Cell rotation graphs of strongly connected orientations of plane graphs with an application*

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Abstract

The cell rotation graph $D(G)$ on the strongly connected orientations of a 2-edge-connected plane graph G is defined. It is shown that $D(G)$ is a directed forest and every component is an in-tree with one root; if T is a component of $D(G)$, the reversions of all orientations in T induce a component of $D(G)$, denoted by T^-, thus (T, T^-) is called a pair of in-trees of $D(G)$; G is Eulerian if and only if $D(G)$ has an odd number of components (all Eulerian orientations of G induce the same component of $D(G)$); the width and height of T are equal to that of T^- respectively. Further it is shown that the pair of directed tree structures on the perfect matchings of a plane elementary bipartite graph G coincide with a pair of in-trees of $D(G)$. Accordingly, such a pair of in-trees on the perfect matchings of any plane bipartite graph have the same width and height.

Keywords Perfect matching, Strongly connected orientation, Eulerian orientation, Ear decomposition, In-tree, Rotation graph, Plane graph

1 Introduction

In connection with directed tree structures on the set of perfect matchings of plane bipartite graphs, in this article we consider the cell rotation of strongly connected orientations of 2-edge connected plane graphs. Investigating relations between Clar aromatic sextet theory and resonance theory, Ohkami et al. (1981) [12, 9] and Chen (1985) [3] established a hierarchical structure on the set of Kekulé patterns (also called

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perfect matchings or 1-factors in graph-theoretic terms) of benzenoid hydrocarbons, which can be represented by a directed rooted tree (in-tree).

The carbon-skeleton of a benzenoid hydrocarbon is the so-called hexagonal system, a 2-connected plane bipartite graph every interior face of which is bounded by a regular hexagon of unit length. In general, most of polycyclic conjugated compounds are represented by plane bipartite graphs. The directed tree structure on the set of perfect matchings was established by Zhang and Guo [16] for generalized hexagonal systems (subgraphs of hexagonal systems), more generally by Zhang and Zhang [19] for plane bipartite graphs. It is known that the directed tree structure of Kekulé patterns of a given hexagonal system H strongly relies on the possible position in the plane where H is placed; that is, there exist two operations, sextet and counter-sextet rotations, on the perfect matchings for producing a pair of directed tree structures, which are not isomorphic in general. There is general interest in seeking some properties or quantities which are the same for the pair of directed tree structures, because they are independent of its position placed and will be invariant for the corresponding hexagonal systems. Along this line Gutman et al. (1989) [6, 7] first observed that the pair of in-trees on the perfect matchings of a hexagonal system have the same width and height. They also made some attempts [8] to correlate these parameters with various physico-chemical properties of benzenoid hydrocarbons.

Orientations of graphs closely relate to their matchings. As early as in 1961, Kasteleyn developed a so-called Pfaffian orientation method for enumerating perfect matchings of plane bipartite graphs. Let G be a bipartite graph with a bipartite partition (W, B) such that every vertex of W and B is regarded as white and black, respectively. Let M be a matching of G. Define an orientation, denoted by ω_M, of G as follows: Orient every edge of M towards the black end-vertex from the white end-vertex; orient every edge of $E(G)\backslash M$ towards the white end-vertex from the black end-vertex. If we can get a Pfaffian orientation [10] in this way, Al-Khnaifes and Sachs [2] showed that the calculation of the number of perfect matching of plane bipartite graphs in question can be simplified greatly. On the other hand, a cycle C of G is an alternating cycle with respect to M if and only if C is a directed cycle following the orientation ω_M. Based on such an orientation, a fast algorithm to determine elementary components of bipartite graphs was proposed [17]. This article is motivated by this relation between matchings and orientations.

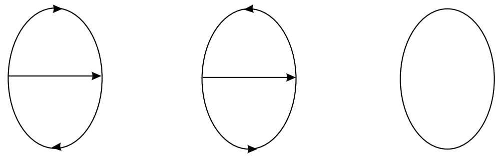
The remainder of this article is organized as follows. In Section 2 some basic terminologies and notations are introduced, and a similar structure, ear decomposition, arising in some kinds of graphs is described. In Section 3 the cell rotation graph $D(G)$ of strongly connected orientations of a 2-edge-connected plane graph G is defined. It is shown that $D(G)$ is a directed forest and every component is an in-tree with one root; if T is a component of $D(G)$, the reversions of orientations corresponding to vertices of T induce a component of $D(G)$, denoted by T^-, (T, T^-) is thus called a pair of in-trees of $D(G)$; all Eulerian orientations of G induce the same component of $D(G)$. As a consequence, a new parity characterization for a plane Eulerian graph is given: G is Eulerian if and only if $D(G)$ has an odd number of components. In Section 4 it is proved that the width and height of T are equal to that of T^{-}, respectively, for any pair of in-trees (T, T^{-}) of $D(G)$. In section 5 it is shown that the pair of directed tree structures on the perfect matchings of a plane elementary bipartite graph G correspond to a pair of in-trees of $D(G)$. As an immediate consequences, we have that the pair of directed tree structures have the same width and height. Accordingly, Gutman et al.'s findings [6, 7, 8] as mentioned above for hexagonal systems are rigorously proved in an extensive sense. Finally, two open problems concerning the cell rotation graphs are proposed.

2 Preliminaries

By a plane graph G we mean an embedding of a planar graph in the plane. This plane graph decomposes the plane into a number of open regions called *faces*; the infinite one is called *exterior face* and the other ones *interior faces*. If f is a face of G, the boundary of f is a subgraph denoted by ∂f. Then G is connected if and only if the boundary of every face is connected; G be a 2-edge connected plane if and only if the boundary of every face admits an Eulerian trail; Further, G is 2-connected if and only if every face is bounded by a cycle. In this article we restrict our consideration to finite 2-edge connected plane graphs, where loops and multiple edges are allowed and regarded as cycles of length 1 and 2 respectively.

Let G be a (di)graph with the vertex-set $V(G)$ and edge-set $E(G)$ (arc-set $A(G)$). For $E' \subseteq E(G)$, let $E'(v)$ denote the set of edges in E' incident to a vertex v. When every edge corresponds exactly to two arcs with distinct directions, a graph also can be regarded as a directed graph. A digraph is called an in-tree if its underlying graph is a tree and contains a unique vertex of out-degree 0 (called the *root*) and there exists a directed path from any other vertex to the root. The reversion of an in-tree is called an *out-tree*. An isolated vertex is a trivial tree. A digraph $\vec{G}$ is called strongly connected if for any vertices $x, y \in V(\vec{G})$ there exists a directed path from x to y. It is known that a connected graph (digraph) is Eulerian if and only if the degree is even (the in-degree equals the out-degree) for every vertex. For other concepts and results about Eulerian (di)graphs and trails and isomorphism of (di)graphs, refer to a recent book [5].

