GROUP-ANTIMAGIC LABELINGS OF GRAPHS

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ABSTRACT. Let A be a non-trivial abelian group. A connected simple graph $G = (V, E)$ is A-**antimagic** if there exists an edge labeling $f : E(G) \to A\backslash\{0\}$ such that the induced vertex labeling $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma\ \{f(u,v) : (u,v) \in E(G)\}$, is a one-to-one map. In this paper, we analyze the group-antimagic property for various classes of graphs.

1. INTRODUCTION

Let G be a connected simple graph. For any non-trivial abelian group A (written additively), let $A^* = A\backslash\{0\}$, where 0 is the additive identity of A (sometimes denoted by 0_A). Let a function $f : E(G) \to A^*$ be an edge labeling of G. Any such labeling induces a map $f^+ : V(G) \to A$, defined by $f^+(v) = \sum_{uv \in E(G)} f(uv)$. If there exists an edge labeling f whose induced map f^+ on $V(G)$ is one-to-one, we say that f is an *A-antimagic labeling* and that G is an *A-antimagic* graph. The *integer-antimagic spectrum* of a graph G is the set $\mathrm{IAM}(G) = \{k : G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$.

The concept of the A-antimagicness property for a graph G naturally arises as a variation of the A-magic labeling problem (where the induced vertex labeling is a constant map). $\mathbb{Z}$-magic (or $\mathbb{Z}_1$-magic) graphs were considered by Stanley [28, 29], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1, 2, 3] and others [7, 9, 15, 16, 25] have studied A-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [4, 6, 8, 10, 11, 12, 13, 14, 17, 18, 19, 20, 26].

2. SOME ALGEBRAIC PROPERTIES OF GROUP-ANTIMAGIC GRAPHS

In this section, we will use the following notation. Let $[G, A]$ denote the class of distinct A-antimagic labelings of G. Note that G is A-antimagic if and only if $[G, A] \neq \varnothing$. For any commutative ring R with unity, $U(R)$ denotes the multiplicative group of units in R.

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Here, we begin to develop an algebraic framework from which group-antimagic graphs can be analyzed.

Theorem 1. *Let A be a non-trivial abelian group, underlying some commutative ring R with unity. If $d \in U(R)$ and $f \in [G, A]$, then $df \in [G, A]$.*

Proof. Suppose that f is an A-antimagic labeling of G. Consider an arbitrary vertex v (having label x under f). Let $|E_i|$ denote the number of edges labeled a_i, which are adjacent to v. Then, $x = \Sigma(a_i|E_i|)$; where $a_i \in A^*$. Let us examine what effect df has on the labeling of v. By multiplying every edge adjacent to v by d, we get the following relationship: $dx = d\Sigma(a_i|E_i|)$. The new induced labeling on v is dx. Also, since $d \in U(R)$, each edge adjacent to v in this new labeling is not equal to 0_A. Furthermore, the map $\mu_d : A \to A$ defined by $a \mapsto da$ is one-to-one. Thus, df induces a vertex labeling which is one-to-one. Hence, df is an A-antimagic labeling of G. □

Corollary 1. *If $d \in U(\mathbb{Z}_k)$ and $f \in [G, \mathbb{Z}_k]$, then $df \in [G, \mathbb{Z}_k]$.*

Proof. Let $A = \mathbb{Z}_n$, the group of integers, modulo n. Now, apply Theorem 1. □

It should be noted that in Theorem 1 and Corollary 1, f and df might yield the same group-antimagic labeling on G.

Theorem 2. *Let A_1 be an abelian group which contains a subgroup isomorphic to A_2. If graph G is A_2-antimagic, then G is A_1-antimagic.*

Proof. Let $H \leq A_1$. Suppose that $f \in [G, A_2]$ and that $\phi : A_2 \to H$ is a group isomorphism. Now, let f induce the label x on a vertex v of G. Let $|E_i|$ denote the number of edges labeled a_i, which are adjacent to v. Then, $x = \Sigma(a_i|E_i|)$; where a_i varies through all the elements of $A_2{}^*$. Now, apply ϕ to the edges which are adjacent to v. Under this new labeling, we get the following relationship: $\phi(x) = \phi[\Sigma(a_i|E_i|)] = \Sigma\phi(a_i)|E_i|$. Since $a_i \neq 0_{A_2}$ and ϕ is a group isomorphism, no edge is labeled 0_{A_1}. The new induced labeling on v is $\phi(x)$. Hence, we have an A_1-magic labeling of G. □

Corollary 2. *Let G be a $\mathbb{Z}_k$-antimagic graph, with $k|n$. Then, G is a $\mathbb{Z}_n$-antimagic graph.*

The reader should observe that the converse of Corollary 2 is not true, for $k \geq |G|$. For example, Figure 1 gives a $\mathbb{Z}_8$-antimagic labeling of $K_{1,3}$. However, it is clear that $K_{1,3}$ is not $\mathbb{Z}_4$-antimagic (as the edges would have to be labeled 1, 2 and 3).

