Graphs with convex labelings

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Abstract

In [2], Harminc and Soták defined an average labeling for simple graphs. They also characterized all graphs with average labelings and all the admissible average labelings for such graphs. For any real number α between 0 and 1, we define an α-convex labeling on a simple graph G as a mapping f from the vertex set of G to the set of all real numbers such that for any non-adjacent vertices u and v having common neighbor w, we have either $f(w) = \alpha f(u) + (1-\alpha)f(v)$ or $f(w) = (1-\alpha)f(u) + \alpha f(v)$. It follows that an average labeling is a $\frac{1}{2}$-convex labeling. In this paper, we also characterize all graphs with an NT convex labeling and obtained all the admissible NT convex labelings for such graphs. We also show that if a graph G admits an NT α-convex labeling, where α is neither $1-\sqrt{2}$ nor $2-\sqrt{2}$, then G admits an NT α-convex labeling for any α between 0 and 1. However, there exists a family of graphs which admit an NT α-convex labeling for $\alpha = \sqrt{2}-1$ or $2-\sqrt{2}$, but not for any other values of α between 0 and 1.

1 Introduction

Harminc [1] characterized the class of linear forests by applying an average valuation of graphs in 1997. Subsequently, Harminc and Soták [2] introduced an average labeling by means of induced sub-paths of graphs and characterized all graphs with an average labeling and all the admissible average labelings for such graphs. In average labeling, each common neighbor of two non-adjacent vertices u and v, if exists, receives a label equal to the "average" of the labels assigned to u and v. Therefore it is natural to consider replacing "average" by "convex combination", leading to convex labeling.

For undefined terms and notations in graph theory, we refer to [3] All graphs considered in this paper are finite, simple and undirected. Let G be a graph. Three vertices u, v and w are called a *triple*, denoted by (u, v, w),

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CONGRESSUS NUMERANTIUM **183** (2006), pp. 33-50

if $\{u, w\} \subset N(v)$ and $uw \notin E(G)$. Let $\mathbb{R}$ denote the set of real numbers. Throughout, we shall use α to denote a number in the open interval $(0, 1)$. Given $\{r_1, r_2, r_3\} \subset \mathbb{R}$ and α, if either $r_2 = \alpha r_1 + (1 - \alpha)r_3$ or $r_2 = (1 - \alpha)r_1 + \alpha r_3$, then we say r_2 is an *α-convex combination* of r_1 and r_3.

Given a graph G, a mapping $f : V(G) \to \mathbb{R}$ is called an *α-convex labeling* of G if for each triple (u, v, w) of G, $f(v)$ is an α-convex combination of $f(u)$ and $f(w)$. Note that the average labeling defined by Harminc and Soták in [2] is simply a $\frac{1}{2}$-convex labeling. A convex labeling f of graph G is *non-trivial*, or simply NT, if there are at least two distinct images under f, otherwise *trivial*. If each component of G is complete, then any mapping of real numbers from $V(G)$ to $\mathbb{R}$ is an α-convex labeling for all α. Therefore, from now on, we only consider non-complete connected graphs. Figure 1 shows a non-complete graph with an NT $\frac{2}{3}$-convex labeling.

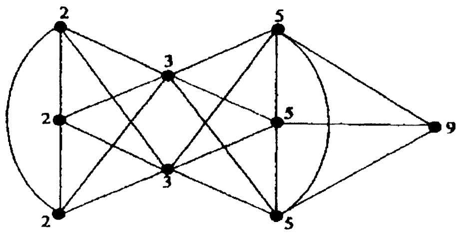

Figure 1. An NT $\frac{2}{3}$-convex labeling of a non-completed graph.

Let f be an NT convex labeling of G. Images under f are called *labels*. The number of distinct labels used is denoted by $\nu(f)$. The *convex labeling number* of G, denoted by $\lambda_c(G)$, is the minimum $\nu(f)$ taken over all NT convex labelings f of G, and is set to ∞ if no such NT convex labeling exists.

In Section 2, we give some notations and basic observations concerning the behavior of NT convex labelings on some subgraphs of non-complete connected graphs. In Section 3, we investigate the isomorphic structures of all graphs which admit NT convex labelings. Based on the isomorphic structures, we show in Section 4 that $\lambda_c(G) = \text{diam}(G) + s$ for some integer $s \geq 1$, where $\text{diam}(G)$ denotes the diameter of non-complete connected graph G. Then, we characterize all the admissible convex labelings of non-complete connected graphs. All results presented by Harminc and Soták in [2] follows from our results.

2 Some notations and preliminary lemmas

Let A be a subgraph of G and let $N_0(A) = V(A)$. For integer $k \geq 1$, the *k-neighborhood* of A, denoted by $N_k(A)$, is $\{v \in V(G) \setminus A : d_G(u,a) = k \text{ for some } a \in A\}$. We use $N(A)$ and $N[A]$ to denote $N_1(A)$ and $N_1(A) \cup A$ respectively. If A is a singleton of vertex x, then $N(A)$ and $N[A]$ will be simply denoted by $N(x)$ and $N[x]$ respectively. A *clique* Q of G is a complete subgraph of G, and is called *maximal* if it is not properly contained in any other clique. We use $P(v_1, v_2, \cdots, v_k)$ and $C(v_1, v_2, \cdots, v_k)$ to denote a path and a cycle, respectively, on vertices v_1 to v_k. We start with a basic fact that is useful in the next section.

Lemma 2.1 *If f is an α-convex labeling of G and $c_0 \in \mathbb{R}$, then the function $f' : V(G) \longrightarrow \mathbb{R}$ defined by $f'(v) = f(v) + c_0$ for all $v \in V(G)$ is also an α-convex labeling of G. Moreover, f is NT if and only if f' is NT.*

Let $U \subseteq V(G)$ and f be a real function on $V(G)$. By $f \equiv c$ on U we mean $f(u) = c$ for all $u \in U$. We also have the following lemmas and corollaries.

Lemma 2.2 *If G has an NT convex labeling f and A is any induced subgraph of G, then f is not constant on $N[A]$.*

Proof. Suppose, to the contrary that $f(u) \equiv c$ on $N[A]$ for some $c \in \mathbb{R}$. Since $N_1(A) \subseteq N[A]$, $f(v) \equiv c$ on $N_1(A)$. We shall prove that $f(v) \equiv c$ on $N_2(A)$. For $v \in N_2(A)$, let w be a neighbor of v in $N_1(A)$ and u a neighbor of w in A. Note that u and v are at distance two through w. Thus $f(w) = \alpha f(u) + (1-\alpha) f(v)$ or $f(w) = \alpha f(v) + (1-\alpha) f(u)$. In either case, we get $f(v) = c$ because $f(u) = f(w) = c$. Therefore, $f(v) \equiv c$ on $N_2(A)$. It follows by induction that f is constant on $N_k(A)$ for all $k \geq 0$. Since $V(G)$ is the union of $N_k(A)$ for all $k \geq 0$, f is constant on $V(G)$, a contradiction. ∎

It is straight forward to prove the following corollaries and lemma.

Corollary 2.3 *Let f be a convex labeling of G. If there is a vertex $u \in V$ such that f is constant on $N[u]$, then f is a trivial convex labeling of G.*

Corollary 2.4 *Let f be a convex labeling of G. If f is constant on some maximal clique Q of G, then f is trivial.*

Lemma 2.5 *Let f be an α-convex labeling of G. Suppose that $P(v_1, v_2, \cdots, v_k)$ is an induced path in G, where $k \geq 3$. Then either $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_k)$ or $f(v_1) \geq f(v_2) \geq \cdots \geq f(v_k)$. Furthermore, if f is NT, then all the inequalities are strict.*

Corollary 2.6 *If f is an NT convex labeling and u is not adjacent to v, then $f(u) \neq f(v)$.*

Let $0 < \alpha < 1$ and $A = \{a_1, a_2, \cdots, a_m\}$ be a set of $m \geq 3$ real numbers. If $a_1, a_2, \cdots, a_m$ is increasing and a_i is an α-convex combination of a_{i-1} and a_{i+1} for $i = 2, \cdots, m-1$, then $a_1, a_2, \cdots, a_m$ is called an α-convex sequence. Clearly, a $\frac{1}{2}$-convex interval is an arithmetical progression. If f is an NT α-convex labeling of G, then by Lemma 2.5, for each induced path $P(v_1, v_2, \cdots, v_k)$ of $k \geq 3$ vertices of G, the set $\{f(v_i) | i = 1, 2, \cdots, k\}$ is an α-convex sequence of length k. Note that, given two real numbers $a < b$ and $0 < \alpha < 1$, we can always extend them to an α-convex sequence $a_1, a_2, \cdots, a_m$ of length m with $a_1 = a$ and $a_2 = b$ (Clearly, $a_1 < a_2 < \cdots < a_m$ by the property of linear convex combination).

