THE ENTIRE EDGE–GRACEFUL SPECTRA OF CYCLES WITH ONE CHORD

W.C. SHIU, M.H. LING, AND RICHARD M. LOW

ABSTRACT. Let G be a connected simple (p, q)–graph and k a non-negative integer. The graph G is said to be k–edge-graceful if the edges can be labeled with $k, k+1, \ldots, k+q-1$ so that the vertex sums are distinct modulo p. The set of all such k where G is k–edge-graceful is called the edge-graceful spectrum of G. In 2004, Lee, Cheng and Wang analyzed the edge-graceful spectra of cycles with one chord for various cases, leaving some cases as open problems. Here, we complete the description of the edge-graceful spectra of cycles with one chord.

1. INTRODUCTION

Let $G = (V, E)$ be a connected simple graph having p vertices and q edges. Given an integer $k \geq 0$, a bijection $f : E(G) \to \{k, k+1, \ldots, k+q-1\}$ is called an *edge labeling* of G. Any such edge labeling induces a map $f^+ : V(G) \to \mathbb{Z}_p$, defined by $f^+(v) = \Sigma f(uv) \pmod p$, where the sum is over all $uv \in E(G)$. If there exists an edge labeling f whose induced map f^+ is a bijection, we say that f is a *k–edge-graceful labeling* of G and that G is *k–edge-graceful*. The set

$$\mathrm{Egsp}(G) = \{k \in \mathbb{N} \cup \{0\} \mid G \text{ is } k\text{–edge-graceful}\}$$

is called the *edge-graceful spectrum* of G. Figure 1 illustrates various k–edge-graceful labelings of K_4.

In 1985, Lo [18] introduced the concept of an edge-graceful (ie, 1–edge-graceful) graph. Since then, a large body of research has emerged in the area of edge-graceful labelings of graphs [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25]. Within this rich area of study, several open problems remain unresolved to this day. Examples of these include Lee's [8] conjecture that all trees of odd order are edge-graceful, as well as Lo's condition being a necessary and sufficient condition for a graph to be edge-graceful. The interested reader is directed to Gallian's [3] excellent

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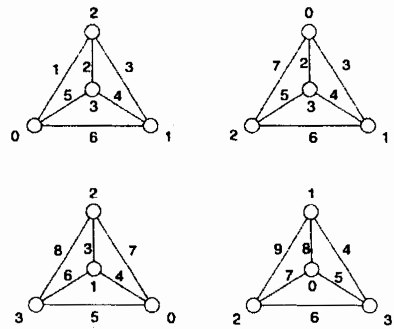

FIGURE 1. $\{1, 2, 3, 4\} \subseteq \text{Egsp}(K_4)$.

survey of general labeling problems, as well as to Wallis' [24] monograph on magic labelings.

The concept of k–edge-graceful labelings was first introduced in 2004 by Lee, Chen and Wang [9]. In their paper, the edge-graceful spectra of two classes of bicyclic graphs, namely the dumbbell graphs and cycles with one chord, were analyzed. For both of these types of graphs, some open cases remained unresolved. In this short note, we complete the analysis on the edge-graceful spectra of cycles with one chord.

2. A NECESSARY CONDITION

In [18], Lo gave a necessary condition for a graph G to be 1–edge-graceful. This can be naturally extended to give the following result.

Theorem A. *If (p, q)–graph G is k–edge-graceful, then*

$$q(q + 2k - 1) \equiv \frac{p(p + 1)}{2}, \quad (\text{mod } p). \tag{2.1}$$

We observe the following for a k–edge-graceful (p, q)–graph:

- If p is odd, (2.1) is equivalent to $q(q + 2k - 1) \equiv 0 \pmod{p}$.
- If p is even, (2.1) is equivalent to $q(q + 2k - 1) \equiv \frac{p^2-p}{2} \equiv \frac{-p}{2} \equiv \frac{p}{2}$ (mod p).

Corollary 1. *If (p, q)–graph G has a k–edge-graceful labeling, then $p \equiv 0, 1, \text{or } 3 \pmod{4}$.*

Proof. Let G be a k–edge-graceful graph and assume that $p \equiv 2 \pmod{4}$. Now, set $p = 4l + 2$. Thus, we have that $q(q + 2k - 1) \equiv 2l + 1 \pmod{p}$. This implies that $q(q + 2k - 1) - (2l + 1) \equiv 0 \pmod{p}$. Since $q(q + 2k - 1)$ is even and $2l + 1$ is odd, $q(q + 2k - 1) - (2l + 1)$ is odd. Thus, we have an odd number which is congruent to 0 (mod p), where p is even. This is impossible and we reach a desired contradiction. □

3. Bicyclic graphs

A $(p, p+1)$–graph G is called a *bicyclic* graph. Figure 2 illustrates a few examples of bicyclic graphs.

