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INTEGER-ANTIMAGIC SPECTRA OF FAN, WHEEL AND GEAR GRAPHS

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Abstract

Let A be a non-trivial abelian group. A connected simple graph $G = (V,E)$ is A-antimagic if there exists an edge labeling $f : E(G) \to A \setminus \{0\}$ such that the induced vertex labeling $f^+ : V(G) \to A$, defined by $f^+(v) = \sum_{uv \in E(G)} f(uv)$, is injective. The integer-antimagic spectrum of a graph G is the set $\mathrm{IAM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$. In this article, we provide a constructive proof for a join graph $G \vee K_1$ obtained from a given graph G with a special edge labeling. In particular, we determine the integer-antimagic spectra of fan and wheel graphs. The integer-antimagic spectrum of gear graph is also determined.

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1. Introduction and Some Known Results

Let G be a connected simple graphs. For any nontrivial abelian group A (written additively), let $A^* = A \setminus \{0\}$, where 0 is the additive identity of A. Let a mapping $f : E(G) \to A^*$ be an edge labeling of G and $f^+ : V(G) \to A$ be its induced labeling, which is defined by $f^+(v) = \sum\limits_{uv \in E(G)} f(uv)$. If there exists an edge labeling f whose induced labeling f^+ on $V(G)$ is injective, then we say that f is an *A-antimagic labeling* and that G is an *A-antimagic* graph. The *integer-antimagic spectrum* of a graph G is the set $\mathrm{IAM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$. Clearly $\mathrm{IAM}(G) \subseteq \{k \mid k \geq |V(G)|\}$.

The concept of A-antimagicness property for a graph G (introduced in [1]) naturally arises as a variation of the A-magic labeling problem (where the induced vertex labeling is a constant map) (for example, see [5, 6]). It is also a variation of anti-magic labeling problem

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(for example, see [10]) and edge-graceful labeling problem (for example, see [11]). The integer-antimagic spectra of some famous classes of graphs were determined [1, 3, 8, 7, 9].

Lemma 1.1 ([1, Lemma 1]). *For $m \geq 1$, a graph of order $4m+2$ is not $\mathbb{Z}_{4m+2}$-antimagic.*

For integers $a \leq b$, let $[a,b]$ denote the set of integers from a to b, inclusive.

Proposition 1.2. *All elements in $[a,b]$ are distinct after taking modulo k for $k \geq b-a+1$.*

Let f be an edge labeling of G and f^+ be its induced vertex labeling. Let

$$I_f(G) = \{f^+(v) \mid v \in V(G)\}.$$

In order to check whether the graph G is $\mathbb{Z}_k$-antimagic, it suffices to check whether $|I_f(G)| = |V(G)|$.

There are antimagic labelings for paths and cycles described in [1] and a minor correction described in [9]. For those labelings, we have the following corollaries.

Corollary 1.3 (Corollary 2.5 [9]). *For $m \geq 1$, there is an edge labeling g for each of the following paths such that $I_g(P_{4m}) = [1,4m]$, $I_g(P_{4m+1}) = [2,4m+2]$, $I_g(P_{4m+2}) = [1,4m+3] \setminus \{2\}$, and $I_g(P_{4m-1}) = [1,4m-1]$.*

Corollary 1.4 (Corollary 2.6 [9]). *For $n \geq 1$, there is an edge labeling f for each of the following cycles such that $I_f(C_{4n-1}) = [3,4n+1]$, $I_f(C_{4n}) = [3,4n+2]$, $I_f(C_{4n+1}) = [2,4n+2]$ and $I_f(C_{4n+2}) = [3,4n+5] \setminus \{4n+2\}$.*

For $S \subset \mathbb{Z}$ and $a \in \mathbb{Z}$, we define the set $a+S = \{a+s \mid s \in S\}$.

