Pancyclism of 3-Domination-Critical Graphs with Small Minimum Degree*

W.C. Shiu
Department of Mathematics, Hong Kong Baptist University
Hong Kong, China

Lian-zhu Zhang†
School of Mathematics, Xiamen University
Xiamen, Fujian 361005, China

ABSTRACT. A graph G is 3-domination-critical if its domination number γ is 3 and the addition of any edge decreases γ by 1. Let G be a connected 3-domination-critical graph of order n. Shao etc. proved that if $\delta(G) \geq 3$ then G is pancyclic, i.e. G contains cycles of each length k, $3 \leq k \leq n$. In this paper, we prove that the number of 2-vertices in G is at most 3. Using this result, we prove that the graph $G - V_1$ is pancyclic, where V_1 is the set of all 1-vertices in G, except G is isomorphic to the graph of order 7 well-defined in the context.

Keywords: 3-domination-critical graphs, pancyclic graphs
MSC(2000): 05C38, 05C69

1 Introduction

The graphs $G = (V(G), E(G))$ in this paper are finite, undirected and simple. Terminologies and notations which are not defined here are referred to [3]. For a vertex $v \in V(G)$ and a subgraph H of G, $N_H(v)$ is the set of neighbors of v contained in H. Set $d_H(v) = |N_H(v)|$. We will write $N(v)$ and $d(v)$ instead of $N_G(v)$ and $d_G(v)$, respectively. $d(v)$ is called the *degree of v in G*. v is also called a *$d(v)$-vertex*. Let $S \subseteq V(G)$. Denote by $G[S]$ the subgraph of G induced by S. Denote by $\omega(G)$ the number of components of G.

Suppose S and T are two vertex sets of G. We say that S *dominates* T, denoted by $S \Rightarrow T$, if every vertex of $T - S$ has at least one neighbor

*The work was partially supported by FRG, Hong Kong Baptist University; and National Natural Science Foundation of China (No. 10571105).

†The work was done while this author was visiting Hong Kong Baptist University.

Utilitas Mathematica 75 (2008), pp. 175-192

in S (when S or T is reduced to one vertex s or t, we simply say that s *dominates* T or S *dominates* t, denoted by $s \Rightarrow T$ or $S \Rightarrow t$, respectively). The set S is a *dominating set* of the graph G if $S \Rightarrow V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. We denote by $\delta(G), \alpha(G)$ and $\kappa(G)$ the minimum degree, the independence number and the connectivity of G, respectively. When no ambiguity can occur, we often simply write δ, α, κ and γ for $\delta(G)$, $\alpha(G)$, $\kappa(G)$ and $\gamma(G)$, respectively. The *diameter* $diam(G)$ of a connected graph G is defined as $\max\{d(u,v) \mid u,v \in V(G)\}$, where $d(u,v)$ is the distance between u and v.

Let k be an integer not less than 2. A graph G is called *k-domination-critical* (abbreviated to *k-critical*) if $\gamma(G) = k$ and $\gamma(G+e) = k-1$ holds for any $e \notin E(G)$. In this paper we consider only connected 3-critical graphs.

By the definition of 3-critical graphs, it is easy to see that if G is a 3-critical graph and $uv \notin E(G)$, then there exists a vertex $w \in V(G) - \{u,v\}$ such that either $\{u,w\}$ dominates $V(G)-\{v\}$ but not v or $\{v,w\}$ dominates $V(G) - \{u\}$ but not u. We adopt the notation in [11] and write $[u,w] \to v$ in the first case and $[v,w] \to u$ in the second case.

Let G be a graph of order n with $n \geq 6$. Let $a+b+c = n-3$, $(a,b,c \geq 1)$ be a partition of $n-3$. If there are three disjoint subsets A, B, C with cardinalities a, b, c, respectively, such that $V(G) = A \cup B \cup C \cup \{u,v,w\}$ with $N(u) = A, N(v) = B$, $N(w) = C$ and $G[A \cup B \cup C]$ is complete. Then G is easily seen to be 3-critical graph. We denote the graph G by $K(a,b,c)$ and call it a *full 3-critical graph*. The full 3-critical graph $K(1,1,1)$ is the unique connected 3-critical graph of order 6.

In this paper, we shall prove that if G is a connected 3-critical graph of order n, then for each k with $3 \leq k \leq n - |V_1(G)|$ the graph $G - V_1(G)$ contains a cycle of length k, where $V_1(G)$ is the set of all 1-vertices in G, except G is isomorphic to the graph G_7 described in Fig. 1. As a consequence of the result, we get that if G is a connected 3-critical graph of order n with $\delta(G) \geq 2$, then G contains a cycle of length k for each k satisfying $3 \leq k \leq n$, except G is isomorphic to G_7.

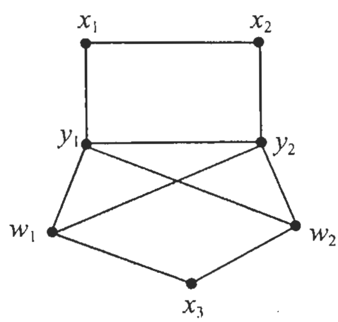

Fig. 1: G_7.

2 Some preliminary theorems

In order to prove our main theorems, we note the following known results, which will be used throughout the following.

Let G be a connected 3-critical graph of order n. Obviously, $n \geq 6$.

Theorem 2.1 ([10]) *G is 2-connected if and only if $\delta(G) \geq 2$.*

Theorem 2.2 ([11]) $2 \leq diam(G) \leq 3$.

Theorem 2.3 ([7]) $3 \leq \alpha(G) \leq \delta(G) + 2$. *Thus if $\delta(G) = 1$, then $\alpha(G) =$* 3.

Theorem 2.4 ([12, 7]) *If $\alpha(G) = \delta(G) + 2 \geq 4$, then*

(a) G has only one vertex, say x_δ, with degree $\delta(G)$;

(b) Every maximum independent set of G contains x_δ, and $G[N(x_\delta)]$ is complete.

Theorem 2.5 ([15]) *If $\delta(G) \geq 2$, then*

(a) $\alpha(G) \leq \kappa(G) + 2$; and

(b) if $\alpha(G) = \kappa(G) + 2$, then G has only one minimum cut-set S, which is the neighborhood of the unique vertex with degree $\delta(G)$.

Theorem 2.6 ([8]) *If $\delta(G) \geq 2$, then $\omega(G - S) \leq |S|$ for any cut-set S of G.*

3 Number of 2-vertices

Let G be a connected 3-critical graph of order $n \geq 7$ with $\delta(G) \geq 2$. By Theorem 2.1, G is 2-connected.

