Shiu 2001B

# A Class of Graphs with $\chi^*$ Close to $\chi - 1$[1]

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**Abstract**

The star chromatic number $\chi^*(G)$ of a graph $G$, a natural generalization of the chromatic number $\chi(G)$ of $G$, was introduced by Vince in 1988. It has been shown that $\chi(G) - 1 < \chi^*(G) \leq \chi(G)$. In this paper, we give a class of graphs $G$ with $\chi^*(G)$ determined. Moreover, this class of graphs may be arbitrarily close to $\chi(G) - 1$.

*Key words: star chromatic number, critical graph.*

*AMS 1991 Subject Classifications:* 05c15, 05c78

## 1. Introduction

In this paper, we consider only finite and simple graphs. Our terminology is standard as in [1]. We repeat the most important definitions. A $k$-coloring of $G$ is a function $\phi$: $V \to \{0, 1, 2, \cdots, k-1\}$ such that if $xy \in E$ then $\phi(x) \neq \phi(y)$.

The chromatic number of $G$, denoted by $\chi(G)$, is the minimum $k$ such that $G$ has a $k$-coloring. If $\chi(G) = k$, then $G$ is said to be $k$-chromatic. For $k \geq 2$, $G$ is said to be $k$-critical if it is $k$-chromatic but every proper subgraph of $G$ has a $(k-1)$-coloring.

In [2], Vince introduced a generalization of the notion of chromatic number in which one allows for possibility of using more than $\chi(G)$ colors but one asks whether the colors assigned to adjacent vertices can be, in some sence, far apart. Let $k$ and $d$ be two positive integers satisfying $2 \leq 2d \leq k$, a $(k, d)$-coloring of $G$ is a function $\phi$: $V \to \{0, 1, 2, \cdots, k-1\}$ such that if $xy \in E$ then $d \leq |\phi(x) - \phi(y)| \leq k - d$. The star chromatic number of $G$ is defined as

$$\chi^*(G) = \inf\{k/d : \text{ there is a } (k, d)\text{-coloring of } G\}$$

Thus a $(k, 1)$-coloring of $G$ is just an ordinary $k$-coloring of $G$.

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On the relation between the star chromatic number and the chromatic number, it was proved that $\chi(G)-1 < \chi^*(G) \leq \chi(G)$ in [2,3]. One extremal case, $\chi^*(G) = \chi(G)$, was investigated in [2,4 - 8]. Another extremal case is $\chi^*(G)$ being close to the lower bound $\chi(G)-1$. Note that the star chromatic number of $G$ maybe viewed as a refinement of the chromatic number. If $G$ is uniquely $k$-colorable, then there is no room to "save" any color, and $\chi^*(G) = \chi(G)$. On the other hand, if $G$ is $k$-critical, then the $k$-th color is barely needed. It seems that in this case there will be room for saving colors. It was shown by Guichard [5] that if $G$ is $k$-critical, and has girth at least $k+1$, then $\chi^*(G) < \chi(G)$. Steffen and Zhu reveals a quantitative relation between the girth and the star chromatic number of a $k$-critical graph in [6]. In this paper, we give a class of graphs $G$ with $\chi^*(G)$ close to $\chi(G)-1$ and determines $\chi^*(G)$ of $G$ by proving the following theorem.

**Theorem 1** *For any $m \geq 1$ and $n \geq 1$ there exists a graph $G^n_m$ which is $(2m+1)$-critical such that*

$$\chi^*(G^n_m) = 2m + \frac{1}{n} = \chi(G^n_m) - 1 + \frac{1}{n}.$$

## 2. The construction of $G^n_m$ and some of its properties

If $H_1, H_2, \cdots, H_p$ are $p$ vertex disjoint graphs, then we use $H_1 * H_2 * \cdots * H_p$ to denote the graph obtained from $H_1, H_2, \cdots, H_p$ by joining every vertex of $H_i$ to every vertex of $H_{i+1}$ for $1 \leq i < p$. We now construct the graph $G^n_m$. Let $H$ be a trivial graph with $V(H) = \{v\}$, and

$$G^n_m = H * K^1_m * K^2_m * \cdots * K^{2n}_m + E^*,$$

where $K^j_m$ is the $j$th complete graph on $m$ vertices $v^j_1, v^j_2, \cdots, v^j_m$, $1 \leq j \leq 2n$ and $E^* = \{vv^{2n}_1, vv^{2n}_2, \cdots, vv^{2n}_m\}$. We shall now establish some properties of $G^n_m$.

