On SD-prime Labeling of Graphs

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Abstract

Let $G = (V(G), E(G))$ be a simple, finite and undirected graph of order n. Given a bijection $f : V(G) \to \{1, \ldots, n\}$, and every edge uv in $E(G)$, one can associate two integers $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$. The labeling f induces an edge labeling $f' : E(G) \to \{0, 1\}$ such that for an edge uv in $E(G)$, $f'(uv) = 1$ if $gcd(S, D) = 1$, and $f'(uv) = 0$ otherwise. Such a labeling is called an SD-prime labeling if $f'(uv) = 1$ for all $uv \in E(G)$. We say G is SD-prime if it admits an SD-prime labeling. A graph G is said to be a *strongly SD-prime graph* if for every vertex v of G, there exists an SD-prime labeling f satisfying $f(v) = 1$. We investigate several results on this newly defined concept. In particular, we give a necessary and sufficient condition for the existence of an SD-prime labeling.

Keywords. Prime labeling, prime cordial labeling, SD-prime labeling

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1 Introduction

Let $G = (V(G), E(G))$ (or simply $G = (V, E)$ for short, if there is no ambiguity) be a simple, finite and undirected graph of order $|V| = n$ and size $|E| = m$. All notation not defined in this paper can be found in [1].

The first paper on graph labeling was introduced by Rosa in 1967. Since then, there have been more than 1500 research papers on graph labelings being published (see the dynamic survey by Gallian [5]).

In [13, 14], the concept of prime graph and prime cordial graphs were introduced.

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Definition 1.1. A bijection $f : V \to \{1,\ldots,n\}$ induces an edge labeling $f' : E \to \{0,1\}$ such that for an edge uv in G, $f'(uv) = 1$ if $gcd(f(u), f(v)) = 1$, and $f'(uv) = 0$ otherwise. Such a labeling is called a *prime labeling* if $f'(uv) = 1$ for all $uv \in E$ We say G is a *prime graph* if it admits a prime labeling.

For an edge labeling $f' : E \to \{0,1\}$ of a graph G, we let $e_{f'}(i)$ be the number of edges labeled with $i \in \{0,1\}$.

Definition 1.2. A bijection $f : V \to \{1,2,3,\ldots,n\}$ induces an edge labeling $f' : E \to \{0,1\}$ such that for an edge uv in G, $f'(uv) = 1$ if $gcd(f(u), f(v)) = 1$, and $f'(uv) = 0$ otherwise. Such a labeling is called a *prime cordial labeling* if $|e_{f'}(1) - e_{f'}(0)| \leq 1$. We say G is *prime cordial* if it admits a prime cordial labeling.

Several results on prime and prime cordial graphs can be found in [2–4, 6, 8–10]. In this paper, we introduce a variant of prime graph labeling which is defined as follows.

Given a bijection $f : V \to \{1,\ldots,n\}$, and every edge uv in E, one can associate two integers $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$.

Definition 1.3. A bijection $f : V \to \{1,\ldots,n\}$ induces an edge labeling $f' : E \to \{0,1\}$ such that for an edge uv in G, $f'(uv) = 1$ if $gcd(S, D) = 1$, and $f'(uv) = 0$ otherwise. Such a labeling is called an *SD-prime labeling* if $f'(uv) = 1$ for all $uv \in E$. We say G is *SD-prime* if it admits an SD-prime labeling.

The following results follow directly from the definitions.

Corollary 1.1. *Every spanning subgraph of an SD-prime graph is also an SD-prime graph.*

Corollary 1.2. *Suppose G admits an SD-prime labeling, then G also admits a prime labeling.*

We first give an obvious necessary condition for G to admit an SD-prime labeling.

Theorem 1.3. *Let f be an SD-prime labeling of G Then $f(u)$ and $f(v)$ have different parity for each edge $uv \in E$*

Proof. Suppose $uv \in E$. Since $gcd(f(u) + f(v), f(u) - f(v)) = 1$, $f(u)$ and $f(v)$ have different parity. $\square$

A vertex v is said to be an *odd* (resp. *even*) *labeled* vertex under a vertex labeling f if $f(v)$ is odd (resp. even). The following is an obvious property

Lemma 1.4. *Suppose $f : V \to \{1, \ldots, n\}$ is a bijective labeling of a graph $G = (V, E)$. Then the number of odd labeled vertices is more than that of even labeled vertices by at most 1.*

Theorem 1.5. *Let G be an SD-prime graph of order greater than 1 There are two sets of vertices X and Y such that (X, Y) is a bipartition of G. Moreover, $0 \leq |X| - |Y| \leq 1$. Hence G is a spanning subgraph of either $K_{m,m}$ or $K_{m,m+1}$ for some $m \geq 1$.*

Proof. Let f be an SD-prime labeling of G. Let X and Y be the sets of odd and even labeled vertices, respectively. By Theorem 1.3, X and Y are independent sets. Hence (X, Y) is a bipartition of G and each component of G is bipartite By Lemma 1.4, $0 \leq |X| - |Y| \leq 1$. □

Note that, the bipartition of a connected bipartite graph is unique up to isomorphism.