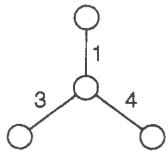

FIGURE 1. $\mathbb{Z}_8$-antimagic labeling of $K_{1,3}$.

3. $\mathbb{Z}_k$-ANTIMAGIC LABELINGS FOR SOME CLASSES OF GRAPHS

Lemma 1. *A graph of order $4m+2$, for all $m \in \mathbb{N}$, is not $\mathbb{Z}_{4m+2}$-antimagic.*

Proof. Let G be a graph of order $4m+2$, and let f and f^+ be a function from $E(G)$ to $\mathbb{Z}^*_{4m+2}$ and the induced map of f from $V(G)$ to $\mathbb{Z}_{4m+2}$, respectively. If f is an $\mathbb{Z}_{4m+2}$-antimagic labeling, then

$$2 \cdot \left[\sum_{e \in E(G)} f(e) \right] \equiv \sum_{v \in V(G)} f^+(v) \equiv \sum_{j=0}^{4m+1} j \equiv 2m+1 \pmod{4m+2},$$

which is impossible. $\square$

Theorem 3. *P_3 is $\mathbb{Z}_k$-antimagic, for all $k \geq 3$, and C_3 is not $\mathbb{Z}_3$-antimagic, but $\mathbb{Z}_k$-antimagic, for all $k \geq 4$.*

Proof. For P_3, label the edges 1 and 2. For C_3, label the edges 1, 2 and 3. C_3 is not $\mathbb{Z}_3$-antimagic because all labels of the three edges must be distinct. $\square$

Theorem 4. *P_{4m+r} and C_{4m+r}, for all $m \in \mathbb{N}$, are $\mathbb{Z}_k$-antimagic, for all $k \geq 4m+r$ if $r = 0, 1, 3$. P_{4m+2} and C_{4m+2}, for all $m \in \mathbb{N}$, are $\mathbb{Z}_k$-antimagic, for all $k \geq 4m+3$.*

Proof. Let $e_1, e_2, \ldots, e_{n-1}$ be edges of P_n, from left to right. A $\mathbb{Z}_k$-antimagic labeling of P_n can be obtained as follows.

Case 1 $n = 4m$:

$$f(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd;} \\ \frac{i}{2} & \text{if } i \text{ is even and } 2 \leq i \leq 2m-2; \\ \frac{i+2}{2} & \text{if } i \text{ is even and } 2m \leq i \leq 4m-2. \end{cases}$$

Case 2 $n = 4m+1$:

$$f(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even;} \\ \frac{i+3}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq 2m-3; \\ \frac{i+5}{2} & \text{if } i \text{ is odd and } 2m-1 \leq i \leq 4m-1. \end{cases}$$

Case 3 $n = 4m+2$:

$$f(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd;} \\ \frac{i+2}{2} & \text{if } i \text{ is even and } 2 \leq i \leq 2m-2; \\ \frac{i+4}{2} & \text{if } i \text{ is even and } 2m \leq i \leq 4m. \end{cases}$$

Case 4 $n = 4m+3$:

$$f(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even;} \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } 1 \le i \le 2m-1; \\ \frac{i+3}{2} & \text{if } i \text{ is odd and } 2m+1 \le i \le 4m+1. \end{cases}$$

Let $e_1, e_2, \ldots, e_n$ be edges of C_n arranged in counter-clockwise direction. A $\mathbb{Z}_k$-antimagic labeling of C_n can be obtained as follows.

