Cubic Graphs with Different Incidence Chromatic Numbers

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Abstract

Incidence coloring of a graph G is a mapping from the set of incidences to a color-set C such that adjacent incidences are assigned distinct colors. In the last ten years, incidence coloring was developed independently. To link up incidence coloring with edge coloring, we proved that if an odd degree regular graph is $(\Delta+1)$-incidence colorable, then it is Δ-edge colorable. This result helps to establish some sufficient conditions for the cubic graphs that are not $(\Delta+1)$-incidence colorable. Moreover, two kinds of cubic graphs will be proved to be 4-incidence colorable.

Keywords : Incidence coloring, Edge coloring, Cubic graph
AMS 2000 MSC : 05C15

1 Introduction

Graph coloring is one of the classical problems in graph theory. No matter which kind of coloring, separating the whole graph into different independent sets is the only objective. The goal of incidence coloring is to determine the minimum number of independent incidence sets which can cover the graph.

Since incidence coloring was introduced, most of the researches were concentrated on minimizing the upper bound of the incidence chromatic number of certain class of graphs, which will be shown in Section 1.2. To diversify the research area, a relation between incidence coloring and edge coloring was established and it will be discussed in Section 2. Moreover, in Section 3, three sufficient conditions will be given to improve the lower bound of the incidence chromatic number of some cubic graphs. Finally, two kinds of cubic graphs will be proved to be 4-incidence colorable in Section 4.

1.1 Necessary Definitions

Unless stated otherwise, all graphs we consider are simple, connected, undirected and finite. Let G be a graph and V, E and Δ be the vertex-set, edge-set and maximum degree of G, respectively. For any vertex $u \in V$, we denote $N_G(u)$(or simply $N(u)$) as the set of neighborhood of u. All notations not defined in this paper are referred to the book [1]. Let

$$I(G) = \{(v, e) \in V \times E \mid v \text{ is incident with } e\}$$

be the set of *incidences* of G. Two incidences (v, e) and (w, f) are said to be *adjacent* if one of the following holds:

(1) $v = w$;
(2) $e = f$; and
(3) the edge $\{v, w\}$ equals to e or f.

Apart from the traditional definition stated in above, Shiu *et al.* in [5] gave an equivalent definition of incidences. Consider G as a digraph by splitting each edge $e(uv) \in E$ into two opposite arcs uv and vu. We use $e = e(uv)(= e(vu))$ to represent the edge, with ends u and v of the undirected graph G. We identify (u, e) with the arc uv and therefore the incidence set $I(G)$ can be identified with the set of all arcs $A(G)$. We say that two distinct arcs uv and xy are *adjacent* provided one of the following holds:

$(\tilde{1})$ $u = x$;
$(\tilde{2})$ $u = y$ and $v = x$;
$(\tilde{3})$ $v = x$ or $y = u$.
Noted that $(\tilde{2})$ becomes redundant.

An *incidence coloring* of G is a mapping $\sigma : I(G) \to C$, where C is a *color-set*, such that adjacent incidences of G are assigned distinct colors. The *incidence chromatic number* of G, denoted by $\chi_i(G)$, is the minimum cardinality of C for which $\sigma : I(G) \to C$ is an incidence coloring. Moreover, let the set of colors assigned to the arcs going into u by $C_G^+(u)$. Similarly, $C_G^-(u)$ represents the set of colors assigned to the arcs going out from u.

1.2 Previous Results

Incidence colorings was first introduced by Brualdi and Massey [2] in 1993. It was initially used to acquire the strong chromatic index of an associated bipartite graph. From the definition of incidence coloring, the following global lower bound is obvious:

Proposition 1.1 *Suppose a graph G with maximum degree Δ, then $\chi_i(G) \geq \Delta + 1$*

In the same paper, Brualdi and Massey proved the following global upper bound:

Theorem 1.2 *For each graph G, we have $\chi_i(G) \leq 2\Delta$.*

Following are some previous results of incidence chromatic number of some classes of graphs:

Theorem 1.3 [2] *For each $n \geq 2$, $\chi_i(K_n) = n = \Delta(K_n) + 1$.*

Theorem 1.4 [5] *Let G be a cubic Hamiltonian graph, then $\chi_i(G) \leq 5$.*

Theorem 1.5 [7] *Let G be a Halin graph with maximum degree $\Delta \geq 5$. Then $\chi_i(G) = \Delta + 2$.*

Theorem 1.6 [3] *Every k-degenerated graph G is $(\Delta + 2k - 1)$-incidence colorable.*

Theorem 1.7 [3] *Every planar graph G is $(\Delta + 7)$-incidence colorable.*

Theorem 1.8 [4] *Every graph G with $\Delta = 3$ is 5-incidence colorable.*

The following lemma is obvious.

