The minimum algebraic connectivity of graphs with a given clique number*

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Abstract The algebraic connectivity of a graph G is the second smallest eigenvalue of its Laplacian matrix. In this paper, it is shown that among all connected graphs with the clique number ω, the minimum value of the algebraic connectivity is attained for a kite graph $PK_{n-\omega,\omega}$, obtained by appending a complete graph K_ω to an end vertex of a path $P_{n-\omega}$. Moreover, some properties for $PK_{n-\omega,\omega}$ are discussed.

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1 Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ (or $N(v)$ for short) be the neighborhood of v in G and $d(v) = |N(v)|$ be the degree of v. For any $e \notin E(G)$, we use $G+e$ to denote the graph obtained by adding e to G. Similarly, for any set W of vertices (edges), $G - W$ and $G + W$ are the graphs obtained by deleting the vertices (edges) in W from G and by adding the vertices (edges) in W to G, respectively. A clique $C \in V(G)$ is a set of mutually adjacent vertices. The clique number of G, denoted by $\omega(G)$ (or ω for short), is the size of the maximum clique in G. Readers are referred to [2] for undefined terms.

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Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$. It is easy to see that $L(G)$ is a symmetric positive semi-definite matrix having 0 as an eigenvalue. Thus, the eigenvalues $\mu_i(G)$'s of $L(G)$ (or the Laplacian eigenvalues of G) satisfy

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0,$$

repeated according to their multiplicities. Fiedler [6] showed that the second smallest Laplacian eigenvalue $\mu_{n-1}(G)$ is 0 if and only if G is disconnected. Thus $\mu_{n-1}(G)$ is popularly known as the algebraic connectivity of G and is usually denoted by $\alpha(G)$.

Let $\boldsymbol{y} \in \mathbb{R}^n$ be a column vector, and y_v denote the entry of $\boldsymbol{y}$ corresponding to the vertex v of G. Such labelings are sometimes called *characteristic valuations* of the vertices of G (see, [15]) and $\boldsymbol{y}$ is called a valuation of G. If $\boldsymbol{x}$ is a unit eigenvector of $L(G)$ corresponding to $\alpha(G)$, we commonly call it a *Fiedler vector* of G. Then we have the following set of equations, known in general as eigenvalue equations:

$$\alpha(G)x_v = d(v)x_v - \sum_{u \in N(v)} x_u \quad \text{for } v \in V(G) \tag{1.1}$$

It is obvious that $\boldsymbol{x}^T\boldsymbol{e} = 0$, where $\boldsymbol{e}$ is an n dimensional all ones column vector, and the following description is well-known:

$$\alpha(G) = (\boldsymbol{x}, L(G)\boldsymbol{x}) = \sum_{v_iv_j \in E(G)} (x_{v_i} - x_{v_j})^2 = \min_{\substack{\boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\} \\ \boldsymbol{y}^T\boldsymbol{e}=0}} \frac{\boldsymbol{y}^T L(G)\boldsymbol{y}}{\boldsymbol{y}^T\boldsymbol{y}}. \tag{1.2}$$

According to Godsil and Royle [7] graphs with small values of $\alpha(G)$ tend to be elongated graphs of large diameter with bridges. There are a lot of literatures concerning the problem of determining the graph with minimum algebraic connectivity in some classes of graphs [4, 5, 9, 11–14, 16, 17]. For more results we refer to de Abreu [1].

Let $\mathscr{G}_{n,\omega}$ be the set of all connected graphs of order n with the clique number ω, where $2 \leq \omega \leq n$. The *kite graph* $PK_{n-\omega,\omega}$ (shown in Fig. 1) is a graph on n vertices obtained from the path $P_{n-\omega}$ and the complete graph K_ω by adding an edge between an end vertex of $P_{n-\omega}$ and a vertex of K_ω. Clearly, $PK_{n-2,2} = P_n$ and $PK_{0,n} = K_n$. In this paper, we will keep the vertex labelings as shown in Fig. 1 for $PK_{n-\omega,\omega}$. Also we shall show that the kite graph $PK_{n-\omega,\omega}$ attains the minimum algebraic connectivity among all graphs in $\mathscr{G}_{n,\omega}$. Moreover, some properties for $PK_{n-\omega,\omega}$ are discussed.