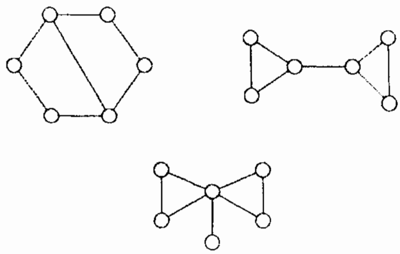

FIGURE 2. Three non-isomorphic $(6,7)$–graphs.

For a bicyclic $(p, p+1)$–graph G, condition (2.1) and Corollary 1 imply the following:

- If p is odd, then $\mathrm{Egsp}(G) \subseteq \{k \in \mathbb{N} \cup \{0\} \mid k \equiv 0 \pmod{p}\} = \{sp \mid s = 0, 1, 2, \ldots\}$.
- If $p = 4n$, then $\mathrm{Egsp}(G) \subseteq \{k \in \mathbb{N} \cup \{0\} \mid k \equiv \frac{p}{4} \pmod{\frac{p}{2}}\} = \{sn \mid s = 1, 3, 5, \ldots\}$.
- If $p \equiv 2 \pmod{4}$, then $\mathrm{Egsp}(G) = \emptyset$.

Furthermore, we note that $k \in \mathrm{Egsp}(G)$ if and only if $k + p \in \mathrm{Egsp}(G)$. Thus to find $\mathrm{Egsp}(G)$, we only need to consider all the values of k between 0 and $p-1$ which satisfy condition (2.1). Hence for a $(p, p+1)$–graph G where p is odd, we only need to determine if G is 0–edge-graceful. Similarly for $p = 4n$, we only need to determine if G is k–edge-graceful for $k = n$, $3n$.

4. Cycles with one chord

The focus of this paper is on the particular class of bicyclic graphs consisting of cycles with one chord. For $p \geq 4$, a *cycle (of order p) with one chord* is a simple graph obtained from a p–cycle by adding a chord. Let the p–cycle be $u_1 u_2 \cdots u_p u_1$. Without loss of generality, we assume that the chord joins u_1 with u_i, where $3 \leq i \leq p-2$. We denote this graph by $C_p(i)$.

In [9], Lee, Chen and Wang studied the edge-graceful spectrum of cycles with one chord. They established the following results, using various clever labeling schemes.

[Lee, Chen and Wang]

- *If p is odd, then $Egsp(C_p(i)) = \{sp \mid s = 0, 1, 2, \ldots\}$.*

- *$Egsp(C_{4n}(n+1)) = \{sn \mid s = 1, 3, 5, \dots\}$.*
- *For $n \geq 2$, $Egsp(C_{4n}(2n+1)) = \{sn \mid s = 1, 3, 5, \dots\}$.*

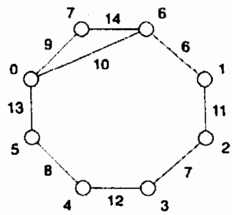

FIGURE 3. A 6–edge-graceful labeling of $C_8(3)$.

However, many unresolved cases were left open. Here, we complete the description of Egsp($C_p(i)$). Specifically, we answer the question: What are the edge-graceful spectra of $C_{4n}(i)$, for $i \neq n+1$, $2n+1$?

Lemma 1. *Let $G = C_{2m+1}$ and $a \in \mathbb{N} \cup \{0\}$. Then, there is an edge labeling $g : E(G) \to \{l \mid a \leq l \leq a+2m\}$ which induces a vertex labeling $g^+ : V(G) \to \{k \mid 2a+m \leq k \leq 2a+3m\}$.*

Proof. Let $G = C_{2m+1} = v_1v_2\cdots v_{2m+1}v_1$. We use the sequence of integers $a, a+1, \dots, a+2m$ to label the edges of G by starting at v_1v_2 and skipping every other edge along the cycle. That is to say, label v_1v_2 with a, v_3v_4 with $a+1$, and so on. The last edge of G is $v_{2m}v_{2m+1}$ and is labeled with $a+2m$. More precisely, this labeling is described by the mapping $g : E(C_{2m+1}) \to \{a, a+1, \dots, a+2m\}$, where $g(v_{2i+1}v_{2i+2}) = a+i$ for $0 \leq i \leq 2m$, where the addition in the subscripts are taken modulo $2m+1$. Then,

$$\begin{aligned}
g^+(v_{2j+1}) &= g(v_{2j}v_{2j+1}) + g(v_{2j+1}v_{2j+2}) \\
&= g(v_{2m+2j+1}v_{2m+2j+2}) + g(v_{2j+1}v_{2j+2}) \\
&= (a+m+j) + (a+j) \\
&= 2a+m+2j, \text{ for } 0 \leq j \leq m, \quad \text{and} \\
g^+(v_{2j}) &= g(v_{2j-1}v_{2j}) + g(v_{2j}v_{2j+1}) \\
&= g(v_{2j-1}v_{2j}) + g(v_{2m+2j+1}v_{2m+2j+2}) \\
&= (a+j-1) + (a+m+j) \\
&= 2a+m+2j-1, \text{ for } 1 \leq j \leq m.
\end{aligned}$$

Thus, the set $\{g^+(v_i) \mid 1 \leq i \leq 2m+1\} = \{k \mid 2a+m \leq k \leq 2a+3m\}$. ☐

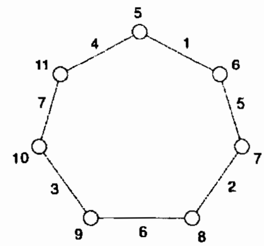

FIGURE 4. An example illustrating Lemma 1, $G = C_7$ and $a = 1$.