Lemma 1.9 *Let $G_1, G_2, \ldots, G_n$ be n disjoint graphs. If $G = G_1 + G_2 + \cdots + G_n$, then $\chi_i(G) = max\{\chi_i(G_j) \mid j = 1, 2, \ldots, n\}$*

Because of Lemma 1.9, the graphs being considered can be reduced to connected.

2 Incidence and Edge Colorings

The following theorem establishes a relation between incidence coloring and edge coloring on regular graph.

Theorem 2.1 *Suppose G is a regular graph, if $\chi_i(G) = \Delta + 1$, then $\chi'(G) \leq \chi'(K_{\Delta+1})$.*

Proof: Suppose Δ is even, then $\chi'(K_{\Delta+1}) = \Delta + 1$. According to Vizing Theorem, the result is achieved. Suppose $\Delta = 2k + 1$. It is known that [1, 8], the chromatic index of K_{2k+2} is equal to $2k+1$. Let σ be a $(2k+2)$-incidence coloring of G and ϕ be a $(2k+1)$-edge coloring of K_{2k+2} with vertices labelled $1, 2, \ldots, 2k+2$. For $1 \leq i < j \leq 2k+2$, let

$$S_{i,j} = \{e(uv) = e(vu) \in E(G) \mid \{\sigma(uv), \sigma(vu)\} = \{i, j\}\}$$

Clearly $\{S_{i,j} \mid 1 \leq i < j \leq 2k+2\}$ is a partition of $E(G)$. Suppose $m \in S_{i,j}$, we define an edge-coloring θ of $E(G)$ by $\theta(m) = \phi(ij)$, where ij is the edge incidence to vertices i and j in K_{2k+2}. Each matching of K_{2k+2} contains non-adjacent edges, that is, for all ij, kl in a matching, $|\{i, j\} \cap \{k, l\}| = 0$. Since $\chi_i(G) = \Delta + 1$ and all vertices of degree Δ, we have $\forall u \in V(G)$,

$|C^+(u)| = 1$. The two distinct edges $m \in S_{i,j}$ and $n \in S_{k,l}$ are adjacent if and only if $|\{i,j\} \cap \{k,l\}| = 1$. As a result, θ is a proper edge coloring. Thus we have $\chi'(G) = \chi'(K_{2k+2}) = 2k+1$. □

Remarks: For Δ is even, the upper bound of the previous theorem is attainable if G is a complete graph. Also, for all $n \equiv 3 \pmod 6$, we have $\chi_i(C_n) = \chi'(C_n) = 3$.

3 Sufficient Conditions for $\chi_i \neq \Delta + 1$

As mentioned before, researches of incidence coloring were concentrated on improving the upper bound. In this section, we will provide three sufficient conditions so as to improve the lower bound of regular graph.

The first condition is obtained from the contrapositive of Theorem 2.1.

Theorem 3.1 *Suppose G is an odd degree regular graphs, if G contains a cut vertex, then it is not $(\Delta+1)$-incidence colorable.*

Proof: From [8, p.284], the chromatic index of G equals to $\Delta+1$. According to the contrapositive of Theorem 2.1, G is not $(\Delta+1)$-incidence colorable. □

Theorem 1.8 revealed that every cubic graph is 5-incidence colorable. Therefore, cubic graph has incidence chromatic number equal to 4 or 5. The following two sufficient conditions are specified for cubic graphs. To prove the first one, Theorem 3.2 should be stated in prior. The theorem is proved by Shiu and Sun [6].

Theorem 3.2 *Suppose $u, v \in V(G)$ with u and v are not adjacent in G and $d(u) = d(v) = \Delta$. Suppose $N(u) = \{x_1, x_2, \ldots, x_{\Delta-2}, y_1, z_1\}$ and $N(v) = \{x_1, x_2, \ldots, x_{\Delta-2}, y_2, z_2\}$ with $d(x_i) = d(y_1) = d(y_2) = \Delta$ for $1 \le i \le \Delta - 2$. In addition, $e(y_1y_2) \in E(G)$. Then G is not $(\Delta+1)$-incidence colorable.*

For cubic graphs, Theorem 3.2 can be simplified into the following corollary:

Corollary 3.3 *Let G be a cubic graph. Suppose G contains a 5 cycle. Then G is not 4-incidence colorable.*

Proof: Consider a 5 cycle C, since G is cubic, there exists two vertices u and v on C that are not adjacent. Let the common neighbor of u and v on C be x_1 and the other neighbor of u and v on C be y_1 and y_2, respectively. Then, they will satisfy the conditions in Theorem 3.2. As a result, G is not 4-incidence colorable. □

The last sufficient condition is in relation with the following figure.