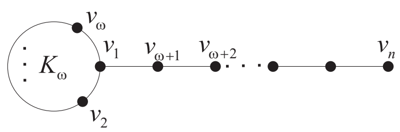

Figure 1: Kite graph $PK_{n-\omega,\omega}$.

2 Preliminaries

For $v \in V(G)$, let $L_v(G)$ be the sub-matrix of $L(G)$ obtained by deleting the row and column corresponding to the vertex v. Similarly, for any subset $V_1 \subset V(G)$, let $L_{V_1}(G)$ be the sub-matrix of $L(G)$ obtained by deleting the rows and columns corresponding to the vertices of V_1. Let B_n and H_n be two matrices of order n. The first one is obtained from $L(P_{n+1})$ by deleting the row and column corresponding to one of end vertices of P_{n+1} and the second one is obtained from $L(P_{n+2})$ by deleting the rows and columns corresponding to two end vertices of P_{n+2}.

For a square matrix M, let $\tau(M)$ be the smallest eigenvalue of M and $\Phi(M) = \Phi(M; x) = \det(xI - M)$ be the characteristic polynomial of M. If $M = L(G)$, we write $\Phi(L(G); x)$ (the Laplacian characteristic polynomial of G) as $\Phi(G)$ or $\Phi(G; x)$ for convenience. The following two lemmas are often used to calculate the Laplacian characteristic polynomial of G.

Lemma 2.1 ([8]) *Let $G = G_1u : vG_2$ be the graph obtained by joining the vertex u of G_1 and the vertex v of G_2 with an edge. Then*

$$\Phi(L(G)) = \Phi(L(G_1))\Phi(L(G_2)) - \Phi(L(G_1))\Phi(L_v(G_2)) - \Phi(L(G_2))\Phi(L_u(G_1)).$$

From the proof of Lemma 2.1 [8, Lemma 8] we can obtain a generalized result as follows:

Lemma 2.2 *Let $M = \begin{pmatrix} A & -E_{11} \\ -E_{11}^T & B \end{pmatrix}$ be a partition matrix over a commutative ring, where A and B are $m \times m$ and $n \times n$ matrices, respectively, and E_{11} is the $m \times n$ matrix whose only nonzero entry is 1 in $(1,1)$-position. Then $\det(M) = \det(A)\det(B) - \det(A_{11})\det(B_{11})$, where A_{11} and B_{11} are matrices obtained from A and B by deleting the first row and the first column, respectively.*

Lemma 2.3 ([9]) *Set $\Phi(P_0) = 0$, $\Phi(B_0) = 1$ and $\Phi(H_0) = 1$. Then we have*

(1) $x\Phi(B_n) = \Phi(P_{n+1}) + \Phi(P_n)$;

(2) $x\Phi(H_n) = \Phi(P_{n+1})$ $(n \geq 1)$.

Lemma 2.4 ([3]) *Let G be a graph and let $G' = G + e$ be the graph obtained from G by adding a new edge e. Then the Laplacian eigenvalues of G and G' interlace, that is*

$$\mu_{i+1}(G') \leq \mu_i(G) \leq \mu_i(G') \text{ for } 1 \leq i \leq n-1.$$

By Lemma 2.4, the following corollary is immediate.

Corollary 2.5 *Let G be a connected graph and v be a pendant vertex of G. Then $\alpha(G) \leq \alpha(G - v)$.*

The inequalities given in Lemma 2.6 are known as Cauchy's inequalities and the whole result is also known as the interlacing theorem [3].

