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Full Friendly Index Sets and Full Product-cordial Index Sets of Twisted Cylinders*

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Abstract

Let $G=(V,E)$ be a connected simple graph. A labeling $f:V\to\mathbb{Z}_2$ induces two edge labelings $f^+, f^*: E\to\mathbb{Z}_2$ defined by $f^+(xy)=f(x)+f(y)$ and $f^*(xy)=f(x)f(y)$ for each $xy\in E$. For $i\in\mathbb{Z}_2$, let $v_f(i)=|f^{-1}(i)|$, $e_{f^+}(i)=|(f^+)^{-1}(i)|$ and $e_{f^*}(i)=|(f^*)^{-1}(i)|$. A labeling f is called friendly if $|v_f(1)-v_f(0)|\le 1$. For a friendly labeling f of a graph G, the friendly index of G under f is defined by $i_f^+(G)=e_{f^+}(1)-e_{f^+}(0)$. The set $\{i_f^+(G)\mid f \text{ is a friendly labeling of } G\}$ is called the full friendly index set of G. Also, the product-cordial index of G under f is defined by $i_f^*(G)=e_{f^*}(1)-e_{f^*}(0)$. The set $\{i_f^*(G)\mid f \text{ is a friendly labeling of } G\}$ is called the full product-cordial index set of G. In this paper, we will determine full friendly index sets and full product-cordial index sets of twisted cylinders.

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1. Introduction

In this paper, all graphs are simple and connected. All undefined symbols and concepts may be looked up from [1]. Let $G=(V,E)$ be a connected simple graph. A labeling $f:V\to\mathbb{Z}_2$ induces two edge labelings $f^+, f^*: E\to\mathbb{Z}_2$ defined by $f^+(xy)=f(x)+f(y)$ and $f^*(xy)=f(x)f(y)$ for each $xy\in E$. For $i\in\mathbb{Z}_2$, let $v_f(i)=|f^{-1}(i)|$, $e_{f^+}(i)=|(f^+)^{-1}(i)|$ and $e_{f^*}(i)=|(f^*)^{-1}(i)|$. A labeling f is called *friendly* if $|v_f(1)-v_f(0)|\le 1$. For a friendly labeling f

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of a graph G, the *friendly index* of G under f is defined by $i_f^+(G) = e_{f^+}(1) - e_{f^+}(0)$. The set

$$\mathrm{FFI}(G) = \{i_f^+(G) \mid \text{is a friendly labeling of } G\}$$

is called the *full friendly index set* of G. The set of the absolute value of all friendly indices of G is called the *friendly index set* of G. Also the *product-cordial index* of G under f is defined by $i_f^*(G) = e_{f^*}(1) - e_{f^*}(0)$. The set

$$\mathrm{FPCI}(G) = \{i_f^*(G) \mid f \text{ is a friendly labeling of } G\}$$

is called the *full product-cordial index set* of G. The set of the absolute value of all product-cordial indices of G is called the *product-cordial index set* of G. Throughout this paper, we will use the term 'labeling' to mean a vertex labeling whose values are taken from $\mathbb{Z}_2$. Note that $i_f^+(G)$ and $i_f^*(G)$ can be extended to any labeling.

Friendly index set was initiated by Lee and Ng in 2004 [6]. More about friendly index sets of graphs can be found in [3,4,9]. Full friendly index set was first introduced by Shiu and Kwong [10] in 2007 (published in 2008). The friendly index sets or full friendly index sets of the graphs $P_m \times P_n$, $C_m \times C_n$ and $C_m \times P_n$ were found [8,10,12–14,16]. Recently Gao determined the full friendly index set of $P_m \times P_n$, but he used the terms 'edge difference set' instead of 'full friendly index set' and 'direct product' instead of 'Cartesian product' in [2]. Friendly index is related to the eigenvalues of a graph (interested readers please see [15]).