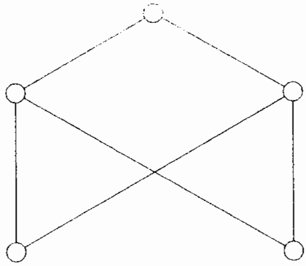
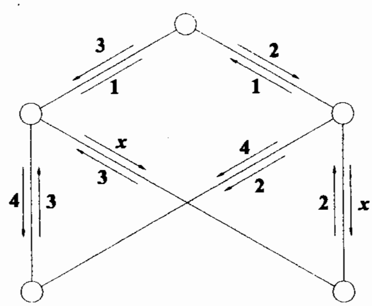


Figure 3.1. 2 overlapping 4 cycles

Theorem 3.4 *Suppose G is a cubic graph containing the graph in Figure 3.1 as a subgraph, which is denoted as H, then G is not 4-incidence colorable.*

Proof: Since each vertex of H is of degree 3 in G. H will be labelled as following:



It is obvious that color x cannot be color 1,2,3 or 4. Therefore, G is not 4-incidence colorable. □

4 Cubic graphs G with $\chi_i(G) = 4$

This section we will prove two kinds of cubic graphs that are 4-incidence colorable.

Theorem 4.1 *Suppose G is the union of n disjoint cycles C_i, $i = 1, 2, \ldots, n$ and a perfect matching F. In addition, suppose each cycle C_i of order p_i with $p_i \equiv 0 \pmod 4$. Let $C_i = u_{i1}u_{i2}\cdots u_{ip_i}$. Suppose there exists* *a renumbering (if necessary) of the vertices u_{iz} such that for all $e \in F$, $e = e(u_{kz}u_{qy})$ with $y \equiv z+2 \pmod 4$, for $1 \le k, q \le n$. Then $\chi_i(G) = 4$.*

Proof: C_i can be colored by the incidence coloring σ as follows:

$$\sigma(u_{iz}u_{i(z+1)}) = \begin{cases} 1 & \text{if} \quad z \equiv 0 \pmod 4, \\ 2 & \text{if} \quad z \equiv 1 \pmod 4, \\ 3 & \text{if} \quad z \equiv 2 \pmod 4, \\ 4 & \text{if} \quad z \equiv 3 \pmod 4, \end{cases}$$

and

$$\sigma(u_{i(z+1)}u_{iz}) = \begin{cases} 4 & \text{if} \quad z \equiv 0 \pmod 4, \\ 1 & \text{if} \quad z \equiv 1 \pmod 4, \\ 2 & \text{if} \quad z \equiv 2 \pmod 4, \\ 3 & \text{if} \quad z \equiv 3 \pmod 4, \end{cases}$$

for all z from 1 to p_i and $u_{i(p_i+1)} = u_{i1}$. This coloring strategy ensures that for all $u_{iz} \in C_i$, $|C^+(u_{iz})| = 1$.

To complete the proof, it remains to color the arcs in F. For all $u_{kz}u_{qy} \in F$, without loss of generality, assume $C^+(u_{kz}) = \{1\}$. From the coloring strategy of the cycles prescribed, we have $C^-(u_{kz}) = \{2, 4\}$. On the other hand, since $y \equiv z+2 \pmod 4$, then $C^+(u_{qy}) = \{3\}$ and $C^-(u_{qy}) = \{2, 4\}$. As a result, we can assign $\sigma(u_{kz}u_{qy}) = 3$ and $\sigma(u_{qy}u_{kz}) = 1$. All other arcs in F can be colored in a similar approach, therefore, G is 4-incidence colorable. Combine with Proposition 1.1, $\chi_i(G) = 4$. □

