Supermagic Labeling of an s-duplicate of $K_{n,n}$*

Wai Chee Shiu†, Peter Che Bor Lam† and Hee Lin Cheng†

Abstract

Let H and s be a graph and a positive integer respectively. An *s-duplicate* of H, denoted by sH, is a graph consisting of s identical components H. In this paper, we shall show that for $n \geq 2$, $sK_{n,n}$ is supermagic if and only if n is even or both s and n are odd.

Key words and phrases : Supermagic, orthogonal Latin square, balanced partition

2000 MSC: 05C78, 05C50

1. Introduction

The graph $G = (V, E)$ is called a (p, q)-graph if $|V| = p$ and $|E| = q$. If there exists a bijection

$$f : E \to \{1, \cdots, q\}$$

such that the *induced mapping* f^+ from V to $\mathbb{Z}$ defined by $f^+(u) = \sum_{uv \in E} f(uv)$ is a constant mapping, then G is called *supermagic* and f a *supermagic labeling* of G. The concept of supermagic was defined by Stewart [6, 7] in 1966. Hartsfield and Ringel [2] had also studied supermagic graphs. In 1998, Shiu, Lam and Lee [4] proved that for $m \geq 2$, $n \geq 2$ but $(m, n) \neq (2, 2)$, the composition of an m-cycle and the null graph on n vertices is edge-magic. Recently, they [5] proved that it is supermagic. All supermagic graphs are edge-magic, but the converse does not necessarily hold. For definition of "edge-magic" and "composition", see [4].

Let H and s be a graph and a positive integer respectively. An *s-duplicate* of H, denoted by sH, is a graph consisting of s identical components H. In this paper, we shall consider the supermagicness of $sK_{n,n}$.

2. The s-duplicate of $K_{n,n}$

We first note that $sK_{n,n}$ and has $2sn$ vertices and sn^2 edges. In the lemma that follows, we give a necessary condition for $sK_{n,n}$ to be supermagic. The main purpose of this paper is to show that this necessary condition is also sufficient with the exception of some trivial cases.

Lemma 2.1: *If $G = sK_{n,n}$ is supermagic, then $n(sn^2 + 1)$ is even.*

Proof: Let f be a supermagic labeling of $G = (V, E)$ and c be constant of the induced mapping f^+. Then

$$\begin{aligned} 2snc &= \sum_{u \in V} f^+(u) = \sum_{u \in V} \sum_{v \in N(u)} f(uv) \\ &= 2 \sum_{e \in E} f(e) = sn^2(sn^2 + 1). \end{aligned}$$

Hence $n(sn^2 + 1) = 2c$ is an even integer. ∎

*Partially supported by Faculty Research Grant, Hong Kong Baptist University.

†Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong.

Note that $n(sn^2+1)$ is even if and only if n is even or both n and s are odd. In the following theorem, we consider the trivial exceptional cases, i.e., when $n \le 2$, where this necessary condition is not sufficient:

Theorem 2.2: *When $n \le 2$, then $sK_{n,n}$ is supermagic if and only if $s = n = 1$.*

Proof: It is clear that $(sK_{1,1})$ is supermagic if and only if $s = 1$. Now $sK_{2,2}$ consists of s components, each of which is a 4-cycle. Because any graph containing a cycle as a component is not supermagic, $sK_{2,2}$ cannot be supermagic. ▮

3. A supermagic labeling of $sK_{n,n}$ when $n \ge 3$

Let $G = (V, E)$ be a simple graph and S be a set. Suppose $f : E \to S$ is a mapping. A *labeling matrix* of G associated with f is a matrix whose rows and columns are named by the vertices of G and the (u, v)-entry is $f(uv)$ if $uv \in E$, and is 0 otherwise. Thus a simple (p, q)-graph $G = (V, E)$ is supermagic if and only if there exists a bijection $f : E \to \{1, 2, \cdots, q\}$ such that the row sums and column sums of the labeling matrix of G associated with f are all equal.

Any labeling matrix of $sK_{n,n}$ may be expressed in the form:

$$\begin{pmatrix} B_1 & O_{2n} & \cdots & O_{2n} & O_{2n} \\ O_{2n} & B_2 & \cdots & O_{2n} & O_{2n} \\ \vdots & & \ddots & & \vdots \\ O_{2n} & O_{2n} & \cdots & B_{s-1} & O_{2n} \\ O_{2n} & O_{2n} & \cdots & O_{2n} & B_s \end{pmatrix} \quad \text{with} \quad B_i = \begin{pmatrix} O_n & A_i \\ A_i^T & O_n \end{pmatrix}, \qquad (1)$$

where A_i is an $n \times n$ matrix, $1 \le i \le s$, and O_j is the $j \times j$ zero matrix.

