The non-edge-magic simple connected cubic graph of order 10

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In memory of my colleague Dr. C.I. Chu (1947 – Jan. 2, 2011)

Abstract

In 2003, Lee, Wang and Wen found a non-edge-magic simple connected cubic graph which satisfying the necessary condition of edge-magicness by using computer search. They asked for a mathematical proof. In this paper, we will provide such a proof.

Keywords: Edge-magic, cubic graph.

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1 Introduction

Let $G = (V, E)$ be a (p, q)-graph, i.e., $|V| = p$ and $|E| = q$. Let $f : E \to \{d, d+1, \ldots, d+q-1\}$ be a bijection for some $d \in \mathbb{Z}$. The *induced mapping* $f^+ : V \to \mathbb{Z}_r$ of f is defined by $f^+(u) = \sum\limits_{uv\in E} f(uv)$ for $u \in V$, the sum is taken in $\mathbb{Z}_r$ for some $r \geq 0$. Note that we denote $\mathbb{Z}$ by $\mathbb{Z}_0$. If f^+ is a constant mapping, then G is called *d-edge-magic over* $\mathbb{Z}_r$. If $d = 1$, then G is simply called *edge-magic over* $\mathbb{Z}_r$, f an *edge-magic labeling of* G *over* $\mathbb{Z}_r$ and the value of f^+ an *edge-magic value of* G *over* $\mathbb{Z}_r$. This concept was first introduced by Shiu and Lee [12] in 2002. Moreover, G being edge-magic over $\mathbb{Z}_p$ or $\mathbb{Z}$ is called *edge-magic* or *supermagic*, the labeling f is called an *edge-magic labeling* or *supermagic labeling*, respectively. These concepts were introduced by Lee, Seah and Tan [6] in 1992 and Stewart [15] in 1966, respectively. Note that edge-magic value is not unique in general.

It is easy to see the following theorem.

Theorem 1.1 ([12]). *Suppose G is d-edge-magic over $\mathbb{Z}_r, r \geq 0$. Then G is d-edge-magic over $\mathbb{Z}_s$ if s is a factor of r.*

It was shown that if a (p,q)-graph is edge-magic then p divides $q(q+1)$ (see [10]). Some edge-magic or supermagic graphs were found [3, 6–16]. More about supermagic graphs can be found in [1,2,4,5]. For regular graph, there is no different between d-edge-magic and edge-magic (see [10,12,13]).

The necessary condition holds for cubic graph only if $p \equiv 2 \pmod 4$. Some cubic multi-graphs and simple disconnected cubic graphs, which satisfy the necessary condition but are not edge-magic, were found [12]. It was conjectured in [12] that *every simple connected cubic graph of order $p \equiv 2$ (mod 4) is edge-magic.*

It is known that the only simple connected cubic graphs of order 6 are $K_{3,3}$ and $C_3 \times K_2$. They are edge-magic [7]. Lee *et al.* [8] showed by using more than three weeks computer time that only the graph described in Fig. 1 is not edge-magic among all simple connected cubic graphs of order 10. So this graph is the smallest non-edge-magic simple connected cubic graph satisfying the necessary condition. That means that the conjecture proposed in [12] is false. They asked for a mathematical proof. In this paper, we shall provide such a proof.

For convenience, we shall use $[n]$ to denote the set $\{1, 2, \dots, n\}$ for a positive integer n. Let S be a set. We use $S \times n$ to denote the multiset of n-copies of S. Note that S may be a multiset itself. From now on, the term "set" means multiset. Set operations are viewed as multiset operations. Let S and T be sets of integers. $S \equiv T \pmod r$ means that two sets are equal after their elements are taken modulo r, where $r \geq 2$.

2 An algorithm for finding cycles

Let G be a simple connected graph of order p with vertex set $[p]$. Following is a depth-first search (DFS) algorithm for finding cycles:

Starts by setting vertex 1 as the root and the current vertex.

Iteration (i is the current vertex): Suppose that there is an unmarked edge, say $\{i,j\}$. If vertex j was already visited by DFS, then mark edge (i,j) as a *back edge* otherwise mark it as a *tree edge* and set i as the parent of j. Set vertex j as the current vertex. Otherwise (all edges incident with

i are marked), set vertex k as the current vertex, where k is the parent of i. If the algorithm returns to the root and all edges incident with the root are marked, then choose another vertex which is incident with some unmarked edges as the root and the current vertex. The algorithm stops when all edges are marked. Then all the tree edges induce a spanning tree of G.

Each back edge defines uniquely a cycle which consists of the back edge (i, j) and the unique (j, i)-path in the spanning tree found by DFS algorithm. It is called a *basic cycle*. The set of all basic cycles is called a *cycle basis*. Suppose G is a connected (p, q)-graph. The number of back edges is always $q - (p - 1)$. It is because that the number of edges in a spanning tree is $p - 1$. So a cycle basis is a set of $q - (p - 1)$ cycles.

