Unicyclic and Bicyclic Graphs of Rank 4 or 5*

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Abstract

The spectrum of a graph G is the collection of eigenvalues of its adjacency matrix $A(G)$. The rank of G, denoted by $r(G)$, is the number of non-zero eigenvalues in its spectrum, or equivalently, the rank of $A(G)$. The nullity of a graph G is the multiplicity of the eigenvalue zero in its spectrum. It is known that the rank is equal to the difference from the order to the nullity of the graph. Hu *et al.* in [On the nullity of bicyclic graphs, *Lin. Algebra Appl.*, **429** (2008), 1387-1391.] characterized bicyclic graphs of order n with nullity $n-4$. That is, they characterized bicyclic graphs of rank 4. But they missed some cases. In this paper, we will complete their proof and characterize unicyclic and bicyclic graphs of rank 5, respectively.

Key words: Eigenvalues (of graphs), unicyclic graph, bicyclic graph, rank, nullity.

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1 Introduction

Let $G = (V, E)$ be a simple undirected graph. For any $v \in V$, denote by $d(v)$ and $N(v)$ the degree and neighborhood of v, respectively. The minimum degree of G is denoted by $\delta(G) = \min_{v \in V}\{d(v)\}$. The disjoint union of two graph G_1 and G_2 is denoted by $G_1 + G_2$. The null graph N_n of order n is the graph without edges. Usually, the star and cycle of order n are denoted by S_n and C_n, respectively. Other undefined notations may be referred to [1].

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Suppose G is a graph of order n. Let $A(G)$ be the adjacency matrix of G. The rank of $A(G)$ is called the *rank* of G and denoted by $r(G)$. The nullity of $A(G)$ is called the *nullity* of G and denoted by $\eta(G)$. It is clear that $r(G) = n - \eta(G)$.

Following is a well-known property:

Theorem 1.1 *If G is a simple graph of order n not isomorphic to N_n, then $\eta(G) \leq n-2$, i.e., $r(G) \geq 2$.*

In [2], Collatz and Sinogowitz first posed the problem of characterizing all graphs which satisfy $\eta(G) > 0$. The nullity of a graph is important in chemistry, because $\eta(G) = 0$ is a necessary (but not sufficient) condition for a so-called conjugated molecule to be chemically stable, where G is the graph representing the carbon-atom skeleton of this molecule. For details and further references see [3].

The following results are obvious.

Proposition 1.2 ([3]) *Let G be a graph of order n. Then $\eta(G) = n$ if and only if G is a null graph.*

Remark 1.1 From Theorem 1.1 and Proposition 1.2, we know that no simple graphs with nullity $n-1$, i.e., rank 1.

Proposition 1.3 ([3]) *Let H be a vertex induced subgraph of G. Then $r(H) \leq r(G)$.*

Proposition 1.4 ([6]) *For a graph G containing a vertex of degree 1, if the induced subgraph H of G is obtained by deleting this vertex together with the vertex adjacent to it, then $\eta(G) = \eta(H)$.*

Proposition 1.5 ([3]) *Let $G = G_1 + G_2 + \cdots + G_t$, where $G_1, G_2, \ldots, G_t$ are t connected components of G. Then $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$.*

Proposition 1.6 ([8, 11]) *For $n \geq 3$,*

$$\eta(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

In 2007, Cheng *et al.* [4] investigated the nullity of graphs and get the following results:

Theorem 1.7 ([4]) *Suppose G is a simple connected graph of order n ($n \geq 2$). Then $\eta(G) = n-2$ if and only if G is isomorphic to a complete bipartite graph.*

Theorem 1.8 ([4]) *Suppose G is a simple connected graph of order n ($n \geq 3$). Then $\eta(G) = n-3$ if and only if G is isomorphic to a complete tripartite graph.*

So the characterization of simple connected graphs of rank 2 and 3 are completely solved, respectively. But simple connected graphs of rank more than or equal to 4 is difficult to characterize. Up to now, only some special classes of graphs of rank 4 are known. In this paper, we will concentrate on connected acyclic, unicyclic and bicyclic graphs.