First, we show the following:

Claim 3.1 *Suppose x_1 and x_2 are two adjacent vertices of degree 2 of G. Then there is no triangle containing the edge x_1x_2.*

Proof: Suppose y is a vertex such that yx_1x_2y forms a triangle. Then y is a cut vertex. By Theorem 2.6 it is impossible. □

Claim 3.2 *Let $P = y_1x_1x_2y_2$ be a path of length 3 in G, where x_1 and x_2 are two vertices of degree 2 of G. Then $y_1y_2 \in E(G)$.*

Proof: Suppose that $y_1y_2 \notin E(G)$. Without loss of generality we may assume that there exists a vertex t such that $[y_1, t] \to y_2$. In order to dominate x_2, t must be x_1. Thus $y_1 \Rightarrow V(G) - \{x_2, y_1, y_2\}$, and hence $\{y_1, x_2\} \Rightarrow V(G)$, a contradiction. □

Claim 3.3 *Let x_1, x_2 be two 2-vertices in G. Then $N(x_1) \cap N(x_2) = \varnothing$.*

Proof: Assume that $N(x_1) \cap N(x_2) \neq \varnothing$. If $x_1x_2 \in E(G)$, let $N(x_1) \cap N(x_2) = \{y\}$, then y is a cut-vertex of G, which contradicts Theorem 2.1. Hence $x_1x_2 \notin E(G)$. Obviously it is impossible that $N(x_1) = N(x_2)$ by Theorem 2.6. Now, suppose that $N(x_1) = \{y_1, y_2\}$ and $N(x_2) = \{y_2, y_3\}$. Set $H = G - \{x_1, x_2, y_1, y_2, y_3\}$. By $n \geq 7$ we have $|H| \geq 2$; and by Theorems 2.3 and 2.4, we have $\alpha(G) = 3$, and hence H is complete.

By Theorem 2.6, one of $N_H(y_1)$ and $N_H(y_2)$ is nonempty. Without loss of generality, we may assume that $N_H(y_1) \neq \varnothing$. Let $u_1 \in H$ such that $y_1u_1 \in E(G)$. Thus y_2y_3, $y_3u_1 \notin E(G)$, otherwise, $\{y_2, u_1\} \Rightarrow V(G)$, a contradiction.

If $N_H(y_3) = \varnothing$, then by 2-connectedness $y_1y_3 \in E(G)$ and y_3 is also a 2-vertex. But $\omega(G - \{y_1, y_2\}) = 3$, which contradicts Theorem 2.6. Let $u_3 \in H$ such that $y_3u_3 \in E(G)$. Similarly, y_1y_2, $y_1u_3 \notin E(G)$.

Suppose $u \in V(H)$ with $y_1u \notin E(G)$. Since $x_1u \notin E(G)$, there exists a vertex t such that $[x_1, t] \to u$ or $[u, t] \to x_1$. In both cases, in order to dominate x_2, $t \in \{x_2, y_2, y_3\}$. For the case $[x_1, t] \to u$ if $t = x_2$, then the vertices of $V(H) - \{u\}$ cannot be dominated; if $t = y_2$, then y_3 cannot be dominated; if $t = y_3$, then u_1 cannot be dominated. Thus we have $[u, t] \to x_1$. Obviously, $t = x_2$ or y_2 is impossible, otherwise y_1 cannot be dominated. Thus we must have that $[u, y_3] \to x_1$, and hence $y_3y_1, uy_2 \in E(G)$. Therefore, each vertex of H not adjacent to y_1 must be adjacent to y_2. Thus $\{y_1, y_2\} \Rightarrow V(G)$, a contradiction. □

Following we shall use n_i to denote the number of i-vertices in G.

Theorem 3.1 *Let G be a connected 3-critical graph of order $n \geq 7$ with $\delta(G) \geq 2$. Then $n_2 \leq 3$. Moreover, if $n_2 = 3$, then G is isomorphic to $K(2, 2, 2)$ or the graph G_7 illustrated in Fig. 1.*

Proof: Suppose that $n_2 \geq 4$. By Theorems 2.3 and 2.4 we have $\alpha(G) = 3$. Denote by $G(2)$ the subgraph of G induced by all the 2-vertices. Thus $\omega(G(2)) \leq 3$. By Claims 3.1 to 3.3 we know that each component of $G(2)$

is either K_1 or K_2. Hence, $G(2)$ must be $2K_2$, $2K_1 \cup K_2$, $2K_2 \cup K_1$ or $3K_2$. The last three cases are impossible by Claims 3.1 to 3.3 and $\alpha(G) = 3$. We only need to consider the case $G(2) = 2K_2$. Let x_1, x_2, x_3, x_4 be the four 2-vertices, and $x_1x_2, x_3x_4 \in E(G)$. Let y_i be the other neighbor of x_i, $i = 1, 2, 3, 4$. By Claim 3.2, we have that $y_1y_2, y_3y_4 \in E(G)$. Since $x_1y_3 \notin E(G)$, there exists a vertex t such that either $[x_1, t] \to y_3$ or $[y_3, t] \to x_1$. In the case $[x_1, t] \to y_3$, in order to dominate x_3, t must be x_4, but y_2 cannot be dominated. In the case $[y_3, t] \to x_1$, in order to dominate x_2, t must be y_2, but x_4 cannot be dominated. Thus we have $n_2 \leq 3$.

Suppose that $n_2 = 3$. Then we have $G(2) = 3K_1$ or $K_1 \cup K_2$. Note that by Theorems 2.3 and 2.4 we have $\alpha(G) = 3$.

(a) Suppose that $G(2) = 3K_1$.

Let x_1, x_2, x_3 be the three 2-vertices in G. Let y_i and z_i be the neighbors of x_i for $1 \leq i \leq 3$. By $\alpha(G) = 3$, we have that $V(G) = A_1 \cup A_2 \cup A_3$, where $A_i = \{x_i, y_i, z_i\}$ for $1 \leq i \leq 3$. In order to dominate x_1, x_2 and x_3 the dominating set must intersect with each A_i. Since G is 3-critical, $G[\{y_1, y_2, y_3, z_1, z_2, z_3\}]$ must be K_6. Therefore, G is isomorphic to $K(2, 2, 2)$.

(b) Suppose that $G(2) = K_1 \cup K_2$.

Let x_1, x_2 and x_3 be the three 2-vertices in G and $x_1x_2 \in E(G)$. Set $N(x_3) = \{w_1, w_2\}$ and let y_1 and y_2 be the other neighbor of x_1 and x_2, respectively. By Claim 3.2, we have $y_1y_2 \in E(G)$. Set $H = G - \{x_1, x_2, x_3, w_1, w_2, y_1, y_2\}$.

Suppose that $V(H) \neq \varnothing$. By $\alpha(G) = 3$, we have

(1) H is a complete graph; and

(2) $y_1 \Rightarrow V(H)$ and $y_2 \Rightarrow V(H)$.

