

# Some sufficient conditions for a planar graph to be of Class 1<sup>1</sup>

Peter Che Bor Lam<sup>2</sup>, Jiazhuang Liu<sup>3</sup>, Wai Chee Shiu<sup>2</sup> and Jianliang Wu<sup>3</sup>

## Abstract

In this paper, we first give some upper bounds on the number of edges for two classes of planar graphs. Then using these upper bounds, we obtain some sufficient conditions for a planar graph to be of Class 1.

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## 1. Introduction

The *chromatic index* of a graph  $G$ , denoted by  $\chi'(G)$ , is the smallest  $k$  for which the edges of  $G$  can be colored by  $k$  colors such that no two adjacent edges have the same color. Vizing (1964) proved that for any simple graph  $G$ ,  $\chi'(G)$  is either  $\Delta$  or  $\Delta + 1$ , where  $\Delta(G)$ , or  $\Delta$  for simplicity, is the maximum degree of  $G$ . A graph  $G$  is said to be of *Class 1* or of *Class 2* if  $\chi'(G) = \Delta$  or  $\Delta + 1$  respectively. However, the problem of classifying a graph is *NP*-complete - see Hoyler (1981). In fact, even planar graphs have not been completely classified. In this paper, we first give some upper bounds on the number of edges for two types of planar graphs. Using these bounds, we obtain some sufficient conditions for a planar graph to be of Class 1.

All graphs considered in this paper are finite, simple planar graphs.  $G = (V, E, F)$  denotes a plane graph, with  $V$ ,  $E$  and  $F$  being the set of vertices, edges and faces of  $G$  respectively. The *degree* of a vertex  $v \in V$ , denoted by  $d_G(v)$ , is the number of vertices in  $G$  adjacent to  $v$ . The *degree* of a face  $f \in F$ , denoted by  $d_G(f)$  is the number of edges incident with  $f$ , where cut edges are counted twice. The order of the sets  $V$ ,  $E$  and  $F$  are denoted by  $v$ ,  $\varepsilon$  and  $\phi$  respectively. Moreover, the number of vertices or faces having degree  $i$  is denoted by  $v_i$  or  $\phi_i$  respectively. Undefined symbols and concepts are referred to [5].

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<sup>2</sup>Department of Mathematics, Hong Kong Baptist University

<sup>3</sup>Institute of Mathematics, Shandong University, Jinan, 250100, P. R. China

Steinberg [2] first considered an analogous problem in vertex coloring. He conjectured that every planar graph without 4- and 5-cycle is 3-colorable. In 1990, Erdős [3] proposed to relax Steinberg's conjecture by asking if there exists an integer  $k \geq 5$  such that every planar graph without  $i$ -cycle, where  $4 \leq i \leq k$ , is 3-colorable. In 1996, Borodin [4] proved that  $k = 9$  is acceptable. However, the smallest value for  $k$  has not yet been determined. We shall tackle the edge coloring problem along this line of thought.

## 2. Lemmas

**Definition [6]** Let  $G$  be a connected graph with  $\chi'(G) = \Delta + 1$ . We say that  $G$  is critical if  $\chi'(G - e) < \chi'(G)$  for any edge  $e$  of  $G$ . A  $\Delta$ -critical graph is one that is  $\Delta(G) = \Delta$  and critical. Obviously, every graph with  $\chi'(G) = \Delta(G) + 1$  has a  $\Delta$ -critical subgraph.

**Lemma 1 [6]** *Let  $G$  be a simple planar graph without loops. If  $G$  is  $\Delta$ -critical, then*

$$\begin{aligned} \varepsilon &\geq (5v + 1)/4 && \text{if } \Delta = 3; \\ \varepsilon &\geq 5v/3 && \text{if } \Delta = 4; \\ \varepsilon &\geq 2v + 1 && \text{if } \Delta = 5; \\ \varepsilon &\geq (9v + 1)/4 && \text{if } \Delta = 6; \\ \varepsilon &\geq 5v/2 && \text{if } \Delta = 7. \end{aligned}$$

Since it is known that every planar graph with  $\Delta \geq 8$  is of Class 1 (see [6]), we only consider graphs with  $3 \leq \Delta \leq 7$  in this paper.