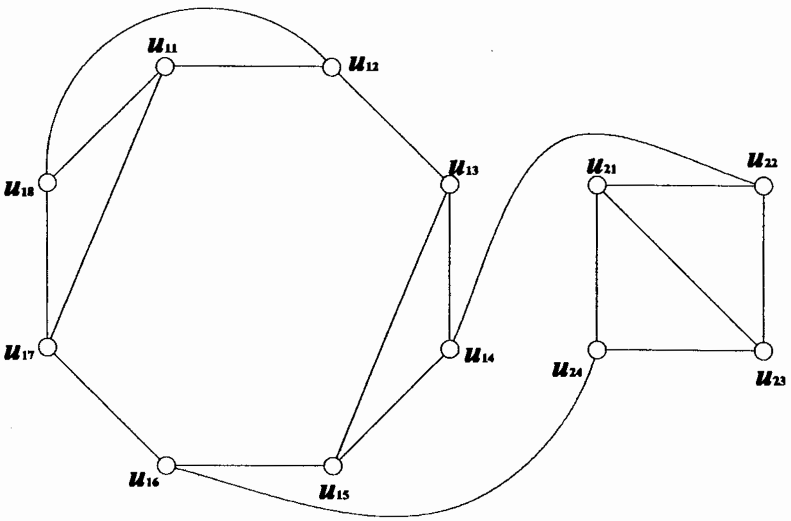


Figure 4.2. Cubic graph composed of a 4-cycle and an 8-cycle

A *triangulated wheel* TW_n is a graph constructed from an n-cycle $C_n = u_1u_2\cdots u_n$, which is called the *inner cycle*, with vertices labelled in counterclockwise. Each vertex u_i is adjacent to a vertex v_i and every vertex v_i is adjacent to two other vertices, w_{2i-1} and w_{2i}, with w_j labelled in counterclockwise. Finally, there is a cycle that join up the vertices w_j. The followings are the diagram of TW_3 and TW_6:

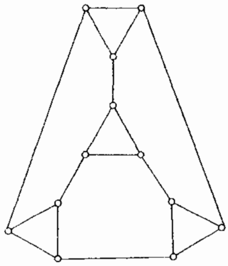


Figure 4.3. Triangulated wheel TW_3

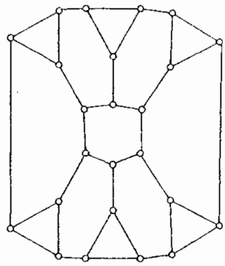


Figure 4.4. Triangulated wheel TW_6

Besides, the graph is named as *triangulated wheel* because it can be viewed as a wheel $W_{n+1} = K_1 \vee C_n$ with all the vertices incidence to the exterior face replaced by a triangle. In addition, the central vertex of W_{n+1} will be replaced by the cycle C_n.

Theorem 4.2 *For each triangulated wheel TW_n with $n \equiv 0 \pmod 3$, $\chi_i(TW_n) = 4$.*

Proof: All notations are the same as defined above. Suppose $C_{3n} = u_1u_2\cdots u_{3n}$ be the inner cycle of TW_{3n}. It can be colored by the incidence coloring σ as follows:

$$\sigma(u_iu_{i+1}) = \begin{cases} 1 & \text{if} \quad i \equiv 0 \pmod 3, \\ 2 & \text{if} \quad i \equiv 1 \pmod 3, \\ 3 & \text{if} \quad i \equiv 2 \pmod 3, \end{cases}$$

and

$$\sigma(u_{i+1}u_i) = \begin{cases} 3 & \text{if} \quad i \equiv 0 \pmod 3, \\ 1 & \text{if} \quad i \equiv 1 \pmod 3, \\ 2 & \text{if} \quad i \equiv 2 \pmod 3, \end{cases}$$

for $i = 1, 2, \ldots, 3n$ with $u_{3n+1} = u_1$. The arcs $w_{2i-1}v_i$, $w_{2i}v_i$ and u_iv_i are colored by 4. All remaining edges are colored by σ as follows:

$$\sigma(v_iu_i) = \begin{cases} 1 & \text{if} \quad i \equiv 1 \pmod 3, \\ 2 & \text{if} \quad i \equiv 2 \pmod 3, \\ 3 & \text{if} \quad i \equiv 0 \pmod 3, \end{cases}$$

$$\sigma(v_iw_{2i-1}) = \begin{cases} 3 & \text{if} \quad i \equiv 1 \pmod 3, \\ 1 & \text{if} \quad i \equiv 2 \pmod 3, \\ 2 & \text{if} \quad i \equiv 0 \pmod 3, \end{cases}$$

and

$$\sigma(v_i w_{2i}) = \begin{cases} 2 & \text{if} \quad i \equiv 1 \pmod 3, \\ 3 & \text{if} \quad i \equiv 2 \pmod 3, \\ 1 & \text{if} \quad i \equiv 0 \pmod 3, \end{cases}$$

for $i = 1, 2, \ldots, 3n$.

$$\sigma(w_j w_{j-1}) = \begin{cases} 1 & \text{if} \quad j \equiv 1 \pmod 3, \\ 3 & \text{if} \quad j \equiv 2 \pmod 3, \\ 2 & \text{if} \quad j \equiv 0 \pmod 3, \end{cases}$$

and

$$\sigma(w_{j-1} w_j) = \begin{cases} 3 & \text{if} \quad j \equiv 1 \pmod 3, \\ 2 & \text{if} \quad j \equiv 2 \pmod 3, \\ 1 & \text{if} \quad j \equiv 0 \pmod 3, \end{cases}$$

for $j = 1, 2, \ldots, 6n$ with $w_0 = w_{6n}$. Therefore, TW_n is 4-incidence colorable for all n. Combining with Proposition 1.1, we have $\chi_i(TW_n) = 4$. □

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