3 Graphs with NT convex labelings

A graph G is *chordal* if every cycle of length at least four has a *chord*, i.e., an edge between two nonconsecutive vertices of the cycle. Many properties of chordal graphs are presented in [4].

Theorem 3.1 *If G has an NT convex labeling, then G is claw-free and chordal.*

Proof. Suppose f is an NT α-convex labeling of G, $0 < \alpha < 1$. If G has a claw H with vertex set $\{u, u_1, u_2, u_3\}$ and edge set $\{uu_i | i = 1, 2, 3\}$. We first show that f is constant on H. For triple (u_1, u, u_2), we may assume by symmetry that $f(u) = \alpha f(u_1) + (1-\alpha) f(u_2)$. For (u_1, u, u_3) and (u_2, u, u_3), if $f(u) = \alpha f(u_1) + (1-\alpha) f(u_3)$ or $f(u) = \alpha f(u_3) + (1-\alpha) f(u_2)$, then $f(u_2) = f(u_3)$ or $f(u_1) = f(u_3)$ respectively and f is constant on H. If $f(u) = (1-\alpha) f(u_1) + \alpha f(u_3)$ and $f(u) = \alpha f(u_2) + (1-\alpha) f(u_3)$, then $f(u), f(u_1), f(u_2)$ and $f(u_3)$ satisfy the equations:

$$\begin{cases} f(u) = \alpha f(u_1) + (1-\alpha) f(u_2) \\ f(u) = \alpha f(u_2) + (1-\alpha) f(u_3) \\ f(u) = \alpha f(u_3) + (1-\alpha) f(u_1) \end{cases}$$

Solving this system, we have $f(u_1) = f(u_2) = f(u_3) = f(u)$ and therefore f is constant on H.

Now suppoe there exists $v \in N(u) \setminus \{u_1, u_2, u_3\}$. If u_i is not adjacent to v for some $i \in \{1, 2, 3\}$, then from (u_i, u, v), we get either $f(u) = \alpha f(v) + (1-\alpha) f(u_i)$ or $f(u) = \alpha f(u_i) + (1-\alpha) f(v)$. This implies $f(v) = f(u)$. If u_i is adjacent to v for all $i \in \{1, 2, 3\}$, then $f(v)$ is an α-convex combination of $f(u_1)$ and $f(u_2)$ and so $f(v) = f(u)$. Consequently, f is constant on $N[u]$. Applying Corollary 2.3, f is trivial on G, a contradiction.

If G is not chordal, then there is a chordless cycle $C(u_1, u_2, \cdots, u_k)$ of length $k \geq 4$. By Lemma 2.5, $f(u) = c$ for some c and for all vertices in $C(u_1, u_2, \cdots, u_k)$. Since f is NT, $N(C(u_1, u_2, \cdots, u_k)) \neq \emptyset$. Suppose $v \in N(C(u_1, u_2, \cdots, u_k))$. If v is adjacent to all vertices on $C(u_1, u_2, \cdots, u_k)$, then $f(v)$ is an α-convex combination of $f(u_1)$ and $f(u_3)$ and consequently $f(v) = c$. Suppose v is adjacent to some, but not all, vertices on $C(u_1, u_2, \cdots, u_k)$. Without loss of generality, we may assume that v is adjacent to u_i but not adjacent to u_{i+1}. Then $f(u_i)$ is an α-convex combination of $f(v)$ and $f(u_{i+1})$. It follows that $f(v) = c$ and consequently f is constant on $N[C(u_1, u_2, \cdots, u_k)]$. By Lemma 2.2, f is constant on G, a contradiction. ∎

Corollary 3.2 *If G is a triangle-free graph with an NT convex labeling, then G is a path.*

To investigate the structure of general graphs with NT convex labelings, we need more information about the relations between maximal cliques.

Lemma 3.3 *Suppose that f is an NT α-convex labeling of G and Q is a maximal clique of G. Then $2 \leq |f(V(Q))| \leq 4$.*

Proof. Since f is NT, $|f(V(Q))| \geq 2$ by Corollary 2.4. Since G is not complete and Q is a maximal clique, there exist two vertices u and v in Q and a vertex w in $N(Q)$ such that $uw \in E(G)$, $vw \notin E(G)$. Then we have, with no loss of generality, $f(u) = \alpha f(w) + (1-\alpha) f(v)$, or $f(w) = \frac{f(u)-(1-\alpha)f(v)}{\alpha}$. Let x be a vertex of Q other than u and v. If x is adjacent to w, then either $f(x) = \alpha f(w) + (1-\alpha) f(v)$ or $f(x) = \alpha f(v) + (1-\alpha) f(w)$. In the former case, we have $f(x) = f(u)$. In the latter case, we have $f(x) = \frac{(1-\alpha)f(u)-(1-2\alpha)f(v)}{\alpha}$. If x is not adjacent to w, then either $f(u) = \alpha f(w) + (1-\alpha) f(x)$ or $f(u) = \alpha f(x) + (1-\alpha) f(w)$. By similar arguments, we have $f(x) = f(v)$ or $f(x) = \frac{(1-\alpha)^2 f(v)-(1-2\alpha)f(u)}{\alpha^2}$. Therefore, $f(V(Q)) \subseteq$

$\{f(u), f(v), \frac{(1-\alpha)f(u)-(1-2\alpha)f(v)}{\alpha}, \frac{(1-\alpha)^2 f(v)-(1-2\alpha)f(u)}{\alpha^2}\}$ and consequently, $|f(V(Q))| \leq 4$. ∎

Let H be a subgraph of G and x a vertex of G not in H. By $N_H(x)$ we denote the set of vertices in H which are adjacent to x. Let γ be a real number and f a vertex labeling of G. The set of vertices of H with label γ under f is denoted by $V(H, f, \gamma)$.

Lemma 3.4 *Suppose Q is a maximal clique and f is an NT α-convex labeling of G. Then*
(a) *If $u \in N(Q)$, then $f(u) \neq f(v)$ for all $v \in N_Q(u)$.*
(b) *If $u \in N(Q)$ is adjacent to $v \in V(Q, f, a)$, then u is adjacent to all vertices in $V(Q, f, a)$.*
(c) *If $u, v \in N(Q)$, $u \neq v$, and $N_Q(u) \cap N_Q(v) \neq \emptyset$, then $N_Q(u) = N_Q(v)$ and $uv \in E(G)$.*

Proof. (a) Suppose $u \in N(Q)$ and $f(u) = f(v)$ for some $v \in N_Q(u)$. Since Q is maximal, there exists $w \in Q$ with $wu \notin E$. Then (u, v, w) is a triple and consequently $f(u) = f(v) = f(w)$. For all $x \in Q \setminus \{w\}$, either (u, x, w) or (u, v, x) is a triple, hence $f \equiv f(u)$ on Q. By Corollary 2.4, we get that f is trivial on G and a contradiction. Therefore $f(u) \neq f(v)$ for all $v \in N_Q(u)$.

(b) Suppose $w \in V(Q, f, h) \setminus \{v\}$. Then $f(v) = f(w) = h$. If $uw \notin E(G)$, then (u, v, w) is a triple, and $f(u) = f(v) = f(w)$, contradicting (a). So u is adjacent to all vertices in $V(Q, f, h)$.

(c) Suppose $u, v \in N(Q)$ such that $u \neq v$ and $N_Q(u) \cap N_Q(v) \neq \emptyset$. To the contrary, assume that $N_Q(u) \neq N_Q(v)$. We have two cases:

Case 1. $N_Q(u) \subset N_Q(v)$ ($N_Q(v) \subset N_Q(u)$ is similar)

Since Q is maximal, there exists $x \in V(Q) \setminus N_Q(v)$. Take $y \in N_Q(u)$ and $z \in N_Q(v) \setminus N_Q(u)$. Then u is adjacent to v, otherwise $\{u, v, x, y\}$ induces a claw in G, contradicting Theorem 3.1. The subgraph induced by $\{x, y, z, u, v\}$ is illustrated in Figure 2(a). For triple (x, y, u), without loss of generality, we assume that $f(y) = \alpha f(x) + (1-\alpha)f(u)$. It follows from (A) and (B) that labels of vertices x, y, z, u and v are pairwisely distinct.

