Edge-magic Indices of $(n, n-1)$-graphs*

Wai Chee Shiu†, Peter, C.B. Lam
Department of Mathematics,
Hong Kong Baptist University,
224 Waterloo Road, Kowloon Tong,
Hong Kong, China.
and
Sin-Min Lee
Department of Mathematics and Computer Science,
San José State University,
One Washington Square, San José, CA 95192, U.S.A.

Abstract

A graph $G = (V, E)$ with p vertices and q edges is called edge-magic if there is a bijection $f : E \to \{1, 2, \ldots, q\}$ such that the induced mapping $f^+ : V \to \mathbb{Z}_p$ is a constant mapping, where $f^+(u) \equiv \sum_{uv \in E} f(uv) \pmod{p}$. A necessary condition of edge-magicness is $p|q(q+1)$. The edge-magic index of a graph G is the least positive integer k such that the k-fold of G is edge-magic. In this paper, we prove that for any multigraph G with n vertices, $n-1$ edges having no loops and no isolated vertices, the k-fold of G is edge-magic if n and k satisfy a necessary condition of edge-magicness and n is odd. For n is even we also have some results on full m-ary trees and spider graphs. Some counterexamples of the edge-magic indices of trees conjecture are given.

Key words and phrases : Edge-magic, edge-magic index, tree, spider graph.

AMS 2000 Mathematics Subject Classification : 05C78, 05C25

1. Introduction

The notion of magic graph was first introduced by Sedláček [18] in 1963. Many other researchers have investigated different forms of magic graphs. For example, see Stewart [28, 29], Kotzig and Rose [10], Stanley [26, 27], Sedláček [19, 20], Dood [2, 3, 4], Muehlbacher [16], Jeurissen [6, 7, 8], Jezny and Trenkler [9], Trenkler [30], Sandorova and Trenkler [17], Hartsfield and Ringel [5].

In this paper, any graph is a finite multigraph (not necessarily connected) having no loop and no isolated vertex. All undefined symbols and concepts may be looked up from [1]. A graph $G = (V, E)$ is a (p, q)-graph if p and q are its order and size respectively, i.e., $|V| = p$ and $|E| = q$. A (p, q)-graph G is called *d-edge-magic* if there exists a bijection $f : E \to \{d, d+1, \ldots, d+q-1\}$ such that the induced mapping $f^+ : V \to \mathbb{Z}_p$ is a constant mapping, where $f^+(u) \equiv \sum_{uv \in E} f(uv)$ $\pmod{p}$ for $u \in V$ and $d \in \mathbb{Z}$. If $d = 1$, then G is simply called *edge-magic* and f an *edge-magic labeling* of G. This concept was introduced by Lee, Seah and Tan [12] in 1992. A stronger concept called supermagic was introduced by Stewart [28, 29] in 1966. Some edge-magic graphs and some supermagic graphs were found in [12, 13, 21, 25] and [5, 23, 24, 29] respectively.

A necessary condition of a (p, q)-graph to be edge-magic is

$$q(q+1) \equiv 0 \pmod{p}. \tag{1.1}$$

*Partially supported by Faculty Research Grant of Hong Kong Baptist University.

†Research done while on sabbatical at Massachusetts Institute of Technology and San José State University in 1998.

But lots of graphs are not edge-magic even though they satisfy the necessary condition of edge-magicness (1.1), which we shall abbreviate as NCEM. For examples, cycles with more than 2 vertices; trees with more than 2 vertices, etc. However, all non edge-magic graphs can be embedded into some edge-magic graphs [12].

Let G be a graph and let k be a positive integer. $G[k]$ is the graph formed by replacing each edge of G with k edges. We call $G[k]$ the *k-fold* of G.

Theorem 1.1 ([11, 12]): *For any (p,q)-graph G, $G[2p]$ is edge-magic. Moreover, if p is odd, then $G[p]$ is edge-magic*

Let $emi(G) = \min\{k \mid G[k] \text{ is edge-magic}\}$. It is called the *edge-magic index* of G. This concept was introduced by Lee, Ho and Tan in [14] in 1996. But this manuscript has not published yet. The first published paper considering this concept is [22] which was written by the authors and was published in 1999. Some results were obtained in [11, 14, 15, 22]. Lee, Ho and Tan [14] conjectured that

Conjecture 1.2 (Edge-magic indices of trees conjecture): For any tree G and $k \geq 2$, if $G[k]$ satisfies NCEM then it is edge-magic.

Theorem 1.3 ([11]): *Suppose G is a (p,q)-graph and $G[k]$ is edge-magic. Then $G[k+2p]$ is edge-magic. Moreover, if p is odd, then $G[k+p]$ is edge-magic*

2. General properties

Let S be a set. We use $S \times n$ to denote the *multiset* of n-copies of S. Note that S may be a multiset itself. From now on, the term "set" means "multiset". The set operations are considered as multiset operations. Let S be a set of qk elements. Let $\mathscr{P}$ be a partition of S such that each class of $\mathscr{P}$ contains k elements. We call $\mathscr{P}$ a *(q,k)-partition* of S. We let $[\boldsymbol{r}] = \{1,\ 2,\ \ldots,\ r\}$ if $r > 0$ and let $[\mathbf{0}] = \varnothing$.

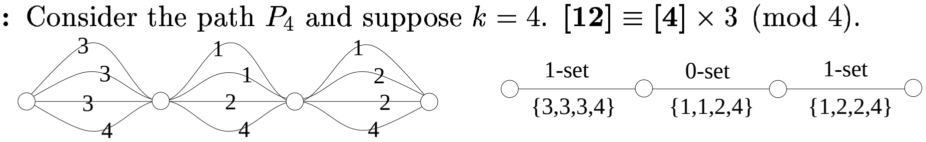
Suppose A is a set consisting of r integers. If the sum modulo n of elements of A is s, then A is called an *$(s;r)$-set* or simply an *s-set*. If $r = 1$, 2 or 3, it is called an *s-singleton*, an *s-doubleton* or an *s-tripleton* respectively. Let A and B be two sets. $A \equiv B \pmod{n}$ means the two sets are the same after taking each element modulo n.

A mapping f is called a *k-fold edge-magic labeling* of a (p,q)-graph G if there is a (q,k)-partition $\mathscr{P}$ of $[\boldsymbol{qk}]$ such that $f : E \to \mathscr{P}$ is a bijection and the induced mapping $f^+ : V \to \mathbb{Z}_p$ is a constant mapping, where

$$f^+(u) \equiv \sum_{uv \in E} \sum_{i \in f(uv)} i \pmod{p}.$$

Thus, finding an edge-magic labeling of $G[k]$ is equivalent to finding a k-fold edge-magic labeling of G.

Example 2.1: Consider the path P_4 and suppose $k = 4$. $[\mathbf{12}] \equiv [\mathbf{4}] \times 3 \pmod{4}$.



□

The following theorem settles our problem when $k = 1$.

Theorem 2.1: *Let G be an $(n, n-1)$-graph. Then G is edge-magic if and only if n is even and* $G \cong K_2 + \overbrace{K_2[2] + \cdots + K_2[2]}^{n/2-1 \text{ times}}$.