The full product-cordial index set was first introduced by Shiu and Kwong in 2011 [11]. They determined full product-cordial index set of torus and cycles. They also showed the relationship between the product-cordial index and the friendly index of regular graphs. More about product-cordial index sets of graphs can be found in [5,7]. Following is a result shown in [11].

Lemma 1.1 ([11, Corollary 2.3]). *Let f be a friendly labeling of G with q edges. If G is an r-regular graph of even order. Then $i_f^*(G) = -(q + i_f^+(G))/2$.*

So, it suffices to deal with either friendly index or product-cordial index of a graph.

For a fixed labeling f, a vertex v is called a *k-vertex* if $f(v) = k$, similarly an edge e is called a *k-edge* if $f^+(e) = k$, and an edge is called an *(i, j)-edge* if it is incident with an i-vertex and a j-vertex. We define the number of (i, j)-edges by $E_f(i, j)$.

Lemma 1.2 ([13, Lemma 2.1]). *Let f be any labeling of a graph G with q edges. If the degree sum of 1-vertices is s, then $i_f^+(G) = 2s - 4E_f(1,1) - q$.*

Lemma 1.3 ([13, Corollary 2.2]). *Let f be any labeling of a graph G with q edges. If the degree sum of 1-vertices is s, then $E_f(0,0) = q - s + E_f(1,1)$.*

Corollary 1.1 ([13, Corollary 2.3]). *Let f be a friendly labeling of a graph G with q edges. If G is regular of even order, then $i_f^+(G) = q - 4E_f(1,1)$ and $E_f(0,0) = E_f(1,1)$.*

2. Extreme Friendly Indices of Twisted Cylinders

A special class of cubic graphs is the class of permutation cubic graphs. For $m \geq 3$, a *permutation cubic graph* on $2m$ vertices is defined by taking two vertex-disjoint cycles on

m vertices and adding a perfect matching between the vertices of the two cycles. Namely, let two cycles be $C = x_1x_2\cdots x_mx_1$ and $C^* = y_1y_2\cdots y_my_1$ and let $\sigma \in \mathfrak{S}_m$, the permutation group on the set $\{1,2,\ldots,m\}$. The permutation cubic graph $\mathcal{P}(m;\sigma) = (V,E)$ is a simple graph with $V = \{x_1,\ldots,x_m,y_1,\ldots,y_m\}$ and $E = E(C)\cup E(C^*)\cup\{x_iy_{\sigma(i)} \mid 1 \le i \le m\}$. Throughout this paper, we shall keep the above notation.

The *twisted cylinder* is a particular permutation cubic graph on $4n$ vertices ($n \ge 2$). Namely, the twisted cylinder $TC(2n) = \mathcal{P}(2n;\sigma)$, where $\sigma = (1,2)(3,4)\cdots(2n-1,2n)$ (the product of n transpositions).

From Corollary 1.1, if f is a friendly labeling of $TC(2n)$, then $i^+_f(TC(2n)) = 6n - 4E_f(1,1)$. If we label x_{2i} and y_{2i} by 1 for $1 \le i \le n$ and others by 0, then all edges of $TC(2n)$ are $(1,0)$-edges. We shall denote this labeling by $f_{\max}$, since it induces the friendly index attaining the maximum value. That is $i^+_{f_{\max}}(TC(2n)) = 6n$. The next question is what the maximum number of $(1,1)$-edges can be made.

We define a friendly labeling of $TC(2n)$ by $f_{\min}(x_i) = f_{\min}(y_i) = 1$ for $1 \le i \le n$ and $f_{\min}(x_i) = f_{\min}(y_i) = 0$ for $n+1 \le i \le 2n$. Then we get that

$$E_{f_{\min}}(1,1) = \begin{cases} 3n-2, & n \text{ is even;} \\ 3n-3, & n \text{ is odd.} \end{cases}$$

Suppose g is a friendly labeling of $TC(2n)$. Let G be the subgraph of $TC(2n)$ induced by all 1-vertices. So G contains $2n$ vertices. Suppose there are v_i vertices of degree i in G, $0 \le i \le 3$. By Handshaking Lemma, we have $2E_g(1,1) = 2n - v_0 + v_2 + 2v_3 \le 2n + v_2 + 2v_3$. In order to maximize $E_g(1,1)$, v_0 must be zero. If all vertices of G lie on C or C^*, then $v_2 = 2n$ and $E_g(1,1) = 2n \le E_{f_{\min}}(1,1)$. Moreover, the equality holds only when $n = 2$.