For each $u \in V(H)$, since $ux_3 \notin E(G)$, there exists a vertex t such that $[x_3, t] \to u$ or $[u, t] \to x_3$. For the case $[x_3, t] \to u$, $t \neq y_1, y_2$ by (2). In order to dominate x_1, $t = x_1$ or x_2, which is impossible, otherwise, y_2 or y_1 cannot be dominated, respectively. Thus we have $[u, t] \to x_3$. In order to dominate x_1, $t = y_1, x_1$ or x_2. But $t = y_1$ is impossible, otherwise, x_2 cannot be dominated. Hence we get $t = x_1$ or x_2. In either of the cases, we always get that $w_1u, w_2u \in E(G)$, and hence we get

(3) $w_1 \Rightarrow V(H)$ and $w_2 \Rightarrow V(H)$.

For each $u \in V(H)$, since $x_1u \notin E(G)$, there exists a vertex t such that $[x_1, t] \to u$ or $[u, t] \to x_1$. In both cases, in order to dominate x_3, $t = x_3, w_1$ or w_2. For the case $[x_1, t] \to u$, by (3) we get $t = x_3$, which is impossible, otherwise, y_2 cannot be dominated. For the case $[u, t] \to x_1$ is also impossible, otherwise, x_2 cannot be dominated. Hence we get that $V(H) = \varnothing$, and hence G is a graph of order 7.

Now we are going to show that $G \cong G_7$. Since $d(y_2) \geq 3$, we have $|N(y_2) \cap \{w_1, w_2\}| \geq 1$. Without loss of generality we assume that $y_2w_1 \in (G)$. Thus $w_1w_2 \notin E(G)$, otherwise, $\{w_1, x_1\} \Rightarrow V(G)$. Since $d(w_2) \geq 3$, we get $w_2y_1, w_2y_2 \in E(G)$. Since $d(w_1) \geq 3$, we get $w_1y_1 \in E(G)$. Hence $G \cong G_7$. It is easy to check that G_7 is 3-critical. □

Remark 3.1 Denote by d_i the number of vertices of degree at most i. Thus $d_i = \sum_{t=1}^{i} n_t$. Sumner *et al.* [11] proved that (a) $d_1 \leq 3$, and 3 is the best possible; (b) $d_2 \leq 5$; (c) $d_3 \leq 8$, and 8 is the best possible. By similar arguments to that of the proof of Theorem 1 in [11], we can prove $d_2 \leq 3$. Moreover, if $d_2 = 3$, then G is isomorphic to $K(1,1,1), K(1,1,2), K(1,2,2), K(2,2,2)$ or G_7.

Remark 3.2 In fact, we can easily prove that G_7 is the only graph with $\delta = 2$ in the family of 3-critical graphs of order at most 7.

4 Pancyclism of 3-critical graphs

A *k-cycle*, denote by C_k, is a cycle of length k. A graph G of order $n \geq 3$ is said to be *pancyclic* if G contains k-cycles for all k, $3 \leq k \leq n$.

As we know, the first result concerning the cyclic structure of connected 3-critical graphs is the following theorem.

Theorem 4.1 ([11]) *Every connected 3-critical graph contains a 3-cycle.*

As for the Hamiltonian properties of 3-critical graphs, Wojcicka proved the following result, which was conjectured by Sumner *et al.* in [11].

Theorem 4.2 ([13]) *Every connected 3-critical graph of order at least 7 has a Hamiltonian path.*

Wojcicka further conjectured that every connected 3-critical graph with $\delta \geq 2$ has a Hamiltonian cycle, i.e., is Hamiltonian.

For a given graph G, let $V_1(G)$ be the set of all 1-vertices in G. Xie *et al.* [14] proved the following.

Theorem 4.3 ([14]) *Let G be a connected 3-critical graph with $\delta(G) = 1$. Then $G - V_1(G)$ is Hamiltonian.*

By Theorem 2.3 and the following two theorems, Wojcicka's conjecture is completely solved.

Theorem 4.4 ([7]) *Let G be a connected 3-critical graph with $\delta(G) \geq 2$. If $\alpha(G) \leq \delta(G) + 1$, then G is Hamiltonian.*

Theorem 4.5 ([12]) *Let G be a connected 3-critical graph with $\delta(G) \geq 2$. If $\alpha(G) = \delta(G) + 2$, then G is Hamiltonian.*

A new and simpler proof of Wojcicka's conjecture is given in [6]. In accordance with the meta-conjecture proposed by Bondy in [2]. Shao *et al.* [9] got the following theorem.

Theorem 4.6 ([9]) *Each connected 3-critical graph with $\delta \geq 3$ is pancyclic.*

In [9], Shao *et al.* constructed the graph G_7 (Fig. 1) to show that $\delta(G) \geq 3$ is the best possible. The graph G_7 contains no C_6.

Note that Theorems 4.3–4.5 can be unified into the following theorem.

Theorem 4.7 *Let G be a connected 3-critical graph. Then $G - V_1(G)$ is Hamiltonian.*

In this paper, we prove that G_7 is, in fact, the only exceptional case for the graph $G - V_1(G)$ to be pancyclic.

Theorem 4.8 *Let G be a connected 3-critical graph. Then $G - V_1(G)$ is pancyclic except G is isomorphic to G_7.*

Corollary 4.9 *Let G be a connected 3-critical graph with $\delta(G) \geq 2$. Then G is pancyclic except G is isomorphic to G_7.*

In order to prove Theorem 4.8, we need the following two well-known results.

Theorem 4.10 ([5]) *If G be a 2-connected graph with $\alpha(G) \leq 2$, then G is pancyclic except C_4 and C_5.*

Theorem 4.11 ([1]) *If G be a 3-connected graph with $\alpha(G) \leq 3$, then G is pancyclic except $K_{3,3}$ and the graph H_8 described in Fig. 2.*

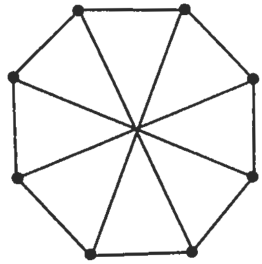

Fig. 2: H_8

Proof of Theorem 4.8: By Theorem 4.6, we may assume that $\delta(G) \leq 2$. It is easy to see that if G is a full 3-critical graph, then the theorem holds. Thus we assume that G is not a full 3-critical graph of order n with $\delta(G) \leq 2$ below.

(I) Suppose that $\delta(G) = 1$.

In [11], Sumner *et al.* proved that if $n_1 = 3$, then G is a full 3-critical graph $K(1,1,1)$ of order 6; and that if $n_1 = 2$, then G is a full 3-critical graph $K(1,1,n-5)$ of order $n \geq 7$. So $n_1 = 1$ and $n \geq 7$. Let x be the unique 1-vertex in G and let $xy \in E(G)$. Set $W = V(G) - (N(y) \cup \{y\})$.

Claim 4.1 ([14]) *(i) $G[N(y) \setminus \{x\}]$ is a complete graph of order at least 2. (ii) For any $u \in N(y) \setminus \{x\}$, $d(u) = n - 3$.*

Proof of Claim 4.1: (i) Suppose $N(y) \setminus \{x\} = \{u\}$ for some $u \in V(G)$. Since $diam(G) \leq 3$ (Theorem 2.2), all vertices not belong to $\{x, y, u\}$ must be adjacent to u. Then $\{y, u\} \Rightarrow V(G)$, a contradiction.

