Invariant Factors of Cartesian Product of Graphs and One Point Unions of Graphs

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Abstract

A matrix called Varchenko matrix associated with a hyperplane arrangement was defined by Varchenko in 1991. Matrices that we shall call q-matrices are induced from Varchenko matrices. Many researchers are interested the invariant factors of these q-matrices. Shiu put this problem to the graph model. In this paper, invariant factors of Cartesian product of graphs will be found.

Keywords : q-matrix, invariant factors, Cartesian product of graphs, one point unions of graphs

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1 Introduction

In 1991, Schechtman and Varchenko defined a square matrix called Varchenko matrix, which is associated with a hyperplane arrangement [7, 10]. A q-matrix, which is a square matrix over the Euclidean domain $\mathbb{Q}[q]$, that is induced from Varchenko matrix. Following is a brief description of it. Please see [3, 8] for detail.

Let $\mathfrak{H}$ be an arrangement of hyperplanes in $\mathbb{R}^n$, $r(\mathfrak{H})$ be the set of regions in $\mathbb{R}^n$ induced by the hyperplanes and B be the Varchenko matrix of $\mathfrak{H}$ [9]. If we set all the weights assigned to hyperplanes to be q, then B becomes a matrix, say Q, whose entries are in $\mathbb{Q}[q]$. Shiu [8] called this matrix Q the q-matrix of $\mathfrak{H}$. Namely, for any $R_i, R_j \in r(\mathfrak{H})$, the (i,j)-entry (or the (R_i, R_j)-entry) of Q is given by $Q_{i,j} = q^{n(R_i,R_j)}$, where $n(R_i, R_j)$ is the number of hyperplanes in $\mathfrak{H}$ which separate R_i from R_j. So we can found the Smith normal form of a q-matrix. Entries appearing in the diagonal of a Smith normal form of the q-matrix are called invariant factors. Applications of invariant factors of a q-matrix can be found in [3].

Given an arrangement of hyperplanes $\mathfrak{H}$ in $\mathbb{R}^n$, we define a graph $G(\mathfrak{H})$ whose vertex set is $r(\mathfrak{H})$, two vertices (regions) are adjacent if their closures have an $(n-1)$-dimensional common boundary. $G(\mathfrak{H})$ is called the *graph* of $\mathfrak{H}$. It is easy to see that $G(\mathfrak{H})$ contains no odd cycles, i.e., $G(\mathfrak{H})$ is a (connected) bipartite graph. For any $R_i, R_j \in r(\mathfrak{H}) = V(G(\mathfrak{H}))$, let $x \in R_i$ and $y \in R_j$. Any connected curve joining x and y must pass through all the hyperplanes in $\mathfrak{H}$ which separate R_i and R_j at least once and there is a connected curve joining x and y passing through those hyperplanes exactly once. Thus $n(R_i, R_j)$ is the distance between R_i and R_j in $G(\mathfrak{H})$.

Shiu [8] generalized the concept of the q-matrix to a graph. Let G be a simple connected graph. Let $D_G = (d_{i,j})$ (or simply D) be the distance matrix of G under an ordering of vertices. Let $Q_G(q) = (q^{d_{i,j}})$ (or simply Q_G), where q is an indeterminate. $Q_G(q)$ is called the *q-matrix of G* (it is unique up to isomorphism). Nonzero invariant factors of $Q_G(q)$ are called *invariant factors* of G. Invariant factors of some graphs have been found in [8].

In this paper all graphs are simple and connected. We shall consider invariant factors of a Cartesian product of graphs and one point unions of graphs. All undefined concept and symbols may be looked up from [1] and [5].

The following example was described in [2, 3, 8].

Example 1.1 For $1 \leq i \leq k$, let $O_i = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_i = 0\}$. Let $\mathfrak{O}_k = \{O_1, \ldots, O_k\}$. Then $r(\mathfrak{O}_k)$ has 2^k regions and can be indexed by vectors $\alpha = (a_1, \ldots, a_k)$, where a_i is either 1 or -1. α corresponds to the region R_α which contains all points $(x_1, \ldots, x_k)$ where $x_i < 0$ if and only if $a_i = -1$. Then the graph of $\mathfrak{O}_k$ is isomorphic to the k-cube Q_k. Note that k-cube is the Cartesian product of k paths of order 2.

