Edge-magic Index Sets of (p, p)-graphs*

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Abstract

A graph $G = (V, E)$ with p vertices and q edges is called *edge-magic* if there is a bijection $f : E \to \{1, 2, \ldots, q\}$ such that the *induced mapping* $f^+ : V \to \mathbb{Z}_p$ is a constant mapping, where $f^+(u) \equiv \sum_{v \in N(u)} f(uv) \pmod p$. The edge-magic index of a graph G is the set of positive integers k for which the k-fold of G is edge-magic. In this paper, we determine the edge-magic index set of graphs whose order and size are equal.

Key words and phrases : Edge-magic, edge-magic index, unicyclic graph
AMS 1991 subject classification : 05C78, 05C25

1. Introduction, Notations and Basic Concepts

In this paper, the term "graph" means finite multigraph (not necessary connected) having no loop and no isolated vertex. All undefined symbols and concepts may be looked up from [1]. A graph $G = (V, E)$ is called a (p, q)-graphs, where p and q are its order and size respectively, i.e., $|V| = p$ and $|E| = q$. A (p, q)-graphs G is called *d-edge-magic* if there exists a bijection

$$f : E \to \{d, d + 1, \ldots, d + q - 1\}$$

such that the *induced mapping* $f^+ : V \to \mathbb{Z}_p$ is a constant mapping, where $f^+(u) \equiv \sum_{v \in N(u)} f(uv) \pmod p$ for $u \in V$, $N(u)$ denotes the neighborhood of

*Partially supported by Faculty Research Grant of Hong Kong Baptist University.

†Research done while on sabbatical at San José State University in 1998.

u and $d \in \mathbb{Z}$. If $d = 1$, then G is simply called *edge-magic* and f an *edge-magic labeling* of G. This concept was introduced by Lee, Seah and Tan in 1992 [2]. Several classes of edge-magic graphs were exhibited, for example [2, 3, 4, 5, 8]. A necessary condition of a (p, q)-graph to be edge-magic is

$$q(q+1) \equiv 0 \pmod{p}. \tag{1.1}$$

Let G be a graph and let k be a positive integer. The *k-fold* of G, denoted by $G[k]$, is a graph made up of k copies of G with the same set of vertices. The disjoint union of k copies of graph, $\underbrace{G + \cdots + G}_{k \text{ times}}$, is denoted by kG.

Let S be a set. We use $S \times n$ to denote the multiset of n-copies of S. Note that S may be a multiset itself. From now on, the term "set" means multiset. Set operations are viewed as multiset operations. Let S be a set of qk elements. Let $\mathscr{P}$ be a partition of S such that each class of $\mathscr{P}$ contains k elements. We call $\mathscr{P}$ a (q, k)-partition of S. Let S and T be sets. The notation $S \equiv T \pmod{p}$ means that two sets are equal after their elements are taken modulo p.

Suppose A is a set consisting of r integers. If the sum modulo p of elements of A is s, then A is called an *$(s; r)$-set.* If $r = 1, 2$ or 3, it is called an *s-singleton*, an *s-doubleton* or an *s-tripleton* respectively.

Let $[\boldsymbol{r}] = \{1, 2, \ldots, r\}$. A mapping f is called a *k-fold edge-magic labeling* of a (p, q)-graph G if there is a *(q, k)-partition* $\mathscr{P}$ of $[\boldsymbol{qk}]$ such that $f : E \to \mathscr{P}$ is a bijection and the *induced mapping* $f^+ : V \to \mathbb{Z}_p$, where $f^+(u) \equiv \sum\limits_{v \in N(u)} \sum\limits_{i \in f(uv)} i \pmod{p}$, is a constant mapping. Thus, finding an edge-magic labeling of G$[k]$ is equivalent to finding a k-fold edge-magic labeling of G.

