Super-edge-graceful labelings of some cubic graphs

Wai Chee SHIU

Department of Mathematics, Hong Kong Baptist University

224 Waterloo Road, Kowloon Tong,

Hong Kong, China

Abstract

The notion of super-edge-graceful graphs was introduced by Mitchem and Simoson in 1994. However, few examples except trees are known. In this paper, we exhibit two classes of infinitely many cubic graphs which are super-edge-graceful. A conjecture is proposed.

Keywords: Super-edge-graceful, cubic graph, permutation cubic graph, permutation Petersen graph, permutation ladder graph

AMS 2000 MSC: 05C78, 05C25

1 Introduction

In this paper all graphs are loopless and connected. A (p,q)-graph $G=(V,E)$ is called *edge-graceful* if there exists a bijection $f: E \to \{1,2,\ldots,q\}$ and the *induced mapping* $f^+: V \to \mathbb{Z}_p = \{0,1,\ldots,p-1\}$ defined by

$$f^+(u) \equiv \sum_{uv\in E} f(uv) \pmod{p}$$

is a bijection. This concept was introduced by Lo [4] in 1985. Lee [1] conjectured that all trees of odd order are edge-graceful. More references about edge-graceful may be found in [6, 8, 2].

Mitchem and Simoson [5] tried to prove the above conjecture and introduce a variation of edge-gracefulness which for trees of odd order implies edge-gracefulness.

Let

$$P = \begin{cases} \{-\frac{p}{2},\ldots,-1,1,\ldots,\frac{p}{2}\} & \text{if } p \text{ is even} \\ \{-\frac{p-1}{2},\ldots,-1,0,1,\ldots,\frac{p-1}{2}\} & \text{if } p \text{ is odd} \end{cases}$$

and

$$Q = \begin{cases} \{-\frac{q}{2},\ldots,-1,1,\ldots,\frac{q}{2}\} & \text{if } q \text{ is even} \\ \{-\frac{q-1}{2},\ldots,-1,0,1,\ldots,\frac{q-1}{2}\} & \text{if } q \text{ is odd} \end{cases}.$$

A (p,q)-graph G is *super-edge-graceful* if there is a pair of mappings (f,f^+) such that $f: E \to Q$ is bijective and $f^+: V \to P$ is also bijective, where $f^+(u) = \sum_{uv\in E} f(uv)$. f is called a *super-edge-graceful labeling of* G.

Partially supported by Faculty Research Grant, Hong Kong Baptist University.

The notions of super-edge-graceful graphs and edge-graceful graphs are different. Mitchem and Simoson [5] showed that the step graph C_6^2 is super-edge-graceful but not edge-graceful. They asked whether edge-gracefulness implies super-edge-gracefulness. It is known that the complete graph K_4 is edge-graceful (see Figure 1). We show below that it is not super-edge-graceful.

Proposition 1.1: *The complete graph K_4 is not super-edge-graceful.*

Proof: Suppose there is a super-edge-graceful labeling $f : E(K_4) \to \{-3,-2,-1,1,2,3\}$ of K_4 such that $f^+ : V(K_4) \to \{-2,-1,1,2\}$ is a bijection. Let the vertices of K_4 be a, b, c, d. Without loss the generality, we may assume $f(ab) = 3$. Since $f^+(a) \le 2$, $f(ad) + f(ac) \le -1$. Similarly, we have

$$f(bd) + f(bc) \le -1. \tag{1}$$

Since $f(ad) + f(ac) + f(bd) + f(bc) \le -2$, $f(cd) \ne -3$. After renaming the vertices if necessary, we may assume $f(ac) = -3$. By (1), $-2 \in \{f(bd), f(bc)\} \subset \{-2,-1,1\}$. Since $f^+(b) \ne 0$, $\{f(bd), f(bc)\} = \{-2, 1\}$. Since $f^+(c) \ge -2$,

$$f(cd) + f(bc) \ge 1. \tag{2}$$

Thus $f(bc) \ne -2$. Hence $f(bc) = 1$ and $f(bd) = -2$. By (2), $f(cd) = 2$. But it is impossible because $f^+(c) \ne 0$. □

Clearly, $K_2[3]$ defined in Figure 2 is not super-edge-graceful.

