Edge-magicness of the composition of a cycle with a null graph[1]

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Abstract

Given two graphs G and H. The composition of G with H is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. In this paper, we prove by construction that the composition of a cycle with a null graph is edge-magic.

Key words and phrases : Edge-magic, composition of graphs, Cayley graph Latin square, group matrix, magic square.

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1. Introduction

The study of edge-magic graphs was initiated by the third author, Seah and Tan.[1] Let $G = (V, E)$ be a (p, q)-graph, i.e., $|V| = p$ and $|E| = q$. If there exists a bijection

$$f : E \to \{k,\ k+1,\ \cdots,\ q-1+k\}$$

for some $k \in \mathbb{Z}$ such that the map $f^+(u) = \sum_{v \in N(u)} f(uv)$ induces a constant map from V to $\mathbb{Z}_p$, then G is called *k-edge-magic* and f is called a *k-edge-magic labeling* of G. If $k = 1$, then G is simply called *edge-magic* and f an *edge-magic labeling* of G. It was shown that a (p, q)-graph is edge-magic only if p divides $q(q + 1)$.

The concept of magic graphs was introduced by Sedláček in 1963,[2,3] and was further developed by several researchers, see [4–7].

The notion of edge-magic graphs can be viewed as the dual concept of edge-graceful graphs which was introduced by Lo in 1985.[8] G is said to be edge-graceful if there exists a bijection

$$f : E \to \{1,\ 2,\ \cdots,\ q\}$$

such that the induce map f^+ (defined above) is a bijection from V onto $\mathbb{Z}_p$. There exist extensive literature on edge-graceful graphs [8–18].

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Given two graphs G and H. The composition of G with H, denoted as $G \circ H$ or $(G[H])$, is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. For example, $C_3 \circ K_2$ is shown in Figure 1.

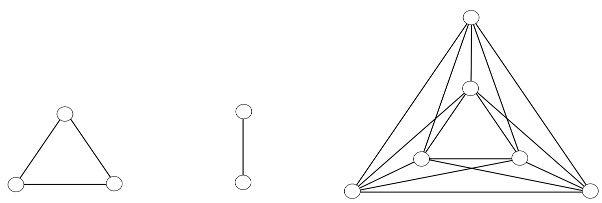

Figure 1

In this paper, we shall prove that for most values of m and n, $C_m \circ N_n$, where C_m is the m-cycle and N_n is the null graph on n vertices, is edge-magic.

The corresponding problem, edge-gracefulness of $C_m \circ N_n$, was considered by Seah and the third author in 1991[14] when m is odd. For other classes of edge-magic graphs, the reader is referred to [19] and [20].

2. Edge-Magicness of Regular Graphs

If $G = (V, E)$ is an r-regular (p, q)-graph, then $2q = pr$. Suppose $f : E \to \{1, 2, \cdots, q\}$ is a bijection. For any integer k, we can define a bijection $g : E \to \{k, k+1, \cdots, k+q-1\}$ by $g(e) = f(e) + k - 1$ for any $e \in E$. Then $g^+(u) = f^+(u) + r(k-1)$ and $\sum_{u \in V} g^+(u) = 2\sum_{e \in E} g(e) = q(q-1+2k)$. If f is edge-magic, then there is a c such that $f^+(u) \equiv c \pmod p$ for each $u \in V$, and

$$\sum_{u \in V} g^+(u) = \sum_{u \in V} [f^+(u) + r(k-1)] \equiv \sum_{u \in V} [c + r(k-1)] \equiv 0 \pmod p.$$

Therefore a regular graph is k-edge-magic for any $k \in \mathbb{Z}$ if and only if it is edge-magic. Moreover, $p | q(q + 2k - 1)$ for any $k \in \mathbb{Z}$. So we have:

Proposition 2.1: *Suppose G is a regular (p, q)-graph. If q is even then $p|q$.*

The above proposition follows from the facts $p|2q$ and $p|q(q + 1)$. This proposition gives a necessary condition for regular (p, q)-graphs with even q to be edge-magic. We have as examples in Figure 2 two graphs which are not edge-magic. When q is odd, the condition $p|q$ is not necessary. For example, the third graph in Figure 2 is a 3-regular graph which is edge-magic.