An orientation ω of a graph G is to assign a direction $\omega(e)$ for every edge e of G; the resulting digraph is denoted by $\vec{G}^{\omega}$. A ω-directed path and cycle mean directed path and cycle of $\vec{G}^{\omega}$. An orientation ω may be viewed as the arc-set of $\vec{G}^{\omega}$; further ω is called *strongly connected* and *Eulerian* if $\vec{G}^{\omega}$ is strongly connected and Eulerian respectively. For $E \subseteq E(G)$, put $\omega(E) = E^{\omega} =: \{\omega(e) : e \in E\}$. The reversion,



ω_1 ω_2 $\omega_1 \oplus \omega_2$

Fig. 1. The symmetric difference of two orientations.

denoted by ω^-, of an orientation ω is to reverse ω-orientation of every edge of G; this is, for every edge e of G $\omega(e) \neq \omega^-(e)$ or $\omega^-(e) = -\omega(e)$.

The symmetric difference of finite sets A and B is denoted by $A \oplus B$. This operation among many finite sets obeys associative and commutative law. We now consider the symmetric difference of two orientations ω_1, ω_2 of a graph G. For every edge $e \in E(G)$, if $\omega_1(e)$ is the reversion of $\omega_2(e)$, both directions are contained in $\omega_1 \oplus \omega_2$; Otherwise, $\omega_1(e) = \omega_2(e) \notin \omega_1 \oplus \omega_2$. Thus $\omega_1 \oplus \omega_2$ may be viewed as a set of $E(G)$. For an example, see Fig. 1. In particular, $\omega_1 \oplus \omega_1^- = E(G)$ and $\omega_1 \oplus E(G) = \omega_1^-$.

It is known [13] that a graph has a strongly connected orientation if and only if it is 2-edge connected; further a graph has an Eulerian orientation if and only if it is Eulerian. From now on we denote $\mathcal{G}$ the family of 2-edge connected plane graphs and $\Omega(G)$ the set of strongly connected orientations of $G \in \mathcal{G}$. The following result is obvious.

Lemma 2.1 . *Let $G \in \mathcal{G}$. Then*
(a) $\omega \in \Omega(G)$ if and only if $\omega^- \in \Omega(G)$;
(b) $\omega \in \Omega(G)$ is Eulerian if and only if ω^- is Eulerian.

It turns out that there exists a similar structure *ear decomposition* for 2-edge connected graphs [14], strongly connected digraphs [4], elementary bipartite graphs and factor-critical graphs [10, 11], etc. We now describe this kind of structures as follows. Let G be a (di)graph. An ear-decomposition of G is a sequence $(G_1, \ldots, G_r)$ of subgraphs of G where $G_r = G$, G_1 is a (directed) cycle and each $G_i (i \geq 2)$ arises from G_{i-1} by adding a (directed) path P_i for which only the end-vertices belong to G_{i-1}. Thus G_i can be expressed as $G_i = G_1 + P_2 + \cdots + P_i (i = 1, 2, \ldots, r)$. For $2 \leq i \leq r$ the P_i are called *(directed) ears* of G_{i-1}; further such an ear P_i is said to be *open* or *closed* according to whether P_i has distinct or the same end-vertices. An ear-decomposition of a graph is said to be *open* if all ears are open.

Lemma 2.2 ([14]). *A graph is 2-edge (vertex) connected if and only if it has an (open) ear decomposition.*

We may say that the above ear-decomposition $G = G_1 + P_2 + \cdots + P_r$ of G *starts* at G_1, where G_1 is allowed to be a subgraph (not necessarily cycle) of G. An *open* ear-decomposition $(G_1, \ldots, G_r)$ of a plane graph G is called a *reducible cell decomposition* if every ear P_i lies in the exterior face of G_{i-1}, $i = 2, \ldots, r$; equivalently, all interior faces of the G_i's remain interior faces of G.

Lemma 2.3. *Every 2-connected plane graph has a reducible cell decomposition.*

Proof. Let G be a 2-connected plane graph. Let G_1 be a cycle that is the boundary of any interior face of G. Choose a maximal subgraph H of G such that every interior face of H is also that of G and H has a reducible cell decomposition $H = G_1 + P_2 + \cdots + P_i (i \geq 1)$. Suppose that H is a proper subgraph of G. Since G is 2-connected, an edge of H together with an edge outside of H lie in one cycle. Hence G has an open ear P_{i+1} of H lying on the exterior face f of H. We choose such an ear P_{i+1} so that the interior region R on f bounded by $P_{i+1} \cup \partial f$ is minimal. By the analogous argument we have that R is also an interior face of G. Thus $H + P_{i+1}$ is a subgraph of G with the above property and larger than H, a contradiction. $\square$

By the analogous arguments as Lemma 2.3, we have the following well-known result.

Lemma 2.4 ([4]). *A digraph is strongly connected if and only if it has a directed ear decomposition, which may start at any subgraph of this digraph.*

Proof. We only show a fact that for any proper subgraph H of a strongly connected digraph G, H has a directed ear: For an arc $(u, v) \in A(G) \setminus A(H)$ incident to a vertex (say u) of H, there exists a directed path from v to u since G is strongly connected. Along this path choosing a part from v to a vertex at which the path first enters H, together with the arc (u, v) we obtain a directed ear of H. $\square$

3 Cell rotation graph of strongly connected orientations

Recall that $\mathcal{G}$ denotes the family of 2-edge connected plane graphs and $\Omega(G)$ the set of strongly connected orientations of a graph $G \in \mathcal{G}$. An *interior* face f of G is called a *cell* of G. In particular, any cycle in the plane has a unique cell. When one traverses the boundary of a cell c, two exactly closed Eulerian trails, *clockwise* and *counterclockwise orientations* of ∂c are determined according as the region c always lies on the right and left sides.

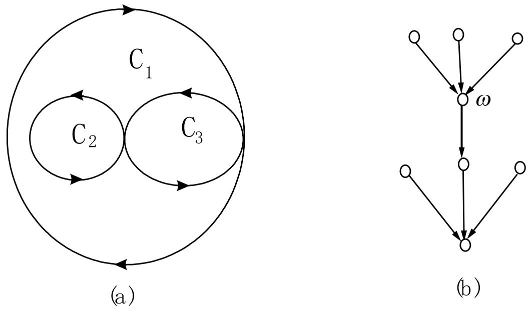


Fig. 2. (a) A plane graph with an orientation ω, ω-clockwise cell c_1 and ω-counterclockwise cells c_2 and c_3, (b) Its cell rotation graph.

Definition 3.1. Let $G \in \mathcal{G}$ and $\omega \in \Omega(G)$. A cell c of G is said to be ω-clockwise (resp. counterclockwise) if ω can determine the clockwise (resp. counterclockwise) orientation of the boundary of c. ω-clockwise and counterclockwise cycles of G can be defined similarly. By ω-directed cell we mean that it is either ω-clockwise or counterclockwise cell. For example, see Fig. 2.

Lemma 3.1. *Let $G \in \mathcal{G}$ and $\omega \in \Omega(G)$. Then*
(a) G has a ω-directed cell whose boundary is a cycle,
(b) For every ω-clockwise (resp. counterclockwise) cycle C, G has a ω-clockwise (resp. counterclockwise) cell in the interior of C.

Proof. (a) Obviously G has a ω-directed cycle. Suppose that C is a ω-directed cycle whose interior region is as small as possible. It is claimed that C is the boundary of a cell. Otherwise, by Lemma 2.4 $\vec{G}^{\omega}$ has a directed ear lying in the interior region of C; a ω-directed cycle containing a smaller interior region than C would be produced, a contradiction.