Proof: Suppose G be edge-magic. Since G contains no loop and no isolated vertex, there exists one component T_1 whose order is greater than its size. T_1 must be a tree. Because every degree one vertex of G must incident to the same edge, $T_1 \cong K_2$ and no other component of G contains a vertex of degree one. Hence $\deg_G(v) \ge 2$ if $v \notin T_1$. From the formula $\sum\limits_{v\in V} \deg_G(v) = 2(n-1)$, we get $\deg_G(v) = 2$ if $v \notin T_1$. So the remaining components of G are cycles. However, if G contains a cycle of length at least 3, then G is not edge-magic. Therefore, G contains exactly one component isomorphic to K_2 and $\frac{n}{2}-1$ identical components each of which is isomorphic $K_2[2]$. It also follows that n is even.

We give an edge-magic labeling of $G \cong K_2 + \overbrace{K_2[2] + \cdots + K_2[2]}^{n/2-1 \text{ times}}$ as follows:
Label the edge of the K_2 by $n-1$. Group the set $[\boldsymbol{n-2}]$ into $\frac{n}{2}-1$ n-doubletons and use elements of each doubleton to label the two edges of each $K_2[2]$, respectively. $\square$

From now on, we assume $k \ge 2$. Let G be an $(n, n-1)$-graph. Then $G[k]$ is an $(n, k(n-1))$-graph. Condition (1.1) becomes $k(n-1)(k(n-1)+1) \equiv 0 \pmod n$ or equivalently

$$k(k-1) \equiv 0 \pmod{n}. \tag{2.1}$$

Theorem 2.2: *Let G be a $(2m+1, 2m)$-graph and let $k \ge 2$. Then $G[k]$ is edge-magic if and only if $k(k-1) \equiv 0 \pmod{2m+1}$.*

Proof: It suffices to find a $(2m, k)$-partition of $[\boldsymbol{2mk}]$ such that each class is a $(0; k)$-set. Since $(k+2m+1)(k+2m) \equiv k(k-1) \pmod{2m+1}$, by Theorem 1.3 if $k \ge 2m+3$ then we reduce the case to $2 \le k \le 2m+2$. Note that we cannot reduce the case $k = 2m+2$ to $k = 1$, since $G[1]$ may not be edge-magic. If $k = 2$, then necessarily $2m+1 = 2$, which is impossible. Also by Theorem 1.1 we can ignore the case $k = 2m+1$. Thus we only need to handle cases when $3 \le k \le 2m+2$ and $k \ne 2m+1$.

Now $[\boldsymbol{2mk}] \equiv [\boldsymbol{2m+1}] \times (k-b) \cup [\boldsymbol{r}] \pmod{2m+1}$ where $r = (2m+1)b - k$, $b = 1$ if $3 \le k \le 2m+1$ and $b = 2$ if $k = 2m+2$. For the last case, $r = 2m$.

Each $[\boldsymbol{2m+1}]$ may be grouped into m 0-doubletons and one 0-singleton.

—	1	2	$\cdots$	$m-1$	m
$2m+1$	$2m$	$2m-1$	$\cdots$	$m+2$	$m+1$

Now we shall deal with $[\boldsymbol{r}]$. It is clear that $[\boldsymbol{2mk}]$ and $[\boldsymbol{2m+1}]$ are 0-sets. Thus $[\boldsymbol{r}]$ is a 0-set too. If $r \ge m+1$ then the set $\{2m+1-r,\ 2m+2-r,\ \ldots,\ r-1,\ r\}$ may be grouped into 0-doubletons. So we only have to deal with $[\boldsymbol{(2m+1)-r-1}] = [\boldsymbol{k-1}]$ if $b = 1$ and $[\boldsymbol{(2m+1)-r-1}] = \varnothing$ if $b = 2$. If $r \le m$ then $k \ge (2m+1)b - m \ge m+1$, and hence $r \le k-1$. Both of these cases may reduce to the case of handling the set $[\boldsymbol{r}]$ where $0 \le r \le k-1$. If $r = 0$, then we need not do anything. So we may assume $1 \le r \le k-1$ and $b = 1$. Now we have a 0-set $[\boldsymbol{r}]$ with $1 \le r \le k-1$, a number of 0-doubletons and a number of 0-singletons. First we would like to combine $[\boldsymbol{r}]$ with some sets to form a $(0; k)$-set.

Case 1: Suppose k is even. Then r is odd. We combine $[\boldsymbol{r}]$ with one 0-singleton and if necessary, an appropriate number of 0-doubletons to form a $(0; k)$-set. The remaining doubletons and singletons can be combined into $(0; k)$-sets.

Case 2: Suppose k is odd. Then r is even. For $k = 3$, $2m+1 = 3$ only. It is the excluded case. Now we assume $5 \le k < 2m+1$. In this case, $b = 1$ only. We first choose 3 copies of $[\boldsymbol{2m+1}]$ and arrange them as follows:

$2m+1$	1	2	$\cdots$	$m-1$	m	$m+1$	$m+2$	$\cdots$	$2m-1$	$2m$
$2m$	$2m-2$	$2m-4$	$\cdots$	2	$2m+1$	$2m-1$	$2m-3$	$\cdots$	3	1
1	2	3	$\cdots$	m	$m+1$	$m+2$	$m+3$	$\cdots$	$2m$	$2m+1$

We have $2m-1$ 0-tripletons, two 0-doubletons and two 0-singletons. After grouping the remaining $[\mathbf{2m+1}]$'s into 0-doubletons and 0-singletons, we have $2m-1$ 0-tripletons, a number of 0-doubletons and $k-2$ 0-singletons. Now we combine $[\boldsymbol{r}]$ with one 0-singleton and if necessary, an appropriate number of 0-doubletons to form a $(0;k)$-set. The remaining doubletons and singletons can be combined into $(0;k)$-sets as in Case 1. $\square$

Let us consider $(2m, 2m-1)$-graphs. There are two cases divided from (2.1). They are $(2m+k-1)k \equiv 0 \pmod{4m}$ and $(2m+k-1)k \equiv 2m \pmod{4m}$.

Theorem 2.3: *Let G be a $(2m, 2m-1)$-graph and let $k \geq 2$ be a positive integer. Then $G[k]$ is edge-magic if $(2m+k-1)k \equiv 0 \pmod{4m}$.*

Theorem 2.3 will be proven in Section 4.

3. Examples and counterexamples

From the last section, there are some unknown cases for graphs having even orders. In this section we shall show that there is no universal answer.

The concept of magic labeling can be found in [26, 27]. We do not recall it here. For the case of edge-magic, we have to modify this concept. For a fixed positive integer n, $f : E \to \mathbb{Z}_n$ is called a *$\mathbb{Z}_n$-magic labeling* of G if the induced mapping $f^+ : V \to \mathbb{Z}_n$ is a constant mapping. It is easy to prove the following theorem.

Theorem 3.1: *Let $G = (V,E)$ be a (p,q)-graph. If $G[k]$ is edge-magic then there is a $\mathbb{Z}_p$-magic labeling f of G such that* $\sum\limits_{e\in E} f(e) \equiv \frac{1}{2}kq(kq+1) \pmod{p}$.