Lemma 2.6 *Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and B be a principal submatrix of A. Let B has eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m$ $(m \leq n)$. Then the inequalities $\lambda_{n-m+i} \leq \rho_i \leq \lambda_i$ hold for $i = 1, 2, \ldots, m$.*

By Lemma 2.6, the following lemma is immediate since $\alpha(P_n) = 4\sin^2\frac{\pi}{2n}$.

Lemma 2.7 *If $k > l \geq 1$, then $\alpha(P_l) > \alpha(P_k)$ and $\tau(B_l) > \tau(B_k)$. Moreover, $\tau(B_n) = \alpha(P_{2n+1})$.*

Let G be a connected graph with at least two vertices, and v be a vertex of G. Suppose that two new paths $P = v(v_{k+1})v_k \cdots v_2v_1$ and $Q = v(u_{l+1})u_l \cdots u_2u_1$ of length k and l ($k \geq l \geq 1$) are attached to G at $v(= v_{k+1} = u_{l+1})$, respectively, to form a new graph $G_{k,l}$ (shown in Fig. 2), where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_l$ are distinct. Let

$$G_{k+1,l-1} = G_{k,l} - u_1u_2 + v_1u_1.$$

We call that $G_{k+1,l-1}$ is obtained from $G_{k,l}$ by grafting an edge (see Fig. 2).

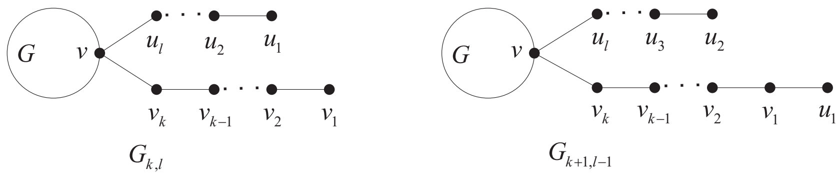

Figure 2: Grafting an edge.

Lemma 2.8 ([10]) *Let $G_{k,l}$ and $G_{k+1,l-1}$ ($k \geq l \geq 1$) be the graphs as defined above. Let $\boldsymbol{x}$ be a Fiedler vector of $G_{k,l}$. Then*

$$\alpha(G_{k,l}) \geq \alpha(G_{k+1,l-1}).$$

Moreover, the inequality is strict if either $x_{v_1} \neq 0$ or $x_{u_1} \neq 0$.

Lemma 2.9 ([9]) *Let u and v be two vertices of G. Suppose that two new paths $P = vv_k \cdots v_2v_1$ and $Q = uu_l \cdots u_2u_1$ of length k and l ($k \geq l \geq 1$) are attached to G at v and u, respectively, to form a new graph $H_{k,l}$, where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_l$ are distinct. Let $\boldsymbol{x}$ be a Fiedler vector of $H_{k,l}$ and let*

$$H'_{k+l} = H_{k,l} - vv_k + u_1v_k \quad \text{and} \quad H''_{k+l} = H_{k,l} - uu_l + v_1u_l.$$

If $x_{v_1}x_{u_1} \geq 0$, then we have $\alpha(H_{k,l}) \geq \min\{\alpha(H'_{k+l}),\, \alpha(H''_{k+l})\}$.

Lemma 2.10 ([16]) *Let G be a connected graph of order n. Suppose that $v_1, \ldots, v_s$ ($s \geq 2$) are non-adjacent vertices of G and $N(v_1) = \cdots = N(v_s)$. Let G_t be a graph obtained from G by adding any t ($0 \leq t \leq \frac{s(s-1)}{2}$) edges among $v_1, \ldots, v_s$. If $\alpha(G) \neq d(v_1)$, then we have $\alpha(G) = \alpha(G_t)$.*

Lemma 2.11 ([15]) *Let μ be a Laplacian eigenvalue of G afforded by eigenvector $\boldsymbol{x}$. If $x_u = x_v$, then μ is also a Laplacian eigenvalue of G' afforded by $\boldsymbol{x}$, where G' is the graph obtained from G by deleting or adding an edge $e = uv$ depending on it is or not an edge of G.*