Let $G = (V, E)$ be a graph. For a positive integer $n, f : E \to \mathbb{Z}_n$ is called a *$\mathbb{Z}_n$-magic labeling* of G if $f^+ : V \to \mathbb{Z}_n$ is a constant mapping, where $f^+(u) \equiv \sum\limits_{v \in N(u)} f(uv) \pmod{n}$. We have

Theorem 1.1 [6]: *Let $G = (V, E)$ be a (p, q)-graph G. If $G[k]$ is edge-magic then there is a $\mathbb{Z}_p$-magic labeling f of G such that*

$$\sum_{e \in E} f(e) \equiv \tfrac{1}{2} kq(kq+1) \pmod{p}.$$

2. Edge-magicness of (p, p)-graphs

In this section, we consider the edge-magicness of a multigraph whose order and size are equal.

Theorem 2.1: *Let G be a (p, p)-graph. G is edge-magic if and only if G is isomorphic to one of the following graph:*
(1) $mK_2[2]$,
(2) $G_1 + mK_2[2]$,
(3) $K_2 + G_2 + mK_2[2]$,
for some m, where G_1 is isomorphic to H_1 or H_2 and G_2 is isomorphic to H_3, H_4, H_5 or H_6 (Figure 2.1).

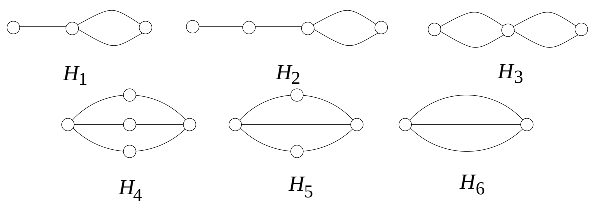


Figure 2.1

Proof: Suppose $G = (V, E)$ is edge-magic. Since the order and the size of G are the same, the labels of edges are distinct. Hence every 1-vertex of G must be incident to the same edge (a vertex of degree d is called a *d-vertex*). Consequently, G contains at most two 1-vertices, and if G contains two 1-vertices then these two vertices induce a component of G isomorphic to K_2. Moreover, if G' is a component of G with two adjacent 2-vertices, then $G' \cong K_2[2]$.

Case 1: Suppose each component of G has the same order and size. In this case, G contains at most one 1-vertex. If G contains no 1-vertex and V^* is a component of G, then from

$$\sum_{v \in V^*} deg(v) = 2e^*, \tag{2.1}$$

where e^* is the size of $G[V^*]$, we have $\deg(v) = 2$ for all $v \in V^*$. Because order of $G[V^*]$ is at least 2, then by last sentence of the first paragraph $G[V^*] \cong K_2[2]$. Therefore $G \cong mK_2[2]$, where $m = \frac{p}{2}$.

Suppose G contains one 1-vertex. From (2.1), the component, say G_1, containing the 1-vertex must contain one 3-vertex and a number of 2-vertices. If the 1-vertex is adjacent to the 3-vertex, then $G_1 \cong H_1$. If the 1-vertex is adjacent to a 2-vertex, then this 2-vertex must be adjacent to the 3-vertex and $G_1 \cong H_2$. Since other components of G contain no

1-vertex, they must be all isomorphic to $K_2[2]$. Hence $G \cong G_1 + mK_2[2]$ for some m, where $G_1 \cong H_1$ or H_2.

Case 2: Suppose G contains a component T whose size is less than its order, then T must be a tree and must contain at least two vertices. Since G contains at most two 1-vertices, T must contain two 1-vertices and must be isomorphic to K_2. Moreover, G has only one such component. By the pigeon hole principle, there is exactly one component, say G_2, whose order is one less than its size. Moreover, the order and the size of each of other components of G are the same. By (2.1) G_2 contains either two 3-vertices or one 4-vertex. If G_2 contains two 3-vertices, then G_2 may contain up to three 2-vertices because 2-vertices not in $K_2[2]$ cannot be adjacent each other. In this case G_2 is isomorphic to $G_2 \cong H_4, H_5, H_6$ or H' (Figure 2.1 and Figure 2.2). If G_2 contains one 4-vertex, then G_2 must contain two 2-vertices. In this case, $G_2 \cong H_3$. Remaining components of G consist only of 2-vertices.