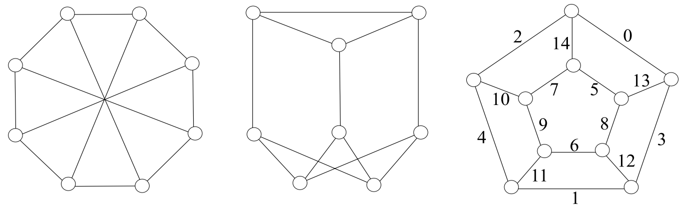

Figure 2

In this paper we shall only consider simple regular graphs, and we shall label the edges of graphs by numbers $0,\ 1,\ \cdots,\ q-1$.

Definition: Let $G = (V, E)$ be a graph and S be a set. Suppose $f : E \to S$ is a mapping. A *labeling matrix* for a labeling f of G is a matrix whose rows and columns are named by the vertices of G and the (u, v)-entry is $f(uv)$ if $uv \in E$, and is $*$ otherwise. The label $f(uv)$ is sometimes written as $f(u, v)$.

Thus a regular (p, q)-graph $G = (V, E)$ is edge-magic if and only if there exists a bijection $f : E \to \{0,\ 1,\ \cdots,\ q-1\}$ such that the row sums and the column sums modulo p of the labeling matrix of G associated with f are all equal. For purposes of these sums, entries with label $*$ are treated as 0.

3. Edge-Magic Labeling of $C_m \circ N_n$

In this section, we shall prove that $C_m \circ N_n$ is edge-magic (we identify C_2 as P_2). For ease of illustration we shall consider $C_m \circ N_n$ as a Cayley graph which is described below.

Let $\mathfrak{C}_m = \langle g \rangle$ be the (multiplicative) cyclic group of order m (≥ 2) generated by g. Let $H = \{h_0 = e,\ h_1,\ \cdots,\ h_{n-1}\}$ be any group of order n, where $n \geq 2$ and e is the identity of H. Throughout this paper we shall use e to denote the identity of a group. Let $\mathfrak{C}_m\{H\}$ denote the Cayley graph of $\mathfrak{C}_m \times H$ generated by $\{g,\ g^{-1}\} \times H$ (for $m = 2$, the generating set is $\{g\} \times H$).

For $m \geq 3$, $\mathfrak{C}_m\{H\}$ is an (mn, mn^2)-graph; and $\mathfrak{C}_2\{H\}$ is an $(2n, n^2)$-graph. Moreover, $\mathfrak{C}_m\{H\}$ is isomorphic to $C_m \circ N_n$ for $m \geq 2$. Note that we may view $\mathfrak{C}_m\{H\}$ as a (simple) graph. For simplicity, we identify $(g^i, x) \in \mathfrak{C}_m \times H$ with $g^i x$ and choose $H = \mathfrak{C}_n = \langle h \rangle$.

When $m = 2$ and $n \geq 2$, $\mathfrak{C}_2\{\mathfrak{C}_n\} \cong K_{n,n}$. We can verify that $K_{2,2}$ is not edge-magic. Since magic square of any order higher than 2 always exists (see [21] and [22]), $K_{n,n}$ is edge-magic for $n \geq 3$. So we may assume that $m \geq 3$ and $n \geq 2$.

We list the elements of $\mathfrak{C}_n$ in the following order $\{e = h^0,\ h^1,\ h^2,\ \cdots,\ h^{n-1}\}$, and list the elements of $\mathfrak{C}_m \times \mathfrak{C}_n$ in the following order: $\{e\mathfrak{C}_n = g^0\mathfrak{C}_n,\ g^1\mathfrak{C}_n,\ \cdots,\ g^{m-1}\mathfrak{C}_n\}$. If $f : E \to S$ is a mapping, then the labeling matrix of f is

$$\begin{pmatrix} * & A_0 & * & \ddots & * & A_{m-1}^T \\ A_0^T & * & A_1 & \ddots & \ddots & \ddots \\ * & A_1^T & * & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ * & * & * & \ddots & * & A_{m-2} \\ A_{m-1} & * & * & \ddots & A_{m-2}^T & * \end{pmatrix}. \tag{3.1}$$