Lemma 2.12 *Let $e = uv$ be an edge of G, and $\boldsymbol{x}$ be a Fiedler vector of G. If $x_u \neq x_v$, then $\alpha(G) > \alpha(G-e)$.*

Proof. From (1.2), we have $\alpha(G) = \boldsymbol{x}^T L(G)\boldsymbol{x} > \boldsymbol{x}^T L(G-e)\boldsymbol{x} \geq \alpha(G-e)$. $\square$

3 Main results

Firstly, we introduce some notation that are used in this section. Let $\mathscr{G}^+_{n,\omega}$ ($2 \leq \omega \leq n$) be the set of all connected graphs which consist of a clique K_ω and ω trees attached at each vertex of K_ω. If

$G \in \mathscr{G}_{n,\omega}^{+}$, then G consists of a complete graph K_ω with vertices $v_1, v_2, \ldots, v_\omega$ and ω trees $T_1, T_2, \ldots, T_\omega$ ($|V(T_1)| \geq |V(T_2)| \geq \cdots \geq |V(T_\omega)| \geq 1$) attached at the vertices $v_1, v_2, \ldots, v_\omega$, respectively. Clearly, $|V(T_1)| + |V(T_2)| + \cdots + |V(T_\omega)| = n$. Then for each $G \in \mathscr{G}_{n,\omega}^{+}$, we write $G = K_\omega(T_1, T_2, \ldots, T_\omega)$. We also write $G = K_\omega(l_1, l_2, \ldots, l_\omega)$ instead of $G = K_\omega(P_{l_1+1}, P_{l_2+1}, \ldots, P_{l_\omega+1})$, where $l_1 \geq l_2 \geq \cdots \geq l_\omega \geq 0$ and $l_1 + l_2 + \cdots + l_\omega + \omega = n$. If $l_i = 0$, then we write $K_\omega(l_1, \ldots, l_{i-1})$ to instead of $K_\omega(l_1, \ldots, l_{i-1}, 0, 0, \ldots, 0)$ for convenience. Clearly, $K_2(n-2) = PK_{n-2,2} = P_n$ and $PK_{n-\omega,\omega} = K_\omega(n-\omega) \in \mathscr{G}_{n,\omega}^{+} \subset \mathscr{G}_{n,\omega}$.

Lemma 3.1 *When $\omega \geq 3$, we have $\alpha(K_\omega(k,l)) > \alpha(PK_{n-\omega,\omega})$, where $k \geq l \geq 1$ and $k + l + \omega = n$.*

Proof. The vertex labelings for $K_\omega(k,l)$ and $PK_{n-\omega,\omega}$ are shown in Fig. 3, respectively.

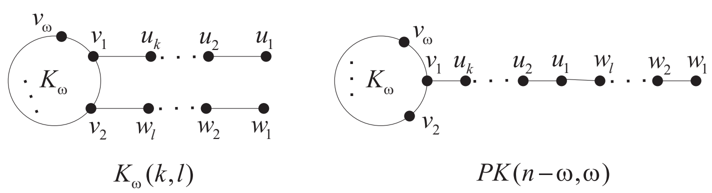

Figure 3: Vertex labelings for $K_\omega(k,l)$ and $PK_{n-\omega,\omega}$.

From Lemma 2.1, we have

$$\Phi(K_\omega(k,l)) = \Phi(PK_{k,\omega})\Phi(P_l) - \Phi(PK_{k,\omega})\Phi(B_{l-1}) - \Phi(L_{v_2}(PK_{k,\omega}))\Phi(P_l),$$
$$\Phi(PK_{n-\omega,\omega}) = \Phi(PK_{k,\omega})\Phi(P_l) - \Phi(PK_{k,\omega})\Phi(B_{l-1}) - \Phi(L_{u_1}(PK_{k,\omega}))\Phi(P_l).$$

Then

$$\Phi(K_\omega(k,l)) - \Phi(PK_{n-\omega,\omega}) = \Phi(P_l)[\Phi(L_{u_1}(PK_{k,\omega})) - \Phi(L_{v_2}(PK_{k,\omega}))].$$