Thus we get the following system of equations:

$$\begin{cases} f(y) = \alpha f(x) + (1-\alpha) f(u) & (1) \\ f(y) = \alpha f(u) + (1-\alpha) f(z) & (2) \\ f(y) = \alpha f(v) + (1-\alpha) f(x) & (3) \\ f(z) = \alpha f(x) + (1-\alpha) f(v) & (4) \\ f(v) = \alpha f(z) + (1-\alpha) f(u) & (5) \end{cases}$$

Note that $\alpha \neq \frac{1}{2}$, otherwise $f(y) = f(z)$ by equations (3) and (4). Summing equations (2) to (5) and simplify, we get $2f(y) = f(x)+f(u)$. With equation (1), we have $(1-2\alpha)f(x) = (1-2\alpha)f(u)$. Because $1-2\alpha \neq 0$, we have $f(x) = f(u)$ and a contradiction.

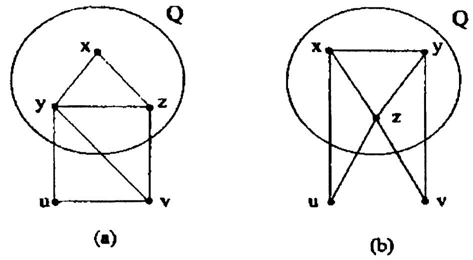

Figure 2. Illustrations for the proof of Lemma 3.4

Case 2. $N_Q(u) \setminus N_Q(v) \neq \emptyset$ and $N_Q(v) \setminus N_Q(u) \neq \emptyset$.

Let $x \in N_Q(u) \setminus N_Q(v)$, $y \in N_Q(v) \setminus N_Q(u)$ and $z \in N_Q(u) \cap N_Q(v)$. If u and v are adjacent, the subgraph induced by $\{x,\ y,\ u,\ v\}$ is a chordless cycle of length 4, contradicting Theorem 3.1. So u and v are not adjacent. The subgraph induced by $\{x,\ y,\ z,\ u,\ v\}$ is illustrated in Figure 2(b). For triple (u, z, v), without loss of generality, we assume that $f(z) = \alpha f(u) + (1-\alpha)f(v)$. It also follows from (a) and (b) that labels of vertices x, y, z, u and v are pairwisely distinct and so we get the following system of equations:

$$\begin{cases} f(z) = \alpha f(u) + (1-\alpha) f(v) & (6) \\ f(z) = \alpha f(y) + (1-\alpha) f(u) & (7) \\ f(z) = \alpha f(v) + (1-\alpha) f(x) & (8) \\ f(x) = \alpha f(u) + (1-\alpha) f(y) & (9) \\ f(y) = \alpha f(x) + (1-\alpha) f(v) & (10) \end{cases}$$

Clearly, $\alpha \neq \frac{1}{2}$. As in Case 1, by summing up equations (7) to (10) and simplying, we get $2f(z) = f(u) + f(v)$. Substituting this to equation (6), we get $f(u) = f(v)$ and a contradiction.

It follows that $N_Q(u) = N_Q(v)$. Furthermore, choose a common neighbor $x \in N_Q(u) = N_Q(v)$ and y in $V(Q) \setminus N_Q(u)$ (the maximality of clique Q implies the existence of vertex y). If $uv \notin E(G)$, then G would have a claw. Hence $uv \in E(G)$, and consequently, assertion (c) holds. ∎

Lemma 3.5 *Suppose that Q is a maximal clique of G and f is an NT convex labeling of G. Then for each $u \in N(Q)$,*
(a) *if $|f(V(Q))| = 2$, then $f(N_Q(u))$ contains only one label; and*
(b) *if $|f(V(Q))| = 3$, then $f(N_Q(u))$ contains at most two labels; and*
(c) *if $|f(V(Q))| = 4$, then $f(N_Q(u))$ contains exactly two labels.*

Proof. By Lemma 3.4(a), Lemma 3.4(b) and maximality of Q, we can see that (a) and (b) hold. To prove (c), let f be an NT α-convex labeling of G. Assume, to the contrary, that there is a vertex $u \in N(Q)$ such that $|f(N_Q(u))| = 1$ or $|f(N_Q(u))| = 3$. If $|f(N_Q(u))| = 1$, we may, by Lemma 3.4(a) and (b), let $f(N_Q(u)) = \{a\}$ and $f(V(Q) \backslash N_Q(u)) = \{b, c, d\}$. Choose $v \in N_Q(u)$ and $x, y, z \in V(Q) \setminus N_Q(u)$ such that $f(v) = a$, $f(x) = b$, $f(y) = c$ and $f(z) = d$. Note that u is at distance two from each of x, y, z. For triple (u, v, x), we assume without loss of generality that $f(v) = \alpha f(u) + (1 - \alpha) f(x)$. Then for triple (u, v, y), we have $f(v) = (1-\alpha) f(u) + \alpha f(y)$. But then for triple (u, v, z), $f(v)$ cannot be an α-convex combination of $f(u)$ and $f(z)$ unless either $b = d$ or $c = d$, which contradicts the assumption. Therefore $|f(N_Q(u))| \neq 1$. The case of $|f(N_Q(u))| = 3$ is similar. ∎

Let $\mathcal{V}_G$ denote the set of all maximal cliques of G. The *clique graph* of G is the graph $\mathcal{C}_G = (\mathcal{V}_G, \mathcal{E}_G)$ with $\mathcal{E}_G = \{QQ' | Q \neq Q' \in \mathcal{V}_G, Q \cap Q' \neq \emptyset\}$. It can be shown that $\mathcal{C}_G$ is chordal if and only if G is chordal.

Theorem 3.6 *Suppose G has an NT convex labeling. Then $\mathcal{C}_G = (\mathcal{V}_G, \mathcal{E}_G)$ is a path of length* $\operatorname{diam}(G) - 1$.

Proof. Let f be an NT α-convex labeling of G. Clearly $\mathcal{C}_G = (\mathcal{V}_G, \mathcal{E}_G)$ is connected because G is. We shall first show that $\mathcal{C}_G$ is a tree. By Theorem 3.1, G, and hence $\mathcal{C}_G$, is a chordal graph. So it is enough to show that $\mathcal{C}_G$ has no triangle. For contradiction, suppose that $\mathcal{C}_G$ contains a triangle $C(Q_1, Q_2, Q_3)$, where $Q_1, Q_2, Q_3 \in \mathcal{V}_G$ are maximal cliques of G. If $V(Q_1) \cap V(Q_2) \cap V(Q_3) \neq \emptyset$, then by Lemma 3.4(c), we have at least two of Q_1, Q_2 and Q_3 equal and a contradiction. Hence $V(Q_1) \cap V(Q_2) \cap V(Q_3) = \emptyset$, but $V(Q_i) \cap V(Q_j) \neq \emptyset$ for any $i \neq j$. Choose $x \in V(Q_1) \cap V(Q_2)$, $y \in V(Q_1) \cap V(Q_3)$ and $z \in V(Q_2) \cap V(Q_3)$. Then $C(x, y, z)$ is a triangle of G. Let Q be a maximal clique of G containing $C(x, y, z)$. Then $V(Q) \cap V(Q_2) \cap V(Q_3) \neq \emptyset$ and is distinct from each of Q_2 and Q_3. Again, we have a contradiction.

It remains to show that each vertex of $\mathcal{C}_G$ has degree at most two. To the contrary, assume $d_{\mathcal{C}_G}(Q) \geq 3$ for some $Q \in \mathcal{V}_G$. Then there must exist three maximal cliques Q_1, Q_2 and Q_3, each of which intersects Q. By the above arguments, such three cliques are pairwise disjoint. First we choose vertices $u \in Q_1 \setminus Q$, $v \in Q_2 \setminus Q$ and $w \in Q_3 \setminus Q$. Then we choose x, y and z in Q such that xu, yv, $zw \in E(G)$. By Lemma 3.4(c), x, y and z are all distinct. The subgraph induced by vertex set $\{x, y, z, u, v, w\}$ is illustrated in Figure 3(a). For triple (u, x, z), we assume without loss of generality that $f(x) = \alpha f(u) + (1-\alpha)f(z)$. Then for triple (u, x, y), $f(x) = \alpha f(y) + (1-\alpha)f(u)$.