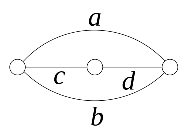


Figure 2.2: H'

Suppose $G_2 \cong H'$. Let the edges of H' be labeled by a, b, c and d as describing in Figure 2.2. Since G is edge-magic, $c = d$, which is impossible. Thus $G \cong K_2 + G_2 + mK_2[2]$ for some m and some $G_2 \cong H_3, H_4, H_5$ or H_6.

Conversely, suppose $G \cong mK_2[2] \cong (mK_2)[2]$. $[\mathbf{2m}]$ can be grouped up as follows:

1	2	3	$\cdots$	m
$2m$	$2m-1$	$2m-2$	$\cdots$	$m+1$

Then we have m 1-doubletons. Hence we have a 2-fold magic labeling of mK_2.

Suppose $G \cong H_1 + mK_2[2]$. $[\mathbf{2m+3}]$ can be grouped as follows:

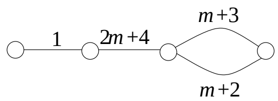
1	2	3	$\cdots$	$m+1$	$2m+3$
$2m+2$	$2m+1$	$2m$	$\cdots$	$m+2$	–

Then we have $m+1$ 0-doubletons and one 0-singleton. We label the edge which is incident with the degree 1 vertex in H_1 by $2m+3$ and the other two edges of H_1 by $m+1$ and $m+2$ respectively. Use elements of each remaining 0-doubleton to label the two edges of each $K_2[2]$ respectively. Then this is an edge-labeling of G.

Suppose $G \cong H_2 + mK_2[2]$. $[\mathbf{2m+4}]$ can be grouped as follows:

2	3	4	$\cdots$	$m+2$	–	1
$2m+3$	$2m+2$	$2m+1$	$\cdots$	$m+3$	$2m+4$	–

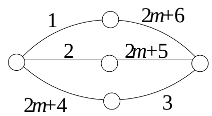
Then we have $m+1$ 1-doubletons, one 1-singleton and one 0-singleton. We label H_2 as follows:



Use elements of each remaining 1-doubleton to label the two edges of each $K_2[2]$ respectively. Then this is an edge-labeling of G.

Suppose $G \cong K_2 + H_3 + mK_2[2]$. Then the order of G is $2m+5$. Similar to the first case, $[\mathbf{2m+5}]$ can be grouped into $m+2$ 0-doubletons and one 0-singleton. Label the edge of K_2 by the 0-singleton and the four edges of H_3 by elements of any two 0-doubletons. The rest is same as the previous case.

Suppose $G \cong K_2 + H_4 + mK_2[2]$. Then the order of G is $2m+7$. Similar to the previous case, $[\mathbf{2m+7}]$ can be grouped into $m+3$ 0-doubletons and one 0-singleton. Label the edge of K_2 by the 0-singleton, i.e., by $2m+7$ and label the edges of H_4 as follows:

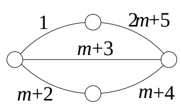


The rest is same as the previous case.

Suppose $G \cong K_2 + H_5 + mK_2[2]$. Then the order of G is $2m+6$. Similar to the previous case, $[\mathbf{2m+6}]$ can be grouped as follows:

1	2	$\cdots$	$m+2$	$m+3$	$-$
$2m+5$	$2m+4$	$\cdots$	$m+4$	$-$	$2m+6$

We have $m+2$ 0-doubletons, one $(m+3)$-singleton and one 0-singleton. We label the edge of K_2 by $2m+6$ and label the edges of H_5 as follows:

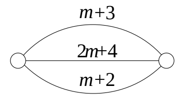


The rest is same as the previous case.

Suppose $G \cong K_2 + H_6 + mK_2[2]$. Then the order of G is $2m+4$. Similar to the previous case, $[\mathbf{2m+4}]$ can be grouped as follows:

2	3	$\cdots$	$m+2$	$-$	1
$2m+3$	$2m+2$	$\cdots$	$m+3$	$2m+4$	$-$

We have $m+1$ 1-doubletons, one 1-singleton and one 0-singleton. We label the edge of K_2 by 1 and label the edges of H_6 as follows :



The rest is same as the previous case. ▮

3. Edge-magicness on k-fold of (p,p)-graphs

In this section, we consider the edge-magicness of the k-fold of a (p,p)-graph.