(b) We only consider a clockwise case. Let C be a ω-clockwise cycle of G. We choose a subgraph $G' \subseteq G$ on the C together with its interior region such that $G' \in \mathcal{G}$ and G' has a ω-clockwise cell c' with the minimal region (in inclusion of sets). We claim that c' is also a cell of G. Otherwise, there exists a ω-directed ear P of G' lying in the c' only the end-vertices of which lie on the $\partial c'$. In any case, P partitions the cell c' of G' into two cells of $G' \cup P$, one must be ω-clockwise, which contradicts the minimality of c'. $\square$

Lemma 3.2. *Let $G \in \mathcal{G}$ and $\omega \in \Omega(G)$. Any distinct ω-clockwise (resp. counterclockwise) cells of G are of edge-disjoint boundaries.*

Proof. Let c_1 and c_2 be distinct ω-directed cells of G. Suppose that ∂c_1 and ∂c_2

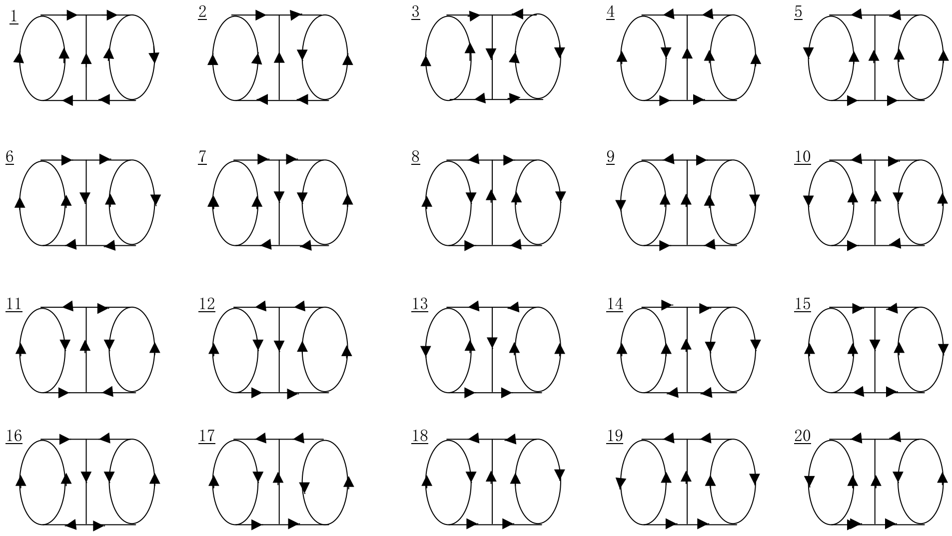


Fig. 3. Strongly connected orientations $\underline{1} - \underline{20}$ of a plane graph G_0.

share an edge e of G. When one traverses the edge e along the orientation $\omega(e)$, one of c_1 and c_2 lies on the right side and the other one does on the left side, which are thus ω-clockwise and ω-counterclockwise respectively, a contradiction. $\square$

Let $G \in \mathcal{G}$ and $\omega \in \Omega(G)$. Let $\mathcal{C}^+(\omega)$ and $\mathcal{C}^-(\omega)$ denote the union of boundaries of ω-clockwise and ω-counterclockwise cells of G respectively. But it should be borne in mind that $\mathcal{C}^+(\omega)$ and $\mathcal{C}^-(\omega)$ can be regarded as the sets of their edges when they appear in symmetric difference operations. Lemma 3.2 guarantees the validation of the following definitions.

Definition 3.2. Let $G \in \mathcal{G}$. Define an operation D for each $\omega \in \Omega(G)$ as follows: $D(\omega)$ is also an orientation of G such that all the clockwise cells of $\vec{G}^\omega$ become simultaneously $D(\omega)$-counterclockwise cells and the orientations of the other edges remain unchanged; i.e., $D(\omega)(e) = \omega^-(e)$ if and only if an edge e lies in the boundary of a ω-clockwise cell of G; symbolically, $D(\omega) := \omega \oplus \mathcal{C}^+(\omega)$.

Lemma 3.3. *Let $G \in \mathcal{G}$. The operation D defines a self-mapping $D : \Omega(G) \to \Omega(G)$.*

Proof. It is obvious that that $D(\omega) \in \Omega(G)$ for any $\omega \in \Omega(G)$. $\square$

Definition 3.3. Let $G \in \mathcal{G}$. The cell rotation graph $D(G)$ of G is defined as a digraph: $\Omega(G)$ is the vertex-set, (ω, ω') is an arc for $\omega, \omega' \in \Omega(G)$ if and only if $D(\omega) = \omega'$.

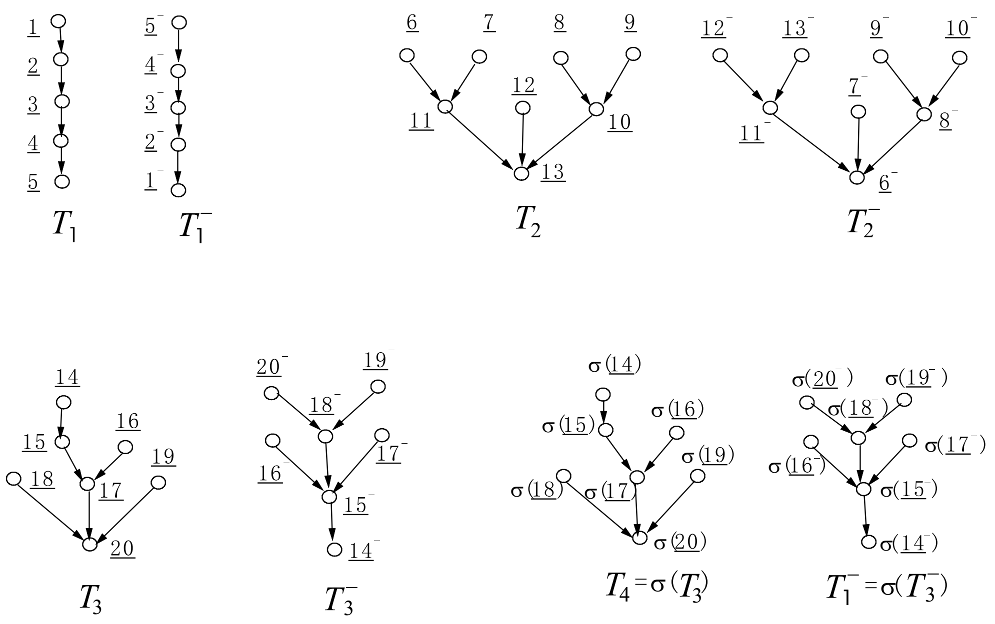


Fig. 4. The cell rotation graph $D(G_0)$ of a plane graph G_0.