Theorem 2.1. *Let G be a connected simple graph. The set of all 2-regular subgraphs of G together with the empty set forms a vector space over $\mathbb{Z}_2$ under the symmetric difference $\bigoplus$ (exclusive OR). Note that, in this vector space, the zero vector is the empty set. Moreover, a cycle basis is a basis of the vector space.*

Proof: It is easy to prove the first statement. So we only prove the second statement. Let T be a spanning tree found by DFS algorithm above. Since there is a bijection between cycle basis and the set of back edges, cycle basis is a linearly independent set. Suppose H is a 2-regular subgraph of G. Let $e_1, \dots, e_k$ be edges of H not in T. That means that $e_1, \dots, e_k$ are back edges. Let α_i be the basic cycle corresponding to e_i, $1 \le i \le k$. Then $H \bigoplus (\bigoplus\limits_{i=1}^{k} \alpha_i)$ does not contain any back edge. Hence it is either the empty set or a 2-regular subgraph of T. But the last case is impossible. Hence we have $H = \bigoplus\limits_{i=1}^{k} \alpha_i$. □

3 Main result

In this section we will prove that the graph G in Fig. 1 is not edge-magic.

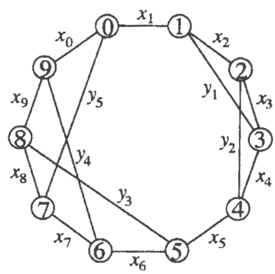

Figure 1: The graph G.

Applying the algorithm on G, we have

edge (1,2): tree	edge (2,3): tree	edge (3,4): tree
edge (4,5): tree	edge (5,6): tree	edge (6,7): tree
edge (7,8): tree	edge (8,9): tree	edge (9,0): tree

edge (0,1): back: (1234567890)	edge (1,3): back: (123)
edge (2,4): back: (234)	edge (5,8): back: (5678)
edge (7,0): back: (7890)	edge (6,9): back: (6789)

We have basis cycles $\alpha_1 = (1234567890)$, $\alpha_2 = (123)$, $\alpha_3 = (234)$, $\alpha_4 = (5678)$, $\alpha_5 = (6789)$ and $\alpha_6 = (7890)$.

Each 2-regular subgraph is formed by $\bigoplus_{i=1}^{6} a_i\alpha_i$, where $a_i \in \mathbb{Z}_2$. So we have totally $2^6 - 1$ 2-regular subgraphs.

$a_1 \cdots a_6$	subgraph	length	mark	$a_1 \cdots a_6$	subgraph	length	mark
100000	α_1	10		000000	$\varnothing$	0	
110000	(134567890)	9		010000	α_2	3	
101000	(124567890)	9		001000	α_3	3	
100100	(12345890)	8		000100	α_4	4	
100010	(12345690)	8		000010	α_5	4	
100001	(12345670)	8		000001	α_6	4	
111000	(2310987654)	10		011000	(1243)	4	
110100	(1345890)	7	*1	010100	(123)(5678)	3+4	*5
110010	(1345690)	7	*2	010010	(123)(6789)	3+4	*6
110001	(1345670)	7	*3	010001	(123)(7890)	3+4	*7
101100	(1245890)	7	*3	001100	(234)(5678)	3+4	*7
101010	(1245690)	7	*2	001010	(234)(6789)	3+4	*6
101001	(1245670)	7	*1	001001	(234)(7890)	3+4	*5
100110	(1234587690)	10		000110	(5698)	4	
100101	(12345870)	8		000101	(567098)	6	
100011	(1234569870)	10		000011	(6709)	4	
111100	(13245890)	8		011100	(1243)(5678)	4+4	
111010	(13245690)	8		011010	(1243)(6789)	4+4	
111001	(13245670)	8		011001	(1243)(7890)	4+4	

110110	(134587690)	9		010110	(123)(5698)	3+4	*8
110101	(1345870)	7	*4	010101	(123)(567098)	3+6	
110011	(134569870)	9		010011	(123)(6709)	3+4	*9
101110	(124587690)	9		001110	(234)(5698)	3+4	*9
101101	(1245870)	7	*4	001101	(234)(567098)	3+6	
101011	(124569870)	9		001011	(234)(6709)	3+4	*8
100111	(1234589670)	10		000111	(569078)	6	
111110	(2310967854)	10		011110	(1243)(5698)	4+4	
111101	(23107854)	8		011101	(1243)(567098)	4+6	
111011	(2310789654)	10		011011	(1243)(6709)	4+4	
110111	(310769854)	9		010111	(123)(569078)	3+6	
101111	(210769854)	9		001111	(234)(569078)	3+6	
111111	(2310769854)	10		011111	(1243)(569078)	4+6	

We fix the label set $S = [\mathbf{15}]$. We shall also call each number in S to be even or odd when that number is even or odd as an integer, respectively. It does not affect the parity of numbers in S if we consider $S \equiv [\mathbf{10}] \cup [\mathbf{5}] \pmod{10}$. Let f be an edge-magic labeling of G with magic value m. And let $f(i-1,i) = x_i$ for $1 \le i \le 9$, $f(9,0) = x_0$, $f(1,3) = y_1$, $f(2,4) = y_2$, $f(5,8) = y_3$, $f(6,9) = y_4$ and $f(7,0) = y_5$ (see Fig. 1). Then we have the following system of equations:

$$x_1 + x_2 + y_1 = m, \quad x_2 + x_3 + y_2 = m, \quad x_3 + x_4 + y_1 = m,$$
$$x_4 + x_5 + y_2 = m, \quad x_5 + x_6 + y_3 = m, \quad x_6 + x_7 + y_4 = m,$$
$$x_7 + x_8 + y_5 = m, \quad x_8 + x_9 + y_3 = m, \quad x_9 + x_0 + y_4 = m,$$
$$x_1 + x_0 + y_5 = m.$$