Case 1 $n = 4m$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \le i \le 2m; \\ 3 + 2(2m - \lceil \frac{i}{2} \rceil) & \text{if } 2m+1 \le i \le 4m. \end{cases}$$

Case 2 $n = 4m+1$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \le i \le 2m; \\ 3 + 2(2m - \lceil \frac{i}{2} \rceil) & \text{if } 2m+1 \le i \le 4m+1. \end{cases}$$

Case 3 $n = 4m+2$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \le i \le 2m+3; \\ 3 + 2(2m - \lceil \frac{i-2}{2} \rceil) & \text{if } 2m+4 \le i \le 4m+2. \end{cases}$$

Case 4 $n = 4m+3$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \le i \le 2m+3; \\ 3 + 2(2m - \lceil \frac{i-3}{2} \rceil) & \text{if } 2m+4 \le i \le 4m+3. \end{cases}$$

□

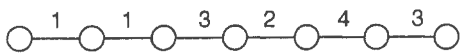

FIGURE 2. $\mathbb{Z}_k$-antimagic labeling of P_7, for $k \ge 7$.

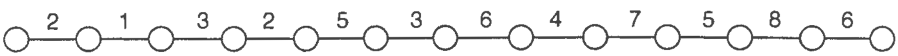

FIGURE 3. $\mathbb{Z}_k$-antimagic labeling of P_{13}, for $k \ge 13$.

Theorem 5. *Let G be a regular Hamiltonian graph of order $4m+r$, $m \in \mathbb{N}$. G is $\mathbb{Z}_k$-antimagic, for all $k \ge 4m+r$ if $r = 0, 1, 3$, and G is $\mathbb{Z}_k$-antimagic, for all $k \ge 4m+3$ if $r = 2$.*

Proof. Let G be a regular Hamiltonian graph of order $4m+r$, and C be a Hamiltonian cycle of G. A group-antimagic labeling of G can be obtained by labeling the edges of C, using the method described in the proof of Theorem 4, and labeling all other edges of G with 1. □

Corollary 3. *All complete graphs and regular complete n-partite graphs of order $4m+r$ ($m \in \mathbb{N}$) are $\mathbb{Z}_k$-antimagic, for all $k \ge 4m+r$ if $r = 0, 1, 3$, and are $\mathbb{Z}_k$-antimagic, for all $k \ge 4m+3$ if $r = 2$.*

Proof. All complete graphs and regular complete n-partite graphs are regular and Hamiltonian. Thus by Theorem 5, the result follows immediately. $\square$

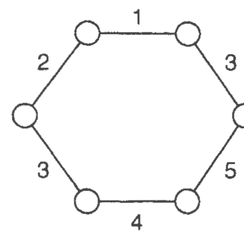

FIGURE 4. $\mathbb{Z}_k$-antimagic labeling of C_6, for $k \geq 7$.

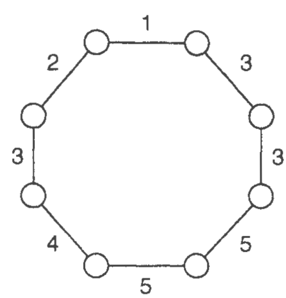

FIGURE 5. $\mathbb{Z}_k$-antimagic labeling of C_8, for $k \geq 8$.

Lemma 2. *Let $N_k = \{1, 2, \ldots, k-1\}$, where $k \geq 3$. Then, there exist r $(2 \leq r < k)$ distinct integers x_i in N_k, with $A_r = \sum_{i=1}^{r} x_i \equiv 0 \pmod{k}$ $\Longleftrightarrow$ (1). k is odd and $r \neq k-2$ OR (2). k is even and $r \neq k-1$.*

Proof. Note that $A_{k-1} = \sum_{i=1}^{k-1} x_i = \frac{k(k-1)}{2}$ is divisible by k if and only if k is odd. Clearly, if k is odd, then $A_{k-2} = \frac{k(k-1)}{2} - x$ (for every $x \in N_k$) is not divisible by k. If k is even, then the $k-2$ distinct terms from $N_k \setminus \{\frac{k}{2}\}$ add up to $\frac{k(k-1)}{2} - \frac{k}{2} = \frac{k(k-2)}{2}$, which is divisible by k. Finally, note that for all $k \geq 5$ $(2 \leq r \leq k-3)$, the sum of the r (r even) distinct terms $1, 2, \ldots, \frac{r}{2}, k - \frac{r}{2}, \ldots, k-2, k-1$ is divisible by k. For all $k \geq 5$ $(2 \leq r \leq k-3)$, the sum of the r (r odd) distinct terms $1, 2, \ldots, \frac{r-1}{2}, \lfloor \frac{k}{2} \rfloor - 1, \lceil \frac{k}{2} \rceil, k - \frac{r-1}{2}, \ldots, k-2$ is divisible by k. $\square$

It follows from Lemma 2 that, given integers r and k with $2 \leq r < k$,

(1) if r is even, there exist distinct integers $x_1, x_2, \ldots, x_r$ in N_k such that $k | \sum_{i=1}^{r} x_i$.