2 Main Results

Theorem 2.1. *If $gcd(a, b) = 1$, then $gcd(a + b, a - b) = 1$ or 2*

Proof. Since $gcd(a, b) = 1$, we must have either both a and b are odd, or else, a and b are of different parity.

If the former holds, then both $a+b$ and $a-b$ are even. So, $gcd(a+b, a-b) = 2k, k \geq 1$. Therefore, $a + b = 2kr$, $a - b = 2ks$, $r, s \geq 1$. This implies that $a = k(r + s)$ and $b = k(r - s)$. Since $gcd(a, b) = 1$, we must have $k = 1$. Hence, $gcd(a + b, a - b) = 2$

If the latter holds, both $a + b$ and $a - b$ are odd. So, $gcd(a + b, a - b) = d \geq 1$. Therefore, $a + b = dr, a - b = ds$, $r, s \geq 1$. This implies that $a = d(r + s)/2, b = d(r - s)/2$. Since $gcd(a, b) = 1$, we must have $d = 1$. Hence, $gcd(a + b, a - b) = 1$. □

We now give a necessary and sufficient condition for the existence of an SD-prime labeling.

Theorem 2.2. *A graph G of order n is SD-prime if and only if G is bipartite and that there exists a labeling $f : V \to \{1, 2, \ldots, n\}$ such that for each edge uv of G, $f(u)$ and $f(v)$ are of different parity and $gcd(f(u), f(v)) = 1$.*

Proof. It follows from Theorems 1.3, 1.5 and the proof of Theorem 2 1. □

Theorem 2.3. *Let G be a graph of order $n \geq 2$ with a vertex of degree $n - 1$. Then G is SD-prime if and only if $G \cong P_2$ or P_3*

Proof. By Theorem 1.5, G is a spanning subgraph of $K_{m,m}$ or $K_{m,m+1}$. Since G has a vertex of degree $n-1 = 2m-1$ or $2m$, it follows that $m = 1$. Clearly, $G \cong P_2$ or P_3. Obviously, P_2 and P_3 are SD-prime. □

Corollary 2.4. *The complete graph K_n is SD-prime if and only if $n = 2$.*

Let $St(n) \cong K_{1,n}$ denote the *star graph*

Corollary 2.5. *The star graph $St(n)$ of order $n+1$ is SD-prime if and only if $n \leq 2$.*

Let Δ be the maximum degree of G.

Theorem 2.6. *Suppose G is of order $n \geq 2$. If G is SD-prime, then $\Delta \leq \lceil \frac{n}{2} \rceil$.*

Proof By Theorem 1 5, G is a spanning subgraph of $K_{m,m}$ or $K_{m,m+1}$. Clearly, $\Delta = m$ or $m+1$. Since $n = 2m$ or $2m+1$, the theorem holds. □

Theorem 2.7. *Let G be an SD-prime graph of order n with $\Delta = \lceil \frac{n}{2} \rceil$. Suppose p is the number of odd integers that are relatively prime to all the even integers in $\{1, 2, \ldots, n\}$ and k is the largest integer such that $2^k \leq n$*

(a) If G is a spanning subgraph of $K_{m,m}$, then G has at most $p+k$ vertices of degree Δ.

(b) If G is a spanning subgrap of $K_{m,m+1}$, then G has at most k vertices of degree Δ.

Moreover, each possible number of vertices of degree Δ is attainable.

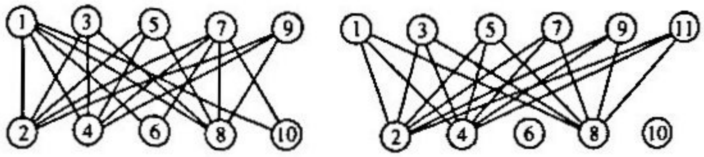
Proof. We prove by constructing an SD-prime graph with maximum number of vertices of degree Δ Let X and Y be the partite set of odd and even labeled vertices respectively.

(a) Let the vertices in X (resp Y) be u_i with $f(u_i) = 2i-1$ (resp. v_i with $f(v_i) = 2i$) for $1 \leq i \leq m$ Join u_j to each v_i if $2j-1$ is an odd integer that is relatively prime to all the even integers in $\{1, 2, \ldots, n\}$, and join v_j to each u_i if $j = 2^s \leq n$ for some integer s. We now have $p+k$ vertices of degree Δ. Observe that each vertex of degree $< \Delta$ has vertex label not relatively prime to at least one vertex label not in the same partite set.

(b) Let the vertices in X (resp. Y) be u_i with $f(u_i) = 2i-1$ for $1 \leq i \leq m+1$ (resp. v_i with $f(v_i) = 2i$ for $1 \leq i \leq m$) Clearly, all the vertices with degree Δ must be in Y. Join v_j to each u_i if $j = 2^s \leq n$ for some integer s Observe that each vertex of degree $< \Delta$ in Y has vertex label not relatively prime to at least one vertex label in X.

Since $gcd(f(u), f(v)) = 1$ for each edge uv such constructed, the graph is SD-prime. Moreover, the construction method allows us to have any possible number of vertices of degree Δ. □

Example 2.1. Here are examples for $n = 10$ and $n = 11$ respectively with maximum number of vertices of degree Δ.