**Lemma 2** *Suppose  $k \geq 4$ . If  $G$  is a connected planar graph without two triangles sharing a common edge and without  $i$ -cycles, where  $3 < i < k$ , then*

$$\varepsilon \leq \frac{3k}{2k - 3}(v - 2).$$

**Proof** We consider a fixed embedding of  $G$  into the plane. Since  $G$  has no two triangles sharing a common edge,  $\varepsilon \geq 3\phi_3$  and

$$2\varepsilon = 3\phi_3 + \sum_{i=k}^{\infty} i\phi_i \tag{1}$$

$$\geq \left( k \sum_{i=3}^{\infty} \phi_i \right) - (k-3)\phi_3 \quad (2)$$

$$\geq k\phi - \frac{k-3}{3}\varepsilon. \quad (3)$$

That is  $\phi \leq \frac{k+3}{3k}\varepsilon$ . Since  $G$  is a plane graph, by Euler's formula, we have

$$2 = v - \varepsilon + \phi \leq v - \varepsilon + \frac{k+3}{3k}\varepsilon,$$

and therefore

$$\varepsilon \leq \frac{3k}{2k-3}(v-2). \quad \blacksquare$$

**Lemma 3** *Suppose  $k \geq 4$ . If  $G$  is a connected planar graph without two 3-cycles sharing a common vertex and without  $i$ -cycles, where  $3 < i < k$ , then*

$$\varepsilon \leq \frac{4k-3}{3(k-2)} - \frac{2k}{k-2}.$$

**Proof** As in Lemma 2, we consider a fixed embedding of  $G$  into the plane. Since  $G$  has no two triangles sharing a common vertex,  $v \geq 3\phi_3$  and similar to (1), we have

$$\begin{aligned} 2\varepsilon &\geq \left( k \sum_{i=3}^{\infty} \phi_i \right) - (k-3)\phi_3 \\ &\geq k\phi - \frac{k-3}{3}v. \end{aligned}$$

That is  $\phi \leq \frac{2}{k}\varepsilon + \frac{k-3}{3k}v$ . Since  $G$  is a plane graph, by Euler's formula, we have

$$2 = v - \varepsilon + \phi \leq v - \varepsilon + \frac{2}{k}\varepsilon + \frac{k-3}{3k}v,$$

and therefore

$$\varepsilon \leq \frac{4k-3}{3(k-2)}v - \frac{2k}{k-2}. \quad \blacksquare$$

### 3. Sufficient Conditions

The following theorem states several sufficient conditions for a planar graph to be of Class 1.

**Theorem 4** *Suppose  $G$  is a connected planar graph. Then  $\chi'(G) = \Delta(G)$  if one of the following conditions is satisfied:*

- 1  $\Delta \geq 6$  and  $G$  has no 4-cycle,
- 2  $\Delta \geq 6$  and no two triangles of  $G$  sharing one common vertex,
- 3  $\Delta \geq 7$  and no two triangles of  $G$  sharing one common edge,
- 4  $\Delta \geq 5$  and  $G$  has no 4-cycle and 5-cycle,
- 5  $\Delta \geq 4$  and  $G$  has no  $i$ -cycle, where  $4 \leq i \leq 14$ ,
- 6  $\Delta \geq 5$  and  $G$  has no 4-cycle and has no two triangles sharing a common vertex,
- 7  $\Delta \geq 4$  and  $G$  has no 4-through 6-cycle and has no two triangles sharing a common vertex.

**Proof** (1) Suppose  $G$  is a graph satisfying conditions of this part and  $\chi'(G) = \Delta + 1$ .

Without loss of generality, we may assume that  $G$  is  $\Delta$ -critical. By Lemma 1, we have

$$\varepsilon \geq \frac{9v+1}{4}.$$

Letting  $k = 5$  in Lemma 2, we have  $\varepsilon \leq 15(v-2)/7$ . If  $v \geq 2$ , then

$$\varepsilon \leq \frac{15}{7}(v-2) < \frac{9v+1}{4} \leq \varepsilon.$$

This contradiction completes the proof of this part.

(2) Again, as in (1), suppose  $G$  is a graph satisfying conditions of this part and is  $\Delta$ -critical. By Lemma 1, we have  $\varepsilon \geq (9v+1)/4$ . Also, since  $G$  has no two triangles sharing a common vertex, by Lemma 3, letting  $k = 4$ , we have  $\varepsilon \leq (13v-24)/6$ . If  $v \geq 2$ , then

$$\varepsilon \leq \frac{13v-24}{6} < \frac{9v+1}{4} \leq \varepsilon.$$

The contradiction completes the proof of this part.

(3) - (6) The proof of these parts is similar to above and therefore is omitted. ■

## References

- [1] V. G. Vizing, On an estimate of the chromatic class of a p-graph (in Russian), *Diskret. Analiz* **3** (1964), 25-30.

- [2] R. Steinberg, The state of the three color problem, Quo Vadis Graph Theory? J. Gimbel, J. W. Kennedy and L. V. Quintas (eds.), *Ann. Discrete Math.* **55** (1993), 211-248.
- [3] P. Erdős, A. L. Rubin and H. Taylor, Choosability in graphs, *Congres. Numer.* **26** (1979), 125-157.
- [4] O. V. Borodin, Structural properties of plane graphs without adjacent triangles and an application to 3-coloring, *Journal of Graph Theory* **21(2)** (1996), 183-186.
- [5] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, Macmillan, London, (1976).
- [6] H. P. Yap, *Some topics in graph theory*, Cambridge University Press, (1986).