Lemma 1.5. *Let G be a graph with a perfect matching. Let $g : E(G) \to \mathbb{Z}$, $a \in \mathbb{Z}$. There is a labeling h such that $I_h(G) = a + I_g(G)$.*

Proof. Let M be a perfect matching of G. Define $h(e) = a + g(e)$ if $e \in M$ and $h(e) = g(e)$ if $e \notin M$. Then h is the required labeling. $\square$

Hence the following results are special cases of Lemma 1.5.

Corollary 1.6 (Lemma 3.2 [8]). *Suppose that $n \geq 2$ and let $g : E(C_{2n}) \to \mathbb{Z}$, $c \in \mathbb{Z}$. There is a labeling h such that $I_h(C_{2n}) = c + I_g(C_{2n})$, where the range of h is a subset of $[1,n+2] \cup [c+2,c+n+1]$.*

Corollary 1.7 (Lemma 3.1 [9]). *Let $g : E(P_{2n}) \to \mathbb{Z}$ be a labeling and $c \in \mathbb{Z}$. There exists a labeling h such that $I_h(P_{2n}) = c + I_g(P_{2n})$, where the range of h is a subset of $[1,n+1] \cup [c+1,c+n]$.*

Theorem 1.8 (Theorem 3.7 [8])**.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph G of order $p \equiv 1 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective. Then, b must be even.*

Theorem 1.9 (Theorem 3.3 [8])**.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph G of order $p \equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $1 \leq b-p < a < b$. Then, $b-a$ is odd.*

Theorem 1.10 (Theorem 3.9 [8])**.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph G of order $p \equiv 3 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective. Then, b must be odd.*

2. Useful Lemmas

In this section, we will construct a group-antimagic join graph from a given graph with a special edge labeling.

Lemma 2.1. *Suppose $f : E(G) \to [1, 4m-2]$ is a labeling of a graph G of order $4m-1$ such that $f^+ : V(G) \to [c, c+4m-2]$ is bijective, where $1 \leq c \leq 4m-1$. Then the join graph $G \vee K_1$ is $\mathbb{Z}_k$-antimagic for $k \geq 4m$.*

Proof. Let u be the vertex of K_1. By Theorem 1.10, c must be odd. Let $c = 2r+1$ for some $0 \leq r \leq 2m-1$. Then $r+2m \in [2r+1, 2r+4m-1]$. There is a unique vertex $v \in V(G)$ such that $f^+(v) = r+2m$. We shall extend the labeling f to the graph $G \vee K_1$ and denote the new labeling by g. That is, $g(e) = f(e)$ for all $e \in E(G)$.

For $w \neq v$, $g(uw) = 2m-r$ if $2r+1 \leq f^+(w) \leq r+2m-1$; and $g(uw) = -2m+r$ if $r+2m+1 \leq f^+(w) \leq 4m-1$; $g(uw) = -1$ if $f^+(w)$ is odd and $f^+(w) \geq 4m$; and $g(uw) = 1$ if $f^+(w)$ is even and $f^+(w) \geq 4m$. Finally, let $g(uv) = r+2m$.

Then one can check that $I_g(G \vee K_1) = [2r+1, 2r+4m]$. Note that $2m-r \not\equiv 0 \pmod k$ for $k \geq 4m$.

Thus by Proposition 1.2, $G \vee K_1$ is $\mathbb{Z}_k$-antimagic for $k \geq 4m$. □

Lemma 2.2. *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph G of order p such that $f^+ : V(G) \to [1, p]$ is bijective. Then the join graph $G \vee K_1$ is $\mathbb{Z}_k$-antimagic for $k \geq p+1$.*

Proof. Let u be the vertex of K_1.

Let $v \in V(G)$ such that $f^+(v) = p$. Then let $g(uv) = 2$ and $g(uw) = 1$ for $w \neq v$. One can check that $I_g(G \vee K_1) = [2, p+2]$.

Alternatively, let $v' \in V(G)$ such that $f^+(v') = 1$. Then let $h(uv') = p-1$ and $h(uw) = -1$ for $w \neq v'$. One can check that $I_h(G \vee K_1) = [0, p]$.