Definition 2.1. Let $DS(a, b)$ denote the *double star graph* obtained from $St(a)$ and $St(b)$ by adding an edge joining the central vertices of the two star graphs.

Theorem 2.8. *For $a \geq b \geq 2$, the double star graph $DS(a, b)$ is SD-prime if and only if $a = b$ or $a = b + 1$.*

Proof. Let the vertex sets of $St(a)$ and $St(b)$ be $\{u\} \cup \{u_j \mid 1 \leq j \leq a\}$ and $\{v\} \cup \{v_j \mid 1 \leq j \leq b\}$, where u and v are the central vertices, respectively.

Suppose $a = b$ or $a = b + 1$ Let $f(v) = 1$, $f(u) = 2$, $f(u_i) = 2i + 1$ for $1 \leq i \leq a$ and $f(v_j) = 2j + 2$ for $1 \leq j \leq b$. By Theorem 2.2, $DS(a, b)$ is SD-prime.

Conversely, suppose $DS(a, b)$ is SD-prime. Since $DS(a, b)$ is a spanning subgraph of $K_{b+1,a+1}$, by Theorem 1.5, $a = b$ or $a = b + 1$. □

Theorem 2.9. *Every path P_n and even cycle C_n are SD-prime.*

Proof. Let $P_n = v_1 \cdots v_n$, and $C_n = v_1 \cdots v_n v_1$. Define $f(v_i) = i$ for both P_n and C_n. By Theorem 2.2, P_n is SD-prime. □

Definition 2.2. A *tadpole graph* $T_{m,l}$ is a simple graph obtained from an m-cycle by attaching a path of length l, where $m \geq 3$ and $l \geq 1$. Let the m-cycle be $u_0 u_1 u_2 \cdots u_{m-1} u_0$ and the attached path is $u_0 u_m u_{m+1} \cdots u_{m+l-1}$.

Theorem 2.10. *The tadpole graph $T_{m,l}$ is SD-prime if and only if $m \geq 4$ is even.*

Proof For even $m \geq 4$, define $f(u_m) = 1$, $f(u_i) = i + 2$ for $0 \leq i \leq m - 1$, and $f(u_i) = i + 1$ for $m + 1 \leq i \leq m + l - 1$. By Theorem 2.2, the theorem holds. □

Theorem 2.11. *The complete bipartite graph $K_{m,n}$ with $1 \leq m \leq n$ is SD-prime if and only if (i) $m = 1$, $n = 1, 2$, or else (ii) $m = 2$, $n = 2, 3$.*

Proof. For $m = 1$, the result follows from Theorem 2.5. So we now assume $m \geq 2$.
(Sufficiency) Since $K_{2,2}$ is a 4-cycle, the result follows from Theorem 2 9 For $K_{2,3}$, we label the odd degree vertices by 2 and 4, and the even degree vertices by 1, 3 and 5. Clearly, the graph is SD-prime.

(Necessity) By Theorem 1.5, $n = m$ or $n = m + 1$. If $m \geq 3$, then there is an edge uv such that $f(u) = 3$ and $f(v) = 6$ so that $f'(uv) = 0$ which implies that $K(m, n)$ is not SD-prime. Hence, $m = 2$ and $n = 2, 3$ □

Definition 2.3. Given $t \geq 3$ paths of order $n_j \geq 2$ with an end vertex $v_{j,1}$ ($1 \leq j \leq t$). A *spider graph* $SP(n_1, n_2, n_3, \ldots, n_t)$ is the one-point union of the t paths at vertex $v_{j,1}$.

Theorem 2.12. *The spider graph $SP(n_1, n_2, n_3, \ldots, n_t)$ is SD-prime if and only if at most 2 of n_j are even.*

Proof. Let v be the vertex of degree t of $SP(n_1, n_2, n_3, \ldots, n_t)$ and all other vertices of the path of order n_j are $v_{j,k}$, $2 \leq k \leq n_j$.

To prove the necessity, we assume that there are k n_j's being even, where $k \geq 0$. Let X (resp. Y) be the set of vertices that are of odd (resp. even) distance from the vertex v. We have that $|X| = \frac{1}{2}(\sum_{j=1}^{t} n_j - t + k)$ and $|Y| = 1 + \frac{1}{2}(\sum_{j=1}^{t} n_j - t - k)$ We see that $|X| - |Y| = k - 1$. Clearly (X, Y) is a bipartition of $SP(n_1, n_2, n_3, \ldots, n_t)$ By Theorem 1 5, $k - 1 \leq 1$ and hence $k \leq 2$

To prove the sufficiency, we consider 3 cases

Case (a). All n_j are odd. Define $f(v) = 1$ and $f(v_{1,k}) = k$ for $2 \leq k \leq n_1$. For $j \geq 2$, let $a_j = \sum_{i=1}^{j-1} n_i - j + 2$. Now, define $f(v_{j,k}) = a_j + k - 1$, $2 \leq k \leq n_j$.

