km-m-k] \subseteq \mathrm{FPCI}(P_{2k}\times P_m)$.

Let γ_1 be the array obtained from of $\beta_{1,m}$ by deleting the first $h+1$ rows and the first k columns. Hence $\gamma_1 = J_{h,k}$. Also we let γ_0 be the array obtained from of $\beta_{1,n}$ by deleting the last $h+1$ rows, the last k columns. Hence $\gamma_0 = O_{h,k}$. We applying Procedure C.

When we swap a 1 with a 0 which lies in the first row of γ_0, the number of 1-edges will decrease by 1. So when we fill up the first row of γ_0 by 1, the number of 1-edges decreases from $2km-m-k$ to $2km-m-k-k=2km-m-2k$. After that, when we fill the 1's row by row, the number of 1-edges does not change until the last h 1's which lie in the last column of γ_1. Finally, when we swap each of the last h 1's, the number of 1-edges increases by 1. So the number of 1-edges increases from $2km-m-2k$ to $2km-m-2k+h=2km-2k-(m+1)/2$. Hence we show that $[2km-m-2k, 2km-2k-(m+1)/2] \subset \mathrm{FPCI}(P_m\times P_{2k})$.

Combining the discussion above we have

Theorem 4.3. *Suppose $m \ge 4k+1$. Then* $\mathrm{FPCI}(P_m\times P_{2k}) = [0, 2km-2k-(m+1)/2]$.

For $2k \le m \le 4k$, let $p=\lceil\sqrt{km}\rceil$. Then $p=\lceil\sqrt{km}\rceil \le \lceil\sqrt{4k^2}\rceil = 2k$. When $p=2k$. It implies that $2k-1 < \sqrt{km} \le 2k$. Hence m is either $4k-1$ or $4k-3$.

Suppose $m=4k-1$, where $k\ge 1$. Then $4k^2-2k+0.25 < 4k^2-k < 4k^2-k+0.0625$. Hence $2k-0.5 < \sqrt{4k^2-k} < 2k-0.25$ and $4k-1 < 2\sqrt{km} < 4k-0.5$. Then $\lceil 2\sqrt{km}\rceil = 4k$. In this case, $2km-\lceil 2\sqrt{km}\rceil = 2km-2k-(m+1)/2$.

Suppose $m=4k-3$, where $k\ge 1$. Then $4k^2-4k+1 < 4k^2-3k < 4k^2-2k+0.25$. Hence $4k-2 < 2\sqrt{km} < 4k-1$. Then $\lceil 2\sqrt{km}\rceil = 4k-1$. In this case, $2km-\lceil 2\sqrt{km}\rceil = 2km-2k-(m+1)/2$.

When $p<2k$. Note that $p-k+1 \le k$ which is the number of zero columns in the array $\beta_{1,m}$. Cases for $km=p^2$, $p^2-p<km$ and $p^2-p\ge km$ are the same as the case when m is even discussed in Subsection 4.1. So we have

Theorem 4.4. *Suppose* $2k \le m \le 4k$. *Then* $\mathrm{FPCI}(P_m \times P_{2k}) = [0, 2km - 2k - \lceil 2\sqrt{km}\rceil]$.

4.3. Odd m and Odd n

In this subsection, we let $m = 2h+1$ and $n = 2k+1$, where $h \ge k \ge 1$.

Procedure D. Let $i = 1$ and $\delta_{1,0} = f_0$. Note that $v_f(0) = v_f(1) + 1$.
Step 1: If $i > h$, then stop. Let $j = 1$.

Step 2: If $j > k+2$, then go to Step 4. If not, then based on $\delta_{i,j-1}$ swap the labels of $u_{2i-1,2j-1}$ and $u_{2i,2j-1}$. Denote the new labeling by $\delta_{i,j}$.

Step 3: Increase j by 1 and repeat Step 2.

Step 4: Let $\delta_{i+1,0} = \delta_{i,k}$. Increase i by 1 and repeat Step 1.