(2) if r is odd, then there exist distinct integers $x_1, x_2, \ldots, x_r$ in N_k such $k | \sum_{i=1}^{r} x_i \iff r \leq k-3$.

Theorem 6. *Let $n \geq 4$ and S_n denote the star graph having $n-1$ leaves. If n is odd, then S_n is $\mathbb{Z}_k$-antimagic, for all $k \geq n$. Otherwise, S_n is $\mathbb{Z}_k$-antimagic, for all $k \geq n+2$; but not $\mathbb{Z}_n$-antimagic nor $\mathbb{Z}_{n+1}$-antimagic.*

Proof. (i). n is odd: Then, $r = n-1$ is even. By Comment (1) following Lemma 2, there exist distinct integers $x_1, x_2, \ldots, x_{n-1} \in \mathbb{Z}_k^*$ (for any $k \geq n$) such that $\sum_{i=1}^{n-1} x_i \equiv 0 \pmod{k}$. Labeling the edges of S_n with $x_1, x_2, \ldots, x_{n-1}$ gives a $\mathbb{Z}_k$-antimagic labeling of S_n, for all $k \geq n$.

(ii). n is even: Then, $r = n-1$ is odd. By Comment (2) following Lemma 2, there exist distinct integers $x_1, x_2, \ldots, x_{n-1} \in \mathbb{Z}_k^*$ such that $\sum_{i=1}^{n-1} x_i \equiv 0 \pmod{k} \iff n-1 \leq k-3 \iff k \geq n+2$. In these cases, labeling the edges of S_n with $x_1, x_2, \ldots, x_{n-1}$ gives a $\mathbb{Z}_k$-antimagic labeling of S_n, for $k \geq n+2$.

Finally, we show that if n is even, then S_n is not $\mathbb{Z}_n$-antimagic nor $\mathbb{Z}_{n+1}$-antimagic. If S_n were $\mathbb{Z}_n$-antimagic, then the central vertex v_0 of S_n (under the induced vertex map) would be labeled $f^+(v_0) = \sum_{x_i \in \mathbb{Z}_n^*} x_i = \frac{n(n-1)}{2} \not\equiv 0 \pmod{n}$ (since n is even). Thus, $f^+(v_0) = f^+(v_j)$, for some leaf v_j of S_n, hence giving us a contradiction. Now, if S_n were $\mathbb{Z}_{n+1}$-antimagic, the central vertex v_0 of S_n (under the induced vertex map) would be labeled $f^+(v_0) = (\sum_{i=1}^{n} i) - x \pmod{n+1}$, where x is the only element in $\mathbb{Z}_{n+1}^*$ not assigned to an edge of S_n. Since n is even, $(\sum_{i=1}^{n} i) - x = \frac{n(n+1)}{2} - x \equiv -x \not\equiv x \pmod{n+1}$, as $n+1$ is odd. Hence, $f^+(v_0) = f^+(v_j)$, for some leaf v_j of S_n, thus giving us a contradiction. $\square$

Theorem 7. *Let T be a tree of order n, having exactly one vertex of even degree. Then,* $\mathrm{IAM}(T) = \{k : k \geq n\}$.

Proof. Let T be a tree of order n with a unique vertex w of even degree. We now view T as a rooted tree with w being the root. Thus, every vertex of T is either a leaf or has an even number of children. Note that $n = 2m+1$, for some $m \in \mathbb{N}$. With the exception of w, all of the vertices of T can be grouped into m pairs of brothers $\{u_i, v_i\}$, for $i = 1, 2, \ldots, m$. Now, take $k \geq n$. Let w_i be the parent of $\{u_i, v_i\}$, for $i = 1, 2, \ldots, m$. Label the edges $u_i w_i$ and $v_i w_i$ with i and $k-i$, respectively. Then, the induced vertex labeling on u_i and v_i are i and $-i \pmod{k}$, respectively, for $i = 1, 2, \ldots, m$. Furthermore, the induced vertex labeling on w is $0 \pmod{k}$. Thus, T is $\mathbb{Z}_k$-antimagic, for all $k \geq n$. $\square$

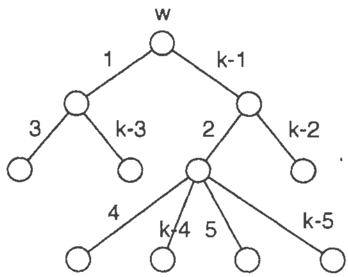

FIGURE 6. $\mathbb{Z}_k$-antimagic labeling of a tree with one vertex of even degree, for $k \geq 11$.