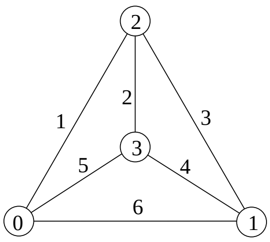

Figure 1: Edge-graceful labeling of K_4

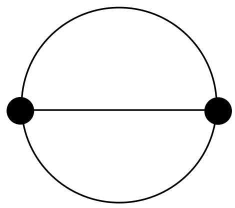

Figure 2: The graph $K_2[3]$

We propose the following conjecture:

Every connected cubic multigraph except K_4 and $K_2[3]$ is super-edge-graceful.

Shiu and Lam [9] found some classes of super-edge-graceful graphs. In this paper, two classes of graphs will be shown to be super-edge-graceful.

2 Permutation cubic graphs and permutation Petersen graphs

A special class of cubic graphs is the class of permutation cubic graphs. For $n \ge 2$, a *permutation cubic graph* on $2n$ vertices is defined by taking two vertex-disjoint cycles on n vertices and adding a perfect matching between the vertices of the two cycles. Namely, let two cycle be $C = u_1u_2\cdots u_nu_1$ and $C^* = v_1v_2\cdots v_nv_1$ and let $\sigma \in S_n$, the permutation group on the set $\{1, 2, \dots, n\}$ letters. The permutation cubic graph $\mathcal{P}(n;\sigma) = (V, E)$ is a simple graph with $V = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and $E = E(C) \cup E(C^*) \cup \{u_iv_{\sigma(i)} \mid 1 \le i \le n\}$. Note that, in this paper a cycle on 2-vertices is defined to be the multigraph consisting of two parallel edges and is denoted by C_2.

Lee [3] conjectured that the permutation cubic graph $\mathcal{P}(n;\sigma)$ is edge-magic if n is odd. It is not suitable to discuss edge-magicness here. The interested reader is referred to [3, 7].

If we renumber the vertices of C^* by $w_i = v_{\sigma(i)}$ and let $\tau = (1\ 2\ \cdots\ n)$ be the n-cycle in S_n, then the edge $v_{\sigma(i)}v_{\sigma(i)+1} = w_iw_{\sigma^{-1}\tau\sigma(i)}$ in $\mathcal{P}(n;\sigma)$.

Now we define a class of cubic graphs called *permutation Petersen graphs* which is generalized from the generalized Petersen graph [10]. Let $\theta \in S_n$ without fixed point, i.e., $\theta(i) \ne i$ for all i, $n \ge 2$. The θ*-Petersen graph* $P(n;\theta)$ is the graph with vertex set $\{u_1, u_2, \dots, u_n\} \cup \{w_1, w_2, \dots, w_n\}$ and edge set $\{u_iu_{i+1}, u_iw_i, w_iw_{\theta(i)} \mid 1 \le i \le n\}$, where the addition is modulo n. Note that if $\theta(i) = j$ and $\theta(j) = i$ for some i, j, then $w_iw_{\theta(i)}$ and $w_jw_{\theta(j)}$ represent two different edges. So $P(n;\theta)$ may not be simple. The cycle $C = u_1u_2\cdots u_nu_1$ is called the *outer cycle* of $P(n;\theta)$ and the subset of edges $\{u_iw_i \mid 1 \le i \le n\}$ is called the *natural perfect matching* of $P(n;\theta)$. The subgraph H of $P(n;\theta)$ induced by vertices $w_1, w_2, \dots, w_n$ is a union of disjoint cycles depend on the cycle decomposition of the permutation θ. Namely, suppose $\theta = \theta_1\theta_2\cdots\theta_r$ is the cycle decomposition of θ. Let $\theta_j = (j_1\ j_2\ \cdots\ j_s)$ and $C_j = w_{j_1}w_{j_2}\cdots w_{j_s}w_{j_1}$. Then $H = \sum\limits_{j=1}^{r} C_j$, the disjoint union of C_j.

Clearly, $\mathcal{P}(n;\sigma) = P(n;\sigma^{-1}\tau\sigma)$, where $\tau = (1\ 2\ \cdots\ n)$. Thus the class of permutation cubic graph is a subclass of the class of permutation Petersen graphs. Note that, $\mathcal{P}(n;\tau) = P(n;\tau)$ is isomorphic to $\mathcal{P}(n;\iota)$, where ι is the identity permutation. Moreover, $\mathcal{P}(n;\iota)$ is isomorphic to $C_n \times P_2$, the Cartesian product of the n-cycle with the path of two vertices. So we identify $P(n;\tau)$, $\mathcal{P}(n;\iota)$ and $C_n \times P_2$.

