Duality in the Bandwidth Problem

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Abstract

The bandwidth is an important invariant in graph theory. However, the problem to determine the bandwidth of a general graph is NP-complete. To get sharp bounds, we propose to pay attention to various duality properties or minimax relations related to the bandwidth problem. This paper presents a summary in this point of view.

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1. Introduction

The bandwidth problem for graphs and matrices originates from sparse matrix computation, circuit layout of VLSI designs and other areas. Mathematically, we can describe it as follows [2]. Let $G=(V,E)$ be a simple graph with vertex set V and edge set E. A *labeling* of G is a bijection $f: V \to \{1, 2, \cdots, |V|\}$. The *bandwidth of a labeling f* for G is defined by

$$B(G,f) = \max\{|f(u) - f(v)| : uv \in E\};$$

and the *bandwidth* of G is defined by

$$B(G) = \min\{B(G,f) : f \text{ is a labeling of } G\}.$$

A labeling that attains this minimum value is called an *optimal labeling*.

The problem of determining $B(G)$ can be viewed as a minimization problem which is NP-hard [8]. So a direction of research is to find sharp bounds for $B(G)$. It is well-known in combinatorial optimization that minimax relations can usually provide good lower or upper bounds. For bandwidth problem, as a minimization problem, we could establish relations with some maximization problems. In this paper, we study this type of duality in several aspects. In Section 2, we illustrate how a duality relation about degrees implies a series of bounds. In Section 3 we discuss the so-called density lower bounds. In Section 4, we describe a method based on the boundary lower bounds.

2. Duality on Degrees

The degree sequence of G is denoted by $d(G) = (d_1, d_2, \ldots, d_n)$, where $d_1 \le d_2 \le \cdots \le d_n$. Suppose that G and G' have the same order n. G is said to be *degree-majorized* by G', denoted by $d(G) \le d(G')$, if $d_i \le d_i'$ for $i = 1, 2, \ldots, n$ (see for example [1]).

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Lemma 2.1: *For any graph G with n vertices,*

$$B(G) \geq \min\{k : d(G) \leq d(P_n^k)\},$$

where P_n^k stands for the k-th power of a path on n vertices.

Proof: Suppose that $B(G) = k$. Then, by definition, G can be embedded in P_n^k, i.e., $G \subseteq P_n^k$. It implies $d(G) \leq d(P_n^k)$. Hence the lower bound follows. ∎

Next, we have the following minimax duality.

Lemma 2.2: *If $d(G) = (d_1, d_2, \ldots, d_n)$ then*

$$\min\{k : d(G) \leq d(P_n^k)\} = \max_{1 \leq i \leq n} \max\left\{d_i - \left\lfloor \tfrac{i-1}{2} \right\rfloor, \left\lceil \tfrac{d_i}{2} \right\rceil\right\}.$$

Proof: It is clear that

$$d(P_n^k) = \begin{cases} (k, k, k+1, k+1, \ldots, 2k, \ldots, 2k) & \text{if} \quad 1 \leq k < \frac{n}{2}, \\ (k, k, k+1, k+1, \ldots, n-1, \ldots, n-1) & \text{if} \quad \frac{n}{2} \leq k < n. \end{cases}$$

If we let $d(P_n^k) = (d_1^*, d_2^*, \ldots, d_n^*)$, then

$$d_i^* = k + \min\left\{\left\lfloor \tfrac{i-1}{2} \right\rfloor, k, n-k-1\right\},$$

and $d(G) \leq d(P_n^k)$ may be written as

$$d_i \leq k + \left\lfloor \tfrac{i-1}{2} \right\rfloor,\ d_i \leq 2k,\ d_i \leq n-1,\ (1 \leq i \leq n).$$

Since the last inequality is redundant, $k \geq \max\left\{d_i - \left\lfloor \tfrac{i-1}{2} \right\rfloor, \left\lceil \tfrac{d_i}{2} \right\rceil\right\}$.

The minimum value of k gives the equality. ∎

By these two lemmas, we have the following minimax-type lower bound due to Chvátal [4].

