Super-edge-graceful labelings of multi-level wheel graphs, fan graphs and actinia graphs

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Abstract

The notion of super-edge-graceful graphs was introduced by Mitchem and Simoson in 1994. However, few examples except trees are known. In this paper, we exhibit three classes of infinitely many graphs including fan graphs, multi-level wheel graphs and actinia graphs, which are super-edge-graceful.

Keywords: super-edge-graceful, wheel graph, fan graph, actinia graph

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1 Introduction

In this paper all graphs are loopless and connected. All undefined symbols and concepts may be looked up from [1]. A (p, q)-graph $G = (V, E)$ is called *edge-graceful* if there exists a bijection $f : E \to \{1, 2, \ldots, q\}$ and the *induced mapping* $f^+ : V \to \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ defined by

$$f^+(u) \equiv \sum_{uv \in E} f(uv) \pmod p$$

is a bijection. This concept was introduced by Lo [5] in 1985. Lee [3] conjectured that all trees of odd order are edge-graceful. More references about edge-graceful may be found in [8, 9].

Mitchem and Simoson [6] tried to prove the above conjecture and introduce a variation of edge-gracefulness which for trees of odd order implies edge-gracefulness.

Let

$$P = \begin{cases} \{-\frac{p}{2}, \ldots, -1, 1, \ldots, \frac{p}{2}\} & \text{if } p \text{ is even} \\ \{-\frac{p-1}{2}, \ldots, -1, 0, 1, \ldots, \frac{p-1}{2}\} & \text{if } p \text{ is odd} \end{cases}$$

and

$$Q = \begin{cases} \{-\frac{q}{2}, \ldots, -1, 1, \ldots, \frac{q}{2}\} & \text{if } q \text{ is even} \\ \{-\frac{q-1}{2}, \ldots, -1, 0, 1, \ldots, \frac{q-1}{2}\} & \text{if } q \text{ is odd} \end{cases}.$$

A (p, q)-graph G is *super-edge-graceful* if there is a pair of mappings (f, f^+) such that $f : E \to Q$ is bijective and $f^+ : V \to P$ is also bijective, where $f^+(u) = \sum_{uv \in E} f(uv)$. f is called a *super-edge-graceful labeling of* G. Such sets P and Q are called the *vertex values set* and *edge labels set* of G, respectively.

The notions of super-edge-graceful graphs and edge-graceful graphs are different. Mitchem and Simoson [6] showed that the step graph C_6^2 is super-edge-graceful but not edge-graceful. Shiu [7] showed that the complete graph K_4 is edge-graceful but not super-edge-graceful.

Mitchem and Simoson [6] showed that

Theorem 1.1 *If G is a super-edge-graceful (p,q)-graph and*

$$q \equiv \begin{cases} -1 \pmod p & \text{if } q \text{ is even,} \\ 0 \pmod p & \text{if } q \text{ is odd.} \end{cases}$$

Then G is also edge-graceful.

Shiu [7] proved that some cubic graphs are super-edge-graceful. Namely he showed that permutation Petersen graphs and some permutation ladder graphs are super-edge-graceful.

In this paper, three classes of graphs are shown to be super-edge-graceful.

2 Fan graphs

The join $K_1 \vee P_{n-1}$ of K_1 and P_{n-1} is called a *fan graph* F_n. The vertex come from K_1 is called the *core*. The edges incident with the core are called *spokes*. Hence F_n has n vertices and $2n-3$ edges. In this section, we will study the super-edge-gracefulness of fan graphs.

First we define some notations on the graph F_{n+1}. Let the core be denoted by c and the vertices on the path P_n be denoted by p_1, p_2, $\ldots$, p_n, respectively. We draw F_{n+1} in the plane as follows:
Draw the P_n horizontally and draw the core under the path P_n. Join the core to each vertex of P_n by straight line. Figure 1 shows the graph F_9.

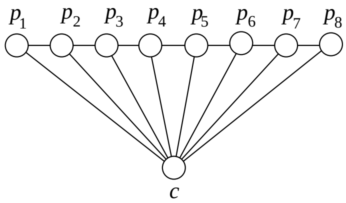

Figure 2.1: The fan graph F_9

Theorem 2.1 *Every fan graphs F_{2n+1} is super-edge-graceful, $n \geq 1$.*

Proof: Here the vertex values set and the edge labels set of F_{2n+1} are

$$P = \{-n, -(n-1), \ldots, -1, 0, 1, 2, \ldots, n\} \text{ and } Q = \{-(2n-1), \ldots, -1, 0, 1, \ldots, 2n-1\},$$

respectively. We shall define an super-edge-graceful labeling $f : E(F_{2n+1}) \to Q$ as follows.

