

# Super-edge-graceful labelings of multi-level wheel graphs, fan graphs and actinia graphs\*

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## Abstract

The notion of super-edge-graceful graphs was introduced by Mitchem and Simoson in 1994. However, few examples except trees are known. In this paper, we exhibit three classes of infinitely many graphs including fan graphs, multi-level wheel graphs and actinia graphs, which are super-edge-graceful.

**Keywords:** super-edge-graceful, wheel graph,  
fan graph, actinia graph

**AMS 2000 MSC:** 05C78, 05C25

## 1 Introduction

In this paper all graphs are loopless and connected. For undefined symbols and concepts please refer to [1]. A  $(p, q)$ -graph  $G = (V, E)$  is called *edge-graceful* if there exists a bijection  $f: E \rightarrow \{1, 2, \dots, q\}$  and the *induced mapping*  $f^+: V \rightarrow \mathbb{Z}_p = \{0, 1, \dots, p-1\}$  defined by

$$f^+(u) \equiv \sum_{uv \in E} f(uv) \pmod{p}$$

is a bijection. This concept was introduced by Lo [5] in 1985. Lee [3] conjectured that all trees of odd order are edge-graceful. More references about edge-graceful may be found in [8, 9].

Mitchem and Simoson [6] tried to prove the above conjecture and introduce a variation of edge-gracefulness which for trees of odd order implies edge-gracefulness.

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\*Partially supported by Faculty Research Grant, Hong Kong Baptist University

Let

$$P = \begin{cases} \left\{ -\frac{p}{2}, \dots, -1, 1, \dots, \frac{p}{2} \right\} & \text{if } p \text{ is even} \\ \left\{ -\frac{p-1}{2}, \dots, -1, 0, 1, \dots, \frac{p-1}{2} \right\} & \text{if } p \text{ is odd} \end{cases}$$

and

$$Q = \begin{cases} \left\{ \frac{q}{2}, \dots, -1, 1, \dots, \frac{q}{2} \right\} & \text{if } q \text{ is even} \\ \left\{ \frac{q-1}{2}, \dots, -1, 0, 1, \dots, \frac{q-1}{2} \right\} & \text{if } q \text{ is odd} \end{cases}$$

A  $(p, q)$ -graph  $G$  is *super-edge-graceful* if there is a pair of mappings  $(f, f^+)$  such that  $f: E \rightarrow Q$  is bijective and  $f^+: V \rightarrow P$  is also bijective, where  $f^+(u) = \sum_{uv \in E} f(uv)$ .  $f$  is called a *super-edge-graceful labeling of  $G$* . Such sets  $P$  and  $Q$  are called the *vertex values set* and *edge labels set* of  $G$ , respectively.

The notions of super-edge-graceful graphs and edge-graceful graphs are different. Mitchem and Simoson [6] showed that the step graph  $C_6^2$  is super-edge-graceful but not edge-graceful. Shiu [7] showed that the complete graph  $K_4$  is edge-graceful but not super-edge-graceful.

Mitchem and Simoson [6] showed that

**Theorem 1.1** *If  $G$  is a super-edge-graceful  $(p, q)$ -graph and*

$$q \equiv \begin{cases} 1 \pmod{p} & \text{if } q \text{ is even,} \\ 0 \pmod{p} & \text{if } q \text{ is odd.} \end{cases}$$

*Then  $G$  is also edge-graceful.*

Shiu [7] proved that some cubic graphs, including permutation Petersen graphs and some permutation ladder graphs, are super-edge-graceful.

In this paper, three classes of graphs are shown to be super-edge-graceful.

## 2 Fan graphs

The join  $K_1 \vee P_{n-1}$  of  $K_1$  and  $P_{n-1}$  is called a *fan graph*  $F_n$ . The vertex in  $K_1$  is called the *core*. The edges incident with the core are

called *spokes*. Hence  $F_n$  has  $n$  vertices and  $2n - 3$  edges. In this section, we will study the super-edge-gracefulness of fan graphs.

First we define some notations. Let the core be denoted by  $c$  and the vertices on the path  $P_n$  be denoted by  $p_1, p_2, \dots, p_n$ , respectively. We draw  $F_{n+1}$  in the plane as follows: Draw the  $P_n$  horizontally and draw the core under the path  $P_n$ . Join the core to each vertex of  $P_n$  by straight line (see Figure 2.1)

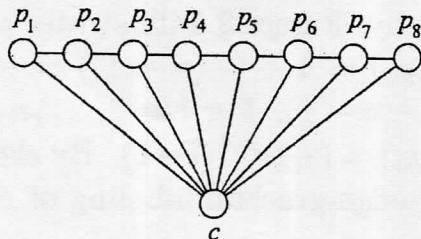


Figure 2.1 The fan graph  $F_9$

**Theorem 2.1** Every fan graphs  $F_{2n+1}$  is super-edge-graceful,  $n \geq 1$ .

**Proof:** Here the vertex values set and the edge labels set of  $F_{2n+1}$  are

$$P = \{-n, -(n-1), \dots, -1, 0, 1, 2, \dots, n\} \text{ and}$$

$$Q = \{-(2n-1), \dots, -1, 0, 1, \dots, 2n-1\},$$

respectively. We shall define an super-edge-graceful labeling  $f : E(F_{2n+1}) \rightarrow Q$  as follows.