After performing Procedure D, let the last labeling be δ.
We can see that $e_{\delta^*_{i,j}}(1) = 2k(i-1) + 2j - 1$ for $1 \le j \le k$ and $e_{\delta^*_{i,k+1}}(1) = 2ki$.
For each $\delta_{i,j}$, where $1 \le i \le h$, $2 \le j \le k$, we swap the labels of $u_{2i-1,1}$ and $u_{2i,1}$. Let this labeling be $\theta_{i,j}$. Then $e_{\theta^*_{i,j}}(1) = e_{\delta^*_{i,j}}(1) - 1 = 2k(i-1) + 2j - 2$.
So each integer $a \in [0, 2kh]$, there is a friendly labeling g such that $e^*_g(1) = a$.

Procedure E. Let $i = 2$. Start from δ.
Step 1: If $i > h$, then stop. For $j = 1$ to n, swap the labels of $u_{2i-1,j}$ and $u_{i,j}$ based on the last labeling and let the new labeling be $\varepsilon_{i,j}$.

Step 2: Increase i by 1. Go to Step 1.

So we have $e_{\varepsilon^*_{i,j}}(1) = 2kh + (i-2)n + j - 1$ if $j \ne n$ and $e_{\varepsilon^*_{i,n}}(1) = 2kh + (i-1)n$.

Procedure E′. For each $\varepsilon_{i,n-1}$, $2 \le i \le h$, we change the label of $u_{m,1}$ from 0 to 1. Then the number of 1-edge of this labeling is 1 more than that of $\varepsilon_{i,n-1}$. That is, it equals to $2kh + (i-1)n - 1$.

After performing Procedures E and E′, we know that for each $a \in [2kh, 4kh - 2k + h - 1]$, there is a friendly labeling g such that $e_{g^*}(1) = a$.

Procedure F. Let $i = 2$. Start from $\varepsilon_{h,n}$.
Step 1: Change the label of $u_{m,1}$ from 0 to 1. Let $j = 3$.

Step 2: If $j > k+1$, then stop. Swap the labels of $u_{m,j}$ and $u_{m,2j-2}$.

Note that, preform each step will increase the number of 1-edge by 1. So
$[4kh - 2k + h, 4kh - k + h - 1] \subset \mathrm{FPCI}(P_{2h+1} \times P_{2k+1})$. Let the last labeling be ζ_0.

Procedure G. Start from ζ_0. For $j = 1$ to $k+1$, let ζ_j be the labeling obtained from ζ_{j-1} by swapping the labels of $u_{m,j}$ and $u_{h+1,j}$.

Then we can see that $e_{\zeta^*_j}(1) = 4kh - k + h - 2 + j$ for $1 \le j \le k$ and $e_{\zeta^*_{k+1}} = 4kh + h$. Finally, based on ζ_{k+1} we swap the labels of $u_{h+1,k+1}$ with $u_{h+1,k+2}$. Then $4kh + h - 1$ is a PC-index of the graph $P_{2h+1} \times P_{2k+1}$.

After performing Procedures D, E, E′, F and G, we have the following lemma.

Lemma 4.2. $[0, 4hk+h] \subseteq \mathrm{FPCI}(P_{2h+1} \times P_{2k+1})$.

Combining with Theorem 3.1, we have

Theorem 4.5. *Suppose* $h \geq 2k+1 \geq 3$. *Then* $\mathrm{FPCI}(P_{2h+1} \times P_{2k+1}) = [0, 4hk+h]$.