For ease of understanding the proof of Theorem 2.2, we start at a simple case.

Theorem 2.1: *The graph $C_{2n} \times P_2$ is super-edge-graceful, where $n \ge 1$.*

Proof: The graph $C_{2n} \times P_2 = P(2n;\tau) = (V, E)$, where $\tau = (1\ 2\ \cdots\ 2n) \in S_{2n}$. We keep the notations of the permutation Petersen graph defined above and assume the outer cycle is C and $C^* = H$ (the inner cycle). We have to find a bijection $f : E \to \{\pm 3n,\ \pm(3n-1),\ \dots,\ \pm 2,\ \pm 1\}$ such that $f^+ : V \to \{\pm 2n,\ \pm(2n-1),\ \dots,\ \pm 2,\ \pm 1\}$ is a bijection.

Firstly, we label the edges of C by using the numbers $\pm(2n+1), \dots, \pm 3n$. The labeling f is defined below:

$$f(u_1u_2) = 2n+1,\ f(u_2u_3) = -(2n+1),\ f(u_3u_4) = 2n+2,\ f(u_4u_5) = -(2n+2), \dots, f(u_{2n-1}u_{2n}) = 3n,$$
$$f(u_{2n}u_1) = -3n,$$

i.e., $f(u_iu_{i+1}) = \begin{cases} 2n + \frac{i+1}{2} & \text{if } i \text{ is odd} \\ -(2n + \frac{i}{2}) & \text{if } i \text{ is even} \end{cases}$, for $1 \le i \le 2n$ with $u_{2n+1} = u_1$.

Secondly, we label the natural perfect matching by using the numbers $-1, -2, \dots, -2n$. The labeling is defined by

$$f(u_1v_1) = -1,\ f(u_2v_2) = -(n+1),\ f(u_3v_3) = -2,$$
$$f(u_4v_4) = -(n+2), \dots, f(u_{2n-1}v_{2n-1}) = -n,\ f(u_{2n}v_{2n}) = -2n,$$

i.e., $f(u_iv_i) = \begin{cases} -\frac{i+1}{2} & \text{if } i \text{ is odd;} \\ -(n + \frac{i}{2}) & \text{if } i \text{ is even,} \end{cases}$ for $1 \le i \le 2n$.

Then $f^+(u_i) = f(u_iu_{i+1}) + f(u_{i-1}u_i) + f(u_iv_i) = \begin{cases} -n & \text{if } i = 1; \\ -\frac{i-1}{2} & \text{if } i \text{ is odd and } 1 < i < 2n; \\ -(n + \frac{i}{2}) & \text{if } i \text{ is even and } 2 \le i \le 2n. \end{cases}$ Hence we have $\{f^+(u_i) \mid 1 \le i \le 2n\} = \{-1, -2, \dots, -2n\}$.

Finally, we have to label the edges of C^* by using the numbers $1, 2, \dots, 2n$. We label the edge v_1v_2 by 1 (the minus of $f(u_1v_1)$); label the edge v_2v_3 by $n+1$ (the minus of $f(u_2v_2)$); label the edge v_3v_4 by 2; continuing this process, finally we label the edge $v_{2n}v_1$ by $2n$. That is, $f(v_iv_{i+1}) = -f(u_iv_i)$ for $1 \le i \le 2n$ with $v_{2n+1} = v_1$.

Then, for $1 \le i \le 2n$, the number

$$f^+(v_i) = f(v_{i-1}v_i) + f(v_iv_{i+1}) + f(u_iv_i) = f(v_{i-1}v_i) = -f(u_{i-1}v_{i-1}),$$

where $v_0 = v_{2n}$ and $u_0 = u_{2n}$. Therefore, $\{f^+(v_i) \mid 1 \le i \le 2n\} = \{1, 2, \dots, 2n\}$. Hence we have a super-edge-graceful labeling of $C_{2n} \times P_2$. $\square$

For the degenerate case, a super-edge-graceful labeling of $C_2 \times P_2$ is shown in Figure 3.

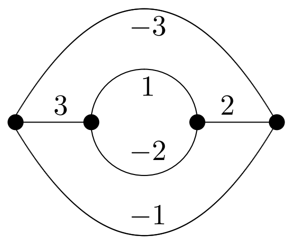

Figure 3: A super-edge-graceful labeling of $C_2 \times P_2$.