2 Some properties on Cartesian product of graphs

Let G and H be two graphs. The *Cartesian product* of G with H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$. The vertex (u_1, v_1) is adjacent to (u_2, v_2) if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. Clearly, $G \times H$ is isomorphic to $H \times G$.

Let G be a graph. Let $d_G(\cdot, \cdot)$ be the distance function defined on G. The following known result can be found in [4, Corollary 1.35].

Theorem 2.1 *Suppose G and H are graphs. Let $(u, v), (x, y) \in V(G \times H)$. Then*

$$d_{G\times H}((u, v), (x, y)) = d_G(u, x) + d_H(v, y).$$

By using the above theorem and some properties of Kronecker product of matrices [6], we have

Theorem 2.2 *Let G and H be graphs. Then $Q_{G\times H} = Q_G \otimes Q_H$, the Kronecker product of Q_G and Q_H.*

Proof: For any $(u,v),(x,y) \in V(G \times H)$. Under the lexicographic order, by Theorem 2.1 we have $(D_{G\times H})_{(u,v),(x,y)} = (D_G)_{u,x} + (D_H)_{v,y}$. Then

$$\begin{aligned}(Q_{G\times H})_{(u,v),(x,y)} &= q^{(D_G)_{u,x}+(D_H)_{v,y}} = q^{(D_G)_{u,x}} q^{(D_H)_{v,y}} \\ &= (Q_G \otimes Q_H)_{(u,v),(x,y)}.\end{aligned}$$

□

3 Finding invariant factors of a Cartesian product of graphs

Let Q be a q-matrix of a graph G of order n. Then Q is an $n \times n$ symmetric matrix over $\mathbb{Q}[q]$. To find the invariant factors of Q, Shiu [8] introduced a concept called pre-invariant factors of G. Following is the definition.

Two matrices A and B over a ring R are called *equivalent* if there are two invertible matrices U and V over R such that $B = UAV$. Let A be a square matrix over a principal ideal domain. If A is equivalent to a diagonal matrix B, then the multiset of entries in the diagonal of B is called a *pre-invariant factor set* of A. If Q is a q-matrix of a graph G, then the set of nonzero pre-invariant factors of Q is called a *pre-invariant factor set* of G. Elements of such set are called *pre-invariant factors* of G. Note that this set is not unique.

It is known [5] that for any matrix A over a principal ideal domain, A is equivalent to a matrix which has the 'diagonal' form

$$\begin{pmatrix} s_1 & & & & & \\ & s_2 & & & 0 & \\ & & \ddots & & & \\ & & & s_r & & \\ & 0 & & & 0 & \\ & & & & & \ddots \end{pmatrix}$$

where $s_i \neq 0$ and $s_i | s_j$ if $i \leq j$. Such s_i are called the nonzero *invariant factors* of A.

In particular, the multiset of invariant factors of a graph G is a pre-invariant factor set of G. The multiset of invariant factors is unique and is denoted by $\mathrm{Inv}(G)$. In fact, the invariant factors and pre-invariant factors can be defined for non-square matrix. From now on, the term 'set' means 'multiset'.

Theorem 3.1 ([8, **Corollary 2.6**]) *Let A be an $m \times n$ matrix of rank r over a principal ideal domain with nonzero invariant factors $s_1, s_2, \ldots, s_r$,* *where $s_i | s_{i+1}, 1 \leq i \leq r-1$. Suppose $\{f_1, \ldots, f_r, 0, \ldots, 0\}$ is a pre-invariant factor set of A, where $f_j \neq 0$. Let ϕ be an irreducible factor of $f_1 f_2 \cdots f_r$. Denote the multiplicities of ϕ in the factors f_j's by $0 \leq a_1 \leq a_2 \leq \cdots \leq a_r$. Then the multiplicity of ϕ in s_j is a_j.*

The sequence $\{a_1, a_2, \ldots, a_r\}$ is called the *multiplicity sequence* of ϕ. For a polynomial Φ, if the multiplicity sequences of all its irreducible factors are the same, then this sequence is called the *multiplicity sequence* of Φ.

Suppose S and T are two subsets of a ring. We define

$$S \cdot T = \{st \mid s \in S, t \in T\}.$$

Theorem 3.2 *Let G and H be graphs. Then a pre-invariant factor set of $G \times H$ is* $\mathrm{Inv}(G) \cdot \mathrm{Inv}(H)$.

