Uniformly pair-bonded trees*

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Abstract

Let $G = (V(G), E(G))$ be a graph with $\delta(G) \geq 1$. A set $D \subseteq V(G)$ is a paired-dominating set if D is a dominating set and the induced subgraph $G[D]$ contains a perfect matching. The paired domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a paired-dominating set of G. The paired bondage number, denoted by $b_p(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma_p(G - E') > \gamma_p(G)$. For any $b_p(G)$ edges $E' \subseteq E$ with $\delta(G - E') \geq 1$, if $\gamma_p(G - E') > \gamma_p(G)$, then G is called uniformly pair-bonded graph. In this paper, we prove that there exists uniformly pair-bonded tree T with $b_p(T) = k$ for any positive integer k. Furthermore, we give a constructive characterization of uniformly pair-bonded trees.

Keywords : Domination number, paired-domination number, paired bondage number, uniformly pair-bonded graph.

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1 Introduction

In this paper, we consider finite undirected simple connected graphs. For all undefined concepts and notations in this paper the reader is referred to [1]. By $V(G)$ and $E(G)$, we mean the vertex set and the edge set of a graph G, respectively. Let $n(G) = |V(G)|$ and $m(G) = |E(G)|$. We write $G[S]$ for the subgraph of G induced by $S \subseteq V(G)$.

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A set $S \subseteq V(G)$ is a *dominating set* of G if each vertex of $V(G) \setminus S$ is adjacent to at least one vertex in S. The cardinality of a minimum dominating set is called the *domination number* of G, denoted by $\gamma(G)$.

The *bondage number* $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $E' \subseteq E$ for which $\gamma(G-E') > \gamma(G)$. Bondage in graphs was introduced by Fink *et al.* [3] and further studied for example in [4, 6].

A graph is called *uniformly bonded*, which was introduced by Hartnell and Rall in [4], if it has bondage number b and the deletion of any b edges results in a graph with increased domination number. Let P_n and C_n denote a path and a cycle with n vertices, respectively. Hartnell and Rall [4] obtained the following result.

Theorem 1.1 *The uniformly bonded graphs with $b(G) = 2$ are C_3 and P_4. The unique uniformly bonded graph with $b(G) = 3$ is C_4. There are no uniformly bonded graphs with $b(G) > 3$.*

A dominating set S is called a *paired-dominating set* if its induced subgraph contains a perfect matching. The cardinality of a minimum paired-dominating set is the *paired-domination number*, denoted by $\gamma_p(G)$. The paired-domination number was introduced by Haynes and Slater [5] and further studied in [7, 2, 8]. A minimum paired-dominating set of G is also called a *γ_p-set of G*.

The *paired bondage number* of G with $\delta(G) \geq 1$, denoted by $b_p(G)$, is the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\delta(G - E') \geq 1$ and $\gamma_p(G - E') > \gamma_p(G)$. In particular, it was defined that $b_p(K_{1,n}) = 0$ for all star graphs $K_{1,n}$. The paired bondage number was introduced by Raczek in [7]. A graph is called *uniformly pair-bonded* if it has paired bondage number $b_p(G)$, and for any subset $E' \subseteq E$ with $\delta(G - E') \geq 1$ and $|E'| = b_p(G)$, the deletion of E' results in a graph with increased paired-domination number. Raczek [7] obtained the following result.

Theorem 1.2 *For any non-negative integer k, there exists a tree with $b_p(T) = k$.*

2 Main results

In this paper, we prove that there exists a uniformly pair-bonded tree T with $b_p(T) = k$ for any positive integer k. Furthermore, we give a constructive characterization of uniformly pair-bonded trees.

Theorem 2.1 *Let G be a uniformly pair-bonded graph. Then $b_p(G) > m(G) - n(G) + \frac{\gamma_p(G)}{2}$.*

Proof: Let S be a γ_p-set of G, and let $E(S, V \setminus S)$ denote the set of edges between S and $V \setminus S$. Define $E_1 \subseteq E(S, V \setminus S)$ such that for each vertex $v \in V - S$, v is incident with exactly one edge of E_1. So $|E_1| = |V \setminus S| = n - \gamma_p(G)$.