Finally, we consider the case when $n \leq m \leq 2n-1$. It is equivalent to $k \leq h \leq 2k$. From Theorem 3.1 we know that the maximum number of 1-edges is

$$M = mn + 1 - \left\lceil 2\sqrt{\frac{mn+1}{2}} \right\rceil,$$

where $p = \lceil \sqrt{(mn+1)/2} \rceil = \lceil \sqrt{2kh+h+k+1} \rceil$. Note that $p = \lceil \sqrt{(mn+1)/2} \rceil < \lceil \sqrt{(2n^2-n+1)/2} \rceil$. So $p \leq n$. By a similar argument as in Subsection 4.1, we have

$$M = \begin{cases} mn+1-2p & \text{if } p^2 - p < 2kh+h+k+1, \\ mn+2-2p & \text{if } p^2 - p \geq 2kh+h+k+1. \end{cases}$$

Consider the array ζ_{k+1}. If we move the last h columns (all 0's) of ζ_{k+1} in front the first column. Then the number of 1-edge is still $4kh+h$. That is, the first h columns are zero columns, the next h columns are all 1 columns and last column is $(O_{1,k}\ J_{1,k+1})^T$. Let us denote this array by ζ. We shall apply a similar procedure (procedure C or its modified procedure) as the case when m is even discussed in Subsection 4.1. The procedure will not affect the last column of ζ. That is, we consider the array ζ' obtained from ζ by deleting the last column. Here ζ' is an $(2k+1) \times (2h)$ array.

When we perform Procedure C or its modified procedure. There is an array γ_1 containing $n-p$ rows or $n-p+1$. In order to not effect the last row of ζ as well as the number of 1-edges, $n-p$ must be less than $k+1$. That is, $p > k$. Since $p^2 \geq 2kh+h+k+1 \geq 2k^2+2k+1$. Then $p > k$. So the condition is satisfied. Note that, since $p^2 \geq 2kh+h+k+1$, the number of entries of γ_0 is greater than that of γ_1.

When $p^2 = (mn+1)/2$ or $p^2 - p < (mn+1)/2$. In this case we can easily prove that $n > p$. After applying Procedure C on ζ', we know that

$$[4kh+2h-p, 4hk+2h+2k+1-2p] = [4kh+2h-p, mn-2p] \subset \mathrm{FPCI}(P_{2h+1} \times P_{2k+1}).$$

Moreover, the index $mn+1-2p$ is attained by the discussion in Section 3 (where $q = (mn+1)/2 - p^2 + p$).

When $p^2 - p \geq (mn+1)/2$. After applying Procedure C2 on ζ', we know that

$$[4kh+2h-p, 4hk+2h+2k+1-2p+1] = [4kh+2h-p, mn+1-2p] \subset \mathrm{FPCI}(P_{2h+1} \times P_{2k+1}).$$

Note that, $[4kh+2h-p, mn+1-2p]$ may be a subset of $[0, 4kh+h]$. Moreover, the index $mn+2-2p$ is attained by the discussion in Section 3 (where $q = (mn+1)/2 - (p-1)^2$).

Combining the discussion above, we have

Theorem 4.6. *Suppose* $k \leq h \leq 2k+1$. *Then*

$$\mathrm{FPCI}(P_{2h+1} \times P_{2k+1}) = [0, 4kh+2h+2k+2 - \lceil 2\sqrt{2kh+h+k+1} \rceil].$$

Example 4.3. Let $m = 7$ and $n = 5$. Then $mn = 35$. That is, $h = 3$, $k = 2$ and $p = 5$. $p^2 - p = 20 > 18$.

$$\underset{\delta_{1,2},\ e(1)=3}{\begin{pmatrix} 0&1&0&1&0&1&0\\ 1&0&1&0&1&0&1\\ 1&0&0&1&0&1&0\\ 1&0&1&0&1&0&1\\ 1&0&0&1&0&1&0 \end{pmatrix}} \quad \underset{\theta_{1,2},\ e(1)=2}{\begin{pmatrix} 0&1&0&1&0&1&0\\ 1&0&1&0&1&0&1\\ 1&0&0&1&0&1&0\\ 0&1&1&0&1&0&1\\ 1&0&0&1&0&1&0 \end{pmatrix}} \quad \underset{\delta_{3,2},\ e(1)=12}{\begin{pmatrix} 1&0&1&0&1&0&0\\ 1&0&1&0&1&0&1\\ 1&0&1&0&1&0&0\\ 1&0&1&0&1&0&1\\ 1&0&1&0&1&0&0 \end{pmatrix}} \quad \underset{\varepsilon_{3,4},\ e(1)=20}{\begin{pmatrix} 1&1&0&1&0&0&0\\ 1&1&1&0&0&0&1\\ 1&1&1&0&0&0&0\\ 1&1&1&0&0&0&1\\ 1&1&1&0&0&0&0 \end{pmatrix}}$$