Now consider triple (v, y, x). If $f(y) = \alpha f(v) + (1-\alpha)f(x)$ then for triple (v, y, z), $f(y) = \alpha f(z) + (1-\alpha)f(v)$. Combining these equations into a system, we have

$$\begin{cases} f(x) = \alpha f(u) + (1-\alpha)f(z) & (11) \\ f(x) = \alpha f(y) + (1-\alpha)f(u) & (12) \\ f(y) = \alpha f(v) + (1-\alpha)f(x) & (13) \\ f(y) = \alpha f(z) + (1-\alpha)f(v) & (14) \end{cases}$$

We may, by Lemma 2.1, let $f(z) = 0$. If $2\alpha = 1$, $f(y) = 0 = f(z)$. If $2\alpha \neq 1$, then solving equations (11) to (14), we get $f(u) = 0 = f(z)$. Both cases lead to contradiction because of Corollary 2.6 and Lemma 3.4. Therefore, for triple (v, y, x), $f(y) = \alpha f(x) + (1-\alpha)f(v)$ and, consequently, for triple (v, y, z), $f(y) = \alpha f(v) + (1-\alpha)f(z)$.

For triple (w, z, y), if $f(z) = \alpha f(w) + (1-\alpha)f(y)$ or $f(z) = (1-\alpha)f(w) + \alpha f(y)$, then for triple (w, z, x), we get $f(z) = (1-\alpha)f(w) + \alpha f(x)$ or $f(z) = \alpha f(x) + (1-\alpha)f(w)$ respectively. Both cases will lead to a system of equations and by similar arguments above, we have $f(y) = f(w)$ or $f(w) = f(x)$ respectively, and contradictions. Hence, for each $Q \in \mathcal{V}_G$, $d_{C_G}(Q) \leq 2$ and consequently C_G is a path. Since $\text{diam}(G) \leq |\mathcal{V}_G|$ and C_G is a path of $|\mathcal{V}_G|$ vertices, thus $|\mathcal{V}_G| = \text{diam}(G)$. ∎

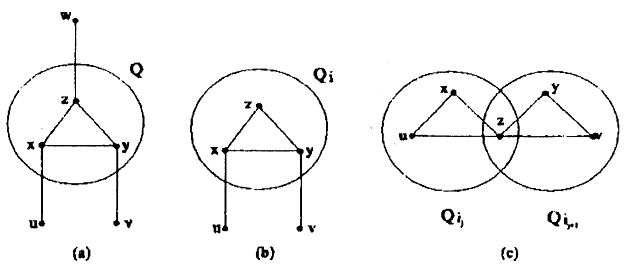

Figure 3. Illustrations for the proofs of Theorems 3.6 and 3.7.

Let $G(v_1, v_2, \cdots, v_n)$ be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and H_i, $i = 1, 2, \cdots, n$ be n graphs. We define the ***substitution*** of G by $(H_1, H_2, \cdots, H_n)$, denoted by $G(H_1, H_2, \cdots, H_n)$, as the graph obtained upon replacing each v_i of G by the graph H_i; and if v_i is adjacent to v_j, then every vertex of H_i is adjacent to every vertex of H_j. In particular, if each H_i is a complete graph K_{m_i} of m_i vertices, $i = 1, 2, \cdots, n$, then $G(K_{m_1}, K_{m_2}, \cdots, K_{m_n})$ is called a *complete substitution* of graph $G(v_1, v_2, \cdots, v_n)$. The complete substitution of the path $P(v_1, v_2, \cdots, v_n)$ was also defined by Harminc and Soták [2] as the *lexicographic extension* of $P(v_1, v_2, \cdots, v_n)$. For example, the graph illustrated in Figure 1 is $P(K_3, K_2, K_3, K_1)$.

Let $t \geq 1$ and $n \geq 3t + 2$ be two integers. We use $\mathcal{P}_t^{\triangle}(v_1, v_2, \cdots, v_n)$ to denote the set of graphs obtained from the path $P(v_1, v_2, \cdots, v_n)$ by adding t nonadjacent edges of the form $v_i v_{i+2}$, $2 \leq i \leq n-3$, such that all resulting triangles are disjoint - (for example, Figure 4). Observe that each graph in $\mathcal{P}_t^{\triangle}(v_1, v_2, \cdots, v_n)$ contains exactly t disjoint triangles.

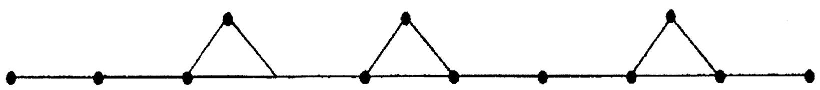

Figure 4. A graph in $\mathcal{P}_3^{\triangle}(v_1, v_2, \cdots, v_{13})$.

Let A, B be two subgraphs of G. By $A \setminus B$, $A \cap B$ and $A \cup B$ we denote the subgraphs induced by $V(A) \setminus V(B)$, $V(A) \cap V(B)$ and $V(A) \cup V(B)$, respectively.

Theorem 3.7 *Let G be a non-complete connected graph with NT convex labelings.*
(1) G has an NT α-convex labeling, $\alpha \neq \sqrt{2}-1$ and $2-\sqrt{2}$, if and only if G is isomorphic to $P(K_{m_1}, K_{m_2}, \cdots, K_{m_n})$ for some positive integers $n \geq 3$, $m_1, m_2, \cdots, m_n$.
(2) G has an NT $(\sqrt{2}-1)$- or $(2-\sqrt{2})$-convex labeling if and only if G is isomorphic to $P(K_{m_1}, K_{m_2}, \cdots, K_{m_n})$ for some positive integers $n \geq 3$, $m_1, m_2, \cdots, m_n$ or isomorphic to a complete substitution of some graph in $\mathcal{P}_t^{\triangle}(v_1, v_2, \cdots, v_n)$ for some positive integers $t \geq 1$ and $n \geq 3t + 2$.

Proof. (1) Suppose G has an NT α-convex labelings with $\alpha \notin \{\sqrt{2}-1, 2-\sqrt{2}\}$. By Theorem 3.6, $\mathcal{C}_G = P(Q_1, Q_2, \cdots, Q_k)$, where $k = |\mathcal{V}_G| \geq 2$. If $k = 2$, then let $K_{m_1} = Q_1 \setminus Q_2$, $K_{m_2} = Q_1 \cap Q_2$ and $K_{m_3} = Q_2 \setminus Q_1$, where

$m_1 = |V(Q_1) \setminus V(Q_2)|$, $m_2 = |V(Q_1) \cap V(Q_2)|$ and $m_3 = |V(Q_2) \setminus V(Q_1)|$. Hence, G is isomorphic to $P(K_{m_1}, K_{m_2}, K_{m_3})$. So assume that $k \geq 3$.

Suppose there exists Q_i, $2 \leq i \leq k-1$, such that $V(Q_i) \setminus V(Q_{i-1}) \cup V(Q_{i+1}) \neq \emptyset$. Take vertices $u \in V(Q_{i-1}) \setminus V(Q_i)$, $v \in V(Q_{i+1}) \setminus V(Q_i)$ and $x \in V(Q_i) \cap V(Q_{i-1})$, $y \in V(Q_i) \cap V(Q_{i+1})$ and $z \in V(Q_i) \setminus (V(Q_{i-1}) \cup V(Q_{i+1}))$ - Figure 3(b). By Lemma 3.4, the labels of five vertices x, y, z, u and v are pairwise distinct.

For triples (u, x, z) and (u, x, y), if $f(x) = \alpha f(u) + (1-\alpha) f(z)$, then $f(x) = \alpha f(y) + (1-\alpha) f(u)$. By the same arguments in the proof of Theorem 3.6, we must have $f(y) = \alpha f(x) + (1-\alpha) f(v)$ and $f(y) = \alpha f(v) + (1-\alpha) f(z)$. Hence, we have the following system of equations:

$$\begin{cases} f(x) = \alpha f(u) + (1-\alpha) f(z) \\ f(x) = \alpha f(y) + (1-\alpha) f(u) \\ f(y) = \alpha f(x) + (1-\alpha) f(v) \\ f(y) = \alpha f(v) + (1-\alpha) f(z) \end{cases}$$

By Lemma 2.1, we may let $f(z) = 0$. Then by the first and last equations, $f(x) = \alpha f(u)$ and $f(y) = \alpha f(v)$. Substituting them to the second and third equations, we have

$$\begin{cases} \alpha f(u) = \alpha^2 f(v) + (1-\alpha) f(u) \\ \alpha f(v) = \alpha^2 f(u) + (1-\alpha) f(v) \end{cases}$$

Then $(\alpha + 1 - \sqrt{2})(\alpha + 1 + \sqrt{2})(\alpha - 1)^2 f(u) = 0$. Since $0 < \alpha < 1$ and $f(u) \neq 0$, therefore $\alpha = \sqrt{2} - 1$, a contradiction. Similarly, if $f(x) = (1-\alpha) f(u) + \alpha f(z)$, then the same argument will give $\alpha = 2 - \sqrt{2}$, also a contradiction. The contradictions show that $V(Q_i) = V(Q_{i-1}) \cup V(Q_{i+1})$ if $2 \leq i \leq k-1$.