2 Acyclic, unicyclic and bicyclic graphs of rank 4

The following theorem gives a concise formula to calculate the rank of acyclic graphs.

Theorem 2.1 ([3]) *If T is an acyclic graph of order n and m is the size of its maximum matchings, then $\eta(T) = n - 2m$, i.e., $r(T) = 2m$.*

Hence there is no acyclic graph with $r(T) = 2k + 1$ for $k \in \mathbb{Z}$.

If an acyclic graph contains a perfect matching, we call it a *PM-acyclic* for convenience. In fact, Theorem 2.1 implies the following corollary.

Corollary 2.2 *Let T be an acyclic graph of order n. The nullity $\eta(T) = 0$ ($r(T) = n$) if and only if T is a PM-acyclic.*

Remark 2.1 For any connected acyclic graph (i.e., tree) T of order n ($n \geq 4$), by Theorem 2.1 we have $r(T) = 4$ if and only if T is isomorphic to one of the following connected acyclic graphs (see Fig. 1).

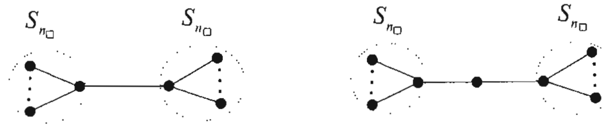

Figure 1: Connected acyclic graphs ($n_1, n_2 \geq 1$).

Let $\mathcal{G}$ be a collection of graphs. A subset N of $\mathbb{N}$ is said to be the *nullity set* of $\mathcal{G}$ if for any $k \in N$, there exists a graph $G \in \mathcal{G}$ such that $\eta(G) = k$.

A *unicyclic graph* is a simple connected graph with equal number of vertices and edges. Denote by $\mathcal{U}_n$ the set of all unicyclic graphs of order n. In this sections, we will characterize the unicyclic graphs with $r(G) = 5$.

For $n \in \mathbb{N}$, let $[0, n] = \{0, 1, 2, \ldots, n\}$.Tan *et al.* [11] investigated the nullity of unicyclic graphs and get the following results.

Theorem 2.3 ([11]) *The nullity set of* $\mathcal{U}_n$ *is* $[0, n-4]$, *where* $n \geq 5$.

Theorem 2.4 ([11]) *Let* $U \in \mathcal{U}_n$ *with* $n \geq 5$. *Then* $r(U) = 4$ *if and only if* $U \cong U_1^*$ *or* $U \cong U_2^*$ *or* $U \cong U_3^*$, *where* U_1^*, U_2^* *and* U_3^* *are shown in Fig. 2.*

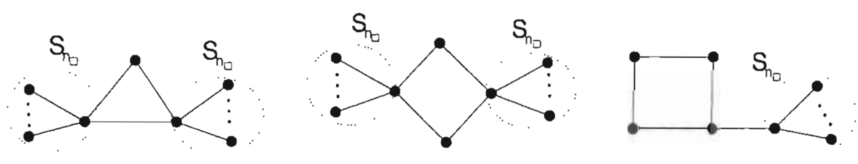

Figure 2: Unicyclic graphs U_1^*, U_2^* and U_3^*.

A *bicyclic graph* is a simple connected graph in which the number of edges equals the number of vertices plus one. By the handshaking lemma [1, Theorem 1.1], it is easy to see that the minimum degree δ of a bicyclic graph is at most 2. Denoted by $\mathcal{B}_n$ the set of all bicyclic graphs of order n.

In 2007, Hu *et al.* [7] characterized the bicyclic graphs with $\eta(G) = n-4$, i.e., $r(G) = 4$.