Hence $G \vee K_1$ is $\mathbb{Z}_k$-antimagic for $k \geq p+1$. □

Lemma 2.3. *Suppose $f : E(G) \to [1, 4m-1]$ is a labeling of a graph G of order $4m$ such that $f^+ : V(G) \to [c, c+4m-1]$ is bijective, where c is even and $2 \le c \le 4m-1$. Then the join graph $G \vee K_1$ is $\mathbb{Z}_k$-antimagic for $k \ge 4m+1$.*

Proof. Let u be the vertex of K_1. Let $c = 2r$ for some $1 \le r \le 2m-1$.

Suppose r is even. Since $r+2m \in [2r+1, 2r+4m-1]$, there is a unique vertex $v \in V(G)$ such that $f^+(v) = r+2m$. Similar to the proof of Lemma 2.1 we extend f to g. For vertex w with $c \le f^+(w) \le r+2m-1$, define $g(uw) = 1$ if $f^+(w)$ is even and $g(uw) = -1$ if $f^+(w)$ is odd; for vertex w with $r+2m+1 \le f^+(w) \le 2r+4m-1$, define $g(uw) = 1$ if $f^+(w)$ is odd and $g(uw) = -1$ if $f^+(w)$ is even. Finally, define $g(uv) = r+2m-1$. Then $g^+(v) = 2r+4m-1$ and $g^+(u) = r+2m$. Other vertex labels cover $[2r, r+2m-1] \cup [r+2m+1, 2r+4m-2] \cup \{2r+4m\}$. Hence $I_g(G \vee K_1) = [2r, 2r+4m]$. Thus g is a $\mathbb{Z}_k$-antimagic labeling for $G \vee K_1$.

Suppose r is odd. There is a unique vertex $v \in V(G)$ such that $f^+(v) = r+2m-1$. For vertex $w \ne v$, define $g(uw) = 1$ if $f^+(w)$ is even and $g(uw) = -1$ if $f^+(w)$ is odd. Finally, define $g(uv) = r+2m+1$. Then $g^+(v) = 2r+4m$ and $g^+(u) = r+2m$. Other vertex labels cover $[2r, r+2m-1] \cup [r+2m+1, 2r+4m-1]$. Hence $I_g(G \vee K_1) = [2r, 2r+4m]$. Thus g is a $\mathbb{Z}_k$-antimagic labeling for $G \vee K_1$. $\square$

Lemma 2.4. *Suppose $f : E(G) \to [1, 4m+1]$ is a labeling of a graph G of order $4m+2$ such that $f^+ : V(G) \to [c, c+4m+2] \setminus \{a\}$ is bijective, where $m \ge 1$, c is odd and $1 \le c < a < c+4m+2$. If $2c \le a$, then the join graph $G \vee K_1$ is $\mathbb{Z}_k$-antimagic for $k \ge 4m+3$.*

Proof. From Theorem 1.9 we know that a is even. Note that, the condition $2c \le a$ implies that $a/2 \in [c, c+4m+2]$.

Let u be the vertex of K_1. There are vertices $y, v_1, v_2 \in V(G)$ such that $f^+(y) = a/2$, $f^+(v_1) = a-1$ and $f^+(v_2) = a+1$. Note that y, v_1 and v_2 are distinct.

Suppose $a/2$ is even. Choose $x \in V(G)$ such that $f^+(x) = a/2-1$. Define $g(ux) = a/2+1$, $g(uy) = -1$, $g(uv_1) = 2$, $g(uv_2) = -2$. For other unlabeled edges, define $g(uw) = 1$ for odd $f^+(w)$ from c to $a-2$ and for even $f^+(w)$ from $a+2$ to $c+4m+2$; otherwise labeled by -1.

Suppose $a/2$ is odd and greater than 1. Choose $x \in V(G)$ such that $f^+(x) = a/2+1$. Define $g(ux) = a/2-1$, $g(uy) = 1$, other unlabeled edges are defined as the previous case.