After performing elementary row operations over the ring $\mathbb{Z}_{10}$, we have

$$\left\{\begin{array}{ll} x_1 = -x_0 - y_5 + m, & x_2 = x_0 - y_1 + y_5, \\ x_3 = -x_0 - y_1 + y_2 - y_5 + m, & x_4 = x_0 - y_2 + y_5, \\ x_5 = -x_0 - y_5 + m, & x_6 = x_0 - y_3 + y_5, \\ x_7 = -x_0 + y_3 - y_4 - y_5 + m, & x_8 = x_0 - y_3 + y_4, \\ x_9 = -x_0 - y_4 + m, & 2y_1 = 2y_2. \end{array}\right\} \tag{3.1}$$

After taking modulo 5 to Eq. (3.1), we have

$$\left\{\begin{array}{ll} x_1 = -x_0 - y_5 + m, & x_2 = x_0 - y_2 + y_5, \\ x_3 = -x_0 - y_5 + m, & x_4 = x_0 - y_2 + y_5, \\ x_5 = -x_0 - y_5 + m, & x_6 = x_0 - y_3 + y_5, \\ x_7 = -x_0 + y_3 - y_4 - y_5 + m, & x_8 = x_0 - y_3 + y_4, \\ x_9 = -x_0 - y_4 + m, & y_1 = y_2 \end{array}\right\} \tag{3.2}$$

Condition 1: We have $x_1 \equiv x_3 \equiv x_5 \equiv a \pmod 5$, $x_2 \equiv x_4 \equiv b \pmod 5$ and $y_1 \equiv y_2 \equiv c \pmod 5$ for some $a, b, c \in \mathbb{Z}_5$. Since $[\mathbf{15}] \equiv [\mathbf{5}] \times 3 \pmod 5$, then a, b, and c must be distinct.

If $y_1 \equiv y_2 \pmod{10}$, then we have $x_1 \equiv x_3 \equiv x_5 \pmod{10}$. It is impossible since $[\mathbf{15}] \equiv [\mathbf{10}] \cup [\mathbf{5}] \pmod{10}$. So we have
Condition 2: $y_1 \not\equiv y_2 \pmod{10}$. Since $y_1 \equiv y_2 \pmod 5$, hence $y_1 \not\equiv y_2 \pmod 2$ or equivalent to $y_2 \equiv y_1 + 5 \pmod{10}$.

Combining Conditions 1 and 2, we have

$$x_1 \equiv x_5 \equiv x_3 + 5 \pmod{10}, \tag{3.3}$$

$$x_2 \equiv x_4 + 5 \pmod{10}, \tag{3.4}$$

$$y_2 \equiv y_1 + 5 \pmod{10}, \tag{3.5}$$

$$x_1, x_2, y_2 \text{ are distinct in } \mathbb{Z}_5. \tag{3.6}$$

Suppose we consider the edge-magicness of G over $\mathbb{Z}_2$. If $m \equiv 0 \pmod 2$, then the subgraph induced by the 1-edges (edges labeled by 1) must be a 2-regular subgraph of order 8. If $m \equiv 1 \pmod 2$, then the subgraph induced by the 0-edges (edges labeled by 0) must be a 2-regular subgraph of order 7. So we have to deal with each 2-regular subgraph of G of order 7 or 8. Since $x_1 \equiv x_5 \not\equiv x_3 \pmod 2$, $x_2 \not\equiv x_4 \pmod 2$ and $y_1 \not\equiv y_2 \pmod 2$, we only have to deal with the subgraph marked by a '$*$' in the table above. All of those are of order 7. That means $m \equiv 1 \pmod 2$. Note that, if the numbers after the '$*$' are the same, then it means that those cases are the same under symmetry. Now if there is no other state, the arithmetics are taken in $\mathbb{Z}_{10}$.
Case 1: We consider the case that when the 2-regular subgraph is the cycle $C = (1345890)$. In this case, edges of C are labeled by $0 \mod 2$, i.e. x_1, y_1, x_4, x_5, y_3, x_9 and x_0 are even. From (3.3) we have $x_1 = x_5 \in \{2, 4\}$.

I. When $x_1 = x_5 = 2$. Then $x_3 = 7$.

A. When $x_2 = 1$. Then $x_4 = 6$.

a. When $y_1 = 4$. Then $y_2 = 9$ and $m = 7$. Hence $x_6 + y_3 = 5 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (5, 10, 1, 4)$ or $(1, 4, 5, 10)$. This implies that $x_7 + y_4 = 2$ or $x_7 + x_8 = 2$, respectively. There is no solution.

b. When $y_1 = 8$. Then $y_2 = 3$ and $m = 1$. Hence $x_6 + y_3 = 9 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (9, 10, 5, 4)$ or $(5, 4, 9, 10)$. This implies that $x_7 + y_4 = 2$ or $x_7 + x_8 = 2$, respectively. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 3$. Hence $x_6 + y_3 = 1 = x_0 + y_5$. There is no solution.