Proof: There are invertible matrices P_1, P_1' such that $P_1 Q_G P_1'$ is equal to a diagonal matrix $\widetilde{Q}_G$, where the entries in the diagonal of $\widetilde{Q}_G$ are invariant factors of G. Similarly, there are invertible matrices P_2, P_2' such that $P_2 Q_H P_2'$ is equal to a diagonal matrix $\widetilde{Q}_H$, where the entries in the diagonal of $\widetilde{Q}_H$ are invariant factors of H. Then

$$\begin{aligned}
\widetilde{Q}_G \otimes \widetilde{Q}_H &= (P_1 Q_G P_1') \otimes (P_2 Q_H P_2') \\
&= (P_1 \otimes P_2)(Q_G \otimes Q_H)(P_1' \otimes P_2') \\
&= (P_1 \otimes P_2) Q_{G \times H} (P_1' \otimes P_2').
\end{aligned}$$

Since $P_1 \otimes P_2$ and $P_1' \otimes P_2'$ are invertible and $\widetilde{Q}_G \otimes \widetilde{Q}_H$ is a diagonal matrix, the entries in the diagonal of $\widetilde{Q}_G \otimes \widetilde{Q}_H$ are pre-invariant factors of $G \times H$. Hence we have the theorem. $\square$

By applying Theorem 3.1 to $\mathrm{Inv}(G) \cdot \mathrm{Inv}(H)$ we can get $\mathrm{Inv}(G \times H)$. We shall provide some examples in next section.

4 Examples

It is known from [8] that the invariant factors set of the n-path P_n is $\{1,\ (1-q^2)\ [n-1 \text{ times}]\}$ and that of the $2s$-cycle C_{2s} is $\{1,\ (1-q^2)\ [s \text{ times}],\ (1-q^2)^2,\ (1-q^2)(1-q^{2s})\ [s-2 \text{ times}]\}$, where $n \geq 1$ and $s \geq 2$. For convenience, we use $f\ [m]$ to mean the factor f appears m times in a set that it belongs to.

Example 4.1 Consider the cylinder $C_{2s} \times P_n$. We use a table to show $\mathrm{Inv}(C_{2s}) \cdot \mathrm{Inv}(P_n)$ as follows:

$1\ [1]$	$(1-q^2)\ [n-1]$
$(1-q^2)\ [s]$	$(1-q^2)^2\ [s(n-1)]$
$(1-q^2)^2\ [1]$	$(1-q^2)^3\ [n-1]$
$(1-q^2)(1-q^{2s})\ [s-2]$	$(1-q^2)^2(1-q^{2s})\ [n(s-2)]$

Note that the first column of the above table describes the invariant factors of C_{2s} and the first row describes the invariant factors of P_n.

In the bipartite case, the number of appearance of the irreducible factor $(1-q)$ in each pre-invariant factor is equal to that of the irreducible factor $(1+q)$. Thus we consider the factor $(1-q^2)$ instead of $(1-q)$ and $(1+q)$. Write $(1-q^{2s}) = (1-q^2)X$. Then the number of appearance of each irreducible factor of X in each pre-invariant factor is the same. Thus we consider the factor X instead of each irreducible factor of it.

The multiplicity sequence of $(1-q^2)$ is

Multiplicity	0	1	2	3
No. of appearance	1	$s+n-1$	$ns-1$	$(n-1)(s-1)$

and the multiplicity sequence of X is

Multiplicity	0	1
No. of appearance	$n(s+2)$	$n(s-2)$

Then the invariant factor set of $C_{2s} \times P_n$ is

$\{1\ [1],\ (1-q^2)\ [s+n-1],\ (1-q^2)^2\ [ns-1],\ (1-q^2)^3\ [n+1-s],$
$(1-q^2)^2(1-q^{2s})\ [n(s-2)]\}$ if $2 \le s \le n+1$;
$\{1\ [1],\ (1-q^2)\ [s+n-1],\ (1-q^2)^2\ [s(n-1)+n],\ (1-q^2)(1-q^{2s})\ [s-n-1]$
$(1-q^2)^2(1-q^{2s})\ [(n-1)(s-1)]\}$ if $2 \le n+1 < s$.