We shall show by examples below that some of the k-fold graphs with even order that satisfy NCEM are edge-magic and some of those are not, even though they have the same parameters.

Example 3.1: Consider the 3-star S_3 for $k=4$. It does not satisfy the hypothesis of Theorem 2.3. If we define $f(e) = [\mathbf{4}]$ for each edge e in S_3. Then f is a 4-fold edge-magic labeling of S_3. $\square$

Example 3.2: Let $S_{3,3}$ be the regular spider graph with 3 legs of length 3 (Figure 3.1). Then $p = 10$, $q = 9$, k can be 5, 6, 10, 11, ... If $S_{3,3}[k]$ is edge-magic then $S_{3,3}$ has a $\mathbb{Z}_{10}$-magic labeling f satisfying the conclusion of Theorem 3.1.

For $k = 5$, 6 or 10. $\frac{1}{2}kq(kq+1) \equiv 5 \pmod{10}$ which does not satisfy the hypothesis of Theorem 2.3. Suppose $f(a_2a_3) = x$. Then $f(b_2b_3) = x = f(c_2c_3)$. Hence $f(a_1a_2) = f(b_1b_2) = f(c_1c_2) = 0$ and $f(v_0a_1) = f(v_0b_1) = f(v_0c_1) = x$ (then $2x \equiv 0 \pmod{10}$). By Theorem 3.1, $6x \equiv 5 \pmod{10}$. Clearly, it is no solution.

$S_{3,3}[11]$ is edge-magic, since $S_{3,3}[11]$ satisfies the hypothesis of Theorem 2.3. Hence $emi(S_{3,3}) = 11$. $\square$

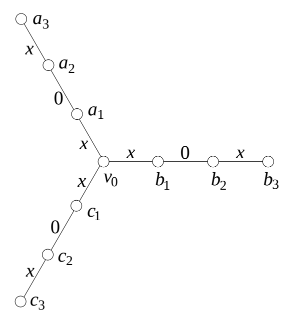


Figure 3.1: Spider graph with 3 legs of length 3.

Example 3.3: Consider the graph G described in Figure 3.2. Then $p = 6$, $q = 5$ and k can be 3, 4, 6, ...

For $k = 3$ or 4, it satisfies the hypothesis of Theorem 2.3. Thus $G[3]$ and $G[4]$ are edge-magic.

For $k = 6$. It does not satisfy the hypothesis of Theorem 2.3. Similarly to Example 3.2, if $G[6]$ is edge-magic, then $4x \equiv 3 \pmod{6}$ has a solution. But it is impossible. $\square$

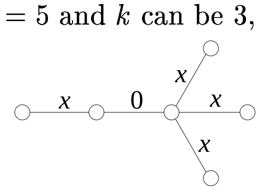


Figure 3.2

Example 3.4: Consider the 5-star S_5 for $k = 6$. Now $p = 6$, $q = 5$ and $kq = 30$. $[\mathbf{30}] \equiv [\mathbf{6}] \times 5 \pmod 6$. Define $f(e) = [\mathbf{6}]$ for each edge e, then f is a 6-fold edge-magic labeling of S_5, i.e., $S_5[6]$ is edge-magic. $\square$

Example 3.2 shows that there is a graph whose edge-magic index is greater than the order. Examples 3.3 and 3.4 show that although two trees have the same parameters p, q, k, they have different edge-magicness. Example 3.3 also shows that $G[k]$ being edge-magic for some smaller k does not imply $G[k']$ being edge-magic for some larger k'. Hence Conjecture 1.2 does not hold for some graphs having even orders.

4. Some useful lemmas

From Section 2, the unsolved case is: For a $(2m, 2m-1)$-graph G, whether $G[k]$ is edge-magic provided $(2m+k-1)k \equiv 2m \pmod{4m}$. By Theorem 1.3, if $k \geq 4m+2$ then we reduce the case to $2 \leq k \leq 4m+1$. By Theorem 1.1 we may assume $k \neq 4m$.

Lemma 4.1: *Let $2 \leq k \leq 4m+1$, $k \neq 4m$ and $m \geq 1$. Elements of $[\boldsymbol{(2m-1)k}]$ are considered in $\mathbb{Z}_{2m}$. If $(2m+k-1)k \equiv 2m \pmod{4m}$, then for any integer z, $0 \leq 2z \leq 2m-2$, $[\boldsymbol{(2m-1)k}]$ has a $(2m-1, k)$-partition containing $2z$ classes being $(0; k)$-sets and $2m-1-2z$ classes being $(m; k)$-sets.*

Proof: $[\boldsymbol{(2m-1)k}] \equiv [\boldsymbol{2m}] \times (k-b) \cup [\boldsymbol{r}] \pmod{2m}$, where

$$b = \begin{cases} 1 & \text{if } 4 \leq k \leq 2m \\ 2 & \text{if } 2m+1 \leq k \leq 4m-1 \\ 3 & \text{if } k = 4m+1 \end{cases}$$

and $r = 2mb - k$. Similar to the proof of Theorem 2.2, each $[\boldsymbol{2m}]$ may be grouped into $m-1$ 0-doubletons, one 0-singleton and one m-singleton.

Case 1: k is even. Before grouping the $[\boldsymbol{2m}]$'s, we choose two copies of $[\boldsymbol{2m}]$ and arrange them as the following array.

$$\boxed{\begin{array}{cccc} 1 & 2 & \cdots & m-1 \\ 2m-1 & 2m-2 & \cdots & m+1 \end{array}}\ \boxed{\begin{array}{c} m \\ m \end{array}}\ \boxed{\begin{array}{cccc} m+1 & m+2 & \cdots & 2m-1 \\ m-1 & m-2 & \cdots & 1 \end{array}}\ \boxed{\begin{array}{c} 2m \\ 2m \end{array}}$$

We swap the i-th entry with the $m+i$ entry of the 1-st row for $z+1 \leq i \leq m-1$. Then the array becomes

$$\boxed{\begin{array}{cccc|cccc} 1 & 2 & \cdots & z & m+z+1 & \cdots & 2m-2 & 2m-1 \\ 2m-1 & 2m-2 & \cdots & 2m-z & 2m-z-1 & \cdots & m+2 & m+1 \end{array}}\ \begin{array}{c} \boxed{m} \\ \boxed{m} \end{array}$$

$$\boxed{\begin{array}{cccc|cccc} m+1 & m+2 & \cdots & m+z & z+1 & \cdots & m-2 & m-1 \\ m-1 & m-2 & \cdots & m-z & m-z-1 & \cdots & 2 & 1 \end{array}}\ \begin{array}{c} \boxed{2m} \\ \boxed{2m} \end{array}$$

We obtain $2z$ 0-doubletons, $(2m-2-2z)$ m-doubletons, two m-singletons and two 0-singletons.

