# Extreme Friendly Indices of $C_m \times C_n$

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**Abstract**

Let $G = (V, E)$ be a connected simple $(p, q)$-graph. A labeling $f : V \to \mathbb{Z}_2$ induces an edge labeling $f^* : E \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |f^{*-1}(i)|$. A labeling $f$ is called friendly if $|v_f(1) - v_f(0)| \leq 1$. For a friendly labeling $f$ of a graph $G$, we define the friendly index of $G$ under $f$ by $i_f(G) = e_f(1) - e_f(0)$. The set $\{i_f(G)|f$ is a friendly labeling of $G\}$ is called the full friendly index set of $G$. In this paper, we will present the maximum and minimum friendly indices of Cartesian product of two cycles.

**Keywords:** vertex labeling, friendly labeling, friendly index set, Cartesian product of two cycles

## 1 Introduction and Notations

In this paper, all graphs are assumed to be loopless and connected. All undefined symbols and concepts can be referred to [1]. Let $G$ be a connected simple $(p, q)$-graph. A labeling $f : V \to \mathbb{Z}_2$ induces an edge labeling $f^* : E \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, define $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |f^{*-1}(i)|$. A labeling $f$ is called *friendly* if $|v_f(1) - v_f(0)| \leq 1$. For a friendly labeling $f$ of a graph $G$, we define the *friendly index* of $G$ under $f$ by $i_f(G) = e_f(1) - e_f(0)$. The set

$$\{|i_f(G)| : f \text{ is a friendly labeling of } G\}$$

is called *the friendly index set* of $G$ [2, 3, 4]. The set

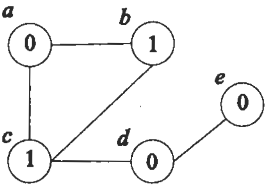
$$\{i_f(G)|f \text{ is a friendly labeling of } G\}$$

is called the *full friendly index set* of $G$ [5].

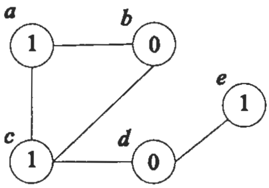
The friendly index set of cycles was determined by Kwong, Lee and Ng [3] and the full friendly index set of $P_2 \times P_n$ was determined by Shiu and Kwong [5]. In this paper, we are interested in the bounds of the full friendly index set of $C_m \times C_n$.

Given cycles $C_m$ and $C_n$ with vertex sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively, the Cartesian product $C_m \times C_n$ is a simple graph with vertex sets consisting of $mn$ vertices labeled $(i, j)$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. Two vertices $(i, j)$ and $(h, k)$ are adjacent in $C_m \times C_n$ if either $i = h$ and $v_j$ is adjacent to $v_k$ in graph $C_n$, or $j = k$ and $v_i$ is adjacent to $v_h$ in graph $C_m$. Note that $C_m \times C_n$ is a graph of order $mn$ and size $2mn$. In this paper, the vertices $(i, j)$ are denoted as $u_{ij}$, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

**Example 1.1** Both labelings $f_1$ and $f_2$ of the graph in Figure 1.1 are friendly. Friendly indices of the graph under $f_1$ and $f_2$ are shown in Figure 1.1.




Labeling $f_1$,
$e_{f_1}(1) - e_{f_1}(0) = 1$




Labeling $f_2$,
$e_{f_2}(1) - e_{f_2}(0) = 3$

Figure 1.1: Friendly labelings of a graph of order 5.

# 2 The upper bounds

For a fixed labeling $f$, a vertex $v$ is called a *k-vertex* if $f(v) = k$ and an edge $e$ is called a *k-edge* if $f^*(e) = k$.