Suppose  $n$  is odd. Firstly we label the first  $n$  spokes  $cp_1, cp_2, \dots, cp_n$  starting from the left by  $1, -2, 3, \dots, n-2, -(n-1)$  and  $-n$ . Secondly, we label the first  $n-1$  edges of  $P_{2n}$  starting from the left by  $-(n+1), n+2, \dots, (2n-2), 2n-1$ . The edge  $p_np_{n+1}$  is labeled by 0. We label the other edges “skew-symmetrically” to the above labeled edges. Namely, we label the last  $n$  spokes  $cp_{2n}, cp_{2n-1}, \dots, cp_{n+1}$  starting from the right by  $1, 2, -3, \dots, -(n-2), n-1$  and  $n$ , and the last edges of  $P_{2n}$  starting from the right by  $n+1, -(n+2), \dots, 2n-2, -(2n-1)$ . So  $f$  is a bijection. Figure 2.2 illustrates the case for  $n = 5$ .

Now we shall show that  $f^+ : V(F_{2n+1}) \rightarrow P$  is a bijection. Clearly,  $f^+(c) = 0$ ,  $f^+(p_1) = 1 + (-(n+1)) = -n$  and  $f^+(p_n) = (-n) + (2n-1) + 0 = n-1$ . For  $i = 2, \dots, n-1$ ,  $f^+(p_i) =$

$(-1)^{i-1}i + (-1)^{i-1}(n+i-1) + (-1)^i(n+i) = (-1)^{i-1}(i-1)$ . Because of skew-symmetry, we know that all the values  $-1, 2, -3, \dots, (n-1)$  and  $n$  are assumed on the right hand side vertices of the fan graph. Hence  $f$  is a super-edge-graceful labeling of  $F_{2n+1}$ .

Suppose  $n$  is even. The labeling is similar. We label the first  $n$  spokes  $cp_1, cp_2, \dots, cp_n$  starting from the left by  $1, -2, 3, \dots, -(n-2), n-1$  and  $n$ , and label the first  $n-1$  edges of  $P_{2n}$  starting from the left by  $(n+1), n+2, \dots, 2n-2, (2n-1)$ . The other side of edges are labeled skew-symmetrically. Figure 2.3 illustrates the case for  $n=4$ . Clearly  $f^+(c) = 0$ ,  $f^+(p_1) = 1 + (-n+1) = -n$  and  $f^+(p_n) = n + ((2n-1)) + 0 = -(n-1)$ . For  $i = 2, \dots, n-1$ , by the same calculation we have  $f^+(p_i) = (-1)^{i-1}(i-1)$ . By skew-symmetrically, we see that  $f$  is a super-edge-graceful labeling of  $F_{2n+1}$ .  $\square$

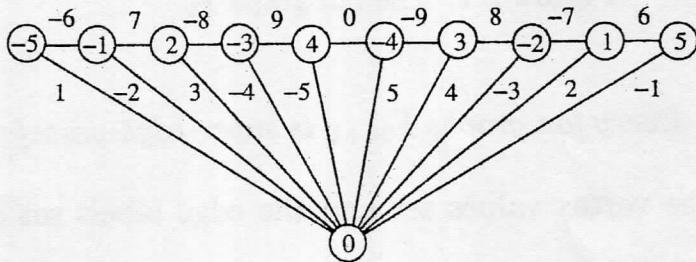


Figure 2.2: A super-edge-graceful labeling for  $F_{11}$

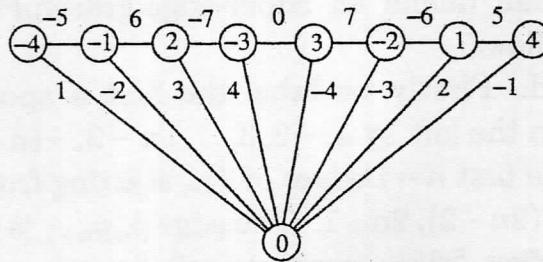


Figure 2.3. A super-edge-graceful labeling for  $F_9$ .

**Proposition 2.2** *The graph  $F_4$  is not super-edge-graceful.*

**Proof:** Suppose there were a super-edge-graceful labeling  $f: E(F_4) \rightarrow \{-2, -1, 1, 2\}$  for  $F_4$ . Then  $f^+: V(F_4) \rightarrow \{-2, -1, 1, 2\}$  is a bijection.