Let $E_2 = E(S, V \setminus S) \setminus E_1$. Then $|E(S, V \setminus S)| = |E_1| + |E_2|$. Let M be a perfect matching of $G[S]$ and $E_3 = E(G[S]) \setminus M$. Then $|E(G[S])| = |M| + |E_3| = \frac{|S|}{2} + |E_3|$. By definition, we have

$$\sum_{v \in V \setminus S} d(v) = 2|E(G[V \setminus S])| + |E(S, V \setminus S)| = 2|E(G[V \setminus S])| + |E_1| + |E_2|$$

and

$$\sum_{v \in S} d(v) = 2|E(G[S])| + |E(S, V \setminus S)| = 2(\frac{|S|}{2} + |E_3|) + |E_1| + |E_2|.$$

Combining the above equalities, we have $m(G) = |E(G[V \setminus S])| + \frac{|S|}{2} + |E_1| + |E_2| + |E_3|$. So,

$$|E(G[V \setminus S])| + |E_2| + |E_3| = m(G) - |E_1| - \frac{\gamma_p(G)}{2}.$$

Thus,

$$|E(G[V \setminus S]) \cup E_2 \cup E_3| = m(G) - n(G) + \frac{\gamma_p(G)}{2}.$$

For any edge set $E \subseteq E(G[V \setminus S]) \cup E_2 \cup E_3$, we have $\delta(G - E) \geq 1$ and $\gamma_p(G - E) \leq \gamma_p(G)$. Since G is a uniformly pair-bonded graph, $b_p(G) > |E(G[V \setminus S]) \cup E_2 \cup E_3| = m(G) - n(G) + \frac{\gamma_p(G)}{2}$. □

Corollary 2.2 *Let T be a uniformly pair-bonded tree. Then*

$$b_p(T) > \frac{\gamma_p(T)}{2} - 1.$$

Let G be a graph. The *open neighborhood* of $v \in V(G)$ in G, denoted by $N_G(v)$, is the set $\{u \in V(G) \mid uv \in E(G)\}$. The *closed neighborhood* of v in G, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. The vertex v is a *leaf* if $|N_G(v)| = 1$. If v is adjacent to a leaf, v is called a *support vertex*.

For any tree T, let $L(T)$ denote the set of leaves of T. If $diam(T) \geq 4$, let $P = v_1 v_2 v_3 v_4 \cdots v_t$ be a longest path in T. Define the edge sets $E_3 = \{uv_3 \mid u \in N_T(v_3),\ u \notin \{v_2, v_4\} \cup L(T)\}$ and $E_4 = \{uv_4 \mid u \in N_T(v_4),\ u \notin \{v_3\} \cup L(T)\}$. (We shall keep these notations up to the end of the proof of Theorem 2.6).

Proposition 2.3 *Let T be a tree with $diam(T) \geq 4$. Suppose that v_3 is not a support vertex and v_4 is a support vertex. Let $F = T - v_3v_4$. Let F_1 and F_2 denote the components of F containing v_3 and v_4, respectively. Then $b_p(T) \leq 1 + b_p(F_2)$.*

Proof: Let E be a minimum edge set of F_2 such that $\delta(F_2 - E) \geq 1$ and $\gamma_p(F_2 - E) > \gamma_P(F_2)$. It is obvious that $\gamma_p(T) \leq \gamma_p(F_1) + \gamma_p(F_2)$. So, $\gamma_p(T) \leq \gamma_p(F_1) + \gamma_p(F_2) < \gamma_p(F_1) + \gamma_p(F_2 - E) = \gamma_p(T - (\{v_3v_4\} \cup E))$. Hence $b_p(T) \leq 1 + b_p(F_2)$. □

Lemma 2.4 *Let T be a tree with $diam(T) \geq 5$. If v_4 is not a support vertex, then $b_p(T) \leq 1 + |E_3| + |E_4|$.*

Proof: Let $F = T - (\{v_2v_3\} \cup E_3 \cup E_4)$. Then F has no isolated vertices. Let F_1 and F_2 denote the components of F containing v_2 and v_3, respectively. Define $F_3 = F - (F_1 \cup F_2)$. It is obvious that

$$\begin{aligned}\gamma_p(T) &\leq \gamma_p(F_1 \cup F_2 \cup \{v_2v_3\}) + \gamma_p(F_3) = 2 + \gamma_p(F_3)\\ &< 4 + \gamma_p(F_3) = \gamma_p(F_1) + \gamma_p(F_2) + \gamma_p(F_3) = \gamma_p(F)\\ &= \gamma_p(T - (\{v_2v_3\} \cup E_3 \cup E_4)).\end{aligned}$$