**Lemma 2.1** *If $f$ is a friendly labeling of a $(p, q)$- graph $G$, then $i_f(G) \leq q$.*

**Proof:** Since the size of the graph $G$ is $q$, we have $e_f(1) \leq q$ and $e_f(0) \geq 0$. Hence, $i_f(G) \leq q$. □

**Corollary 2.2** *If $f$ is a friendly labeling of the graph $C_m \times C_n$, then $i_f(C_m \times C_n) \leq 2mn$.*

**Lemma 2.3** *An odd cycle $C$ in a graph with labeling $f$ contains at least one* 0*-edge.*

**Proof:** Since $\sum_{e\in E(C)} f^*(e) = 2\sum_{v\in V(C)} f(v) \equiv 0 \pmod 2$, there exists at least one 0-edge in the odd cycle $C$. $\square$

**Theorem 2.4** *If $f$ is a friendly labeling of the graph $C_m \times C_n$, then $i_f(C_m \times C_n) \leq 2mn - 2m$ when $m$ is even and $n$ is odd.*

**Proof:** The graph $C_m \times C_n$ contains at least $m$ edge disjoint odd cycles. By Lemma 2.3, we have $e_f(0) \geq m$ and so $e_f(1) \leq 2mn - m$. Hence, $i_f(C_m \times C_n) \leq 2mn - 2m$. $\square$

Note that $i_f(C_m \times C_n) \leq 2mn - 2n$ when $m$ is odd and $n$ is even. Therefore, we consider the case when $m$ is even and $n$ is odd only.

**Theorem 2.5** *If $f$ is a friendly labeling of the graph $C_m \times C_n$, then $i_f(C_m \times C_n) \leq 2mn - 2m - 2n$ when $m$ and $n$ are odd.*

**Proof:** Using similar arguments above, the graph $C_m \times C_n$ contains at least $m+n$ edge disjoint odd cycles. By Lemma 2.3, we have $e_f(0) \geq m+n$ and so $e_f(1) \leq 2mn - m - n$. Hence, $i_f(C_m \times C_n) \leq 2mn - 2m - 2n$. $\square$

From the above theorems, the upper bounds of friendly indices of $C_m \times C_n$ are respectively $2mn$, $2mn - 2m$ and $2mn - 2m - 2n$ according to different combinations of the parity of $m$ and $n$.

For $1 \leq i \leq m, 1 \leq j \leq n$, let $f(u_{ij}) = i + j \pmod 2$. It is easy to see that $f$ is a friendly labeling of $C_m \times C_n$. For each edge $u_{ab}u_{cd} \in E(C_m \times C_n)$, either $a = c$ and $b \equiv d \pm 1 \pmod n$, or $b = d$ and $a \equiv c \pm 1 \pmod m$.

$$\begin{aligned} f^*(u_{ab}u_{cd}) &= f(u_{ab}) + f(u_{cd}) = a + b + c + d \\ &\equiv \begin{cases} 0 \pmod 2 & \text{if} \quad a = c, b = 1 \text{ and } d = n \text{ is odd}, \\ 0 \pmod 2 & \text{if} \quad b = d, a = 1 \text{ and } c = m \text{ is odd}, \\ 1 \pmod 2 & \text{if} \quad \text{otherwise}. \end{cases} \end{aligned}$$

Then

$$e_f(0) = \begin{cases} 0 & \text{if} \quad m, n \text{ are even}, \\ m & \text{if} \quad m \text{ is even and } n \text{ is odd}, \\ m+n & \text{if} \quad m, n \text{ are odd}. \end{cases}$$

and hence

$$i_f(C_m \times C_n) \equiv \begin{cases} 2mn & \text{if } m, n \text{ are even,} \\ 2mn - 2m & \text{if } m \text{ is even and } n \text{ is odd,} \\ 2mn - 2m - 2n & \text{if } m, n \text{ are odd.} \end{cases}$$

Therefore, the maximum friendly indices of $C_m \times C_n$ is respectively $2mn$ when $m$ and $n$ are even, $2mn - 2m$ when $m$ is even and $n$ is odd, and $2mn - 2m - 2n$ when $m$ and $n$ are odd. Hence, the bounds of Corollary 2.2, Theorem 2.4 and Theorem 2.5 are sharp.