We consider the set $[\boldsymbol{r}]$. Note that $[\boldsymbol{r}]$ is a 0-set if and only if b is odd. If $r \geq m$ then the set $\{2m-r, 2m-r+1, \ldots, r-1, r\}$ may be grouped into a number of 0-doubletons and one m-singleton. The residual set $[\boldsymbol{2m-r-1}]$ is an m-set if and only if b is odd. Note that $0 \leq 2m-r-1 = 2m-2mb+k-1 \leq k-1$. Let $N \equiv [\boldsymbol{2m-r-1}] \cup \{(b+1)m\} \pmod{2m}$. If $r \leq m-1$ then $0 \leq r \leq k-2$ (since r is even). Let $N \equiv [\boldsymbol{r}] \cup \{bm, 2m\} \pmod{2m}$. For both case, N is an m-set with even cardinality k or less.

Up to now since $[\boldsymbol{(2m-1)k}]$ is an m-set, we have $(2m-2-2z)$ m-doubletons, a number of 0-doubletons, an even number of m-singletons, an even number of 0-singletons and an m-set N.

Similar to the proof of Theorem 2.2, these sets can be combined into $(2m-2-2z)$ $(m;k)$-sets and $2z$ $(0;k)$-sets.

Case 2: k is odd. Before grouping each $[\boldsymbol{2m}]$ into $m-1$ 0-doubletons, two 0-singletons and two m-singletons, we choose 3 sets of $[\boldsymbol{2m}]$ and arrange them as the following array.

$$\begin{array}{||cccc||c||cccc||c||}
\hline
1 & 2 & \cdots & m-1 & m & m+1 & m+2 & \cdots & 2m-1 & 2m \\
2m-1 & 2m-3 & \cdots & 3 & 1 & 2m-2 & 2m-4 & \cdots & 2 & 2m \\
m & m+1 & \cdots & 2m-2 & 2m-1 & 1 & 2 & \cdots & m-1 & 2m \\
\hline
\end{array}$$

We swap the i-th entry with the $m+i$ entry of the 1-st row for $1 \leq i \leq z$. Then the array becomes

$$\begin{array}{||cccc|ccc||c||}
\hline
m+1 & m+2 & \cdots & m+z & z+1 & \cdots & m-1 & m \\
\hline
2m-1 & 2m-3 & \cdots & 2m-2z+1 & 2m-2z-1 & \cdots & 3 & 1 \\
m & m+1 & \cdots & m+z-1 & m+z & \cdots & 2m-2 & 2m-1 \\
\hline
\end{array}$$

$$\begin{array}{||cccc|ccc||c||}
\hline
1 & 2 & \cdots & z & m+z+1 & \cdots & 2m-1 & 2m \\
\hline
2m-2 & 2m-4 & \cdots & 2m-2z & 2m-2z-2 & \cdots & 2 & 2m \\
\hline
1 & 2 & \cdots & z & z+1 & \cdots & m-1 & 2m \\
\hline
\end{array}$$

We obtain $2z$ 0-tripletons, $(2m-2-2z)$ m-tripletons, one m-singletons and three 0-singletons. The rest is similar to Case 1. □

We also have a similar lemma for the case $(2m+k-1)k \equiv 0 \pmod{4m}$.

Lemma 4.2: *Let $2 \leq k \leq 4m+1$ (in fact $k=2$ and $k=4m+1$ are not the cases) and $m \geq 1$. Elements of $[\boldsymbol{(2m-1)k}]$ are considered in $\mathbb{Z}_{2m}$. If $(2m+k-1)k \equiv 0 \pmod{4m}$, then for any integer z, $0 \leq 2z+1 \leq 2m-1$, $[\boldsymbol{(2m-1)k}]$ has a $(2m-1,k)$-partition containing $2z+1$ classes being $(0;k)$-sets and $2m-2-2z$ classes being $(m;k)$-sets.*

Proof: If $k=3$ then $2m=2$ or 6. When $2m=2$, it is trivial.
When $2m=6$, $[\mathbf{15}] = [\mathbf{6}] \times 2 \cup [\mathbf{3}]$.
For $2z+1=1$, then $\{2,\ 2,\ 2\}$, $\{1,\ 1,\ 1\}$, $\{3,\ 3,\ 3\}$, $\{4,\ 5,\ 6\}$ and $\{4,\ 5,\ 6\}$ is a required partition.
For $2z+1=3$, then $\{2,\ 2,\ 2\}$, $\{3,\ 4,\ 5\}$, $\{3,\ 4,\ 5\}$, $\{1,\ 1,\ 1\}$ and $\{3,\ 6,\ 6\}$ is a required partition.
For $2z+1=5$, then $\{1,\ 2,\ 3\}$, $\{1,\ 2,\ 3\}$, $\{3,\ 4,\ 5\}$, $\{1,\ 5,\ 6\}$ and $\{2,\ 4,\ 6\}$ is a required partition.
For the case of $4 \leq k \leq 4m$, the proof is similar to the proof of Lemma 4.1. □

Proof of Theorem 2.3: By Theorem 1.3 we can reduce the case to $2 \leq k \leq 4k+1$. The theorem follows from Lemma 4.2 by choosing $2z+1 = 2m-1$. □

Theorem 4.3: *Let G be a $(2m, 2m-1)$-graph. Suppose $k \geq 2$ and $k(k-1) \equiv 0 \pmod{2m}$. If G has a $\mathbb{Z}_2$-magic labeling f such that the pre-image of 1 has odd cardinality, then $G[k]$ is edge-magic.*

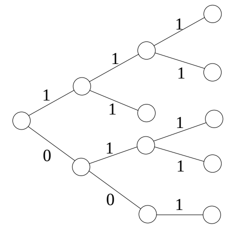
Proof: If $(2m+k-1)k \equiv 0 \pmod{4m}$, then by Theorem 2.3 the theorem follows. If $(2m+k-1)k \equiv 2m \pmod{4m}$, then change all the label 1's to m's. Then f becomes a $\mathbb{Z}_{2m}$-magic labeling of G. By Lemma 4.1, we have $|f^{-1}(1)|$ m-sets. Each edge which is labeled by m is assigned by an m-set. The other edge is assigned by a 0-set. The we have a k-fold edge-magic labeling of G. □

Corollary 4.4: *Let G be a $(2m, 2m-1)$-graph. Suppose $k \geq 2$ and $k(k-1) \equiv 0 \pmod{2m}$. If G has no vertices of even degree, then $G[k]$ is edge-magic.*

Proof: We labeling all edges by 1. Then this is a $\mathbb{Z}_2$-magic labeling. By Theorem 4.3, the corollary follows. □

The following example illustrates the proof of Theorem 4.3.

Example 4.1: Consider the following tree. Then $m=6$ ($p=12$ and $q=11$). Let $k=4$. Then $[\mathbf{44}] \equiv ([\mathbf{12}] \times 3) \cup [\mathbf{8}]$.



A $\mathbb{Z}_2$-magic labeling

Thus we need nine $(6;4)$-sets and two $(0;4)$-sets. Consider two **[12]**'s. They are arranged as follows:

1	*8*	*9*	*10*	*11*	*6*	*7*	*2*	*3*	*4*	*5*	*12*
11	*10*	*9*	*8*	*7*	*6*	*5*	*4*	*3*	*2*	*1*	*12*

The other **[12]** is arranged as follows:

	1	*2*	*3*	*4*	*5*	*6*
12	*11*	*10*	*9*	*8*	*7*	

Consider the set **[8]**. It is partition as $N = \{1, 2, 3\}$, $\{6\}$, $\{4, 8\}$ and $\{5, 7\}$. Therefore, we have one 6-tripleton N, eight 6-doubletons, four 6-singletons, three 0-singletons and seven 0-doubletons. Combine N with $\{12\}$ to obtain a $(6;4)$-set. Combine all the 6-singletons and 0-singletons to 0-doubletons and join with other doubletons as the following result.