Theorem A ([7]) *Let* $B \in \mathcal{B}_n$ ($n \geq 6$ *and* $n \neq 7$). *Then* $r(B) = 4$ *if and only if* $B \cong B_n^i$ ($1 \leq i \leq 4$), *where* B_n^i ($1 \leq i \leq 6$) *are shown in Fig. 3. Specifically, let* $B \in \mathcal{B}_7$, *then* $r(B) = 4$ *if and only if* $B \cong B_7^i$ ($1 \leq i \leq 5$), *where* B_n^i ($1 \leq i \leq 5$) *are shown in Fig. 4.*

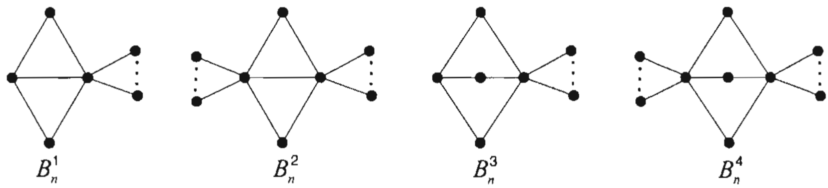

Figure 3: Bicyclic graphs B_n^i $(1 \leq i \leq 4,\ n \geq 6,\ n \neq 7)$.

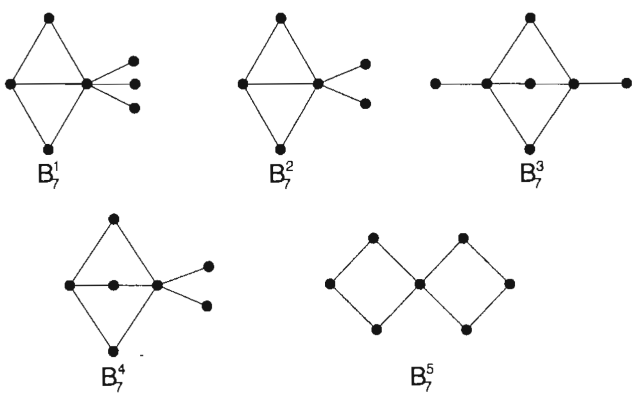

Figure 4: Bicyclic graphs B_7^i $(1 \leq i \leq 5)$.

We give a briefing on their proof. They considered the following two cases.

Case 1: $\delta(B) = 2$.

In this case, they got only $B = B_7^5$ satisfies $r(B) = 4$.

Case 2: $\delta(B) = 1$.

Let x be a pendant vertex in B, and $N(x) = \{y\}$. Delete x, y from B, the resulting graph is denoted by $G_1 = G - \{x, y\} = G_{11} + G_{12} + \cdots + G_{1t}$, where $G_{11}, G_{12}, \ldots, G_{1t}$ are t connected components of G_1.

Firstly, they proved that G_1 contains only one nontrivial component, say G_{11}. That is, $G_1 = G - \{x, y\} = G_{11} + N_{n-n_1-2}$, where $|V(G_{11})| = n_1$.

Furthermore, they considered the following two subcases.

Subcase 2.1: $\delta(G_{11}) = 1$.

In this subcase, they got $r(B) = 4$ if and only if $B \cong B_n^i$, $1 \leq i \leq 4$.

Subcase 2.2: $\delta(G_{11}) = 2$. In the original proof of this subcase, Hu *et al.*

claimed that there is no bicyclic graphs of rank 4. But, in fact, there are some bicyclic graphs of rank 4 with $\delta(G_{11}) = 2$.

Following we correct Theorem A as Theorem 2.5 and give an amended proof.

Theorem 2.5 *Let $B \in \mathcal{B}_n$ $(n \geq 6)$. Then $r(B) = 4$ if and only if $B = B_7^5$ or $B \cong H_n^i$ $(1 \leq i \leq 6)$, where H_n^i $(1 \leq i \leq 6)$ are shown in Fig. 5.*