This matrix is visualized as an $m \times m$ matrix $X = (x_{ij})$, each entry of which is occupied by an $n \times n$ matrix. For $i = 1,\ 2,\ \cdots, m$, the entry $x_{i,i+1}$ is the $n \times n$ matrix A_{i-1} and the entry $x_{i+1,i}$ is A_{i-1}^T, where $i+1$ is taken to be 1 if $i = m$ and $A_i = \left(a_{i'j'}^{(i)}\right)$. For $1 \leq i',\ j' \leq n$, the entry $a_{i'j'}^{(i)}$ of A_i corresponds to the $(g^i h^{i'-1}, g^i h^{j'-1})$-entry of the labeling matrix, and has the label $f(g^i h^{i'-1}, g^i h^{j'-1})$. Each remaining entry of X is occupied by an $n \times n$ matrix of $*$'s, meaning that no edge connects the corresponding vertices.

We shall use $S \times n$ to denote the multi-set which is an n-copies of a set S. Note that S may be a multi-set itself. Thus $f : E \to \{0,\ 1, \cdots,\ mn-1\} \times n$ is an edge-magic labeling of $\mathfrak{C}_m\{\mathfrak{C}_n\}$ if and only if row sums modulo mn and column sums modulo mn of the above matrix are constant, and the problem is reduced to determining whether we can assign $\{0,\ 1, \cdots,\ mn-1\} \times n$ into entries of m matrices A_i such that row sums modulo mn and column sums modulo mn of the above matrix are constant.

Let S be a multi-set whose elements are numbers. For $m,\ n \geq 2$, if there is a partition of S containing m classes such that each class has n elements and whose sum in each class is the same, then we call S has an (m, n)-*balance partition.*

Lemma 3.1: *If n is even, then $\{0,\ 1, \cdots,\ mn-1\}$ has an (m, n)-balance partition.*

Proof: Put $A_\uparrow = (0,\ 1,\ \cdots,\ m-1)$ and $A_\downarrow = (m-1,\ m-2,\ \cdots,\ 1,\ 0)$. Let

$$S_j = \begin{cases} A_\uparrow + mj\vec{\mathbf{1}} & \text{if } j \text{ is even}, \\ A_\downarrow + mj\vec{\mathbf{1}} & \text{if } j \text{ is odd}, \end{cases}$$

for $0 \leq j \leq n-1$, where $\vec{\mathbf{1}} = (1,\ 1,\ \cdots,\ 1)$. Construct an $n \times m$ matrix whose rows are $S_0,\ S_1,\ \cdots,\ S_{n-1}$ respectively. Then each column of this matrix is a required class. ∎

Lemma 3.2: *If m is odd, then $\{0,\ 1,\ \cdots,\ m-1\} \times 3$ has an $(m, 3)$-balance partition.*

Proof: Define three row vectors as follows: $A_\uparrow = (0,\ 1,\ \cdots,\ m-1)$, $B = (b_0,\ b_1,\ \cdots, b_{n-1})$ and $C = (c_0,\ c_1,\ \cdots, c_{n-1})$, where $b_i = i + \frac{m-1}{2}$, $c_i = m - 1 - 2i$, $0 \le i \le m-1$, and the arithmetic are taken in $\mathbb{Z}_m = \{0,\ 1,\ \cdots,\ m-1\}$. Columns of the $3 \times m$ matrix whose rows are $A_\uparrow$, B and C respectively are the required classes. ▮

Lemma 3.3: *If both n and m are odd, then $\{0,\ 1, \cdots,\ mn-1\}$ has an (m,n)-balance partition.*

Proof: From the proof of Lemma 3.1, we get $S_0,\ S_1,\ \cdots,\ S_{n-4}$ (omit this process if $n = 3$). Define $A_\uparrow$, B and C as in Lemma 3.2. Let $S_{n-i} = A_i + (n-i)m\vec{1}$, where $1 \le i \le 3$, $A_3 = A_\uparrow$, $A_2 = B$, $A_1 = C$ and $\vec{1} = (1,\ 1,\ \cdots,\ 1)$. Columns of the $n \times m$-matrix whose rows are $S_0,\ S_1,\ \cdots,\ S_{n-1}$ respectively are the required classes. ▮