Suppose that $u_1, u_2 \in N(y) \setminus \{x\}$ with $u_1u_2 \notin E(G)$. Assume, without loss of generality, that there exists a vertex t such that $[u_1, t] \to u_2$. In order to dominate x, we have $t = x$ or y. Thus $W \subseteq N(u_1)$, and hence $\{y, u_1\} \Rightarrow V(G)$, a contradiction.

(ii) Obviously, $W \not\subseteq N(u)$, otherwise, $\{y, u\} \Rightarrow V(G)$. Let v be a vertex in W with $uv \notin E(G)$. Thus there exists t such that $[u, t] \to v$ or $[v, t] \to u$. Note that the case $[v, t] \to u$ is impossible, otherwise, we have that $t = x$ and thus $W \subseteq N(v)$, and hence $\{y, v\} \Rightarrow V(G)$. Hence we get that $[u, t] \to v$, and that $t = x$ or y in order to dominate x. In either case we have $W - \{v\} \subseteq N(u)$. Thus $d(u) = n - 3$ by (i). ■

Now we are going to prove the pancyclism of $G - \{x\}$. Let $G^* = G - \{x, y\}$. Obviously, $\alpha(G^*) = 2$. Suppose that v is a cut-vertex of G^*. Set $G^* - \{v\} = R_1 \cup R_2$. By Claim 4.1 (i), without loss of generality we may assume that $N(y) - \{x\} \subseteq V(R_1) \cup \{v\}$. Let $u \in N(y) - \{x\}$ with $u \neq v$. By Claim 4.1 (ii), u has exactly one non-adjacent vertex in R_2, and hence $|R_2| = 1$. This contradicts that $n_1 = 1$. Thus $\kappa(G^*) \geq 2$. It is easy to see that G^* is neither C_4 nor C_5. Therefore, by Theorem 4.10, the graph G^* is pancyclic, and hence so is $G - \{x\}$.

(II) Suppose that $\delta(G) = 2$. By Theorem 3.1 and Remark 3.2, we may assume that $n_2 \leq 2$ and $n \geq 8$.

Case 1. Suppose that $n_2 = 2$. By Theorems 2.4 and 2.6 we have $\alpha(G) = 3$ and G is Hamiltonian. In this case the subgraph $G(2)$ induces by all the 2-vertices of G is either K_2 or $2K_1$.

Subcase 1.1. Suppose $G(2) = K_2$.

Let x_1 and x_2 be the 2-vertices and let, $N(x_1) = \{x_2, y_1\}$ and $N(x_2) = \{x_1, y_2\}$. By Claim 3.2, $y_1y_2 \in E(G)$. Let $G^* = G - \{x_1, x_2\}$. Obviously, $2 \leq \alpha(G^*) \leq 3$. Suppose that $\alpha(G^*) = 3$. Let I_3 be the maximum independent set of G^*. If $y_1 \notin I_3$, then $I_3 \cup \{x_1\}$ is an independent set of G, a contradiction. If $y_1 \in I_3$, then $y_2 \notin I_3$. In this case, $I_3 \cup \{x_2\}$ is an independent set of G, also a contradiction. Thus $\alpha(G^*) = 2$. It is easy to see that $\kappa(G^*) \geq 2$.

By Theorem 4.1, G contains a cycle of length 3, and so does G^*. Thus, G^* is neither C_4 nor C_5. By Theorem 4.10, the graph G^*, i.e. $G - \{x_1, x_2\}$ is pancyclic. In order to prove that G is pancyclic, it suffices to prove G contains an $(n-1)$-cycle.

Let $C_n = y_1x_1x_2y_2y_3 \cdots y_{n-2}y_1$ be a Hamiltonian cycle of G. Set $y_{n-1} = y_1$.

Since $\alpha(G) = 3$, the set $\{x_1, y_2, y_4, y_6\}$ is not independent. Then either $y_2y_4 \in E(G)$, $y_4y_6 \in E(G)$ or $y_2y_6 \in E(G)$. For the first two cases, G contains an $(n-1)$-cycle. Then the theorem holds. Suppose not, that means $y_2y_4 \notin E(G)$, $y_4y_6 \notin E(G)$ but $y_2y_6 \in E(G)$. Since $d(y_4) \geq 3$, there is a vertex y_i such that $y_4y_i \in E(G)$, where $7 \leq i \leq n-1$. Recall that $y_{n-1} = y_1$.

If $n = 8$, then $y_4y_7 \in E(G)$. Hence $y_1x_1x_2y_2y_6y_5y_4y_7$ is an 7-cycle. If $n \geq 9$, then consider the set $\{x_2, y_3, y_5, y_7\}$. We get that either G contains an $(n-1)$-cycle or $y_3y_7 \in E(G)$. Consider the set $\{x_2, y_4, y_6, y_8\}$. We get that $y_4y_8 \in E(G)$. Hence $y_1x_1x_2y_2y_6y_7y_3y_4y_8 \cdots y_{n-2}y_{n-1}$ is an $(n-1)$-cycle.

Subcase 1.2. Suppose $G(2) = 2K_1$.

Let x_1 and x_2 be the 2-vertices, and let $N(x_1) = \{y_1, y_2\}$ and $N(x_2) = \{y_3, y_4\}$. By Claim 3.3, the vertices y_1, y_2, y_3, y_4 are distinct. Let $H = G - \{x_1, x_2, y_1, y_2, y_3, y_4\}$. Obviously, H is a complete graph of order $n-6$. First we show the following six Claims.

Claim 4.2 *For any $y_i, 1 \leq i \leq 4$ and any $u \in V(H)$, if $y_i u \notin E(G)$, then there exists a vertex t such that $[y_i, t] \to u$.*

Proof of Claim 4.2: Suppose not, there exists a vertex t such that $[u, t] \to y_i$. It is easy to see that x_1 or x_2 cannot be dominated. ■

Claim 4.3 *For each $1 \leq i \leq 4$, y_i has at least $|H| - 2$ neighbors in H.*

Proof of Claim 4.3: Suppose not, we assume that u_1, u_2 and u_3 are vertices in H which are not adjacent to y_1 in G. By Claim 4.2 there exists a vertex t_1 such that $[y_1, t_1] \to u_1$. Since u_2 must be dominated, $t_1 \neq x_2$. In order to dominate x_2, $t_1 = y_3$ or y_4, say $t_1 = y_3$, and hence $y_3u_2, y_3u_3 \in E(G)$. Similarly, we have that $[y_1, t_2] \to u_2$, and $t_2 = y_4$. Thus $y_4u_1, y_4u_3 \in E(G)$. Now considering y_1u_3, $[y_1, t_3] \to u_3$ is impossible since $y_3u_3, y_4u_3 \in E(G)$. Therefore, y_1 has at least $|H| - 2$ neighbors in H. Similarly, we obtain that each of y_2, y_3 and y_4 has at least $|H| - 2$ neighbors in H. ■

Remark 4.1 From the proof above we can see that if H contains exactly two vertices not adjacent to y_i, say $i = 1$ or 2, then $u_iy_3, u_i'y_4 \in E(G)$, (or $u_iy_4, u_i'y_3 \in E(G)$).