Suppose  $f(cp_2) = 0$ . Without loss of the generality, we may assume  $f(p_1p_2) = 2$ . Since  $f^+(p_2) \leq 2$  and  $f(xp_2) = 0$ ,  $f(p_2p_3) < 0$ . Since  $f^+(p_2) \neq 0$ ,  $f(p_2p_3) = -1$ . Since  $f^+(p_3) \neq 0$ ,  $f(cp_3) \neq 1$ . So  $f(cp_3) = -2$ . But it is impossible for  $f^+(p_3) = -3$ .

Suppose  $f(cp_2) \neq 0$ . Without loss of the generality, we may assume  $f(p_1p_2) = 0$ . Then  $f^+(p_1) = f(cp_1)$ . Since  $f^+$  is a bijection,  $f(cp_2) + f(cp_3) \neq 0$ . Since  $f$  is a bijection,  $|f(cp_2)| \neq |f(cp_3)|$ . Since  $f$  is a bijection again,  $|f(p_2p_3)| = |f(cp_2)|$  or  $|f(p_2p_3)| = |f(cp_3)|$ . This will imply that  $f^+(p_2) = 0$  or  $f^+(p_3) = 0$ . It is impossible.  $\square$

The super-edge-gracefulness of  $F_{2n}$  is still open.

### 3 Multi-level wheel Graphs

In this section, we will show that some kind of multi-level wheel graphs are super-edge-graceful. We consider the simplest case first.

For  $n > 2$ , the graph  $K_1 \vee C_n$  is called the *wheel graph* of order  $n+1$ , denoted by  $W_{n+1}$ . Note that  $C_2$  is a multi-graph consisting of two vertices and two parallel edges. The vertex in  $K_1$  is called the *core* and is denoted by  $c$ . We draw  $W_{n+1}$  in the plane in the following way. Draw the cycle  $C_n$  as an  $n$ -polygon and then put the core in the center of the polygon. Join the core to each vertex of  $C_n$  by straight line. Vertices lying on the polygon are denoted by  $u_1, \dots, u_n$  in clockwise. The edges  $cu_i$ ,  $1 \leq i < n$ , are called *spokes*. The cycle is also called the *ring* of the wheel. Hence  $W_{n+1}$  has  $n+1$  vertices and  $2n$  edges.

**Theorem 3.1** For  $n > 1$ ,  $W_{2n+1}$  is super-edge-graceful.

**Proof:** In this case  $P = \{-n, \dots, -1, 0, 1, \dots, n\}$  and  $Q = \{2n, \dots, 1, 1, \dots, 2n\}$ . We will define a labeling  $f: E(W_{2n+1}) \rightarrow Q$ . First we label the spokes from  $cu_1$  to  $cu_{2n}$  by  $1, 1, 2, 2, \dots, -n, n$  in clockwise. That is  $f(cu_{2i-1}) = -i$  and  $f(cu_{2i}) = i$  for  $1 \leq i \leq n$ . Then we label the edges of the cycle from  $u_{2n}u_1$  to  $u_{2n-1}u_{2n}$  by  $n+1, -(n+1), \dots, 2n, -2n$  in clockwise. That is  $f(u_{2i-2}u_{2i-1}) = n+i$  and  $f(u_{2i-1}u_{2i}) = -(n+i)$ , for  $1 \leq i < n$  (for convenience we let  $u_0 = u_{2n}$ ).

It is clear that  $f^+(c) = 0$ . For  $1 \leq i \leq n$ ,

$$f^+(u_{2i-1}) = f(u_{2i-2}u_{2i-1}) + f(u_{2i-1}u_{2i}) + f(cu_{2i-1})$$

$$= n+i - (n+i) - i = -i, \text{ and}$$

$$f^+(u_{2i-2}) = f(u_{2i-3}u_{2i-2}) + f(u_{2i-2}u_{2i-1}) + f(cu_{2i-2})$$

$$= -(n+i-1) + (n+i) + (i-1) = i.$$

So  $f$  is a super-edge-graceful labeling.  $\square$

Figure 3.1 shows a super-edge-graceful labeling for  $W_9$ .

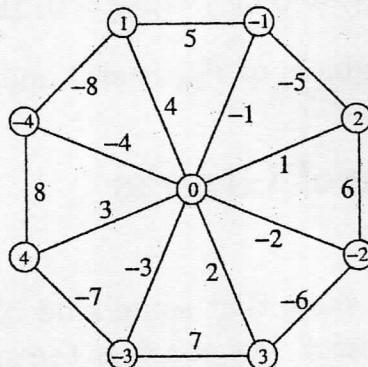


Figure 3.1: A super-edge-graceful labeling for  $W_9$

It is known that  $W_4 \equiv K_4$  is not super-edge-graceful. The super-edge-gracefulness of  $W_{2n}$  is still open.