	1	*8*	*9*	*10*	*11*	2	3	4	5	*2*	*4*
	2	*10*	*9*	*8*	*7*	*4*	*3*	*2*	*1*	*10*	*8*
	3	*1*	6	4	5	*7*	*12*	6	1	*3*	*5*
	12	*11*	*6*	8	7	5	*12*	6	*11*	*9*	*7*
Sum	6	6	6	6	6	6	6	6	6	0	0

□

5. Edge-magicness on k-fold of a tree

Although there are no universal results on graphs having even orders, we have some results on trees. We shall view any tree as a rooted tree.

Before the consideration, we introduce some concept and terminologies. Let G be a graph. A vertex $u \in V(G)$ is called a *pendant* if $\deg_G(u) = 1$. Let $v \in V(G)$. Let P be a path originating from v to a pendant of G. If all internal vertices of P are of degree 2 in G, then $V(P) \cup E(P) \setminus \{v\}$ is called a *leg* of v. The number of edges ℓ in P is called the *length* of the leg. The leg will also be called an *ℓ-leg.*

Lemma 5.1: *Let T be a tree and let f be a $\mathbb{Z}_n$-magic labeling of T for some n. If there is an edge e which is incident with a pendant of T and $f(e) = 0$, then $f = 0$.*

Proof: Let $e = uw$ and u be a pendant, since $f^+(u) = f(e) = 0$, $f^+ = 0$. Also f is a $\mathbb{Z}_n$-magic labeling of $T - u$, since $T - u$ is also a tree and $f^+ = 0$. There is an edge e' incident with a pendant w in $T - u$. Then $f(e') = f^+(w) = 0$. By induction f is a zero mapping. □

Lemma 5.2: *If T is a tree of order $2m$, then T has a nonzero $\mathbb{Z}_{2m}$-magic labeling and a unique nonzero $\mathbb{Z}_2$-magic labeling.*

Proof: First we prove that T has a nonzero $\mathbb{Z}_2$-magic labeling. If $m = 1$ then it is obvious. Suppose the lemma holds when the order of T, which is even, is less than $2m$, $m \geq 2$. Now we assume the order of T is $2m$.

Suppose T has two pendants v_1 and v_2 of distance 2. Let u be the vertex adjacent to v_1 and v_2. Since $T - \{v_1, v_2\}$ has a nonzero $\mathbb{Z}_2$-magic labeling f, by Lemma 5.1 $f^+ = 1$. We label uv_1 and uv_2 by 1. Clearly this extended labeling is a $\mathbb{Z}_2$-magic labeling of T.

Suppose the distance of any two pendants is greater than 2. Then T has a 2-leg of some vertex, say u. Let the two vertices lying in the leg be v_1 and v_2, and let uv_1 and v_1v_2 be edges. By induction $T - \{v_1,\ v_2\}$ has a nonzero $\mathbb{Z}_2$-magic labeling. We label uv_1 by 0 and v_1v_2 by 1. Clearly this extended labeling is a $\mathbb{Z}_2$-magic labeling of T.

Suppose there are two nonzero $\mathbb{Z}_2$-magic labelings f and g of T. Since f and g are nonzero, $f^+ = 1 = g^+$. Since $(f-g)^+(u) = f^+(u) - g^+(u) = 0$ for all $u \in V(T)$, by Lemma 5.1 $f-g$ is a zero mapping.

By the same proof of Theorem 4.3, we have a nonzero $\mathbb{Z}_{2m}$-labeling g of T. In fact, $g^+ = m$. □

Lemma 5.3: *Suppose f is a $\mathbb{Z}_n$-magic labeling of a graph G for some n. Let L be a leg of v, $v \in V(G)$. Let u be the end vertex of L, i.e., u is a pendant in G. If $f^+(u) = x$, then all edges of L are labeled by x and 0 alternatively.*

Let v be a vertex and r be the root of T. Let ℓ be the distance between v and r, then v is called an ℓ-th *level vertex* of T. We say that an edge e lies on the ℓ-th *layer* if e is incident with an $(\ell-1)$-th level vertex and an ℓ-th level vertex. A tree is called a *regular m-ary tree* if each vertex has either m children or no children. A *full m-ary tree of height h*, denoted by $T_{m,h}$, is a regular tree such that each pendant is a h-th level vertex.

The order of $T_{m,h}$ is $1 + m + \cdots + m^h$. Thus if m or h is even, then the order of $T_{m,h}$ is odd.

Theorem 5.4: *Let m and h be odd. Suppose $T_{m,h}[k]$ satisfies NCEM. If $h \equiv 1 \pmod 4$, then $T_{m,h}[k]$ is edge-magic.*

Proof: Let f be the nonzero $\mathbb{Z}_2$-magic labeling of $T_{m,h}$. Then $f(e) = 1$ if e lies on the h-th layer. Since m is odd, $f(e) = 0$ if e lies on the $(h-1)$-th layer. Similarly, $f(e) = 1$ if and only if e lies on the $(h-2i)$-th layer, $0 \le i \le \frac{h-1}{2}$. Thus, $|f^{-1}(1)| = m^h + m^{h-2} + \cdots + m^3 + m \equiv 1 \pmod 2$ By Theorem 4.3 we have the theorem. □

A graph G is called a *spider graph with Δ legs* if it is obtained from the union of Δ paths with one of the end vertices of each path identified. The identified vertex is called the *center* of the spider graph. If all legs are ℓ-legs, then G is called a *regular spider graph with Δ legs of length ℓ*, and denoted by $S_{\Delta,\ell}$. Note that $S_{\Delta,\ell}$ is a $(\Delta\ell+1, \Delta\ell)$-graph and $S_{\Delta,1}$ is the star S_Δ.

Theorem 5.5: *Let G be a spider graph with maximum degree Δ, $\Delta \ge 2$. Let $\ell_1^{(0)}, \ldots, \ell_{r_0}^{(0)}$, $\ell_1^{(1)}, \ldots, \ell_{r_1}^{(1)}$, $\ell_1^{(2)}, \ldots, \ell_{r_2}^{(2)}$, $\ell_1^{(3)}, \ldots, \ell_{r_3}^{(3)}$ be the length of its legs, where $r_0 + r_1 + r_2 + r_3 = \Delta$ and $\ell_{j_i}^{(i)} \equiv i \pmod 4$, $0 \le i \le 3$, $1 \le j_i \le r_i$. Suppose $G[k]$ satisfies NCEM. If $r_1 \not\equiv r_2 \pmod 2$, then $G[k]$ is edge-magic.*

Proof: The order of G is $n = 1 + \displaystyle\sum_{i=0}^{3}\sum_{j=1}^{r_i} \ell_j^{(i)}$. Then $n \equiv 1 + r_1 + r_3 \pmod 2$.

If $r_1 \equiv r_3 \pmod 2$, then G has odd order. By Theorem 2.2, we have the theorem.