For the case $n \ge 3$, in order to maximize $E_g(1,1)$ by finding a friendly labeling g, we may assume that both C and C^* contain 1-vertices. Thus, there are at least two 1-vertices, which lie on C (respectively C^*), of degree less than 3 in G. Hence $v_3 \le 2n-4$. So we have to maximize $E_g(1,1) = n + v_2/2 + v_3$ subject to $0 \le v_2 + v_3 \le 2n$ and $v_3 \le 2n-4$. It is easy to see that $E_g(1,1) = 3n-2$ is the algebraically maximum value only when $v_3 = 2n-4$ and $v_2 = 4$.

Suppose g is a friendly labeling with $E_g(1,1) = 3n-2$, $n \ge 3$. Then both C and C^* contain 1-vertices, $v_2 = 4$ and $v_3 = 2n-4$. Hence the graphs $H = G\cap C$ and $H^* = G\cap C^*$ contain exactly two vertices of degree one, respectively, where G was defined above. That means H and H^* are paths. Without loss of generality, we may assume $H = x_1x_2\cdots x_k$ and $H^* = y_1y_2\cdots y_h$, where $2n > k \ge h > 0$ and $h+k = 2n$. Since x_k is of degree 2 in G and $k \ge h$, x_k is adjacent to y_{k-1} in G. This implies $k-1 \le h$ and hence $k = h = n$. Since x_n is adjacent to y_{n-1}, n must be even.

So when n is even, the labeling $f_{\min}$ defined above attaches the maximum number of $(1,1)$-edges. Hence we have

Theorem 2.1. *Suppose n is even. Then the maximum and minimum values of the friendly index of $TC(2n)$ are $6n$ and $-6n+8$, respectively.*

Now suppose n is odd. By the discussion before Theorem 2.1, $E_g(1,1) \le 3n-3$ for every friendly labeling g. It is attached by putting $g = f_{\min}$. So we have

Theorem 2.2. *Suppose n is odd. Then the maximum and minimum values of the friendly index of $TC(2n)$ are $6n$ and $-6n+12$, respectively.*

3. Full Friendly Index Sets of Twisted Cylinders

In this section, we will determine the full friendly index sets of twisted cylinders. Firstly, we show a non-existence case.

Lemma 3.1 ([10, Corollary 2 or 5] [13, Lemma 2.9]). *In a cycle, the number of* 1*-edges must be even under any labeling.*

Corollary 3.1. *In an even cycle, the numbers of* 1*-edges and* 0*-edges are even under any labeling.*

Theorem 3.1. *Suppose g is a friendly labeling of $TC(2n)$ for $n \geq 2$. Then $E_g(1,1) \neq 1$.*

*Proof*Suppose g is a friendly labeling of $TC(2n)$ such that $E_g(1,1) = 1$. Note that $E_g(1,1) = E_g(0,0) = 1$ (Corollary 1.1). Let this edge be e. Hence e is a 0-edge.

Case 1. Suppose $e \in E(C)$ or $e \in E(C^*)$. Without loss of generality, we may assume $e = x_1x_2 \in E(C)$ or $e = x_2x_3 \in E(C)$. For the first case, since C and $x_1x_2y_1y_2x_1$ are even cycles with only common edge e, by Corollary 3.1 each of them contains another 0-edge, which must be a $(0,0)$-edge. For the last case, since C and $x_2x_3y_4y_3y_2y_1x_2$ are even cycles with only common edge e, we obtain a similar result. Then $E_g(0,0) \geq 2$ which is a contradiction.