Now we are going to consider the general case.

Theorem 2.2: *For each $\theta \in S_{2n}$ without fixed point, the permutation Petersen graph $P(2n; \theta)$ is super-edge-graceful, where $n \ge 1$.*

Proof: Keep the notations defined before Theorem 2.1. Firstly, the labeling f for the edges of the outer cycle and the natural perfect matching of $P(2n;\theta)$ is the same as the labeling for those of $C_{2n} \times P_2$.

Let $\theta = \theta_1\theta_2\cdots\theta_r$ be the cycle decomposition, for some $r > 0$. Let $C_j = w_{j_1}w_{j_2}\cdots w_{j_s}w_{j_1}$ be defined before Theorem 2.1, $1 \le j \le r$. Let $f(w_{j_i}w_{j_{i+1}}) = -f(u_{j_i}w_{j_i})$ for $1 \le i \le s$, where $w_{j_{s+1}} = w_{j_1}$. Then $f^+(w_{j_i}) = f(w_{j_{i-1}}w_{j_i}) = -f(u_{j_{i-1}}w_{j_{i-1}})$, where $w_{j_0} = w_{j_s}$ and $u_{j_0} = u_{j_s}$. Since C_j are disjoint cycles, $\{f^+(w_i) \mid 1 \le i \le 2n\} = \{1, 2, \dots, 2n\}$. Hence we have a super-edge-graceful labeling of $P(2n;\theta)$. $\square$

Remark 2.1: To label the edges of C_j's in the above proof, we provide another labeling which will be used in the next section. Let $f(w_{j_i}w_{j_{i+1}}) = -f(u_{j_{i+1}}w_{j_{i+1}})$ for $1 \le i \le s$, where $w_{j_{s+1}} = w_{j_1}$. Then $f^+(w_{j_i}) = -f(u_{j_{i+1}}w_{j_{i+1}})$. Suppose $\theta(1) = 2, \theta(2n) = 1$. Then we have Figure 7a in the next section.

Now we are going to consider the graphs $P(2n+1;\theta)$ for $n \ge 1$. We only prove the simple case. The general case is similarly to prove.

Theorem 2.3: *The graph $C_{2n+1} \times P_2$ is super-edge-graceful, where $n \ge 1$.*

Proof: Keep the notations defined before Theorem 2.1. We have to label the edges of the graph by using the numbers $\pm(3n+1),\ \pm 3n,\ \dots,\ \pm 1,\ 0$ such that $f^+ : V \to \{\pm 2n,\ \pm(2n-1),\ \dots,\ \pm 2,\ \pm 1\}$ is a bijection.

Firstly, we label the edges of the outer cycle. Let
$f(u_1u_2) = -(2n+2)$, $f(u_2u_3) = 2n+3$, $f(u_3u_4) = -(2n+4)$, $\dots$, $f(u_nu_{n+1}) = (-1)^n(3n+1)$, $f(u_{n+1}u_{n+2}) = (-1)^{n+1}(3n+1)$, $f(u_{n+2}u_{n+3}) = (-1)^{n+2}(3n)$, $\dots$, $f(u_{2n}u_{2n+1}) = 2n+2$, $f(u_{2n+1}u_1) = 0$, i.e.,

$$f(u_iu_{i+1}) = (-1)^i(2n+1+i),\ f(u_{n+i}u_{n+i+1}) = (-1)^{n+i}(3n+2-i) \text{ for } 1 \le i \le n, \text{ and } f(u_{2n+1}u_1) = 0.$$

For $1 \le i \le n$, $f(u_{2n+1-i}u_{2n+2-i}) = (-1)^{2n+1-i}[3n+2-(n+1-i)] = (-1)^{i-1}(2n+i+1) = -f(u_iu_{i+1})$. Since $2n+2 \le |f(u_iu_{i+1})| \le 3n+1$ for $1 \le i \le n$,

$$\{f(u_iu_{i+1}) \mid 1 \le i \le 2n+1\} = \{\pm(3n+1),\ \pm 3n,\ \dots,\ \pm(2n+2),\ 0\}.$$

Secondly, we label the natural perfect matching. Let
$f(u_jw_j) = -\{2n+3-2j+\frac{1}{2}[1+(-1)^j]\}$, $f(u_{2n+2-j}w_{2n+2-j}) = -\{2n+3-2j+\frac{1}{2}[1-(-1)^j]\}$ for $2 \le j \le n$, $f(u_{n+1}w_{n+1}) = -2$, $f(u_{2n+1}w_{2n+1}) = -1$, and $f(u_1w_1) = 2n+1$.