Theorem 3.1: *Let G be a $(2m+1, 2m+1)$-graph and let $k \geq 2$. Then $G[k]$ is edge-magic.*

Proof: The theorem follows from the following lemma. ▮

Lemma 3.2: *Suppose $k \geq 2$. $[\mathbf{2m+1}] \times k$ has a $(2m+1, k)$-partition such that each class of this partition is a $(0;k)$-set.*

Proof: Each $[\mathbf{2m+1}]$ may be grouped into m 0-doubletons and one 0-singleton as follows:

1	2	$\cdots$	$m-1$	$-$
$2m$	$2m-1$	$\cdots$	$m+1$	$2m+1$

Case 1: k is even.

$[\mathbf{2m+1}] \times k$ can be grouped into mk 0-doubletons and k 0-singletons. These 0-singletons can be grouped into $\frac{k}{2}$ 0-doubletons. Combine these $\frac{1}{2}k(2m+1)$ 0-doubletons we have a required partition of $[\mathbf{2m+1}] \times k$.

Case 2: k is odd.

$[\mathbf{2m+1}] \times 3$ can be grouped into $2m+1$ 0-tripletons as follows:

$2m+1$	1	2	$\cdots$	$m-1$	m
$2m$	$2m-2$	$2m-4$	$\cdots$	2	$2m+1$
1	2	3	$\cdots$	m	$m+1$

$m+1$	$m+2$	$\cdots$	$2m-1$	$2m$
$2m-1$	$2m-3$	$\cdots$	3	1
$m+2$	$m+3$	$\cdots$	$2m$	$2m+1$

The remaining $k-3$ copies of $[\mathbf{2m+1}]$ are grouped into $\frac{1}{2}(k-3)(2m+1)$ 0-doubletons as in Case 1. Similar to Case 1, we combine $\frac{1}{2}(k-3)$ of these

0-doubletons with a 0-tripleton to obtain a $(0;k)$-set. Hence we have a required partition. ∎

Lemma 3.3: *Let $k \geq 2$ be an even integer. For any integer z, $0 \leq z \leq m$, $[\mathbf{2m}] \times k$ has a $(2m,k)$-partition consisting of $2z$ $(0;k)$-sets and $2m-2z$ $(m;k)$-sets.*

Proof: Note that $\frac{1}{2}(2mk)(2mk+1) \equiv 0 \pmod{2m}$. Two copies of $[\mathbf{2m}]$ can be grouped into $2z$ 0-doubletons and $2m-2z$ m-doubletons as follows:

1	2	$\cdots$	z	$m+z+1$	$\cdots$	$2m-2$	$2m-1$	$2m$
$2m-1$	$2m-2$	$\cdots$	$2m-z$	$2m-z-1$	$\cdots$	$m+2$	$m+1$	m

$m+1$	$m+2$	$\cdots$	$m+z$	$z+1$	$\cdots$	$m-2$	$m-1$	m
$m-1$	$m-2$	$\cdots$	$m-z$	$m-z-1$	$\cdots$	2	1	$2m$

The remaining $(k-2)$ copies of $[\mathbf{2m}]$ can be grouped into $(k-2)m$ 0-doubletons. Combining these doubletons suitably we obtain a required partition. ∎

Lemma 3.4: *Let $k \geq 3$ be an odd integer. For any integer z, $0 \leq z \leq m-1$, $[\mathbf{2m}] \times k$ has a $(2m,k)$-partition consisting of $2z+1$ $(0;k)$-sets and $2m-2z-1$ $(m;k)$-sets.*

Proof: Note that $\frac{1}{2}(2mk)(2mk+1) \equiv m \pmod{2m}$. Three copies of $[\mathbf{2m}]$ can be grouped into $2z+1$ 0-tripletons and $2m-2z-1$ m-tripletons as follows:

1	2	$\cdots$	z	$m+z+1$	$\cdots$	$2m-2$	$2m-1$	$2m$
$2m-2$	$2m-4$	$\cdots$	$2m-2z$	$2m-2z-2$	$\cdots$	4	2	$2m$
1	2	$\cdots$	z	$z+1$	$\cdots$	$m-2$	$m-1$	$2m$

$m+1$	$m+2$	$\cdots$	$m+z$	$z+1$	$\cdots$	$m-2$	$m-1$	m
$2m-1$	$2m-3$	$\cdots$	$2m-2z+1$	$2m-2z-1$	$\cdots$	5	3	1
m	$m+1$	$\cdots$	$m+z-1$	$m+z$	$\cdots$	$2m-3$	$2m-2$	$2m-1$

The rest is same as the proof of Lemma 3.3. ∎

Applying Lemmas 3.3 and 3.4 we have the following theorems respectively.

Theorem 3.5: *Let G be a $(2m,2m)$-graph. Suppose $k \geq 2$ and even. Then $G[k]$ is edge-magic.*

Theorem 3.6: *Let G be a $(2m,2m)$-graph. Suppose $k \geq 3$ and odd. Then $G[k]$ is edge-magic if G has a $\mathbb{Z}_2$-magic labeling f such that $\left|f^{-1}(1)\right|$ is odd.*

Proof: Change all the label 1's to m's , then f becomes a $\mathbb{Z}_{2m}$-magic labeling of G. Note that f has only two values 0 and m. By Lemma 3.4, G has a k-fold edge-magic labeling. ∎

Corollary 3.7: *Let G be a $(4m+2, 4m+2)$-graph with a perfect matching. Then $G[k]$ is edge-magic for $k \geq 2$.*

4. Edge-magic Index Sets of Connected Unicyclic Graphs

The set $I_m(G) = \{k \geq 1 \mid G[k] \text{ is edge-magic}\}$ is called the *edge-magic index set* of G. The least integer in $I_m(G)$ is called *edge-magic index* of G. Edge-magic index sets of some graph were found in [6, 7]. In this section, we shall consider the edge-magic index set of connected $(2m, 2m)$-graphs. A connected $(2m, 2m)$-graph G is called a *unicyclic graph*, since $G = T + e$ for some tree T and an additional edge $e \notin E(T)$. Note that G may not be simple. It is easy to prove the following theorem.

Theorem 4.1: *Suppose G is a (p, p)-graph, not necessary connected. G is edge-magic if and only if $I_m(G) = \mathbb{P}$, where $\mathbb{P}$ is the set of all positive integers.*

Lemma 4.2 [6]: *Let T be a tree. T has a (unique) non-zero $\mathbb{Z}_2$-magic labeling if and only if the order of T is even.*

Corollary 4.3: *Suppose $G = T + e$ of even order greater than 2. If T has the non-zero $\mathbb{Z}_2$-magic labeling f such that $\left|f^{-1}(1)\right|$ is odd, then $I_m(G) = \mathbb{P} \setminus \{1\}$.*

Proof: Define $f(e) = 0$. Then f is a $\mathbb{Z}_2$-magic labeling of G. By Theorems 3.5 or 3.6, $G[k]$ is edge-magic for $k \geq 2$. By Theorem 2.1 G is not edge-magic. Hence the corollary follows. ∎

Remark 4.4: Suppose T is a tree that does not satisfy the conditions of Corollary 4.3. $T + e$ may have the conclusion of Corollary 4.3. See the following examples.

Example 4.1: Consider T as follows:

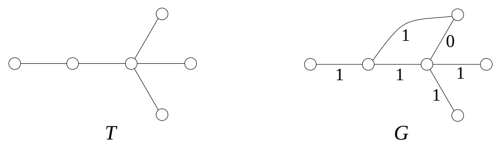


Figure 4.1

T does not satisfy the conditions of Corollary 4.3 but G satisfies the conditions of Theorem 3.6. By Theorems 3.5 or 3.6, $I_m(G) = \mathbb{P} \setminus \{1\}$. ♠

Example 4.2: Consider the regular spider graph (it is called regular superstar in some articles) $S_{3,3}$ with 3 legs of length 3 (Figure 4.2 (a)). $S_{3,3}[k]$ is not edge-magic when $k \equiv 5, 6, 10 \pmod{20}$ and is edge-magic when $k \equiv 11 \pmod{20}$ (see [6]). Using the same argument in [6] one can show that $S_{3,3}[k]$ is not edge-magic when $k \equiv 21 \pmod{20}$ and $S_{3,3}[k]$ is edge-magic when $k \equiv 15, 16,$ or $20 \pmod{20}$. In this example, $G[k]$ is not edge-magic when k is odd, where $G = S_{3,3} + e$ for any additional edge $e \notin E(S_{3,3})$.

