Construction of group-magic graphs and some A-magic graphs with A of even order

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Abstract

Let A be an abelian group. An A-magic of a graph $G = (V, E)$ is a labeling $l : E(G) \to A \setminus \{0\}$ such that the sum of the labels of the edges incident with $u \in V$ is a constant, where 0 is the identity element of the group A. In this paper, we will show that some classes of graphs are A-magic for all abelian group A of even order other than 2. Also, we prove that product and composition of A-magic graphs are also A-magic.

Keywords : Group-magic, A-magic
AMS 2000 MSC : 05C78

1 Introduction

Let A be an (additive) abelian group with identity 0 (called zero) and let $G = (V, E)$ be a graph. G is said to be *A-magic* if there exists a mapping $l : E \to A \setminus \{0\}$ such that the *induced mapping* $l^+ : V \to A$ defined by $l^+(u) = \sum\limits_{uv \in E} l(uv)$, for all $u \in V$, is a constant mapping and l is called an *A-magic labeling* of G. The value of the constant mapping, denoted by l^+, is called the *A-magic value* of G corresponding to l. It is straight-forward to determine whether a graph is $\mathbb{Z}_2$-magic or not. So we shall only investigate A-magic graphs with $|A| \geq 3$.

Lee et al. [3] found some graphs that are V_4-magic, where $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. In this paper, we extend the result of [3] to any abelian groups of even order greater than 2. It is well known that any even order abelian group A can be decomposed into two groups such that $A \cong H_1 \times H_2$, where H_1 is a group whose order is a power of 2 and H_2 is a group of odd order. Note that H_2 may be the trivial group. By the fundamental theorem of abelian groups, H_1 contains a subgroup isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_4$, and if $|H_2| > 1$ then H_2 contains a subgroup isomorphic to $\mathbb{Z}_p$, where p is an odd prime number. Moreover, if $|H_1| \geq 4$, then H_1 contains a subgroup isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$. So if $|A|$ is even and greater than 2, then A contains a subgroup isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_p$, where p is a prime. Note that when p is an odd prime. Then $\mathbb{Z}_2 \times \mathbb{Z}_p \cong \mathbb{Z}_{2p}$.

Let A be an abelian group of order $2n$ with $n \geq 2$. To prove that a graph is A-magic, it suffices to prove that the graph is both $\mathbb{Z}_4$-magic and $(\mathbb{Z}_2 \times \mathbb{Z}_p)$-magic for p is a prime.

Moreover, we will prove that the *Cartesian product* and *composition* of A-magic graphs are A-magic also.

If G is an A-magic graph, then each of its component is also A-magic. Thus in this paper we consider connected graph only. All undefined symbols and concepts may be found in [1].

2 Construction of A-magic graphs

Lemma 2.1 *Suppose A is an abelian group. Let G and H be two edge-disjoint A-magic graphs having the same vertex set. The union of G and H, i.e., $G \cup H$ is also an A-magic graph.*

Proof: Suppose G and H are A-magic graphs with A-magic labelings such that a and b are the corresponding A-magic values, respectively. Then label the edges of $G \cup H$ with the same labeling as those in G and H. As a result, $a + b$ is the A-magic value of $G \cup H$ corresponding to this labeling. □

The *Cartesian product* of graphs G and H, denoted by $G \times H$, is a graph with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $v = v'$ and $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

Theorem 2.2 *Let G and H be A-magic graphs of order m and n, respectively. The product of G and H, i.e., $G \times H$, is also an A-magic graph.*

Proof: Let G' be a graph with vertex set $V(G) \times V(H)$ and two vertices (u, v) and (u', v') in $V(G')$ is adjacent if and only if $v = v'$ and $uu' \in E(G)$. H' is another graph with vertex set $V(G) \times V(H)$ and two vertices (u, v) and (u', v') in $V(G')$ is adjacent if and only if $u = u'$ and $vv' \in E(H)$.