Since $\{f(u_jw_j),\ f(u_{2n+2-j}w_{2n+2-j})\} = \{-(2n+4-2j),\ -(2n+3-2j)\}$ for $2 \le j \le n$,

$$\{f(u_jw_j) \mid 1 \le j \le 2n+1\} = \{-2n,\ -(2n-1),\ \dots,\ -2,\ -1,\ 2n+1\}.$$

Now for $2 \le i \le n$,

$$
\begin{aligned}
f^+(u_i) =& f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iw_i) \\
=& (-1)^{i-1}(2n+i) + (-1)^i(2n+1+i) - \{2n+3-2i+\tfrac{1}{2}[1+(-1)^i]\} \\
=& (-1)^i - \{2n+3-2i+\tfrac{1}{2}[1+(-1)^i]\} \\
=& -\{2n+3-2i+\tfrac{1}{2}[1-(-1)^i]\} = f(u_{2n+2-i}w_{2n+2-i}).
\end{aligned}
$$

Similarly, we have $f^+(u_{2n+2-i}) = f(u_iw_i)$ for $2 \le i \le n$. Clearly, $f^+(u_1) = -1$, $f^+(u_{2n+1}) = 2n+1$ and $f^+(u_{n+1}) = -2$. Thus,

$$\{f^+(u_i) \mid 1 \le i \le 2n+1\} = \{-2n,\ -(2n-1),\ \dots,\ -2,\ -1,\ 2n+1\}.$$

Finally, similar to the proof Theorem 2.1, by letting $f(w_iw_{i+1}) = -f(u_iw_i)$ for all i we have

$$\{f^+(w_i) \mid 1 \le i \le 2n+1\} = \{2n,\ 2n-1,\ \dots,\ 2,\ 1,\ -(2n+1)\}.$$

Hence the proof is complete. $\square$

By a similar proof of Theorem 2.3 we have

Theorem 2.4: *For each $\theta \in S_{2n+1}$ without fixed point, the graph $P(2n+1;\theta)$ is super-edge-graceful, where $n \ge 1$.*

We give two figures below to illustrate the proof of the theorems.

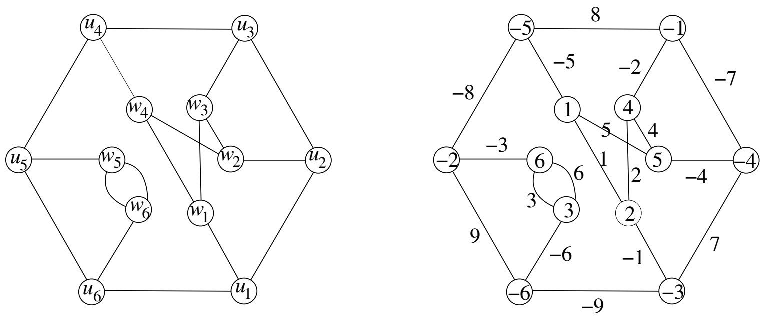

Figure 4: A super-edge-graceful labeling of $P(6;\theta)$, where $\theta = (1423)(56)$.

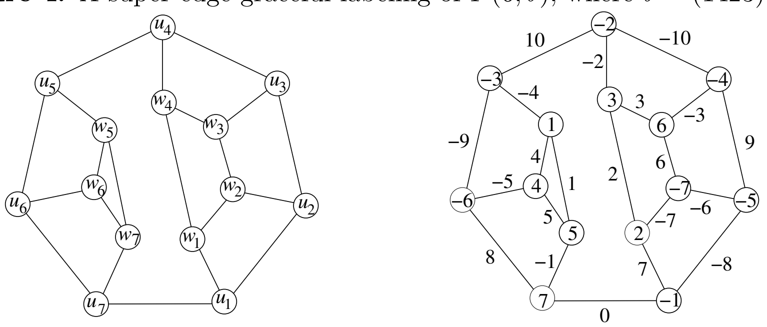

Figure 5: A super-edge-graceful labeling of $P(7;\theta)$, where $\theta = (1234)(567)$.

