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[GTN XXXVII:3] EDGE GAME-COLORING OF GRAPHS†

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Abstract

Corresponding to the game chromatic number of graphs, in this paper we consider the game chromatic index χ'_g of graphs. This is defined similarly, except that edges, instead of vertices of graphs are colored. Upper bounds for this index for trees and wheels are given.

1. Introduction

In 1991 Bodlaender **[1]** introduced the game-coloring problem of graphs. Let G be a graph and let $X = \{1, \ldots, k\}$ be a set of colors. Consider a two-person game on G as follows: Players 1 and 2 make alternate moves with player 1 moving first. Each move consists of choosing an uncolored vertex, and coloring it with a color from X, so that in the subgraph H of G induced by the colored vertices, adjacent vertices have distinct colors. The game ends when one of the two players can no longer execute a move. Player 1 wins if all the vertices of G are colored, otherwise player 2 wins. A graph G is called *k-game-colorable* if player 1 has a winning strategy for $|X| = k$, and the *game-chromatic number* $\chi_g(G)$ of G is the least integer k such that G is k-game-colorable. Upper bounds for the game chromatic number of some classes of graphs are given in **[2]–[4]**.

In this paper, we consider the edge-version of the game-coloring of graphs. This is defined similarly, except that the two players color the edges of a graph G instead of its vertices. In this case, a move consists of choosing an uncolored edge, and choosing a color from X, so that in the subgraph H of G induced by the colored edges, adjacent edges receive distinct colors. A color that can be assigned to an edge to constitute a move is called a *feasible* color. The smallest number k such that player 1 has a winning strategy with k colors in edge game-coloring G is called the *edge game-chromatic index* of G, and is denoted by $\chi'_g(G)$. An obvious lower bound on $\chi'_g(G)$ is the edge chromatic number of a graph G, $\chi'(G)$. This lower bound can be achieved because $\chi'_g(K_{1,r}) = \chi'(K_{1,r}) = r$. Note that $\chi'_g(G)$ can be strictly greater than $\chi'(G)$ because $\chi'_g(P_n) = 3$ but $\chi'(P_n) = 2$ when $n \geq 5$. $K_{1,r}$ and P_n, denote a star of order $r+1$ and a path of order n, respectively. Each edge is adjacent to at most $2\Delta(G) - 2$ distinct edges, where $\Delta(G)$ is the maximum degree of G. Thus, $2\Delta(G) - 1$ is a trivial upper bound on $\chi'_g(G)$. It is easy to see that, for any cycle graph C, $\chi'_g(C) = 3$.

In this paper all graphs are finite and simple. Symbols and concepts not defined here can be found in **[5]**. In Section 2, we shall establish an upper bound on the game chromatic index for trees. In Section 3, we obtain this index for wheels.

2. Game-Chromatic Index for Trees

First we give an upper bound on χ'_g for trees using a method due to Faigle, Kern, Kierstead and Trotter **[2]**.

Theorem 1: $\chi'_g(T) \leq \Delta(T) + 2$ for each tree T.

Proof: We give a winning strategy for Player 1 using $\Delta(T) + 2$ colors.

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Initially, player 1 chooses an arbitrary edge $e = v_0v_1$ of T, where $\deg_T(v_0) = 1$, and assigns a color to this edge. Let $T^* = \{e\}$. Henceforth, T is regarded as a digraph with v_0 as its root.

Suppose that player 2 has just moved by coloring an edge e_1. Let P be the directed path from v_0 to e_1 in T, and let e^* be the last edge P has in common with T^*. We update T^* to $T^* \cup \{P\}$, i.e., $T^* := T^* \cup \{P\}$.

If e^* is uncolored, then we assign a feasible color to e^*. If e^* is colored and T^* contains an uncolored edge f, then assign a feasible color to f. Otherwise, we color any edge f that is adjacent to at least one edge of T^* and let $T^* := T^* \cup \{f\}$.

Suppose $f = \overrightarrow{uv}$ is the last edge of the directed path in T from v_0 to v. If at most one arc out of v has been colored, then the total number of colored arcs incident with f is at most $\Delta(G)$. As soon as a second outgoing arc of v has been colored, player 1 will color f unless it has previously been colored. At this stage, at most $\Delta(G) + 1$ colored arcs are incident with f. With $\Delta(G) + 2$ colors available, player 1 can always find a feasible color for f. ■

3. Game-Chromatic Index for Wheels

A *wheel* W_n is a graph obtained from a cycle C_n by adding a new vertex and joining it to each of the vertices of C_n. The new vertex is called the *center* of the wheel, the subgraph C_n is called the *rim*, and the edges that join the center to vertices on the rim are called *spokes*. It is clear that $\chi'_g(W_n) \le n + 2$.

Theorem 2: $\chi'_g(W_3) = 5$ and $\chi'_g(W_n) = n + 1$ when $n \ge 4$.

Proof: $\chi'_g(W_3) = 5$ can be verified directly.

Suppose $n \ge 4$.