Then G' is a graph with n components and each component is isomorphic to G. Therefore, G' is A-magic because G is A-magic. Similarly, H' is a graph with m components and each component is isomorphic to H. Thus H' is A-magic. Since G' and H' are edge-disjoint having the same vertex set, by Lemma 2.1 $G' \cup H' = G \times H$ is a also A-magic. $\square$

Suppose $A(G)$ is the adjacency matrix of a simple graph $G = (V, E)$ and l is a labeling of G. The *labeling matrix* for l, denoted by $\mathcal{L}(G)$, is obtained from $A(G) = (a_{u,v})$ by replacing $a_{u,v}$ with $l(uv)$ or $*$ if $a_{u,v} = 1$ or $a_{u,v} = 0$, respectively. This concept was first introduced in [4]. Using matrix presentation Shiu et al. did some labeling problem on some classes of graphs [4, 5, 6, 7]. Moreover, if G is an A-magic graph, then the sum of each row (column as well) of $\mathcal{L}(G)$ are all equal to the A-magic value l^+. For purposes of these row sums, entries with symbol $*$ are treated as 0.

Let G and H be simple graphs with order g and h, respectively. The *composition* or *lexicographic product* of graphs G and H denoted by $G \circ H$ is a graph with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. For example, $C_3 \circ K_2$ is shown in the figure below. Under the lexicographic order, the adjacency matrix of $G \circ H$ is equal to the following formula:

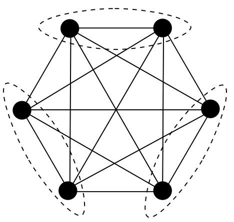
$$A(G \circ H) = A(G) \otimes J_h + I_g \otimes A(H)$$

where J_h is a $h \times h$ matrix whose entries are 1 and $\otimes$ is the Kronecker product operator.

Theorem 2.3 *Let G and H be A-magic connected simple graphs with order g and h, respectively. Then $G \circ H$ is also an A-magic graph.*

Proof: Let $\mathcal{L}(G)$ and $\mathcal{L}(H)$ be labeling matrices corresponding to some A-magic labelings, respectively. Let p and q be A-magic values of G and H accordingly. Let $\mathcal{L}(G \circ H) = \mathcal{L}(G) \otimes J_h + I_g \otimes \mathcal{L}(H)$. The above formula involving multiplication of an element $a \in A$ and a matrix $J_h \in M_{h,h}(\mathbb{Z})$, which is defined as $aJ_h = K$, where K is a $h \times h$ matrix whose entries are a. Obviously, the locations of the non-zero entry of $A(G) \otimes J_h$ are different from those of the matrix $I_g \otimes A(H)$. Therefore, $\mathcal{L}(G \circ H)$ does not contain zero entries. Moreover, it is clear that $\mathcal{L}(G \circ H)$ is a labeling matrix of an A-magic labeling of $G \circ H$ with magic value $hp + q$. $\square$

Example 2.1 $\mathcal{L}(C_3) = \begin{pmatrix} * & 2 & 2 \\ 2 & * & 2 \\ 2 & 2 & * \end{pmatrix}$ and $\mathcal{L}(K_2) = \begin{pmatrix} * & 1 \\ 1 & * \end{pmatrix}$ are $\mathbb{Z}_4$-magic labeling matrices of C_3 and K_2, respectively. Then $\mathcal{L}(C_3) \otimes J_2 + I_2 \otimes \mathcal{L}(K_2) = \left(\begin{array}{cc|cc|cc} * & 1 & 2 & 2 & 2 & 2 \\ 1 & * & 2 & 2 & 2 & 2 \\ \hline 2 & 2 & * & 1 & 2 & 2 \\ 2 & 2 & 1 & * & 2 & 2 \\ \hline 2 & 2 & 2 & 2 & * & 1 \\ 2 & 2 & 2 & 2 & 1 & * \end{array}\right)$ is a $\mathbb{Z}_4$-magic labeling matrix of $C_3 \circ K_2$. The $*$ denotes that there is no edge incident with the corresponding vertices. $\square$



$C_3 \circ K_2$

At the 35th Southeastern International Conference on Combinatorics, Graph Theory and Computing (2004), S-M. Lee and R. Low informed that the above results have been concurrently obtained by them [2]. However, their proofs are different from ours.