3 Ladder graphs and permutation ladder graphs

A ladder graph $L(n)$ of order $2n$ is a cubic multigraph obtained from $P_n \times K_2$ by duplicating the edges which are incident with vertices of degree 2 (see Figure 6).

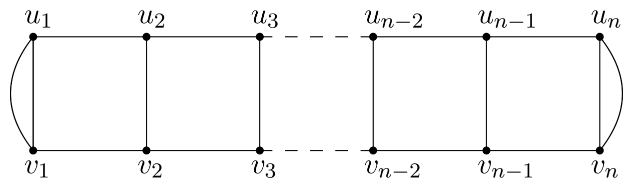

Figure 6: The ladder graph $L(n)$.

Let $\sigma \in S_n$. The $\sigma L(n)$ is defined by taking a cycle $u_1u_2\dots u_nv_nv_{n-1}\dots v_1u_1$ and adding a perfect matching $M = \{u_iv_{\sigma(i)} \mid 1 \le i \le n\}$. If σ is the identity permutation, then $\sigma L(n)$ is the ladder graph $L(n)$. If we renumber the vertices by $w_i = v_{\sigma(i)}$ and let $\tau = (1\ 2\ \cdots\ n) \in S_n$, then $v_{\sigma(i)}v_{\sigma(i)+1} = w_iw_{\sigma^{-1}\tau\sigma(i)}$.

Now we define a class of cubic multigraphs called *permutation ladder graphs*. Let $\theta \in S_n$ having the properties that $\theta(n) = 1$, $\theta(1) \ne n$ and $\theta(i) \ne i$ for $2 \le i \le n-1$, where $n \ge 3$. The θ*-ladder graph* $L(n;\theta)$ is the graph with the vertex set

$$\{u_1, u_2, \dots u_n, w_1, w_2, \dots, w_n\}$$

and the edge set (multiset)

$$\{u_1u_2, u_2u_3, \dots, u_{n-1}u_n, u_1w_1, u_1'w_1', u_2w_2, u_3w_3, \dots, u_{n-1}w_{n-1}, u_nw_n, u_n'w_n', w_1w_{\theta(1)}, \dots, w_{n-1}w_{\theta(n-1)}\},$$

where $u_1' = u_1$, $u_n' = u_n$, $w_1' = w_1$ and $w_n' = w_n$. Note that the conditions $\theta(n) = 1$ and $\theta(1) \ne n$ guarantee $\deg(w_1) = 3 = \deg(w_n)$. The condition $\theta(i) \ne i$ guarantee the graph is loopless.

Suppose $\sigma \in S_n$ with $\sigma(1) = 1$ and $\sigma(n) = n$. Then $\sigma L(n) = L(n; \sigma^{-1}\tau\sigma)$, where $\tau = (1\ 2\ \cdots\ n)$.

Now we shall apply the super-edge-graceful labeling of $P(n;\theta)$ described in the previous section to construction a super-edge-graceful labeling of the permutation ladder graph with u_2w_2 being an edge. For this case, $\theta(1) = 2$.

Suppose $\theta \in S_{2n}$ without fixed point having the properties that $\theta(1) = 2$ and $\theta(2n) = 1$, where $n \ge 2$. Consider the permutation Petersen graph $P(2n;\theta)$. Let f be the super-edge-graceful labeling of $P(2n;\theta)$ described in Remark 2.1. So we have Figure 7a.

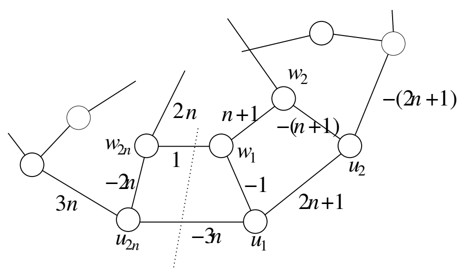

Figure 7a: Another super-edge-graceful labeling of $P(2n;\theta)$ with $\theta(1) = 2, \theta(2n) = 1$.