We can see that $sK_{n,n}$ is supermagic if and only if there exist $n \times n$ matrices $A_1, \cdots, A_s$ with distinct entries taken from the set $\{1, 2, \cdots, sn^2\}$, and with all row sums and all column sums are the same. Let S be a set containing sn elements. Let $\mathscr{P}$ be a partition of S. If each class of $\mathscr{P}$ contains n elements, and the sum of numbers in each class is a constant, then $\mathscr{P}$ is called an (s, n)-*balanced partition* of S. $\mathscr{P}$ is usually denoted by $\{C_1, \cdots, C_s\}$, where $C_i = (c_{i1}, c_{i2}, \cdots, c_{in})^T$ for $1 \le i \le s$. We shall first quote the following lemmas:

Lemma 3.1 [4]: *Suppose $s, n \ge 2$. If n is even or both s and n are odd, then $\{0, 1, \cdots, sn-1\}$ has an (s, n)-balanced partition.*

Lemma 3.2 [1, 3, 8]: *There exists a pair of orthogonal Latin squares of order n if $n \ge 3$ and $n \ne 6$.*

Suppose all elements in the first column of a square matrix X of order n are distinct, and any other column of X is a permutation of the first column. Let C be a column n-vector whose elements are all distinct. We shall denote by $X(C)$ a matrix whose first column is C, and all other columns are permutations of C in exactly the same manner as X. Note that if X is a Latin square, so is $X(C)$. The following theorem is the main result of this paper:

Theorem 3.3: *Let $n \ge 3$, $s \ge 1$ Then $sK_{n,n}$ is supermagic.*

Proof: Case (I): $n \ne 6$. By Lemma 3.1 and Lemma 3.2, we can find an (s, n)-balanced partition $\{C_1, \cdots, C_s\}$ of $\{0, 1, \cdots, sn-1\}$ and two orthogonal $n \times n$ Latin squares X and Y. Let $C = (1, 2. \cdots, n)^T$ and $A_i = nY(C_i) + X(C)$ for $i = 1, \cdots, s$. We then substitute A_i into (1).

Suppose the sum of elements in each C_i and in C is k_1 and k_0 respectively. Since both $nY(C_i)$ and $X(C)$ are Latin squares, the sum of elements each row and in each column of A_i is nk_1+k_0. Let

$x \in \{1, 2, \cdots, sn^2\}$, x can be written as $x = nl + a$ uniquely, where $0 \le l \le sn - 1$ and $1 \le a \le n$. Therefore, there exists a unique i such that $l \in C_i$. Since $nY(C_i)$ and $X(C)$ are orthogonal, the ordered pair (l, a) appears exactly once in A_i. Hence, all elements of $\{1, 2, \cdots, sn^2\}$ appear only in one of the matrices A_i, and exactly once that matrix.

Case (II): $n = 6$. Since a pair of 6×6 orthogonal Latin squares does not exist, the method in Case (I) does not work. Nevertheless, M. Suzuki[‡] found the following pair of orthogonal matrices:

$$Z = \begin{pmatrix} 1 & 6 & 6 & 1 & 6 & 1 \\ 5 & 5 & 2 & 2 & 5 & 2 \\ 3 & 4 & 4 & 4 & 3 & 3 \\ 4 & 3 & 3 & 3 & 4 & 4 \\ 2 & 2 & 5 & 5 & 2 & 5 \\ 6 & 1 & 1 & 6 & 1 & 6 \end{pmatrix} \text{ and } Z^T = \begin{pmatrix} 1 & 5 & 3 & 4 & 2 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 2 & 4 & 3 & 5 & 1 \\ 1 & 2 & 4 & 3 & 5 & 6 \\ 6 & 5 & 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}. \tag{2}$$