We shall construct an *m-level wheel* recurrently for  $m > 1$ . The wheel graph is 1-level wheel. Suppose we have an  $(m-1)$ -level wheel graph. An *m-level wheel graph* is a graph obtained from the  $(m-1)$ -level graph by appending an even number of edges (called the *m-th level spokes*) to the ring of the wheel (the outer cycle) and append a new ring (called the *m-th level ring*) to the most exterior spokes. We shall use the notation  $W(n_1, n_2, \dots, n_m)$  to denotes an *m-level wheel graph* which contains  $n_1$  spokes in the 1-st level,  $n_2$  spokes in the 2-nd level,  $\dots$ ,  $n_m$  spokes in the *m-th level*. Hence  $n_2, \dots, n_m$  must be even. It is clear that  $W(n_1, n_2, \dots, n_m)$  is planar. So we will view it as a plane graph.

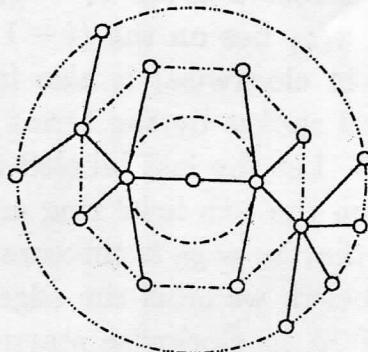


Figure 3.2: 3-level wheel graph  $W(2, 8, 6)$ .

Note that there may be many  $m$ -level wheel graphs having the same parameters  $n_1, n_2, \dots, n_m$ . In the following are some examples.

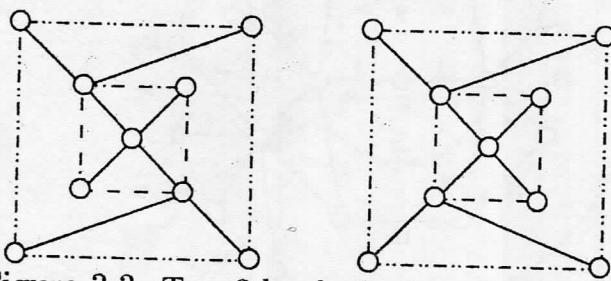


Figure 3.3: Two 2-level wheel graphs  $W(4, 4)$

We shall show that each multi-level wheel graph with even number of spokes in the first level is super-edge-graceful. Before proving this, we use the following example to illustrate the idea of the proof.

**Example 3.1** We consider the graph  $W(2, 8, 6)$  described in Figure 3.2. This is a  $(17, 32)$ -graph. So  $P = \{-8, 7, \dots, 1, 0, 1, \dots, 7, 8\}$  and  $Q = \{16, -15, \dots, 1, 1, \dots, 15, 16\}$ . There are 16 spokes and 16 edges in rings. First we partition  $Q$  into two sequences  $S = \{1, 1, -2, 2, \dots, -8, 8\}$  and  $R = \{9, -9, 10, -10, \dots, 16, 16\}$ . According to the parameters, we partition  $S$  into three disjoint subsequences  $S_1, S_2, S_3$  such that  $S_1$  consists of the first 2 terms of  $S$ ,  $S_2$  consists of the next 8 terms of  $S$ ,  $S_3$  consists of the last 6 terms of  $S$ . Similarly we partition  $R$  into three subsequences  $R_1, R_2, R_3$ . Namely,  $S_1 = \{-1, 1\}$ ,  $S_2 = \{-2, 2, -3, 3, -4, 4, -5, 5\}$ ,  $S_3 = \{-6, 6, -7, 7, -8, 8\}$ ,  $R_1 = \{9, -9\}$ ,  $R_2 = \{10, -10, 11, -11, 12, -12, 13, 13\}$  and  $R_3 = \{14, 14, 15, -15, 16, -16\}$ .

For the  $i$ -th level, we choose a spoke  $e_i = u_i x_{i-1}$  such that  $u_i$  lies on the  $i$ -th level ring,  $x_{i-1}$  lies on the  $(i-1)$ -st level ring and the next spoke (counting in clockwise) is also incident with  $x_{i-1}$ . First we label the  $i$ -th level spokes by the terms of  $S_i$  in clockwise starting from  $e_i$ ,  $i = 1, 2, 3$ . Let the last labeled  $i$ -th level spoke be  $e'_i = v_i y_{i-1}$ , where  $v_i$  lies on the  $i$ -th level ring and  $y_{i-1}$  lies on the  $(i-1)$ -st level ring. Note that  $x_0 = y_0$  is the core  $c$ .

After all spokes are labeled, we label the edges lying in the  $i$ -th level ring by the terms of  $R_i$  in clockwise starting from  $u_i v_i$ ,  $i = 1, 2, 3$ . So we get

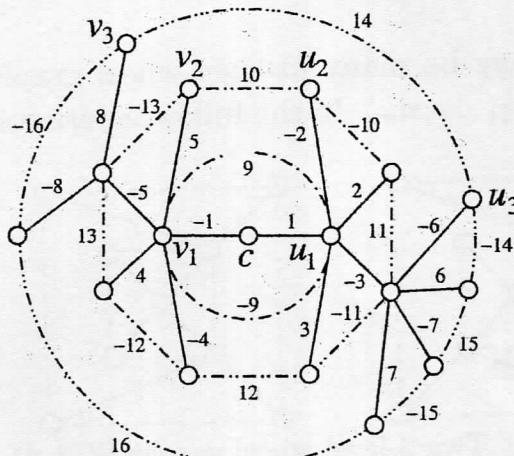


Figure 3.4. A super-edge-graceful labeling for  $W(2, 8, 6)$

Then the induced labels for the vertices lying on the 1-st level ring are 1 and  $-1$ ; lying on the 2-nd level ring are  $-2, 3, -3, 4, -4, 5, -5, 2$ ; lying on the 3-rd level ring are  $-6, 7, -7, 8, -8, 6$ , respectively. And the induced label for the core is 0.