Suppose n is odd. Firstly we label the first n spokes $cp_1, cp_2, \ldots, cp_n$ starting from the left by $1, -2, 3, \ldots, n-2, -(n-1)$ and $-n$. Secondly, we label the first $n-1$ edges of P_{2n} starting from the left by $-(n+1), n+2, \ldots, -(2n-2), 2n-1$. For the edge p_np_{n+1}, we label it by 0. We label the other edges "skew-symmetrically" to the above labeled edges. Namely, we label the last n spokes $cp_{2n}, cp_{2n-1}, \ldots, cp_{n+1}$ starting from the right by $-1, 2, -3, \ldots, -(n-2), n-1$ and n, and the last edges of P_{2n} starting from the right by $n+1, -(n+2), \ldots, 2n-2, -(2n-1)$. So f is a bijection. Figure 2.2 is a demonstration for $n = 5$.

Now we are going to show that $f^+ : V(F_{2n+1}) \to P$ is a bijection. Clearly, $f^+(c) = 0$, $f^+(p_1) = 1 + (-(n+1)) = -n$ and $f^+(p_n) = (-n) + (2n-1) + 0 = n-1$. For $i = 2, \dots, n-1$, $f^+(p_i) = (-1)^{i-1}i + (-1)^{i-1}(n+i-1) + (-1)^i(n+i) = (-1)^{i-1}(i-1)$. By skew-symmetrically, we know that all the values $-1, 2, -3, \dots, -(n-1)$ and n are assumed on the right hand side vertices of the fan graph. Hence f is a super-edge-graceful labeling of F_{2n+1}.

Suppose n is even. The labeling is similarly. We label the first n spokes $cp_1, cp_2, \dots, cp_n$ starting from the left by $1, -2, 3, \dots, -(n-2), n-1$ and n, and label the first $n-1$ edges of P_{2n} starting from the left by $-(n+1), n+2, \dots, 2n-2, -(2n-1)$. The other side of edges are labeled skew-symmetrically. Figure 2.3 is a demonstration for $n = 4$. Clearly $f^+(c) = 0$, $f^+(p_1) = 1 + (-(n+1)) = -n$ and $f^+(p_n) = n + (-(2n-1)) + 0 = -(n-1)$. For $i = 2, \dots, n-1$, by the same calculation we have $f^+(p_i) = (-1)^{i-1}(i-1)$. By skew-symmetrically, we see that f is a super-edge-graceful labeling of F_{2n+1}. $\square$

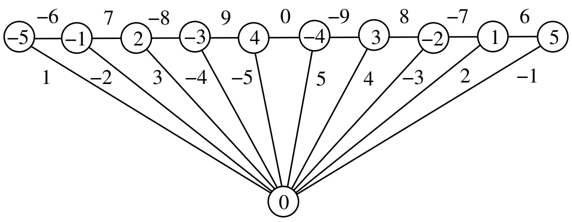

Figure 2.2: A super-edge-graceful labeling for F_{11}.

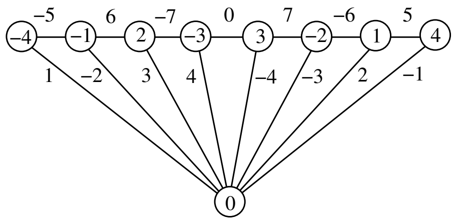

Figure 2.3: A super-edge-graceful labeling for F_9.

Proposition 2.2 *The graph F_4 is not super-edge-graceful.*

Proof: We keep the notations defined as the beginning of this section. Suppose there were a super-edge-graceful labeling $f : E(F_4) \to \{-2, -1, 0, 1, 2\}$ for F_4. Then $f^+ : V(F_4) \to \{-2, -1, 1, 2\}$ is a bijection.

Suppose $f(cp_2) = 0$. Without loss of the generality, we may assume $f(p_1p_2) = 2$. Since $f^+(p_2) \le 2$ and $f(xp_2) = 0$, $f(p_2p_3) < 0$. Since $f^+(p_2) \ne 0$, $f(p_2p_3) = -1$. Since $f^+(p_3) \ne 0$, $f(cp_3) \ne 1$. So $f(cp_3) = -2$. But it is impossible for $f^+(p_3) = -3$.