Theorem 2.3: *Suppose $d_1, d_2, \ldots, d_n$ is the degree sequence of a graph G. Then*

$$B(G) \geq \max\left\{\left\lceil \tfrac{\Delta(G)}{2} \right\rceil, \max_{1 \leq i \leq n}\left\{d_i - \left\lfloor \tfrac{i-1}{2} \right\rfloor\right\}\right\}.$$

The survey [2] said "It is not so clear which graphs realize the bound". By the foregoing lemmas, it is clear that the bound is attained when $G = K_{1,n}$, P_n^k and K_n. Furthermore, we have an improvement of this lower bound when a part of vertices have been labeled. We need this kind of lower bounds in the branch-and-bound algorithms, for example.

Theorem 2.4: *Suppose $v_1, v_2, \ldots, v_j$ have been labeled $1, 2, \ldots, j$ respectively and the remaining vertices have degree sequence $d_{j+1} \leq d_{j+2} \leq \cdots \leq d_n$. Then*

$$B(G) \geq \begin{cases} \max\{b_1, b_2, b_3\} & \text{if} \quad j < \frac{n}{2} \\ \max\{b_1, b_2'\} & \text{if} \quad j \geq \frac{n}{2}, \end{cases}$$

where

$$\begin{aligned} b_1 &= \max_{1 \leq i \leq j} \max\left\{d(v_i) - i + 1, \left\lceil \tfrac{d(v_i)}{2} \right\rceil\right\}, \\ b_2 &= \max_{1 \leq i \leq j} \max\left\{d_{j+i} - i + 1, \left\lceil \tfrac{d_{j+i}}{2} \right\rceil\right\}, \end{aligned}$$

$$b_2' = \max_{1\le i\le n-j} \max\left\{d_{j+i} - i + 1, \left\lceil \frac{d_{j+i}}{2} \right\rceil\right\},$$
$$b_3 = \max_{2j+1\le i\le n} \max\left\{d_i - \left\lfloor \frac{i-1}{2} \right\rfloor, \left\lceil \frac{d_i}{2} \right\rceil\right\}.$$

Proof: Suppose that $B(G) = k$. Embed G in P_n^k with $v_1, v_2, \ldots, v_j$ as its first j vertices. Then

$$d(v_i) \le \min\{k + i - 1, 2k, n - 1\},\ 1 \le i \le j,$$

and $B(G) = k \ge b_1$. When $j < \frac{n}{2}$ and $k \ge \frac{n}{2}$ the degree sequence for the remaining vertices of P_n^k will be $k, k+1, \ldots, k+j-1, k+j, k+j, k+j+1, k+j+1, \ldots, n-1, \ldots, n-1$. It follows from the first j terms of the sequence that $B(G) \ge b_2$ (as in the case of b_1), from the last $n \pm 2j$ terms of the sequence that $B(G) \ge b_3$ (as in Theorem 2.3). Therefore $B(G) \ge \max\{b_1, b_2, b_3\}$. A similar argument will lead to the same conclusion for other values of k. When $j \ge \frac{n}{2}$, the remaining degree sequence of P_n^k will be $k, k+1, k+2, \ldots, n \pm 1$. So, we have $B(G) \ge b_2'$ and $B(G) \ge \max\{b_1, b_2'\}$. ▮

The efficiency of these minimax-type lower bounds can be illustrated by the fact that they can imply most lower bounds in terms of other graph invariants listed in the survey [2]. We mention some typical examples as follows.

Corollary 2.5:

(1) *For* $n = |V(G)|,\ m = |E(G)|,\ B(G) \ge n \pm \dfrac{1 + \sqrt{(2n \pm 1)^2 - 8m}}{2}$;

(2) *For maximum degree* $\Delta(G)$, $B(G) \ge \left\lceil \dfrac{\Delta(G)}{2} \right\rceil$;

(3) *For minimum degree* $\delta(G)$, $B(G) \ge \delta(G)$;

(4) *For connectivity* $\kappa(G)$, $B(G) \ge \kappa(G)$;

(5) *For chromatic number* $\chi(G)$, $B(G) \ge \chi(G) \pm 1$;

(6) *For vertex-independence number* $\beta_0(G)$, $B(G) \ge \left\lceil \dfrac{|V(G)|}{\beta_0(G)} \right\rceil \pm 1$.