Hence $b_p(T) \leq 1 + |E_3| + |E_4|$. □

Lemma 2.5 *Let T be a tree with $diam(T) \geq 4$. If both v_3 and v_4 are support vertices, then $b_p(T) \leq 2 + |E_3|$.*

Proof: Let $F = T - v_3v_4$. Then F has no isolated vertices. Let F_1 and F_2 denote the components of F containing v_3 and v_4, respectively. It is obvious that $\gamma_p(T) \leq \gamma_p(F_1) + \gamma_p(F_2)$. Let S be a γ_p-set of T, and let M be a perfect matching of $T[S]$. Since v_2, v_3 and v_4 are support vertices of T, it follows that $v_2, v_3, v_4 \in S$.

If $v_3v_4 \in M$, then there exists a vertex $u \in N(v_2) \cap L(T)$ such that $uv_2 \in M$. Say $v \in N(v_4) \cap L(T)$. Then $v \notin S$. Let $S' = (S \setminus \{u\}) \cup \{v\}$. Then S' is a γ_p-set of T and v_3v_4 does not belong to any perfect matching of $T[S']$. So, without loss of generality, we may assume that $v_3v_4 \notin M$. Then $S \cap V(F_1)$ and $S \cap V(F_2)$ are paired-dominating sets of F_1 and F_2, respectively. Hence, $\gamma_p(F_1) \leq |S \cap V(F_1)|$ and $\gamma_p(F_2) \leq |S \cap V(F_2)|$. So $\gamma_p(F_1) + \gamma_p(F_2) \leq |S \cap V(F_1)| + |S \cap V(F_2)| = |S| = \gamma_p(T)$. Therefore, $\gamma_p(T) = \gamma_p(F_1) + \gamma_p(F_2)$.

Since $\gamma_p(F_1) < \gamma_p(F_1 - (\{v_2v_3\} \cup E_3))$, $\gamma_p(T) = \gamma_p(F_1) + \gamma_p(F_2) < \gamma_p(F_1 - (\{v_2v_3\} \cup E_3)) + \gamma_p(F_2)$. Thus, $\gamma_p(T) < \gamma_p(T - (\{v_2v_3, v_3v_4\} \cup E_3))$. Hence, $b_p(T) \leq 2 + |E_3|$. □

It is easy to see that for any tree T, if $b_p(T) \geq 2$, then $diam(T) \geq 4$.

Theorem 2.6 *Let T be a uniformly pair-bonded tree with $b_p(T) = k \geq 2$. Let T_1 and T_2 denote the two components of $T - v_2v_3$, where $v_2 \in V(T_1)$ and $v_3 \in V(T_2)$. Then T_2 is a uniformly pair-bonded tree with $b_p(T_2) = k - 1$.*

Proof: Since $b_p(T) = k$, it follows that $\gamma_p(T - v_2v_3) = \gamma_p(T)$. So, $\gamma_p(T_1) + \gamma_p(T_2) = \gamma_p(T)$. For any edge set $E \subseteq E(T_2)$ with $|E| = k - 1$ and $\delta(T_2 - E) \geq 1$, $\gamma_p(T - E - v_2v_3) > \gamma_p(T)$. So $\gamma_p(T_1) + \gamma_p(T_2 - E) > \gamma_p(T_1) + \gamma_p(T_2)$. Hence $\gamma_p(T_2 - E) > \gamma_p(T_2)$. So $b_p(T_2) \leq k - 1$. If there exists an edge set $E' \subseteq E(T_2)$ with $|E'| < k - 1$, $\delta(T_2 - E') \geq 1$ and $\gamma_p(T_2 - E') > \gamma_p(T_2)$, then $\gamma_p(T_1) + \gamma_p(T_2 - E') > \gamma_p(T_1) + \gamma_p(T_2) = \gamma_p(T - v_2v_3)$. That is, $\gamma_p(T - E' - v_2v_3) > \gamma_p(T - v_2v_3) = \gamma_p(T)$. Hence $b_p(T) \leq k - 1$, which is a contradiction. Hence, T_2 is a uniformly pair-bonded tree with $b_p(T_2) = k - 1$. □

Let $K_{1,r}$ denote a star with r leaves. The vertex of $K_{1,r}$ with degree r is called the *central vertex*. Let $S(k,l)$ be obtained from stars $K_{1,k}$ and $K_{1,l}$ by joining an edge between the central vertices. $S(k,l)$ is called a *double star*. By Corollary 2.2, we have the following result.