**Theorem 3.2** For  $m \geq 1$  and  $n_1$  even, the  $m$ -level wheel graph  $W(n_1, n_2, \dots, n_m)$  is super-edge-graceful.

**Proof:** Let  $n_i = 2l_i$  for some positive integer  $l_i$ . Let

$$S = \{-1, 1, 2, 2, \dots, -\sum_{i=1}^m l_i, \sum_{i=1}^m l_i\} \text{ and}$$

$$R = \{1 + \sum_{i=1}^m l_i, -1 - \sum_{i=1}^m l_i, 2 + \sum_{i=1}^m l_i, -2 - \sum_{i=1}^m l_i, \dots, 2 \sum_{i=1}^m l_i, -2 \sum_{i=1}^m l_i\}$$

be two sequences. According to the parameters  $2l_1, 2l_2, \dots, 2l_m$ , we partition  $S$  into  $m$  subsequences  $S_1, \dots, S_m$  and  $R$  into  $m$  subsequences  $R_1, \dots, R_m$ . Namely, for  $i = 1, \dots, m$ , let

$$S_i = \{ 1 + \sum_{j=1}^{i-1} l_j, 1 + \sum_{j=1}^{i-1} l_j, \dots, -\sum_{j=1}^i l_j, \sum_{j=1}^i l_j \} \text{ and}$$

$$R_i = \{ 1 + \sum_{j=1}^{i-1} l_j + \sum_{j=1}^m l_j, -1 - \sum_{j=1}^{i-1} l_j - \sum_{j=1}^m l_j, \dots,$$

$$\sum_{j=1}^i l_j + \sum_{j=1}^m l_j, -\sum_{j=1}^i l_j - \sum_{j=1}^m l_j \}$$

We shall prove the following predicate by induction on  $m$ .

$P(m)$  = "There is a super-edge-graceful labeling for  $W(n_1, \dots, n_m)$  such that the  $i$ -th level spokes are labeled by terms of  $S_i$  and the edges lying on the  $i$ -th level ring are labeled by terms of  $R_i$ , for each  $i$ ,  $1 \leq i < m$ ."

When  $m = 1$ , the graph is  $W_{2l_1+1}$ . By Theorem 3.1  $P(1)$  is true.

Suppose  $P(k-1)$  is true for  $k \geq 2$ . Now we consider  $P(k)$ . We have subsequences  $S_1, \dots, S_k$  and  $R_1, \dots, R_k$ . Since  $W(n_1, \dots, n_{k-1})$  is a subgraph of  $W(n_1, \dots, n_k)$ . By induction assumption, there is a super-edge-graceful labeling  $f_{k-1}$  for  $W(n_1, \dots, n_{k-1})$  such that the  $i$ -th level spokes are labeled by terms of  $S_i$  and the edges lying on the  $i$ -th level ring are labeled by terms of  $R_i$ , for each  $i$ ,  $1 \leq i \leq k-1$ . It suffices to extend the labeling  $f_{k-1}$  to be a super-edge-graceful labeling of the whole graph. That is, we need to label the  $k$ -th level spokes and the edges lying on the  $k$ -th level ring by  $S_k$  and  $R_k$  respectively such that the labeling becomes a super-edge-graceful labeling. Note that  $\{f_{k-1}^+(w) \mid w \in V(W(n_1, \dots, n_{k-1}))\} = \{0\} \cup \left( \bigcup_{j=1}^{k-1} S_j \right)$ .

Choose a  $k$ -th level spoke  $e = ux$  which is the first (counting in clockwise) edge incident with a vertex in  $(k-1)$ -st level ring, where  $u$  lies on the  $k$ -th level ring and  $x$  lies on the  $(k-1)$ -st level ring. Label the  $k$ -th level spokes by the terms of  $S_k$  in clockwise starting from  $e$ . Let the last labeled  $k$ -th level spoke be  $e' = vy$ , where  $v$  lies on the  $k$ -th level ring and  $y$  lies on the  $(k-1)$ -st level ring. Label

the edges lying in the  $k$ -th level ring by the terms of  $R_k$  in clockwise starting from  $uv$ . Let this extension labeling be  $f_k$ .

It is clear that  $f_{k-1}^+(w) = f_k^+(w)$  when  $w$  does not lie on the  $k$ -th or  $(k-1)$ -st level ring. For vertex  $w$  which lies on the  $(k-1)$ -st level ring, since  $w$  is incident with some pairs of  $k$ -level spokes whose labels sum is 0,  $f_{k-1}^+(w) = f_k^+(w)$ .