If $r_1 \not\equiv r_3 \pmod 2$, then n is even. By Lemma 5.2, there exists a nonzero 2-magic labeling f of G. By Lemma 5.3 $\left|f^{-1}(1)\right| \equiv r_1 + r_2 \equiv 1 \pmod 2$. By Theorem 4.3, $G[k]$ is edge-magic. □

Corollary 5.6: *Suppose $G = S_{\Delta,\ell}[k]$ satisfies NCEM. Then G is edge-magic if either $\Delta\ell$ is even or $\ell \equiv 1 \pmod 4$.*

Proof: If $\Delta\ell$ is even, then the order of G is odd. By Theorem 2.2 we have the corollary.

If $\Delta\ell$ is odd then Δ is odd. Since $\ell \equiv 1 \pmod 4$, $r_0 = r_2 = r_3 = 0$ and $\Delta = r_1$, where r_i are defined in Theorem 5.5. Thus $r_1 \not\equiv r_2 \pmod 2$. By Theorem 5.5 the corollary holds. □

Corollary 5.7: *$S_\Delta[k]$ is edge-magic if it satisfies NCEM.*

Lemma 5.8: *Let G be a spider graph with even order n. Let f be the nonzero $\mathbb{Z}_2$-magic labeling of G. If $G[k]$ is edge-magic, then the system*

$$\begin{cases} |f^{-1}(1)|x \equiv \frac{1}{2}(n-1+k)k \pmod{n} \\ (r_1+r_3-1)x \equiv 0 \pmod{n} \end{cases} \tag{5.1}$$

has a solution, where r_1 and r_3 are defined in Theorem 5.5.

Proof: By Theorem 3.1 G has a $\mathbb{Z}_n$-magic labeling g such that

$$2\sum_{e\in E(G)} g(e) \equiv k(n-1)[k(n-1)+1] \pmod{2n}$$

Since n is even, $n^2 \equiv 0 \pmod{2n}$, hence

$$2\sum_{e\in E(G)} g(e) \equiv (n-1+k)k \pmod{2n}. \tag{5.2}$$

If $g=0$ then $x=0$ is a solution of (5.1).
If $g \neq 0$ then let $g^+ = x_0$ for some $x_0 \in \mathbb{Z}_n \setminus \{0\}$. By Lemma 5.3 $g(e)=x_0$ if and only if $f(e)=1$. Thus (5.2) becomes $2|f^{-1}(1)|x_0 \equiv (n-1+k)k \pmod{2n}$.

Let c be the center of G. Since $g^+(c) = x_0$, $(r_1+r_3)x_0 \equiv x_0 \pmod{n}$. Thus x_0 is a solution of (5.1). □

Corollary 5.9: *Let $\Delta\ell$ be odd. If $S_{\Delta,\ell}[k]$ is edge-magic, then*

$$\begin{cases} \Delta(\ell+1)x \equiv (\Delta\ell+k)k \pmod{2(\Delta\ell+1)} \\ (\Delta-1)x \equiv 0 \pmod{\Delta\ell+1} \end{cases} \tag{5.3}$$

has a solution.

Suppose the order of a tree T is $2m$ and $k(k-1) \equiv 0 \pmod{2m}$. By Lemma 5.2 there exists a unique nonzero $\mathbb{Z}_2$-magic labeling f. From Theorem 4.3 the sufficient condition of $T[k]$ being edge-magic is that

$$|f^{-1}(1)| \text{ is odd.} \tag{5.4}$$

We shall give a method to check whether a tree satisfies (5.4).

Snapping Method: Let T be a tree. There are 4 snapping processes below:

(a) If T contains a 4-leg L of some vertex, then remove the leg L from T, i.e., the resulting tree is $T \setminus L$.

(b) If T contains two 2-legs of some vertices (not necessarily distinct), then remove these two legs from T.

(c) If T contains a vertex having two 1-legs, then remove these two legs from T.

(d) If T contains a vertex v of degree 3 having one 1-leg and one 2-leg, then remove these two legs and v from T.

Suppose f is a nonzero $\mathbb{Z}_2$-magic labeling of T. By Lemma 5.1 $f^+ = 1$. Clearly f is also a nonzero $\mathbb{Z}_2$-magic labeling of the resulting tree which is obtained by applying a snapping process. Since we remove even number of edges which are labeled 1, the snapping process preserves the parity of $|f^{-1}(1)|$.

Let T' be a tree obtained from a tree T by applying a finite number of snapping processes, then we write $T \to T'$. Clearly, "$\to$" is a transitive relation (in fact, it is an equivalent relation since for each snapping process we can defined an inverse process — grafting process).

Suppose T is a tree of order n that snapping processes cannot be applied on it. Then T is not a path. There is a vertex of T whose degree is greater than 2. We choose one such vertex, say v, farthest from the root of T. If $\deg(v) \geq 4$, then v has either two 1-legs or two legs of length at least 2. In this case, process (c) or (b) can be applied on T. So $\deg(v) = 3$ and v has one 1-leg and one 3-leg. If there were another vertex u of degree greater than 2, then we consider v as the root of the tree T. By the same argument above, we have $\deg(u) = 3$ and u has one 1-leg and one 3-leg. In this case, process (b) can be applied on T. Therefore, T must be a spider. If $n \geq 5$, then the snapping process can be continuously applied.

Thus, the Snapping Method can be applied on any tree until the resulting tree T' containing at most 4 vertices. Then T' is K_1, K_2, P_3, P_4 or S_3. It is easy to see that K_1 and P_3 has no $\mathbb{Z}_2$-magic labeling, and $S_3 \to K_2$. Thus combining with Lemma 5.2 we have

Theorem 5.10: *Let T be a tree. T has a nonzero $\mathbb{Z}_2$-magic labeling if and only if T is of even order. Moreover, suppose T is of even order then T satisfies* (5.4) *if and only if $T \to K_2$.*

Corollary 5.11: *Let T be a tree of even order. Suppose $T[k]$ satisfies NCEM. If $T \to K_2$ then $T[k]$ is edge-magic.*

Example 5.1:
(a)

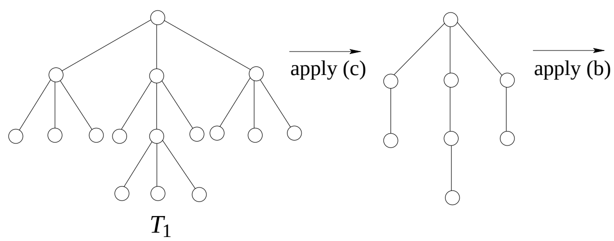


Figure 5.1 (a): A ternary tree.

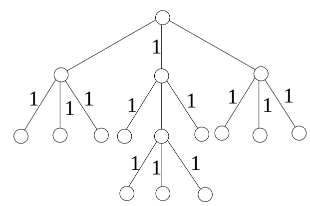


Figure 5.1(b): The 2-magic labeling.