We delete the edges u_1u_{2n} and w_1w_{2n} from $P(2n;\theta)$ and add two additional edges $u_1'w_1'$ and $u_{2n}'w_{2n}'$ on the resulting graph, where $u_1' = u_1$, $u_{2n}' = u_{2n}$, $w_1' = w_1$ and $w_{2n}' = w_{2n}$. Then we obtain $L(2n;\theta)$. We define a labeling g on $L(2n;\theta)$ same as the labeling f on $P(2n;\theta)$ except the deleted edges and the added edges. Define $g(u_1'w_1') = -3n$ and $g(u_{2n}'w_{2n}') = 1$ (see Figure 7b). Then $g^+(w_1) = (n+1) + (-1) + (-3n) = -2n = f^+(u_{2n})$, $g^+(u_{2n}) = 3n + (-2n) + 1 = n+1 = f^+(w_1)$ and $g^+(x) = f^+(x)$ for $x \neq u_{2n}$ and $x \neq w_1$. Hence g is a super-edge-graceful labeling of $L(2n;\theta)$.

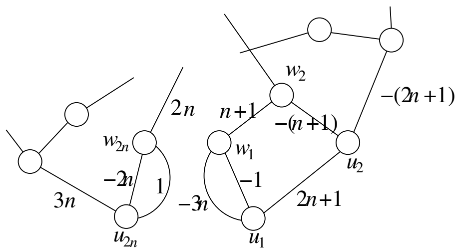

Figure 7b: A partial super-edge-graceful labeling of $L(2n;\theta)$ with $\theta(1) = 2$ and $\theta(2n) = 1$.

We state the theorem as follows:

Theorem 3.1: *For $n \geq 1$, suppose $\theta \in S_{2n}$ without fixed point and suppose $\theta(1) = 2$ and $\theta(2n) = 1$. Then $L(2n;\theta)$ is super-edge-graceful.*

Proof: For $n \geq 2$, we have just proved. For $n = 1$, then $\theta = (12)$ and $L(2;(12)) = L(2)$. Following is a super-edge-graceful labeling of $L(2)$.

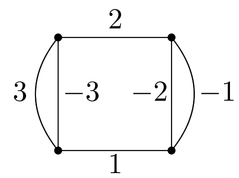

Figure 8: A super-edge-graceful labeling of $L(2)$.

□

Suppose $\theta \in S_{2n+1}$ without fixed point and has the properties that $\theta(1) = 2$ and $\theta(2n+1) = 1$. For applying the labeling of $P(2n+1;\theta)$ we make a reflection on the graph $L(2n+1;\theta)$ and shift the numbering. Namely, after making reflection we rename u_i to $u_{3-i \pmod{2n+1}}$ and w_i to $w_{3-i \pmod{2n+1}}$ for all i. Then the condition $\theta(1) = 2$ and $\theta(2n+1) = 1$ become $\theta(2) = 3$ and $\theta(1) = 2$ (see Figure 9).

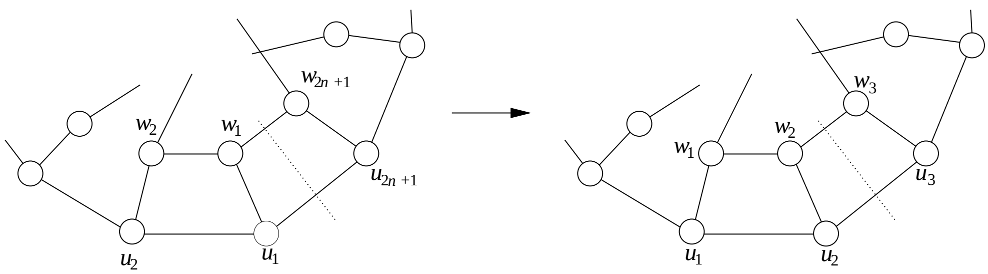

Figure 9: Rename the vertices of $L(2n+1;\theta)$.

So it is equivalent to consider the permutation ladder graph $L(2n+1;\theta)$, where θ has no fixed point and $\theta(1)=2$ and $\theta(2)=3$.

For $n\ge 2$, let f be the super-edge-graceful labeling of $P(2n+1;\theta)$ which is used to prove Theorem 2.4. Similar to $L(2n;\theta)$ we delete the edges u_2u_3 and w_2w_3 from $P(2n+1,\theta)$ and add two additional edges $u_2'w_2'$ and $u_3'w_3'$, where $u_2'=u_2$, $u_3'=u_3$, $w_2'=w_2$ and $w_3'=w_3$. Then we obtain $L(2n+1;\theta)$.