Theorem 2.4 *Let A be an abelian group of even order. A graph G is A-magic if degrees of its vertices are either all odd or all even.*

Proof: Let a be an element of A of order 2. We simply label all the edges of G by a. Then $l^+(v) = a$ or 0 for all $v \in V(G)$ as the degree of v is odd or even, respectively. □

3 $\mathbb{Z}_4$-magic graphs

In this section, we will prove that some graphs considered in [3] are $\mathbb{Z}_4$-magic. All arithmetic for the labeling l are taken in $\mathbb{Z}_4$.

Theorem 3.1 *The complete bipartite graph $K_{m,n}$ is $\mathbb{Z}_4$-magic for $m, n \geq 2$.*

Proof: Let (X, Y) be a bipartition of $K_{m,n}$, where $X = \{u_i \mid 1 \leq i \leq m\}$ and $Y = \{v_j \mid 1 \leq j \leq n\}$. Consider the following two cases

Case 1. Suppose m and n have the same parity. By Theorem 2.4, $K_{m,n}$ is $\mathbb{Z}_4$-magic.

Case 2. Without loss of generality, assume m is odd and n is even. We label the edges u_iv_j, where $1 \leq i \leq m$ and $1 \leq j \leq n$, according to the following strategy:

$$l(u_iv_j) = \begin{cases} 1, & \text{for } i = 1, 2 \text{ and } j \text{ is odd} \\ 3, & \text{for } i = 1, 2 \text{ and } j \text{ is even} \\ 2, & \text{otherwise} \end{cases}$$

Then $l^+ = 0$. □

Theorem 3.2 *$K_n - e$, the complete graph with one edge removed, is $\mathbb{Z}_4$-magic for $n \geq 4$.*

Proof: Let $V(K_n) = \{v_1, \ldots, v_n\}$. Without loss of generality we may assume $e = v_1v_2$. Consider the following two cases:

Case 1. Suppose n is odd, that is, $n = 2k + 1$ for some $k \geq 1$. We label the edges in the following way

$$l(v_iv_j) = \begin{cases} 1, & \text{for } i = 1 \text{ and } j = 3, 4; \\ 3, & \text{for } i = 2 \text{ and } j = 3, 4; \\ 2, & \text{otherwise.} \end{cases}$$

With the above labeling, we have

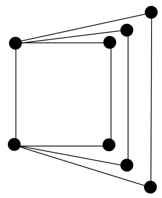
$$l^+(v_i) = \begin{cases} 1 + 1 + 2(2k - 3), & \text{for } i = 1; \\ 3 + 3 + 2(2k - 3), & \text{for } i = 2 \\ 1 + 3 + 2(2k - 2), & \text{for } i = 3, 4; \\ 2(2k), & \text{otherwise.} \end{cases}$$

That is, $l^+ = 0$.

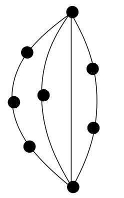
Case 2. Suppose $n = 2k$ for some $k \geq 2$. With similar labeling strategy, we have $l^+ = 2$. □

The *n-gon book of k pages*, denoted by $B(n, k)$, is formed by attaching k copies of C_n to a common edge. $B(n, k)$ is a special case of generalized theta graph whose definition is as follows.

Let $a_1, a_2, \ldots, a_k$ be k natural numbers and let v_1 and v_2 be two vertices. For $k \geq 2$, the *generalized theta graph* $\Theta(a_1, a_2, \ldots, a_k)$ is obtained by connecting v_1 and v_2 by k parallel and non-intersecting paths of length $a_1, a_2, \ldots, a_k$. As a result, $\deg(v_1) = \deg(v_2) = k$ and all other vertices are of degree 2. Thus $B(n, k) = \Theta(1, \overbrace{n - 1, n - 1, \ldots, n - 1}^{k \text{ times}})$, for $k \geq 1$ and $n \geq 2$.



Book $B(4,3)$



Generalized theta graph $\Theta(3,1,2,4)$

Theorem 3.3 *Suppose $k \geq 2$. The generalized theta graph $\Theta(a_1, a_2, \ldots, a_k)$ is $\mathbb{Z}_4$-magic for any natural numbers $a_1, a_2, \ldots, a_k$.*

Proof: Consider the following two cases:

Case 1. Suppose k is even. By Theorem 2.4, $\Theta(a_1, a_2, \ldots, a_k)$ is $\mathbb{Z}_4$-magic.