B. When $x_2 = 3$. Then $x_4 = 8$.

a. When $y_1 = 4$. Then $y_2 = 9$ and $m = 9$. Hence $x_6 + y_3 = 7 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (1, 6, 3, 4)$ or $(3, 4, 1, 6)$. This implies that $x_7 + y_4 = 8$ or $x_7 + x_8 = 8$, respectively. There is no solution.

b. When $y_1 = 6$. Then $y_2 = 1$ and $m = 1$. Hence $x_6 + y_3 = 9 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (9, 10, 5, 4)$ or $(5, 4, 9, 10)$. This implies that $x_7 + y_4 = 2$ or $x_7 + x_8 = 2$, respectively. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 5$. Hence $x_6 + y_3 = 3 = x_0 + y_5$. There is no solution.

C. When $x_2 = 5$. Then $x_4 = 10$.

a. When $y_1 = 4$. Then $y_2 = 9$ and $m = 1$. Hence $x_6 + y_3 = 9 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (5, 4, 3, 6)$ or $(3, 6, 5, 4)$. This implies that $x_7 + y_4 = 6$ or $x_7 + x_8 = 6$, respectively. There is no solution.

b. When $y_1 = 6$. Then $y_2 = 1$ and $m = 3$. Hence $x_6 + y_3 = 1 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (5, 8, 9, 4)$ or $(9, 4, 5, 8)$. This implies that $x_7 + y_4 = 8$ or $x_7 + x_8 = 8$, respectively. There is no solution.

c. When $y_1 = 8$. Then $y_2 = 3$ and $m = 5$. Hence $x_6 + y_3 = 3 = x_0 + y_5$. There is no solution.

D. When $x_2 = 9$. Then $x_4 = 4$.

a. When $y_1 = 6$. Then $y_2 = 1$ and $m = 7$. Hence $x_6 + y_3 = 5 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (5, 10, 1, 4)$ or $(1, 4, 5, 10)$. This implies that $x_7 + y_4 = 2$ or $x_7 + x_8 = 2$, respectively. There is no solution.

b. When $y_1 = 8$. Then $y_2 = 3$ and $m = 9$. Hence $x_6 + y_3 = 7 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (1, 6, 3, 4)$ or $(3, 4, 1, 6)$. This implies that $x_8 + y_9 = 3$ or $x_9 + y_4 = 3$, respectively. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 1$. Hence $x_6 + y_3 = 9 = x_0 + y_5$. So $(x_6, y_3) = (1, 8)$, $(3, 6)$ or $(5, 4)$.

1. When $(x_6, y_3) = (1, 8)$. This implies that $x_8 + y_9 = 3$. There is no solution.
2. When $(x_6, y_3) = (3, 6)$. This implies that $x_8 + x_9 = 5$. Hence $x_8 = 1$ and $x_9 = 4$. This implies $x_0 + y_4 = 7$. There is no solution.
3. When $(x_6, y_3) = (5, 4)$. This implies that $x_7 + y_4 = 6$. Hence $x_7 = y_4 = 3$. This implies $x_8 + y_5 = 8$. There is no solution.

II. When $x_1 = x_5 = 4$. Then $x_3 = 9$.

A. When $x_2 = 1$. Then $x_4 = 6$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 7$. Hence $x_6 + y_3 = 3 = x_0 + y_5$. So $(x_6, y_3) = (1, 2)$, $(5, 8)$ or $(3, 10)$.

1. When $(x_6, y_3) = (1, 2)$. This implies that $x_7 + y_4 = 6$ and $x_8 + y_9 = 5$. Then $x_4 = x_7 = 3$, $x_8 = 5$ and $x_9 = 10$. This implies $y_5 = 9$. It is impossible.
2. When $(x_6, y_3) = (5, 8)$. This implies that $x_7 + y_4 = 2$ and $x_8 + y_9 = 9$. Then $\{y_4, x_7\} = \{10, 2\}$, $x_8 = 7$ and $x_9 = 2$. This implies $x_0 + y_4 = 5$. There is no solution.
3. When $(x_6, y_3) = (3, 10)$. This implies that $x_7 + y_4 = 4$ and $x_8 + y_9 = 7$. Then $\{y_4, x_7\} = \{1, 3\}$, $x_8 = 5$ and $x_9 = 2$. This implies $x_0 + y_4 = 5$. There is no solution.

b. When $y_1 = 8$. Then $y_2 = 3$ and $m = 3$. Hence $x_6 + y_3 = 9 = x_0 + y_5$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 5$. Hence $x_6 + y_3 = 1 = x_0 + y_5$. There is no solution.

B. When $x_2 = 3$. Then $x_4 = 8$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 9$. Hence $x_6 + y_3 = 5 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (5, 10, 3, 2)$ or $(3, 2, 5, 10)$. This implies that $x_7 + y_4 = 4$ or $x_7 + x_8 = 4$, respectively. There is no solution.

b. When $y_1 = 6$. Then $y_2 = 1$ and $m = 3$. Hence $x_6 + y_3 = 9 = x_0 + y_5$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 7$. Hence $x_6 + y_3 = 3 = x_0 + y_5$. We have $(x_6, y_3, y_5, x_0) = (7, 6, 1, 2)$ or $(1, 2, 7, 6)$. This implies that $x_8 + x_9 = 1$ or $x_9 + y_4 = 1$, respectively. There is no solution.