Case 2. Suppose $e \notin E(C) \cup E(C^*)$. Without loss of generality, we may assume $e = x_1y_2$. Two even cycles $x_1x_2y_1y_2x_1$ and $x_1x_2x_3x_4y_3y_2x_1$ have two common edges e and x_1x_2. Since x_1x_2 is not a $(0,0)$-edge, similar to Case 1 each of these two even cycles contains another 0-edge, which must be a $(0,0)$-edge. Hence $E_g(0,0) \geq 2$ which is a contradiction. □

Lemma 3.2 ([13, Lemma 2.7]). *Let f be a labeling of a graph H such that $E_f(1,1) = k$. Suppose $uv \notin E(H)$, $f(u) = 1$ and $f(v) = 0$. Let u is adjacent to a 1-vertices, and v is adjacent to b 1-vertices. Let g be a labeling obtained from f by swapping the labels of u and v. Then $E_g(1,1) = k-a+b$.*

Lemma 3.3 ([13, Lemma 2.8]). *Let f be a labeling of a graph H such that $E_f(1,1) = k$. Suppose $uv \in E(H)$, $f(u) = 1$ and $f(v) = 0$. Let u is adjacent to a 1-vertices, and v is adjacent to b 1-vertices. Let g be a labeling obtained from f by swapping the labels of u and v. Then $E_g(1,1) = k-a+b-1$.*

Procedure A. Start from the initial labeling $f_{\max}$, for $1 \leq i \leq n$, swap the labels of y_{2i-1} and y_{2i} to get the labeling f_i.

Then by Lemma 3.3 $E_{f_1}(1,1) = 2$, $E_{f_i}(1,1) = E_{f_{i-1}}(1,1) + 1$ for $2 \leq i \leq n-1$, and $E_{f_n}(1,1) = n$.

Procedure B. Start from the labeling f_n, for $1 \leq i \leq n$, swap the labels of y_{2i-1} and x_{2i-1} to get the labeling g_i.

Then by Lemma 3.2 $E_{g_1}(1,1) = n+1$, $E_{g_i}(1,1) = E_{g_{i-1}}(1,1) + 1$ for $2 \leq i \leq n$. Note that $E_{g_n}(1,1) = 2n$.

Procedure C. Start from the labeling g_n, for $1 \le i \le n$, swap the labels of y_i and x_{n+i} to get the labeling h_i.

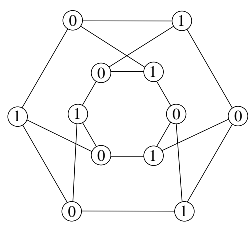
Then by Lemma 3.2 $E_{h_1}(1,1) = 2n-1$, $E_{h_i}(1,1) = E_{h_{i-1}}(1,1)+1$ for $2 \le i \le n-1$. And $E_{h_n}(1,1) = 3n-2$ if n is even, $E_{h_n}(1,1) = 3n-3$ if n is odd.

After performing Procedures A, B and C, we obtain the following theorem.

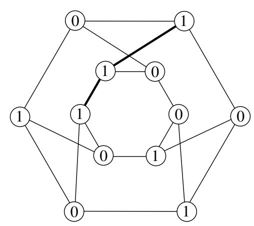
Theorem 3.2. *The full friendly index set of $TC(2n)$ is*

$$\{6n-4j \mid 0 \le j \le 3n-2,\, j \ne 1\} \text{ if } n \text{ is even,}$$
$$\{6n-4j \mid 0 \le j \le 3n-3,\, j \ne 1\} \text{ if } n \text{ is odd.}$$

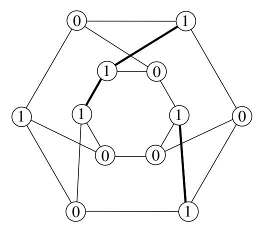
Example 3.1. Consider the graph $TC(6)$. Following the procedures A, B and C we have