Let $K_{m_1} = Q_1 \setminus Q_2$, $K_{m_n} = Q_k \setminus Q_{k-1}$ and $K_{m_i} = Q_{i-1} \cap Q_i$ for $i = 2, 3, \cdots, n-1$, where $n = k+1$, $m_1 = |V(Q_1 \setminus Q_2)|$, $m_n = |V(Q_k \setminus Q_{k-1})|$ and $m_i = |V(Q_{i-1} \cap Q_i)|$ for $i \neq 1, n$. Then G is isomorphic to $P(K_{m_1}, K_{m_2}, \cdots, K_{m_n})$ for $n = k+1 \geq 4$.

Suppose that there exist positive integers $n \geq 3$, $m_1, m_2, \cdots, m_n$, such that G is isomorphic to $P(K_{m_1}, K_{m_2}, \cdots, K_{m_n})$. For any $0 < \alpha < 1$, we define a labeling f of G as follows. If $\alpha = \frac{1}{2}$, then for $1 \leq i \leq n$, let $f(u) = i$, $u \in V(K_{m_i})$. If $\alpha \neq \frac{1}{2}$, then for $1 \leq i \leq n$,

$$f(u) = \alpha^{i-1}(1-\alpha)^{n-i}, u \in V(K_{m_i}).$$

It is straightforward to check that f is NT and α-convex.

(2) Suppose G has an NT $(\sqrt{2}-1)$-convex labeling f (the other case is similar). In the same way as in (1) we can prove that, if $k=2$ or $k \geq 3$ and for each $2 \leq i \leq k-1$, $V(Q_i) = (V(Q_{i-1}) \cap V(Q_i)) \cup (V(Q_i) \cap V(Q_{i+1}))$, then G is isomorphic to $P(K_{m_1}, K_{m_2}, \cdots, K_{m_n})$ for some positive integers $n \geq 3$, $m_1, m_2, \cdots, m_n$. So assume $k \geq 3$ and there are $t(\geq 1)$ Q_i, $2 \leq i \leq k-1$, such that $V(Q_i) \supset (V(Q_{i-1}) \cap V(Q_i)) \cup (V(Q_i) \cap V(Q_{i+1}))$. Let such t cliques be $Q_{i_1}, Q_{i_2}, \cdots, Q_{i_t}$, $2 \leq i_1 < i_2 < \cdots < i_t \leq k-1$. Then by Lemmas 3.4 and 3.5, $|f(V(Q_{i_j}))| = 3$ and $|f(V(Q_{i_j-1}) \cap V(Q_{i_j}))| = |f(V(Q_{i_j}) \cap V(Q_{i_j+1}))| = 1$ for each $1 \leq j \leq t$.

We now show that $i_{j+1} - i_j \geq 2$ for each $1 \leq j \leq t-1$, that is, Q_{i_j} and $Q_{i_{j+1}}$ are nonadjacent in $\mathcal{C}_G$ or disjoint in G for $j = 1, 2, \cdots, t-1$. Suppose to the contrary that there are two Q_{i_j} and $Q_{i_{j+1}}$ such that $V(Q_{i_j}) \cap V(Q_{i_{j+1}}) \neq \phi$. Clearly, $i_{j+1} = i_j + 1$ for $\mathcal{C}_G$ is a path. Now consider different vertices $u \in V(Q_{i_j-1} \cap Q_{i_j})$, $v \in V(Q_{i_{j+1}} \cap Q_{i_{j+1}+1})$, $x \in V(Q_{i_j}) \setminus (V(Q_{i_j-1} \cap Q_{i_j}) \cup V(Q_{i_j} \cap Q_{i_{j+1}}))$, $y \in V(Q_{i_{j+1}}) \setminus (V(Q_{i_j} \cap Q_{i_{j+1}}) \cup V(Q_{i_{j+1}} \cap Q_{i_{j+1}+1}))$ and $z \in V(Q_{i_j} \cap Q_{i_{j+1}})$. The subgraph induced by vertices x, y, z and u, v in G is demonstrated in Figure 3(c). For triple (x, z, y), if $f(z) = (\sqrt{2}-1)f(x) + (2-\sqrt{2})f(y)$, then consider triples (x, z, v), (u, z, y) and (u, z, v), we obtain the following system of equations:

$$\begin{cases} f(z) = (\sqrt{2}-1)f(x) + (2-\sqrt{2})f(y) \\ f(z) = (\sqrt{2}-1)f(v) + (2-\sqrt{2})f(x) \\ f(z) = (\sqrt{2}-1)f(y) + (2-\sqrt{2})f(u) \\ f(z) = (\sqrt{2}-1)f(u) + (2-\sqrt{2})f(v) \end{cases}$$

Solving this system, we must have $f(x) = f(y) = f(z) = f(u) = f(v)$, a contradiction. Similarly, if $f(z) = (\sqrt{2}-1)f(y) + (2-\sqrt{2})f(x)$, we can also get the same contradiction.

Let $P(v_1, \cdots, v_{i_1}, x_{i_1}, v_{i_1+1}, \cdots, v_{i_2}, x_{i_2}, v_{i_2+1}, \cdots, x_{i_t}, v_{i_t+1}, \cdots, v_{k+1})$ be a path of $k+t+1$ vertices. By $P_t^{\triangle}(v_1, \cdots, v_{k+1})$, we denote the graph obtained by adding t edges $v_{i_j}v_{i_j+1}$, $j = 1, 2, \cdots, t$, to the above path. Let $K_{m_1} = Q_1 \setminus Q_2$, $K_{m_{k+1}} = Q_k \setminus Q_{k-1}$ and $K_{m_i} = Q_{i-1} \cap Q_i$ for $i = 2, 3, \cdots, k$, $K'_{n_j} = Q_{i_j} \setminus (Q_{i_j-1} \cup Q_{i_j+1})$, $j = 1, 2, \cdots, t$, where $m_1 = |V(Q_1 \setminus Q_2)|$, $m_{k+1} = |V(Q_k \setminus Q_{k-1})|$ and $m_i = |V(Q_{i-1} \cap Q_i)|$, $i = 2, 3, \cdots, k$, and $n_j = |V(Q_{i_j} \setminus (Q_{i_j-1} \cup Q_{i_j+1}))|$, $j = 1, 2, \cdots, t$. Thus G is isomorphic to

$$P_t^{\triangle}(K_{m_1}, \cdots, K_{m_{i_1}}, K'_{n_{i_1}}, K_{m_{i_1+1}}, \cdots, K_{m_{i_t}}, K'_{n_{i_t}}, K_{m_{i_t+1}}, \cdots, K_{m_{k+1}}).$$

As before, we can define NT $(\sqrt{2}-1)$- and $(2-\sqrt{2})$-convex labelings of G if G is isomorphic to $P(K_{m_1}, \cdots, K_{m_n})$ for some positive integers $n \geq 3$,

$m_1, \cdots, m_n$. Therefore, we assume G is isomorphic to the substitution $P_t^{\triangle}(K_{m_1}, \cdots, K_{m_{i_1}}, K'_{n_{i_1}}, K_{m_{i_1+1}}, \cdots, K_{m_{i_t}}, K'_{n_{i_t}}, K_{m_{i_t+1}}, \cdots, K_n)$ of $P_t^{\triangle}(v_1, \cdots, v_{i_1}, x_{i_1}, v_{i_1+1}, \cdots, v_{i_t}, x_{i_t}, v_{i_t+1}, \cdots, v_n)$, with t edges $v_{i_j}v_{i_j+1}$, $j = 1, 2, \cdots, t$, for some positive integers t, $n \geq 3t+2$, $m_1, m_2, \cdots, m_n$ and n_j, $j = 1, 2, \cdots, t$. We first define an NT $(\sqrt{2}-1)$-convex labeling g of $P_t^{\triangle}(v_1, \cdots, v_{i_1}, x_{i_1}, \cdots, v_{i_2}, x_{i_2}, v_{i_2+1}, \cdots, v_{i_t}, x_{i_t}, v_{i_t+1}, \cdots, v_n)$ and then obtain an NT $(\sqrt{2}-1)$-convex labeling f of G by g.