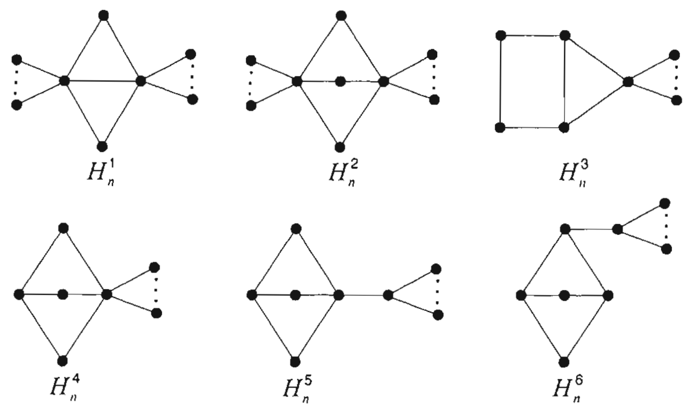

Figure 5: Bicyclic graphs H_n^i $(1 \leq i \leq 6,\ n \geq 6)$.

Before making an amendment on the proof of Hu *et al.*, we combine some kind of graphs described in Theorem A. Now we use H_n^1 in Fig. 5 to denote the graph B_n^1 or B_n^2 in Fig. 3 and H_n^2 in Fig. 5 to denote B_n^3 or B_n^4 in Fig. 3.

Proof. From Case 1 up to Subcase 2.1, the proof of Hu *et al.* is correct. So we only make an amendment on the Subcase 2.2.

From Propositions 1.5 and 1.4, $n - 4 = \eta(B) = \eta(G_1) = \eta(G_{11}) + n - n_1 - 2$. Hence we have $\eta(G_{11}) = n_1 - 2$. By Theorem 1.7, $G_{11} \cong K_{t_1,t_2}$ with $t_1 + t_2 = n_1$. Without loss of generality we assume that $t_1 \leq t_2$. Since B is bicyclic, G_{11} is either a unicyclic graph or a bicyclic graph. That is, $n_1 \leq |E(G_{11})| \leq n_1 + 1$. Since $G_{11} \cong K_{t_1,t_2}$, we have $2 = t_1 \leq t_2 \leq 3$. Then $G_{11} \cong K_{2,2} = C_4$ or $G_{11} \cong K_{2,3}$.

If $G_{11} \cong K_{2,2} = C_4$, then $G_1 = C_4 + N_{n-6}$. Adding x and y back to G_1, we need to insert edges from y to each of $n - 6$ isolated vertices of G_1.

This gives a star S_{n-4}. Since $B \in \mathcal{B}_n$, two edges must be added from the center y of S_{n-4} to C_4. Then $B \cong H_n^3$ or $B \cong H_n^4$.

If $G_{11} \cong K_{2,3}$, then $G_1 = K_{2,3} + N_{n-7}$. Adding x and y back to G_1, we need to insert edges from y to each of $n-7$ isolated vertices of G_1. This gives a star S_{n-5}. Since $B \in \mathcal{B}_n$, only one edge must be added from the center y of S_{n-5} to $K_{2,3}$. Then $B \cong H_n^5$ or $B \cong H_n^6$. $\square$

3 Unicyclic and bicyclic graphs of rank 5

In this section, we shall determine all unicyclic and bicyclic graphs of rank 5. Similar to the last section, it is easy to see that the minimum degree of a unicyclic graph is at most 2.

Theorem 3.1 *Let $U \in \mathcal{U}_n$ with $n \geq 5$. Then $r(U) = 5$ if and only if $U \cong U^*$ (shown in Fig. 6) or $U = C_5$.*

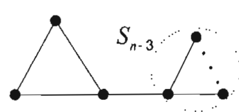

Figure 6: Unicyclic graph U^*.

Proof. If $U \cong U^*$ or $U = C_5$, then it is easy to check that $r(U) = 5$. For the "only if" part, we consider the following two cases:

Case 1: $\delta(U) = 2$.

Since $U \in \mathcal{U}_n$, $U = C_n$. By Proposition 1.6, $r(C_n) = 5$ if and only if $n = 5$.

Case 2: $\delta(U) = 1$.