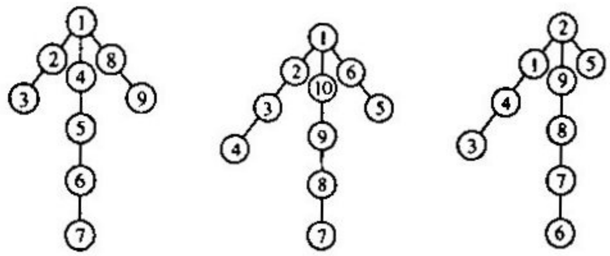
Case (b). n_1 is even and n_j is odd for $j \geq 2$. Define $f(v) = 1$ and $f(v_{1,k}) = k$ for $2 \leq k \leq n_1$. For $j \geq 2$ and a_j as defined in Case (a), define $f(v_{j,k}) = a_j + n_j - k + 1, 2 \leq k \leq v_j$

Case (c). n_1 and n_2 are even and n_j is odd for $j \geq 3$ Define
$f(v) = 2$, $f(v_{1,2}) = 1$;
$f(v_{1,k}) = n_1 - k + 3$ for $3 \leq k \leq n_1$;

$f(v_{j,k}) = a_j + n_j - k + 1$ for $j \geq 2$, $2 \leq k \leq v_j$ and a_j as defined in Case (a).

For each of the above case, it is easy to verify that the end-vertex labels of all possible edges are relatively prime. By Theorem 2.2, the labeling is SD-prime. □

Example 2.2. Here are the labeling of (a) $SP(3,5,3)$, (b) $SP(4,3,5)$ and (c) $SP(4,2,5)$ according to the proof above.

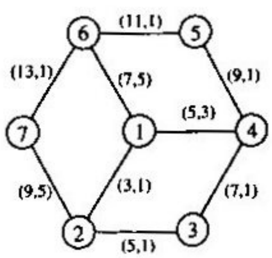


Definition 2.4. Let $G_n, n \geq 2$, denote the *gear graph* obtained from a wheel graph of order $2n+1$ by deleting n spokes where no two of the spokes are consecutive.

Theorem 2.13. *The graph G_n is SD-prime for all $n \geq 2$.*

Proof. Let $V(G_n) = \{v, v_1, v_2, v_3, \ldots, v_{2n}\}$ such that $\deg(v_k) = 3$ for even k. Define $f(v) = 1$, $f(v_1) = 2n+1$ and $f(v_k) = k$ for $k \neq 1$. It is easy to verify that the end-vertex labels of all possible edges are relatively prime. By Theorem 2.2, the labeling is SD-prime □

Example 2.3. Here is the labeling of G_3 according to the proof above.



A *bicyclic graph* is a connected (simple) graph of order p and size $p+1$, where $p \geq 3$. It is known that the bicyclic graph without pendant is either a one point union of two cycles, a long dumbbell graph or a cycle with a long chord (also called a theta graph) [11,12]. The definitions and notation of those graphs will be listed in this paper later one by one

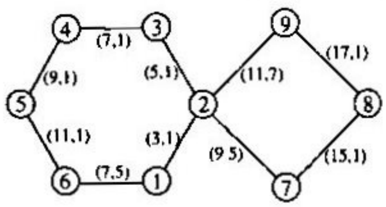
Definition 2.5. For $t \geq 2$, a *one point union* of t cycles is a graph obtained from t cycles, say C_{n_i} for $1 \leq i \leq t$, by identifying one vertex from each cycle. We denote such a graph by $U(n_1, \ldots, n_t)$

Theorem 2.14. *The graph $U(n_1, \ldots, n_t)$ is SD-prime if and only if $t = 2$ and n_1, n_2 are both even.*

Proof. Let $C_{n_i} = v v_{i,1} \cdots v_{i,n_i-1} v$, where v is the common vertex of each cycle. Define $f(v_{1,n_1-1}) = 1$, $f(v) = 2$, $f(v_{1,j}) = j + 2$ for $1 \leq j \leq n_1 - 2$ and $f(v_{2,j}) = n_1 + j$ for $1 \leq j \leq n_2 - 1$ Now it is easy to verify that for any two adjacent vertices, their end-vertex labels are relatively prime By Theorem 2.2, the sufficiency holds.

To prove the necessity, we first note that all n_k are even. Let X (resp. Y) be the set of vertices that are of odd (resp. even) distance from the vertex v. We have that $|X| = \sum\limits_{k=1}^{t} \frac{n_k}{2}$ and $|Y| = 1 + \sum\limits_{k=1}^{t} \frac{n_k - 2}{2}$. We see that $|X| - |Y| = t - 1$. Clearly (X, Y) is a bipartition of $U(n_1, \ldots, n_t)$. By Theorem 1.5, $t \leq 2$. Since we assume that $t \geq 2$, $t = 2$. □

Example 2.4. Here is the labeling of $U(6, 4)$ according to the proof above



Definition 2.6. A *cycle with a long chord* (or *theta graph*) is a graph obtained from a cycle C_m, $m \geq 4$, by adding a chord of length l where $l \geq 1$. Namely, let $C_m = u_0 u_1 \cdots u_{m-1} u_0$. Without loss of generality, we may assume the long chord joins u_0 with u_a, where $2 \leq a \leq m - 2$ That is, $u_0 u_m u_{m+1} \cdots u_{m+l-2} u_a$ is the chord. We denote this graph by $C_m(a, l)$ Note that when $l = 1$, the chord is $u_0 u_a$.