Case 2. Suppose k is odd. Without loss of generality, we may assume a_1 and a_2 have the same parity. Then label the first two paths (of lengths a_1 and a_2, respectively) as $(1,3)$-paths. Other edges of the graph are labeled by 2.

With the above labeling l, we have

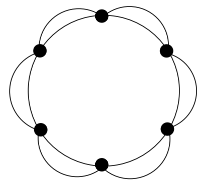
$$l^+(v_i) = \begin{cases} 1+1+2(k-2), & \text{for } i=1; \\ 3+3+2(k-2), & \text{for } i=2 \text{ if both } a_1 \text{ and } a_2 \text{ are even;} \\ 1+1+2(k-2), & \text{for } i=2 \text{ if both } a_1 \text{ and } a_2 \text{ are odd.} \end{cases}$$

For each of other vertices u, $l^+(u)$ is either $2+2=0$ or $1+3=0$.
Therefore $l^+ = 0$. $\square$

Let A be an abelian group and suppose $m, n \in A$. An *(m,n)-path* is a path whose edges are labeled as m and n alternately with the first edge being labeled by m.

Corollary 3.4 *For $n \geq 2$ and $k \geq 1$, the n-gon book of k pages $B(n,k)$ is $\mathbb{Z}_4$-magic.*

Given a cycle C_n, for each pair of adjacent vertices, paste a path of length $m-1$ on it by identifying the end vertices of the path with these adjacent vertices respectively, where $m, n \geq 2$ (here C_2 is a graph obtained by joining two edges to two vertices). The resulting graph is called a *flower* graph and is denoted by $C_m@C_n$.

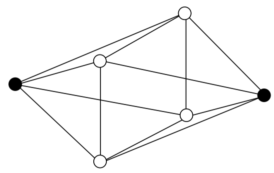


Flower $C_2@C_6$

Theorem 3.5 *For $m, n \geq 2$, the flower $C_m@C_n$ is $\mathbb{Z}_4$-magic.*

Proof: It is obvious that every flower graph is an Eulerian graph. By Theorem 2.4, $C_m@C_n$ is $\mathbb{Z}_4$-magic.$\square$

The *join* of the graphs G and H, denoted by $G \vee H$, is obtained from the disjoint union $G + H$ by adding the edges $\{xy \mid x \in V(G), y \in V(H)\}$. The join graph $C_n \vee N_k$, where N_k is the null graph of order k, is called *k-pyramid* and is denoted by $kP(n)$. The 2-pyramid graph $C_n \vee N_2$ is called *bipyramid graph* and is denoted by $BP(n)$. The 1-pyramid graph $C_n \vee N_1$ is the wheel graph W_n.



Bipyramid $C_4 \vee N_2$

Theorem 3.6 *The k-pyramid graph $kP(n)$ is $\mathbb{Z}_4$-magic with $k \geq 2$ and $n \geq 3$.*

Proof: Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of C_n and $v_1, v_2, v_3, \ldots, v_k$ be the vertices of N_k. Then, we have $\deg(u_i) = k+2$ and $\deg(v_i) = n$. Consider the following three cases:

Case 1. Suppose n and k have the same parity. By Theorem 2.4, $kP(n)$ is $\mathbb{Z}_4$-magic.

Case 2. Suppose n is even and k is odd.

For $k=1$, define $l(u_iv_1) = 1$ for $1 \leq i \leq n$ and $l(e) = 2$ for other edge e.

For $k \geq 3$, define

$$l(e) = \begin{cases} 1, & \text{if } e = u_i v_j, \text{ for } i \text{ is odd, } j = 1,2; \\ 3, & \text{if } e = u_i v_j, \text{ for } i \text{ is even, } j = 1,2; \\ 2, & \text{otherwise} \end{cases}$$

Then, $l^+ = 0$.

