The bounds of Full Friendly Index Sets of $C_m \times C_n$

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Outline

- ⋆ Labeling and induced labeling
- ⋆ Background and some known results
- ⋆ General properties
- ⋆ Upper bounds
- ⋆ Lower bounds
- ⋆ Further studies

Labeling and its induced labeling

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For each $i \in A$, let $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |f^{*-1}(i)|$.

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A graph G is ***A-cordial*** if there is an A-cordial labeling of G.

This concept was introduced by Hovey in 1991 which is a generalization of cordial graphs introduced by Cahit (when $A = \mathbb{Z}_2$).

Some known results in 1987

1987 Cahit ($A = \mathbb{Z}_2$):

1. Every tree is cordial.
2. K_n is cordial if and only if $n \leq 3$.
3. $K_{m,n}$ is cordial for all m, n.
4. Wheel $W_n = K_1 \vee C_{n-1}$ is cordial if and only if $n \not\equiv 0 \pmod 4$.
5. C_n is cordial if and only if $n \not\equiv 2 \pmod 4$.

Some known results in 1989-1999

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1. Characterized cordial generalized Petersen graphs completely.
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1999 Seoud and Abdel proved certain cylinder graphs are cordial.

Some known results in 2006-2007

2006, Chartrand, Lee and Zhang introduced the ***friendly index set*** of G:

$$\mathrm{FI}(G) = \{|i_f(G)| \mid f \text{ is friendly}\},$$

where $i_f(G) = e_f(1) - e_f(0)$.

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Salehi and Lee found the friendly index sets of paths.

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Hence G is cordial if and only if $\{-1, 0, 1\} \cap \mathrm{FFI}(G) \neq \varnothing$.

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$\mathrm{FFI}(P_2 \times P_{2n+1}) = \{2i - 1 - 6n \mid 3 \leq i \leq 6n + 1, i \neq 6n\}$,

General properties

For a fixed labeling f, a vertex v is called a k-***vertex*** if $f(v) = k$ and an edge e is called a k-***edge*** if $f^*(e) = k$.

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Lemma 1 *If f is a labeling of a (p, q)-graph G, then $i_f(G) \leq q$.*

Lemma 2 *An odd cycle C in a graph with labeling f contains at least one 0-edge.*