**Example 2.1** Labelings $f$ of $C_6 \times C_4$, $C_6 \times C_3$ and $C_5 \times C_3$ illustrate the proof of Corollary 2.2, Theorem 2.4 and Theorem 2.5.

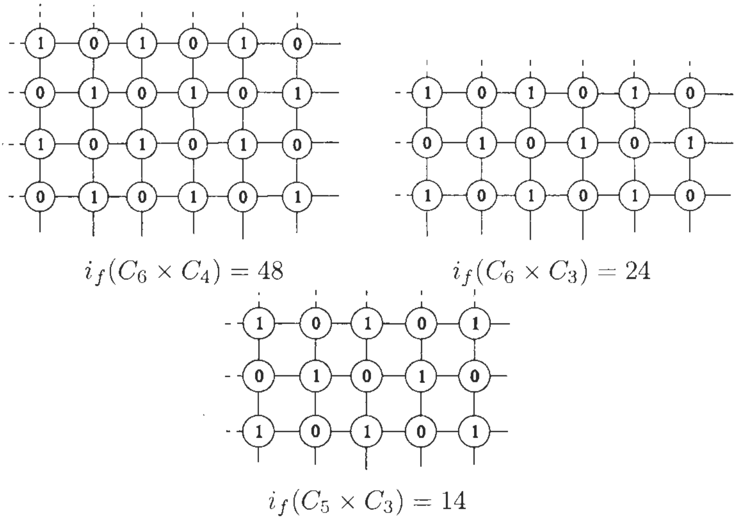



Figure 2.2: Labelings of $C_6 \times C_4$, $C_6 \times C_3$ and $C_5 \times C_3$

# 3 The lower bounds

Let $f$ be any labeling of a graph containing a cycle $C$ as its subgraph. The cycle is called *mixing* (under $f$), if there is two vertices $u, v \in V(C)$ such that $f(u) = 1$ and $f(v) = 0$. Clearly, a mixing cycle contains at least one 1-edge. The cycle $C$ is called 1-*full cycle* (under $f$), if $f(u) = 1$ for any vertex $u \in V(C)$. The cycle $C$ is called *0-full cycle* (under $f$), if $f(u) = 0$ for any vertex $u \in V(C)$. Note that the content in the bracket will be omitted if there is no ambiguity.

**Lemma 3.1** [5] *For any labeling, the number of* 1*-edges in a mixing cycle is even.*

Now we consider the graph $C_m \times C_n$. Due to isomorphic, we may assume that $n \le m$. For $1 \le i \le m$, the cycle $u_{i1}u_{i2}\ldots u_{in}u_{i1}$ is called *vertical cycle* and for $1 \le j \le n$, the cycle $u_{1j}u_{2j}\ldots u_{mj}u_{1j}$ is called *horizontal cycle.*

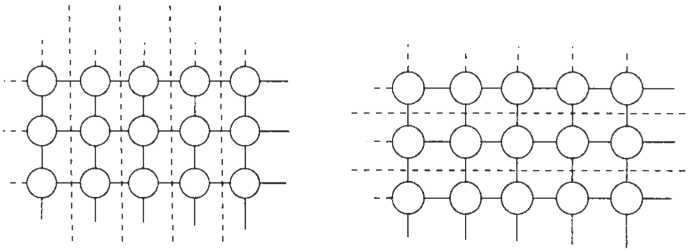


Figure 3.3: vertical cycles and horizontal cycles

**Theorem 3.2** *Let $f$ be a friendly labeling of the graph $C_m \times C_n$. If $m$ is even, then $i_f(C_m \times C_n) \ge 4n - 2mn$.*