If $a/2 = 1$, then $c = 1$. Let $v_1, v_2 \in V(G)$ such that $f^+(v_1) = 1$ and $f^+(v_2) = 3$. Define $g(uv_1) = 2$ and $g(uv_2) = -1$; for $w \notin \{v_1, v_2\}$, define $g(uw) = 1$ for even $f^+(w)$ and $g(uw) = -1$ for odd $f^+(w)$. $\square$

Lemma 2.5. *Suppose $f : E(G) \to [1, 4m]$ is a labeling of a graph G of order $4m+1$ such that $f^+ : V(G) \to [c, c+4m]$ is bijective, where $2 \le c \le 4m$. Then the join graph $G \vee K_1$ is $\mathbb{Z}_k$-antimagic for $k \ge 4m+3$.*

Proof. By Theorem 1.8 we know that $c = 2r$ for some $r \geq 1$. Let u be the vertex of K_1.

Suppose r is even. Let $v_1, v_2 \in V(G)$ such that $f^+(v_1) = r+2m$ and $f^+(v_2) = 2r+4m$. Define $g(uv_1) = r+2m$ and $g(uv_2) = 2$ first. Then $g^+(v_1) = 2r+4m$ and $g^+(v_2) = 2r+4m+2$. Let $w_1, w_2, w_3 \in V(G)$ such that $f^+(w_1) = r+2m+1$, $f^+(w_2) = r+2m+2$ and $f^+(w_3) = 2r+4m-1$. Define $g(uw_1) = g(uw_2) = -1$ and $g(uw_3) = 2$. Then $g^+(w_1) = r+2m$, $g^+(w_2) = r+2m+1$, $g^+(w_3) = 2r+4m+1$. For vertex $w \notin \{v_1, v_2, w_1, w_2, w_3\}$, define $g(uw) = 1$ for even $f^+(w)$ with $2r \leq f^+(w) \leq r+2m-2$; $g(uw) = -1$ for odd $f^+(w)$ with $2r+1 \leq f^+(w) \leq r+2m-1$, and $g(uw) = -1$ for even $f^+(w)$ with $r+2m+4 \leq f^+(w) \leq 2r+4m-2$; $g(uw) = 1$ for odd $f^+(w)$ with $r+2m+3 \leq f^+(w) \leq 2r+4m-3$. Then $g^+(u) = r+2m+2$ and the set of those $g^+(w)$'s is $[2r, r+2m-1] \cup [r+2m+3, 2r+4m-2]$. Hence $I_g(G \vee K_1) = [2r, 2r+4m+2] \setminus \{2r+4m-1\}$.

Suppose r is odd. Let $v_1, v_2 \in V(G)$ such that $f^+(v_1) = r+2m-1$ and $f^+(v_2) = 2r+4m$. Define $g(uv_1) = r+2m+1$ and $g(uv_2) = 2$ first. Then $g^+(v_1) = 2r+4m$ and $g^+(v_2) = 2r+4m+2$. Let $w_1, w_2, w_3 \in V(G)$ such that $f^+(w_1) = r+2m$, $f^+(w_2) = r+2m+1$ and $f^+(w_3) = r+2m+2$. Define $g(uw_1) = 2$, $g(uw_2) = -1$ and $g(uw_3) = -3$. Then $g^+(w_1) = r+2m+2$, $g^+(w_2) = r+2m$, $g^+(w_3) = r+2m-1$. For vertex $w \notin \{v_1, v_2, w_1, w_2, w_3\}$, define $g(uw) = 1$ for even $f^+(w)$ and $g(uw) = -1$ for odd $f^+(w)$. Then $g^+(u) = r+2m+1$ and the set of those $g^+(w)$'s is $[2r, r+2m-2] \cup [r+2m+3, 2r+4m-1]$. Hence $I_g(G \vee K_1) = [2r, 2r+4m+2] \setminus \{2r+4m+1\}$.