By a similar proof of Lemma 2.1 (see [8]), we have

$$\Phi(L_{u_1}(PK_{k,\omega})) = \Phi(K_\omega)\Phi(B_{k-1}) - \Phi(K_\omega)\Phi(H_{k-2}) - \Phi(L_{v_1}(K_\omega))\Phi(B_{k-1}). \tag{3.1}$$

It is well-known that $\Phi(K_\omega) = x(x-\omega)^{\omega-1}$, $\Phi(L_{v_1}(K_\omega)) = \Phi(L_{v_2}(K_\omega)) = (x-1)(x-\omega)^{\omega-2}$, $\Phi(L_{\{v_1,v_2\}}(K_\omega)) = (x-2)(x-\omega)^{\omega-3}$ (please see [7]). Also Lemma 2.3 implies that $x\Phi(B_{k-1}) - x\Phi(H_{k-2}) = \Phi(P_k)$. Then by applying Lemma 2.2, Eq. (3.1) becomes

$$\begin{aligned}\Phi(L_{u_1}(PK_{k,\omega})) &= (x-\omega)^{\omega-1}(x\Phi(B_{k-1}) - x\Phi(H_{k-2})) - \Phi(L_{v_1}(K_\omega))\Phi(B_{k-1}) \\ &= (x-\omega)^{\omega-1}\Phi(P_k) - \Phi(L_{v_1}(K_\omega))\Phi(B_{k-1}),\end{aligned}$$

Similarly we have

$$\begin{aligned}\Phi(L_{v_2}(PK_{k,\omega})) &= \Phi(L_{v_2}(K_\omega))\Phi(P_k) - \Phi(L_{v_2}(K_\omega))\Phi(B_{k-1}) - \Phi(L_{\{v_1,v_2\}}(K_\omega))\Phi(P_k) \\ &= (x-\omega)^{\omega-3}[x^2 - (\omega+2)x + \omega + 2]\Phi(P_k) - \Phi(L_{v_2}(K_\omega))\Phi(B_{k-1}).\end{aligned}$$

Therefore,

$$\Phi(K_\omega(k,l)) - \Phi(PK_{n-\omega,\omega}) = (\omega-2)(\omega+1-x)(x-\omega)^{\omega-3}\Phi(P_l)\Phi(P_k).$$

Let $\alpha = \alpha(PK_{n-\omega,\omega})$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the Laplacian eigenvalues of $K_\omega(k,l)$. By Lemma 2.4, we have $\alpha < \alpha(P_l)$ and $\alpha < \alpha(P_k)$. Then $\omega + 1 - \alpha > 0$. Moreover, Lemma 2.4 implies that $\alpha \leq \mu_{n-2}(PK_{n-\omega,\omega} - u_1 w_l) = \mu_{n-2}(K_\omega(k,l) - v_2 w_l) \leq \mu_{n-2}(K_\omega(k,l))$. Therefore,

$$\begin{aligned}\Phi(K_\omega(k,l);\alpha) - \Phi(PK_{n-\omega,\omega};\alpha) =& \alpha(\alpha - \mu_{n-1})\cdots(\alpha - \mu_1) \\ =& (\omega-2)(\omega+1-\alpha)(\alpha-\omega)^{\omega-3}\Phi(P_l;\alpha)\Phi(P_k;\alpha).\end{aligned}$$

i.e.,

$$\begin{aligned}&\alpha(\mu_{n-1} - \alpha)\cdots(\mu_1 - \alpha) \\ =&(-1)^4\alpha^2(\omega-2)(\omega+1-\alpha)(\omega-\alpha)^{\omega-3}\underbrace{\prod_{i=1}^{l-1}(\mu_i(P_l)-\alpha)}_{>0}\underbrace{\prod_{i=1}^{k-1}(\mu_i(P_k)-\alpha)}_{>0} > 0.\end{aligned}$$