Let x be a pendent vertex in U, and $N(x) = \{y\}$. Let $U_1 = U - \{x, y\}$. In fact, U_1 does not contain vertex of degree 1. Otherwise, let v be a vertex of degree 1 in U_1 and $N(v) = \{u\}$. Let $U_2 = U_1 - \{u, v\}$. By Proposition 1.4, we have $\eta(U) = \eta(U_1) = \eta(U_2) = n - 5$. But $|V(U_2)| = n - 4$. Hence, $\eta(U_2) = |V(U_2)| - 1$. By Theorem 1.1, $U_2 \cong N_{n-4}$. But it is impossible.

By the above discussion and U is unicyclic, $U_1 = C_{n_1} + N_{n-2-n_1}$. By Propositions 1.5 and 1.4, $n - 5 = \eta(U_1) = \eta(C_{n_1}) + (n - 2 - n_1)$, we have

$\eta(C_{n_1}) = n_1 - 3$. By Theorem 1.8 we have C_{n_1} is a complete tripartite graph. Then $n_1 = 3$. So $U_1 = C_3 + N_{n-5}$. Recovering x and y to U_1, we get $U \cong U^*$. □

A *one-point union of two cycles* is a simple graph obtained from two cycles C_l and C_k, where $l \geq k \geq 3$, by identifying one vertex from each cycle. We denote this graph by $U(l,k)$. In some articles (for example, [8]), such graph is called an ∞*-graph* and denoted by $B(k,1,l)$.

A *long dumbbell graph* is a simple graph obtained from two cycles C_l and C_k, by joining a path of length $q-1$ for $l \geq k \geq 3$ and $q \geq 2$. We denote this graph by $D(l,k;q-1)$. In some articles (for example, [8]), such graph is also called an ∞*-graph* and denoted by $B(k,q,l)$.

A *cycle with a long chord* is a simple graph obtained from an m-cycle, $m \geq 4$, by adding a chord of length p where $p \geq 1$. Let the m-cycle be $u_0u_1\cdots u_{m-1}u_0$. Without loss of generality, we may assume the chord joins u_0 with u_l, where $2 \leq l \leq \lfloor \frac{m}{2} \rfloor$ and $1 \leq p \leq l$. That is, $u_0u_mu_{m+1}\cdots u_{m+p-2}u_l$ is the chord. We denote this graph by $C_m(l;p)$. In some articles (for example, [8]), such graph is called a θ*-graph* and denoted by $P(l,p,q)$, where $l+q=m$.

Figure 7: $U(l,k)$ (∞-graph $B(k,q,l)$ when $q=1$), $D(l,k;q)$ (∞-graph $B(k,q,l)$ when $q \geq 2$) and $C_m(l;p)$ (θ-graph $P(l,p,q)$, where $l+q=m$).

Proposition 3.2 ([10]) *A simple connected bicyclic graph without pendant is either a one-point union of two cycles, a long dumbbell graph or a cycle with a long chord.*

Denoted by U_n^+ the sets of all graphs of order n that are one-point union of two cycles with trees attached; D_n^+ the sets of all graphs of order n that a long dumbbell graph with trees attached; and Θ_n^+ the sets of all graphs of order n that a cycle with a long chord with trees attached. Hence $\mathcal{B}_n = U_n^+ + D_n^+ + \Theta_n^+$.

Li *et al.* [8] investigated the nullity of these three types of bicyclic graphs, and gave a characterization of these three types of bicyclic graphs with extremal nullity, respectively.

Lemma 3.3 ([8]) *The nullity set of D_n^+ $(n \geq 7)$ is $[0, n-6]$.*

Lemma 3.4 ([8]) *The nullity set of U_n^+ $(n \geq 8)$ is $[0, n-6]$.*

Theorem 3.5 *Let $B \in \mathcal{B}_n$ $(n \geq 5)$. Then $r(B) = 5$ if and only if $B \cong U(3,3)$, $U(3,4)$, $C_5(2;2)$ or $B \cong B_i^*$ $(i = 1,2,3)$, where B_i^* $(i = 1,2,3)$ are shown in Fig. 8.*