Example 4.2 Consider the torus $C_{2s} \times C_{2t}$. Then $\mathrm{Inv}(C_{2s}) \cdot \mathrm{Inv}(C_{2t})$ is

$1\ [1]$	$(1-q^2)\ [s]$
$(1-q^2)\ [t]$	$(1-q^2)^2\ [st]$
$(1-q^2)^2\ [1]$	$(1-q^2)^3\ [s]$
$(1-q^2)(1-q^{2t})\ [t-2]$	$(1-q^2)^2(1-q^{2t})\ [s(t-2)]$

$(1-q^2)^2\ [1]$	$(1-q^2)(1-q^{2s})\ [s-2]$
$(1-q^2)^3\ [t]$	$(1-q^2)^2(1-q^{2s})\ [t(s-2)]$
$(1-q^2)^4 [1]$	$(1-q^2)^3(1-q^{2s})\ [s-2]$
$(1-q^2)^3(1-q^{2t})\ [t-2]$	$(1-q^2)^2(1-q^{2s})(1-q^{2t})\ [(s-2)(t-2)]$

Note that the first row of the first table describes the first part invariant factors of C_{2s}, the first row of the second table describes the last part invariant factors of C_{2s}, and the first column of the first table describes the invariant factors of C_{2t}.

Similar to Example 4.1 the multiplicity sequence of the factor $(1-q^2)$ is listed as follows:

Multiplicity	0	1	2	3	4
No. of appearance	1	$s+t$	$st+s+t-2$	$2st-s-t$	$(s-1)(t-1)$

It is known that $q^n - 1 = \prod_{d|n} \Phi_d(q)$, where Φ_d is the d-th cyclotomic polynomial [11]. Let $g = \text{g.c.d}(s,t)$ and let $X = \prod_{\substack{d|2s \\ d\nmid 2g}} \Phi_d$, $Y = \prod_{\substack{d|2t \\ d\nmid 2g}} \Phi_d$ and $Z = \prod_{\substack{d|2g \\ d\neq 1,2}} \Phi_d$. Note that $\Phi_1(q) = q-1$ and $\Phi_2(q) = q+1$. So $1-q^{2s} = (1-q^2)XZ$ and $1-q^{2t} = (1-q^2)YZ$.

Any irreducible factor of X appears in each pre-invariant factor of the same times. It is similarly for Y and Z. Therefore, we consider the factors X, Y and Z instead of each irreducible factor.

If $X \neq 1$, then the multiplicity sequence of the factor X is are

Multiplicity	0	1
No. of appearance	$2t(s+2)$	$2t(s-2)$

Similarly if $Y \neq 1$, then the multiplicity sequence of the factor Y is

Multiplicity	0	1
No. of appearance	$2s(t+2)$	$2s(t-2)$

If $Z \neq 1$, then the multiplicity sequence of the factor Z is

Multiplicity	0	1	2
No. of appearance	$2(s+t)+st+4$	$2st-8$	$(s-2)(t-2)$

For convenience, when $X = 1$, $Y = 1$ and $Z = 1$, we adapt the above results as degenerated cases.

Without loss of generality, we let $s \geq t$.

Case 1. Suppose $s = t = 2$. In this case $X = Y = Z = 1$. It is easy to see that

$$\begin{aligned}\text{Inv}(C_4 \times C_4) = \{&1\ [1],\ (1-q^2)\ [4],\ (1-q^2)^2\ [6],\ (1-q^2)^3\ [4],\\ &(1-q^2)^4\ [1]\}.\end{aligned}$$

Case 2. Suppose $s > t = 2$. In this case, Y is either 1 if $g = 2$ or $\Phi_4 = (q^2+1)$ if $g = 1$. For the last case, s is odd and Y does not appear as a factor of any pre-invariant factor in $\mathrm{Inv}(C_s) \cdot \mathrm{Inv}(C_4)$.