That is $\mu_{n-1} > \alpha$. The proof is completed. □

Theorem 3.2 *Among all graphs in $\mathscr{G}^+_{n,\omega}$, $2 \leq \omega \leq n$, the minimum algebraic connectivity is attained uniquely at $PK_{n-\omega,\omega}$.*

Proof. For each $K_\omega(T_1, T_2, \ldots, T_\omega) \in \mathscr{G}^+_{n,\omega}$, let $|V(T_i)| = l_i + 1$ for $i = 1, 2, \ldots, \omega$, where $l_1 \geq l_2 \geq \cdots \geq l_\omega \geq 0$ and $l_1 + l_2 + \cdots + l_\omega = n - \omega$.

If $l_2 = 0$ and $K_\omega(T_1, T_2, \ldots, T_\omega)$ is not isomorphic to $PK_{n-\omega,\omega}$, then Lemma 2.8 implies that $\alpha(K_\omega(T_1, T_2, \ldots, T_\omega)) \geq \alpha(K^+_\omega(i))$ for some i ($1 \leq i \leq n-2$), where $K^+_\omega(i)$ is shown in Fig. 4. Let $\boldsymbol{x}$ be a Fiedler vector of $K^+_\omega(i)$. Then $x_{v_n} \neq 0$ or $x_{v_{n-1}} \neq 0$. (Otherwise, (1.1) implies that $x_{v_1} = x_{v_{\omega+1}} = \cdots = x_{v_{n-1}} = x_{v_n} = 0$. Then by Lemma 2.11, we have $\alpha(K^+_\omega(i))$ is a Laplacian eigenvalue of K_ω. This is impossible since $\alpha(K^+_\omega(i)) \leq 1$ (c.f. [6])). Thus Lemma 2.8 implies that $\alpha(K^+_\omega(i)) > \alpha(PK_{n-\omega,\omega})$. The result follows.

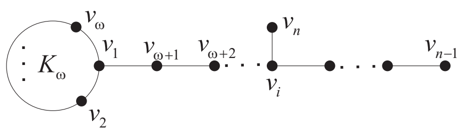

Figure 4: Graph $K^+_\omega(i)$, where $i = 1, \omega+1, \ldots, n-2$.

If $l_2 \neq 0$, then Lemma 2.8 implies that $\alpha(K_\omega(T_1, T_2, \ldots, T_\omega)) \geq \alpha(K_\omega(l_1, l_2, \ldots, l_\omega))$. Moreover, by Lemmas 2.9 and 3.1, we have

$$\alpha(K_\omega(l_1, l_2, \ldots, l_\omega)) \geq \alpha(K_\omega(l_1 + l_3 + \cdots + l_\omega, l_2)) > \alpha(K_\omega(l_1 + l_2 + \cdots + l_\omega)) = \alpha(PK_{n-\omega,\omega}).$$

The result follows. □

Theorem 3.3 *For $\omega \leq n-1$, we have* $\min\left\{\frac{(\omega+1)-\sqrt{(\omega+1)^2-4}}{2}, \alpha(P_{2(n-\omega)-1})\right\} \leq \alpha(PK_{n-\omega,\omega}) \leq \alpha(P_{n-\omega+2})$.

Proof. Since $\omega \leq n-1$, Lemma 2.10 implies that $\alpha(PK_{n-\omega,\omega}) = \alpha(PS_{n-\omega,\omega})$, where $PS_{n-\omega,\omega}$ (shown in Fig. 5) is a graph of order n obtained from the path $P_{n-\omega}$ and the star S_ω by adding an edge between an end vertex of $P_{n-\omega}$ and the center of S_ω. Moreover, by Corollary 2.5, we have

$$\alpha(PS_{n-\omega,\omega}) \leq \alpha(PS_{n-\omega,\omega} - \{v_3, \ldots, v_\omega\}) = \alpha(P_{n-\omega+2}).$$

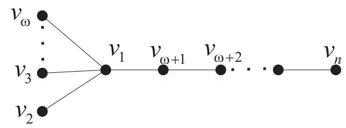

Figure 5: Graph $PS_{n-\omega,\omega}$.