Let k be odd. Arithmetic is taken in $\mathbb{Z}_{10}$ in this example. If $G[k]$ is edge-magic, then there is a $\mathbb{Z}_{10}$-magic labeling f of G such that

$$\sum_{e' \in E} f(e') = \frac{1}{2}(10k)(10k+1) = 5.$$

Let $f^+ = x$ and $f(e) = y$. Without loss of generality, we may assume $e = a_i b_j$ or $e = va_i$, $1 \le i \le j \le 3$ (Figure 4.2(b) shows the case of $i = 2, j = 3$). For each case, we have $\sum\limits_{e' \in E} f(e') = 6x$. But $6x = 5$ has no solution.

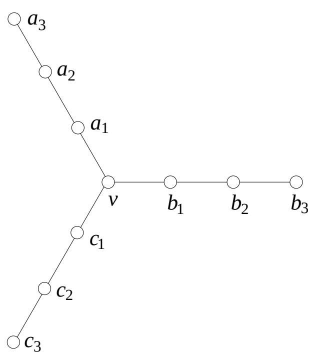


Figure 4.2 (a)

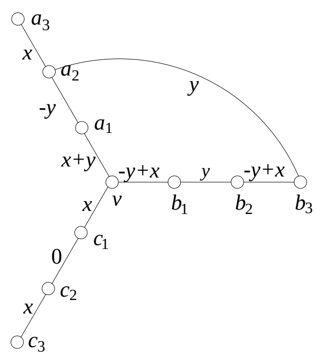


Figure 4.2 (b)

♠

5. Edge-magic Index Sets of Cycles

In this section, we consider the edge-magic index set of an n-cycle C_n. When n is odd, by Theorem 3.2 $I_m(C_n) = \mathbb{P} \setminus \{1\}$. Thus we only need to consider an even cycle.

Lemma 5.1: *$I_m(C_{2s}) = \mathbb{P} \setminus \{1\}$ for $s \geq 2$.*

Proof: Clearly, C_{2s} has a $\mathbb{Z}_2$-magic labeling f such that $|f^{-1}(1)| = s$. If s is odd then the lemma follows from Theorems 3.5 and 3.6. If s is even, then by Theorem 3.5 we only have to handle $k \in \mathbb{P} \setminus \{1\}$ being odd. If x be odd. Then $sx \equiv s \pmod{2s}$. By changing all the labels "1" of f to "x", we obtain a $\mathbb{Z}_{2s}$-magic labeling of C_{2s}. Applying Lemma 5.2 below, we obtain the lemma. ▮

Lemma 5.2: *Let k and x be odd integers. $[\mathbf{2s}] \times k$ has a $(2s, k)$-partition consisting of s $(0; k)$-sets and s $(x; k)$-sets.*

Proof: In this proof, all arithmetic are taken in $\mathbb{Z}_{2s}$. For any $a \in \mathbb{Z}_{2s}$, there is a unique $b \in \mathbb{Z}_{2s}$, such that $a + b = x$. Since x is odd, $a \neq b$. Clearly, this correspondence is a bijection. Thus, $[\mathbf{2s}]$ can be grouped into m x-doubletons. The remaining $k - 1$ copies of $[\mathbf{2s}]$ can be grouped into a number of 0-doubletons. Similar to the proof of Lemma 3.1, we have a required partition. ▮

Thus we have the following theorem.

Theorem 5.3: *$I_m(C_n) = \mathbb{P} \setminus \{1\}$ for $n \geq 3$.*

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