It is easy to compute that the multiplicity sequence of the factor $(1-q^2)$ is

Multiplicity	0	1	2	3	4
No. of appearance	1	$s+2$	$3s$	$3s-2$	$s-1$

and the multiplicity sequence of the factor X is

Multiplicity	0	1
No. of appearance	$4s+8$	$4s-8$

Then

$$\begin{aligned}\mathrm{Inv}(C_{2s} \times C_4) = \{&1\ [1],\ (1-q^2)\ [s+2],\ (1-q^2)^2\ [3s],\\ &(1-q^2)^3\ [5],\ (1-q^2)^2(1-q^{2s})\ [3s-7],\\ &(1-q^2)^3(1-q^{2s})\ [s-1]\}.\end{aligned}$$

Case 3. Suppose $s \geq t \geq 3$. Then the multiplicity sequence of the factor X is

Multiplicity	0	1
No. of appearance	$2t(s+2)$	$2t(s-2)$

the multiplicity sequence of the factor Y is

Multiplicity	0	1
No. of appearance	$2s(t+2)$	$2s(t-2)$

and the multiplicity sequence of the factor Z is

Multiplicity	0	1	2
No. of appearance	$2(s+t)+st+4$	$2st-8$	$(s-2)(t-2)$

Then the invariant factors of $C_{2s} \times C_{2t}$ are shown below:

Invariant factor in terms of X, Y, Z	Invariant factor	Number of appearance
1	1	1
$(1-q^2)$	$(1-q^2)$	$s+t$
$(1-q^2)^2$	$(1-q^2)^2$	$st+s+t-2$
$(1-q^2)^3$	$(1-q^2)^3$	5
$(1-q^2)^3 Z$	$(1-q^2)^2(1-q^{2g})$	$(s+2)(t-2)$
$(1-q^2)^3 XZ$	$(1-q^2)^2(1-q^{2s})$	$4(s-t)$
$(1-q^2)^3 XYZ$	$\dfrac{(1-q^2)^2(1-q^{2s})(1-q^{2t})}{(1-q^{2g})}$	$st-3s+t-1$
$(1-q^2)^4 XYZ$	$\dfrac{(1-q^2)^3(1-q^{2s})(1-q^{2t})}{(1-q^{2g})}$	$s+t-3$
$(1-q^2)^4 XYZ^2$	$(1-q^2)^2(1-q^{2s})(1-q^{2t})$	$(s-2)(t-2)$

In particular, when $s = t \geq 3$ we have

$$\begin{aligned}\operatorname{Inv}(C_{2s} \times C_{2s}) = \{&1\ [1],\ (1-q^2)\ [2s],\ (1-q^2)^2\ [s^2+2s-2],\\ &(1-q^2)^3\ [5],\ (1-q^2)^2(1-q^{2s})\ [2s^2-2s-5],\\ &(1-q^2)^3(1-q^{2s})\ [2s-3],\\ &(1-q^2)^2(1-q^{2s})^2\ [(s-2)^2]\}.\end{aligned}$$

Example 4.3 Consider the k-dimensional grid $P_{n_1} \times \cdots \times P_{n_k}$. Since any pre-invariant factor in $\operatorname{Inv}(P_{n_1}) \cdots \operatorname{Inv}(P_{n_k})$ is of the form $(1-q^2)^j$ for some $j \geq 0$, $\operatorname{Inv}(P_{n_1} \times \cdots \times P_{n_k}) = \operatorname{Inv}(P_{n_1}) \cdots \operatorname{Inv}(P_{n_k})$. We shall use the generating function $f_n = 1 + (n-1)x$ as an enumerator of $\operatorname{Inv}(P_n)$. Then the generating function of $\operatorname{Inv}(P_{n_1}) \cdots \operatorname{Inv}(P_{n_k})$ is $f_{n_1} \cdots f_{n_k} = 1 + \sum_{j=1}^{k} a_j x^j$, where $a_j = \sum_{1 \leq i_1 < \cdots < i_j \leq k} \left(\prod_{l=1}^{j} (n_{i_l} - 1) \right)$.

Note that if we replace P_{n_i} by T_{n_i}, a tree of order n_i, then we will get the same result. Suppose $n_1 = \cdots = n_k = n$. Then the invariant factor set of $\underbrace{P_n \times \cdots \times P_n}_{k \text{ times}}$ is

$$\{1\ [1], (1-q^2)\ [\tbinom{k}{1}(n-1)],\ \ldots, (1-q^2)^j\ [\tbinom{k}{j}(n-1)^j], \ldots, (1-q^2)^k\ [(n-1)^k]\}.$$

In particular,

$$\operatorname{Inv}(Q_k) = \{1\ [1],\ (1-q^2)\ [k],\ \ldots, (1-q^2)^j\ [\binom{k}{j}], \ldots, (1-q^2)^k\ [1]\}.$$

So the invariant factors of the hyperplane arrangement described in Example 1.1 has been found.

