Edge-face Total Chromatic Number of 3-Regular Halin Graphs*

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Abstract

A Halin graph is a plane graph $H = T \cup C$, where T is a plane tree with no vertex of degree two and at least one vertex of degree three or more, and C is a cycle connecting the end vertices of T in the cyclic order determined by a plane embedment of T. In this paper, we show that if G is a 3-regular Halin graph, then $4 \le \chi_{ef}(G) \le 5$; and these bounds are sharp.

Key Words: Edge-face total chromatic number, 3-regular Halin graph.

1991 AMS subject Classifications: 05C15.

1 Introduction

In this paper, G denotes a simple, finite plane graph with vertex-, edge- and face-set V, E and F respectively. Vertices and edges on the boundary of a face f are said to be incident with it. For undefined terms of graphs, we refer to [1].

A *Halin* graph G is a plane graph consisting of a plane embedment of a tree T whose order is at least 4 and each interior vertex of which is of degree 3 or more, and of a cycle C^* obtained from T by connecting all end vertices of T. The tree T is called the *characteristic* tree of G, and C^* is called the *adjoint* cycle of G. Vertices and edges on the cycle C^* are called the *outer vertices* and *outer edges* respectively. Other vertices and edges are called *inner vertices* and *inner edges* respectively. A path consisting of inner edges is called an *inner path*. The face incident with all outer vertices and outer edges is called the *outer* face and is denoted by f_0. All other faces are called *inner* faces. Faces of degree 3 are sometimes called *triangles*. Note that an inner face is bounded by one outer edge and an inner path. Two end vertices of the characteristic tree of a Halin graph are called *neighboring vertices* if they are linked by an edge of the adjoint cycle C^*. Two inner faces are *neighbors* of each other, or *neighboring faces*, if they are incident with a common outer vertex.

Definition 1.1 A *proper k-edge-face total* coloring, or simply a *k-EFT coloring*, of a loopless plane graph G is an assignment of k colors $c_1, c_2, \cdots, c_k$ to all edges and faces in $E \cup F$ such that no two adjacent or incident elements have the same color. A graph G is *k-edge-face total colorable*, or simply *k-EFT colorable*, if there exists a *k-EFT coloring* on G. Moreover,

$$\chi_{ef}(G) = \min\{k \mid G \textit{ is } k\textit{-EFT colorable}\}$$

is called the *edge-face total chromatic* number, or simply the *EFT chromatic number*, of G.

*Partially Supported by Faculty Research Grant, Hong Kong Baptist University; and Research Grant Council Grant, Hong Kong

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In 1987, Borodin [2] first showed that for any planar graph G, we have $\chi_{ef}(G) \leq \Delta(G) + x$, where x = 1 or 3 if $\Delta(G) \geq 17$ or 11 respectively. Hu and Zhang [4] conjectured in 1992 that $\chi_{ef}(G) \leq \Delta(G)+3$ for any planar graph G. Lin, Hu and Zhang [5] also established the conjecture for $\Delta = 3$. In 1995, Chang *et al* [3] and also Wang [6] proved that if G is an outerplanar graphs without cut-vertex and $\Delta(G) \geq 6$, then $\chi_{ef}(G) = \Delta(G)$. In this paper, we shall show that if G is a 3-regular Halin graph, then $4 \leq \chi_{ef}(G) \leq 5$; and these bounds are sharp.

2 Main Results

The following lemma can be obtained from Definition 1.1.

Lemma 2.1 *If G is a plane graph consisting of odd faces, then $\chi_{ef}(G) \geq 4$.*

Proof Let f be an odd face in a plane graph. The boundary of f is an odd cycle and the edges on an odd cycle should receive at least three different colors. Together with the color assigned to f itself, the EFT chromatic number of the graph containing f must be at least 4. ▮

Theorem 2.2 *Suppose G is a 3-regular Halin graph. Then $4 \leq \chi_{ef}(G) \leq 5$.*

Proof We shall first show that any 3-regular Halin graph G consists of at least one triangle. Suppose $D = u_0u_1u_2\cdots u_n$ be a diameter of T, the characteristic tree of G. Clearly $d_G(u_1) = 3$ and there exists $u' \in N_G(u_1) \setminus \{u_0, u_2\}$. Because D is a diameter, u' must be an end vertex of T and u_0u_1u' is a triangle of G. By Lemma 2.1, we have $\chi_{ef}(G) \geq 4$.

We shall now construct a 5-EFT coloring of G by the following steps.

1. Choose any inner vertex of G and assign colors c_1, c_2 and c_3 to the three edges incident with that vertex in a clockwise direction. An inner vertex whose three incident edges have been assigned colors is marked as *labeled*. An inner vertex which has not been marked as labeled are called *unlabeled*.

2. If there are unlabeled vertices remaining, then choose an unlabeled vertex v adjacent to a labeled vertex u. Without loss of generality, we may assume that c_1 has been assigned to the edge uv, and that colors have been assigned to the three edges incident to u in a clockwise direction. We then assign the remaining two colors, c_2 and c_3 to edges incident to v in an anti-clockwise direction and mark v as a labeled vertex. This process will continue until all inner vertices have been marked as labeled.