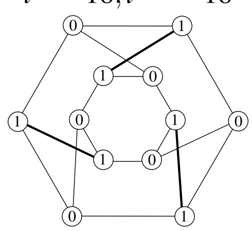
$E_{f_{\max}}(1,1) = 0$
$i^+ = 18, i^* = -18$



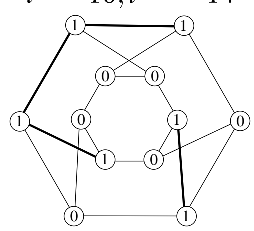
$E_{f_1}(1,1) = 2$
$i^+ = 10, i^* = -14$



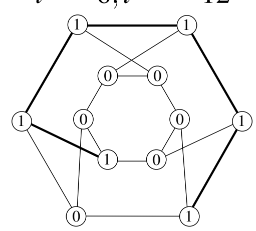
$E_{f_2}(1,1) = 3$
$i^+ = 6, i^* = -12$



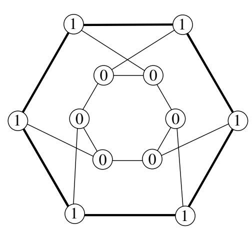
$E_{f_3}(1,1) = 3$
$i^+ = 6, i^* = -12$



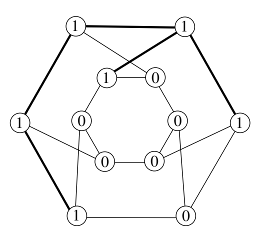
$E_{g_1}(1,1) = 4$
$i^+ = 2, i^* = -10$



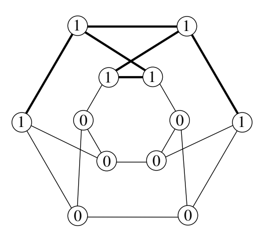
$E_{g_2}(1,1) = 5$
$i^+ = -2, i^* = -8$



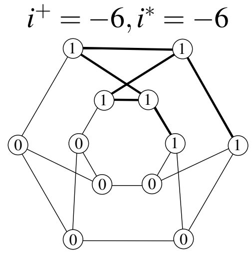
$E_{g_3}(1,1) = 6$
$i^+ = -6, i^* = -6$



$E_{h_1}(1,1) = 5$
$i^+ = -2, i^* = -8$



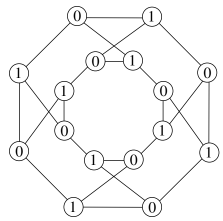
$E_{h_2}(1,1) = 6$
$i^+ = -6, i^* = -6$



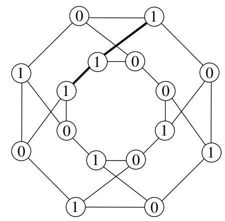
$E_{h_3}(1,1) = 6$
$i^+ = -6, i^* = -6$

So $FFI(TC(6)) = \{-6,-2,2,6,10,18\}$ and
$FPCI(TC(6)) = \{-18,-14,-12,-10,-8,-6\}$. □

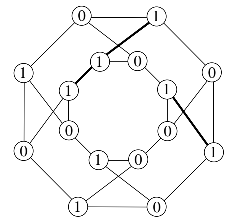
Example 3.2. Consider the graph $TC(8)$. Following the procedures A, B and C we have



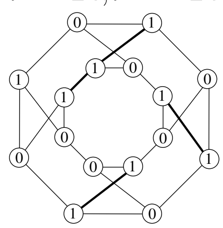
$E_{f_{\max}}(1,1) = 0$
$i^+ = 24, i^* = -24$



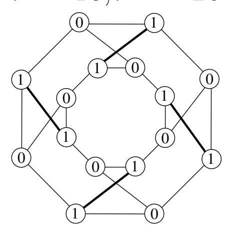
$E_{f_1}(1,1) = 2$
$i^+ = 16, i^* = -20$