Suppose  $w$  is a vertex lying on the  $k$ -level ring. Then  $\deg(w) = 3$ . Let  $w$  be incident with a  $k$ -level spokes  $wz$  and two other edges  $\epsilon_1$  and  $\epsilon_2$  in clockwise, where  $z$  lies on the  $(k-1)$ -th level ring. If  $f_k(wz) = -a \in S_k$  for some  $a > 0$ , then  $f_k(\epsilon_1) = -b$  and  $f_k(\epsilon_2) = b$  for some  $b > 0$ . Hence  $f_k^+(w) = -a$ . Note that  $-\sum_{j=1}^k l_j \leq -a < 1 - \sum_{j=1}^{k-1} l_j$ . If  $f_k(wz) = a \in S_k$  for some  $a > 0$  and  $w \neq v$ , then  $f_k(\epsilon_1) = -b$  and  $f_k(\epsilon_2) = b+1$  for some  $b > 0$ . Hence  $f_k^+(w) = a+1$ . Since  $w \neq v$ ,  $\sum_{j=1}^{k-1} l_j \leq a < \sum_{j=1}^k l_j$ . So  $f_k^+(w) = a+1 \in S_k$ . If  $w = v$ , then

$$f_k^+(v) = \sum_{j=1}^k l_j + (1 + \sum_{j=1}^{k-1} l_j + \sum_{j=1}^k l_j) + (-2 \sum_{j=1}^k l_j) = 1 + \sum_{j=1}^{k-1} l_j \in S_k$$

Thus  $\{f_k^+(w) \mid w \text{ lies on the } k\text{-level ring}\} = S_k$ . Hence  $\{f_k^+(w) \mid w \in V(W(n_1, \dots, n_{k-1}, n_k))\} = \{0\} \cup \left( \bigcup_{j=1}^k S_j \right)$ . Thus  $P(k)$  is true.  $\square$

**Remark 3.1** If  $W_{2n}$  is super-edge-graceful, for some  $n$ , then by using the same argument of the proof above, we can prove that the complicated-wheel graph  $W(2n-1, n_2, \dots, n_m)$  is super-edge-graceful. But whether  $W_{2n}$  is super-edge-graceful or not for  $n \geq 3$  is still open.

**Remark 3.2** We may define a more complicated multi-level wheel graphs. Namely, we may append any number of spokes to each level of ring. We would like to ask the super-edge-gracefulness of such graphs.

## 4 Actinia graphs

Frucht and Harary [2] have the following construction of graphs. Given two graph  $G$  and  $H$ , the *corona of  $G$  with  $H$* , denoted by  $G \odot H$ , is the graph with

$$V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i),$$

$$E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} \left( E(H_i) \cup \{iu_i \mid u_i \in V(H_i)\} \right),$$

where  $V(H_i) = V(H)$  and  $E(H_i) = E(H)$  for all  $i \in V(G)$ .

Suppose integers  $m \geq 2$  and  $n > 1$ . The graph  $C_m \odot N_n$  is called an *regular actinia graph* and denoted by  $A(m, n)$ , where  $N_n$  is the null graph of order  $n$ . Following are some actinia graphs.

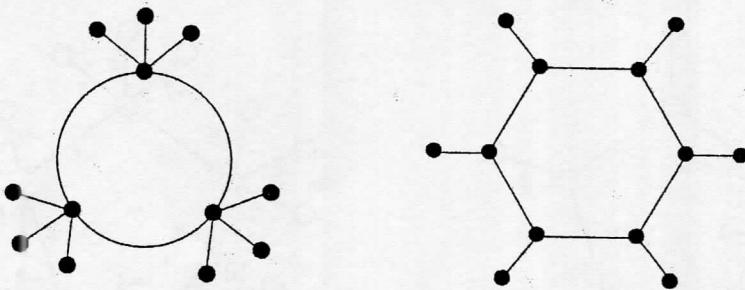


Figure 4.1. Regular actinia graphs  $A(3, 3)$  and  $A(6, 1)$

The graph  $A(m, n)$  is a unicyclic graph with  $m + mn$  vertices.

**Lemma 4.1** Suppose  $G$  is a super-edge-graceful  $(p, p)$ -graph (not necessarily connected). Two extra vertices are added on the graph  $G$  and these two vertices are joined by edges with a common vertex of  $G$ . Then the resulting graph  $G'$  is still super-edge-graceful.