We first consider examples. The cell rotation graph of a plane graph shown in Fig. 2(a) is an in-tree (see Fig. 2(b)). Another example is somewhat complicated. Let $G_0 \in \mathcal{G}$ be a plane graph with strongly connected orientations $\underline{1} - \underline{20}$ illustrated in Fig. 3. The other strongly connected orientations of G_0 can be generated from them. If $\underline{\text{i}}$ denotes an orientation of G_0, $\underline{\text{i}}^-$ is the reversion of $\underline{\text{i}}$. Let σ be a rotation about the geometry center of plane graph G_0 by 180^o. Let $\sigma(\underline{\text{i}})$ denote an orientation of G_0 obtained from $\vec{G}^{\underline{\text{i}}}$ by a rotation σ. G_0 has a total of 54 strongly connected orientations, which can be expressed as

$$\Omega(G_0) = \{\underline{1}, \dots, \underline{20}\} \cup \{\underline{1}^-, \dots, \underline{20}^-\} \cup \{\sigma(\underline{14}), \dots, \sigma(\underline{20})\} \cup \{\sigma(\underline{14}^-), ..., \sigma(\underline{20}^-)\}.$$

For any orientation $\omega \in \Omega(G_0)$ without ω-clockwise cell, by Lemmas 2.4 and 3.1 it follows that the boundary of G_0 is ω-counterclockwise. Accordingly, G_0 has exactly 8 orientations without clockwise cells, which are the roots of $D(G)$. Further $D(G_0)$ consists of four pairs of in-trees $(T_1, T_1^-), (T_2, T_2^-), (T_3, T_3^-), (T_4, T_4^-)$, which are referred to Fig. 4. In general, we have the following result.

Theorem 3.4. *Let $G \in \mathcal{G}$. Then $D(G)$ is a directed forest and every component is a non-trivial in-tree (thus we call a component of $D(G)$ an in-tree).*

To prove Theorem 3.4 we first introduce the following concepts, then establish some lemmas. Let $G \in \mathcal{G}$. The dual of G is denoted G^*. For a face f of G, the *depth* $d(f)$ of f is defined as the length of shortest path between two vertices of G^*

corresponding to f and the exterior face of G. Then $d(f) \geq 1$ for every interior face (cell) f of G.

Lemma 3.5. *Let $G \in \mathcal{G}$. Then $D(G)$ has no directed cycle.*

Proof. By contrary, suppose that $D(G)$ has a directed cycle as follows: $\omega, D(\omega), D^2(\omega), \cdots, D^k(\omega)$, where $\omega \in \Omega(G), D^k(\omega) = D^0(\omega) = \omega$ and $D^i(\omega) \neq D^j(\omega)$ for $1 \leq i < j \leq k$. Let f_0 be a cell of G with $d(f_0) = \min\{d(f) : f$ is a $D^i(\omega)$-clockwise cell of G for some $1 \leq i \leq k\}$. Since $d(f_0) \geq 1$, there exists a face f of G such that $d(f) = d(f_0) - 1$ and ∂f and ∂f_0 have an edge e in common. Without loss of generality, suppose that f_0 is a ω-clockwise cell. Then f_0 is $D(\omega)$-counetclockwise and $D(\omega)(e) = -\omega(e)$. Since f is not $D^i(\omega)$-clockwise cell for all $0 \leq i \leq k$, by induction method we can see that $D^i(\omega)(e) = -\omega(e)$ and f_0 is not $D^i(\omega)$-clockwise for all $1 \leq i \leq k$. But $D^k(\omega) = \omega$; i.e., $D^k(\omega)(e) = \omega(e)$, a contradiction. □

Proof of Theorem 3.4. It is claimed that the underlying graph of $D(G)$ is a forest. Otherwise, suppose that C is a cycle of the underlying graph of $D(G)$. By Lemma 3.5 the corresponding orientation $\vec{C}$ in $D(G)$ is not a directed cycle, which implies that $\vec{C}$ has a vertex with in-degree 2 and a vertex with out-degree 2. The latter contradicts that the out-degrees of all vertices of $D(G)$ are no more than 1. Hence $D(G)$ has $|\Omega(G)| - r$ arcs, where r denotes the number of orientations $\omega \in \Omega(G)$ with $D(\omega) = \omega$. So $D(G)$ has exactly $r(\geq 1)$ components and every component of $D(G)$ is an in-tree with one root. Let ω_0 be the root of an in-tree of $D(G)$. By Lemma 3.1 $\mathcal{C}^+(\omega_0) = \emptyset$ implies $\mathcal{C}^-(\omega_0) \neq \emptyset$. Let $\omega_1 := \omega_0 \oplus \mathcal{C}^-(\omega_0)$. It is obvious that $\omega_1 \in \Omega(G)$ and $D(\omega_1) = \omega_0$. Thus every component of $D(G)$ is non-trivial. □

Lemma 3.6. *Let $G \in \mathcal{G}$. For $\omega, \omega' \in \Omega(G)$, suppose that $\omega \oplus \omega' = \cup_{i=1}^r C_i (r \geq 1)$, where the C_i's are the boundaries of both ω-clockwise and ω'-counterclockwise cells. Then $D(\omega) \oplus D(\omega')$ either is empty or consists of the boundaries of both $D(\omega)$-clockwise and $D(\omega')$-counterclockwise cells.*

Proof. Let $\mathcal{C}_0 := \cup_{i=1}^r C_i = \omega \oplus \omega'$. Then $\mathcal{C}_0 \subseteq \mathcal{C}^+(\omega)$ and $\mathcal{C}_0 \subseteq \mathcal{C}^-(\omega')$. Let $\mathcal{C}_1 := \mathcal{C}^+(\omega) \backslash \mathcal{C}_0$. It is obvious that $\mathcal{C}_1 \subseteq \mathcal{C}^+(\omega')$. Put $\mathcal{C}_0' := \mathcal{C}^+(\omega') \backslash \mathcal{C}_1$. Further

$$
\begin{aligned}
D(\omega) \oplus D(\omega') &= (\omega \oplus \mathcal{C}^+(\omega)) \oplus (\omega' \oplus \mathcal{C}^+(\omega')) \\
&= (\omega \oplus \omega') \oplus (\mathcal{C}^+(\omega) \oplus \mathcal{C}^+(\omega')) \\
&= \mathcal{C}_0 \oplus (\mathcal{C}_0 \oplus \mathcal{C}_1) \oplus (\mathcal{C}_0' \oplus \mathcal{C}_1)) \\
&= \mathcal{C}_0'.
\end{aligned}
$$

If $\mathcal{C}_0' \neq \emptyset$, every component of $\mathcal{C}_0'$ is the boundary of a ω'-clockwise cell, thus both $D(\omega')$-counterclockwise and $D(\omega)$-clockwise cell. □

Lemma 3.7. *Let $G \in \mathcal{G}$ and $\omega, \omega' \in \Omega(G)$. By $\omega \sim \omega'$ we mean that ω and ω' belong to the same component of $D(G)$. Assume that $\omega \neq \omega'$. Then the following statements are equivalent:*

(a) $\omega \sim \omega'$,

(b) $\Omega(G)$ has a sequence $\omega_1(=\omega), \omega_2, \cdots, \omega_k(=\omega')$ such that $\omega_i \oplus \omega_{i+1}$ is the boundary of a either ω_i- or ω_{i+1}-clockwise cell of G, for all $1 \leq i < k$,

(c) $\omega \oplus \omega'$ is the union of mutually edge-disjoint ω-directed cycles of G.