5 Finding invariant factors of one point unions of graphs

Suppose two graphs G and H have exactly one common vertex. Then the union of these two graphs is called a *one point union* of G and H and denoted by $G \cdot H$. Clearly, $G \cdot H$ is isomorphic to $H \cdot G$. Suppose G_i are graphs, $1 \leq i \leq n$. We use the notation $G_1 \cdot G_2 \cdots G_n$ to denote $(G_1 \cdot G_2 \cdots G_{n-1}) \cdot G_n$, which is called a *one point unions* of graphs $G_1, \dots, G_n$.

In this section we will extend the following theorem which is described in [8].

Theorem 5.1 ([8, Theorem 4.2]) *Suppose H and K are graphs. Then* $\mathrm{Inv}(H) \cup \mathrm{Inv}(K) \setminus \{1\}$ *is a pre-invariant factor set of $H \cdot K$.*

Corollary 5.2 *Let H_i be graphs, $1 \leq i \leq n$. Then a pre-invariant factor set of $H_1 \cdot H_2 \cdots H_n$ is* $\left(\bigcup_{i=1}^{n} \mathrm{Inv}(H_i)\right) \setminus \{1\ [n-1]\}$.

Proof: Suppose $n = 3$. By Theorem 5.1 $S = \mathrm{Inv}(H_1) \cup \mathrm{Inv}(H_2) \setminus \{1\}$ is a pre-invariant factor set of $H_1 \cdot H_2$. From S we can get $\mathrm{Inv}(H_1 \cdot H_2)$. By Theorem 5.1 again, $\mathrm{Inv}(H_1 \cdot H_2) \cup \mathrm{Inv}(H_3) \setminus \{1\}$ is a pre-invariant factor set of $H_1 \cdot H_2 \cdot H_3$. Since the multiplicity sequence of each irreducible factor is unique, we can replace $\mathrm{Inv}(H_1 \cdot H_2)$ by S to be a part of pre-invariant set. Then $S \cup \mathrm{Inv}(H_3) \setminus \{1\} = \mathrm{Inv}(H_1) \cup \mathrm{Inv}(H_2) \cup \mathrm{Inv}(H_3) \setminus \{1, 1\}$ is also a pre-invariant factor set of $H_1 \cdot H_2 \cdot H_3$.

For the general case is just applying Theorem 5.1 repeatedly. □

Example 5.1 Consider $C_{2s} \cdot C_{2t}$ for $s \geq t \geq 2$. By Theorem 5.1 a pre-invariant set of $C_{2s} \cdot C_{2t}$ is
$\{1\ [1],\ (1-q^2)\ [s+t],\ (1-q^2)^2\ [2],\ (1-q^2)(1-q^{2s})\ [s-2],\ (1-q^2)(1-q^{2t})\ [t-2]\}$.

Let X, Y, Z be defined as in Example 4.2. Similar to the previous examples, we have the multiplicity sequences of the above factors.

The multiplicity sequence of the factor $(1-q^2)$ is

Multiplicity	0	1	2
No. of appearance	1	$s+t$	$s+t-2$

The multiplicity sequence of the factor X is

Multiplicity	0	1
No. of appearance	$s+2t+1$	$s-2$

The multiplicity sequence of the factor Y is

Multiplicity	0	1
No. of appearance	$t+2s+1$	$t-1+2$

The multiplicity sequence of the factor Z is

Multiplicity	0	1
No. of appearance	$s+t+3$	$s+t-4$

If $s \geq t = 2$, then Y does not appear as a factor of any pre-invariant factor. It is easy to find that

$$\{1\ [1],\ (1-q^2)\ [s+2],\ (1-q^2)^2\ [2],\ (1-q^2)(1-q^{2s})\ [s-2]\}$$

is the invariant factor set of $C_{2s} \cdot C_4$.