But for $c = 2$, we can extend f to a simpler labeling h as follows:
Let $v' \in V(G)$ such that $f^+(v') = 4m+2$. Define $h(uv') = 3$ and $h(uw) = 1$ for $w \neq v'$. Then $I_h(G \vee K_1) = [3, 4m+5] \setminus \{4m+4\}$. By Proposition 1.2 we have the lemma. □

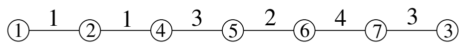
3. Applications

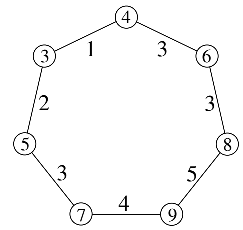
The *fan graph* F_n of order n is the join graph $P_{n-1} \vee K_1$. The *wheel graph* W_n of order n is the join graph $C_{n-1} \vee K_1$.

Theorem 3.1. *For $m \geq 1$, the fan graph F_{4m} and the wheel graph W_{4m} are $\mathbb{Z}_k$-antimagic for $k \geq 4m$.*

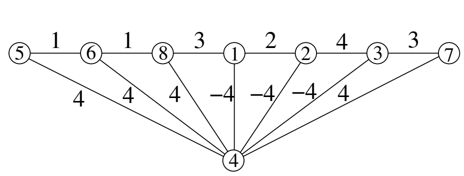
Proof. Combining Lemma 2.1, and Corollary 1.3 or 1.4, we have the theorem. □

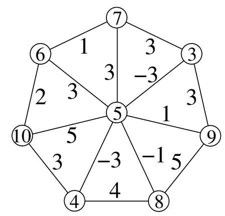
Example 3.1. Following are the labelings of P_7 and C_7 defined in [1], respectively.



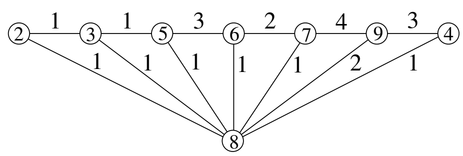


According to the proof of Lemma 2.1 we have labelings of F_8 and W_8 below, respectively.





Following is a labeling for F_8 defined in the proof of Lemma 2.2.



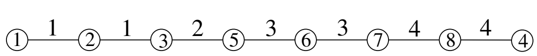
These labelings respectively induce $\mathbb{Z}_k$-antimagic labelings for F_8 and W_8, for $k \geq 8$. ■

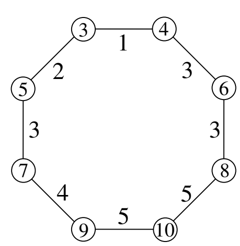
Theorem 3.2. *For $m \geq 1$, the fan graph F_{4m+1} and the wheel graph W_{4m+1} are $\mathbb{Z}_k$-antimagic for $k \geq 4m+1$.*

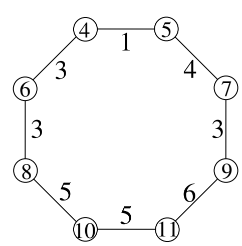
Proof. Combining Lemma 2.2 and Corollary 1.3 we show that F_{4m+1} is $\mathbb{Z}_k$-antimagic for all $k \geq 4m+1$.

From Corollary 1.4 and Corollary 1.6, there is a labeling h such that $I_h(C_{4m}) = [4, 4m+3]$. By Lemma 2.3 we obtain that W_{4m+1} is $\mathbb{Z}_k$-antimagic for all $k \geq 4m+1$. □

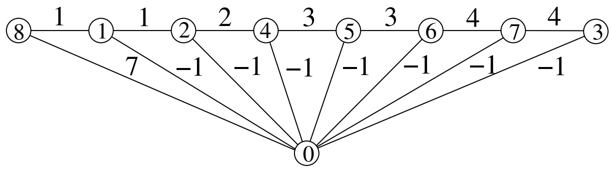
Example 3.2. Following are the original labelings of P_8 and C_8 defined in [1], and a modified labeling of C_8, respectively.