$$\underset{\substack{\text{Apply Procedure E}'\\ \text{on } \varepsilon_{3,4},\ e(1)=21}}{\begin{pmatrix} 1&1&0&1&0&0&0\\ 1&1&1&0&0&0&1\\ 1&1&1&0&0&0&0\\ 1&1&1&0&0&0&1\\ 1&1&1&0&0&0&1 \end{pmatrix}} \quad \underset{\varepsilon_{3,5},\ e(1)=22}{\begin{pmatrix} 1&1&1&0&0&0&0\\ 1&1&1&0&0&0&1\\ 1&1&1&0&0&0&0\\ 1&1&1&0&0&0&1\\ 1&1&1&0&0&0&0 \end{pmatrix}} \quad \underset{\substack{\text{Apply Procedure F}\\ \text{on } \varepsilon_{3,5},\ e(1)=23}}{\begin{pmatrix} 1&1&1&0&0&0&0\\ 1&1&1&0&0&0&1\\ 1&1&1&0&0&0&0\\ 1&1&1&0&0&0&1\\ 1&1&1&0&0&0&1 \end{pmatrix}} \quad \underset{\zeta_0,\ e(1)=24}{\begin{pmatrix} 1&1&1&0&0&0&0\\ 1&1&1&0&0&0&0\\ 1&1&1&0&0&0&1\\ 1&1&1&0&0&0&1\\ 1&1&1&0&0&0&1 \end{pmatrix}}$$

$$\underset{\zeta_3,\ e(1)=27}{\begin{pmatrix} 1&1&1&0&0&0&0\\ 1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0\\ 1&1&1&1&0&0&0\\ 1&1&1&1&0&0&0 \end{pmatrix}} \quad \underset{\substack{\text{Last swapping after}\\ \text{Procedure G},\ e(1)=26}}{\begin{pmatrix} 1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0\\ 1&1&1&0&0&0&0\\ 1&1&1&1&0&0&0\\ 1&1&1&1&0&0&0 \end{pmatrix}}$$

In this case $[4kh + 2h - p, mn + 1 - 2p] \subset [0, 4kh + h]$. We have got 28 labelings corresponding to all integers in $[0, 27]$. Following we just want to illustrate the application of Procedure C2 for ζ.

$$\underset{\zeta,\ e(1)=27}{\left(\begin{array}{c|cc|ccc|c} 0&0&0&1&1&1&0\\ \cline{2-6} 0&0&0&1&1&1&0\\ 0&0&0&1&1&1&1\\ 0&0&0&1&1&1&1\\ 0&0&0&1&1&1&1 \end{array}\right)} \quad \underset{e(1)=26}{\left(\begin{array}{c|cc|ccc|c} 0&0&0&0&1&1&0\\ \cline{2-6} 0&0&0&1&1&1&0\\ 0&0&0&1&1&1&1\\ 0&0&0&1&1&1&1\\ 0&0&1&1&1&1&1 \end{array}\right)} \quad \underset{e(1)=25}{\left(\begin{array}{c|cc|ccc|c} 0&0&0&0&0&1&0\\ \cline{2-6} 0&0&0&1&1&1&0\\ 0&0&0&1&1&1&1\\ 0&0&0&1&1&1&1\\ 0&1&1&1&1&1&1 \end{array}\right)}$$

$$\underset{e(1)=26}{\left(\begin{array}{c|cc|ccc|c} 0&0&0&0&0&0&0\\ \cline{2-6} 0&0&0&1&1&1&0\\ 0&0&0&1&1&1&1\\ 0&0&1&1&1&1&1\\ 0&1&1&1&1&1&1 \end{array}\right)}$$ □

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