Theorem 2.15. *For $a \geq 2$, $l \geq 1$, the theta graph $C_m(a; l)$ is SD-prime if and only if both m and $a + l$ are even*

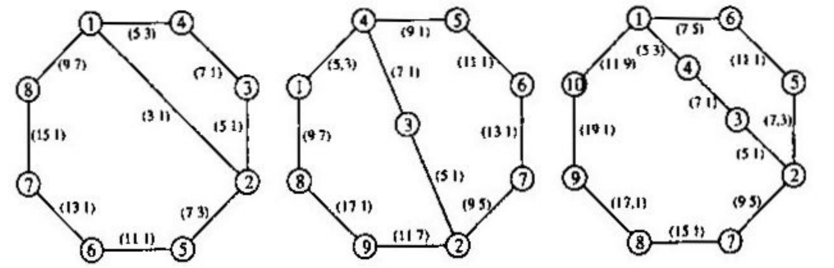
Proof. The necessity follows from Theorem 2.2 We now prove the sufficiency. We keep the notation defined in the definition.

If $l \geq 1$ is odd, we define
$f(u_0) = 1$, $f(u_a) = 2$, $f(u_j) = a + l + 1 - j$ for $1 \leq j \leq a - 1$;
$f(u_j) = l + j$ for $a + 1 \leq j \leq m - 1$; and
$f(u_j) = l + m + 1 - j$ for $m \leq j \leq m + l - 2$.

If $l \geq 2$ is even, we define
$f(u_{m-1}) = 1$, $f(u_a) = 2$, $f(u_j) = j + l + 2$ for $0 \leq j \leq a - 1$;
$f(u_j) = m + a + l - j$ for $a + 1 \leq j \leq m - 2$; and
$f(u_j) = l + m - j + 1$ for $m \leq j \leq m + l - 2$

In each of the labelings above, it is easy to verify that for any two adjacent vertices, their end-vertex labels are relatively prime. By Theorem 2.2, the sufficiency holds $\square$

Example 2.5. Here are the SD-prime labelings of $C_8(3,1)$, $C_8(4,2)$ and $C_8(3,3)$ according to the above proof, respectively.



Definition 2.7. A *long dumbbell graph* is a graph obtained from two cycles C_a and C_b, by joining a path P_{k+1} of length k for $a, b \geq 3$ and $k \geq 1$. Without loss of generality, we may assume

$$C_a = u_1 \cdots u_a u_1, \quad P_{k+1} = u_1 w_1 \cdots w_{k-1} v_1 \quad \text{and } C_b = v_1 \cdots v_b v_1.$$

This graph is denoted by $D(a, b; k)$. When $k = 1$, $P_2 = u_1 v_1$ and $D(a, b; k)$ is called a *dumbbell graph*.

Theorem 2.16. *For $a \geq b \geq 4$, a long dumbbell graph $D(a, b; k)$ is SD-prime if and only if both a and b are even.*

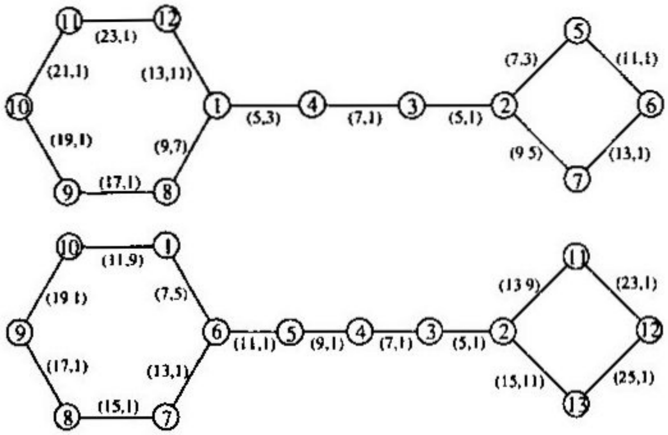
Proof. The necessity follows from Theorem 2.2. We now prove the sufficiency. We keep the notation defined in the definition.

If $k \geq 1$ is odd, we define
$f(u_1) = 1$, $f(u_j) = b + k + j - 1$ for $2 \leq j \leq a$,
$f(v_1) = 2$, $f(v_j) = k + j$ for $2 \leq j \leq b$; and
$f(w_j) = k - j + 2$ for $1 \leq j \leq k - 1$.