C. When $x_2 = 5$. Then $x_4 = 10$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 1$. Hence $x_6 + y_3 = 7 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (1, 6, 5, 2)$ or $(5, 2, 1, 6)$. This implies $x_7 + y_4 = 10$ or $x_7 + x_8 = 10$. There is no solution.

b. When $y_1 = 8$. Then $y_2 = 3$ and $m = 7$. Hence $x_6 + y_3 = 3 = x_0 + y_5$. So $(x_6, y_3) = (1, 2)$ or $(7, 6)$. It implies $x_7 + y_4 = 6$ or 10, respectively. It is no solution.

c. When $y_1 = 6$. Then $y_2 = 1$ and $m = 5$. Hence $x_6 + y_3 = 1 = x_0 + y_5$. There is no solution.

D. When $x_2 = 7$. Then $x_4 = 2$.

a. When $y_1 = 6$. Then $y_2 = 1$ and $m = 7$. Hence $x_6 + y_3 = 3 = x_0 + y_5$. So $(x_6, y_3) = (1, 2)$, $(3, 10)$ or $(5, 8)$.

1. When $x_6 = 1$. Then $x_7 + y_4 = 6$. Hence $x_7 = y_4 = 7$ (since they are odd). So we get $x_8 + y_5 = 4$ which is no solution.

2. When $x_6 = 3$, i.e. $y_3 = 10$. Then $x_8 + x_9 = 7$. Hence $x_8 = 5$ and $x_9 = 2$. It implies $x_7 + y_5 = 2$ which is no solution.

3. When $x_6 = 5$. Then $x_7 + y_4 = 2$ which is no solution.

b. When $y_1 = 8$. Then $y_2 = 3$ and $m = 9$. Hence $x_6 + y_3 = 5 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (3, 2, 5, 10)$ or $(5, 10, 3, 2)$. This implies that $x_7 + x_8 = 4$ or $x_7 + y_4 = 4$, resectively. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 1$. Hence $x_6 + y_3 = 7 = x_0 + y_5$. So $(x_6, y_3, y_5, x_0) = (1, 6, 5, 2)$ or $(5, 2, 1, 6)$. This implies that $x_7 + y_4 = 10$ or $x_7 + x_8 = 10$, respectively. There is no solution.

Therefore, there is no edge-magic labeling such that the edges of $C = (1345890)$ are labeled by even numbers.

Case 2: We consider the case that when the 2-regular subgraph is the cycle $C = (1345690)$. If we swap the location of vertices 6 and 8, then the cycle is referred to Case 1.

Case 3: We consider the case that when the 2-regular subgraph is the cycle $C = (1345670)$. If we swap the location of vertices 6 and 8, and swap the location of vertices 7 and 9, then the cycle is referred to Case 1.

Case 4: We consider the case that when the 2-regular subgraph is the cycle $C = (1345870)$. If we swap the location of vertices 7 and 9, then the cycle is referred to Case 1.

Case 5: We consider the case that when the 2-regular subgraph is the 3+4 cycle $C = (123)(5678)$. In this case, x_2, x_3, y_1, x_6, x_7, x_8 and y_3 are even. From (3.3) we have $x_1 = x_5 \in \{1, 3, 5\}$.

I. When $x_1 = x_5 = 1$. Then $x_3 = 6$.

A. When $x_2 = 2$. Then $x_4 = 7$.

a. When $y_1 = 4$. Then $y_2 = 9$ and $m = 7$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. So $(y_5, x_0) = (3, 3)$ and $\{x_6, y_3\} = \{2, 4\}$. This implies that $x_7 + x_8 = 4$ which is no solution.

b. When $y_1 = 8$. Then $y_2 = 3$ and $m = 1$. Hence $x_6 + y_3 = 10 = x_0 + y_5$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 3$. Hence $x_6 + y_3 = 2 = x_0 + y_5$. So $\{x_0, y_5\} = \{3, 9\}$ and $\{x_6, y_3\} = \{4, 8\}$. This implies that $x_9 + y_4 = 10$ or $x_7 + x_8 = 10$. There is no solution.

B. When $x_2 = 4$. Then $x_4 = 9$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 7$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. So $(y_5, x_0) = (3, 3)$ and $\{x_6, y_3\} = \{2, 4\}$. This implies that $x_7 + x_8 = 4$ which is no solution.

b. When $y_1 = 8$. Then $y_2 = 3$ and $m = 3$. Hence $x_6 + y_3 = 2 = x_0 + y_5$. So $\{y_5, x_0\} = \{5, 7\}$ and $\{x_6, y_3\} = \{2, 10\}$. This implies that $x_7 + y_4 = 1$ or $x_8 + x_9 = 1$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 5$. Hence $x_6 + y_3 = 4 = x_0 + y_5$. There is no solution.