Proof. (b)$\Rightarrow$(a). It suffices to prove that $\omega_i \sim \omega_{i+1}$ for all $1 \leq i \leq k-1$. Without loss of generality assume that $\omega_i \oplus \omega_{i+1}$ is the boundary of a ω_i-clockwise and ω_{i+1}-counterclockwise cell. By Theorem 3.4 there exists a positive integer t such that $D^{t+1}(\omega_i) = D^t(\omega_i)$, that is, G has no $D^t(\omega_i)$-clockwise cells and $D^t(\omega_i)$ is a root. Therefore by Lemma 3.6 there exists a positive integer j such that $D^j(\omega_i) \oplus D^j(\omega_{i+1}) = \emptyset$, i.e., $D^j(\omega_i) = D^j(\omega_{i+1})$, which implies that $\omega_i \sim \omega_{i+1}$ for all i.

(c)$\Rightarrow$(b). Suppose that $\omega \oplus \omega'$ is the union of mutually edge-disjoint ω-directed cycles $C_1, C_2, \ldots, C_m (m \geq 1)$ of G. Let $\omega'_{i+1} := \omega'_i \oplus C_i, i = 1, \ldots, m$, where $\omega'_1 = \omega, \omega'_{m+1} = \omega'$. Without loss of generality assume that C_1 is ω-clockwise. We now prove that $\Omega(G)$ has a required sequence between $\omega'_1(=\omega)$ and ω'_2 by induction on the number of faces of G lying in the cell c_1 of the cycle C_1. If c_1 is a cell of G, it is trivial; Otherwise, by Lemma 2.4 $\vec{C}_1^\omega$ has a ω-directed ear P in c_1. Then $G_0 := P \cup C_1 \in \mathcal{G}$ forms two cells R_1 and R_2 of G_0 such that $\partial R_1 \cap \partial R_2 = P$. Then either R_1 or R_2 (say R_1) is ω-clockwise cell of G_0. Let $\omega^* := \omega \oplus \partial R_1$. Then R_2 is ω^*-clockwise. Since both R_1 and R_2 contain fewer cells of G than c_1, by induction hypothesis we have that $\Omega(G)$ has a required sequence $\omega \cdots \omega^* \cdots \omega'_2$. By the same reason $\Omega(G)$ has a required sequence $\omega'_i \cdots \omega'_{i+1}$ for $i = 1, \ldots, m$, as expected.

(a)$\Rightarrow$(c). Suppose that $\omega \sim \omega'$ for distinct $\omega, \omega' \in \Omega(G)$. It is sufficient to prove that every component of $\vec{G}^\omega[\omega \oplus \omega']$ is an Eulerian digraph, i.e., the out-degree of any vertex v equals its in-degree in this digraph. We proceed by induction on $d(\omega, \omega')$, which denotes the length of a path between ω and ω' in the underlying graph of $D(G)$. If $d(\omega, \omega') = 1$, suppose that $D(\omega) = \omega'$. Then $\omega \oplus \omega'$ is the union of edge-disjoint boundaries of some ω-clockwise cells by Definition 3.2. Hence the result holds.

In what follows, suppose that $d(\omega, \omega') = n + 1 \geq 2$. Then there exists $\omega^* \in \Omega(G)$ such that $d(\omega, \omega^*) = n$ and $d(\omega^*, \omega') = 1$. Let $E_1 := \omega \oplus \omega^* \subseteq E(G)$ and $E_2 := \omega^* \oplus \omega' \subseteq E(G)$. By induction hypothesis we have that both $\omega(E_1)$ and $\omega'(E_2)$ form digraphs every component of which is Eulerian. It follows that

$$\omega \oplus \omega' = (\omega \oplus \omega^*) \oplus (\omega^* \oplus \omega') = E_1 \oplus E_2 = \cup_{v \in V(G)}(E_1(v) \oplus E_2(v)).$$

For any given vertex v of G, it is obvious that $(E_1 \oplus E_2)(v) = E_1(v) \oplus E_2(v)$. Put $\Gamma := E_1(v) \cap E_2(v)$, and $\Gamma_i := E_i(v) \backslash \Gamma$ for i=1 and 2. Then $E_1(v) \oplus E_2(v) = \Gamma_1 \cup \Gamma_2$. For every edge e of Γ, we have that $\omega(e) = \omega'(e)$; for every edge e of $\Gamma_i, i = 1, 2$, we have that $\omega(e) \neq \omega'(e)$.

For $E \subseteq E(G)$, let $d(E^\omega(v))$ denote the number of out-arcs of E^ω incident to v minus the number of in-arcs incident to v. Put $d_0 := d(\Gamma^\omega) = d(\Gamma^{\omega'})$. Then $d(E_1^\omega(v)) = d(\Gamma^\omega) + d(\Gamma_1^\omega) = 0$ and $d(E_2^{\omega'}(v)) = d(\Gamma^{\omega'}) + d(\Gamma_2^{\omega'}) = 0$ imply that

$$d((E_1 \oplus E_2)^\omega(v)) = d(\Gamma_1^\omega(v)) + d(\Gamma_2^\omega(v)) = d(\Gamma_1^\omega(v)) - d(\Gamma_2^{\omega'}(v)) = -d_0 + d_0 = 0.$$

Thus every component of $\vec{G}^\omega[\omega \oplus \omega']$ is Eulerian. $\square$

Corollary 3.8. *Let $G \in \mathcal{G}$ and $\omega_1, \omega_2 \in \Omega(G)$. Then $\omega_1 \sim \omega_2$ if and only if $\omega_1^- \sim \omega_2^-$.*

Proof. Since $\omega_1 \oplus \omega_2 = \omega_1^- \oplus \omega_2^-$, Lemma 3.7 implies the result. $\square$

Theorem 3.9. *Let $G \in \mathcal{G}$ and T an in-tree of $D(G)$ with root ω_0. Then $\{\omega^- : \omega \sim \omega_0, \omega \in \Omega(G)\}$ also induces an in-tree of $D(G)$, denoted by T^- and called the* dual *of T.*

Proof. Theorem 3.4 and Corollary 3.8 imply the theorem. $\square$

Let T_ω denote the component of $D(G)$ containing $\omega \in \Omega(G)$. The following result is immediate from Theorem 3.9.

Corollary 3.10. *Let $G \in \mathcal{G}$. Then*
(a) $(T^-)^- = T$, where T an in-tree of $D(G)$,
(b) T_{ω^-} is the dual of T_ω for any $\omega \in \Omega(G)$.

Lemma 3.11. *Let $G \in \mathcal{G}$ and $\omega \in \Omega(G)$. Then $T_\omega = T_{\omega^-}$ if and only if ω is an Eulerian orientation of G.*

Proof. If $T_\omega = T_{\omega^-}$, $\omega \sim \omega^-$. Since $\omega \oplus \omega^- = E(G)$, by Lemma 3.7 we have that $\vec{G}^\omega$ is an Eulerian digraph. Conversely, assume that $\vec{G}^\omega$ is an Eulerian digraph. Then $E(G) = \omega \oplus \omega^-$ is the union of edge-disjoint ω-directed cycles. By Lemma 3.7 we have that $\omega \sim \omega^-$, i.e. $T_\omega = T_{\omega^-}$. $\square$

Theorem 3.12. *Let $G \in \mathcal{G}$ be Eulerian. Then all the Eulerian orientations of G induce an in-tree of $D(G)$.*