Vertical and horizontal cycles of $C_m \times C_n$

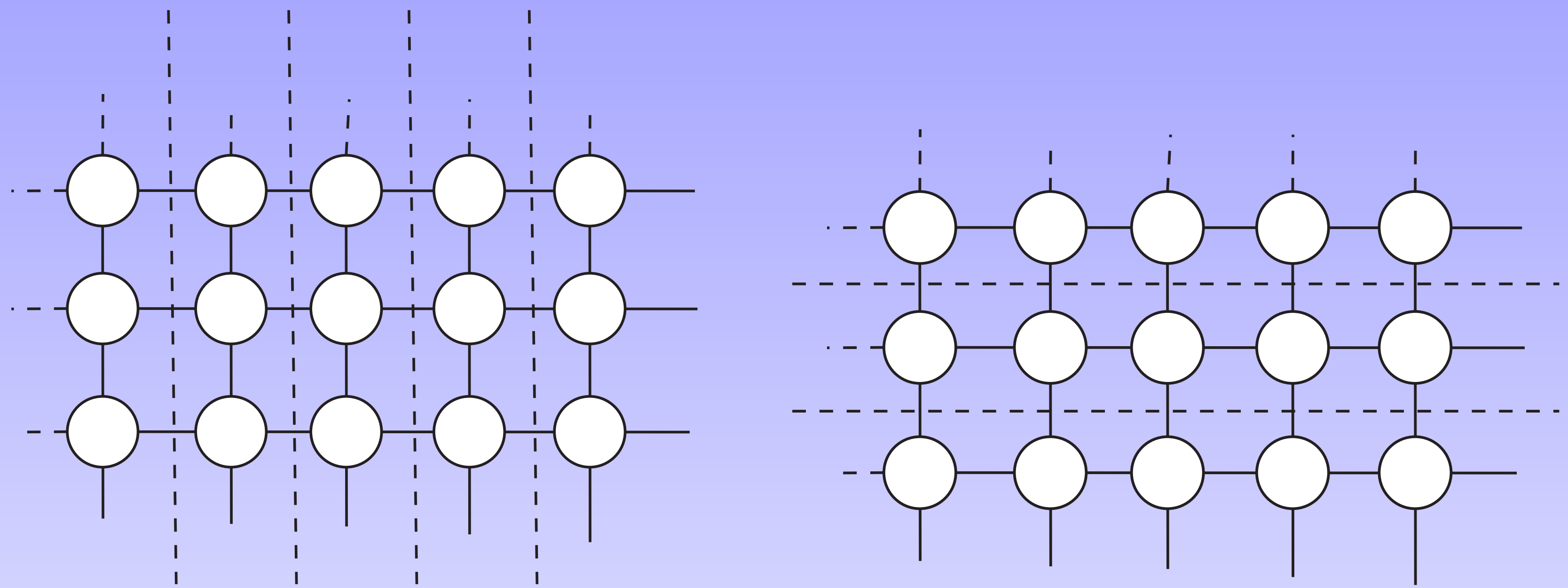

Figure 1: 5 vertical cycles and 3 horizontal cycles of $C_5 \times C_3$.

Upper bounds

Corollary 3 *If f is a (friendly) labeling of the graph $C_m \times C_n$, then $i_f(C_m \times C_n) \leq 2mn$.*

Upper bounds

Theorem 4 *If f is a (friendly) labeling of the graph $C_m \times C_n$, then $i_f(C_m \times C_n) \leq 2mn - 2m$ for even m and odd n.*

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Proof: $C_m \times C_n$ contains at least m edge disjoint odd (vertical) cycles.
By Lemma 2, we have $e_f(0) \geq m$ and so
$e_f(1) \leq 2mn - m$. Hence,