Case 3. Suppose n is odd and k is even. Define

$$l(e) = \begin{cases} 1, & \text{if } e = u_i v_j, \text{ for } i = 1,2, \ j \text{ is even;} \\ 3, & \text{if } e = u_i v_j, \text{ for } i = 1,2, \ j \text{ is odd;} \\ 2, & \text{otherwise} \end{cases}$$

Then, $l^+ = 0$. □

Corollary 3.7 *The bipyramid graph $BP(n)$ and the wheel graph W_n are $\mathbb{Z}_4$-magic for $n \geq 3$.*

4 $\mathbb{Z}_{2p}$ -magic graphs

Consider the group $\mathbb{Z}_{2p}$, where p is an odd prime (actually p can be any positive odd integer greater than 1). Then $\mathbb{Z}_{2p}$ contains at least 5 nonzero elements, namely p, 2, $p+2$, $2p-2 = -2$, $p-2$. In this section, we will prove that some graphs considered in [3] are $\mathbb{Z}_{2p}$-magic by using these 5 elements. All arithmetic for the labeling l are taken in $\mathbb{Z}_{2p}$.

Theorem 4.1 *The complete bipartite graph $K_{m,n}$ is $\mathbb{Z}_{2p}$-magic for $m, n \geq 2$.*

Proof: Notation of vertices and edges are the same as Theorem 3.1. Consider the following two cases:

Case 1. Suppose m and n have the same parity, by Theorem 2.4, $K_{m,n}$ is $\mathbb{Z}_{2p}$-magic.

Case 2. Without loss of generality, assume m is odd and n is even. We label the edges $u_i v_j$, with $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$, according to the following strategy:

$$l(u_i v_j) = \begin{cases} 2, & \text{for } i = 1 \text{ and } j \text{ is even;} \\ p-2, & \text{for } i = 2 \text{ and } j \text{ is even;} \\ -2, & \text{for } i = 1 \text{ and j is odd;} \\ p+2, & \text{for } i = 2 \text{ and j is even;} \\ p, & \text{otherwise.} \end{cases}$$

Then $l^+ = 0$. □

Theorem 4.2 *$K_n - e$, the complete graph with one edge removed, is $\mathbb{Z}_{2p}$-magic for $n \geq 4$.*

Proof: Notation of vertices and edges are the same as Theorem 3.2. Consider the following two cases:

Case 1. Suppose n is odd. We define the labeling l as follows:

$$l(v_i v_j) = \begin{cases} 2, & \text{for } i = 1 \text{ and } j = 3; \\ p-2, & \text{for } i = 1 \text{ and } j = 4; \\ -2, & \text{for } i = 2 \text{ and } j = 3; \\ p+2, & \text{for } i = 2 \text{ and } j = 4; \\ p, & \text{otherwise.} \end{cases}$$

We can easily check that $l^+ = 0$. For example,
$l^+(v_1) = 2 + (p-2) + p(n-4) = p(n-3) = 0$.

Case 2. Suppose $n = 2k$. With similar labeling strategy, we will have $l^+ = p$. □

Theorem 4.3 *The generalized theta graph $\Theta(a_1, a_2, \ldots, a_k)$ is $\mathbb{Z}_{2p}$-magic for any natural numbers $a_1, a_2, \ldots, a_k$.*

Proof: Consider the following two cases:

Case 1. Suppose k is even. By Theorem 2.4, $\Theta(a_1, a_2, \ldots, a_k)$ is $\mathbb{Z}_{2p}$-magic.

Case 2. Suppose k is odd. Without loss of generality, we may assume a_1 and a_2 have the same parity. Then label the first two paths (of lengths a_1 and a_2, respectively) as a $(2, -2)$-path and a $(p-2, p+2)$-path, respectively. All other edges are labelled by p.

With the above labeling, we can easily check that $l^+ = 0$. For example,
$l^+(v_2) = (p+2) + (-2) + p(k-2) = 0$ if both a_1 and a_2 are even.
$l^+(v_2) = 2 + (p-2) + p(k-2) = 0$ if both a_1 and a_2 are odd. □

Corollary 4.4 *For $n \geq 2$ and $k \geq 1$, the n-gon book of k pages $B(n, k)$ is $\mathbb{Z}_{2p}$-magic.*

Theorem 4.5 *For $m, n \geq 2$, the flower $C_m@C_n$ is $\mathbb{Z}_{2p}$-magic.*

Proof: By Theorem 2.4, $C_m@C_n$ is $\mathbb{Z}_{2p}$-magic. □

Theorem 4.6 *The k-pyramid graph $kP(n)$ is $\mathbb{Z}_{2p}$-magic with $k \geq 2$.*

Proof: Notation of vertices and edges are the same as Theorem 3.6. Consider the following three cases:

Case 1. Suppose n and k have the same parity. By Theorem 2.4, $kP(n)$ is $\mathbb{Z}_{2p}$-magic.