Remark 4.2 For $n \geq 11$, we construct a 3-critical graph F_n satisfying that each $y_i, 1 \leq i \leq 4$, has exactly $|H| - 2$ neighbors in H as follows. Given a complete graph K_4 with the vertex set $V(K_4) = \{y_i \mid 1 \leq i \leq 4\}$ and a complete graph K_{n-6} with the vertex set $V(K_{n-6}) = \{u_j \mid 1 \leq j \leq n-6\}$. Let

$$V(F_n) = \{x_1, x_2\} \cup V(K_4) \cup V(K_{n-6}),$$

$$E(F_n) = \{x_1y_1, x_1y_2, x_2y_3, x_2y_4, y_1u_3, y_1u_4, y_2u_1, y_2u_2, y_3u_2, y_3u_4, y_4u_1, y_4u_3\} \cup \{y_iu_j \mid 1 \leq i \leq 4,\ 5 \leq j \leq n-6\} \cup E(K_4) \cup E(K_{n-6}).$$

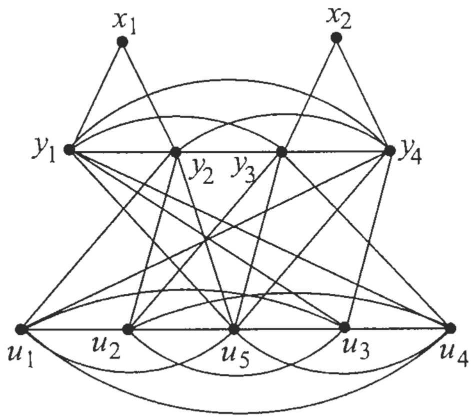

Fig. 3. F_{11}

Claim 4.4 *If $y_1y_2, y_3y_4 \in E(G)$, then $G[\{y_1, y_2, y_3, y_4\}]$ is complete.*

Proof of Claim 4.4: Suppose not, without loss of generality we may assume that $y_1y_3 \notin E(G)$, and that there exists a vertex t such that $[y_1, t] \to y_3$. In order to dominate x_2, $t \in \{x_2, y_4\}$, which is impossible since $x_2, y_4 \in N(y_3)$. ■

First, for $n \geq 7$, we construct a connected 3-critical graph, denoted by G_n, as follows.

$$\begin{aligned} V(G_n) =& \{x_1, x_2, y_1, y_2, y_3, y_4\} \cup V(K_{n-6}); \\ E(G_n) =& \{x_1y_1, x_1y_2, x_2y_3, x_2y_4, y_1y_3, y_2y_3\} \\ & \cup \{y_iu \mid 1 \leq i \leq 4, \; u \in V(K_{n-6})\} \cup E(K_{n-6}). \end{aligned}$$

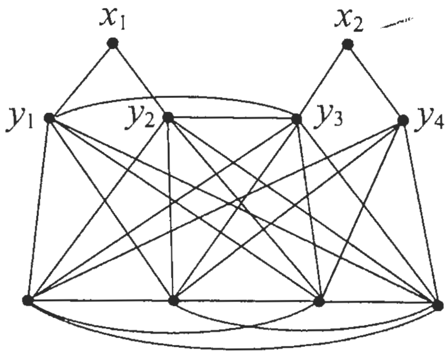

Fig. 4. G_{10}

Note that the graph G_n of order 7 is isomorphic to the graph described in Fig. 1. It is exactly the reason why we denote the graph by G_n. It is easy to check that G_n is 3-critical. Obviously, for $n \geq 8$, G_n is pancyclic.

By Theorem 2.2, there must be at least one edge between $\{y_1, y_2\}$ and $\{y_3, y_4\}$. Without loss of generality following we assume that $y_2y_3 \in E(G)$.

Claim 4.5 *If $y_1 \Rightarrow V(H)$ or $y_4 \Rightarrow V(H)$, then G is isomorphic to the graph G_n.*

Proof of Claim 4.5: By symmetric, we may assume $y_1 \Rightarrow V(H)$. Obviously, we have that $y_1y_2, y_1y_4, y_3y_4 \notin E(G)$ by $\gamma(G) = 3$.

Suppose there is a vertex $u \in V(H)$ such that $y_4u \notin E(G)$. Then by Claim 4.2 there exists a vertex t such that $[y_4, t] \to u$. In order to dominate x_1, $t = x_1$ or y_2. If $t = x_1$ then y_3 cannot be dominated; and if $t = y_2$ then y_1 cannot be dominated. Thus $y_4 \Rightarrow V(H)$. Since $\gamma(G) = 3$, at least one of y_1y_3 and y_2y_4 does not belong to $E(G)$. Without loss of generality we assume that $y_1y_3 \notin E(G)$.

By a similar argument, we have $y_2 \Rightarrow V(H)$ and $y_3 \Rightarrow V(H)$.

Finally, since $y_1y_3 \notin E(G)$, there exists a vertex t such that $[y_1, t] \rightarrow y_3$ or $[y_3, t] \rightarrow y_1$. If $[y_1, t] \rightarrow y_3$, then, in order to dominate x_2, $t = x_2$ or y_4. Since $x_2y_3 \in E(G)$, $t = y_4$. Hence $y_2y_4 \in E(G)$. If $[y_3, t] \rightarrow y_1$, then, in order to dominate x_1, $t = x_1$ or y_2. Since $x_1y_1 \in E(G)$. Hence we have also $y_2y_4 \in E(G)$.

Therefore, G is isomorphic to G_n. ■

Claim 4.6 *Let G be a connected 3-critical graph of order 8 with $G(2) = 2K_1$. Then G is isomorphic to G_8.*

Proof of Claim 4.6. By Claim 4.5, it suffices to prove that $y_1 \Rightarrow V(H)$ or $y_4 \Rightarrow V(H)$. Suppose not, we set $V(H) = \{u_1, u_2\}$ and consider the following two cases.

Case A. Suppose $|N_H(y_1)| = 0$ or $|N_H(y_4)| = 0$.

Without loss of generality we assume that $|N_H(y_1)| = 0$. Since $y_1u_1, y_1u_2 \notin E(G)$, by Remark 4.1, without loss of generality we have that $y_3u_1, y_4u_2 \in E(G)$, and hence $y_4u_1 \notin E(G)$. Otherwise, we have $y_4 \Rightarrow V(H)$. Since $d(u_1) \geq 3$, we get that $u_1y_2 \in E(G)$. Thus $y_1y_2, y_1y_4 \notin E(G)$, (otherwise, we have $\{y_2, y_4\} \Rightarrow V(G)$). Therefore, $N(y_1) \subseteq \{x_1, y_3\}$, which contradicts $d(y_1) \geq 3$.