Theorem 2. *Let $n \geq 1$. Then, $C_{4n}(i)$ is n–edge-graceful and $3n$–edge-graceful.*

Proof. Consider the cycle $C_{2(2n-1)+1} = C_{4n-1} = v_1 v_2 \cdots v_{4n-1} v_1$ with $m = 2n - 1$ and $a = 1$. Using Lemma 1, we label C_{4n-1} with g, where the edge label set is $\{l \mid 1 \leq l \leq 4n-1\}$ and the set of vertex sums is $\{k \mid 2n+1 \leq k \leq 6n-1\}$. Moreover, note that $g(v_{2n-1}v_{2n}) = n$ and that $g(v_{2n}v_{2n+1}) = 3n$.

First, we construct an n–edge-graceful labeling of C_{4n}. We subdivide edge $v_{2n-1}v_{2n}$ by adding a vertex v to obtain a $4n$–cycle. The two new edges $v_{2n-1}v$ and vv_{2n} are each labeled with n. Then, we add a chord joining v to any other non-adjacent vertex in the new cycle and label the chord 0. Note that the vertex sum for v is $2n$. For this labeling of $C_{4n}(i)$, the edge muti-set is $\{0, 1, 2, \ldots, n, n, n+1, \ldots, 4n-1\}$ and the vertex set is $\{2n, 2n+1, 2n+2, \ldots, 6n-1\}$. After reduction modulo $4n$, the vertex set is $\mathbb{Z}_{4n}$. By re-labeling some of the edges (by adding $4n$), we obtain an n–edge-graceful labeling of $C_{4n}(i)$.

To construct a $3n$–edge-graceful labeling of C_{4n}, we proceed in a similar fashion. From the initial labeling of C_{4n-1}, we subdivide edge $v_{2n}v_{2n+1}$ by adding a vertex v to obtain a $4n$–cycle. The two new edges $v_{2n}v$ and vv_{2n+1} are each labeled with $3n$. Then, we add a chord joining v to any other non-adjacent vertex in the new cycle and label the chord 0. Note that the vertex sum for v is $6n$. For this labeling of $C_{4n}(i)$, the edge muti-set is $\{0, 1, 2, \ldots, 3n, 3n, 3n+1, \ldots, 4n-1\}$ and the vertex set is $\{2n+1, 2n+2, \ldots, 6n-1, 6n\}$. After reduction modulo $4n$, the vertex set is $\mathbb{Z}_{4n}$. By re-labeling some of the edges (by adding $4n$), we obtain a $3n$–edge-graceful labeling of $C_{4n}(i)$. □

In Figure 5, an example is given to illustrate the construction of an n–edge-graceful labeling of $C_{4n}(i)$.

Corollary 2. *$Egsp(C_{4n}(i)) = \{sn \mid s = 1, 3, 5, \ldots\}$.*

Proof. This follows immediately from Theorem 2 and the remarks made in Section 3. □

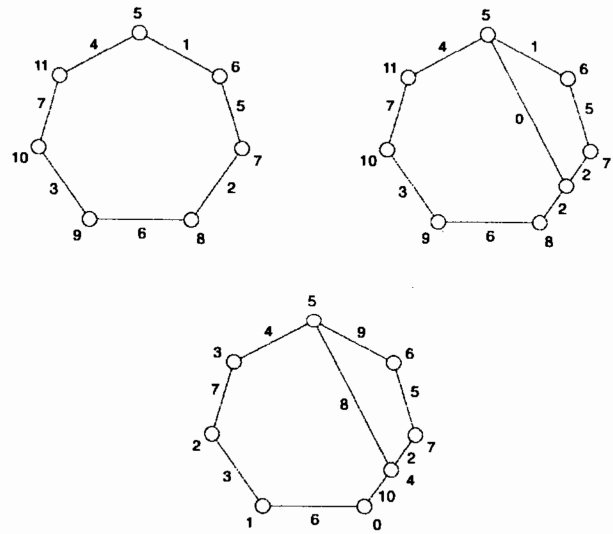

FIGURE 5. A 2–edge-graceful labeling for $C_8(4)$.

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Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Kowloon Tong, Hong Kong, China

E-mail address: `wcshiu@hkbu.edu.hk`

Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Kowloon Tong, Hong Kong, China

E-mail address: `mhling@math.hkbu.edu.hk`

Department of Mathematics, San Jose State University, San Jose, CA 95192, USA

E-mail address: `low@math.sjsu.edu`