C. When $x_2 = 8$. Then $x_4 = 3$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 1$. Hence $x_6 + y_3 = 10 = x_0 + y_5$. There is no solution since x_6 and y_3 are even.

b. When $y_1 = 4$. Then $y_2 = 9$ and $m = 3$. Hence $x_6 + y_3 = 2 = x_0 + y_5$. So $\{y_5, x_0\} = \{5, 7\}$ and $\{x_6, y_3\} = \{2, 10\}$. This implies that $x_7 + y_4 = 3$ or $x_8 + x_9 = 3$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 9$. Hence $x_6 + y_3 = 8 = x_0 + y_5$. So $\{y_5, x_0\} = \{3, 5\}$ and $(x_6, y_3) = (4, 4)$. This implies that $x_7 + y_4 = 5$. There is no solution.

D. When $x_2 = 10$. Then $x_4 = 5$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 3$. Hence $x_6 + y_3 = 2 = x_0 + y_5$. So $\{y_5, x_0\} = \{3, 9\}$ and $\{x_6, y_3\} = \{4, 8\}$. This implies that $x_9 + y_4 = 10$ or $x_7 + x_8 = 10$. There is no solution.

b. When $y_1 = 4$. Then $y_2 = 9$ and $m = 5$. Hence $x_6 + y_3 = 4 = x_0 + y_5$. There is no solution.

c. When $y_1 = 8$. Then $y_2 = 3$ and $m = 9$. Hence $x_6 + y_3 = 8 = x_0 + y_5$. So $\{y_5, x_0\} = \{3, 5\}$ and $(x_6, y_3) = (4, 4)$. This implies that $x_7 + y_4 = 5$. There is no solution.

II. When $x_1 = x_5 = 3$. Then $x_3 = 8$.

A. When $x_2 = 2$. Then $x_4 = 7$.

a. When $y_1 = 4$. Then $y_2 = 9$ and $m = 9$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. So $\{y_5, x_0\} = \{1, 5\}$, and $\{x_6, y_3\} = \{2, 4\}$ or $\{6, 10\}$. This implies $x_7 + x_8 = 8$ or $x_9 + y_4 = 8$. Since either 2 or 6 will be occupied by x_6 or y_3, there is no solution.

b. When $y_1 = 6$. Then $y_2 = 1$ and $m = 1$. Hence $x_6 + y_3 = 8 = x_0 + y_5$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 5$. Hence $x_6 + y_3 = 2 = x_0 + y_5$. There is no solution.

B. When $x_2 = 4$. Then $x_4 = 9$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 9$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. So $\{y_5, x_0\} = \{1, 5\}$, and $\{x_6, y_3\} = \{2, 4\}$ or $\{6, 10\}$. This implies $x_7 + x_8 = 8$ or $x_9 + y_4 = 8$. Since either 2 or 6 will be occupied by x_6 or y_3, there is no solution.

b. When $y_1 = 6$. Then $y_2 = 1$ and $m = 3$. Hence $x_6 + y_3 = 10 = x_0 + y_5$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 7$. Hence $x_6 + y_3 = 4 = x_0 + y_5$. There is no solution.

C. When $x_2 = 6$. Then $x_4 = 1$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 1$. Hence $x_6 + y_3 = 8 = x_0 + y_5$. There is no solution.

b. When $y_1 = 4$. Then $y_2 = 9$ and $m = 3$. Hence $x_6 + y_3 = 10 = x_0 + y_5$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 9$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. $\{y_5, x_0\} = \{1, 5\}$, and $\{x_6, y_3\} = \{2, 4\}$. This implies $x_7 + y_4 = 7$ or $x_8 + x_9 = 7$. There is no solution.

D. When $x_2 = 10$. Then $x_4 = 5$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 5$. Hence $x_6 + y_3 = 2 = x_0 + y_5$, which is no solution.

b. When $y_1 = 4$. Then $y_2 = 9$ and $m = 7$. Hence $x_6 + y_3 = 4 = x_0 + y_5$, which is no solution.

c. When $y_1 = 6$. Then $y_2 = 1$ and $m = 9$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. This implies $\{x_6, y_3\} = \{2, 4\}$. So $x_7 + y_4 = 5$ and $x_8 + x_9 = 7$ or $x_7 + y_4 = 7$ and $x_8 + x_9 = 5$. Hence $\{x_7, x_8\} = \{2, 4\}$ and $\{x_9, y_4\} = \{1, 5\}$. This implies $x_0 = 3$ which is impossible.

III. When $x_1 = x_5 = 5$. Then $x_3 = 10$.

A. When $x_2 = 2$. Then $x_4 = 7$.

a. When $y_1 = 4$. Then $y_2 = 9$ and $m = 1$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. So $x_0 = y_5 = 3$ and $\{x_6, y_3\} = \{2, 4\}$. This implies $x_7 + x_8 = 8$ and $x_9 + y_4 = 8$. There is no solution.

b. When $y_1 = 6$. Then $y_2 = 1$ and $m = 3$. Hence $x_6 + y_3 = 8 = x_0 + y_5$. There is no solution.

c. When $y_1 = 8$. Then $y_2 = 3$ and $m = 5$. Hence $x_6 + y_3 = 10 = x_0 + y_5$. This implies that $\{x_6, y_3\} = \{4, 6\}$. Then $x_7 + y_4 = 1$ or $x_8 + x_9 = 1$. There is no no solution.