We shall use these matrices, in stead of orthogonal Latin squares, and the balanced partition to construct a supermagic labeling matrix of $sK_{6,6}$. Now $\{0, 1, \cdots, 6s - 1\}$ has an $(s, 6)$-balanced partition $\{C_1, \cdots, C_6\}$, where for $i = 1, 2, \cdots, s$,

$$C_i = (i - 1,\ 2s - i,\ 2s + i - 1,\ 4s - i,\ 4s + i - 1,\ 6s - i)^T.$$

Note that the sum of all elements of C_i is $18s - 3$. As in case (I), we put $A_i = nZ(C_i) + (Z(C))^T$ for $i = 1, \cdots, s$ and then substitute A_i into (1). The sum of each column of A_i is $n(18s - 3) + 21 = 3n(s - 1) + 21$. Since the elements of each row of $Z(C_i)$ are exactly two numbers of C_i whose sum is $6s - 1$, and each of these two numbers appears three times, the row sum of each A_i is also $3n(6s - 1) + 21$, same as the column sums. Let $x \in \{1, 2, \cdots, 36s\}$, then x can be written uniquely as $x = 6l + a$, where $0 \le l \le 6s - 1$ and $1 \le a < 6$. As in case (I), there exists a unique i such that $l \in C_i$. Since $nZ(C_i)$ and $(Z(C))^T$ are orthogonal, all elements of $\{1, 2, \cdots, sn^2\}$ appear only in one of the matrices A_i, and exactly once that matrix. ∎

Example 3.1: Let $s = 3$, $n = 4$ and $C = (1, 2, 3, 4)^T$. A (3,4)-balanced partition of $(0, 1, \cdots, 11)^T$ is $C_1 = (0, 5, 6, 11)^T$, $C_2 = (1, 4, 7, 10)^T$, and $C_3 = (2, 3, 8, 9)^T$.

$$X = \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} a & b & c & d \\ c & d & a & b \\ d & c & b & a \\ b & a & d & c \end{pmatrix}$$

are two orthogonal Latin squares. Then

$$X(C) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad Y(C_1) = \begin{pmatrix} 0 & 5 & 6 & 11 \\ 6 & 11 & 0 & 5 \\ 11 & 6 & 5 & 0 \\ 5 & 0 & 11 & 6 \end{pmatrix},$$

$$Y(C_2) = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 7 & 10 & 1 & 4 \\ 10 & 7 & 4 & 1 \\ 4 & 1 & 10 & 7 \end{pmatrix}, \quad Y(C_3) = \begin{pmatrix} 2 & 3 & 8 & 9 \\ 8 & 9 & 2 & 3 \\ 9 & 8 & 3 & 2 \\ 3 & 2 & 9 & 8 \end{pmatrix}.$$

[‡]Mutsumi Suzuki, see homepage
http://www.pse.che.tohoku.ac.jp/~msuzuki/MagicSquare.even.html

Therefore the required assignment is given by:

$$A_1 = 4Y(C_1) + X(C) = \begin{pmatrix} 1 & 22 & 27 & 48 \\ 26 & 45 & 4 & 23 \\ 47 & 28 & 21 & 2 \\ 24 & 3 & 46 & 25 \end{pmatrix},$$

$$A_2 = 4Y(C_2) + X(C) = \begin{pmatrix} 5 & 18 & 31 & 44 \\ 30 & 41 & 8 & 19 \\ 43 & 32 & 17 & 6 \\ 20 & 7 & 42 & 29 \end{pmatrix},$$

$$A_3 = 4Y(C_3) + X(C) = \begin{pmatrix} 9 & 14 & 35 & 40 \\ 34 & 37 & 12 & 15 \\ 39 & 36 & 13 & 10 \\ 16 & 11 & 38 & 33 \end{pmatrix}.$$

Example 3.2: Suppose $s = 4$ and $n = 6$. Let Z be given as in (2). Following the proof of Case (II) of Theorem 3.3, we obtain the following (4,6)-balanced partition of $\{0, 1, 2, \cdots, 23\}$: $C_1 = (0, 7, 8, 15, 16, 23)^T$, $C_2 = (1, 6, 9, 14, 17, 22)^T$, $C_3 = (2, 5, 10, 13, 18, 21)^T$ and $C_4 = (3, 4, 11, 12, 19, 20)^T$. Then