Theorem 3.4: *Suppose $m \ge 3$ and $n \ge 2$. If n is even or both m and n are odd, then $\mathfrak{C}_m\{\mathfrak{C}_n\}$ is edge-magic.*

Proof: Use the m classes of $\{0,\ 1,\ \cdots,\ mn-1\}$ constructed from Lemma 3.1 or 3.3 to construct m Latin squares with entries in the corresponding classes. Let these Latin squares be $A_0,\ A_1,\ \cdots,\ A_{m-1}$. Substituting into (3.1), we obtain a labeling matrix of $\mathfrak{C}_m\{\mathfrak{C}_n\}$ and an edge-magic labeling of $\mathfrak{C}_m\{\mathfrak{C}_n\}$. ▮

A Latin square is a square matrix in which each row and each column consists of the same set of entries without repetition.[23] A Latin square can be constructed from a group matrix or a submatrix of a group matrix whose rows and columns are named by elements of two cosets. The following definition of group matrix can be found in [24] and [25].

Let $\alpha : G \to S$ be a mapping from a finite group $G = \{g_1 = e,\ g_2,\ \cdots,\ g_n\}$ to a set S. A *group matrix* of G associated with α is an $n \times n$ matrix whose (i,j)-th entry (or (g_i, g_j)-entry) is $\alpha(g_i^{-1}g_j)$. This group matrix is in effect formed by renaming entries of the multiplication table of G under α.

The simplest way to construct a Latin square is by putting $G = \{e = g^0,\ g^1,\ \cdots,\ g^{n-1}\}$, obtaining a cyclic Latin square (see [26]).

Example 3.1: Consider $\mathfrak{C}_3\{\mathfrak{C}_6\}$, i.e. $m = 3$ and $n = 6$. We have a matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 5 & 4 & 3 \\ 6 & 7 & 8 \\ 11 & 10 & 9 \\ 12 & 13 & 14 \\ 17 & 16 & 15 \end{pmatrix}.$$

and three classes $\{0,\ 5,\ 6,\ 11,\ 12,\ 17\}$, $\{1,\ 4,\ 7,\ 10,\ 13,\ 16\}$, and $\{2,\ 3,\ 8,\ 9,\ 14,\ 15\}$. A labeling matrix for an edge-magic labeling of $\mathfrak{C}_3\{\mathfrak{C}_6\}$ is:

$$\left(\begin{array}{c|cccccc|cccccc|cccccc}
e & \ast & \ast & \ast & \ast & \ast & \ast & 0 & 5 & 6 & 11 & 12 & 17 & 2 & 8 & 3 & 9 & 14 & 15 \\
h & \ast & \ast & \ast & \ast & \ast & \ast & 6 & 0 & 5 & 17 & 11 & 12 & 3 & 2 & 8 & 15 & 9 & 14 \\
h^2 & \ast & \ast & \ast & \ast & \ast & \ast & 5 & 6 & 0 & 12 & 17 & 11 & 8 & 3 & 2 & 14 & 15 & 9 \\
h^3 & \ast & \ast & \ast & \ast & \ast & \ast & 11 & 17 & 12 & 0 & 6 & 5 & 9 & 15 & 14 & 2 & 3 & 8 \\
h^4 & \ast & \ast & \ast & \ast & \ast & \ast & 12 & 11 & 17 & 5 & 0 & 6 & 14 & 9 & 15 & 8 & 2 & 3 \\
h^5 & \ast & \ast & \ast & \ast & \ast & \ast & 17 & 12 & 11 & 6 & 5 & 0 & 15 & 14 & 9 & 3 & 8 & 2 \\
\hline
g & 0 & 6 & 5 & 11 & 12 & 17 & \ast & \ast & \ast & \ast & \ast & \ast & 1 & 4 & 7 & 10 & 13 & 16 \\
gh & 5 & 0 & 6 & 17 & 11 & 12 & \ast & \ast & \ast & \ast & \ast & \ast & 7 & 1 & 4 & 16 & 10 & 13 \\
gh^2 & 6 & 5 & 0 & 12 & 17 & 11 & \ast & \ast & \ast & \ast & \ast & \ast & 4 & 7 & 1 & 13 & 16 & 10 \\
gh^3 & 11 & 17 & 12 & 0 & 5 & 6 & \ast & \ast & \ast & \ast & \ast & \ast & 10 & 16 & 13 & 1 & 7 & 4 \\
gh^4 & 12 & 11 & 17 & 6 & 0 & 5 & \ast & \ast & \ast & \ast & \ast & \ast & 13 & 10 & 16 & 4 & 1 & 7 \\
gh^5 & 17 & 12 & 11 & 5 & 6 & 0 & \ast & \ast & \ast & \ast & \ast & \ast & 16 & 13 & 10 & 7 & 4 & 1 \\
\hline
g^2 & 2 & 3 & 8 & 9 & 14 & 15 & 1 & 7 & 4 & 10 & 13 & 16 & \ast & \ast & \ast & \ast & \ast & \ast \\
g^2h & 8 & 2 & 3 & 15 & 9 & 14 & 4 & 1 & 7 & 16 & 10 & 13 & \ast & \ast & \ast & \ast & \ast & \ast \\
g^2h^2 & 3 & 8 & 2 & 14 & 15 & 9 & 7 & 4 & 1 & 13 & 16 & 10 & \ast & \ast & \ast & \ast & \ast & \ast \\
g^2h^3 & 9 & 15 & 14 & 2 & 8 & 3 & 10 & 16 & 13 & 1 & 4 & 7 & \ast & \ast & \ast & \ast & \ast & \ast \\
g^2h^4 & 14 & 9 & 15 & 3 & 2 & 8 & 13 & 10 & 16 & 7 & 1 & 4 & \ast & \ast & \ast & \ast & \ast & \ast \\
g^2h^5 & 15 & 14 & 9 & 8 & 3 & 2 & 16 & 13 & 10 & 4 & 7 & 1 & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}\right),$$

where the first column gives names of rows, and names of columns follow the same order. Note that Latin squares in the above matrix are obtained from a group matrix of the dihedral group $D = \langle x, y|\ x^3 = y^2 = e,\ xyx = y\rangle = \{e,\ x,\ x^2,\ y, yx,\ yx^2\}$.

Example 3.2: Consider $\mathfrak{C}_4\{\mathfrak{C}_4\}$, i.e. $m = 4$ and $n = 4$. We have a matrix

$$\left(\begin{array}{cccc}
0 & 1 & 2 & 3 \\
7 & 6 & 5 & 4 \\
8 & 9 & 10 & 11 \\
15 & 14 & 13 & 12
\end{array}\right).$$

and four classes $\{0,\ 7,\ 8,\ 15\}$, $\{1,\ 6,\ 9,\ 14\}$, $\{2,\ 5,\ 10,\ 13\}$ and $\{3,\ 4,\ 11,\ 12\}$. A labeling matrix for an edge-magic labeling of $\mathfrak{C}_4\{\mathfrak{C}_4\}$ is:

$$\left(\begin{array}{c|cccc|cccc|cccc|cccc}
e & \ast & \ast & \ast & \ast & 0 & 7 & 8 & 15 & \ast & \ast & \ast & \ast & 3 & 12 & 11 & 4 \\
h & \ast & \ast & \ast & \ast & 15 & 0 & 7 & 8 & \ast & \ast & \ast & \ast & 4 & 3 & 12 & 11 \\
h^2 & \ast & \ast & \ast & \ast & 8 & 15 & 0 & 7 & \ast & \ast & \ast & \ast & 11 & 4 & 3 & 12 \\
h^3 & \ast & \ast & \ast & \ast & 7 & 8 & 15 & 0 & \ast & \ast & \ast & \ast & 12 & 11 & 4 & 3 \\
\hline
g & 0 & 15 & 8 & 7 & \ast & \ast & \ast & \ast & 1 & 6 & 9 & 14 & \ast & \ast & \ast & \ast \\
gh & 7 & 0 & 15 & 8 & \ast & \ast & \ast & \ast & 14 & 1 & 6 & 9 & \ast & \ast & \ast & \ast \\
gh^2 & 8 & 7 & 0 & 15 & \ast & \ast & \ast & \ast & 9 & 14 & 1 & 6 & \ast & \ast & \ast & \ast \\
gh^3 & 15 & 8 & 7 & 0 & \ast & \ast & \ast & \ast & 6 & 9 & 14 & 1 & \ast & \ast & \ast & \ast \\
\hline
g^2 & \ast & \ast & \ast & \ast & 1 & 14 & 9 & 6 & \ast & \ast & \ast & \ast & 2 & 5 & 10 & 13 \\
g^2h & \ast & \ast & \ast & \ast & 6 & 1 & 14 & 9 & \ast & \ast & \ast & \ast & 13 & 2 & 5 & 10 \\
g^2h^2 & \ast & \ast & \ast & \ast & 9 & 6 & 1 & 14 & \ast & \ast & \ast & \ast & 10 & 13 & 2 & 5 \\
g^2h^3 & \ast & \ast & \ast & \ast & 14 & 9 & 6 & 1 & \ast & \ast & \ast & \ast & 5 & 10 & 13 & 2 \\
\hline
g^3 & 3 & 4 & 11 & 12 & \ast & \ast & \ast & \ast & 2 & 13 & 10 & 5 & \ast & \ast & \ast & \ast \\
g^3h & 12 & 3 & 4 & 11 & \ast & \ast & \ast & \ast & 5 & 2 & 13 & 10 & \ast & \ast & \ast & \ast \\
g^3h^2 & 11 & 12 & 3 & 4 & \ast & \ast & \ast & \ast & 10 & 5 & 2 & 13 & \ast & \ast & \ast & \ast \\
g^3h^3 & 4 & 11 & 12 & 3 & \ast & \ast & \ast & \ast & 13 & 10 & 5 & 2 & \ast & \ast & \ast & \ast
\end{array}\right).$$

Example 3.3: Consider $\mathfrak{C}_5\{\mathfrak{C}_3\}$, i.e. $m = 5$ and $n = 3$. We have a matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 7 & 8 & 9 & 5 & 6 \\ 14 & 12 & 10 & 13 & 11 \end{pmatrix}.$$

and five classes $\{0,\ 7,\ 14\}$, $\{1,\ 8,\ 10\}$, $\{2,\ 9,\ 10\}$, $\{3,\ 5,\ 13\}$, and $\{4,\ 6,\ 11\}$. We may use these classes to construct 5 Latin squares and get a labeling matrix for an edge-magic labeling of $\mathfrak{C}_5\{\mathfrak{C}_3\}$ as in the previous two examples.

When n is odd and m is even, m does not divide $\frac{1}{2}mn(mn-1)$ and $\{0,\ 1,\ 2,\ \cdots,\ mn-1\}$ does not have an (m, n)-balance partition. However, in this case, the existence of an (m, n)-balance partition for $\{0.\ 1,\ 2,\ \cdots,\ mn-1\}$ is not necessary for constructing an edge-magic labeling of $\mathfrak{C}_m\{\mathfrak{C}_n\}$. It suffices to have two types of $\frac{m}{2}$ Latin squares, where row sums and column sums of the same type of Latin squares are all equal.

Let S be a multi-set whose elements are numbers, and $m,\ n \geq 2$. Suppose there is a partition of S into m classes with n elements in each class. If the sums of elements in $\frac{m}{2}$ of the classes are all equal to one value, and the sums of elements in the remaining classes are all equal to another value, then we call S has an (m, n)-*semi-balance partition.*

Lemma 3.5: *If m is even, then $\{0,\ 1,\ \cdots,\ m-1\} \times 3$ has an $(m, 3)$-semi-balance partition.*