Proof:

(1) Let k be the lower bound of Theorem 2.3. Then, by Lemma 2.2, $d(P_n^k) \ge d(G)$. By $\sum_{i=1}^{n} d_i^* \ge \sum_{i=1}^{n} d_i$ and the degree sequence of P_n^k (see the proof of Lemma 2.2.), we have $k^2 - (2n \pm 1)k + 2m \le 0$. Thus

$$B(G) \ge k \ge \frac{(2n \pm 1) - \sqrt{(2n \pm 1)^2 - 8m}}{2}.$$

(2) Follows from Theorem 2.3.

(3) For $i = 1$, $B(G) \ge d_1 = \delta(G)$.

(4) $B(G) \ge \delta(G) \ge \kappa(G)$.

(5) For the chromatic number of G, there must be a critical subgraph H such that $\delta(H) \ge \chi(G) \pm 1$. Hence $B(G) \ge B(H) \ge \delta(H) \ge \chi(G) \pm 1$.

(6) $B(G) \ge \chi(G) \pm 1 \ge \left\lceil \dfrac{|V(G)|}{\beta_0(G)} \right\rceil \pm 1$. ▮

Note that any graph which attains one of the six bounds in Corollary 2.5 also attains the one in Theorem 2.3.

3. Duality on Diameters

Theorem 3.1: *Let u and v be any two vertices in G. If there is a (u, v)–path of length k, then*

$$B(G,f) \geq \frac{|f(u)-f(v)|}{k}.$$

Proof: Let $P = v_0v_1v_2\ldots v_k$ be a path between u and v in G (where $u = v_0$, $v = v_k$). Then

$$B(G,f) \geq \max\left\{|f(x)-f(y)| : xy \in E(P)\right\}$$
$$\geq \frac{1}{k}\sum_{i=0}^{k-1} |f(v_{i+1})-f(v_i)| \geq \frac{|f(u)-f(v)|}{k}.$$

∎

From this, we have the following useful lower bound due to V. Chvátal [4].

Corollary 3.2: *For any connected graph G with diameter D(G),*

$$B(G) \geq \left\lceil \frac{|V(G)|-1}{D(G)} \right\rceil.$$

Proof: It follows by choosing u and v satisfying $f(u)=1$ and $f(v)=|V(G)|$ for any labeling *f*. ∎

To strengthen this bound, we can establish a minimax form as follows [3,7].

Corollary 3.3: *For any connected graph G*

$$B(G) \geq \max_k \max\left\{\frac{|S|-1}{D(S)} : S \subseteq V(G),\ D(S)=k\right\}$$

where the diameter of subset S is defined by $D(S) = \max\left\{d_G(x,y) : x,y \in S\right\}$.

Proof: It follows by choosing u and v satisfying $f(u) = \min\left\{f(x) : x \in S\right\}$ and $f(v) = \max\left\{f(x) : x \in S\right\}$ for any labeling *f*. ∎

This minimax lower bound is sharp for many classes of special graphs. Two examples are given below (see [9] and [7]).

A caterpillar is a tree which yields a path (the spine) when all its pendant vertices are removed.

Proposition 3.4: *For a caterpillar T with spine* $u_1u_2\ldots u_m$, *let* T_{ij} *be the subtree induced by* $u_i, u_{i+1}, \ldots, u_j$ *and their neighbors. Then*

$$B(T) = \max_{1\leq i\leq j\leq m} \left\lceil \frac{|V(T_{ij})|-1}{j-i+2} \right\rceil.$$

A tree of diameter 4 consists of a central star $S_0 \cong K_{1,r}$ and r stars $S_1, S_2, \ldots, S_r$, each of which has an end vertex of S_0 as its center. We denote this tree T by $(S_0; S_1, S_2, \ldots, S_r)$.