Case 2. Suppose n is even and k is odd.

For $k = 1$, define

$$l(e) = \begin{cases} 2, & \text{if } e = u_i v_1, \text{ for } i \text{ is odd;} \\ p-2, & \text{if } e = u_i v_1, \text{ for } i \text{ is even;} \\ p, & \text{otherwise.} \end{cases}$$

For $k \geq 3$, define

$$l(e) = \begin{cases} 2, & \text{if } e = u_i v_j, \text{ for } i \text{ is odd, } j = 1; \\ p-2, & \text{if } e = u_i v_j, \text{ for } i \text{ is odd; } j = 2; \\ -2, & \text{if } e = u_i v_j, \text{ for } i \text{ is even, } j = 1; \\ p+2, & \text{if } e = u_i v_j, \text{ for } i \text{ is even, } j = 2; \\ 2, & \text{otherwise.} \end{cases}$$

Then $l^+ = 0$.

Case 3. Suppose n is odd and k is even.

$$l(u_i v_j) = \begin{cases} 2, & \text{for } i = 1; \ j \text{ is even} \\ p-2, & \text{for } i = 2; \ j \text{ is even} \\ -2, & \text{for } i = 1; \ j \text{ is odd} \\ p+2, & \text{for } i = 2; \ j \text{ is odd} \\ 2, & \text{otherwise} \end{cases}$$

Then $l^+ = 0$. □

Corollary 4.7 *The Bipyramid graph $BP(n)$ and the wheel W_n are $\mathbb{Z}_{2p}$-magic for $n \geq 3$.*

k-pyramid graph has not been considered in [3]. So we have to show that it is also V_4-magic.

Theorem 4.8 *The k-pyramid graph $kP(n)$ is V_4-magic with $k \geq 2$.*

Proof: Notation of vertices and edges are the same as Theorem 3.6. Consider the following three cases:

Case 1. Suppose n and k have the same parity. Label all the edges by (1,0), then $l^+ = (1, 0)$ or (0,0) according to the degrees being odd and even respectively. Therefore $kP(n)$ is V_4-magic.

Case 2. Suppose n is even and k is odd. Define

$$l(u_iv_j) = \begin{cases} (1,0), & \text{for } i \text{ is odd; } j = 1,2 \\ (0,1), & \text{for } i \text{ is even; } j = 1,2 \\ (1,1), & \text{otherwise} \end{cases}$$

Then $l^+ = 0$ (i.e., $l^+ = (0,0)$).

Case 3. Suppose n is odd and k is even. Define

$$l(u_iv_j) = \begin{cases} (1,0), & \text{for } i = 1,2; \ j \text{ is even} \\ (0,1), & \text{for } i = 1,2; \ j \text{ is odd} \\ (1,1), & \text{otherwise} \end{cases}$$

Then $l^+ = 0$. □

Hence, combining the results from Sections 3, 4 and [3], we conclude that all graphs described in Sections 3 and 4 are A-magic, where A is an abelian group of even order greater than 2.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph theory with applications*, New York: Macmillan Ltd. Press, (1976).

[2] S-M. Lee and R.M. Low, On the Products of Group-Magic Graphs, submitted to *ARS Combin.*, (2003).

[3] S-M. Lee, F. Saba, E. Salehi, H. Sun, On the V_4-magic graphs, *Congressus Numerantium*, **156** (2002), 59-67.

[4] W.C. Shiu, P.C.B. Lam, S-M. Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium*, **132** (1998), 9-18.

[5] W.C. Shiu, P.C.B. Lam, H.L Cheng, Supermagic labeling of an s-duplication of $K_{n,n}$, *Congressus Numerantium*, **146** (2000), 119-124.

[6] W.C. Shiu, P.C.B. Lam, S-M. Lee, On a construction of supermagic graphs, *JCMCC*, **42** (2002), 147-160.

[7] W.C. Shiu, P.C.B. Lam, H.L. Cheng, Edge-gracefulness of the composition of paths with a null graphs, *Discrete Mathematics*, **253** (2002), 63-76.