If $k \geq 2$ is even, we define
$f(u_a) = 1$, $f(u_j) = k + j + 1$ for $1 \leq j \leq a - 1$;
$f(v_1) = 2$, $f(v_j) = a + k + j - 1$ for $2 \leq j \leq b$; and
$f(w_j) = k - j + 2$ for $1 \leq j \leq k - 1$

In each of the labelings above, it is easy to verify that for any two adjacent vertices, their end-vertex labels are relatively prime. By Theorem 2.2, the sufficiency holds. □

Example 2.6. Here are the labelings of $D(6,4;3)$ and $D(6,4,4)$ according to the proof above



Hence, we have completely determined the SD-primality of all bicyclic graph without pendant

3 Strongly SD-prime Graphs

Definition 3.1. A graph G is said to be a *strongly SD-prime graph* if for every vertex v of G there exists an SD-prime labeling f satisfying $f(v) = 1$

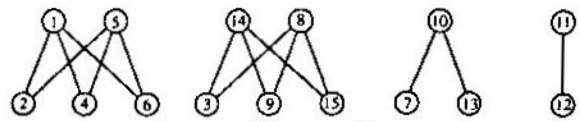
Corollary 3.1. *The spanning subgraph of a strongly SD-prime graph is also a strongly SD-prime graph.*

Corollary 3.2. *If G is strongly SD-prime, then G is also SD-prime.*

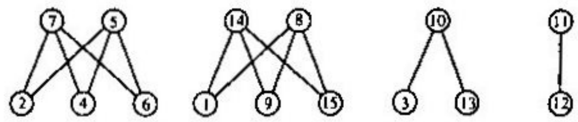
Theorem 3.3. *A connected graph G is strongly SD-prime only if it is SD-prime of even order Moreover, G is a spanning subgraph of $K_{m,m}$ for some $m \geq 1$.*

Proof. From Theorem 1.5, we may assume G is bipartite with bipartition (X, Y). Since G is strongly SD-prime, for $x \in X$ and $y \in Y$, there are two SD-prime labelings f and g such that $f(x) = 1$ and $g(y) = 1$. Then X is the set of all odd labeled vertices under f and is the set of all even labeled vertices under g. By the proof of Theorem 1.5, we have $|X| = |Y|$ So we have the theorem □

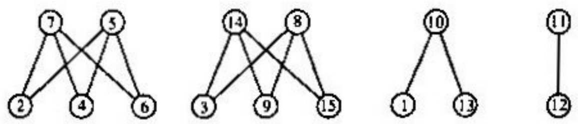
Example 3.1. For disconnected graph, the above theorem does not hold. Following we show that $K_{2,3} + K_{2,3} + K_{1,2} + K_{1,1}$ (the disjoint union) is strongly SD-prime disconnected graph. By symmetry, there are only 5 cases we have to deal with. We show them below:



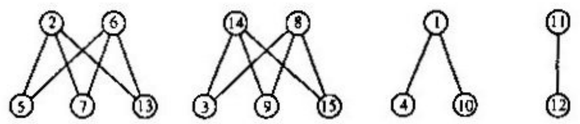
Label 1 in $K_{2,3}$ smaller part.



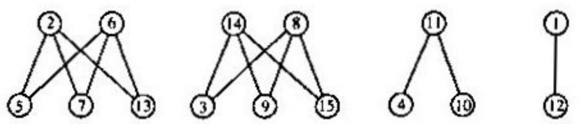
Label 1 in $K_{2,3}$ larger part



Label 1 in $K_{1,2}$ larger part



Label 1 in $K_{1,2}$ smaller part



Label 1 in $K_{1,1}$.

By Corollary 2.4, Theorems 2 11 and 3.3, we have

Corollary 3.4.

(i) The complete graphs and star graphs are strongly SD-prime if and only if they are isomorphic to K_2.

(ii) The complete bipartite graph $K_{m,n}$ is strongly SD-prime if and only if $m = n = 1$ or 2.

In what follows, we use the notation established in Section 2, whenever the same families of graphs are considered.

Theorem 3.5. *The path P_n and the cycle C_n are strongly SD-prime if and only if $n \geq 2$ is even.*

Proof. The necessity for P_n follows from Theorem 3 3.

Suppose $n \geq 2$ is even. For $1 \leq a \leq n/2$, the function given by $f(v_a) = 1$, $f(v_j) = j-a+1$ for $a+1 \leq j \leq n$, and $f(v_j) = n-a+j+1$ for $1 \leq j \leq a-1$ produces an SD-prime labeling. By symmetry, P_n is strongly-SD prime

Since C_n is SD-prime if and only if n is even, by symmetry, it is also strongly SD-prime for even n. □

Theorem 3.6. *The double star $DS(a,b)$ is strongly SD-prime if and only if $a = b \geq 2$.*

Proof. The necessity follows from Theorem 3 3. To prove the sufficiency, we let $n = 2a+2$ and $p = 2k-1$ be the largest prime in $\{1, 2, \ldots, n\}$. By symmetry, we only need to show that there exist two SD-prime labelings f and g such that $f(u_1) = 1$ and $g(v) = 1$. For the latter case, we have done in the proof of Theorem 2.8. So we only deal with the former case. Define $f(u) = 2$, $f(v) = p$, $f(u_i) = 2i-1$ for $i = 1, 2, 3, \ldots, k-1, k+1, \ldots, a+1$, and $f(v_i) = 2i+2$ for $i = 1, 2, \ldots, a$.