If $s \geq t \geq 3$, then

Invariant factor in terms of X, Y, Z	Invariant factor	Number of appearance
1	1	1
$(1-q^2)$	$(1-q^2)$	$s+t$
$(1-q^2)^2$	$(1-q^2)^2$	2
$(1-q^2)^2 Z$	$(1-q^2)(1-q^{2g})$	$t-2$
$(1-q^2)^2 XZ$	$(1-q^2)(1-q^{2s})$	$s-t$
$(1-q^2)^2 XYZ$	$(1-q^2)(1-q^{2s})(1-q^{2t})/(1-q^{2g})$	$t-2$

Example 5.2 Consider the graph $\overbrace{C_{2s} \cdot C_{2s} \cdots C_{2s}}^{n \text{ times}}$, where $s \geq 2$ and $n \geq 1$. Then the multiplicity sequence of $(1-q^2)$ is $\{0\ [1],\ 1\ [ns],\ 2\ [n(s-1)]\}$ and that of X is $\{0\ [ns+n+1],\ 1\ [n(s-2)]\}$. Therefore, the invariance factor set of $\underbrace{C_{2s} \cdot C_{2s} \cdots C_{2s}}_{n \text{ times}}$ is

$$\{1\ [1],\ (1-q^2)\ [ns],\ (1-q^2)^2\ [n],\ (1-q^2)(1-q^{2s})\ [n(s-2)]\}.$$

Even though the complete graph K_n $(n \geq 3)$ is not bipartite, we can also consider its invariant factors. From [8], it is known that for $n \geq 2$,

$$\mathrm{Inv}(K_n) = \{1\ [1],\ (1-q)\ [n-2],\ (1-q)(1+(n-1)q)\ [1]\}.$$

Example 5.3 Consider the graph $K_{m_1} \cdot K_{m_2} \cdots K_{m_n}$. Suppose $n_1 \geq n_2 \geq \cdots \geq n_t$ are the multiplicities of the distinct values $r_1, r_2, \ldots, r_t$

of $m_1, m_2, \dots, m_n$, respectively. Then

$$\begin{aligned}
&\bigcup_{i=1}^{n} \operatorname{Inv}(K_{m_i}) \setminus \{1\ [n-1]\} \\
&= \{1, (1-q)\ [\sum_{i=1}^{n}(m_i - 2)],\ (1-q)(1+(m_1-1)q)\ [1],\ \dots, \\
&\quad (1-q)(1+(m_n-1)q)\ [1]\} \\
&= \{1, (1-q)\ [(\sum_{i=1}^{n} n_i r_i) - 2n],\ (1-q)(1+(r_1-1)q)\ [n_1],\ \dots, \\
&\quad (1-q)(1+(r_t-1)q)\ [n_t]\}
\end{aligned}$$

Now we have the multiplicity sequence of $(1-q)$ is

$$\{0\ [1],\ 1\ [n_1 + \cdots + n_t r_t - n]\}$$

and that of $1 + (r_i - 1)q$ is

$$\{0\ [n_1 r_1 + \cdots + n_t r_t - n + 1 - n_i],\ 1\ [n_i]\},$$

for $1 \le i \le t$. Therefore, the invariant factor set of $K_{m_1} \cdot K_{m_2} \cdots K_{m_n}$ is

$$\begin{aligned}
&\{1\ [1],\ (1-q)\ [n_1 r_1 + \cdots + n_t r_t - n - n_1], \\
&\ (1-q)(1+(r_1-1)q)\ [n_1 - n_2], \\
&\ (1-q)(1+(r_1-1)q)(1+(r_2-1)q)\ [n_2 - n_3], \\
&\qquad \vdots \\
&\ (1-q)(1+(r_1-1)q)\cdots(1-(r_{t-1}-1)q)\ [n_{t-1} - n_t], \\
&\ (1-q)(1+(r_1-1)q)\cdots(1-(r_{t-1}-1)q)(1-(r_t-1)q)\ [n_t]\}.
\end{aligned}$$

For example, consider the graph $K_4 \cdot K_5 \cdot K_6$. Then $r_1 = 4$, $r_2 = 5$, $r_3 = 6$ and $n_1 = n_2 = n_3 = 1$. We have

$$\operatorname{Inv}(K_4 \cdot K_5 \cdot K_6) = \{1\ [1],\ (1-q)\ [11],\ (1-q)(1+3q)(1+4q)(1+5q)\ [1]\}.$$

Consider the graph $K_5 \cdot K_5 \cdot K_6$. Then $r_1 = 5$, $r_2 = 6$ $n_1 = 2$ and $n_2 = 1$. We have

$$\begin{aligned}
\operatorname{Inv}(K_5 \cdot K_5 \cdot K_6) = \{&1\ [1],\ (1-q)\ [11],\ (1-q)(1+4q)\ [1], \\
&(1-q)(1+4q)(1+5q)\ [1]\}.
\end{aligned}$$

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