   So far all inner edges were assigned one of the colors c_1, c_2 and c_3. Moreover, colors assigned to adjacent edges are all distinct. Note that any pair of neighboring vertices of G are connected by an inner path, which is also a path in T. We shall show that this inner path is colored alternately by two colors. We call such a path an *alternate path*. Let u and v be two neighboring vertices, and let $P = v_0v_1\dots v_n$ be the path in T connecting u and v, where $n \geq 2$, $v_0 = u$ and $v_n = v$. If $n = 2$, P must be an alternate path. Suppose $n \geq 3$ and $1 \leq i \leq n-2$. Because c_1, c_2 and c_3 are assigned to edges incident with v_i and to edges incident with v_{i+1} in opposite directions, colors assigned to $v_{i-1}v_i$ must be the same as that assigned to $v_{i+1}v_{i+2}$. Therefore P is an alternate path.

3. Designate a certain triangle f^* of G as the *first face*. Renaming the colors if necessary, we assume the inner path of f^* to be colored with c_1 and c_2. Color the outer face, the outer edge uv of f^* and the triangle f^* with c_1, c_3 and c_4 respectively. Let f' be that neighboring face of f^* with inner path colored in c_2 and c_3. We call f' the *second face*. Starting from the outer edge of f', assign the colors c_4 and c_5 alternately to edges of $C^* - uv$ until we have assigned color to the outer edge of f'', the other neighboring face of f^*. We call f'' the *last face*.

4. For each inner face f whose inner path is colored by c_2 and c_3, we choose a color from $\{c_4, c_5\} \setminus \{x\}$, where x is the color assigned to the outer edge of f. For each remaining inner face whose inner path is colored by c_1 and c_i, where $i = 2$ or 3, we color it with a color from $\{c_2, c_3\} \setminus \{c_i\}$.

It is clear that (i) at each vertex, all incident edges have distinct colors, (ii) the color of the outer face is distinct those of all inner faces and edges, and (iii) the color of each inner face is distinct from those of its incident edges.

From (3), both the first face f^* and the outer edge of the second face f' are colored with c_4. From (4), the second face is colored with c_5. Since the inner path of the last face f'' is colored with c_1 and c_3, the last face will be colored with c_2. Therefore faces adjacent to f^* are colored differently from that of f^*. Let f be an inner face other than f^*. If the inner path of f is colored by c_2 and c_3, then f is colored with either c_4 or c_5. Moreover, the inner path of each inner face adjacent to f is colored by c_1 and one of the two colors c_2 and c_3. Therefore an inner face adjacent to f will be colored with either c_3 or c_2. If the inner path of f is colored by c_1 and c_2, then f is colored with c_3. Moreover, the inner path of each inner face adjacent to f is colored by c_3 and one of the two colors c_1 and c_2. Therefore an inner face adjacent to f will not be colored with c_3. Similarly, If the inner path of f is colored by c_1 and c_3, then f is colored with c_2 and an inner face adjacent to f will not be colored with c_2.

It follows that $\chi_{ef}(G) \leq 5$. ∎

From the above theorem, we know that there are two classes of 3-regular Halin graphs with EFT chromatic number equal to 4 and 5. In the rest of this section, give two examples to show that the bounds given in Theorem 2.2 are sharp.

Example 2.3 *Let G be a 3-regular Halin graph with exactly one inner vertex - Figure 1(a). Then $\chi_{ef}(G) = 5$.*

Proof Because of Theorem 2.2, it is sufficient to show that G is not 4-EFT colorable. Suppose that G is EFT colorable by the color set $\{c_1, c_2, c_3, c_4\}$ and the outer face is colored by c_1. Since the edges and face of one triangle needs all four colors, one of the inner edge of that triangle must be colored by c_1. We can see that the face tuv must be colored by c_3, and consequently the edge tv must be colored by c_4. Then the edge uv cannot be colored properly. Therefore $\chi_{ef}(G) = 5$.

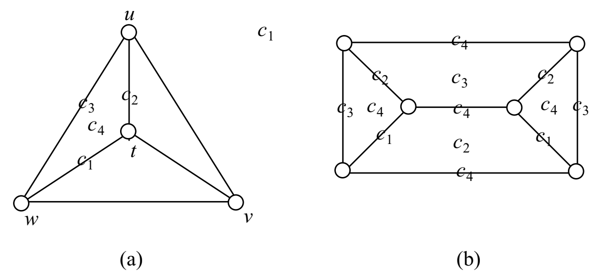

Figure 1

Example 2.4 *Let G be a 3-regular Halin graph with exactly two inner vertices - Figure 1(b). Then $\chi_{ef}(G) = 4$.*

Proof A 4-EFT coloring of G is indicated in Figure $1(b)$.

3 Discussion

Theorem 2.2 in fact provides a linear time algorithm to obtain a proper 5-EFT coloring of any 3-regular Halin graphs. It would be interesting to find out what 3-regular Halin graphs is 4-EFT colorable. So we conclude this paper by posting the following problem.

Problem: *Characterize 3-regular Halin graphs with EFT chromatic number equal to 4.*

References

[1] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, New York: Macmillan Ltd. Press, (1976).

[2] O. V. Borodin, Simultaneous colorings of graphs on the plane (Russian), *Metody Diskretn. Anal.*, **45** (1987), 21-27.

[3] C. F. Chang, J. X. Chang, P. C. B. Lam, and J. F. Wang, Edge-face total chromatic number of outerplanar graphs with $\Delta(G) = 6$, *Lecture Notes in Computer Science*, **959** (1995), 396-399, Springer.

[4] G. Z. Hu and Z. F. Zhang, The edge-face total colouring of planar graphs, *Journal of Tsinghua University*, **32**(**3**) (1992), 18-23.

[5] C. Lin, G. Z. Hu, and Z. F. Zhang, A six-colour theorem for the edge-face total colouring of planar graphs, *Manuscript* (1992).

[6] W. Wang, On the colorings of outerplanar graphs, *Discrete Math.*, **147** (1995), 257-269.