Definition. Let $m \geq 2$. A rooted tree T is *full m-ary* if every vertex of T is either a leaf or has exactly m children.

Corollary 4. *All full $2r$-ary trees of order n are $\mathbb{Z}_k$-antimagic, for all $k \geq n$.*

Proof. In a full $2r$-ary tree, there is exactly one vertex of even degree. Thus, the claim follows immediately from Theorem 7. □

Theorem 8. *Let T be a tree of order n, having exactly two vertices of even degree. Then, T is $\mathbb{Z}_k$-antimagic, for all $k \geq n+1$.*

Proof. Let T be a tree of order n with even-degree vertices v and w. Since the number of odd vertices must be even, $n = 2m$ for some $m \in \mathbb{N}$. Viewing T as a rooted tree (with root w), we see that v has an odd number of child(ren) while each of the vertices in $V(T) \setminus \{v\}$ is either a leaf or has an even number of children. Let v_0 be a particular son of v. Then, vertices in $V(T) \setminus \{w, v_0\}$ can be grouped into $m-1$ pairs of brothers $\{u_i, v_i\}$, for $i = 1, 2, \ldots, m-1$. Now, take $k \geq n+1$. Let w_i be the parent of $\{u_i, v_i\}$, for $i = 1, 2, \ldots, m-1$. Without loss of generality, set $v_1 = v$. Label the edges $u_i w_i$ and $v_i w_i$ with i and $k-i$, respectively, and label vv_0 with $\lceil \frac{k}{2} \rceil$. Then, the induced vertex labelings on u_i and v_i are i and $-i$ (mod k), respectively, for $i = 1, 2, \ldots, m-1$. Furthermore, the induced vertex labelings on u_1, v, v_0 and w are 1, $\lceil \frac{k}{2} \rceil - 1$, $\lceil \frac{k}{2} \rceil$ and 0 (mod k), respectively. Thus, T is $\mathbb{Z}_k$-antimagic, for all $k \geq n+1$. □

Definition. A tree is called a *double-star* if it has exactly 2 non-pendant vertices. Let x and y be the 2 non-pendant vertices of a double-star. We denote the double-star $S_{r,s}$, where r and s are the degrees of x and y respectively. x and y are called *centers* of $S_{r,s}$.

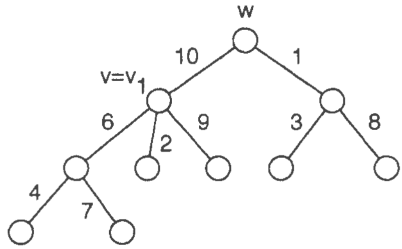

FIGURE 7. $\mathbb{Z}_{11}$-antimagic labeling of a tree with two vertices of even degree.

Theorem 9. *Let $S_{r,s}$ be a double-star of order n, where $r \le s$. If $n \equiv 2 \pmod 4$, then* $\mathrm{IAM}(S_{r,s}) = \{k : k \ge n+1\}$. *Otherwise,* $\mathrm{IAM}(S_{r,s}) = \{k : k \ge n\}$.

Proof. Let $S_{r,s}$ be a double-star of order n, $r \le s$, and having centers x and y. Note that $r + s = n$. Let $\{x_1, \dots, x_{r-1}\}$ and $\{y_1, \dots, y_{s-1}\}$ be the two sets of leaves adjacent to x and y, respectively.

Case 1 $n \equiv 1$ or $3 \pmod 4$:
Here, exactly one vertex (x or y) in $S_{r,s}$ is of even degree. Thus, by Theorem 7, we see that $\mathrm{IAM}(S_{r,s}) = \{k : k \ge n\}$.

Case 2 $n \equiv 2 \pmod 4$ and both r and s are even:
By Lemma 1, $S_{r,s}$ is not $\mathbb{Z}_n$-antimagic. Since r and s are both even, we see that the centers x and y are the only vertices of even degree in $S_{r,s}$. Thus by Theorem 8, we see that $\mathrm{IAM}(S_{r,s}) = \{k : k \ge n+1\}$.