Proof. Let $\omega_1, \omega_2 \in \Omega(G)$. Suppose that ω_1 is an Eulerian orientation of G. It suffices to prove that $\omega_1 \sim \omega_2$ if and only if ω_2 is Eulerian. Put $E := E(G)$ and $E_0 := \omega_1 \oplus \omega_2$. For any given vertex v of G, let $E_1(v) := \{e \in E(v) : \omega_1(e) = \omega_2(e)\}$. Then $E_0(v) = \omega_1(E(v)) \oplus \omega_2(E(v)) = E(v) \backslash E_1(v)$. Adopting the notation in the proof of Lemma 3.7, we denote $d_0 := d(E_1^{\omega_i}(v))$ for $i = 1, 2$. Since ω_1 is an Eulerian orientation of G, then $d(E_0^{\omega_1}(v)) = -d_0$; Further $d(E_0^{\omega_2}(v)) = -d(E_0^{\omega_1}(v)) = d_0$. Hence

$$d(E^{\omega_2}(v)) = 2d_0 = 0 \Leftrightarrow d(E_1^{\omega_2}(v)) = 0 \Leftrightarrow d(E_0^{\omega_1}(v)) = 0.$$

Since ω_1 is Eulerian, moreover, by Lemma 3.7 we have that $\omega_1 \sim \omega_2$ if and only if for every vertex v of G, $d(E_0^{\omega_1}(v)) = d(E_1^{\omega_1}(v)) = 0$; equivalently, $d(E^{\omega_2}(v)) = 0$, i.e. ω_2 is Eulerian. $\square$

By Lemma 3.11 and Theorem 3.12, we immediately have the following new parity characterization of plane Eulerian graphs.

Theorem 3.13. *Let $G \in \mathcal{G}$. Then $D(G)$ has an odd number of in-trees if and only if G is Eulerian.*

In Fig. 2, a connected cell rotation graph is illustrated. The following result gives a general characterization for $D(G)$ being an in-tree.

Theorem 3.14. *Let $G \in \mathcal{G}$. Then $D(G)$ is an in-tree if and only if every block of G is a cycle.*

Proof. Suppose that every block of G is a cycle. For any $\omega \in \Omega(G)$, since every block of G is a ω-directed cycle, ω is an Eulerian orientation of G. By Theorem 3.12 we have that $D(G)$ is an in-tree.

Conversely, suppose that $D(G)$ is an in-tree. Theorem 3.13 implies that G is an Eulerian graph. We assert that every block of G is a cycle. Otherwise, suppose that G has a block B of G different from cycles. By Lemma 2.3 B has a reducible cell decomposition $B = C_1 + P_2 + \cdots + P_r (r \geq 2)$, which implies that $B_2 := C_1 + P_2$ have exactly two cells c_1 and c_2 that are also cells of B, where $C_1 = \partial c_1$. Then $P := \partial c_1 \cap \partial c_2$ is a path of length ≥ 1. Let $B - P$ denote the subgraph obtained from B by removing the interior of P. Then $B - P = C + P_3 + \cdots + P_r$, where $C := B_2 - P$ is a cycle, which implies that $B - P$ is 2-connected by Lemma 2.2. Let $\omega^* \in \Omega(B - P) \neq \emptyset$. The path P can be orientated in two different ways to get directed paths, denoted by $\omega_1^*(P)$ and $\omega_2^*(P)$. Then $\omega_i' := \omega^* \cup \omega_i^*(P) \in \Omega(B), i = 1, 2$. For any other block B_j of G different from B, let $\omega_j(B_j) \in \Omega(B_j)$. We now construct orientations ω_1 and ω_2 of G as follows: $\omega_i(E(G)) := \omega_i' \cup (\cup \omega_j(B_j)), i = 1, 2$. Since an orientation of G is strongly connected if and only if its restriction on each block of G is strongly connected, $\omega_i \in \Omega(G)$ for $i = 1, 2$. But $\omega_1 \oplus \omega_2$ forms the path P, which implies by Lemma 3.7 that ω_1 and ω_2 belong to distinct in-trees in $D(G)$, a contradiction. $\square$

4 Properties for a pair of In-trees

¿From Fig. 4, we know that an in-tree of $D(G)$ is not necessarily isomorphic to its dual. We now turn to consider the properties which are the same for any pair of in-trees T and T^- of $D(G)$. Obviously, T and T^- have the same number of vertices. A vertex ω of an in-tree T of $D(G)$ with in-degree 0 is called a *leaf* of T; further called a *main leaf* if G has no ω-counterclockwise cycle.

Lemma 4.1. *Let $G \in \mathcal{G}$. Then any in-tree of $D(G)$ has a unique root and a unique main leaf.*

Proof. It is known that any in-tree of $D(G)$ has a unique root from Theorem 3.4. Let T be an in-tree of $D(G)$. By Lemma 3.1(b) it is easily seen that $\omega \in \Omega(G)$ is a main leaf of T if and only if ω^- is a root of T^-, which imply the lemma. □

The number of leaves of an in-tree T is called the *width* of T, denoted by $w(T)$. The largest length of directed paths between leaves and the root of T is called the *height* of T, denoted by $h(T)$. We have the following main results of this article.

Theorem 4.2. *Let $G \in \mathcal{G}$ and T an in-tree of $D(G)$. Then $h(T) = h(T^-)$.*

Theorem 4.3. *Let $G \in \mathcal{G}$ and T an in-tree of $D(G)$. Then $w(T) = w(T^-)$.*

Let $\omega \in \Omega(G)$. The height $h(\omega)$ of ω is the distance between ω and the root of T_ω. It is obvious that $h(T) = \max_{\omega \in T} h(\omega)$. We first obtain some lemmas as follows.

Lemma 4.4. *Let $G \in \mathcal{G}$. For distinct $\omega, \omega' \in \Omega(G)$, assume that $\omega \oplus \omega'$ is the union of mutually edge-disjoint ω-clockwise cycles. Then $\Omega(G)$ has a sequence $\omega_1(=\omega), \omega_2, \cdots, \omega_k(=\omega')$ such that $\omega_i \oplus \omega_{i+1}$ is the boundary of a ω_i-clockwise cell of G for $i = 1, 2, \ldots, k-1$.*

Proof. It follows by the proof of Lemma 3.7 (part (c) $\Rightarrow$ (b)). □

Lemma 4.5. *Let $G \in \mathcal{G}$ and $\omega_1, \omega_2 \in \Omega(G)$. Suppose that $\omega_1 \sim \omega_2$ and $\omega_1 \oplus \omega_2$ is the union of the boundaries of both ω_1-clockwise and ω_2-counterclockwise cells. Then*

$$h(\omega_2) + 1 \geq h(\omega_1) \geq h(\omega_2).$$

Proof. Let i be the minimum non-negative integer such that $D^i(\omega_2)$ is the root of T_{ω_2}. For $0 \leq j \leq i-1, D^j(\omega_1)$ is not the root of T_{ω_2}; Otherwise, by Lemma 3.6 $D^j(\omega_1) \oplus D^j(\omega_2) = \emptyset$, which would imply that $D^j(\omega_2)$ is the root of T_{ω_2}, a contradiction. Further, by Lemma 3.6 $D^i(\omega_1) \oplus D^i(\omega_2)$ is either empty or consists of the boundaries of both $D^i(\omega_1)$-clockwise and $D^i(\omega_2)$-counterclockwise cells. If $D^i(\omega_1) \oplus D^i(\omega_2) = \emptyset$, then $h(\omega_1) = h(\omega_2)$; Otherwise, since G has no $D^i(\omega_2)$-clockwise cycles, it follows that $\mathcal{C}^+(D^i(\omega_1)) = D^i(\omega_1) \oplus D^i(\omega_2)$. Thus $D^{i+1}(\omega_1) = D^i(\omega_2)$ by Definition 3.2, which implies that $h(\omega_1) = h(\omega_2) + 1$. $\square$