Take arbitrary two real numbers $a_1 < a_2$, and then extend them to obtain a $(\sqrt{2}-1)$-convex sequence $\{a_1, a_2, \cdots, a_{i_1}\}$ of length i_1. Let $g(v_s) = a_s$, $s = 1, 2, \cdots, i_1$, and let

$$\begin{aligned} g(x_{i_1}) &= \tfrac{a_{i_1}-(\sqrt{2}-1)a_{i_1-1}}{2-\sqrt{2}}, \\ g(v_{i_1+1}) &= \tfrac{\sqrt{2}a_{i_1}-2(\sqrt{2}-1)a_{i_1-1}}{2-\sqrt{2}}, \\ g(v_{i_1+2}) &= \tfrac{2a_{i_1}-\sqrt{2}a_{i_1-1}}{2-\sqrt{2}}. \end{aligned}$$

We see that g is NT and $(\sqrt{2}-1)$-convex on the subgraph of $P_t^{\triangle}(v_1, \cdots, v_{i_1}, x_{i_1}, v_{i_1+1}, \cdots, v_n)$ induced by the vertices $v_1, \cdots, v_{i_1}, x_{i_1}, v_{i_1+1}, v_{i_1+2}$. Similarly, again using two real numbers $a_{i_1+1} = g(v_{i_1+1})$ and $a_{i_1+2} = g(v_{i_1+2})$ (Clearly, $a_{i_1+1} < a_{i_1+2}$ by Lemma 2.5), we extend them to get an α-convex sequence $\{a_{i_1+1}, a_{i_1+2}, \cdots, a_{i_2}\}$ of length $i_2 - i_1$. Then similarly, let $g(v_s) = a_s$, $s = i_1+1, i_1+2, \cdots, i_2$, and let

$$\begin{aligned} g(x_{i_2}) &= \tfrac{a_{i_2}-(\sqrt{2}-1)a_{i_2-1}}{2-\sqrt{2}}, \\ g(v_{i_2+1}) &= \tfrac{\sqrt{2}a_{i_2}-2(\sqrt{2}-1)a_{i_2-1}}{2-\sqrt{2}}, \\ g(v_{i_2+2}) &= \tfrac{2a_{i_2}-\sqrt{2}a_{i_2-1}}{2-\sqrt{2}}. \end{aligned}$$

We get an NT $(\sqrt{2}-1)$-convex labeling g of the subgraph of $P_t^{\triangle}(v_1, \cdots, v_{i_2}, x_{i_2}, v_{i_2+1}, \cdots, v_n)$ induced by the vertices $v_1, \cdots, v_{i_2}, x_{i_2}, v_{i_2+1}, v_{i_2+2}$. We can continue in this way to extend g to an NT $(\sqrt{2}-1)$-convex labeling on $P_t^{\triangle}(v_1, \cdots, v_{i_1}, x_{i_1}, v_{i_1+1}, \cdots, v_n)$.

Finally, define a labeling f of G as follows:

$$\begin{aligned} f(u) &= g(v_i), u \in K_{m_i}, i = 1, 2, \cdots, n; \\ f(u) &= g(x_{i_j}), u \in K'_{n_j}, j = 1, 2, \cdots, t. \end{aligned}$$

Since g is an NT $(\sqrt{2}-1)$-convex labeling of G, so is f. ∎

Corollary 3.8 *Let G be a non-complete connected graph.*

(1) *If G admits an NT α-convex labeling, where $\alpha \neq \sqrt{2}-1, 2-\sqrt{2}$, then G admits an NT α-convex labeling for any α between 0 and 1.*

(2) *If G is isomorphic to a complete substitution of some graph in $\mathcal{P}_t^{\triangle}(v_1, v_2, \cdots, v_n)$ for some positive integers $t \geq 1$ and $n \geq 3t+2$, then G admits an NT α-convex labeling, when α is either $\sqrt{2}-1$ or $2-\sqrt{2}$, but not for any other values of α.*

4 *NT* convex labelings

In this section, we attempt to the characterize NT convex labelings of non-complete connected graphs. Let G be a non-complete connected graph. A vertex of graph G is called *simplicial* if all vertices adjacent to it induce a clique in G. A maximal clique $Q \in \mathcal{V}_G$ of G is called an *end-clique* if it intersects only one maximal clique in G and called a *mid-clique* of G otherwise. By Theorem 3.6, if G has NT convex labelings, then there are exactly two end-cliques in G corresponding to two end-vertices of the path $\mathcal{C}_G$ and other cliques are mid-cliques corresponding to those vertices of degree 2 of $\mathcal{C}_G$. Each vertex only contained in end-cliques is simplicial.

Lemma 4.1 *Let f be an NT α-convex labelingof G, $Q \neq Q' \in \mathcal{V}_G$ and $V(Q) \cap V(Q') \neq \emptyset$. If Q and Q' are not both end-cliques, then $f(V(Q) \cap V(Q'))$ contains only one label.*

Proof. By contradiction, assume that there are two maximal cliques Q, Q' of G that are not both end-cliques, such that $f(V(Q) \cap V(Q'))$ contains at least two different labels. Assume Q' is a mid-clique. Then by Theorem 3.6, let Q'' be another maximal clique adjacent to Q' in $\mathcal{C}_G$. Now, choose vertices $x, y \in V(Q) \cap V(Q')$ with $f(x) \neq f(y)$, and $z \in V(Q') \cap V(Q'')$, $u \in V(Q) \setminus V(Q')$, $v \in V(Q'') \setminus V(Q')$. The subgraph induced by vertices x, y, z, u, v is illustrated in Figure 5(a). By Lemma 2.1, we may let $f(u) = 0$. Then $\{f(x), f(y)\} = \{\alpha f(z), (1-\alpha)f(z)\}$ by considering two triples (u, x, z) and (u, y, z). Without loss of generality, let $f(x) = \alpha f(z)$ and $f(y) = (1-\alpha)f(z)$. For triple (x, z, v), two possible α-convex combinations imply $f(v) \in \{(1+\alpha)f(z), \frac{\alpha^2-\alpha+1}{\alpha}f(z)\}$. From triple (y, z, v), we have $f(v) \in \{(2-\alpha)f(z), \frac{\alpha^2-\alpha+1}{1-\alpha}f(z)\}$. Therefore, $\{(1+\alpha)f(z), \frac{\alpha^2-\alpha+1}{\alpha}f(z)\} \cap \{(2-\alpha)f(z), \frac{\alpha^2-\alpha+1}{1-\alpha}f(z)\} \neq \phi$. This forces $\alpha = \frac{1}{2}$, from which it follows that $f(x) = f(y) = \frac{1}{2}f(z)$, contradicting our assumption. ∎

Lemma 4.2 *Let G be a non-complete connected graph and f an NT convex labeling. If $diam(G) \geq 3$, then $diam(G) + s \leq \nu(f) \leq diam(G) + s + 2$*

for some integer $s \geq 1$; if diam$(G) = 2$, then diam$(G) + 1 \leq \nu(f) \leq$ diam$(G) + 2$.

Proof. Suppose diam$(G) \geq 3$. By Theorem 3.7, if G is isomorphic to $P(K_{m_1}, K_{m_2}, \cdots, K_{m_n})$ for some integers $n \geq 3, m_1, m_2, \cdots, m_n \geq 0$, then applying Lemma 2.5, $\nu(f) \geq n = \text{diam}(G) + 1$ since G contains an induced path of length diam(G). On the other hand, since diam$(G) \geq 3$, $|\mathcal{V}_G| \geq 3$. So there is at least one mid-clique in G. Lemma 4.1 and Lemma 3.5(3) imply that for each mid-clique $Q \in \mathcal{V}_G$, $|f(V(Q))| = 2$ and, for each end-clique $Q \in \mathcal{V}_G$, $2 \leq |f(V(Q))| \leq 3$. Thus $\nu(f) \leq n + 2 = \text{diam}(G) + 3$. Hence diam$(G)+s \leq \nu(f) \leq$ diam$(G)+s+2$ for $s = 1$. Similarly, if G is isomorphic to the complete substitution of some graph in $\mathcal{P}_t^{\triangle}(v_1, v_2, \cdots, v_n)$ for some integers $t \geq 1$ and $n \geq 3t+2$, then $\nu(f) \geq n+t = \text{diam}(G)+t+1$ since G contains an induced subgraph isomorphic to some graph in $\mathcal{P}_t^{\triangle}(v_1, v_2, \cdots, v_n)$. On the other hand, note that, among all mid-cliques of G, there are t cliques each labeled with three real numbers by f and others each labeled with two numbers by f, while for each Q of two end-cliques, $2 \leq |f(V(Q))| \leq 3$. It follows that $\nu(f) \leq n + 2 = \text{diam}(G) + t + 3$. Thus diam$(G) + s \leq \nu(f) \leq$ diam$(G) + s + 2$ for $s = t + 1 \geq 2$.