$i_f(C_m \times C_n) \leq 2mn - 2m$. $\square$

Note that $i_f(C_m \times C_n) \leq 2mn - 2n$ when m is odd and n is even. Therefore, we consider the case when m is even and n is odd only.

Upper bounds

Theorem 5 *If f is a (friendly) labeling of the graph $C_m \times C_n$, then $i_f(C_m \times C_n) \leq 2mn - 2m - 2n$ for odd m and n.*

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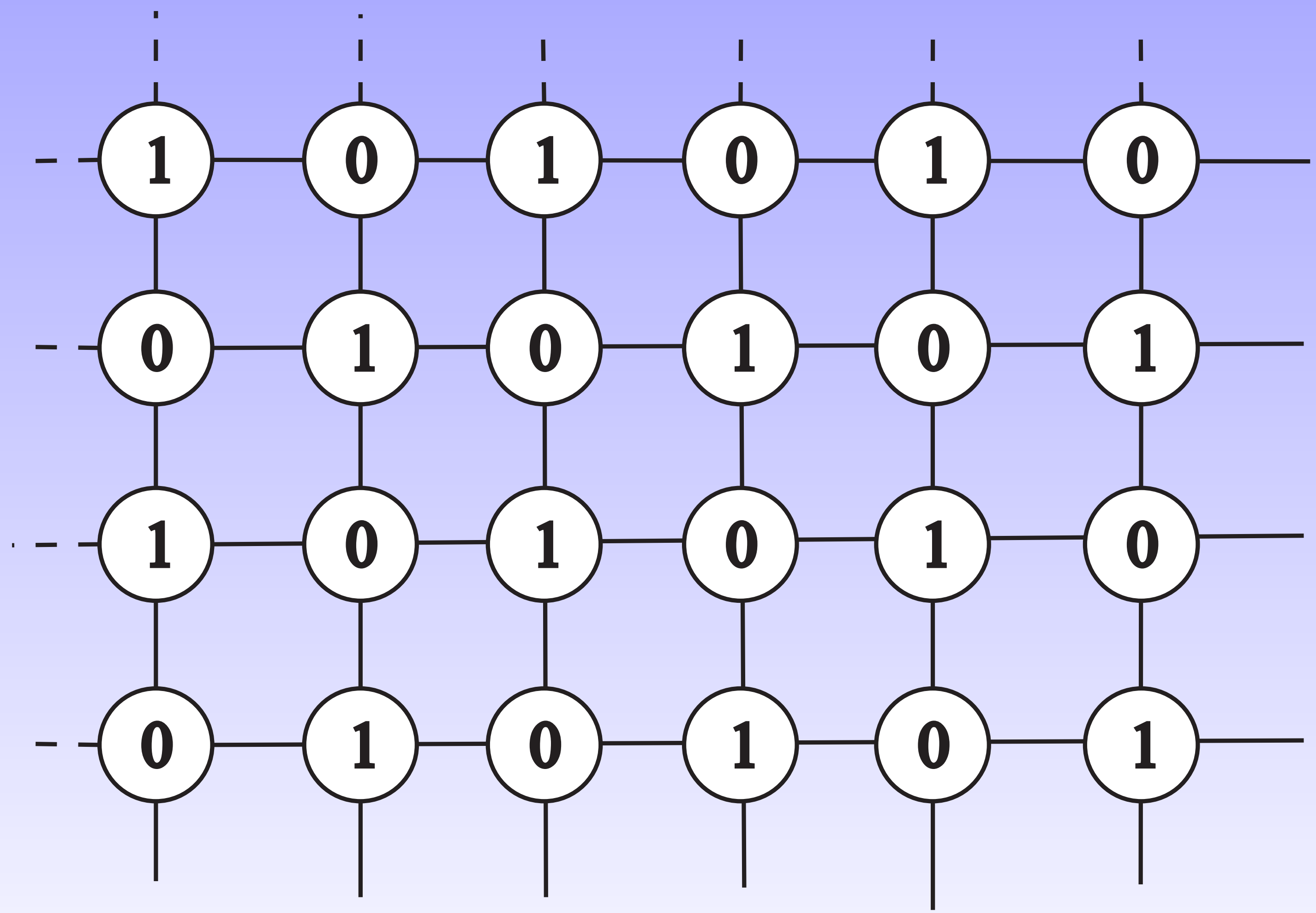
Proof: $C_m \times C_n$ contains at least $m + n$ edge disjoint odd cycles (m vertical and n horizontal). By Lemma 2, we have $e_f(0) \geq m + n$ and so $e_f(1) \leq 2mn - m - n$. Hence, $i_f(C_m \times C_n) \leq 2mn - 2m - 2n$. □