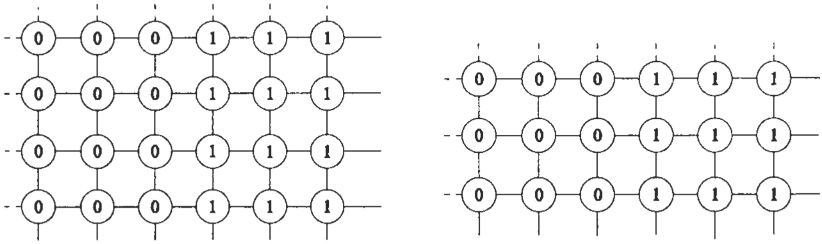
**Proof:** Let $r$ be the number of horizontal 1-full cycles and $s$ be the number of horizontal 0-full cycles. By the property of friendly labeling, we have $0 \le r, s \le \frac{n}{2}$. If $r = s = 0$, then all horizontal cycles are mixing and hence the number of edge disjoint mixing cycles in $C_m \times C_n$ is at least $n$. If either $r = 0$ or $s = 0$, then, without loss of generality, we may assume $r \ne 0$ and $s = 0$. In this case, the number of horizontal mixing cycles is $n - r$, and hence there exist at least $\lceil \frac{mn/2}{(n-r)} \rceil$ vertical mixing cycles since $\frac{mn}{2}$ 0-vertices lie in $n - r$ horizontal cycles. Therefore, there are at least $n - r + \lceil \frac{mn}{2(n-r)} \rceil$ edge disjoint mixing cycles in $C_m \times C_n$. Note that $n - r + \lceil \frac{mn}{2(n-r)} \rceil \ge n - \frac{n}{2} + \frac{m}{2}\frac{n}{(n-r)} \ge \frac{m+n}{2} \ge n$. If $r \ne 0$ and $s \ne 0$, then there exist $m$ vertical mixing cycles. Hence, there are at least $n$ edge disjoint mixing cycles in $C_m \times C_n$. For each case, the number of edge disjoint mixing cycles in $C_m \times C_n$ is at least $n$. By Lemma 3.1, we get $e_f(1) \geq 2n$ and $e_f(0) \leq 2mn - 2n$. Hence, $i_f(C_m \times C_n) \geq 4n - 2mn$. □

Suppose $m$ is even. Let $f(u_{ij}) = 0$ for $1 \leq i \leq \frac{m}{2}, 1 \leq j \leq n$ and $f(u_{ij}) = 1$ for $\frac{m}{2} + 1 \leq i \leq m, 1 \leq j \leq n$. It is easy to see that $f$ is a friendly labeling of $C_m \times C_n$. For each edge $u_{ab}u_{cd} \in E(C_m \times C_n)$, either $a = c$ and $b \equiv d \pm 1 \pmod n$, or $b = d$ and $a \equiv c \pm 1 \pmod m$.

$$\begin{aligned} f^*(u_{ab}u_{cd}) &= f(u_{ab}) + f(u_{cd}) = a + b + c + d \\ &\equiv \begin{cases} 1 \pmod 2 & \text{if } \; b = d \text{ and } a = c - 1 = m/2, \\ 1 \pmod 2 & \text{if } \; b = d, a = 1 \text{ and } c = m, \\ 0 \pmod 2 & \text{if } \; \text{otherwise}. \end{cases} \end{aligned}$$

Hence, $e_f(1) = 2n$ and the minimum friendly index of $C_m \times C_n$ is $4n - 2mn$. That is, the bound of Theorem 3.2 is sharp.

**Example 3.1** Labelings $f$ of $C_6 \times C_4$ and $C_6 \times C_3$ illustrate the proof of Theorem 3.2.




$i_f(C_6 \times C_4) = -32$ $\qquad$ $i_f(C_6 \times C_3) = -24$

Figure 3.4: Labelings of $C_6 \times C_4$ and $C_6 \times C_3$

**Theorem 3.3** *Let $f$ be a friendly labeling of the graph $C_m \times C_n$. If $m$ is odd, then $i_f(C_m \times C_n) \geq 4n + 4 - 2mn$.*