Lemma 4.6. *Let $G \in \mathcal{G}$ and T an in-tree of $D(G)$. Suppose that ω is the main leaf of T. Then $h(\omega) \geq h(\omega')$ for any vertex ω' of T.*

Proof. Let ω be the main leaf of T and ω' any other vertex of T. Since $\omega \sim \omega'$ and G has no ω-counterclockwise directed cycles, by Lemma 3.7 $\omega \oplus \omega'$ consists of mutually edge-disjoint ω-clockwise (ω'-counterclockwise) cycles. By Lemma 4.4 we know that $\Omega(G)$ has a sequence $\omega_1(= \omega)\omega_2 \cdots \omega_k(= \omega')$ such that $\omega_i \oplus \omega_{i+1}$ is the boundary of a ω_i-clockwise cell ($1 \leq i < k$). By Lemma 4.5 it follows that $h(\omega_1) \geq h(\omega_2) \geq \cdots \geq h(\omega_k)$. $\square$

Proof of Theorem 4.2. Let ω_0 be the root of T and ω_1 the main leaf of T. Then ω_1^- is the root of T^- and ω_0^- is the main leaf of T^-. Since $h(T) = h(w_1)$ and $h(T^-) = h(w_0^-)$, it is sufficient to prove that $h(\omega_1) = h(\omega_0^-)$. Put $h := h(\omega_1)$. Let $\omega_1 D(\omega_1) \cdots D^h(\omega_1)(= \omega_0)$ denote the path of T from the main leaf ω_1 to the root ω_0. Then, for all $0 \leq i < h$, $D^i(\omega_1) \oplus D^{i+1}(\omega_1) = D^i(\omega_1)^- \oplus D^{i+1}(\omega_1)^-$ consists of the boundaries of all $D^i(\omega_1)$-clockwise cells, which are all $D^{i+1}(\omega_1)^-$-clockwise cells. Thus by Lemma 4.5 we have that

$$h(D^i(\omega_1)^-) + 1 \geq h(D^{i+1}(\omega_1)^-) \geq h(D^i(\omega_1)^-),$$

namely,

$$1 \geq h(D^{i+1}(\omega_1)^-) - h(D^i(\omega_1)^-) \geq 0, \tag{1}$$

for $i = 0, 1, \ldots, h - 1$. Summing up the inequalities in (1), we have that

$$h(D^h(\omega_1)^-) = \sum_{i=0}^{h-1}(h(D^{i+1}(\omega_1)^-) - h(D^i(\omega_1)^-)) \leq h.$$

That is, $h(\omega_1) \geq h(\omega_0^-)$. By Corollary 3.10(a) or similar arguments we have that $h(\omega_0^-) \geq h(\omega_1)$, which implies that $h(\omega_0^-) = h(\omega_1)$. The proof is completed. $\square$

Proof of Theorem 4.3. We define a mapping between the non-leaves of T and T^- as $f : \omega \mapsto D(\omega^-)$ for every non-leave ω of T. Since any in-tree of $D(G)$ is not trivial, $D(\omega^-)$ is a non-leaf of T^-. We now prove that f^2 is an identity mapping on the set of non-leaves of T and T^-. It is obvious that $\mathcal{C}^+(\omega^-) = \mathcal{C}^-(\omega)$ and $\mathcal{C}^-(\omega^-) = \mathcal{C}^+(\omega)$. By Definition 3.2 we have that $D(\omega^-) \oplus \omega^- = \mathcal{C}^+(\omega^-)$, which implies that

$$D(\omega^-)^- \oplus \omega = \mathcal{C}^+(\omega^-) = \mathcal{C}^-(\omega). \tag{2}$$

Since ω is a non-leaf of T, it is easily seen that every component of $\mathcal{C}^+(\omega)(=\mathcal{C}^-(\omega^-))$ shares an edge with a component of $\mathcal{C}^-(\omega)(=\mathcal{C}^+(\omega^-))$, which implies that $\mathcal{C}^+(\omega^-) = \mathcal{C}^-(D(\omega^-))$ by Definition 3.2. Then $\mathcal{C}^+(D(\omega^-)^-) = \mathcal{C}^-(\omega)$ and

$$D(D(\omega^-)^-) \oplus D(\omega^-)^- = \mathcal{C}^-(\omega). \tag{3}$$

Combining Eqs. (2) and (3) we have that

$$D(D(\omega^-)^-) \oplus \omega = \{D(D(\omega^-)^-) \oplus D(\omega^-)^-\} \oplus \{D(\omega^-)^- \oplus \omega\} = \mathcal{C}^-(\omega) \oplus \mathcal{C}^-(\omega) = \emptyset,$$

which means that $D(D(\omega^-)^-) = \omega$; namely, $f^2(\omega) = \omega$. Thus f is a bijection between the non-leaves of T and T^-. So T and T^- have the same number of leaves. $\square$

5 An application to perfect matchings

Recall first some concepts and notations. Let G be a plane bipartite graph. A *perfect matching* of G is a set of independent edges which cover all vertices of G. An edge of G is called *fixed single* if it belongs to none of its perfect matchings. G is *elementary* if it is connected and every edge is contained in its perfect matching. It is known that elementary bipartite graphs with more than one edge are 2-connected. Other properties on elementary (plane) bipartite graphs can be found in [10, 20].

Let G be a plane bipartite graph with a perfect matching M. A cycle C of G is said to be M-alternating if the edges of C appear alternatively in M and $E(G)\backslash M$. An M-alternating cycle C is said to be *proper* (*improper*) if every edge of C belonging to M goes from the white (black) end-vertex to the black (white) end-vertex by the clockwise orientation of C; Further proper (improper) M-alternating cycle C is said to be *minimal* if only fixed single edges in the interior of C are incident to vertices of C. When G is elementary, in particular, every minimal M-alternating cycle of G is the boundary of a cell.

Let $\mathcal{C}^+_{g'}(M)$ ($\mathcal{C}^-_{g'}(M)$) denote the union of all minimal proper (improper) M-alternating cycles of G, which are pairwise disjoint. Let $M(G)$ denote the set of perfect matchings of G.

Definition 5.1. For $M \in M(G)$, define pairs of operations $R_{g'}$ and $\overline{R}_{g'}$ as $R_{g'}(M) := M \oplus \mathcal{C}^+_{g'}(M) \in M(G)$ and $\overline{R}_{g'}(M) := M \oplus \mathcal{C}^-_{g'}(M) \in M(G)$.

Definition 5.2. Define pairs of digraphs $R_{g'}(G)$ and $\overline{R}_{g'}(G)$ on the $M(G)$ as follows: for $M, M' \in M(G)$, $M \neq M'$, (M, M') is an arc of $R_{g'}(G)$ if and only if $R_{g'}(M) = M'$; and (M, M') is an arc of $\overline{R}_{g'}(G)$ if and only if $\overline{R}_{g'}(M) = M'$.