Then $T_1 \not\to K_2$.
(b)

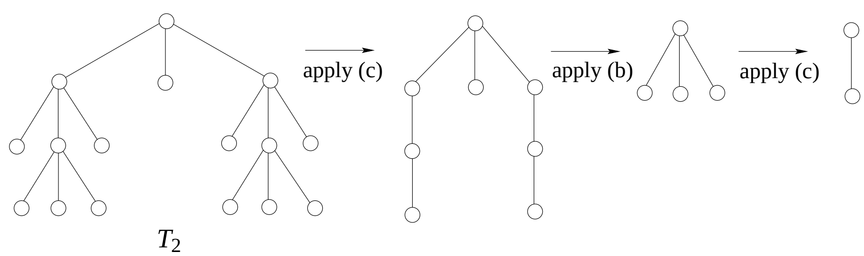


Figure 5.2 (a): A ternary tree.

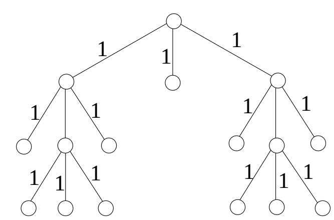


Figure 5.2(b): The 2-magic labeling.

Then $T_2 \to K_2$. $\square$

Example 5.1 shows that even though T_1 and T_2 are regular ternary trees with same order and same height, T_2 satisfies (5.4) and T_1 does not. In fact, if $(15 + k)k \equiv 16 \pmod{32}$, for example $k = 16$, then $T_1[16]$ is not edge-magic. It is because if $T_1[16]$ is edge-magic, then by Theorem 3.1 there is a $\mathbb{Z}_{16}$-magic labeling f such that $f^+ = x$ for some x. Thus x would be a solution of the following system

$$\begin{cases} 12x \equiv 8 \pmod{16} \\ 3x \equiv x \pmod{16}. \end{cases}$$

However, the system has no solution.

We are not sure whether the converse of Lemma 5.8 holds or not. But the following examples show a positive message. It also shows that the converse of Corollary 5.11 does not hold.

Example 5.2: Consider the graph $S_{5,3}$ for $k = 16$. It does not satisfy the hypothesis of Corollary 5.6. But equation (5.3) has solutions. In this case, equation (5.3) is

$$\begin{cases} 20x \equiv 496 \pmod{32} \\ \;\;4x \equiv 0 \pmod{16}, \end{cases}$$

The solution is $x = 4$ or $x = 12$ in $\mathbb{Z}_{16}$.

We consider $x = 4$. Then $[\mathbf{240}] \equiv [\mathbf{16}] \times 15 \pmod{16}$. It suffices to find a partition of $[\mathbf{240}]$ such that it contains ten $(4;16)$-sets and five $(0;16)$-sets. Using $[\mathbf{16}] \times 2 \cup \{8,\ 16\}$ we have

$$\Big\{\{1,\ 3\},\ \{5,\ 15\},\ \{7,\ 13\},\ \{9,\ 11\},\ \{8,\ 12\},\ \{2,\ 14\},\ \{6,\ 10\}\Big\} \times 2 \bigcup \Big\{\{4,\ 4,\ 8,\ 16\}, \{16,\ 16\}\Big\}.$$

There are ten 4-doubletons, four 0-doubletons, one $(0;\,4)$-set and one extra 0-doubleton $\{16,\ 16\}$. Using the rest of the numbers, we obtain 103 0-doubletons. Combining a suitable number of 0-doubletons with the above subsets we have a required partition of S. Hence $S_{5,3}[16]$ is edge-magic. When $x = 12$ it is similar, since we have 12 pairs of $\{8,\ 16\}$ for adjustment. $\square$

Example 5.3: Consider the graph G described in Figure 5.3. Then $p = 16$ and $q = 15$. For $k = 16$, it satisfies $kq(kq+1) \equiv p \pmod{2p}$. It also satisfies the conclusion of Theorem 3.1. A $\mathbb{Z}_{16}$-magic labeling is described in Figure 5.3, where x can be 4 or 12. In this example we only consider $x = 4$. The case $x = 12$ is left to the interested reader as an exercise. Thus $G[16]$ is edge-magic if and only if there is a $(15,16)$-partition of $[\mathbf{16}] \times 15$ consisting with thirteen $(4;16)$-sets, one $(8;16)$-set and one $(12;16)$-set.

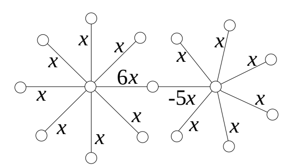


Figure 5.3.

Use two sets of $[\mathbf{16}]$ to form the table below:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	2	1	16	15	14	13	12	11	10	9	8	7	6	5	4

Break up the three highlighted columns $\{4,\ 16\}$, $\{8,\ 12\}$ and $\{16,\ 4\}$. Using these six numbers, we have $\{12,\ 16\}$, $\{4,\ 4\}$, $\{16\}$ and $\{8\}$. Now we have finished the main requirement, i.e., we have 13 4-doubletons, one 8-doubleton, one 12-doubleton and some (in this case is two) singletons. The rest is similar to the proof of previous theorems or examples. We have a required partition. $\square$

6. Conclusion

Suppose G is a $(2m, 2m-1)$-graph and k is a positive integer satisfying the conclusion of Theorem 3.1. Let f be a $\mathbb{Z}_{2m}$-magic labeling of G. Let $A = \{f(e) \mid e \in E\}$. The requirement for proving $G[k]$ being edge-magic is to find a $(2m-1;k)$-partition of $[\boldsymbol{(2m-1)k}] \equiv ([\boldsymbol{2m}] \times (k-b)) \cup [\boldsymbol{r}]$ for some $1 \le b \le 3$ (we assume $4 \le k \le 4m+1$) such that the set of sums of its classes is same as A, where $r = 2mb - k$. Normally we use two or three copies of $[\boldsymbol{2m}]$, $[\boldsymbol{r}]$, some 0's and some m's to obtain some doubletons (respectively, tripletons) and a set such that the set of sums of these doubletons (respectively, tripletons) and the set is equal to A when k is even (respectively, odd). Then the rest procedure is easy to do.

In our experience, there are many rooms for us to arrange the numbers to obtain such doubletons (tripletons) and a set. Thus we believe that the following conjectures are true. Note that the unsolved case is when n is even and $\frac{1}{2}(n+k-1)k \equiv \frac{1}{2}n \pmod{n}$. Example 5.3 verifies Conjecture 6.1.

Conjecture 6.1: Let $G = (V, E)$ be an $(n, n-1)$-graph and let $2 \le k \le 2n+1$ $(k \ne 2n)$, where n is even. If there is an (nonzero) $\mathbb{Z}_n$-magic labeling f of G such that $\sum_{e \in E} f(e) \equiv \frac{1}{2}n \pmod{n}$, then $G[k]$ is edge-magic.

Conjecture 6.1 may be more complicated, since the nonzero $\mathbb{Z}_n$-magic labeling may have more than two values (see Example 5.3). For the spider graph the nonzero $\mathbb{Z}_n$-magic labeling is only two values if any. This will be an easier case. We state the conjecture as follows:

Conjecture 6.2: Let $G = (V, E)$ be a spider graph with even order n and let $2 \le k \le 2n+1$ $(k \ne 2n)$. Let f be the nonzero $\mathbb{Z}_2$-magic labeling of G such that $|f^{-1}(1)|$ is even. If the system

$$\begin{cases} |f^{-1}(1)|x \equiv \frac{1}{2}n & \pmod{n} \\ (r_1 + r_3 - 1)x \equiv 0 & \pmod{n} \end{cases}$$

has solution then $G[k]$ is edge-magic, where r_1 and r_3 are defined in Theorem 5.5.