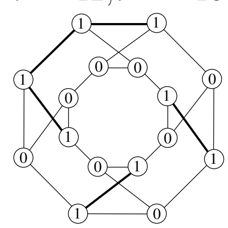
$E_{f_2}(1,1) = 3$
$i^+ = 12, i^* = -18$



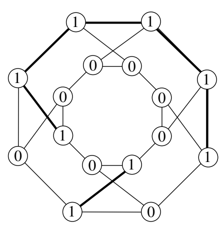
$E_{f_3}(1,1) = 4$
$i^+ = 8, i^* = -16$



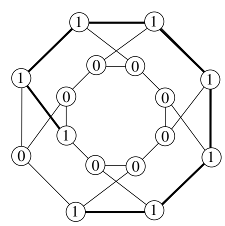
$E_{f_4}(1,1) = 4$
$i^+ = 8, i^* = -16$



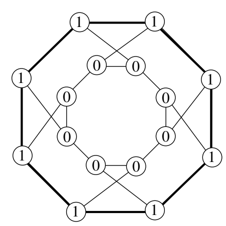
$E_{g_1}(1,1) = 5$
$i^+ = 4, i^* = -14$



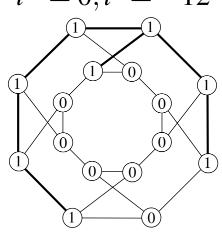
$E_{g_2}(1,1) = 6$
$i^+ = 0, i^* = -12$



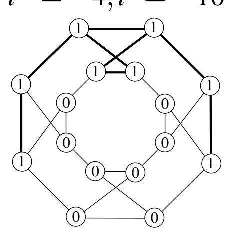
$E_{g_3}(1,1) = 7$
$i^+ = -4, i^* = -10$



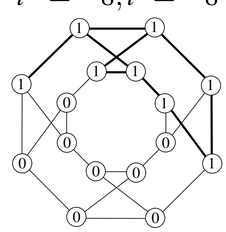
$E_{g_4}(1,1) = 8$
$i^+ = -8, i^* = -8$



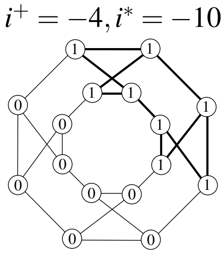
$E_{h_1}(1,1) = 7$
$i^+ = -4, i^* = -10$



$E_{h_2}(1,1) = 8$
$i^+ = -8, i^* = -8$



$E_{h_3}(1,1) = 9$
$i^+ = -12, i^* = -6$



$E_{h_4}(1,1) = 10$
$i^+ = -16, i^* = -4$

So $FFI(TC(8)) = \{-16,-12,-8,-4,0,4,8,12,16,24\}$,
$FPCI(TC(8)) = \{-24,-20,-18,-16,-14,-12,-10,-8,-6,-4\}$. $\square$

By Lemma 1.1 we have

Corollary 3.2. *The full product-cordial index set of $TC(2n)$ is*

$$\{-6n+2j \mid 0 \le j \le 3n-2, j \ne 1\} \text{ if } n \text{ is even,}$$
$$\{-6n+2j \mid 0 \le j \le 3n-3, j \ne 1\} \text{ if } n \text{ is odd.}$$

Corollary 3.3. *The friendly index set of $TC(2n)$ is*

$$\{6n-4j \mid 0 \le j \le 3n/2, j \ne 1\} \text{ if } n \text{ is even,}$$
$$\{6n-4j \mid 0 \le j \le (3n-1)/2, j \ne 1\} \text{ if } n \text{ is odd.}$$

Corollary 3.4. *The product-cordial index set of $TC(2n)$ is*

$$\{6n-2j \mid 0 \le j \le 3n-2, j \ne 1\} \text{ if } n \text{ is even,}$$
$$\{6n-2j \mid 0 \le j \le 3n-3, j \ne 1\} \text{ if } n \text{ is odd.}$$

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