$$Z(C_1) = \begin{pmatrix} 0 & 23 & 23 & 0 & 23 & 0 \\ 16 & 16 & 7 & 7 & 16 & 7 \\ 8 & 15 & 15 & 15 & 8 & 8 \\ 15 & 8 & 8 & 8 & 15 & 15 \\ 7 & 7 & 16 & 16 & 7 & 16 \\ 23 & 0 & 0 & 23 & 0 & 23 \end{pmatrix}, Z(C_2) = \begin{pmatrix} 1 & 22 & 22 & 1 & 22 & 1 \\ 17 & 17 & 6 & 6 & 17 & 6 \\ 9 & 14 & 14 & 14 & 9 & 9 \\ 14 & 9 & 9 & 9 & 14 & 14 \\ 6 & 6 & 17 & 17 & 6 & 17 \\ 22 & 1 & 1 & 22 & 1 & 22 \end{pmatrix},$$

$$Z(C_3) = \begin{pmatrix} 2 & 21 & 21 & 2 & 21 & 2 \\ 18 & 18 & 5 & 5 & 18 & 5 \\ 10 & 13 & 13 & 13 & 10 & 10 \\ 13 & 10 & 10 & 10 & 13 & 13 \\ 5 & 5 & 18 & 18 & 5 & 18 \\ 21 & 2 & 2 & 21 & 2 & 21 \end{pmatrix}, Z(C_4) = \begin{pmatrix} 3 & 20 & 20 & 3 & 20 & 3 \\ 19 & 19 & 4 & 4 & 19 & 4 \\ 11 & 12 & 12 & 12 & 11 & 11 \\ 12 & 11 & 11 & 11 & 12 & 12 \\ 4 & 4 & 19 & 19 & 4 & 19 \\ 20 & 3 & 3 & 20 & 3 & 20 \end{pmatrix}.$$

Then $A_i = 6Z(C_i) + (Z(C))^T$ for $i = 1, 2, 3, 4$ are:

$$A_1 = \begin{pmatrix} 1 & 143 & 141 & 4 & 140 & 6 \\ 102 & 101 & 46 & 45 & 98 & 43 \\ 54 & 92 & 94 & 93 & 53 & 49 \\ 91 & 50 & 52 & 51 & 95 & 96 \\ 48 & 47 & 99 & 100 & 44 & 97 \\ 139 & 2 & 3 & 142 & 5 & 144 \end{pmatrix}, A_2 = \begin{pmatrix} 7 & 137 & 135 & 10 & 134 & 12 \\ 108 & 107 & 40 & 39 & 104 & 37 \\ 60 & 86 & 88 & 87 & 59 & 55 \\ 85 & 56 & 58 & 57 & 89 & 90 \\ 42 & 41 & 105 & 106 & 38 & 103 \\ 133 & 8 & 9 & 136 & 11 & 138 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 13 & 131 & 129 & 16 & 128 & 18 \\ 114 & 113 & 34 & 33 & 110 & 31 \\ 66 & 80 & 82 & 81 & 65 & 61 \\ 79 & 62 & 64 & 63 & 83 & 84 \\ 36 & 35 & 111 & 112 & 32 & 109 \\ 127 & 14 & 15 & 130 & 17 & 132 \end{pmatrix}, A_4 = \begin{pmatrix} 19 & 125 & 123 & 22 & 122 & 24 \\ 120 & 119 & 28 & 27 & 116 & 25 \\ 72 & 74 & 76 & 75 & 71 & 67 \\ 73 & 68 & 70 & 69 & 77 & 78 \\ 30 & 29 & 117 & 118 & 26 & 115 \\ 121 & 20 & 21 & 124 & 23 & 126 \end{pmatrix}.$$

These are the matrices of the labeling matrix of $4K_{6,6}$.

References

[1] R. C. Bose, S. S. Shrikhande and E. T. Parker, Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture, *Canadian Journal of Mathematics*, **12** (1960), 189-203.

[2] N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *Journal of Recreational Mathematics*, **21** (1989), 107-115.

[3] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, Cambridge, 251-260, 1992.

[4] W. C. Shiu, P. C. B. Lam and S. M. Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium*, **132** (1998), 9-18.

[5] W. C. Shiu, P. C. B. Lam and S. M. Lee, On a Construction of Supermagic Graphs, preprint, 2000.

[6] B. M. Stewart, Magic graphs, *Canadian Journal of Mathematics*, **18** (1966), 1031-1059.

[7] B. M. Stewart, Supermagic Complete Graphs, *Canadian Journal of Mathematics*, **19** (1967), 427-438.

[8] D. R. Stinson, A Short Proof of the Nonexistence of a Pair of Orthogonal Latin Squares of Order Six, *Journal of Combonatorial Theory, Series A*, **36** (1984), 373-376.