Case B. Suppose $|N_H(y_1)| = 1$ and $|N_H(y_4)| = 1$.

Suppose that $y_1u_1, y_4u_2 \in E(G)$ and $y_1u_2, y_4u_1 \notin E(G)$. We may assume, without loss of generality, that there exists a vertex $t \in \{y_1, y_2, u_1, u_2\}$ such that $[x_1, t] \rightarrow x_2$. If $t = y_1$ or $t = u_1$, then u_2 or y_4 cannot be dominated, respectively. Suppose that $t = y_2$. Thus $u_1, u_2, y_4 \in N(y_2)$ and hence we get $y_2, y_3, y_4 \notin N(y_1)$ by $\gamma(G) = 3$. Therefore $d(y_1) = 2$, a contradiction. Now we get $t = u_2$ and hence $u_2y_3 \in E(G)$. Since $\gamma(G) = 3$, we have that $y_3y_4, y_1y_4 \notin E(G)$. By $d(y_4) \geq 3$, we get $y_4y_2 \in E(G)$, and hence $y_1y_3 \notin E(G)$. Otherwise we have $\{y_1, y_3\} \Rightarrow V(G)$. Similarly, we get $y_1y_2 \in E(G)$, and hence $y_2u_1 \notin E(G)$. Finally, we get $y_3u_1 \in E(G)$. Thus $\{y_2, y_3\} \Rightarrow V(G)$, a contradiction.

Now suppose that $y_1u_1, y_4u_1 \in E(G)$ and $y_1u_2, y_4u_2 \notin E(G)$. Since $d(u_2) \geq 3$, we have that $y_2u_2, y_3u_2 \in E(G)$, hence y_1y_2, $y_3y_4, y_1y_4 \notin E(G)$ by $\gamma(G) = 3$. Since $x_1u_2 \notin E(G)$, there exists a vertex t such that $[x_1, t] \rightarrow u_2$ or $[u_2, t] \rightarrow x_1$. In the case $[x_1, t] \rightarrow u_2$, $t = x_2$ or y_4 in order to dominate x_2. If $t = x_2$, then u_1 cannot be dominated; and if $t = y_4$, then y_3 cannot be dominated. Thus we get $[u_2, t] \rightarrow x_1$. In order to dominate x_2, $t = x_2$, y_3 or y_4. If $t = x_2$ or y_4, then y_1 cannot be dominated; and if $t = y_3$, then y_4 cannot be dominated. ∎

By Claim 4.6, from now on we assume that $n \geq 9$.

Claim 4.7 *Suppose $n \geq 9$,*

(i) there are two distinct vertices $u_1, u_2 \in V(H)$ such that $y_1u_1, y_4u_2 \in E(G)$;

(ii) there are two distinct vertices $w_1, w_2 \in V(H)$ such that $y_1w_1, y_2w_2 \in E(G)$, or $y_3w_1, y_4w_2 \in E(G)$.

Proof of Claim 4.7: (i) By Claim 4.2, $|N_H(y_1)| \geq |H| - 2$ and $|N_H(y_4)| \geq |H| - 2$. Thus if $n \geq 10$, then by Hall's Theorem [4, pp. 25], the conclusion holds. Suppose that $n = 9$ (i.e., $|H| = 3$). We may assume that $|N_H(y_1)| = 1$ and let $y_1u_1 \in E(G)$, where $u_1 \in V(H)$. In this case, y_1 is not adjacent to exact two vertices u_2 and u_3 of H. By Remark 4.1, there exists a vertex, say $u_2 \in V(H)$ such that, $y_4u_2 \in E(G)$. This shows that (i) holds.

(ii) The proof is similar to that of (i). ∎

Now we prove the pancyclism of G.

(1) Since $H \cong K_{n-6}$, G has a cycle of length k for each k satisfying $3 \leq k \leq n - 6$.

(2) By Claim 4.7(i), we get a Hamiltonian cycle $C_n = y_1x_1y_2y_3x_2y_4u_2u_3 \cdots u_{n-6}u_1y_1$ of G, where u_i's are vertices of H. Then $C_n \cup H$ contains a cycle of length k for each k satisfying $8 \leq k \leq n$.

(3) By Claim 4.7(ii), we get an $(n-3)$-cycle $C_{n-3} = y_1x_1y_2w_2w_3 \cdots w_{n-6} w_1y_1$ (or $C_{n-3} = y_3x_2y_4w_2w_3 \cdots w_{n-6}w_1y_3$) of G, where w_i's are vertices of H. Then $C_{n-3} \cup H$ contains a cycle of length k for each k satisfying $5 \leq k \leq n - 3$.

By (1)-(3), in order to prove the pancyclism of G, it suffices to prove that when $n = 9$, G contains a 4-cycle and a 7-cycle.

First we show that G contains a 7-cycle.

If $y_3u_3 \in E$, then $C_1 = y_3u_3u_2u_1y_1x_1y_2y_3$ is a 7-cycle;

If $y_3u_2 \in E$, then $C_2 = y_3u_2u_3u_1y_1x_1y_2y_3$ is a 7-cycle;

If $y_1u_2 \in E$, then $C_3 = y_1u_2y_4x_2y_3y_2x_1y_1$ is a 7-cycle;

Therefore, we may assume that y_3u_3, y_3u_2, $y_1u_2 \notin E(G)$. By Claim 4.2, it follows that $y_1u_3 \in E(G)$. Similarly, we get that $y_4u_3 \in E(G)$. Thus we get $C_7 = y_1x_1y_2y_3x_2y_4u_3y_1$, a 7-cycle in G.

Now we show that G contains a 4-cycle.

If $y_1u_3 \in E$, then $C_1 = u_3y_1u_1u_2u_3$ is a 4-cycle;

If $y_1u_2 \in E$, then $C_2 = u_2y_1u_1u_3u_2$ is a 4-cycle;

If $y_1y_3 \in E$, then $C_3 = y_1y_3y_2x_1y_1$ is a 4-cycle;

If $y_1y_4 \in E$, then $C_4 = y_1y_4u_3u_1y_1$ is a 4-cycle;

Therefore, we may assume that y_1u_3, y_1u_2, y_1y_3, $y_1y_4 \notin E(G)$.

Since $d(y_1) \geq 3$, it follows that $y_1y_2 \in E(G)$. Similarly, we get that $y_3y_4 \in E(G)$. Thus $G[\{y_1, y_2, y_3, y_4\}]$ is complete by Claim 4.4. But it is impossible since y_1y_3,$y_1y_4 \notin E(G)$.

Case 2. Suppose $n_2 = 1$, i.e., $G(2) = K_1$.

Let x be the unique 2-vertex in G, and let $N(x) = \{y_1, y_2\}$. Set $H = G - \{x, y_1, y_2\}$. By Theorems 2.3 and 2.4, $3 \leq \alpha(G) \leq 4$.