Case 3 $n \equiv 2 \pmod 4$ and both r and s are odd:
By Lemma 1, $S_{r,s}$ is not $\mathbb{Z}_n$-antimagic. Now, let $k \ge n+1$.
If $n = 6$, then label xy with 1, $\{xx_1, xx_2\}$ with $\{1, 5\}$, and $\{yy_1, yy_2\}$ with $\{2, 3\}$.
If $n \ge 10$, then label

(a) xy with 1;
(b) $\{xx_1, xx_2, \dots, xx_{r-1}\}$ with $\{1, k-2\} \cup \{3, 4, \dots, \frac{r+1}{2}\} \cup \{k - \frac{r+1}{2}, k - \frac{r-1}{2}, \dots, k-3\}$;
(c) $\{yy_1, yy_2, \dots, yy_{s-1}\}$ with $\{2, \frac{n}{2}-1, \frac{n}{2}, k-\frac{n}{2}\} \cup \{\frac{r+3}{2}, \frac{r+5}{2}, \dots, \frac{n}{2} - 2\} \cup \{k - (\frac{n}{2}-2), k - (\frac{n}{2}-3), \dots, k - \frac{r+3}{2}\}$.

Case 4 $n \equiv 0 \pmod 4$ and both r and s are even:
Let $k \ge n$. Label

(a) xy with $\frac{n}{2}$;
(b) $\{xx_1, xx_2, \dots, xx_{r-1}\}$ with $\{k - \frac{n}{2}\} \cup \{1, 2, \dots, \frac{r}{2} - 1\} \cup \{k - (\frac{r}{2}-1), k - (\frac{r}{2}-2), \dots, k-1\}$;

(c) $\{yy_1, yy_2, \ldots, yy_{s-1}\}$ with $(\{\frac{r}{2}, \frac{r}{2}+1, \ldots, \frac{n}{2}-1\} \cup \{k-(\frac{n}{2}-1), k-(\frac{n}{2}-2), \ldots, k-\frac{r}{2}\}) \setminus \{\frac{n}{4}\}$.

Case 5 $n \equiv 0 \pmod 4$ and both r and s are odd:

Let $k \geq n$. Label

(a) xy with 1;

(b) $\{xx_1, xx_2, \ldots, xx_{r-1}\}$ with $\{1, k-2\} \cup \{3, 4, \ldots, \frac{r+1}{2}\} \cup \{k-\frac{r+1}{2}, k-\frac{r-1}{2}, \ldots, k-3\}$;

(c) $\{yy_1, yy_2, \ldots, yy_{s-1}\}$ with $(\{2, \frac{n}{2}, k-1\} \cup \{\frac{r+3}{2}, \frac{r+5}{2}, \ldots, \frac{n}{2}-1\} \cup \{k-(\frac{n}{2}-1), k-(\frac{n}{2}-2), \ldots, k-\frac{r+3}{2}\}) \setminus \{\frac{n}{4}+1\}$.

□

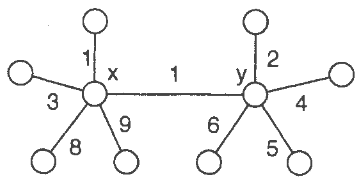

FIGURE 8. $\mathbb{Z}_{11}$-antimagic labeling of $S_{5,5}$.

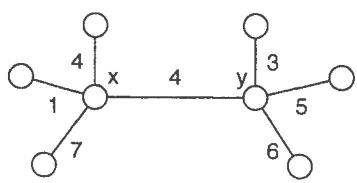

FIGURE 9. $\mathbb{Z}_8$-antimagic labeling of $S_{4,4}$.

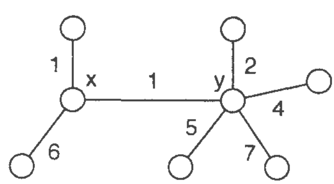

FIGURE 10. $\mathbb{Z}_8$-antimagic labeling of $S_{3,5}$.

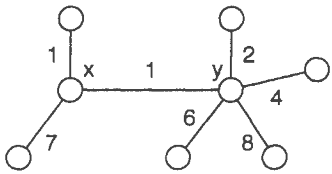

FIGURE 11. $\mathbb{Z}_9$-antimagic labeling of $S_{3,5}$.