**Property 1** $\alpha(G^n_m) = n$. *Where $\alpha(G)$ denotes the independence number of $G$.*

**Property 2** $\chi(G^n_m) = 2m+1$.

**Proof:** By the definition of $G^n_m$, $K^j_m * K^{j+1}_m$ is a complete subgraph of order $2m$ in $G^n_m$ for $j = 1, 2, \cdots, 2n-1$. We shall prove that $\chi(G^n_m) \geq 2m+1$. If not, suppose there exists a $2m$-coloring $f: V \to \{1, 2, \cdots, 2m\}$ of $G^n_m$. Without loss of generality, let the vertices of $K^1_m$ and $K^2_m$ be assigned colors $\{1, 2, \cdots, m\}$ and $\{m+1, m+2, \cdots, 2m\}$ respectively. Thus $\chi(G^n_m) = 2m$ implies that the vertices in $K^{2j-1}_m$ are assigned colors $\{1, 2, \cdots, m\}$ and the vertices in $K^{2j}_m$ are assigned colors $\{m+1, m+2, \cdots, 2m\}$ for $j = 2, 3, \cdots, n$. That is, the vertices of $K^1_m$ and $K^{2n}$ receive colors $\{1, 2, \cdots, m\}$ and $\{m+$

$1, m+2, \cdots, 2m\}$ respectively. However the vertex $u$, the vertices in $K_m^1$ and the vertices in $K_m^{2n}$ must all receive distinct colors, a contradiction. Therefore $\chi(G_m^n) \geq 2m+1$. It is clear that $\chi(G_m^n) \leq 2m+1$, therefore $\chi(G_m^n) = 2m+1$. ∎

**Property 3** *$G_m^n$ is $(2m+1)$-critical.*

**Property 4** *There exists a $(2mn+1, n)$-coloring of $G_m^n$.*

**Proof:** Let $v_i^j$ be $i$th vertex of $K_m^j (i = 1, 2, \cdots, m; j = 1, 2, \cdots, 2n)$ in $G_m^n$. We define a mapping $f$ on $G_m^n$ to $\mathbb{Z}$ as follows:

$$\begin{aligned} f(v_i^j) &= [i-1+(j-1)m]n \quad \text{for} \quad i = 1, 2, \cdots, m; \; j = 1, 2, \cdots, 2n; \\ f(v) &= 2n^2m. \end{aligned}$$

For each vertex $u \in G_m^n$, we assign the color number $\phi(u)$, where $0 \leq \phi(u) \leq 2mn$ and $\phi(u) \equiv f(u) \mod 2mn+1$. The (13,2)-coloring $f$ of $G_3^2$ is shown in Figure 1.

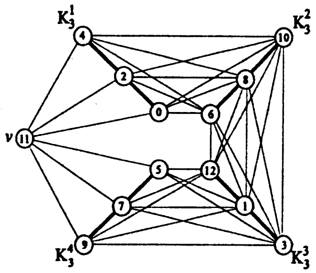


**Figure 1.**

For any $x = v_i^j \in V(K_m^j)$ and $x' = v_{i'}^{j+1} \in V(K_m^{j+1})$ $j = 1, 2, \cdots, 2n-1$, and $1 \leq i, i' \leq m$, we have

$$\begin{aligned} f(x') - f(x) &= [i-1+jm]n - [i'-1+(j-1)m]n \\ &= (m+i-i')n. \\ (2mn+1)-(n+1) &\geq |\phi(x') - \phi(x)| \geq n. \end{aligned} \tag{1}$$

For any $x = v_i^{2n} \in (K_i^{2n})$, $1 \leq i \leq m$ we have

$$\begin{aligned} f(v) - f(x) &= 2n^2m - [i-1+(2n-1)m]n \\ &= (m-i+1)n. \\ (2mn+1)-(mn+1) &\geq |\phi(v) - \phi(x)| \geq n. \end{aligned} \tag{2}$$

For any $x = v_i^1 \in V(K_m^1)$, $1 \leq i \leq m$, we have

$$\begin{aligned} f(v) - f(x) &= 2n^2m - [i-1]n \\ &= (2nm+1)n - in. \\ (2nm+1) - n &\geq |\phi(v) - \phi(x)| \geq (2nm+1) - mn. \end{aligned} \tag{3}$$

It follows from (1), (2) and (3) that $\phi$ is a $(2mn+1, n)$-coloring of $G_m^n$. ∎

## 3. Proof of Theorem 1

By Properties 4 and 2 we have

$$\chi^*(G_m^n) \le \frac{2mn+1}{n} = 2m + \frac{1}{n} = \chi(G_m^n) - 1 + \frac{1}{n}.$$

From Theorem 5 in [7], we have for any graph $G$,

$$\chi^*(G) \ge \chi(G) - 1 + \frac{1}{\alpha(G)}.$$

Hence by Properties 1 and 2 we have

$$\chi^*(G_m^n) \ge \chi(G_m^n) - 1 + \frac{1}{n} = 2m + \frac{1}{n}.$$

Therefore

$$\chi^*(G_m^n) = 2m + \frac{1}{n} = \chi(G_m^n) - 1 + \frac{1}{n}.$$

By Property 3, $G_m^n$ is $(2m+1)$-critical and the theorem follows. ∎

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