**Proof:** In this case the vertex values set  $P$  of  $G$  and the edge labels set  $Q$  of  $G$  are the same, and those of  $G'$  are  $P \cup \{-\lfloor \frac{p}{2} \rfloor - 1, \lfloor \frac{p}{2} \rfloor + 1\}$ . So keep the super-edge-graceful labeling of  $G$  and label the two extra edges by  $-\lfloor \frac{p}{2} \rfloor - 1$  and  $\lfloor \frac{p}{2} \rfloor + 1$ . Then the extended labeling is a super-edge-graceful labeling of  $G'$ .  $\square$

**Proposition 4.2** For  $m > 2$ ,  $A(m, 1)$  is super-edge-graceful.

**Proof:** The label set is  $Q = \{\pm 1, \pm 2, \dots, \pm m\}$ . Let the vertices lying clockwise in the cycle  $C_m$  be  $u_1, \dots, u_m$ . Let the vertex adjacent with  $u_i$  be  $v_i$ ,  $1 < i < m$ . We label the edges of  $C_m$  by

$1, 2, \dots, -m$  clockwise starting at  $u_1u_2$ , and then label the edge  $u_iu_i$  by  $i$ ,  $1 \leq i \leq m$ . Denote this labeling by  $f$ . Then  $f^+(v_i) = i$ ,  $f^+(u_i) = -(i-1)$ ,  $i+i = -(i-1)$  if  $2 \leq i \leq m$  and  $f^+(u_1) = -m$ ,  $1+1 = -m$ . Hence  $f$  is a super-edge-graceful labeling of  $A(m, 1)$ .  $\square$

**Proposition 4.3** For  $m \geq 2$  and odd  $n \geq 1$ ,  $A(m, n)$  is super-edge-graceful.

**Proof:** By Proposition 4.2 we have a super-edge-graceful labeling for the subgraph  $A(m, 1)$ . Applying Lemma 4.1 repeatedly,  $A(m, n)$  is super-edge-graceful.  $\square$

In the following is an example to illustrate the proof above.

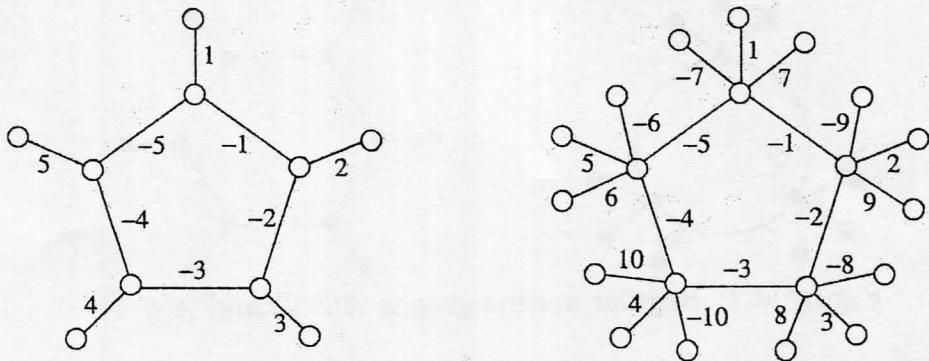


Figure 4.2: Super-edge-graceful labelings for  $A(5, 1)$  and  $A(5, 3)$

**Proposition 4.4** For odd  $m \geq 3$  and even  $n \geq 2$ ,  $A(m, n)$  is super-edge-graceful.

**Proof:** By means of Lemma 4.1 it suffices to show that  $C_m$  is super-edge-graceful. But it was proved by Mitchem and Simoson [6].  $\square$

Following from Theorem 1.1 we have

**Corollary 4.5** For odd  $m \geq 3$  and even  $n \geq 2$ ,  $A(m, n)$  is edge-graceful.

For  $m$  even, we do not know whether  $C_m$  is super-edge-graceful. Mitchem and Simoson [6] pointed out that  $C_4$  and  $C_6$  are not super-edge-graceful but  $C_8$  is. So we cannot simply prove that  $A(m, n)$  is

super-edge-graceful when both  $m$  and  $n$  are even. In order to show this result, we have to show  $A(m, 2)$  is super-edge-graceful first for  $m$  even.

**Proposition 4.6** *For even  $m > 2$ ,  $A(m, 2)$  is super-edge-graceful.*

**Proof:** Let  $m = 2k$  for some  $k \geq 1$ . Let  $u_1, \dots, u_{2k}$  be vertices lying clockwise on the cycle  $C_{2k}$ . Let  $x_i$  and  $y_i$  be two vertices outside the cycle adjacent with  $u_i$ . Let  $f$  be an edge labeling of  $A(2k, 2)$  defined by

$$f(u_i x_i) = \begin{cases} i & \text{if } 1 \leq i \leq k, \\ -i + k & \text{if } k+1 \leq i \leq 2k, \end{cases}$$

$$f(u_i y_i) = \begin{cases} 2k+1-i & \text{if } 1 \leq i \leq k, \\ -3k+i-1 & \text{if } k+1 \leq i \leq 2k, \end{cases}$$

$$f(u_{2j-1} u_{2j}) = -2k-j \quad \text{if } 1 < j < k,$$

$$f(u_{2j-2} u_{2j-1}) = 3k+1-j \quad \text{if } 1 \leq j \leq k$$

Note that  $u_{2k} = u_0$ . Clearly,  $f$  is a bijection. Also we have  $f(u_i x_i) + f(u_i y_i) = 2k+1$  if  $1 \leq i \leq k$  and  $f(u_i x_i) + f(u_i y_i) = -(2k+1)$  if  $k+1 \leq i \leq 2k$ .