Upper bounds are sharp

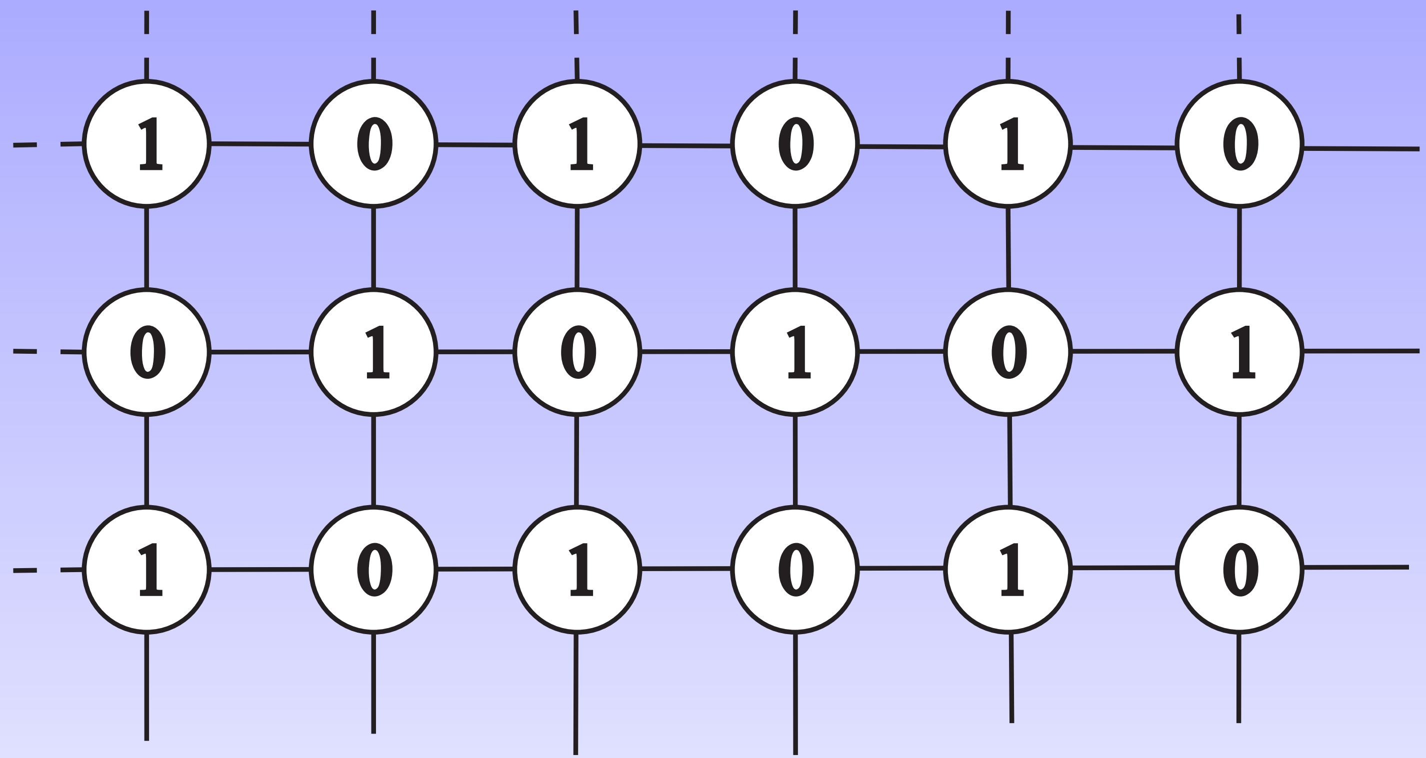
Example 1



$$i_f(C_6 \times C_4) = 48$$

Upper bounds are sharp

Example 1



$$i_f(C_6 \times C_3) = 24$$

Upper bounds are sharp

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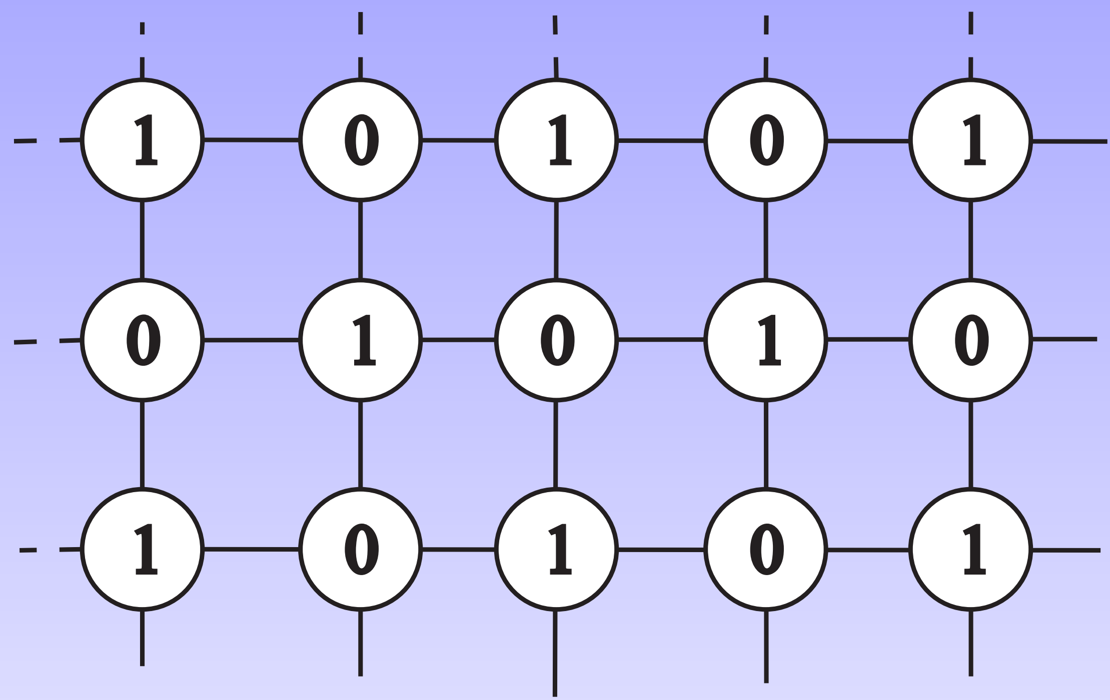

$$i_f(C_5 \times C_3) = 14$$

Lower bounds

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Clearly, a mixing cycle contains at least one 1-edge.