Suppose $f(cp_2) \ne 0$. Without loss of the generality, we may assume $f(p_1p_2) = 0$. Then $f^+(p_1) = f(cp_1)$. Since f^+ is a bijection, $f(cp_2) + f(cp_3) \ne 0$. Since f is a bijection, $|f(cp_2)| \ne |f(cp_3)|$. Since f is a bijection again, $|f(p_2p_3)| = |f(cp_2)|$ or $|f(p_2p_3)| = |f(cp_3)|$. This will imply that $f^+(p_2) = 0$ or $f^+(p_3) = 0$. It is impossible. $\square$

The super-edge-gracefulness of F_{2n} is still open.

3 Multi-level wheel Graphs

In this section, we will show that some kind of multi-level wheel graphs are super-edge-graceful. We consider the simple case first.

For $n \ge 2$, the graph $K_1 \vee C_n$ is called the *wheel graph* of order $n+1$ and denoted by W_{n+1}. Note that C_2 is a multi-graph consisting of two vertices and two parallel edges. The vertex come from K_1 is called the *core* and denoted by c. We draw W_{n+1} in the plane in the following way. Draw the cycle C_n as an n-polygon and then put the core in the center of the polygon. Join the core to each vertex of C_n by straight line. Vertices lying on the polygon are denoted by $u_1, \dots, u_n$ in clockwise. The edges cu_i, $1 \le i \le n$, are called *spokes*. The cycle is also called the *ring* of the wheel. Hence W_{n+1} has $n+1$ vertices and $2n$ edges.

Theorem 3.1 *For $n \ge 1$, W_{2n+1} is super-edge-graceful.*

Proof: In this case $P = \{-n, \dots, -1, 0, 1, \dots, n\}$ and $Q = \{-2n, \dots, -1, 1, \dots, 2n\}$. We will define a labeling $f : E(W_{2n+1}) \to Q$. First we label the spokes from cu_1 to cu_{2n} by -1, 1, -2, 2, $\dots$, $-n$, n in clockwise. That is $f(cu_{2i-1}) = -i$ and $f(cu_{2i}) = i$ for $1 \le i \le n$. Then we label the edges of the cycle from $u_{2n}u_1$ to $u_{2n-1}u_{2n}$ by $n+1, -(n+1), \dots, 2n, -2n$ in clockwise. That is $f(u_{2i-2}u_{2i-1}) = n+i$ and $f(u_{2i-1}u_{2i}) = -(n+i)$, for $1 \le i \le n$ (for convenience we let $u_0 = u_{2n}$).

It is clearly that $f^+(c) = 0$. For $1 \le i \le n$,

$f^+(u_{2i-1}) = f(u_{2i-2}u_{2i-1}) + f(u_{2i-1}u_{2i}) + f(cu_{2i-1}) = n + i - (n+i) - i = -i,$ and

$f^+(u_{2i-2}) = f(u_{2i-3}u_{2i-2}) + f(u_{2i-2}u_{2i-1}) + f(cu_{2i-2}) = -(n+i-1) + (n+i) + (i-1) = i.$

So f is a super-edge-graceful labeling. $\square$

Figure 3.1 shows a super-edge-graceful labeling for W_9.

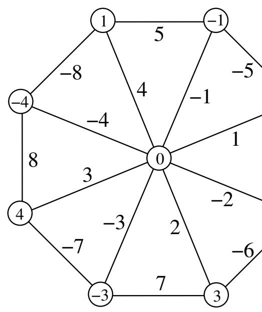

Figure 3.1: A super-edge-graceful labeling for W_9

It is known that $W_4 \equiv K_4$ is not super-edge-graceful. The super-edge-gracefulness of W_{2n} is still open.

We shall construct an *m-level wheel* recurrently for $m \ge 1$. The wheel graph is 1-level wheel. Suppose we have an $(m-1)$-level wheel graph. An m-level wheel graph is a graph obtained from the $(m-1)$-level graph by appending a numbers of pair edges (called the *m-th level spokes*) to the ring of the wheel (the outer cycle) and append a new ring (called the *m-th level ring*) to the most exterior spokes. We shall use the notation $W(n_1, n_2, \dots, n_m)$ to denotes an m-level wheel graph which contains n_1 spokes in the 1-st level, n_2 spokes in the 2-nd level, $\dots$, n_m spokes in the m-th level. Hence $n_2, \dots, n_m$ must be even. It is clear that $W(n_1, n_2, \dots, n_m)$ is a planar. So we will view it as a plane graph.