Suppose that $\alpha(G) = 4$. By Theorem 2.4(ii), every maximum independent set of G contains x, and hence y_1 and y_2 do not belong to any independent set of G. Thus $\alpha(H) = 3$ and hence $\alpha(G^*) = 3$, where $G^* = G - x$. By Theorem 2.5, G has exactly one minimum cut-set $\{y_1, y_2\}$. Hence, $\kappa(G^*) \geq 3$. It is easy to see that G^* is neither $K_{3,3}$ nor the graph H_8. By Theorem 4.11, G^* is pancyclic, and hence so does G.

Suppose that $\alpha(G) = 3$. Then we have $\alpha(H) = 2$.

Subcase 2.1. Suppose $\kappa(H) \geq 2$.

Since the order of H is at least 6, by Theorem 4.10 H is pancyclic. Hence it suffices to show that G contains an $(n-2)$-cycle and an $(n-1)$-cycle. Let $C_n = y_1xy_2y_3\cdots y_{n-1}y_1$ be a Hamiltonian cycle of G. Let $y_n = y_1$.

Case 2.1.1. We shall show that G contains an $(n-1)$-cycle first. Suppose not, we have the following conditions:

(0) $y_1y_2 \notin E(G)$,

(1) $y_iy_{i+2} \notin E(G)$ for each $i = 2, 3, \ldots n-2$, and

(2) at least one of y_iy_j and $y_{i+1}y_{j+2}$ does not belong to $E(G)$, for each $2 \leq i \leq n-1$, $j \neq i-1, i, i+1$. (Otherwise, we get an $(n-1)$-cycle $y_1xy_2\cdots y_iy_jy_{j-1}\cdots y_{i+2}y_{j+2}\cdots y_{n-1}y_1$ if $i < j$ and $y_1xy_2\cdots y_jy_iy_{i-1}\cdots y_{j+2}y_{i+1}\cdots y_{n-1}y_1$ if $i > j$.)

Considering the set $\{x, y_3, y_5, y_7\}$, by $\alpha(G) = 3$ and (1), we get $y_3y_7 \in E(G)$. If $n = 8$, then by (2), we have $y_5y_1, y_5y_2 \notin E(G)$, and hence $d(y_5) = 2$, a contradiction. Similarly, if $n = 9$, we get $y_3y_7, y_5y_1 \in E(G)$. Since the set $\{x, y_4, y_6, y_8\}$ is not independent, we get $y_4y_8 \in E(G)$ by (1). Hence we get $C_8 = y_1xy_2y_3y_7y_8y_4y_5y_1$, a contradiction. We assume that $n \geq 10$. Considering the sets $\{x, y_5, y_7, y_9\}$ and $\{x, y_4, y_6, y_8\}$ we get $y_4y_8, y_5y_9 \in E(G)$. Thus we get an $(n-1)$-cycle $y_1xy_2y_3y_7y_8y_4y_5y_9y_{10}\cdots y_{n-1}y_1$, a contradiction.

Case 2.1.2. Finally we shall show that G contains an $(n-2)$-cycle. Suppose not, we have the following conditions:

(3) $y_iy_{i+3} \notin E(G)$, $(i = 2, 3, \ldots n-3)$; and $y_1y_3, y_2y_{n-1} \notin E(G)$, and

(4) at least one of y_iy_j and $y_{i+2}y_{j+2}$ does not belong to $E(G)$ (otherwise similar to (2) $G - \{y_{i+1}, y_{j+1}\}$ contains an $(n-2)$-cycle), for each $i = 2, 3, \ldots, n-1$, $j \neq i-1, i, i+1$.

Case a. Suppose $y_1y_2 \notin E(G)$. Suppose $n \geq 12$. Since $\alpha(G) = 3$, by considering the sets $\{x, y_3, y_6, y_9\}$, $\{x, y_5, y_8, y_{11}\}$ we have y_3y_9, $y_5y_{11} \in E(G)$. This contradicts (4). So we only need to consider the cases when $8 \leq n \leq 11$.

Suppose $y_iy_{i+2} \in E(G)$ for some i with $2 \leq i \leq n-1$. Let $G_1 = G - y_{i+1}$. Rename $y_j = z_j$ for $1 \leq j \leq i$ and $y_j = z_{j-1}$ for $i+2 \leq j \leq n-1$. Then $z_1xz_2\cdots z_{n-2}z_1$ is a Hamiltonian cycle of G_1. By assumption, G_1 does not contain any $(n-2)$-cycle. By Theorem 4.10, $\alpha(G_1) = 3$. By the proof of Case 2.1.1, we get that $n - 1 \leq 7$, i.e., $n = 8$. Since $d(y_1) \geq 3$, $y_1y_4 \in E(G_1)$ or $y_1y_6 \in E(G_1)$. Suppose $y_1y_4 \in E(G_1)$. Since $d(y_5) \geq 3$, $y_5y_3 \in E(G_1)$ or $y_5y_7 \in E(G_1)$. If $y_5y_3 \in E(G_1)$, then $y_1y_4y_3y_5y_6y_7y_1$ is a 6-cycle, a contradiction. Thus $y_5y_7 \in E(G_1)$. Since $d(y_2) \geq 3$, $y_2y_6 \in E(G_1)$ or $y_2y_4 \in E(G_1)$. If $y_2y_4 \in E(G_1)$, then $y_1xy_2y_4y_5y_7y_1$ is a 6-cycle, a contradiction. Thus $y_2y_6 \in E(G_1)$. But now $y_2y_6y_7y_5y_4y_3y_2$ is a 6-cycle, a contradiction. Therefore, $y_1y_6 \in E(G_1)$. By considering the vertex y_3, we have $y_3y_5 \in E(G_1)$ or $y_3y_7 \in E(G_1)$. For both cases, we will get a 6-cycle, a contradiction again.

Suppose $y_iy_{i+2} \in E(G)$ for each i with $2 \leq i \leq n-1$. That is, conditions (1), (3) and (4) hold. Suppose $n \geq 10$. By considering the sets $\{x, y_3, y_5, y_7\}$ and $\{x, y_5, y_7, y_9\}$ we get $y_3y_7, y_5y_9 \in E(G)$. This contradicts (4). So we only need to consider the cases when $n = 8$ and 9. If $n = 9$, by (1), (3) and $d(y_5) \geq 3$, $y_1y_5 \in E(G)$. Consider the set $\{x, y_3, y_5, y_7\}$ we have $y_3y_7 \in E(G)$. Hence

$y_1xy_2y_3y_7y_6y_5y_1$ is a 7-cycle. If $n = 8$, by (1) and (3) we get that $d(y_5) = 2$, a contradiction.