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Clearly, a mixing cycle contains at least one 1-edge.

C is called 1-***full cycle*** (under f), if any vertex of C is 1-vertex.

C is called 0-***full cycle*** (under f), if any vertex of C is 0-vertex.
The bracket will be omitted if there is no ambiguity.

Lower bounds

Lemma 6 (Shiu and Kwong, 2007) *For any labeling, the number of* 1*-edges in a mixing cycle is even.*

Due to isomorphic, we may assume that $n \leq m$.

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Theorem 7 *Let* f *be a friendly labeling of* $C_m \times C_n$*. If* m *is even, then* $i_f(C_m \times C_n) \geq 4n - 2mn$*.*

Lower bounds

Proof of Theorem 7

r: number of horizontal 1-full cycles
s: number of horizontal 0-full cycles

By the property of friendly labeling, $0 \le r, s \le \frac{n}{2}$.

Lower bounds

Proof of Theorem 7

r: number of horizontal 1-full cycles
s: number of horizontal 0-full cycles

Suppose $r = s = 0$.

$C_m \times C_n$ contains at least n edge disjoint mixing cycles.

Lower bounds

Proof of Theorem 7
r: number of horizontal 1-full cycles
s: number of horizontal 0-full cycles

Suppose either $r = 0$ or $s = 0$, without loss of generality, we may assume $r \neq 0$ and $s = 0$.

Lower bounds

Proof of Theorem 7
r: number of horizontal 1-full cycles
s: number of horizontal 0-full cycles

Suppose either $r = 0$ or $s = 0$, without loss of generality, we may assume $r \neq 0$ and $s = 0$.
$\Rightarrow$ the number of horizontal mixing cycles is $n - r$, and hence there exist at least $\lceil \frac{mn/2}{(n-r)} \rceil$ vertical mixing cycles since $\frac{mn}{2}$ 0-vertices distribute in $n - r$ horizontal cycles.

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$\Rightarrow$ at least
$n - r + \lceil \frac{mn}{2(n-r)} \rceil \geq n - \frac{n}{2} + \frac{m}{2} \frac{n}{(n-r)} \geq \frac{m+n}{2} \geq n$ edge disjoint mixing cycles.

Lower bounds

Proof of Theorem 7

r: number of horizontal 1-full cycles
s: number of horizontal 0-full cycles

If $r \neq 0$ and $s \neq 0$, then there exist $m \geq n$ vertical mixing cycles.

Lower bounds

Proof of Theorem 7
r: number of horizontal 1-full cycles
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For each case, the number of edge disjoint mixing cycles in $C_m \times C_n$ is at least n.