Theorem 2.7 *Let T be a tree with $b_p(T) = 1$. Then T is a uniformly pair-bonded tree if and only if T is a double star.*

In the following, we define two operations on T when T is either a star or a double star.

- **Operation 1:** If T is a star, we attach to each vertex of T at least one leaf.
- **Operation 2:** If T is a double star, we attach to each leaf of T at least one leaf.

Let τ_1 be the family of all trees obtained from stars by Operation 1, and let τ_2 be the family of all trees obtained from double stars by Operation 2.

Theorem 2.8 *Suppose that T is obtained from the star $K_{1,r}$ by Operation 1. Then T is a uniformly pair-bonded tree with $b_p(T) = r$.*

Proof: Let E denote the edge set of star $K_{1,r}$. It is obvious that $\gamma_p(T) = 2r$ and $\gamma_p(T - E) = 2r + 2$. So, we have $b_p(T) \leq r$. For any $E' \subset E$, we have $\gamma_p(T - E') = \gamma_p(T)$. Hence, $b_p(T) = r$. Since E is the unique set of edges of T such that $|E| = r$ and $\delta(T - E) \geq 1$, T is a uniformly pair-bonded tree. □

Theorem 2.9 *Suppose that T is obtained from the double star $S(r,s)$ by Operation 2. Then T is a uniformly pair-bonded tree with $b_p(T) = r + s$.*

Proof: Suppose that u and v are the central vertices of the double star $S(r,s)$. Let E denote the edge set of the double star $S(r,s)$. It is easy to prove that $\gamma_p(T) = 2r + 2s$ and $\gamma_p(T - E + uv) = 2r + 2s + 2$. So, $b_p(T) \leq r + s$. For any $E' \subset E$ with $|E'| < |E| - 1$ and $\delta(T - E') \geq 1$, we have $\gamma_p(T - E') = \gamma_p(T)$. Hence, $b_p(T) = r + s$. Since $E \setminus \{uv\}$ is the unique set of edges such that $|E \setminus \{uv\}| = r + s$ and $\delta(T - E + uv) \geq 1$, T is a uniformly pair-bonded tree. $\square$

Theorem 2.10 *If T is a uniformly pair-bonded tree, then T is a double star or $T \in \tau_1 \cup \tau_2$.*

Proof: We shall prove the theorem by induction on $b_p(T)$. If T is a uniformly pair-bonded tree with $b_p(T) = 1$, by Theorem 2.7, T is a double star.

Suppose that T is a uniformly pair-bonded tree with $b_p(T) = 2$. Let $v_1v_2v_3v_4 \cdots v_t$ be a longest path of T. We write v_2v_3 as e. Let T_1 and T_2 be the two components of $T - e$, where $v_2 \in V(T_1)$ and $v_3 \in V(T_2)$. Then T_1 is a star with the central vertex v_2. By Theorem 2.6, T_2 is a uniformly pair-bonded tree with $b_p(T_2) = 1$. By Theorem 2.7, T_2 is a double star.

Let u and v denote the central vertices of T_2. By symmetry, we may assume that $v_3 \in N_{T_2}[v] \setminus \{u\}$. Suppose that $v_3 = v$. Then T is obtained from the star $K_{1,2}$ by Operation 1. Hence, $T \in \tau_1$. Suppose that $v_3 \in N_{T_2}(v) \setminus \{u\}$. If $|N(v) \cap L(T_2)| \geq 2$, then $b_p(T) = 1$, which yields a contradiction. Thus $|N(v) \cap L(T_2)| = 1$. Then T is obtained from the double star $S(1,1)$ by Operation 2. Hence, $T \in \tau_2$. Therefore, $T \in \tau_1 \cup \tau_2$.

For $k \geq 3$, we assume that if T' is a uniformly pair-bonded tree with $b_p(T') = k - 1$, then $T' \in \tau_1 \cup \tau_2$.