Lemma 5.1 [19]. *Let G be a plane elementary bipartite graph. Then $R_{g'}(G)$ and $\overline{R}_{g'}(G)$ are in-trees.*

Let G be a plane bipartite graph. For any $M \in M(G) \neq \emptyset$, an orientation ω_M of G associated with M is defined as in Section 1. It is known that a cycle C of G is proper (improper) M-alternating if and only if C is a ω_M-clockwise (counterclockwise) directed cycle; and G is elementary if and only if ω_M is a strongly connected orientation of G [17].

Theorem 5.2. *Let G be a plane elementary bipartite graph with at least one cycle. Then $(R_{g'}(G), \overline{R}_{g'}(G))$ corresponds to a pair of in-trees of $D(G)$.*

Proof. Define a mapping $f : M(G) \to \Omega(G)$ as $f(M) = \omega_M$ for $M \in M(G)$. It is obvious that f is injective. For $M, M' \in M(G)$, $M \oplus M' = \omega_M \oplus \omega_{M'}$ is the union of disjoint M-alternating and thus ω_M-directed cycles. By Lemma 3.7 we have that $\omega_M \sim \omega_{M'}$. Let $\omega \in \Omega(G)$ with $\omega \sim \omega_M$. By Lemma 3.7 we have that $\Omega(G)$ has a sequence $\omega_1(= \omega_M), \omega_2, \ldots, \omega_k(= \omega)(k \geq 2)$ such that $c_i = \omega_i \oplus \omega_{i+1}$ is the boundary of a ω_i-directed cell, $i = 1, \ldots, k-1$. Since G is 2-connected, the c_i's are ω_i-directed cycles. Then $M_2 := M \oplus c_1 \in M(G)$ and $f(M_2) = \omega_2$. By repeating the above procedure we have that $M_i := M \oplus c_1 \oplus \cdots \oplus c_{i-1} \in M(G)$ and $f(M_i) = \omega_i (2 \leq i \leq k)$, that is, $\omega = \omega_{M_k}$. So f is a bijection between $M(G)$ and the vertex-set of T_{ω_M} for $M \in M(G)$. It is easily seen that (M, M') is an arc of $R_{g'}(G)$ if and only if $(\omega_M, \omega_{M'})$ is an arc of T_{ω_M}. Thus f induces an isomorphism between $R_{g'}(G)$ and T_{ω_M}. Similarly, it follows that $\overline{f} : M \mapsto \omega_M^-$ induces an isomorphism between $\overline{R}_{g'}(G)$ and $T_{\omega_M^-}$. □

Combining Theorems 4.2 and 4.3 with Theorem 5.2, we immediately have

Corollary 5.3. *Let G be a plane elementary bipartite graph. Then $R_{g'}(G)$ and $\overline{R}_{g'}(G)$ have the same width and height.*

Let $T_1, \ldots, T_n$ be in-trees. Define the digraph $T := T_1 \otimes \cdots \otimes T_n$ as follows: $V(T) := V(T_1) \times \cdots \times V(T_n) = \{(v_1, \ldots, v_n) : v_i \in V(T_i), i = 1, \ldots, n\}$; for $v = (v_1, \ldots, v_n), v' = (v'_1, \ldots, v'_n) \in V(T)$, $(v, v') \in A(T)$ if and only if for $\emptyset \neq I \subseteq \{1, \ldots, n\}, (v_i, v'_i) \in A(T_i)$ if $i \in I$ and $v_i = v'_i$ is the root of T_i if $i \notin I$. Then T is also an in-tree.

Removing all fixed single edges from G, every component of the resultant subgraph is a plane elementary bipartite graph, which is thus called an *elementary component*.

Theorem 5.4 [18]. *Let G be a plane bipartite graph with distinct perfect matchings. Let $G_1, \ldots, G_n (n \geq 1)$ denote the elementary components of G. Then*

$$R_{g'}(G) \cong R_{g'}(G_1) \otimes \cdots \otimes R_{g'}(G_n) \text{ and } \overline{R}_{g'}(G) \cong \overline{R}_{g'}(G_1) \otimes \cdots \otimes \overline{R}_{g'}(G_n).$$

Furthermore both $R_{g'}(G)$ and $\overline{R}_{g'}(G))$ are in-trees.

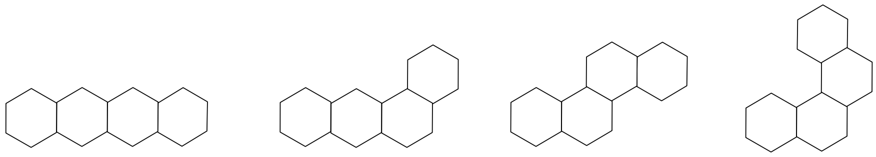


Fig. 5. Homeomorphic hexagonal chains.

We have the following main result in this section.

Theorem 5.5. *Let G be a plane bipartite graph with perfect matchings. Then $w(R_{g'}(G)) = w(\overline{R}_{g'}(G))$ and $h(R_{g'}(G)) = h(\overline{R}_{g'}(G))$.*

Proof. Let $T_1, \ldots, T_n$ be non-trivial in-trees and $T = T_1 \otimes \cdots \otimes T_n$. It is easy to see that

$$h(T) = \max_{1 \leq i \leq n} h(T_i).$$

Since a vertex $v = (v_1, \ldots, v_n)$ of T is non-leaf of T if and only if every v_i is also a non-leaf of T_i, we have the following relation

$$nl(T) = \prod_{1 \leq i \leq n} nl(T_i),$$

where $nl(T)$ denotes the number of non-leaves of in-tree T. For every elementary component G_i, by Corollary 5.3 we have that $w(R_{g'}(G_i)) = w(\overline{R}_{g'}(G_i))$ and $h(R_{g'}(G_i)) = h(\overline{R}_{g'}(G_i))$. The above relations together with Theorem 5.4 imply the required equations. □

For a hexagonal system H with a perfect matching M, although H is not necessarily elementary (normal), any M-alternating cycle of H contains no fixed single edges in its interior [20]; that is, every minimal M-alternating cycle must be a hexagon (sextet). Hence $R_{g'}(H)$ and $\overline{R}_{g'}(H)$ coincide with the pair of directed rooted structures on the perfect matchings of H produced by sextet and counter-sextet rotations. Accordingly, Gutman et al.'s findings [6, 7, 8]: such a pair of in-trees have the same width and height, are rigorously proved in an extensive sense (Theorems 4.2, 4.3 and 5.5).

6 Open problems

Finally we would like to mention that homeomorphic plane graphs have isomorphic cell rotation graphs. For example, the cell rotation graphs of the four homeomorphic hexagonal chains in Fig. 5 are isomorphic to $D(G_0)$ (see Fig. 4). On the other hand, this cell rotation graph consists only of pairs of the directed tree structures on the perfect matchings of these hexagonal chains. We conclude this paper with proposing the following open problems.

Problem 6.1. Characterize such plane graphs whose cell rotation graphs consists only of pairs of in-tree structures on the sets of perfect matchings of plane bipartite graphs?

Problem 6.2. For any 2-edge connected plane graph, do the heights of in-trees of its cell rotation graph possess the interpolation property: the heights of in-trees compose of consecutive positive integers?

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