Case b. Suppose $y_1y_2 \in E(G)$. Then G has a $(n-1)$-cycle $C_{n-1} = y_1y_2y_3 \cdots y_{n-1}y_1$. In this case, we may assume that (1) and (2) also hold for C_{n-1}. By (1) and (3), we can see that $n \geq 9$. If $n = 9$, then by (1) and (3) we have $y_1y_5, y_4y_8 \in E(G)$. Then $y_1xy_2y_3y_7y_6y_5y_1$ is a 7-cycle, a contradiction. So we assume $n \geq 10$. Considering the sets $\{x, y_3, y_5, y_7\}$, $\{x, y_4, y_6, y_8\}$ and $\{x, y_5, y_7, y_9\}$ we have $y_3y_7, y_4y_8, y_5y_9 \in E(G)$. This contradicts to (4).

Subcase 2.2. Suppose $\kappa(H) \leq 1$.

We have $\kappa(H) = 1$, otherwise $\omega(G - \{y_1, y_2\}) = 3$, which contradicts Theorem 2.6. Let y be a cut vertex of H, and H_1 and H_2 be the components of $H - \{y\}$. Set $V(H_1) = \{x_1, x_2, \ldots, x_p\}$ and $V(H_2) = \{z_1, z_2, \ldots, z_q\}$, where $p + q = n - 4 \geq 4$.

Since $\alpha(G) = 3$, we have the following

(a) H_1 and H_2 are complete.

(b) Either $G[V(H_1) \cup \{y\}]$ or $G[V(H_2) \cup \{y\}]$ is complete.

(c) $p, q \geq 2$. Otherwise, suppose that $V(H_1) = \{x_1\}$. By $d(x_1) \geq 3$, we have that $x_1y_1, x_1y_2 \in E(G)$. Since $x_1z_q \notin E(G)$, there exists a vertex t such that $[x_1, t] \to z_q$ or $[z_q, t] \to x_1$. In the case $[x_1, t] \to z_q$, in order to dominate x we have $t \in \{x, y_1, y_2\}$. If $t = x$, then the vertices of $V(H_2) - \{z_q\}$ cannot be dominated. If $t = y_1$ then $y_1 \Rightarrow V(H_2) - \{z_q\}$. Hence $\{y_2, z_1\} \Rightarrow V(G)$ which contradicts $\gamma(G) = 3$. If $t = y_2$ then $y_2 \Rightarrow V(H_2) - \{z_q\}$. Hence $\{y_1, z_1\} \Rightarrow V(G)$ which contradicts $\gamma(G) = 3$. In the case $[z_q, t] \to x_1$, $t = x$ in order to dominate x, i.e., $[z_q, x] \to x_1$. Hence $\{z_q, y_1\} \Rightarrow V(G)$.

From (b) without loss of generality we may assume $G[V(H_1) \cup \{y\}]$ is complete. Without loss of generality, we may assume there is a vertex in H_i adjacent to y_i, $i = 1, 2$. Otherwise, G is not 2-connected. If y_2 and y are adjacent with exactly one vertex in H_2. Then G is not Hamiltonian, which contradicts Theorem 4.4. Thus we may assume $y_1x_1, y_2z_q, yz_1 \in E(G)$. There is a path of length k_1 from y to y_1 in $G[V(H_1) \cup \{y, y_1\}]$ with $2 \leq k_1 \leq p+1$. There is a path of length k_2 from y_2 to y in $G[V(H_2) \cup \{y, y_2\}]$ with $3 \leq k_2 \leq q+1$. Thus we have a cycle of length ℓ with $7 \leq \ell \leq p+q+4 = n$. By (b) and (c), there is a 3-cycle. Thus, in the following we need to find cycles of lengths 4,5,6.

Since $x_1z_q \notin E(G)$, by symmetric we may assume that there exists a vertex t such that $[x_1, t] \to z_q$. In order to dominate x, $t \in \{x, y_1, y_2\}$. Since $y_2z_q \in E(G)$, $t \neq y_2$. In order to dominate z_1, $t \neq x$. Thus $t = y_1$ and hence $y_1z_1 \in E(G)$. Then $G - \{x, y_2, z_2, \ldots, z_q\}$ contains a cycle of length ℓ, where $4 \leq \ell \leq p + 3$ and $G - \{x_1, x_2, \ldots, x_p, y\}$ contains a cycle of length ℓ, where $5 \leq \ell \leq q + 3$. Since $p, q \geq 2$, we have a 4-cycle and a 5-cycle. We also get a 6-cycle unless $p = q = 2$. For $p = q = 2$, in order to dominate y_2, either $y_1y_2 \in E(G)$ or $x_1y_2 \in E(G)$. Then either $x_1yz_1z_2y_2y_1x_1$ or $x_1x_2yz_1z_2y_2x_1$ is a 6-cycle in G.

The proof of Theorem 4.8 is complete. □

Acknowledgment : The authors thank the referee and Prof. Feng Tian for valuable comments and suggestions.

References

[1] D. Amar, I. Fournier and A. Germa, Pancyclism in Chvátal-Erdős' graphs, *Graphs and Combin.* **7** (1991) 101-112.

[2] J.A. Bondy, Pancyclic graphs, *Proc. 2nd Louisiana Conf. Combin., Graph Theory, Computing; Baton Rouge* (1971), 167-172.

[3] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London, 1976.

[4] V. Bryant, *Aspects of Combinatorics*, Cambridge University Press, 1993.

[5] N. Chakroun and D. Sotteau, A Chvátal-Erdős Condition for Pancyclability in digraphs with stability number at most 3, *Proceedings of the workshop Cycles and Rays*, Montréal, Eds. Hahn, Sabidussi, Woodrow (1990), 75-86.

[6] Y.J. Chen and F. Tian, A new proof of Wojcicka's conjecture, *Discrete Appl. Math.*, **127** (2003), 545-554.

[7] O. Favaron, F. Tian and L. Zhang, Independence and hamiltonicity in 3-domination-critical graphs, *J. of Graph Theory*, **25** (1997), 173-184.

[8] E. Flandrin, F. Tian, B. Wei and L. Zhang, Some properties of 3-domination-critical graphs, *Discrete Math.*, **205** (1999), 65-76.

[9] B. Shao and B. Wei, Pancyclism of 3-domination-critical graphs, *Thesis of Master of Sci.*, Institute of Systems Science, The Chinese Academy of Sciences, 2001.

[10] D.P. Sumner, Critical concepts in domination, *Discrete Math.*, **86** (1990), 33-46.

[11] D.P. Sumner and P. Blitch, Domination critical graphs, *J. Combin. Theory (B)*, **34** (1983), 65-76.

[12] F. Tian, B. Wei and L. Zhang, Hamiltonicity in 3-domination-critical graphs with $\alpha = \delta + 2$, *Discrete Appl. Math.*, **92** (1999), 57-70.

[13] E. Wojcicka, Hamiltonian properties of domination-critical graphs, *J. Graph Theory*, **14** (1990), 205-215.

[14] Y.F. Xue and Z.Q. Chen, Hamilton cycles in domination-critical graphs, *J. Nanjing University* (Natural Science Edition, Special Issue on Graph Theory), **27** (1991), 58-62.

[15] L. Z. Zhang and F. Tian, Independence and connectivity in 3-domination-critical graphs, *Discrete Math.*, **259** (2002), 227-236.