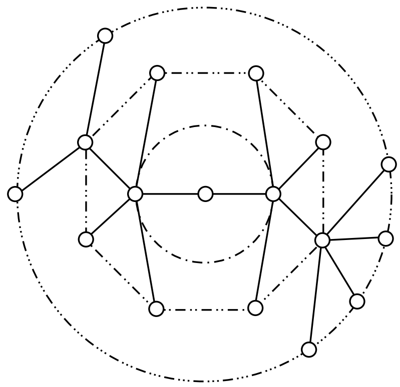

Figure 3.2: 3-level wheel graph $W(2,8,6)$.

Note that there may be many m-level wheel graphs have the same parameter $n_1, n_2, \dots, n_m$. Following are some examples.

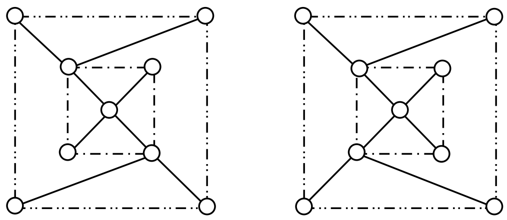

Figure 3.3: Two 2-level wheel graphs $W(4,4)$.

We shall show that each multi-level wheel graph with even number of spokes in the first level is super-edge-graceful. Before proving this result, we use the following example to illustration the idea of the proof.

Example 3.1 We consider the graph $W(2,8,6)$ described in Figure 3.2. This is a $(17,32)$-graph. So $P = \{-8,-7,\dots,-1,0,1,\dots,7,8\}$ and $Q = \{-16,-15,\dots,-1,1,\dots,15,16\}$. There are 16 spokes and 16 edges in rings. First we partition Q into two sequences $S = \{-1,1,-2,2,\dots,-8,8\}$ and $R = \{9,-9,10,-10,\dots,16,-16\}$. According to the parameters, we partition S into three disjoint subsequences S_1, S_2, S_3 such that S_1 consists of the first 2 terms of S, S_2 consists of the next 8 terms of S, S_3 consists of the last 6 terms of S. Similarly we partition R into three subsequences R_1, R_2, R_3. Namely, $S_1 = \{-1,1\}$, $S_2 = \{-2,2,-3,3,-4,4,-5,5\}$, $S_3 = \{-6,6,-7,7,-8,8\}$, $R_1 = \{9,-9\}$, $R_2 = \{10,-10,11,-11,12,-12,13,-13\}$ and $R_3 = \{14,-14,15,-15,16,-16\}$.

For the i-th level, we choose a spoke $e_i = u_i x_{i-1}$ such that u_i lies on the i-th level ring, x_{i-1} lies on the $(i-1)$-st level ring and the next spoke (counting in clockwise) is also incident with x_{i-1}. First we label the i-th level spokes by the terms of S_i in clockwise starting from e_i, $i = 1,2,3$. Let the last labeled i-th level spoke be $e_i' = v_i y_{i-1}$, where v_i lies on the i-th level ring and y_{i-1} lies on the $(i-1)$-st level ring. Note that $x_0 = y_0$ is the core c.

After all spokes are labeled, we label the edges lying in the i-th level ring by the terms of R_i in clockwise starting from $u_i v_i$, $i = 1,2,3$. So we get

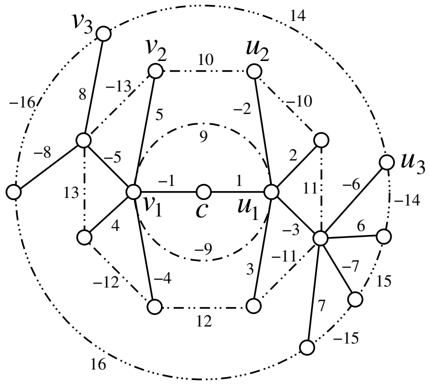

Figure 3.4: A super-edge-graceful labeling for $W(2, 8, 6)$.

Then the induced labels for the vertices lying on the 1-st level ring are 1 and -1; lying on the 2-nd level ring are $-2, 3, -3, 4, -4, 5, -5, 2$; lying on the 3-rd level ring are $-6, 7, -7, 8, -8, 6$, respectively. And the induced label for the core is 0.