On the other hand, Lemma 2.6 implies that $\tau(L_{v_{\omega+1}}(PK_{n-\omega,\omega})) \leq \alpha(PK_{n-\omega,\omega})$, where $v_{\omega+1}$ is described in Fig. 5. Since

$$L_{v_{\omega+1}}(PK_{n-\omega,\omega}) = L_{V_1}(PK_{n-\omega,\omega}) \oplus B_{n-\omega-1},$$

where $\oplus$ is the direct sum of matrices and $V_1 = \{v_{\omega+1}, \ldots, v_n\}$, and

$$\Phi(L_{V_1}(PK_{n-\omega,\omega})) = (x-\omega)^{\omega-2}[x^2 - (\omega+1)x + 1],$$

$\tau(L_{v_{\omega+1}}(PK_{n-\omega,\omega})) = \min\left\{\frac{(\omega+1)-\sqrt{(\omega+1)^2-4}}{2}, \tau(B_{n-\omega-1})\right\}$.
Moreover, by Lemma 2.7, we have $\tau(B_{n-\omega-1}) = \alpha(P_{2(n-\omega)-1})$. Thus the result follows. $\square$

Lemma 3.4 *When $\omega \leq n-1$, for any $e \notin E(PK_{n-\omega,\omega})$, we have $\alpha(PK_{n-\omega,\omega}) < \alpha(PK_{n-\omega,\omega}+e)$.*

Proof. If $\omega = n-1$, since $\alpha(PK_{1,n-1}) = 1 < \alpha(PK_{1,n-1}+e) = 2$ for any $e \notin E(PK_{1,n-1})$, then the result follows. Now, we consider $\omega < n-1$. Let $e = v_iv_j$, where $i, j = 1, 2, \ldots, n$ and $i < j$. The vertex labeling for $PK_{n-\omega,\omega}$ is shown in Fig. 1.

If $i = 1$ and $j = \omega+2, \ldots, n$ or $i = 2, 3, \ldots, \omega$ and $j = n$, then Lemma 2.4 and Theorem 3.2 imply that

$$\alpha(PK_{n-\omega,\omega} + v_iv_j) \geq \alpha(PK_{n-\omega,\omega} + v_iv_j - v_{\omega+1}v_{\omega+2}) > \alpha(PK_{n-\omega,\omega})$$

since $PK_{n-\omega,\omega} + v_iv_j - v_{\omega+1}v_{\omega+2} \in \mathscr{G}^+_{n,\omega}$ and $PK_{n-\omega,\omega} + v_iv_j - v_{\omega+1}v_{\omega+2}$ is not isomorphic to $PK_{n-\omega,\omega}$.

If $i, j = \omega+1, \ldots, n$, then the same reasoning implies that

$$\alpha(PK_{n-\omega,\omega} + v_iv_j) \geq \alpha(PK_{n-\omega,\omega} + v_iv_j - v_{i+1}v_{i+2}) > \alpha(PK_{n-\omega,\omega}).$$

If $i = 2, 3, \ldots, \omega$ and $j = \omega+2, \ldots, n-1$, then the same reasoning implies that

$$\alpha(PK_{n-\omega,\omega} + v_iv_j) \geq \alpha(PK_{n-\omega,\omega} + v_iv_j - v_1v_{\omega+1}) > \alpha(PK_{n-\omega,\omega}).$$