**Proof:** We adopt the notations defined in the proof of Theorem 3.2. If $r = s = 0$, then all horizontal cycles are mixing and the number of horizontal mixing cycle is $n$. There is at least one vertical mixing cycle since $m$ is odd and $f$ is balanced. Therefore, the number of edge disjoint mixing cycles in $C_m \times C_n$ is at least $n+1$. If either $r = 0$ or $s = 0$, using the same argument of the proof of Theorem 3.2, there are $n - r + \lceil \frac{mn}{2(n-r)} \rceil$ edge disjoint mixing cycles in $C_m \times C_n$. Note that $n - r + \lceil \frac{mn}{2(n-r)} \rceil \geq \frac{n}{2} + \lceil \frac{m}{2} \rceil \geq \frac{n}{2} + \frac{m+1}{2} \geq n + \frac{1}{2} > n$. If $r \neq 0$ and $s \neq 0$, then there exist $m$ vertical mixing cycles. Hence, there are at least $n+1$ edge disjoint mixing cycles in $C_m \times C_n$. For each case, the number of edge disjoint mixing cycles in $C_m \times C_n$ is at least $n+1$. By Lemma 3.1, we get $e_f(1) \geq 2n+2$ and $e_f(0) \leq 2mn - 2n - 2$, and $i_f(C_m \times C_n) \geq 4n + 4 - 2mn$. $\square$

Suppose $m$ is odd. Let

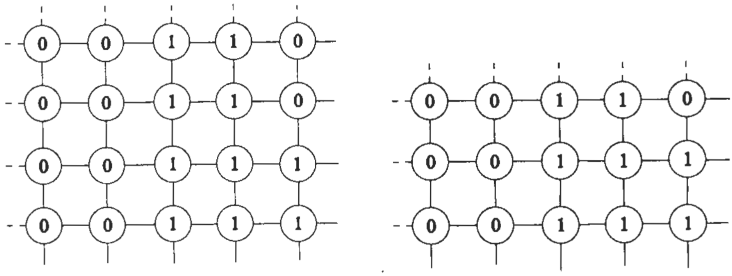
$$f(u_{ij}) = \begin{cases} 0 & \text{if} \quad 1 \leq i \leq \lfloor \frac{m}{2} \rfloor, 1 \leq j \leq n, \\ 1 & \text{if} \quad \lceil \frac{m}{2} \rceil \leq i \leq m-1, 1 \leq j \leq n, \\ 0 & \text{if} \quad i = m, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor, \\ 1 & \text{if} \quad i = m, \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n. \end{cases}$$

For each edge $u_{ab}u_{cd} \in E(C_m \times C_n)$, either $a = c$ and $b \equiv d \pm 1 \pmod{n}$, or $b = d$ and $a \equiv c \pm 1 \pmod{m}$. $f^*(u_{ab}u_{cd}) = f(u_{ab}) + f(u_{cd}) = a + b + c + d$

$$\equiv \begin{cases} 1 \pmod 2 & \text{if} \quad b = d \text{ and } a = c - 1 = \lfloor \frac{m}{2} \rfloor, \\ 1 \pmod 2 & \text{if} \quad 1 \leq b = d \leq \lfloor \frac{n}{2} \rfloor \text{ and } a = c - 1 = m - 1, \\ 1 \pmod 2 & \text{if} \quad \lfloor \frac{n}{2} \rfloor + 1 \leq b = d \leq n, a = 1 \text{ and } c = m, \\ 1 \pmod 2 & \text{if} \quad a = c = m \text{ and } b = d - 1 = \lfloor \frac{n}{2} \rfloor, \\ 1 \pmod 2 & \text{if} \quad a = c = m, b = 1 \text{ and } d = n, \\ 0 \pmod 2 & \text{if} \quad \text{otherwise}. \end{cases}$$

Hence, $e_f(1) = 2n + 2$ and the minimum friendly index of $C_m \times C_n$ is $4n + 4 - 2mn$. That is, the bound of Theorem 3.3 is sharp.

**Example 3.2** Labelings $f$ of $C_5 \times C_4$ and $C_5 \times C_3$ illustrate the proof of Theorem 3.3.




$i_f(C_5 \times C_4) = -20$ $\qquad$ $i_f(C_5 \times C_3) = -14$

Figure 3.5: Labelings of $C_5 \times C_4$ and $C_5 \times C_3$

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