Proposition 3.5: *For a tree of diameter 4,* $T=(S_0;S_1,S_2,\ldots,S_r)$*, let* n_i *and* n_{0i} *be the vertex numbers of* S_i *and* $S_0\cup S_i$ *respectively, and n is the vertex number of T. Then*

$$B(T)=\max\left\{\max_{0\le i\le r}\left\lceil\frac{n_i\pm1}{2}\right\rceil,\max_{1\le i\le r}\left\lceil\frac{n_{0i}\pm1}{3}\right\rceil,\left\lceil\frac{n\pm1}{4}\right\rceil\right\}.$$

4. Duality on Boundaries

For a subset $S\subseteq V(G)$, the *interior* and *outer boundaries* of *S* are defined respectively as $\partial(G)=\{u\in S:\exists v\in V\setminus S \text{ such that } uv\in E(G)\}$ and $N(G)=\{u\in V\setminus S:\exists v\in S \text{ such that } u\in E(G)\}$. The latter is also called the *neighboring* set of *S*. L.H. Harper [5] first established the following duality relation.

Theorem 4.1: *For any connected graph G,* $B(G)\ge\max_{1\le k\le n}\min\{|\partial(G)|:S\subseteq V(G),|S|=k\}$.

By symmetry, we have $B(G)\ge\max_{1\le k\le n}\min\{|N(G)|:S\subseteq V(G),|S|=k\}$.

This lower bound is of significance in solving the bandwidth problem for some special graphs, for example *n*-cubes, $P_m\times P_n$ and others (see [2,5,6]). Nevertheless, we can make it sharper in some cases.

Suppose a labeling *f* is given. Let $u_i=f^{-1}(i)$, $1\le i\le n$, and

$$S_k(f)=\{u_1,u_2,\ldots,u_k\}=f^{-1}(\{1,2,\ldots,k\}).$$

We have the following.

Theorem 4.2: *For any connected graph G,*

$$B(G)\ge\min_f\max_{1\le k\le n}\left|N\left(S_k(f)\right)\right|,$$

where the minimum is taken over all labelings.

The proof is similar to that of Theorem 4.1 (see [2, p.225]). However, this lower bound is greater than or equal to that of Theorem 4.1. In fact, for any labeling *f* and any integer *k*,

$$\left|N\left(S_k(f)\right)\right|\ge\min\{|N(S)|:S\subseteq V(G),\,|S|=k\}.$$

Hence

$$\min_f\max_{1\le k\le n}\left|N\left(S_k(f)\right)\right|\ge\max_{1\le k\le n}\min\{|N(S)|:S\subseteq V(G),\,|S|=k\}.$$

This is a quite good minimax duality relation since the equality holds for many families of graphs. We do not know whether the equality will always hold.

From the above discussion, we may introduce two new invariants:

$$H(G)=\max_{1\le k\le n}\min\{|N(S)|:S\subseteq V(G),\,|S|=k\}\ \text{and}\ H'(G)=\min_f\max_{1\le k\le n}\left|N\left(S_k(f)\right)\right|.$$

Recently, J. Yuan [10] proved that $H'(G)$ is in fact the pathwidth of G. Furthermore, we may combine the interior and outer boundaries to make

$$H^*(G) = \min_f \max_{1\le k\le n} \max\left\{\left|N\left(S_k(f)\right)\right|, \left|\partial\left(S_k(f)\right)\right|\right\}.$$

Then we have

Theorem 4.3: *For any connected graph G,* $B(G) \ge H^*(G) \ge H'(G) \ge H(G)$.

Here, the lower bound $H^*(G)$ is sometimes greater than $H(G)$. For example, $G = K_{1,3}$, $H(G) = 1$, $B(G) = H^*(G) = 2$. As an application of lower bound $H^*(G)$, let us see the bandwidth of the "*fan meshes*" (grid graphs in a fan) $F_{m,n}$, which is shown in Figure 1. We originally studied this problem in [11]. Now, here is a revised version.