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Proof of Theorem 7
r: number of horizontal 1-full cycles
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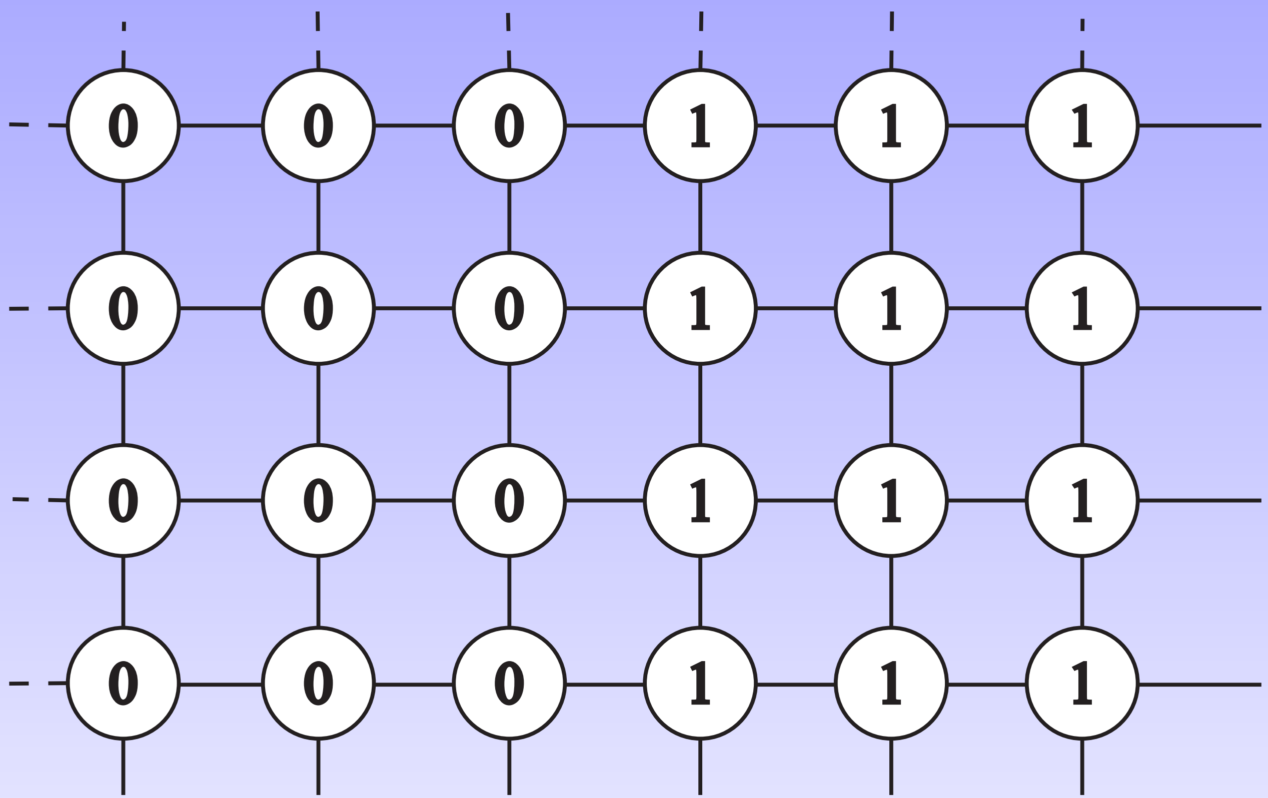
For each case, the number of edge disjoint mixing cycles in $C_m \times C_n$ is at least n.
By Lemma 6, we get $e_f(1) \geq 2n$ and $e_f(0) \leq 2mn - 2n$.
Hence, $i_f(C_m \times C_n) \geq 4n - 2mn$. $\square$

Lower bounds is sharp

Example 2



$$i_f(C_6 \times C_4) = -32$$

Lower bounds is sharp

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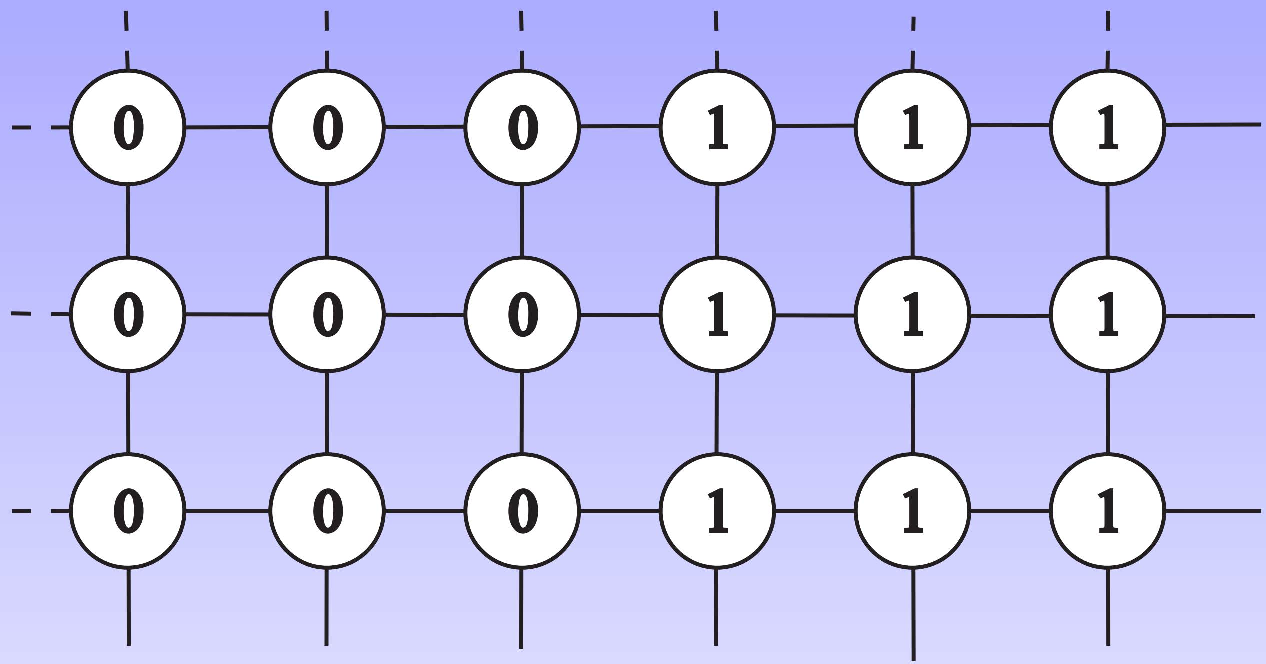