Finally we hold out a last conjecture.

Conjecture 6.3: Let $G = (V, E)$ be a (p, q)-graph and $k \ge 2$. If there is a $\mathbb{Z}_p$-magic labeling f of G such that $\sum_{e \in E} f(e) \equiv \frac{1}{2}kq(kq+1) \pmod{p}$ then $G[k]$ is edge-magic.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, 1976.

[2] M. Dood, On the construction of magic graphs, *Proc. Fifth S.E. Conference on Combinatorics, Graph Theory and Computing* (1974), 361-374.

[3] M. Dood, Generalizations of magic graphs, *J. Combin. Theory, Ser. B*, **17** (1974), 205-217.

[4] M. Dood, Characterizations of regular magic graphs, *J. Combin. Theory, Ser. B*, **25** (1978), 94-104.

[5] N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *Journal of Recreational Mathematics*, **21** (1989), 107-115.

[6] R.H. Jeurissen, The incidence matrix and labelings of a graph, *J. Combin. Theory, Ser. B*, **30** (1981), 290-301.

[7] R.H. Jeurissen, Disconnected graphs with magic labelings, *Discrete Math.*, **43** (1983), 47-53.

[8] R.H. Jeurissen, Pseudo-magic graphs, *Discrete Math.*, **43** (1983), 207-214.

[9] S. Jezny and M. Trenkler, Characterization of magic graphs, *Czechoslovak Math. J.*, **33 (108)** (1983), 435-438.

[10] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), 451-461.

[11] P.C.B. Lam, W.C. Shiu and S-M Lee, Edge-magic Indices of the Second Power of Paths, *Technical Report*, **218** (1999), Dept. of Math., Hong Kong Baptist University.

[12] S-M Lee, E. Seah and S.K. Tan, On edge-magic graphs, *Congr. Numer.*, **86** (1992), 179-191.

[13] S-M Lee, W.M. Pigg and T.J. Cox, On edge-magic cubic graphs conjecture, *Congr. Numer.*, **105** (1994), 214-222.

[14] S-M Lee, Y.S. Ho and S.K. Tan, On edge-magic Indices of Graphs which are Stars, manuscript, 1996.

[15] S-M Lee, Y.S. Ho and S.K. Tan, Edge-magic Indices of One-point Union of Complete Graphs, manuscript, 1997.

[16] J. Muehlbacher, Magische Quadrate und ihre Verallgemeinerung: ein graphen theoretisches Problem, *Appl. Comput. Sci., Ber. Prakt. Inf.*, **13** (1979), 45-64.

[17] L. Sandorova and M. Trenkler, On a generalization of magic graphs, in "Combinatorics 1987", *Proc., 7th Hungray Colloq. Eger/Hung. Colloquia Mathematica Societatis Janos Boylai*, **52** (1988), 447-452.

[18] J. Sedláček, Some properties of interchange graphs, in "Theory of Graphs and Its Applications", *Proc. Symp. Smolenice, Prague, 1963* (1964), 145-150.

[19] J. Sedláček, On magic graphs, *Math. Slov.*, **26** (1976), 329-335.

[20] J. Sedláček, Some properties of magic graphs, in "Graphs, Hypergraph, Block Syst. 1976, *Proc. Symp. Comb. Anal., Zielona Gora* (1976), 247-253.

[21] W.C. Shiu, P.C.B. Lam and S-M Lee, Edge-magicness of the composition of a cycle with a null graph, *Congr. Numer.*, **132** (1998), 9-18.

[22] W.C. Shiu, P.C.B. Lam and S-M Lee, Edge-magic Index Sets of (p, p)-graphs, *Congr. Numer.*, **137** (1999), 97-107.

[23] W.C. Shiu, P.C.B. Lam and S-M Lee, On a Construction of Supermagic of Graphs, presented at the 14th Mid-West Conference on Combinatorics, Crytography and Computing, Oct. 26-28, 2000.

[24] W.C. Shiu, P.C.B. Lam and H.L. Cheng, Supermagic Labeling on an s-duplicate of $K_{n,n}$, *Congr. Numer.*, **146** (2000), 119-124.

[25] W.C. Shiu and S-M Lee, Some Edge-magic Cubic Graphs, to appear in *J. Combin. Math. Combin. Comput.* **40** (Feb. 2002).

[26] R.P. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.*, **40** (1973), 607-632.

[27] R.P. Stanley, Magic labelings of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings, *Duke Math. J.*, **43** (1976), 511-531.

[28] B. M. Stewart, Magic graphs, *Canad. J. Math.*, **18** (1966), 1031-1059.

[29] B. M. Stewart, Supermagic Complete Graphs, *Canad. J. Math.*, **19** (1967), 427-438.

[30] M. Trenkler, Some results on magic graphs, in "Graphs and other Combinatorial Topics", *Proc. of the 3rd Czechoskovak Symp., Prague, 1983*, edited by M. Fieldler, Teubner-Texte zur Mathematik Band, **59** (1983), Leipzig, 328-332.

Edge-magic Indices of $(n, n-1)$-graphs*

Wai Chee Shiu[†], **Peter, C.B. Lam**
Department of Mathematics,
Hong Kong Baptist University,
224 Waterloo Road, Kowloon Tong,
Hong Kong, China.
and
Sin-Min Lee
Department of Mathematics and Computer Science,
San José State University,
One Washington Square, San José, CA 95192, U.S.A.

*Partially supported by Faculty Research Grant of Hong Kong Baptist University.
[†]Research done while on sabbatical at Massachusetts Institute of Technology and San José State University in 1998.

Running Title

Edge-magic indices of $(n, n-1)$-graphs

All correspondence should be sent to:

Wai Chee SHIU
Department of Mathematics,
Hong Kong Baptist University,
224 Waterloo Road, Kowloon,
Hong Kong.

e-mail address: wcshiu@math.hkbu.edu.hk

Abstract

A graph $G=(V,E)$ with p vertices and q edges is called edge-magic if there is a bijection $f:E\to\{1,\ 2,\ \ldots,\ q\}$ such that the induced mapping $f^+:V\to\mathbb{Z}_p$ is a constant mapping, where $f^+(u)\equiv\sum_{uv\in E}f(uv)\pmod p$. A necessary condition of edge-magicness is $p|q(q+1)$. The edge-magic index of a graph G is the least positive integer k such that the k-fold of G is edge-magic. In this paper, we prove that for any multigraph G with n vertices, $n-1$ edges having no loops and no isolated vertices, the k-fold of G is edge-magic if n and k satisfy a necessary condition of edge-magicness and n is odd. For n is even we also have some results on full m-ary trees and spider graphs. Some counterexamples of the edge-magic indices of trees conjecture are given.

Key words and phrases	:	Edge-magic, edge-magic index, tree, spider graph.
AMS 1991 subject classification	:	05C78, 05C25