Thus  $\{f^+(x_i) \mid 1 < i \leq 2k\} = \{\pm 1, \dots, \pm k\}$ ,  $\{f^+(y_i) \mid 1 \leq i < 2k\} = \{\pm(k+1), \dots, \pm 2k\}$ .

For  $1 \leq j \leq \lceil \frac{k}{2} \rceil$ ,  $f^+(u_{2j-1}) = f(u_{2j-1} x_{2j-1}) + f(u_{2j-1} y_{2j-1}) + f(u_{2j-1} u_{2j}) + f(u_{2j-2} u_{2j-1}) = (2k+1) + (3k+1-j) + (-2k-j) = 3k-2j+2$ . For  $1 + \lceil \frac{k}{2} \rceil \leq j \leq k$ ,  $f^+(u_{2j-1}) = f(u_{2j-1} x_{2j-1}) + f(u_{2j-1} y_{2j-1}) + f(u_{2j-1} u_{2j}) + f(u_{2j-2} u_{2j-1}) = (2k+1) + (3k-j+1) + (-2k-j) = -k-2j$ .

Similarly, for  $1 \leq j < \lfloor \frac{k}{2} \rfloor$ ,  $f^+(u_{2j}) = (2k+1) + (3k+1-j-1) + (-2k-j) = 3k+1-2j$ , for  $1 + \lfloor \frac{k}{2} \rfloor \leq j \leq k-1$ ,  $f^+(u_{2j}) = -(2k+1) + (3k+1-j-1) + (-2k-j) = -k-1-2j$ , and  $f^+(u_{2k}) = (2k+1) + (-2k-k) + f(u_{2k} u_1) = -2k-1-2k-k+3k = -2k-1$ .

It is easy to check that the set  $\{f^+(u_i) \mid 1 \leq i \leq 2k\}$  is equal to  $\{\pm(2k+1), \dots, \pm 3k\}$ . Hence  $f$  is a super-edge-graceful labeling for  $A(2k, 2)$   $\square$

In the following are two examples showing some super-edge-graceful labelings.

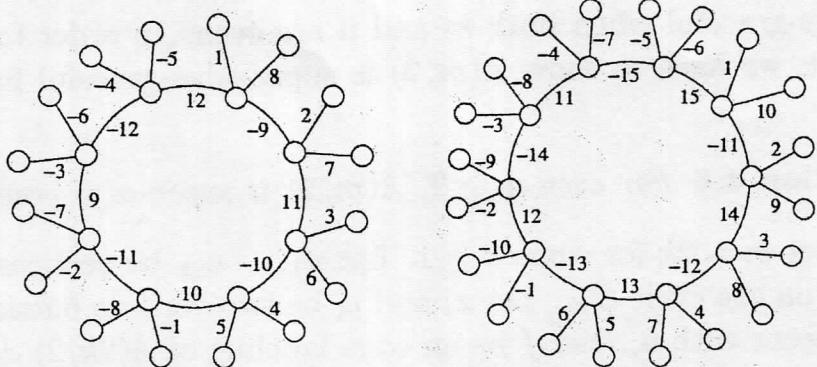


Figure 4.3: Super-edge-graceful labelings for  $A(8, 2)$  and  $A(10, 2)$

Hence we have

**Proposition 4.7** For  $m \geq 2$  and even  $n \geq 2$ ,  $A(m, n)$  is super-edge-graceful.

Combining Propositions 4.3, 4.4 and 4.7, we have

**Theorem 4.8** For  $m > 2$  and  $n \geq 1$ ,  $A(m, n)$  is super-edge-graceful.

Now we can consider a more general case of actinia graphs. Let  $C_m$  be the  $m$ -cycle with vertices  $v_1, v_2, \dots, v_m$  in clockwise,  $m \geq 2$ . Let  $n_1, n_2, \dots, n_m > 0$ . An actinia graph  $A(m; n_1, n_2, \dots, n_m)$  is a graph obtained from  $C_m$  by attaching  $n_i$  edges to the vertex  $v_i$ ,  $1 \leq i \leq m$ .

By using Lemma 4.1 we may show that

**Theorem 4.9** An actinia graph  $A(m, n_1, n_2, \dots, n_m)$  is super-edge-graceful if all  $n_i$ 's are positive and of the same parity. Moreover, when  $m$  is odd the positivity of  $n_i$ 's may be omitted.

Lee et al. [4] studied the super edge-gracefulness of some  $A(m; n_1, n_2, \dots, n_m)$ . They called them typical ring-worm graphs. All about these graphs are covered in this paper except the following:

**Theorem 4.10** [4] An actinia graph  $A(m, n_1, n_2, \dots, n_m)$  is super-edge-graceful if  $m$  is even and all except one  $n_i$ 's are even.

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