B. When $x_2 = 4$. Then $x_4 = 9$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 1$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. So $x_0 = y_5 = 3$. This implies $x_7 + x_8 = 8$ and $x_9 + y_4 = 8$. There is no solution.

b. When $y_1 = 6$. Then $y_2 = 1$ and $m = 5$. Hence $x_6 + y_3 = 10 = x_0 + y_5$. This implies that $\{x_0, y_5\} = \{3, 7\}$. Then $x_7 + x_8 = 8$ or $x_9 + y_4 = 8$. There is no solution.

c. When $y_1 = 8$. Then $y_2 = 3$ and $m = 7$. Hence $x_6 + y_3 = 2 = x_0 + y_5$. There is no solution.

C. When $x_2 = 6$. Then $x_4 = 1$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 3$. Hence $x_6 + y_3 = 8 = x_0 + y_5$. There is no solution.

b. When $y_1 = 4$. Then $y_2 = 9$ and $m = 5$. Hence $x_6 + y_3 = 10 = x_0 + y_5$. We have $\{x_6, y_3\} = \{2, 8\}$ and $\{x_0, y_5\} = \{3, 7\}$. Then $x_9 + y_4 = 2$ or $x_7 + x_8 = 2$. There is no solution.

c. When $y_1 = 8$. Then $y_2 = 3$ and $m = 9$. Hence $x_6 + y_3 = 4 = x_0 + y_5$. Then $x_6 = y_3 = 2$ and $\{y_5, x_0\} = \{1, 3\}$. This implies $x_7 + y_4 = 7$ or $x_8 + x_9 = 7$. There is no solution.

D. When $x_2 = 8$. Then $x_4 = 3$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 5$. Hence $x_6 + y_3 = 10 = x_0 + y_5$. Then $\{x_6, y_3\} = \{4, 6\}$ and $\{y_5, x_0\} = \{1, 9\}$. This implies $x_7 + y_4 = 1$ or $x_8 + x_9 = 1$. There is no solution.

b. When $y_1 = 4$. Then $y_2 = 9$ and $m = 7$. Hence $x_6 + y_3 = 2 = x_0 + y_5$, which is no solution.

c. When $y_1 = 6$. Then $y_2 = 1$ and $m = 9$. Hence $x_6 + y_3 = 4 = x_0 + y_5$. This implies $x_6 = y_3 = 2$ and $\{x_0, y_5\} = \{1, 3\}$. So $x_7 + y_4 = 7$ and $x_8 + x_9 = 7$, which is no solution.

Therefore, there is no edge-magic labeling such that the edges of $C = (123)(5678)$ are labeled by even numbers.

Case 6: We consider the case that when the 2-regular subgraph is the 3+4 cycle $C = (123)(6789)$. In this case, x_2, x_3, y_1, x_7, x_8, x_9 and y_4 are even. From (3.3) we have $x_1 = x_5 \in \{1, 3, 5\}$.

I. When $x_1 = x_5 = 1$. Then $x_3 = 6$. For those subcases raised from this case we will get that $x_6 + y_3 = x_0 + y_5$ and it is equal to the value corresponding to each subcase of the subcase I in Case 5. Since these unknowns are odd, we can easy check that there is no solution.

II. When $x_1 = x_5 = 3$. Then $x_3 = 8$.

A. When $x_2 = 2$. Then $x_4 = 7$.

a. When $y_1 = 4$. Then $y_2 = 9$ and $m = 9$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. So $\{x_6, y_3\} = \{y_5, x_0\} = \{1, 5\}$.

1. Suppose $x_6 = 1$. Then $x_7 + y_4 = 8$ and hence $\{x_7, y_4\} = \{2, 6\}$. From Eq. (3.1) we have $x_8 + y_5 = m - x_7$ and $x_9 + x_0 = m - y_4$. Then we get $\{x_8 + y_5, x_9 + x_0\} = \{3, 7\}$. It is no solution.

2. Suppose $x_6 = 5$. Then $x_7 + y_4 = 4$ and hence $\{x_7, y_4\} = \{4, 10\}$. From Eq. (3.1) we have $x_8 + y_5 = m - x_7$ and $x_9 + x_0 = m - y_4$. Then we get $\{x_8 + y_5, x_9 + x_0\} = \{5, 9\}$. It is no solution.

b. When $y_1 = 6$. Then $y_2 = 1$ and $m = 1$. Hence $x_6 + y_3 = 8 = x_0 + y_5$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 5$. Hence $x_6 + y_3 = 2 = x_0 + y_5$. There is no solution.

B. When $x_2 = 4$. Then $x_4 = 9$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 9$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. So $\{x_6, y_3\} = \{y_5, x_0\} = \{1, 5\}$. The argument is the same as the subcase II-A-a.

b. When $y_1 = 6$. Then $y_2 = 1$ and $m = 3$. Hence $x_6 + y_3 = 10 = x_0 + y_5$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 7$. Hence $x_6 + y_3 = 4 = x_0 + y_5$. There is no solution.

C. When $x_2 = 6$. Then $x_4 = 1$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 1$. Hence $x_6 + y_3 = 8 = x_0 + y_5$. There is no solution.

b. When $y_1 = 4$. Then $y_2 = 9$ and $m = 3$. Hence $x_6 + y_3 = 10 = x_0 + y_5$. There is no solution.

c. When $y_1 = 10$. Then $y_2 = 5$ and $m = 9$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. $\{y_5, x_0\} = \{1, 5\}$ and $\{x_6, y_3\} = \{7, 9\}$, or $\{x_6, y_3\} = \{1, 5\}$ and $\{y_5, x_0\} = \{7, 9\}$.