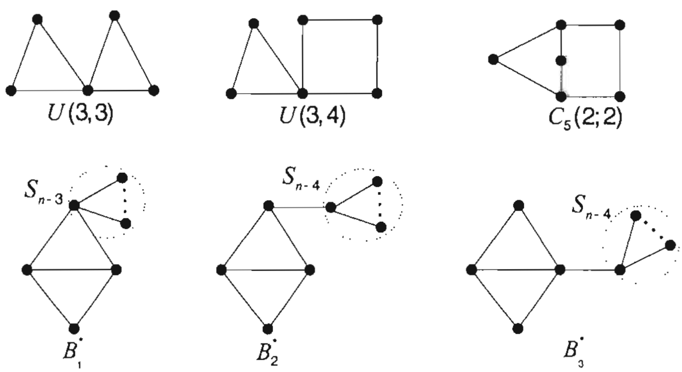

Figure 8: Bicyclic graphs $U(3,3)$, $U(3,4)$, $C_5(2;2)$ and B_i^* $(i = 1,2,3)$.

Proof. It is easy to check that those graphs described in Fig. 8 are of rank 5. For the "only if" part, we consider the following three cases:

Case 1: $B \in D_n^+$.

Since $B \in D_n^+$, $n \geq 6$. By the contrapositive of Lemma 3.3, we have $n \leq 6$. Hence $n = 6$ and $B \cong G_1$ (see Fig. 9). It is easy to get that $r(G_1) = 6$. So there is no graph in D_n^+ with rank 5.

Case 2: $B \in U_n^+$.

Since $B \in U_n^+$, $n \geq 5$. By the contrapositive of Lemma 3.4, we have $n \leq 7$. That is, $n = 5, 6$ or 7. Since $B \in U_n^+$, B contains a vertex-induced subgraph $U(l,k)$ for some $l \geq k \geq 3$. Since $l+k \leq n+1 \leq 8$, $3 \leq k \leq l \leq 5$. Then either $B \cong U(3,3)$, $B \cong U(3,4)$, or B contains one of the graphs G_i $(2 \leq i \leq 9)$ (see Fig. 9) as its vertex-induced subgraph. It is easy to check that $U(3,3)$ and $U(3,4)$ are of rank 5 and the others are of rank greater than 5. So $B \cong U(3,3)$ or $B \cong U(3,4)$.

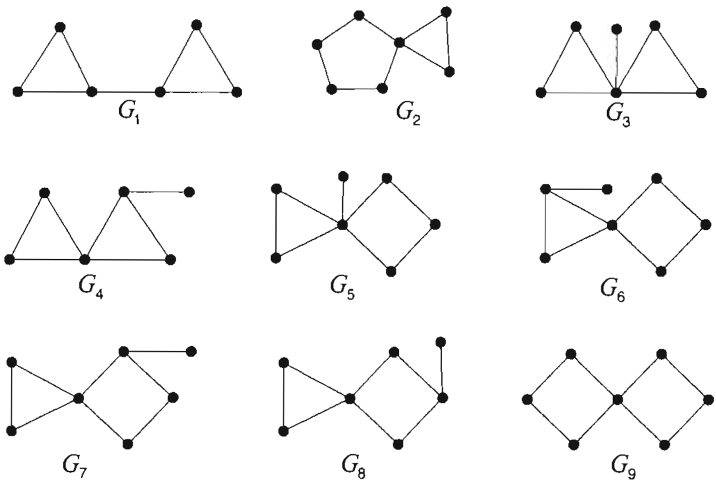

Figure 9: Bicyclic graphs G_i $(i = 1, 2, \ldots, 9)$.

Case 3: $B \in \Theta_n^+$.