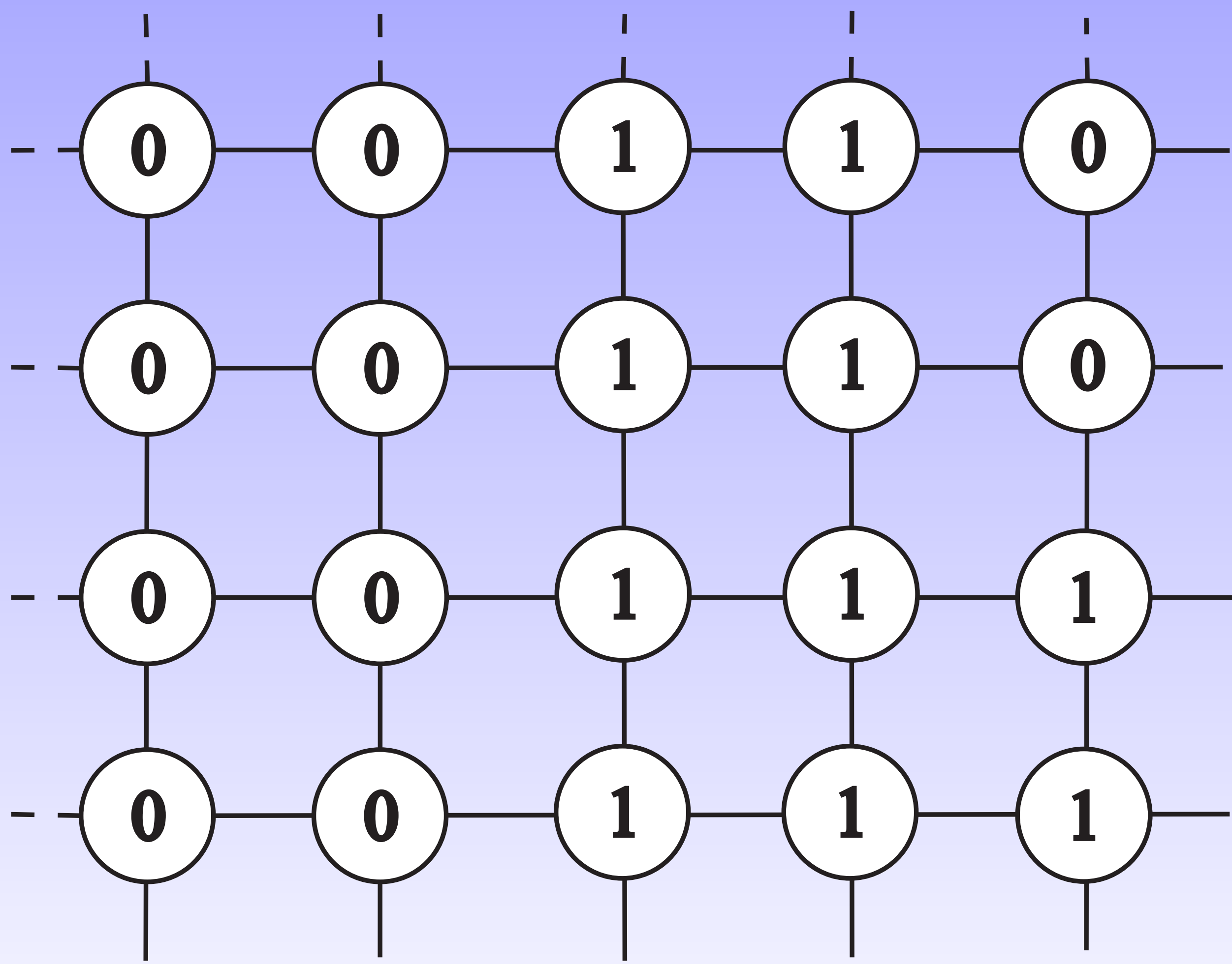
$$i_f(C_6 \times C_3) = -24$$

Lower bounds

Theorem 8 *Let f be a friendly labeling of the graph $C_m \times C_n$. If m is odd, then $i_f(C_m \times C_n) \geq 4n + 4 - 2mn$.*