Now, let T be a uniformly pair-bonded tree with $b_p(T) = k$. Let $v_1v_2v_3v_4 \cdots v_t$ be a longest path of T. We write $e = v_2v_3$. Let T_1 and T_2 be the two components of $T - e$, where $v_2 \in V(T_1)$ and $v_3 \in V(T_2)$. Then T_1 is a star with the central vertex v_2. By Theorem 2.6, T_2 is a uniformly pair-bonded tree with $b_p(T_2) = k - 1$. By the induction assumption, $T_2 \in \tau_1 \cup \tau_2$. We will show in the following that in each of the cases $T_2 \in \tau_1$ and $T_2 \in \tau_2$, $T \in \tau_1 \cup \tau_2$.

Case 1: Suppose $T_2 \in \tau_1$. Since $b_p(T_2) = k - 1$, T_2 is obtained from a star $K_{1,k-1}$ by Operation 1. Let E denote the set of edges of $K_{1,k-1}$, and let c be the central vertex of the star. Then we consider the following four subcases.

Subcase 1: Suppose $v_3 \in N_{T_2}(c) \cap L(T_2)$. If $|N_{T_2}(c) \cap L(T_2)| \geq 2$, then $\gamma_p(T) = 2k$. It is easy to see that $\gamma_p(T - E) > \gamma_p(T)$. Hence, $b_p(T) \leq |E| = k - 1$, which is a contradiction. Since $N_{T_2}(c) \cap L(T_2) = \{v_3\}$, T is a tree obtained from the double star $S(k-1,1)$ by Operation 2. Hence, $T \in \tau_2$.

Subcase 2: Suppose $v_3 \in N_{T_2}(c) \setminus L(T_2)$. Then $v_1v_2v_3cu_1u_2$ is a longest path of T, for some $u_1, u_2 \in V(T_2)$. Applying Lemma 2.5 to this path, we have $b_p(T) \leq 2 + 0 = 2$. It is a contradiction.

Subcase 3: Suppose $v_3 \in L(T_2) \setminus N_{T_2}(c)$. Then $v_1v_2v_3u_3cu_4u_5$ is a longest path of T, for some $u_3, u_4, u_5 \in V(T_2)$. If $|N_{T_2}(u_3) \cap L(T_2)| \geq 2$, then $\gamma_p(T - E) = 2k + 2 > \gamma_p(T)$. Hence, $b_p(T) \leq |E| = k - 1$, which is a contradiction. If $|N_{T_2}(u_3) \cap L(T_2)| = 1$, by Lemma 2.4, it follows that $b_p(T) \leq 1 + 0 + 1 = 2$. It is a contradiction.

Subcase 4: Suppose $v_3 = c$. Then T is obtained from the star $K_{1,k}$ by Operation 1. Hence $T \in \tau_1$.

Combining all the subcases, we have, in Case 1, that $T \in \tau_1 \cup \tau_2$.

Case 2: Suppose $T_2 \in \tau_2$. Since $b_p(T_2) = k - 1$, T_2 is obtained from a double star $S(s, t)$ by Operation 2, where $s + t = k - 1$. Let c_1 and c_2 be the central vertices of the double star. Let T_3 be the component of $T_2 - c_1c_2$ containing c_1. Without loss of generality, we may assume that $\deg_{T_3}(c_1) = s$. Since $v_3 \in V(T_2)$, by symmetry we may assume that $v_3 \in V(T_3)$. Hence we have the following two subcases.

Subcase 1: Suppose $v_3 = c_1$. Then T is obtained from a double star $S(s + 1, t)$ by Operation 2. Hence, $T \in \tau_2$.

Subcase 2: Suppose $v_3 \neq c_1$. Then either $w_3w_2c_2c_1w_1v_3v_2v_1$ or $w_3w_2c_2c_1v_3v_2v_1$ is a longest path of T, for some $w_1, w_2, w_3 \in V(T_2)$, depending on v_3 being a leaf of T_2 or not. Applying Lemma 2.4 to this path, we have $b_p(T) \leq 1 + (t - 1) + s = k - 1$. It is a contradiction.

Therefore, in Case 2, we also have that $T \in \tau_1 \cup \tau_2$. □

By Theorems 2.7 to 2.10, we obtain the following corollary.

Corollary 2.11 *T is a uniformly pair-bonded tree if and only if T is a double star or $T \in \tau_1 \cup \tau_2$.*

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