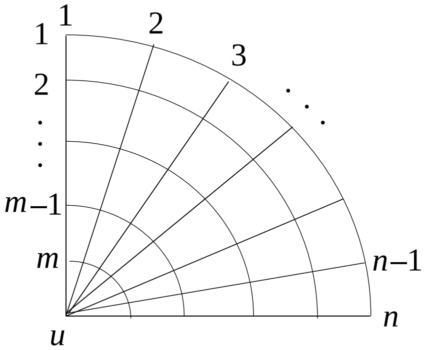

Figure 1

Proposition 4.4: *Let* $F_{m,n}$ *be a mesh* $P_m \times P_n$ *with an extra vertex u joined to n vertices of* P_n *at one side. Then* $B(F_{m,n}) = \min\left\{n, \left\lceil \frac{n}{2} \right\rceil + m \pm 1\right\}$.

Proof: Let $R_1, R_2, \ldots, R_m$ be the vertex sets of *m* copies of P_n; $T_1, T_2, \ldots, T_n$ be those of *n* copies of P_m in $F_{m,n}$ respectively. For any subset $S \subseteq V$, denote

$$\varphi(S) = \left|\left\{R_i : R_i \cap S \ne \emptyset\right\}\right| \text{ and } \psi(S) = \left|\left\{T_j : T_j \cap S \ne \emptyset\right\}\right|.$$

For a given labeling *f*, suppose $k = f(u)$.

Case 1 If $\varphi\left(S_k(f)\right) < m$, then every T_j has at least one vertex in $N\left(S_k(f)\right)$. Thus $B(G, f) \ge \left|N\left(S_k(f)\right)\right| \ge n$.

Case 2 If $\varphi\left(S_k(f)\right) = m$ and $\psi\left(S_k(f)\right) = n$, then every T_j has at least one vertex in $\partial\left(S_{k-1}(f)\right)$. Thus $B(G, f) \ge \left|\partial\left(S_{k-1}(f)\right)\right| \ge n.$

Case 3 Suppose that $\varphi\left(S_k(f)\right) = m$ and $\psi\left(S_k(f)\right) < n$. If $\left|R_m \cap S_k(f)\right| \le \left\lfloor \frac{n}{2} \right\rfloor$ then $R_m \setminus S_k(f) \subseteq N\left(S_k(f)\right)$, so $B(G, f) \ge \left|N\left(S_k(f)\right)\right| \ge \left\lceil \frac{n}{2} \right\rceil + m \pm 1$. If $\left|R_m \cap S_k(f)\right| \ge \left\lceil \frac{n}{2} \right\rceil$ then $B(G, f) \ge \left|\partial\left(S_{k-1}(f)\right)\right| \ge \left\lceil \frac{n}{2} \right\rceil + m - 1$.

Summing up the above cases, we have

$$B(G) \ge \min\left\{n, \left\lceil \frac{n}{2} \right\rceil + m - 1\right\}.$$

On the other hand, we can construct a labeling *f* attaining the lower bound $\beta = \min\left\{n, \left\lceil \frac{n}{2} \right\rceil + m - 1\right\}$. For clarity, we may write down all labels in an $m \times n$ table with an extra row (row $m + 1$) for $f(u)$. (see examples below). If $\beta = n$, we may simply label the table row by row. Assume in the sequel that $\beta = \lceil \frac{n}{2} \rceil + m - 1$. Denote $r = \lceil \frac{n}{2} \rceil - m$. We partition the table into levels:

$$
\begin{aligned}
L_1 &= \{(1,1),\ldots,(1,r+1)\} \\
L_2 &= \{(2,1),\ldots,(2,r+2),(1,r+2)\} \\
&\vdots \\
L_m &= \{(m,1),\ldots,(m,r+m),(m-1,r+m),\cdots,(1,r+m)\}
\end{aligned}
$$

where (i, j) stands for the element of row i and column j. Then, take the row m+1 as L_{m+1}. The remaining $\lfloor \frac{n}{2} \rfloor$ columns are partitioned symmetrically. After that, we put the labels $1,2,\cdots,mn,mn+1$ into the table level by level (where $f(u) = m\lceil \frac{n}{2} \rceil + 1$). It is easy to check that $B(G, f) = \lceil \frac{n}{2} \rceil + m - 1$. This completes the proof. ▮

Example: The optimal labelings for $F_{3,6}$ and $F_{3,7}$ are as follows.