If diam$(G) = 2$, then G has only two maximal cliques, that is, two end-cliques. Let Q and Q' be such two end-cliques. Let $A_1 = V(Q) \setminus V(Q')$, $A_2 = V(Q) \cap V(Q')$, $A_3 = V(Q') \setminus V(Q)$. Then $|f(A_i)| \leq 2$ for each $i \in \{1, 2, 3\}$. To prove $\nu(f) \leq \text{diam}(G) + 2$, it suffices to show that there is at most one i, $i \in \{1, 2, 3\}$, such that $|f(A_i)| = 2$. Let f be α-convex for some $0 < \alpha < 1$.

If $|f(A_1)| = |f(A_2)| = 2$ (the case of $|f(A_2)| = |f(A_3)| = 2$ is similar), then there are $x_1, x_2 \in A_1$ and $y_1, y_2 \in A_2$ such that $f(x_1) \neq f(x_2)$ and $f(y_1) \neq f(y_2)$. Let $z \in A_3$. The subgraph induced by vertices x_1, x_2, y_1, y_2, z is demonstrated in Figure 5(b). Clearly, the labels of vertices x_1, x_2, y_1, y_2, z are pairwise different. We may let $f(z) = 0$ by Lemma 2.1. Then $\{f(y_1), f(y_2)\} = \{\alpha f(x_1), (1-\alpha) f(x_1)\} = \{\alpha f(x_2), (1-\alpha) f(x_2)\}$. Thus $\alpha f(x_1) = (1-\alpha) f(x_2)$ and $\alpha f(x_2) = (1-\alpha) f(x_1)$. This must imply that $\alpha = \frac{1}{2}$ and then $f(x_1) = f(x_2)$, a contradiction.

If $|f(A_1)| = |f(A_3)| = 2$, then there are $x_1, x_2 \in A_1$ and $y_1, y_2 \in A_3$ such that $f(x_1) \neq f(x_2)$ and $f(y_1) \neq f(y_2)$. Let $z \in A_2$. The subgraph induced by vertices x_1, x_2, y_1, y_2, z is demonstrated in Figure 5(c). Obviously, these five vertices have pairwise distinct labels. We may also assume $f(z) = 0$. Then $\{f(y_1), f(y_2)\} = \{\frac{1-\alpha}{\alpha} f(x_1), \frac{\alpha}{1-\alpha} f(x_1)\} = \{\frac{1-\alpha}{\alpha} f(x_2), \frac{\alpha}{1-\alpha} f(x_2)\}$. Since $f(x_1) \neq f(x_2)$, $\frac{1-\alpha}{\alpha} f(x_1) = \frac{\alpha}{1-\alpha} f(x_2)$ and $\frac{\alpha}{1-\alpha} f(x_1) = \frac{1-\alpha}{\alpha} f(x_2)$. It follows that $\alpha = \frac{1}{2}$ and $f(y_1) = f(y_2)$, which contradicts our assumption.
∎

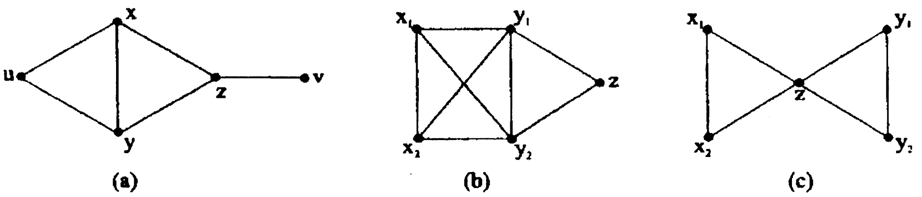

Figure 5. Demonstrations for the proofs of Lemma 4.1 and Lemma 4.2.

The following theorem follows by Lemma 4.2 and the proofs of sufficiency in Theorem 3.7 immediately.

Theorem 4.3 *For any non-complete connected graph G, $\lambda_c(G) = \infty$ or $\lambda_c(G) = diam(G) + s$ for some integer $s \geq 1$.*

Now we present characterizations of NT convex labelings of G. Suppose f is an NT α-convex labeling of G with $f(V(G)) = \{r_1, r_2, \cdots, r_m\}$, $r_1 < r_2 < \cdots < r_m$. If $\mathrm{diam}(G) = 2$, then G is isomorphic to a complete substitution of a path of length 2. Thus by Lemma 4.2, it is easily seen that $\{r_1, r_2, \cdots, r_m\}$ is an α-convex sequence of length 3 (in this case, $m = \mathrm{diam}(G)+1 = 3$), or $\{r_1, r_2, \cdots, r_m\} \backslash \{r_p\}$ is an α-convex sequence of length 3 for $p = 1, 2$, or $p = 2, 3$, or $p = 3, 4$ (in this case, $m = \mathrm{diam}(G) + 2 = 4$). For general cases $\mathrm{diam}(G) \geq 3$, we have the following theorem.

Theorem 4.4 *Let G be a non-complete connected graph with diameter greater than 2 and f an NT α-convex labeling, $0 < \alpha < 1$. Suppose $f(V(G)) = \{r_1, r_2, \cdots, r_m\}$ with $r_1 < r_2 < \cdots < r_m$. Then there must exist $t \geq 0$ distinct labels $r_{i_1}, r_{i_2}, \cdots, r_{i_t}$, where $3 \leq i_1 < i_2 < \cdots < i_t \leq m-2$ and $i_{j+1} - i_j \geq 3$, $j = 1, 2, \cdots, t-1$, the following hold:*
(1) $\{r_1, r_2, \cdots, r_m\} \backslash \{r_{i_1}, r_{i_2}, \cdots, r_{i_t}\}$ is an α-convex sequence, or $\{r_1, \cdots, r_m\} \backslash \{r_{i_1}, \cdots, r_{i_t}, r_p\}$ is an α-convex sequence for $p = 1, 2$ or $p = m-1, m$, or $\{r_1, r_2, \cdots, r_m\} \setminus \{r_{i_1}, r_{i_2}, \cdots, r_{i_t}, r_p, r_q\}$ is an α-convex sequence for $p = 1, 2$ and $q = m-1, m$.
(2) For each j, $1 \leq j \leq t-1$, $\{r_{i_j - 1}, r_{i_j}, \cdots, r_{i_{j+1}}, r_{i_{j+1}+1}\} \setminus \{r_p, r_q\}$ for $p = i_j - 1, i_j$ and $q = i_{j+1}, i_{j+1}+1$, is an α-convex sequence.
(3) $\{r_1, \cdots, r_{i_1}, r_{i_1+1}\} \setminus \{r_q\}$ is an α-convex sequence for $q = i_1, i_1+1$, or $\{r_1, \cdots, r_{i_1}, r_{i_1+1}\} \setminus \{r_p, r_q\}$ is an α-convex sequence for $p = 1, 2$ and $q = i_1, i_1+1$.
(4) $\{r_{i_t - 1}, r_{i_t}, \cdots, r_m\} \setminus \{r_p\}$ is an α-convex sequence for $p = i_t - 1, i_t$, or $\{r_{i_t-1}, r_{i_t}, \cdots, r_m\} \setminus \{r_p, r_q\}$ is an α-convex sequence for $p = i_t - 1, i_t$ and $q = m-1, m$.