REFERENCES

[1] M. Doob, On the construction of magic graphs, *Proc. Fifth S.E. Conference on Combinatorics, Graph Theory and Computing* (1974), 361–374.

[2] M. Doob, Generalizations of magic graphs, *Journal of Combinatorial Theory, Series B*, **17** (1974), 205–217.

[3] M. Doob, Characterizations of regular magic graphs, *Journal of Combinatorial Theory, Series B*, **25** (1978), 94–104.

[4] M.C. Kong, S–M Lee, and H. Sun, On magic strength of graphs, *Ars Combinatoria*, **45** (1997), 193–200.

[5] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), 451–461.

[6] S–M Lee, Yong–Song Ho and R.M. Low, On the integer–magic spectra of maximal planar and maximal outerplanar graphs, *Congressus Numerantium*, **168** (2004), 83–90.

[7] S–M Lee, A. Lee, Hugo Sun, and Ixin Wen, On group–magic graphs, JCMCC, **38** (2001), 197–207.

[8] S–M Lee and F. Saba, On the integer–magic spectra of two–vertex sum of paths, *Congressus Numerantium*, **170** (2004), 3–15.

[9] S–M Lee, F. Saba, E. Salehi, and H. Sun, On the V_4–group magic graphs, *Congressus Numerantium*, **156** (2002), 59–67.

[10] S–M Lee, F. Saba, and G. C. Sun, Magic strength of the k–th power of paths, *Congressus Numerantium*, **92** (1993), 177–184.

[11] S–M Lee and E. Salehi, Integer–magic spectra of amalgamations of stars and cycles, *Ars Combinatoria*, **67** (2003), 199–212.

[12] S–M Lee, E. Salehi and H. Sun, Integer–magic spectra of trees with diameters at most four, *JCMCC*, **50** (2004), 3–15.

[13] S–M Lee, L. Valdes, and Yong–Song Ho, On group–magic spectra of trees, double trees and abbreviated double trees, *JCMCC*, **46** (2003), 85–95.

[14] R.M. Low and S–M Lee, On the integer–magic spectra of tessellation graphs, *Australas. J. Combin.*, **34** (2006), 195–210.

[15] R.M. Low and S–M Lee, On the products of group–magic graphs, *Australas. J. Combin.*, **34** (2006), 41–48.

[16] R.M. Low and S–M Lee, On group–magic eulerian graphs, *JCMCC*, **50** (2004), 141–148.

[17] R.M. Low and L. Sue, Some new results on the integer–magic spectra of tessellation graphs, *Australas. J. Combin.*, **38** (2007), 255–266.

[18] E. Salehi, Zero–sum magic graphs and their null sets, *Ars Combinatoria*, **82** (2007), 41–53.

[19] E. Salehi, On zero–sum magic graphs and their null sets, *Bulletin of the Institute of Mathematics, Academia Sinica*, **3** (2008), 255–264.
[20] E. Salehi and P. Bennett, On integer–magic spectra of caterpillars, *JCMCC*, **61** (2007), 65–71.
[21] J. Sedlácek, On magic graphs, *Math. Slov.*, **26** (1976), 329–335.
[22] J. Sedlácek, Some properties of magic graphs, in Graphs, Hypergraph, and Bloc Syst. 1976, *Proc. Symp. Comb. Anal., Zielona Gora* (1976), 247–253.
[23] W.C. Shiu, P.C.B. Lam and S–M. Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium*, **132** (1998), 9–18.
[24] W.C. Shiu, P.C.B. Lam and S–M. Lee, On a Construction of Supermagic Graphs, *JCMCC*, **42** (2002), 147–160.
[25] W.C. Shiu and R.M. Low, Group–magicness of complete N–partite graphs, *JCMCC*, **58** (2006), 129–134.
[26] W.C. Shiu and R.M. Low, Integer–magic spectra of sun graphs, *J. Comb. Optim.*, **14** (2007), 309–321.
[27] W.C. Shiu and R.M. Low, Z_k-magic labelings of fans and wheels with magic-value zero, *Australas. J. Combin.*, **45** (2009), 309–316.
[28] R.P. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.*, **40** (1973), 607–632.
[29] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen–Macaulay rings, *Duke Math. J.*, **40** (1976), 511–531.
[30] W.D. Wallis, *Magic Graphs*, Birkhauser Boston, (2001).

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