1	4	9	13	17	19
2	3	8	12	16	18
5	6	7	11	14	15
10					

$B(F_{3,6}) = 5\ \ (r = 0)$

1	2	6	12	16	20	22
3	4	5	11	15	19	21
7	8	9	10	14	17	18
13						

$B(F_{3,7}) = 6\ \ (r = 1)$

5. Concluding Remarks

How to determine the bandwidth of a graph? There may be two approaches as follows. (1) a static approach: first, give a sharp lower bound; then, construct a labeling attaining the bound. (2) a dynamic approach: during the labeling searching process, the lower bounds are improved successively until a bound is attained by a labeling. The former is used to get explicit solutions for some special graphs; the latter is used to get numerical solutions (in branch-and-bound algorithms or dynamic programming algorithms). In either case, the greater a lower bound is found, the better. Therefore, it is natural to set up a lower bound through a maximization problem.

Furthermore, the bandwidth minimization problem can be stated as an integer programming as follows. Let $x_{ij} = 1$ if $f(v_i) = j$ and 0 otherwise. Then the label of v_i is $f(v_i) = \sum_{k=1}^{n} kx_{ik}$, $1 \le i \le n$, and we have the following IP problem:

$$
\begin{aligned}
&\text{Minimize} && y && \\
&\text{subject to} && \sum_{k=1}^{n} k(x_{ik} - x_{jk}) \le y && \text{for } v_i v_j \in E, && (1)\\
& && \sum_{j=1}^{n} x_{ij} = 1 && \text{for } i = 1,\ldots,n, && (2)\\
& && \sum_{i=1}^{n} x_{ij} = 1 && \text{for } j = 1,\ldots,n, && (3)\\
& && x_{ij} \ge 0 \quad \text{integer.} && && (4)
\end{aligned}
$$

We may apply the method of Lagrange relaxation with respect to (1) so as to include these constraints into the objective function as a term of “penalty”. The reformulation will then be the well-known assignment problem (optimal bipartite matching problem). Thus we can obtain lower bounds by solving the dual of the assignment problem (a maximization problem). This is the so-called Lagrange relaxation and duality procedure, which would be considered as an efficient way in the algorithmic study of the bandwidth problem.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications.* American Elsevier, New York, 1976.

[2] P.Z. Chinn, J. Chvátalová, A.K. Dewdney, and N.E. Gibbs, “The Bandwidth problem for graphs and matrices - a survey”, *J. of Graph Theory*, **6** (1982), 223-254.

[3] F.R.K. Chung, “Labelings of graphs”, in L.W. Beineke and R.J. Wilson (ed.), *Seclected Topics in Graph Theory*, Vol. **3** (1988), 151-168.

[4] V. Chvátal, “A remark on a problem of Harary”, *Czechoslovak Math. J.*, **20** (1970), 109-111.

[5] L.H. Harper, “Optimal numberings and isoperimetric problems on graphs”, *J. Combin. Theory*, **1** (1966), 385-393.

[6] Y. Lin, “On the Harper’s method for the bandwidth problem”, *Chinese J. of Opns. Res.*, **2:2** (1983), 11-17.

[7] Y. Lin, “A level structure approach on the bandwidth problem for special graphs”, *Ann. of the New York Academy of Sciences*, **576** (1989), 344-357.

[8] C.H. Papadimitriou, “The NP-completeness of the bandwidth minimization problem”, *Computing*, **16** (1976), 263 - 270.

[9] M.M. Systo and J. Zak, The bandwidth problem: critical subgraphs and the solution for caterpillars. *Ann. of Discrete Math.* **16** (1982) 281 - 286.

[10] J. Yuan, “Quasi-bandwidth and forward bandwidth”, *Science in China* (A), **25:12** (1995).

[11] K. Zhang and Y. Lin, “Bandwidth of grid graphs on a fan”, *J. of Zhengzhou Univ.*, **25:4** (1993), 11-15.