It is easy to verify that for any two adjacent vertices, their end-vertex labels are relatively prime. By Theorem 2 2, the sufficiency holds □

Theorem 3.7. *A tadpole graph $T_{m,l}$ is strongly SD-prime if and only if both m and l are even.*

Proof The necessity follows from Theorem 3 3 To prove the sufficiency, it suffices to show that there exist SD-prime labeling such that $f(u_i) = 1$ for each $i \in \{0, 1, 2, \ldots, m/2, m, m+1, \ldots, m+l-1\}$. We consider two cases

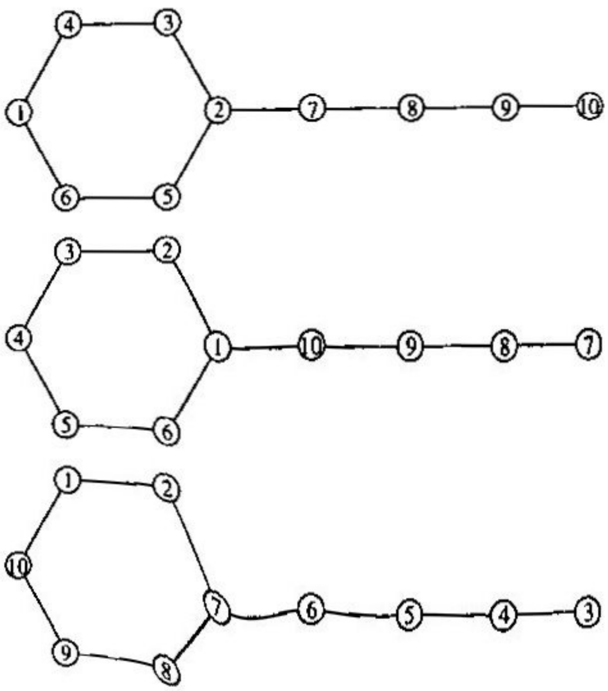
Case (a). $i \in \{0, 1, 2, \ldots, m/2\}$.
If i is odd, then define $f(u_i) = 1$;
$f(u_j) = 2+j$ for $0 \leq j \leq i-1$;
$f(u_j) = m-j+i+1$ for $i+1 \leq j \leq m-1$; and
$f(u_j) = j+1$ for $m \leq j \leq m+l-1$.
If $i = 0$, then define $f(u_j) = j+1$ for $0 \leq j \leq m-1$, and $f(u_j) = 2m+l-j$ for $m \leq j \leq m+l-1$.
If $i > 0$ is even, then define $f(u_i) = 1$, $f(u_0) = i+l+1$;
$f(u_j) = j+1$ for $1 \leq j \leq i-1$,
$f(u_j) = m+l+i+1-j$ for $i+1 \leq j \leq m-1$; and
$f(u_j) = m+l+i-j$ for $m \leq j \leq m+l-1$

Case (b). $i \in \{m, m+1, \ldots, m+l-1\}$

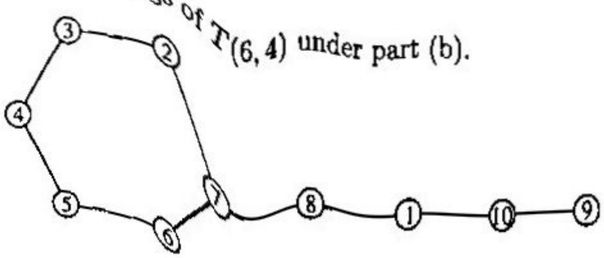
If i is odd, then define $f(u_i) = 1$;
$f(u_0) = m+1$, $f(u_j) = j+1$ for $1 \le j \le m-1$;
$f(u_j) = j+2$ for $m \le j \le i-1$; and
$f(u_j) = m+l+i+1-j$ for $i+1 \le j \le m+l-1$
If i is even, then define $f(u_i) = 1$;
$f(u_j) = j+2$ for $0 \le j \le m-1$;
$f(u_j) = m+i-j+1$ for $m \le j \le i-1$; and
$f(u_j) = j+1$ for $i+1 \le j \le m+l-1$

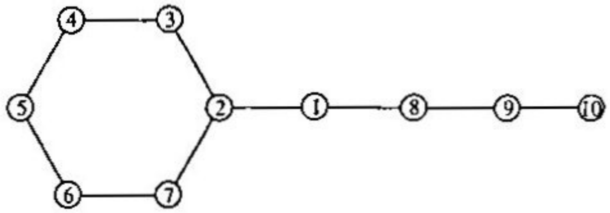
Clearly, each function above is an SD-prime labeling. Hence, the theorem holds. □

Example 3.2. Here are three labelings of $T(6,4)$ under part (a).



Here are two labelings of $T(6,4)$ under part (b).