Theorem 3.2 *For $m \geq 1$ and n_1 even, the m-level wheel graph $W(n_1, n_2, \ldots, n_m)$ is super-edge-graceful.*

Proof: Let $n_i = 2l_i$ for some positive integer l_i. Let $S = \{-1, 1, -2, 2, \ldots, -\sum_{i=1}^{m} l_i, \sum_{i=1}^{m} l_i\}$ and $R = \{1+\sum_{i=1}^{m} l_i, -1-\sum_{i=1}^{m} l_i, 2+\sum_{i=1}^{m} l_i, -2-\sum_{i=1}^{m} l_i, \ldots, 2\sum_{i=1}^{m} l_i, -2\sum_{i=1}^{m} l_i\}$ be two sequences. According to the parameters $2l_1, 2l_2, \ldots, 2l_m$, we partition S into m subsequences $S_1, \ldots, S_m$ and R into m subsequences $R_1, \ldots, R_m$. Namely, for $i = 1, \ldots m$, let $S_i = \{-1-\sum_{j=1}^{i-1} l_j, 1+\sum_{j=1}^{i-1} l_j, \ldots, -\sum_{j=1}^{i} l_j, \sum_{j=1}^{i} l_j\}$ and $R_i = \{1+\sum_{j=1}^{i-1} l_j + \sum_{j=1}^{m} l_j, -1-\sum_{j=1}^{i-1} l_j - \sum_{j=1}^{m} l_j, \ldots, \sum_{j=1}^{i} l_j + \sum_{j=1}^{m} l_j, -\sum_{j=1}^{i} l_j - \sum_{j=1}^{m} l_j\}$.

We shall prove the following predicate by induction on m:

$P(m) =$"There is a super-edge-graceful labeling for $W(n_1, \ldots, n_m)$ such that the i-th level spokes are labeled by terms of S_i and the edges lying on the i-th level ring are labeled by terms of R_i, for each i, $1 \leq i \leq m$."

When $m = 1$, the graph is W_{2l_1+1}. By Theorem 3.1 $P(1)$ is true.

Suppose $P(k-1)$ is true for $k \geq 2$. Now we consider $P(k)$. We have subsequences $S_1, \ldots, S_k$ and $R_1, \ldots, R_k$. Since $W(n_1, \ldots, n_{k-1})$ is a subgraph of $W(n_1, \ldots, n_k)$. By induction assumption, there is a super-edge-graceful labeling f_{k-1} for $W(n_1, \ldots, n_{k-1})$ such that the i-th level spokes are labeled by terms of S_i and the edges lying on the i-th level ring are labeled by terms of R_i, for each i, $1 \leq i \leq k-1$. It suffices to extend the labeling f_{k-1} to be a super-edge-graceful labeling of the whole graph. That is, we need to label the k-th level spokes and the edges lying on the k-th level ring by S_k and R_k respectively such that the labeling becomes a super-edge-graceful labeling. Note that $\{f_{k-1}^{+}(w) \mid w \in V(W(n_1, \ldots, n_{k-1}))\} = \{0\} \cup \left(\bigcup_{j=1}^{k-1} S_j\right)$.

Choose a k-th level spoke $e = ux$ which is the first (counting in clockwise) edge incident with a vertex in $(k-1)$-st level ring, where u lies on the k-th level ring and x lies on the $(k-1)$-st level ring. Label the k-th level spokes by the terms of S_k in clockwise starting from e. Let the last labeled k-th level spoke be $e' = vy$, where v lies on the k-th level ring and y lies on the $(k-1)$-st level ring. Label the edges lying in the k-th level ring by the terms of R_k in clockwise starting from uv. Let this extension labeling be f_k.

It is clear that $f_{k-1}^+(w) = f_k^+(w)$ when w does not lie on the k-th or $(k-1)$-st level ring. For vertex w which lies on the $(k-1)$-st level ring, since w is incident with some pairs of k-level spokes whose labels sum is 0, $f_{k-1}^+(w) = f_k^+(w)$.