Suppose $\delta(B) = 2$. Then $B = C_m(l;p)$ for some m, p, l with $1 \leq p \leq l$ and $2 \leq l \leq \lfloor \frac{m}{2} \rfloor$. Note that $n = m + p - 1$. From $n \geq 5$ and these constraints, it is easy to see that $m \geq 4$. For convenience, we let $s = m+p-1$ and $k = p + l$. Then B contains two vertex-induced subgraphs C_s and C_k. It is easy to see that $s \geq k$. Since $n = m + p - 1 \geq 5$, we have $s = m + p - l \geq \lceil \frac{m}{2} \rceil + p \geq 4$. Since $5 = r(B) \geq r(C_s)$, by Proposition 1.6 we have $s \leq 5$.

Suppose $s = 5$. If $p = 1$, then B contains G_{10} or G_{11} (see Fig. 10) as its vertex-induced subgraph. If $p \geq 2$, then B is isomorphic to $C_5(2,2)$ or contains G_{11} (see Fig. 10) as its vertex-induced subgraph. We can easy to check that $C_5(2,2)$ is of rank 5 and the others are of rank greater than 5. So $B \cong C_5(2,2)$.

Suppose $s = 4$. B is one of the graphs G_{12}, G_{13} and G_{14} (see Fig. 10). Since $r(G_{12}) = 6$, $r(G_{13}) = 2$, $r(G_{14}) = 4$, in this case no graphs are of rank 5.

Consequently, for every $B \in \Theta_n^+$ with $r(B) = 5$, if $\delta(B) = 2$, then only $B \cong C_5(2,2)$.

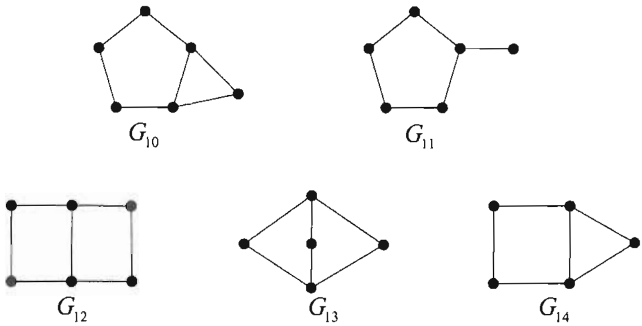

Figure 10: Bicyclic graphs G_i $(10 \leq i \leq 14)$.

Suppose $\delta(B) = 1$. Let x be a pendant vertex in B, and $N(x) = \{y\}$. Let $B_1 = B - \{x, y\}$. By the same argument in the proof of Case 2 of Theorem 3.1, B_1 does not contain vertex of degree 1. Since B is bicyclic, B_1 contains exactly one nontrivial component. That is, $B_1 = B_{11} + N_{n-2-n_1}$, and $|V(B_{11})| = n_1 \geq 2$. By Propositions 1.5 and 1.4. $n - 5 = \eta(B) = \eta(B_1) = \eta(B_{11}) + n - 2 - n_1$, we have $\eta(B_{11}) = n_1 - 3$. Then B_{11} must be a complete tripartite graph K_{t_1,t_2,t_3} by Theorem 1.8, where $t_1 + t_2 + t_3 = n_1$. Since $B \in \Theta_n^+$, $B_{11} = K_{1,1,1} \cong C_3$ or $B_{11} = K_{1,1,2}$. Hence, B_1 is isomorphic to $C_3 + N_{n-5}$ or $K_{1,1,2} + N_{n-6}$.

Suppose $B_1 = C_3 + N_{n-5}$. Adding x and y back to B_1 we need to insert edges from y to each of $n-5$ isolated vertices of B_1. This gives a star S_{n-3}. In order to assume that there is a cycle with a long chord as a subgraph in B, two edges must be added from the center of S_{n-3} to C_3. Thus $B \cong B_1^*$.

Suppose $B_1 = K_{1,1,2} + N_{n-6}$. Adding x and y back to B_1 we need to insert edges from y to each of $n - 6$ isolated vertices of B_1. This gives a star S_{n-4}. Since $K_{1,1,2}$ is already a a cycle with a long chord, one edge must be added from the center of S_{n-4} to $K_{1,1,2}$. Thus B is isomorphic to B_2^* or B_3^*.

This completes the proof. □

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