Lower bounds are sharp

Example 3



$$i_f(C_5 \times C_4) = -20$$

Lower bounds are sharp

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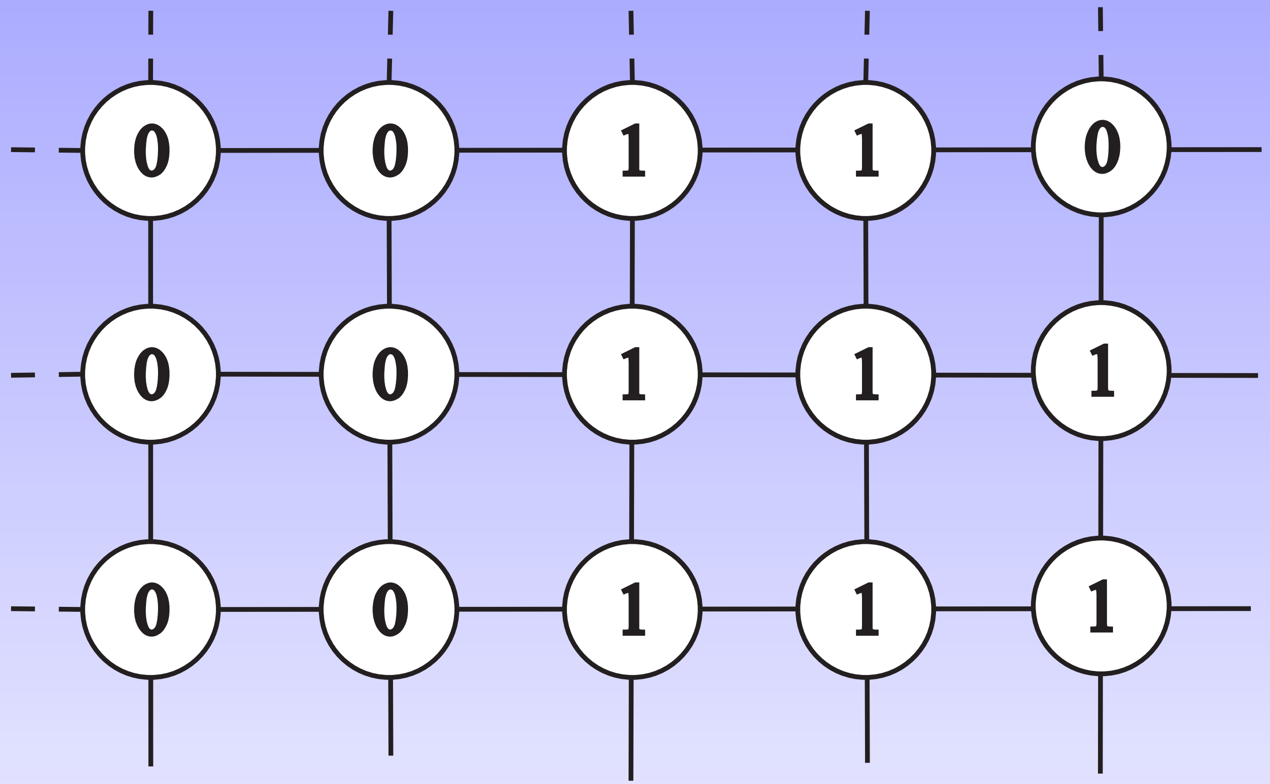

$$i_f(C_5 \times C_3) = -14$$

Further studies

- ⋆ Determine the FFI$(C_m \times C_n)$
- ⋆ $P_m \times P_n$ for $m \geq 3$.
- ⋆ Unicylic graphs.
- ⋆ Composition graphs, for example $C_m \circ N_n$.
- ⋆ Cylinder graph, i.e., $C_m \times P_n$.
- ⋆ Other products graphs.

END

Thank you