Suppose w is a vertex lying on the k-level ring. Then $\deg(w) = 3$. Let w be incident with a k-level spokes wz and two other edges ϵ_1 and ϵ_2 in clockwise, where z lies on the $(k-1)$-th level ring. If $f_k(wz) = -a \in S_k$ for some $a > 0$, then $f_k(\epsilon_1) = -b$ and $f_k(\epsilon_1) = b$ for some $b > 0$. Hence $f_k^+(w) = -a$. Note that $-\sum\limits_{j=1}^{k} l_j \le -a \le -1 - \sum\limits_{j=1}^{k-1} l_j$. If $f_k(wz) = a \in S_k$ for some $a > 0$ and $w \neq v$, then $f_k(\epsilon_1) = -b$ and $f_k(\epsilon_1) = b+1$ for some $b > 0$. Hence $f_k^+(w) = a+1$. Since $w \neq v$, $\sum\limits_{j=1}^{k-1} l_j \le a < \sum\limits_{j=1}^{k} l_j$. So $f_k^+(w) = a+1 \in S_k$. If $w = v$, then $f_k^+(v) = \sum\limits_{j=1}^{k} l_j + (1 + \sum\limits_{j=1}^{k-1} l_j + \sum\limits_{j=1}^{k} l_j) + (-2\sum\limits_{j=1}^{k} l_j) = 1 + \sum\limits_{j=1}^{k-1} l_j \in S_k$. Thus $\{f_k^+(w) \mid w \text{ lies on the } k\text{-level ring}\} = S_k$. Hence $\{f_k^+(w) \mid w \in V(W(n_1,\dots,n_{k-1},n_k))\} = \{0\} \cup \left(\bigcup\limits_{j=1}^{k} S_j\right)$. Thus $P(k)$ is true. □

Remark 3.1 If W_{2n} is super-edge-graceful, for some n, then by using the same argument of the proof above, we can prove that the complicated-wheel graph $W(2n-1, n_2, \dots, n_m)$ is super-edge-graceful. But it is still open that whether W_{2n} is super-edge-graceful, $n \ge 3$.

Remark 3.2 We may define a more complicated multi-level wheel graphs. Namely, we may append any number of spokes to each level of ring. We would like to ask for the super-edge-gracefulness of such graph.

4 Actinia graphs

Frucht and Harary [2] have the following construction of graphs. Given two graph G and H, the *corona of G with H*, denoted by $G \odot H$, is the graph with

$$V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i),$$
$$E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} \Big(E(H_i) \cup \{iu_i \mid u_i \in V(H_i)\}\Big),$$

where $V(H_i) = V(H)$ and $E(H_i) = E(H)$ for all $i \in V(G)$.

Suppose integers $m \ge 2$ and $n \ge 1$. The graph $C_m \odot N_n$ is called an *regular actinia graph* and denoted by $A(m,n)$, where N_n is the null graph of order n. Following are some actinia graphs.

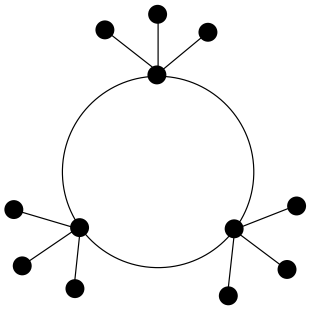

Figure 4.1: Regular actinia graph $A(3,3)$.

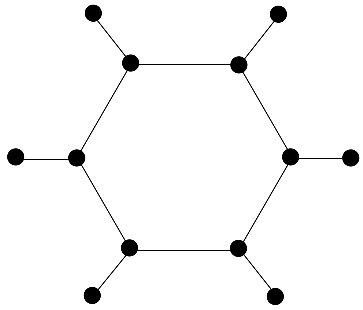

Figure 4.2: Regular actinia graph $A(6,1)$.

The graph $A(m,n)$ is a unicyclic graph with $m+mn$ vertices.

Lemma 4.1 *Suppose G is a super-edge-graceful (p,p)-graph (not necessarily connected). Two extra vertices are added on the graph G and these two vertices are joined by edges with a common vertex of G. Then the resulting graph G' is still super-edge-graceful.*

Proof: In this case the vertex values set P of G and the edge labels set Q of G are the same, and those of G' are $P \cup \{-\lfloor \frac{p}{2} \rfloor - 1, \lfloor \frac{p}{2} \rfloor + 1\}$. So keep the super-edge-graceful labeling of G and label the two extra edges by $-\lfloor \frac{p}{2} \rfloor - 1$ and $\lfloor \frac{p}{2} \rfloor + 1$. Then the extended labeling is a super-edge-graceful labeling of G'. $\square$