1. Suppose $\{x_6, y_3\} = \{7,9\}$. Then $x_7 + y_4 = 2$ or 10. But there is no solution.

2. Suppose $\{y_5, x_0\} = \{7,9\}$. Then $x_7 + x_8 = 2$ or 10. But there is no solution.

D. When $x_2 = 10$. Then $x_4 = 5$.

a. When $y_1 = 2$. Then $y_2 = 7$ and $m = 5$. Hence $x_6 + y_3 = 2 = x_0 + y_5$, which is no solution.

b. When $y_1 = 4$. Then $y_2 = 9$ and $m = 7$. Hence $x_6 + y_3 = 4 = x_0 + y_5$, which is no solution.

c. When $y_1 = 6$. Then $y_2 = 1$ and $m = 9$. Hence $x_6 + y_3 = 6 = x_0 + y_5$. The argument is the same as the subcase II-C-c.

III. When $x_1 = x_5 = 5$. Then $x_3 = 10$. For those subcases raised from this case we will get that $x_6 + y_3 = x_0 + y_5$ and it is equal to the value corresponding to each subcase of the subcase III in Case 5. Since these unknowns are odd, we can easy check that there is no solution.

Therefore, there is no edge-magic labeling such that the edges of $C = (123)(6789)$ are labeled by even numbers.

Case 7: We consider the case that when the 2-regular subgraph is the 3+4 cycle $C = (123)(7890)$. In this case, x_2, x_3, y_1, x_8, x_9, x_0 and y_5 are even. From (3.3) we have $x_1 = x_5 \in \{1, 3, 5\}$.

If we make the following changes on the unknowns: $x_1 \to a_5$, $x_2 \to a_2$, $x_3 \to a_3$, $x_4 \to a_4$, $x_5 \to a_1$, $x_6 \to a_0$, $x_7 \to a_9$, $x_8 \to a_8$, $x_9 \to a_7$, $x_0 \to a_6$, $y_1 \to b_1$, $y_2 \to b_2$, $y_3 \to b_5$, $y_4 \to b_4$ and $y_5 \to b_3$. Since $a_1 = a_5$, we get a system same as Eq. (3.1). By the same proof of Case 5, we get that there is no edge-magic labeling such that the edges of $C = (123)(7890)$ are labeled by even numbers.

Case 8: We consider the case that when the 2-regular subgraph is the 3+4 cycle $C = (123)(5698)$. If we swap the location of vertices 5 and 7, then the subgraph is referred to Case 6.
Case 9: We consider the case that when the 2-regular subgraph is the 3+4 cycle $C = (123)(6709)$. If we swap the location of vertices 6 and 8, then the subgraph is referred to Case 7.

So we get the following result.

Theorem 3.1. *The graph G described in Fig. 1 is not edge-magic.*

References

[1] S. Drajnová, J. Ivančo and A. Semaničová, Number of edges in supermagic graphs, *J. Graph Theory*, **52** (2005), 15-26.

[2] J.A. Gallian, A dynamic survey of graph labeling, Electronic J. Combin., **17** (2010), #DS6.

[3] N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *Journal of Recreational Mathematics*, **21** (1989), 107-115.

[4] J. Ivančo, On supermagic regular graphs, *Mathematica Bohemica*, **125** (2000), 99-114.

[5] J. Ivančo, Z. Lastivková and A. Semaničová, On magic and supermagic line graphs, *Mathematica Bohemica*, **129** (2004), 33-42.

[6] S-M. Lee, Eric Seah and S.K. Tan, On edge-magic graphs, *Congressus Numerantium*, **86** (1992), 179-191.

[7] S-M. Lee, W.M. Pigg and T.J. Cox, On edge-magic cubic graphs conjecture, *Congressus Numerantium*, **105** (1994), 214-222.

[8] S-M. Lee, L. Wang and Y. Wen, On the edge-magic cubic graphs and multigraphs, *Congressus Numerantium*, **165** (2003), 145-160.

[9] A. Semaničová, On magic and supermagic circulant graphs, *Discrete Math.*, **306** (2006), 2263-2269.

[10] W.C. Shiu, P.C.B. Lam and S-M. Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium*, **132** (1998), 9-18.

[11] W.C. Shiu, P.C.B. Lam and H.L. Cheng, Supermagic labeling of an s-duplicate of $K_{n,n}$, *Congressus Numerantium*, **146** (2000), 119-124.

[12] W.C. Shiu and S-M. Lee, Some edge-magic cubic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **40** (2002), 115-127.

[13] W.C. Shiu, P.C.B. Lam and S-M. Lee, On a construction of supermagic graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **42** (2002), 147-160.

[14] W.C. Shiu, Edge-magic labeling matrices of the composition of paths with null graphs, *Congressus Numerantium*, **187** (2007), 55-69.

[15] B. M. Stewart, Magic graphs, *Canadian Journal of Mathematics*, **18** (1966), 1031-1059.

[16] B. M. Stewart, Supermagic complete graphs, *Canadian Journal of Mathematics*, **19** (1967), 427-438.