Corollary 3.8. *The graph $U(n_1, n_2, \ldots, n_t)$ is not strongly SD-prime for all $t \geq 2$.*

Lemma 3.9. *The graph $C_m(a; 1)$ is strongly SD-prime if and only if m is even and a is odd.*

Proof. Without loss of generality, we assume that $m \geq 2a \geq 6$. The necessity follows from Theorem 3 3 By symmetry and Theorem 2.15, we need to show that there exist SD-prime labeling such that $f(u_i) = 1$ for $i \in \{1, 2, \ldots, a-1\} \cup \{a+1, a+2, \ldots, m-1\}$. Observe that the reflection $u_j \mapsto u_{a-j}$ for $0 \leq j \leq a$ and $u_j \mapsto u_{m+a-j}$ for $a+1 \leq j \leq m-1$ is an isomorphism for $C_m(a; 1)$. Hence, we only need to consider odd i

We first consider $i \in \{1, 3, 5, \ldots, a-2\}$. Define $f(u_i) = 1$, $f(u_j) = j+2$ for $0 \leq j \leq i-1$, and $f(u_j) = m+i+1-j$ for $i+1 \leq j \leq m-1$. Clearly, f is an SD-prime labeling.

An SD-prime labeling for $i \in \{a+2, a+4, \ldots, m-1\}$ can be obtained similarly. This completes the proof. □

Theorem 3.10. *For odd a, l, $C_{a+l}(a, l)$ is strongly SD-prime if*

(1) $a = 3$;

(2) $a = 2^k + 5, k \geq 1$;

(3) $a - 5 \not\equiv 0 \pmod p$ *where p is any prime factor of $l+2$.*

Proof. Note that $C_{a+l}(a, l)$ has 2 edge-disjoint paths of same length. By Theorem 2.15 and symmetry under a suitable reflection as in the proof of Lemma 3.9, we only need to show that there exist SD-prime labeling such that $f(u_i) = 1$ for $i \in \{1, 3, 5, \ldots, a-2\} \cup \{a+2, a+4, \ldots, a+l-1\}$. We first consider $f(u_i) = 1$ for $i \in \{1, 3, 5, \ldots, a-2\}$. We define a labeling f as follows

$f(u_i) = 1$, $f(u_0) = 2$;

$f(u_j) = j - a - l + 3$ for $a + l \leq j \leq a + 2l - 2$

$f(u_j) = a + l + 2 - j$ for $i + 1 \leq j \leq a$;

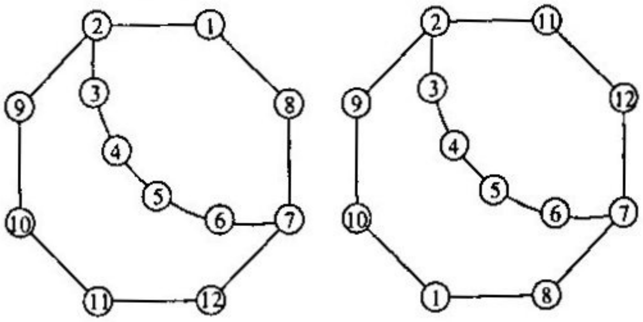
$f(u_j) = a + l + 1 - i + j$ for $1 \le j \le i - 1$;

$f(u_j) = 2a + 2l - j$ for $a + 1 \le j \le a + l - 1$.

It can be checked easily that under this labeling, $f(u)$ is relatively prime to $f(v)$ for $uv \neq u_a u_{a+1}$. For edge $u_a u_{a+1}$, we have $f(u_a) = l + 2$ and $f(u_{a+1}) = a + 2l - 1$. Now, $gcd(l + 2, a + 2l - 1) = gcd(l + 2, a - 5) = 1$ if one of the provided conditions is satisfied. Hence, the labeling is SD-prime.

We now consider $f(u_i) = 1$ for $i \in \{a+2, a+4, \ldots, a+l-1\}$. In a similar way, we can also define a labeling f such that $f(u)$ is relatively prime to $f(v)$ for $uv \neq u_{a-1}u_a$, and that $f(u_{a-1}) = a + 2l - 1$, $f(u_a) = l + 2$. By the same argument as above, f is also an SD-prime labeling. The proof is complete. $\square$

Example 3.3. Here are labelings of $C_8(3,5)$ with $f(u_1) = 1$ and $f(u_5) = 1$ respectively, according to the proof above.



Corollary 3.11. *For $a \not\equiv 5 \pmod 7$, the graph $C_{2a}(a;a)$ is strongly SD-prime if and only if a is odd.*

Proof. By an argument similar to that in Theorem 3.10, we only need to consider $f(u_i) = 1$ for $i \in \{1, 3, \ldots, a-2\}$. Suppose $a \not\equiv 5 \pmod 7$. By the same labeling as above, we only need to consider edge $u_a u_{a+1}$ with $f(u_a) = a + 2$ and $f(u_{a+1}) = 3a - 1$. This gives $gcd(a + 2, 3a - 1) = gcd(a + 2, a - 5) = gcd(7, a - 5) = 1$ since $a \not\equiv 5 \pmod 7$. Thus, the labeling is SD-prime. The proof is complete. $\square$

Problem 3.1. *Determine the strongly SD-primality of $C_m(a;l)$*

It is clear that if $D(a, b; k)$ is strongly SD-prime, then both a and b are even and k is odd. We end this paper with the following problem.

Problem 3.2. *Study the strongly SD-primality of $D(a, b; k)$ for even a, b and odd k.*

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