We define a labeling g on $L(2n+1;\theta)$ same as the labeling f on $P(2n+1;\theta)$ except the deleted edges and the added edges. Define $g(u_2'w_2')=2n+3$ and $g(u_3'w_3')=2n$ (see Figure 10). Then $g^+(w_2)=-(2n+1)+(-2n)+(2n+3)=-2n+2=f^+(u_3)$, $g^+(u_3)=-(2n+4)+(-2n+3)+2n=-2n+1=f^+(w_2)$ and $g^+(x)=f^+(x)$ for $x\ne u_3$ and $x\ne w_2$. Hence g is a super-edge-graceful labeling of $L(2n+1;\theta)$.

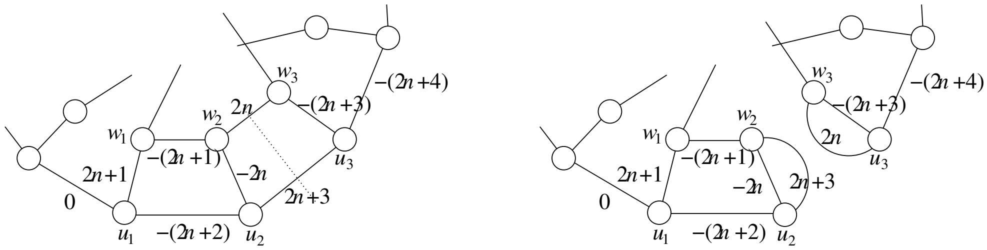

Figure 10: A partial super-edge-graceful labeling of $L(2n+1;\theta)$ with $\theta(1)=2$ and $\theta(2)=3$.

So we have the following theorem.

Theorem 3.2: *For $n\ge 1$, suppose $\theta\in S_{2n+1}$ without fixed point and suppose $\theta(1)=2$ and $\theta(2)=3$. Then $L(2n+1;\theta)$ is super-edge-graceful.*

Proof: For $n\ge 2$, we have just proved. For $n=1$, then $\theta=(123)$ and $L(3;\theta)=L(3)$. Following is a super-edge-graceful labeling of $L(3)$.

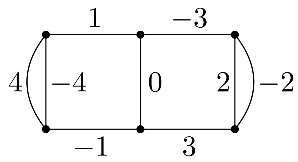

Figure 11: A super-edge-graceful labeling of $L(3)$.

□

Combine Theorems 3.1 and 3.2 we have

Theorem 3.3: *For $n\ge 2$, suppose $\theta\in S_n$ without fixed point and suppose $\theta(1)=2$ and $\theta(n)=1$. Then $L(n;\theta)$ is super-edge-graceful.*

We leave two unsolved cases here:

1. Could $L(n;\theta)$ be super-edge-graceful for each $n\ge 4$ when the condition $\theta(1)=2$ is removed?

2. Could $\sigma L(n)$ be super-edge-graceful for each $n\ge 3$ when the condition $\sigma(1)=1$ and $\sigma(n)=n$ is removed?

References

[1] S-M. Lee, A conjecture on edge-graceful trees, *Scientia*, **3** (1989), 45-57.

[2] S-M. Lee, New direction in the theory of edge-graceful graphs, *Proc. 6th Caribbean Conf. Comb. and Computing* (1991), 216-231.

[3] S-M. Lee, W.M. Pigg and T.J. Cox, On Edge-magic Cubic Graphs Conjecture, *Congressus Numerantium*, **105** (1994), 214-222.

[4] S.P. Lo, On edge-graceful labelings of graphs, *Congressus Numerantium*, **50** (1985), 231-241.

[5] J. Mitchem and A. Simoson, On Edge-Graceful and Super-Edge-Graceful Graphs, *Ars Combin.*, **37** (1994), 97-111.

[6] W.C. Shiu, S-M. Lee and K. Schaffer, Some k-fold Edge-graceful Labelings of $(p, p-1)$-graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **38** (2001), 81-95.

[7] W.C. Shiu and S-M. Lee, Some Edge-magic Cubic Graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **40** (2002), 115-127.

[8] W.C. Shiu, P.C.B. Lam and H.L. Cheng, Edge-gracefulness of the composition of paths with null graphs, *Discrete Math.*, **253** (2002), 63-76.

[9] W.C. Shiu and P.C.B. Lam, Super-edge-graceful labelings of multi-level wheel graphs, fan graphs and actinia graphs, to appear in *Congressus Numerantium*, (2005).

[10] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, 1996.