Suppose $i = 2, 3, \ldots, \omega$ and $j = \omega+1$. Let $\boldsymbol{x}$ be a Fiedler vector of $PK_{n-\omega,\omega} + v_iv_{\omega+1}$. If $x_{v_{\omega+1}} \neq x_{v_i}$ (or $x_{v_{\omega+1}} \neq x_{v_1}$), then Lemma 2.12 implies that $\alpha(PK_{n-\omega,\omega} + v_iv_{\omega+1}) > \alpha(PK_{n-\omega,\omega})$ (or $\alpha(PK_{n-\omega,\omega} + v_iv_{\omega+1}) > \alpha(PK_{n-\omega,\omega} + v_iv_{\omega+1} - v_1v_{\omega+1}) = \alpha(PK_{n-\omega,\omega})$), the result follows; if $x_{v_{\omega+1}} = x_{v_i}$, $x_{v_{\omega+1}} = x_{v_1}$ and $\alpha(PK_{n-\omega,\omega} + v_iv_{\omega+1}) = \alpha(PK_{n-\omega,\omega})$, then Lemma 2.4 implies that $\alpha(PK_{n-\omega,\omega}) \leq \mu_{n-2}(PK_{n-\omega,\omega} - v_1v_{\omega+1}) = \alpha(P_{n-\omega})$. On the other hand, since $x_{v_{\omega+1}} = x_{v_i}$ and $x_{v_{\omega+1}} = x_{v_1}$, Lemma 2.11 implies that $\alpha(PK_{n-\omega,\omega} + v_iv_{\omega+1}) = \alpha(PK_{n-\omega,\omega})$ is also a Laplacian eigenvalue of $PK_{n-\omega,\omega} - v_1v_{\omega+1} = K_\omega \cup P_{n-\omega}$. That is $\alpha(PK_{n-\omega,\omega} + v_iv_{\omega+1}) = \alpha(PK_{n-\omega,\omega}) = \alpha(P_{n-\omega})$. Moreover, by Theorem 3.3, we have $\alpha(P_{n-\omega}) \leq \alpha(P_{n-\omega+2})$. This contradicts the fact that $\alpha(P_{n-\omega}) > \alpha(P_{n-\omega+2})$, which has been proved in Lemma 2.7. $\square$

Lemma 3.5 $\alpha(PK_{n-\omega,\omega}) > \alpha(PK_{n-\omega+1,\omega-1})$ *for $3 \leq \omega \leq n$.*

Proof. When $\omega = n$, it is easy to see that $\alpha(PK_{0,n}) = n > \alpha(PK_{1,n-1}) = 1$. When $3 \leq \omega < n$, then Lemmas 3.4 and 2.4 imply that $\alpha(PK_{n-\omega+1,\omega-1}) < \alpha(PK_{n-\omega+1,\omega-1} + e) \leq \alpha(PK_{n-\omega,\omega})$, where $e \notin E(PK_{n-\omega+1,\omega-1})$. Thus the result follows. $\square$

Theorem 3.6 *Among all graphs in $\mathscr{G}_{n,\omega}$, $2 \leq \omega \leq n$, the minimum algebraic connectivity is attained uniquely at $PK_{n-\omega,\omega}$.*

Proof. If $\omega = n$, then only one graph $K_n \in \mathscr{G}_{0,n}$; if $\omega = n-1$, the result follows from Lemma 3.4; if $\omega = 2$, since it is known that the path P_n is the graph with minimum algebraic connectivity among all graphs of order n [16], the result follows. In what follows, we consider $2 < \omega < n-1$.

If $G \in \mathscr{G}^{+}_{n-\omega,\omega}$ and G is not isomorphic to $PK_{n-\omega,\omega}$, then the result follows from Theorem 3.2.

Suppose $G \in \mathscr{G}_{n-\omega,\omega}$ and $G \notin \mathscr{G}^{+}_{n-\omega,\omega}$. Let G' be a graph obtained from G by deleting some edges such that $G' \in \mathscr{G}^{+}_{n-\omega,\omega}$. If $G' \cong PK_{n-\omega,\omega}$, then the result follows from Lemma 3.4. If G' is not isomorphic to $PK_{n-\omega,\omega}$, then the result follows from Theorem 3.2. $\square$

Finally, applying Lemma 3.5 and Theorem 3.6, the following corollary is obtained immediately.

Corollary 3.7 *Among all connected graphs of order n, the minimum algebraic connectivity is attained uniquely at P_n.*

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