Proof: Let $A_\uparrow = (a_0,\ a_1,\ \cdots,\ a_{m-1}) = (0,\ 1,\ \cdots,\ m-1)$. Put $D = (d_0,\ d_1,\ \cdots,\ d_{m-1})$, with $d_{2i} = m-1-i$ and $d_{2i+1} = \frac{m}{2}-1-i$ for $0 \leq i \leq \frac{m}{2}-1$. We have $\{d_0,\ d_1,\ \cdots,\ d_{m-1}\} = \{0,\ 1,\ \cdots,\ m-1\}$. Since $a_{2i} + 2d_{2i} = 2(m-1)$ and $a_{2i+1} + 2d_{2i+1} = m-1$, the required classes are the columns of the $3 \times m$ matrix whose rows are $A_\uparrow$, D and D. ∎

Lemma 3.6: *If n is odd and m is even, then $\{0,\ 1,\ \cdots,\ mn-1\}$ has an (m, n)-semi-balance partition.*

Proof: From the proof of Lemma 3.1, we get $S_0,\ S_1,\ \cdots, S_{n-4}$ (omit this process if $n = 3$). Let $A_\uparrow$ and D be defined as in Lemma 3.5. Let $S_{n-3} = A_\uparrow + (n-3)m\vec{\mathbf{1}}$, $S_{n-2} = D + (n-2)m\vec{\mathbf{1}}$ and $S_{n-1} = D + (n-1)m\vec{\mathbf{1}}$. Construct an $n \times m$ matrix whose rows are $S_0,\ S_1,\ \cdots, S_{n-1}$ respectively. Then columns of this matrix are the required classes. ∎

Theorem 3.7: *Suppose $m \geq 3$ and $n \geq 2$. If n is odd and m is even, then $\mathfrak{C}_m\{\mathfrak{C}_n\}$ is edge-magic.*

Proof: Suppose X is the matrix constructed from Lemma 3.6. Let A_j be a Latin squares with entries of the j-th column of X. Substituting A_j into (3.1), we obtain a labeling matrix for an edge-magic labeling of $\mathfrak{C}_m\{\mathfrak{C}_n\}$. ∎

The following main result of this paper follows from Theorems 3.4 and 3.7.

Theorem 3.8: *If* $m,\ n \geq 2$ *and* $(m,n) \neq (2,2)$, *then* $C_m \circ N_n$ *is edge-magic.*

We conclude this paper with the following:

Example 3.4: Consider $\mathfrak{C}_4\{\mathfrak{C}_3\}$, i.e. $m = 4$ and $n = 3$. We have a matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 7 & 5 & 6 & 4 \\ 11 & 9 & 10 & 8 \end{pmatrix}.$$

and four classes $\{0,\ 7,\ 11\}$, $\{1,\ 5,\ 9\}$, $\{2,\ 6,\ 10\}$, and $\{3,\ 4,\ 8\}$. An edge-magic labeling of $\mathfrak{C}_4\{\mathfrak{C}_3\}$ is represented by:

$$\left(\begin{array}{c|ccc|ccc|ccc|ccc}
e & * & * & * & 0 & 7 & 11 & * & * & * & 3 & 8 & 4 \\
h & * & * & * & 11 & 0 & 7 & * & * & * & 4 & 3 & 8 \\
h^2 & * & * & * & 7 & 11 & 0 & * & * & * & 8 & 4 & 3 \\
\hline
g & 0 & 11 & 7 & * & * & * & 1 & 5 & 9 & * & * & * \\
gh & 7 & 0 & 11 & * & * & * & 9 & 1 & 5 & * & * & * \\
gh^2 & 11 & 7 & 0 & * & * & * & 5 & 9 & 1 & * & * & * \\
\hline
g^2 & * & * & * & 1 & 9 & 5 & * & * & * & 2 & 6 & 10 \\
g^2h & * & * & * & 5 & 1 & 9 & * & * & * & 10 & 2 & 6 \\
g^2h^2 & * & * & * & 9 & 5 & 1 & * & * & * & 6 & 10 & 2 \\
\hline
g^3 & 3 & 4 & 8 & * & * & * & 2 & 10 & 6 & * & * & * \\
g^3h & 8 & 3 & 4 & * & * & * & 6 & 2 & 10 & * & * & * \\
g^3h^2 & 4 & 8 & 3 & * & * & * & 10 & 6 & 2 & * & * & *
\end{array}\right).$$

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