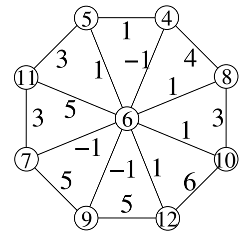






According to the proofs of Lemmas 2.2 and 2.3 we have labelings of F_9 and W_9 below, respectively.





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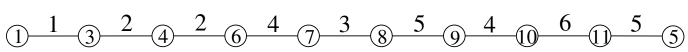
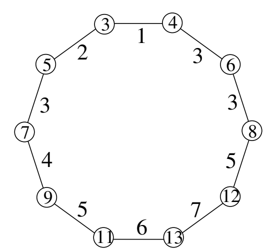
Theorem 3.3. *For $m \geq 1$, the fan graph F_{4m+3} and the wheel graph W_{4m+3} are $\mathbb{Z}_k$-antimagic for $k \geq 4m+3$.*

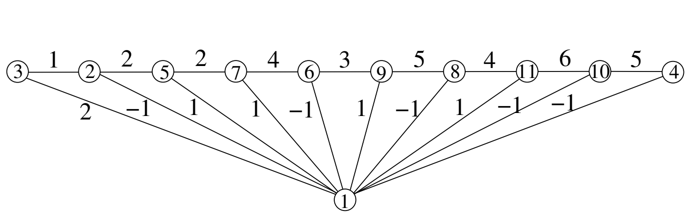
Proof. Combining Corollary 1.3 or 1.4 and Lemma 2.4 we have the theorem. □

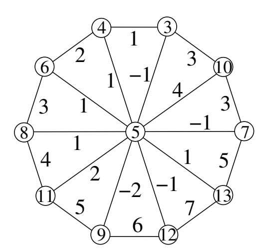
Example 3.3. Following are the labelings of P_{10} and C_{10} defined in [1], respectively.





According to the proof of Lemma 2.4 we have labelings of F_{11} and W_{11} below, respectively.





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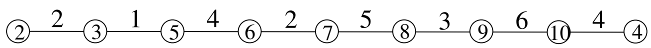
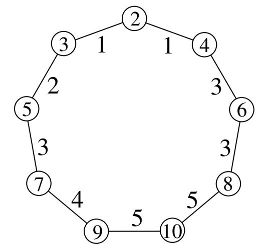
Theorem 3.4. *For $m \geq 1$, the fan graph F_{4m+2} and the wheel graph W_{4m+2} are $\mathbb{Z}_k$-antimagic for $k \geq 4m+3$.*

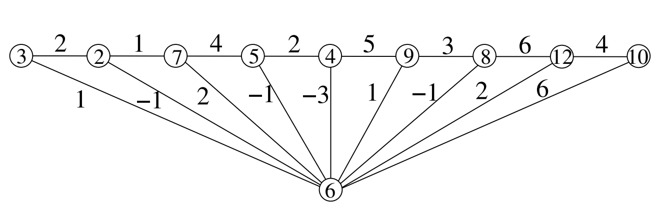
Proof. Combining Corollary 1.3 or 1.4 and Lemma 2.5 we have the theorem. □

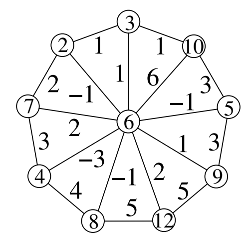
Example 3.4. Following are the labelings of P_9 and C_9 defined in [1], respectively.