It is sufficient to show that player 1 can color all spokes with $n + 1$ colors when $n \ge 4$. Any rim edge is incident with two spokes and two other rim edges. Hence $n + 1 \ge 5$ assures that there will be a feasible color for any rim edge after all spokes have been colored. Let c_k denote the k^{th} color introduced during the game. Player 1 plans to color r spokes with r distinct colors, for $1 \le r \le n - 2$. Initially, he colors an arbitrarily chosen spoke with color c_1. Suppose $r - 1$ spokes have been colored with $r - 1$ colors when it is the turn of player 2, where $2 \le r \le n - 2$, so that there are at least three uncolored spokes.

If player 2 colors a spoke, he would be helping player 1 to accomplish his goal. If player 2 colors a rim edge, he can at his best prevent at most two of the uncolored spokes from being colored with c_r by player 1, but there are at least 3 uncolored spokes.

When player 1 colors spoke $n - 2$, he should ensure that no pair of uncolored spokes remain that are both adjacent to the same rim edge. This can be done by choosing to color an uncolored spoke, if there are two uncolored spokes adjacent to the same rim edge; or by choosing to color the middle spoke, if there are three of uncolored spokes that are adjacent to two rim edges. With this play, at best, player 2 can color the rim incident with one of the two remaining uncolored spokes using color c_{n-1}. However, player 1 can color the other spoke using color c_{n-1}. Hence, no matter how player 2 moves, player 1 can color the last uncolored spoke with c_n or c_{n+1}.

If player 2 chooses to color the spoke $n - 2$ with c_{n-2}, then player 1 can simply color one of the two uncolored spokes with c_{n-1}. No move by player 2 can then prevent player 1 from coloring the last uncolored spoke with c_n or c_{n+1}.

Subsequent to the submission of this paper the authors were alerted to a preprint from Cai and Zhu **[6]** that obtains a much more general result: If a graph G has degeneracy k, then the chromatic index of G is at most $\Delta(G) + 3k - 1$. Since a forest has degeneracy 1, Theorem 2 is a special case of this result.

4. Discussion

For each graph G, $E(G)$ can be partitioned into $k = \chi'(G)$ matchings $E_1, E_2, \ldots, E_k$. We call each of these matchings a color-class of G. It is obvious that in the edge game-coloring of graph G, player 1 should always color the edges in a color-class with the same color, and player 2 should try to do the opposite, i.e., try to color each edge in a matching with distinct colors. Consider the game-chromatic index of K_5. It is known that $\chi'(K_5) = 5$ and each matching of K_5 contains at most two edges. Let $X = \{1, 2, \ldots, 6\}$. Initially, player 1 colors an arbitrary edge. Suppose player 2 has just colored an edge e with color $j \in X$. If an uncolored edge remains, then player 1 choose an edge e' that is not adjacent to e and colors e' with j also. This guarantees that there are eight edges of G that are colored with four colors from X. Therefore, player 1 has a winning strategy using 6 colors.

It is known that there is no upper bound for $\chi_g(G)$ as a function of $\chi(G)$ [2] because there exist bipartite graphs with arbitrarily large game-chromatic numbers. This may not be true when we consider the relation between $\chi'_g(G)$ and $\chi'(G)$. It is perhaps true that there is a constant c such that $\chi'_g(G) \le \Delta(G) + c$ for all graphs. Additionally, $\chi'_g(G) \ge \chi'(G)$ and this lower bound can be achieved. How many graphs are there that satisfy $\chi'_g(G) = \chi'(G)$? Is it true that almost all graphs satisfy $\chi'_g(G) > \chi'(G)$? We conclude this paper with two open questions:

Question 1: Is there a constant $c \ge 2$ such that $\chi'_g(G) \le \Delta(G) + c$ for each graph G? If this is true, is $c = 2$ sufficient? □

Question 2: Let $\mathcal{G}_n$ be the set of graphs of order n and $\mathcal{G}_n^* = \{G \in \mathcal{G}_n : \chi'_g(G) > \chi'(G)\}$. Is it true that

$$\lim_{n \to \infty} \frac{|\mathcal{G}_n^*|}{|\mathcal{G}_n|} = 1 ?$$ □

References

[1] H.L. Bodlaender; On the complexity of some coloring games, *Lecture Notes in Computer Science,* **484**, 30–40, Springer (1991).

[2] U. Faigle, U. Kern, H. Kierstead and W.T. Trotter; On the game chromatic number of some classes of graphs, *Ars Combin.*, **35**, 143–150 (1993).

[3] H.A. Kierstead and W.T. Trotter; Planar graph coloring with an uncooperative partner, *J. Graph Theory,* **18**(6), 569–584 (1994).

[4] T. Dinski and X. Zhu; Game chromatic number of graphs, *Discrete Math.* —In press.

[5] J.A. Bondy and U.S.R. Murty; *Graph Theory with Applications*, Macmillan Press, New York (1976).

[6] L. Cai and X. Zhu; Game chromatic index and game coloring index of graphs. —Preprint.

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