Proposition 4.2 *For $m \geq 2$, $A(m,1)$ is super-edge-graceful.*

Proof: The label set is $Q = \{\pm 1, \pm 2, \ldots, \pm m\}$. Let the vertices lying clockwise in the cycle C_m be $u_1, \ldots, u_m$. Let the vertex adjacent with u_i be v_i, $1 \leq i \leq m$. We label the edges of C_m by $-1, -2, \ldots, -m$ clockwise starting at u_1u_2. And then label the edge u_iv_i by i, $1 \leq i \leq m$. Let this labeling be denoted by f. Then $f^+(v_i) = i$, $f^+(u_i) = -(i-1) - i + i = -(i-1)$ if $2 \leq i \leq m$ and $f^+(u_1) = -m - 1 + 1 = -m$. Hence f is a super-edge-graceful labeling of $A(m,1)$. $\square$

Proposition 4.3 *For $m \geq 2$ and odd $n \geq 1$, $A(m,n)$ is super-edge-graceful.*

Proof: By Proposition 4.2 we have a super-edge-graceful labeling for the subgraph $A(m,1)$. Applying Lemma 4.1 repeatedly, $A(m,n)$ is super-edge-graceful. $\square$

Following is an example to illustrate the proof above.

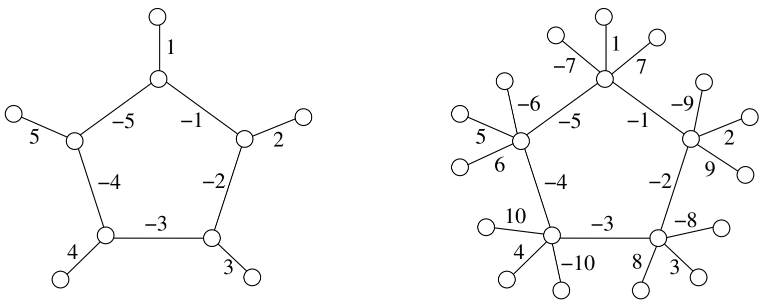

Figure 4.3: Super-edge-graceful labelings for $A(5,1)$ and $A(5,3)$.

Proposition 4.4 *For odd $m \geq 3$ and even $n \geq 2$, $A(m,n)$ is super-edge-graceful.*

Proof: By means of Lemma 4.1 it suffices to show that C_m is super-edge-graceful. But it was proved by Mitchem and Simoson [6]. $\square$

Following from Theorem 1.1 we have

Corollary 4.5 *For odd $m \geq 3$ and even $n \geq 2$, $A(m,n)$ is edge-graceful.*

For m even, we do not know whether C_m is super-edge-graceful. Mitchem and Simoson [6] pointed out that C_4 and C_6 are not super-edge-graceful but C_8 is. So we cannot simply prove that $A(m,n)$ is super-edge-graceful when both m and n are even. In order to show this result, we have to show $A(m,2)$ is super-edge-graceful first for m even.

Proposition 4.6 *For even $m \geq 2$, $A(m,2)$ is super-edge-graceful.*

Proof: Let $m = 2k$ for some $k \geq 1$. Let $u_1, \ldots u_{2k}$ be vertices lying clockwise on the cycle C_{2k}. Let x_i and y_i be two vertices outside the cycle adjacent with u_i. Let f be an edge labeling of $A(2k,2)$ defined by:

$$f(u_i x_i) = \begin{cases} i & \text{if } 1 \leq i \leq k, \\ -i+k & \text{if } k+1 \leq i \leq 2k, \end{cases}$$

$$f(u_i y_i) = \begin{cases} 2k+1-i & \text{if } 1 \leq i \leq k, \\ -3k+i-1 & \text{if } k+1 \leq i \leq 2k, \end{cases}$$

$$f(u_{2j-1}u_{2j}) = -2k-j \qquad \text{if } 1 \leq j \leq k,$$

$$f(u_{2j-2}u_{2j-1}) = 3k+1-j \qquad \text{if } 1 \leq j \leq k.$$

Note that $u_{2k} = u_0$. Clearly, f is a bijection. Also we have $f(u_i x_i) + f(u_i y_i) = 2k+1$ if $1 \leq i \leq k$ and $f(u_i x_i) + f(u_i y_i) = -(2k+1)$ if $k+1 \leq i \leq 2k$.

Thus $\{f^+(x_i) \mid 1 \leq i \leq 2k\} = \{\pm 1, \ldots, \pm k\}$, $\{f^+(y_i) \mid 1 \leq i \leq 2k\} = \{\pm(k+1), \ldots, \pm 2k\}$.