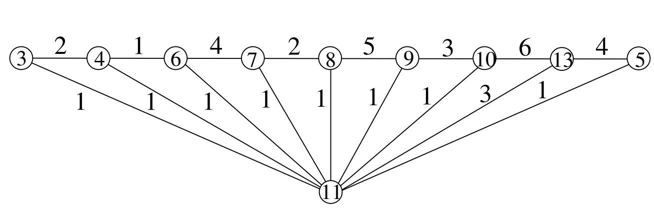


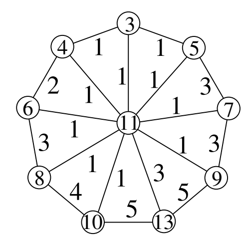
According to the proof of Lemma 2.5 we have (general) labelings of F_{10} and W_{10} below, respectively.





According to the proof of Lemma 2.5 we have simpler labelings of F_{10} and W_{10} below, respectively.





■

Combining the above results, we summarize as follows:

Theorem 3.5. *For $n \geq 4$,*

$$\mathrm{IAM}(W_n) = \mathrm{IAM}(F_n) = \begin{cases} [n, \infty), & \text{if } n \not\equiv 2 \pmod 4; \\ [n+1, \infty), & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Note that $\mathrm{IAM}(F_3) = \mathrm{IAM}(C_3) = [4, \infty)$ (see [1]).

4. Further Results

Let $V(W_p) = \{u, v_1, \ldots, v_{p-1}\}$, where $v_1v_2 \cdots v_{p-1}v_1$ is a $(p-1)$-cycle and u is the *hub* (or the *center*) of the wheel, i.e., $\deg(u) = p-1$. The edge uu_i, $1 \le i \le p-1$ is called a *spoke* of the wheel. Let $S = \{uv_i \mid 1 \le i \le p-1\}$ be the set of all spokes. Let $\varnothing \ne A \subset S$. The graph $W_p(A) = W_p - (S \setminus A)$ is called a *broken wheel graph* (or *broken wheel*, for short) [4]. Let k be a factor of $p-1$ with $k \ge 2$. Let $A_k = \{uv_{ik+1} \mid 0 \le i \le (p-1)/k - 1\}$. The graph $RW_p(k) = W_p(A_k)$ is called a *regular broken wheel*. Here, we only focus on $RW_{2n+1}(2)$. In some articles, for example [2], $RW_{2n+1}(2)$ is also called a *gear graph* and denoted by G_n. For simplicity, we shall use this notation for the rest of this paper.

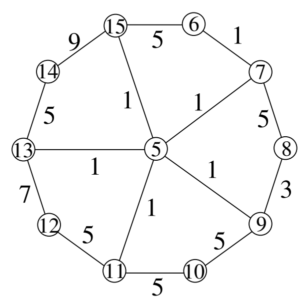
Theorem 4.1. *For $n \ge 2$,* $\mathrm{IAM}(G_n) = [2n+1, \infty)$.

Proof. For convenience, let $v_{2n+1} = v_1$. For $1 \le i \le 2n$, define

$$f(uv_i) = 1 \text{ for even } i \text{ and } f(v_iv_{i+1}) = \begin{cases} i, & i \text{ is odd;} \\ n, & i \text{ is even.} \end{cases}$$

Then $f^+(u) = n$ and $f^+(v_i) = n + i$ for $1 \le i \le 2n$. Thus $I_f(G_n) = [n, 3n]$ and hence G_n is $\mathbb{Z}_k$-antimagic for $k \ge 2n+1$. □

Example 4.1. Following is the labeling of G_5.



■

Corollary 4.2. *For $n \ge 2$,* $\mathrm{IAM}(G_n \vee K_1) = [2n+2, \infty)$ *for odd n and* $\mathrm{IAM}(G_n \vee K_1) = [2n+3, \infty)$ *for even n.*

Proof. This follows from Lemmas 2.1 and 2.5. □

Let G and H be connected simple graphs. Let $u \in V(G)$ and $v \in V(H)$. The graph $G^{uv}H$ is obtained from G and H by add a new edge (bridge) uv. By using the constructions described in [9, 8] we may construct many $\mathbb{Z}_k$-antimagic graphs of the form $G^{uv}H$, where H is either a path, a cycle, or a complete graph.

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