Proof. By Theorem 3.6, we let $C_G = P(Q_1, Q_2, \cdots, Q_k)$, where $k = \text{diam}(G) \geq 3$. Lemma 2.5 implies that $r_1 \in f(V(Q_1))$ and $r_m \in f(V(Q_k))$ or $r_1 \in f(V(Q_k))$ and $r_m \in f(V(Q_1))$. Without loss of generality, assume that $r_1 \in f(V(Q_1))$ and $r_m \in f(V(Q_k))$. By Lemma 4.2, $\text{diam}(G) + s \leq m \leq \text{diam}(G) + s + 2$ for some integer $s \geq 1$. Furthermore, if G is isomorphic to a complete substitution of a path of length diam(G), then $\text{diam}(G) + 1 \leq m \leq \text{diam}(G) + 3$. Note that, each shortest path between two simplicial vertices in two end-cliques Q_1 and Q_k respectively, is an induced path of $\text{diam}(G) + 1$ vertices. If $m = \text{diam}(G) + 1$, then $|f(V(Q_i))| = 2$ for $i = 1, 2, \cdots, k$. Thus, all vertices of such an induced path must be assigned labels $r_1, r_2, \cdots, r_m$ in order from one end-vertex to the other end-vertex, respectively. Hence, $\{r_1, r_2, \cdots, r_m\}$ is an α-convex sequence. If $m = \text{diam}(G) + 2$, then $|f(V(Q_1))| = 3$ or $|f(V(Q_k))| = 3$, and $|f(V(Q_i))| = 2$ for $i \neq 1, k$. If $f(V(Q_1 \setminus Q_2)) = \{r_1, r_2\}$ or $f(V(Q_k \setminus Q_{k-1})) = \{r_{m-1}, r_m\}$, then for $p = 1, 2$ or $p = m-1, m$, $\{r_1, r_2, \cdots, r_m\} \setminus \{r_p\}$ is an α-convex sequence since it must be the label set of an induced path from a simplicial vertex in Q_1 to a simplicial vertex in Q_k. If $m = \text{diam}(G) + 3$, then $f(V(Q_1 \setminus Q_2)) = \{r_1, r_2\}$, $f(V(Q_k \setminus Q_{k-1})) = \{r_{m-1}, r_m\}$, and $f(V(Q_i) = \{r_{i+1}, r_{i+2}\}$ for $i \neq 1, k$. For any $r_p \in \{r_1, r_2\}$ and $r_q \in \{r_{m-1}, r_m\}$, there must exist an induced path from a simplicial vertex in Q_1 to a simplicial vertex in Q_k whose label set is $\{r_1, r_2, \cdots, r_m\} \setminus \{r_p, r_q\}$, thus $\{r_1, r_2, \cdots, r_m\} \setminus \{r_p, r_q\}$ is an α-convex sequence.

Now suppose that G is isomorphic to a complete substitution of some graph in $\mathcal{P}_t^{\triangle}(v_1, \cdots, v_n)$ for some integer $t \geq 1$ and $n \geq 3t + 2$. Then $\alpha = \sqrt{2} - 1$ or $2 - \sqrt{2}$ and $\text{diam}(G) + t + 1 \leq m \leq \text{diam}(G) + t + 3$. Accordingly, there are exactly t mid-cliques $Q_{i_1}, Q_{i_2}, \cdots, Q_{i_t}$, where $i_{j+1} - i_j \geq 2$, $1 \leq j \leq t-1$, such that $|f(V(Q_{i_j}))| = 3$ for each $1 \leq j \leq t$. So, by Lemma 4.1, we can let $f(V(Q_{i_j} \setminus (Q_{i_j - 1} \cup Q_{i_j + 1}))) = \{r_{i_j}\}$, $j = 1, 2, \cdots, t$. Then $f(V(Q_{i_j} \cap Q_{i_j - 1})) = \{r_{i_j - 1}\}$ and $f(V(Q_{i_j} \cap Q_{i_j + 1})) = \{r_{i_j + 1}\}$, $j = 1, 2, \cdots, t$. Note that, $G' = G \setminus (\cup_{j=1}^{t} (Q_{i_j} \setminus (Q_{i_j - 1} \cup Q_{i_j + 1})))$ is isomorphic to a complete substitution of a path of length diam(G). The restriction $f|_{G'}$ of f on G' is also an α-convex labeling of G' with label set $\{r_1, \cdots, r_m\} \setminus \{r_{i_1}, \cdots, r_{i_t}\}$. Thus $\{r_1, r_2, \cdots, r_m\} \setminus \{r_{i_1}, r_{i_2}, \cdots, r_{i_t}\}$ is an α-convex sequence $m = \text{diam}(G) + t + 1$, or $\{r_1, r_2, \cdots, r_m\} \setminus \{r_{i_1}, r_{i_2}, \cdots, r_{i_t}, r_p\}$ is an α-convex sequence for $p = 1, 2$ or $p = m - 1, m$ if $m = \text{diam}(G) + t + 2$, or $\{r_1, r_2, \cdots, r_m\} \setminus \{r_{i_1}, r_{i_2}, \cdots, r_{i_t}, r_p, r_q\}$ is an α-convex sequence for $p = 1, 2$ and $q = m - 1, m$ $m = \text{diam}(G) + t + 3$. Similarly, since for $j = 1, 2, \cdots, t-1$, each induced subgraph $G_j = \cup_{i=i_j}^{i_{j+1}} Q_i$ is isomorphic to a complete substitution of a path of length $i_{j+1} - i_j + 1$ and the restriction $f|_{G_j}$ of f on G_j is an α-convex labeling of G_j with label set $\{r_{i_j - 1}, r_{i_j}, \cdots, r_{i_{j+1}}, r_{i_{j+1}+1}\}$, $\{r_{i_j - 1}, r_{i_j}, \cdots, r_{i_{j+1}}, r_{i_{j+1}+1}\} \setminus \{r_p, r_q\}$ for $p = i_j - 1, i_j$ and $q = i_{j+1}, i_{j+1} + 1$, is an α-convex sequence.

Finally, let $G_0 = \cup_{i=1}^{i_1} Q_i$ and $G_{t+1} = \cup_{i=i_t}^{k} Q_i$. G_0 is isomorphic to a complete substitution of a path of length i_1 and G_{t+1} is isomorphic to a complete substitution of a path of length $k - i_t + 1$. $f(V(G_0)) = \{r_1, \cdots, r_{i_1}, r_{i_1+1}\}$ and $f(V(G_{t+1})) = \{r_{i_t-1}, r_{i_t}, \cdots, r_m\}$. If $m = \mathrm{diam}(G)+t+1$, then $\{r_1, \cdots, r_{i_1}, r_{i_1+1}\}\setminus\{r_q\}$ for $q = i_1, i_1+1$ and $\{r_{i_t-1}, r_{i_t}, \cdots, r_m\}\setminus \{r_p\}$ for $p = i_t - 1, i_t$ are α-convex sequences. If $m = \mathrm{diam}(G) + t + 2$, then $\{r_1, \cdots, r_{i_1}, r_{i_1+1}\}\setminus\{r_p, r_q\}$ for $p = 1, 2$, $q = i_1, i_1+1$, or $\{r_{i_t-1}, r_{i_t}, \cdots, r_m\}\setminus \{r_p\}$ for $p = i_t - 1, i_t$, $q = m - 1, m$, is an α-convex sequence. If $m = \mathrm{diam}(G)+t+3$, then $\{r_1, \cdots, r_{i_1}, r_{i_1+1}\} \setminus \{r_p, r_q\}$ for $p = 1, 2$, $q = i_1, i_1+1$, and $\{r_{i_t-1}, r_{i_t}, \cdots, r_m\} \setminus \{r_p\}$ for $p = i_t - 1, i_t$, $q = m - 1, m$, are α-convex sequences. ∎

Corollary 4.5 *Assume* $0 < \alpha < 1$. *Let* $n, m_1, \cdots, m_n$ *be positive integers and let* $\{a_1, \cdots, a_n\}$ *be a set of* n *real numbers with* $a_1 < \cdots < a_n$. *If* $\{a_1, \cdots, a_n\}$ *satisfies one of the following conditions:*
(1) $\{a_1, a_2, \cdots, a_n\}$ *is an* α*-convex sequence, or*
(2) $\{a_1, a_2, \cdots, a_n\} \setminus \{a_i\}$ *is an* α*-convex sequence for each* $i = 1, 2$ *or* $i = n - 1, n$ *, or*
(3) $\{a_1, a_2, \cdots, a_n\} \setminus \{a_i, a_j\}$ *is an* α*-convex sequence for any* $i = 1, 2$ *and* $j = n - 1, n$,
then there exists a non-complete connected graph G *of* $\sum_{i=1}^{n} m_i$ *vertices with* α*-convex labeling* f *such that* m_i *vertices are labeled by the real number* a_i, $i \in \{1, 2, \cdots, n\}$. *Such graph* G *is uniquely determined up to the isomorphism.*

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