For $1 \leq j \leq \lceil \frac{k}{2} \rceil$, $f^+(u_{2j-1}) = f(u_{2j-1}x_{2j-1}) + f(u_{2j-1}y_{2j-1}) + f(u_{2j-1}u_{2j}) + f(u_{2j-2}u_{2j-1}) = (2k+1) + (3k+1-j) + (-2k-j) = 3k-2j+2$. For $1 + \lceil \frac{k}{2} \rceil \leq j \leq k$, $f^+(u_{2j-1}) = f(u_{2j-1}x_{2j-1}) + f(u_{2j-1}y_{2j-1}) + f(u_{2j-1}u_{2j}) + f(u_{2j-2}u_{2j-1}) = -(2k+1) + (3k-j+1) + (-2k-j) = -k-2j$.

Similarly, for $1 \leq j \leq \lfloor \frac{k}{2} \rfloor$, $f^+(u_{2j}) = (2k+1) + (3k+1-j-1) + (-2k-j) = 3k+1-2j$; for $1 + \lfloor \frac{k}{2} \rfloor \leq j \leq k-1$, $f^+(u_{2j}) = -(2k+1) + (3k+1-j-1) + (-2k-j) = -k-1-2j$; and $f^+(u_{2k}) = -(2k+1) + (-2k-k) + f(u_{2k}u_1) = -2k-1-2k-k+3k = -2k-1$.

It is easy to check that the set $\{f^+(u_i) \mid 1 \leq i \leq 2k\}$ is equal to $\{\pm(2k+1), \ldots, \pm 3k\}$. Hence f is a super-edge-graceful labeling for $A(2k,2)$. $\square$

Following are two examples showing some super-edge-graceful labelings.

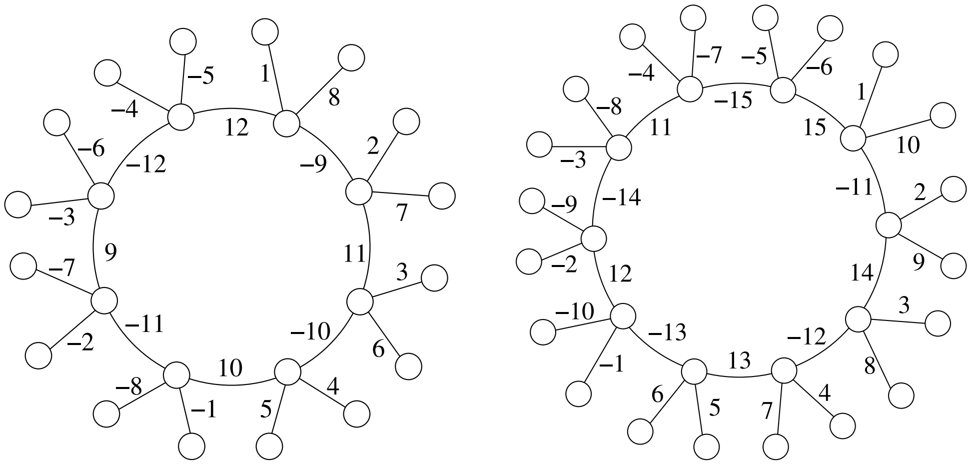

Figure 4.4: Super-edge-graceful labelings for $A(8,2)$ and $A(10,2)$.

Hence we have

Proposition 4.7 *For $m \geq 2$ and even $n \geq 2$, $A(m,n)$ is super-edge-graceful.*

Combining Propositions 4.3, 4.4 and 4.7, we have

Theorem 4.8 *For $m \geq 2$ and $n \geq 1$, $A(m,n)$ is super-edge-graceful.*

Now we can consider a more general case of actinia graph. Let C_m be the m-cycle with vertices $v_1, v_2, \ldots, v_m$ in clockwise, $m \geq 2$. Let $n_1, n_2, \ldots, n_m \geq 0$. An actinia graph $A(m; n_1, n_2, \ldots, n_m)$ is a graph obtained from C_m by attaching n_i edges to the vertex v_i, $1 \leq i \leq m$.

By using Lemma 4.1 we may show that

Theorem 4.9 *An actinia graph $A(m; n_1, n_2, \ldots, n_m)$ is super-edge-graceful if all n_i's are positive and of the same parity. Moreover, when m is odd the positivity of n_i's may be omitted.*

Lee *et al.* [4] studied the super edge-gracefulness of some $A(m; n_1, n_2, \ldots, n_m)$. They called them typical ring-worm graphs. All but one results about these graphs are covered in this paper. The exceptional result is stated below.

Theorem 4.10 *An actinia graph $A(m; n_1, n_2, \ldots, n_m)$ is super-edge-graceful if m is even and all n_i's are even except one.*

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