The number of spanning trees of composite graphs*

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Abstract

In this paper, some formulae for computing the numbers of spanning trees of the corona and the join of graphs are deduced.

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1 Introduction

Let G be a simple connected graph with edge set $E(G)$ and vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The number of spanning trees of G, denoted by $t(G)$, is the total number of distinct spanning subgraphs of G that are trees. Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$, and the Laplacian characteristic polynomial $\Phi(G, x)$ of G is defined as $\Phi(G, x) = \det(xI - L(G))$. It is easy to see that $L(G)$

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is a symmetric positive semidefinite matrix having 0 as an eigenvalue. The Laplacian spectrum of G is

$$S(G) = (\mu_1(G), \mu_2(G), \ldots, \mu_n(G)),$$

where $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$, are the eigenvalues of $L(G)$ (or the Laplacian eigenvalues of G) arranged in non-increasing order. When one graph is under discussion, we may write μ_i instead of $\mu_i(G)$. For a connected graph G of order n, it has been proven [1, p.284] that:

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i. \tag{1.1}$$

This formula can be used to obtain some sharp upper bounds for $t(G)$ in terms of graph structural parameters such as the number of vertices, the number of edges, maximum vertex degree, minimum vertex degree, connectivity, chromatic number and matching number in [2]. In this paper, we mainly use this formula to compute the number of spanning trees of the corona and the join of graphs, respectively.

2 Preliminaries

Let G and H be two graphs. The corona $G \circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the ith copy of H to the ith vertex of G, $i = 1, 2, \ldots, |V(G)|$. The vertex-disjoint union of the graphs G and H is denoted by $G \cup H$. The join $G \vee H$ is obtained from $G \cup H$ by adding all possible edges from vertices of G to vertices of H, i.e., $G \vee H = \overline{\overline{G} \cup \overline{H}}$, where $\overline{G}$ is the complement of a graph G.

Lemma 2.1 ([4]) *Let G and H be two graphs of order r and s, respectively. If $S(G) = (\mu_1(G), \mu_2(G), \ldots, \mu_r(G))$ and $S(H) = (\mu_1(H), \mu_2(H), \ldots, \mu_s(H))$, then the Laplacian eigenvalues of $G \circ H$ are:*

(a) $\frac{\mu_i(G)+s+1 \pm \sqrt{(s+1)^2 - 4\mu_i(G)}}{2}$ *with multiplicity* 1 *for* $i = 1, 2, \ldots, r$, *and*

(b) $\mu_j(H) + 1$ *with multiplicity* r *for* $j = 1, 2, \ldots, s-1$.

Lemma 2.2 ([5]) *Let G and H be two connected graphs of order r and s, respectively. If*
$S(H) = (\mu_1(H), \mu_2(H), \ldots, \mu_s(H))$, then the Laplacian polynomial of $G \circ H$ can be expressed as follows

$$\Phi(G \circ H, x) = \left(\prod_{i=1}^{s-1}(x - 1 - \mu_i(H))^r\right)\begin{vmatrix} -L(G) & -(x-s-1)I_r \\ xI_r & (x-1)I_r \end{vmatrix}.$$

Lemma 2.3 ([3]) *Let G and H be two graphs of order r and s, respectively. If $S(G) = (\mu_1(G), \mu_2(G), \ldots, \mu_r(G))$ and $S(H) = (\mu_1(H), \mu_2(H), \ldots, \mu_s(H))$, then the Laplacian spectrum of $G \vee H$ is $S(G \vee H) = (r + s,\ \mu_1(G) + s,\ \mu_2(G) + s,\ \ldots,\ \mu_{r-1}(G) + s,\ \mu_1(H) + r,\ \mu_2(H) + r,\ \ldots,\ \mu_{s-1}(H) + r,\ 0)$.*

3 Main results

Let G and H be two graphs of orders r and s, respectively. In this section, the number of spanning trees of $G \circ H$ and $G \vee H$ are computed, respectively.

3.1 $G \circ H$

If G is connected, then $G \circ H$ is also connected. $\frac{\mu_r(G)+s+1-\sqrt{(s+1)^2-4\mu_r(G)}}{2} = 0$ since $\mu_r(G) = 0$. Combining with Lemma 2.1, we have

Theorem 3.1 *Let G be a connected graph of order r, and H be a graph of order s. If*
$S(G) = (\mu_1(G), \ldots, \mu_{r-1}(G), 0)$ and $S(H) = (\mu_1(H), \ldots, \mu_{s-1}(H), 0)$, then

$$t(G \circ H) = \frac{\prod\limits_{i=1}^{r-1}\left[\mu_i^2(G) + 2(s+3)\mu_i(G)\right]\prod\limits_{i=1}^{s-1}(\mu_i(H)+1)^r}{r\,4^{r-1}}. \tag{3.2}$$

Proof. By Lemma 2.1 and Eq.(1.1), we have

$$\begin{aligned} t(G \circ H) &= \frac{(s+1) \prod_{i=1}^{r-1} \frac{(\mu_i(G)+s+1)^2-(s+1)^2+4\mu_i(G)}{4} \prod_{i=1}^{s-1} (\mu_i(H)+1)^r}{(s+1)r} \\ &= \frac{\prod_{i=1}^{r-1} \left[\mu_i^2(G) + 2(s+3)\mu_i(G)\right] \prod_{i=1}^{s-1} (\mu_i(H)+1)^r}{r\, 4^{r-1}}. \end{aligned}$$

This completes the proof of (3.2). □

The expression (3.2) is somewhat complicated. In what follows, we will give a simpler expression for $t(G \circ H)$ in terms of $t(G)$ and $t(H \vee K_1)$.

Theorem 3.2 *Let G be a connected graph of order r, and H be a graph of order s with*
$S(H) = (\mu_1(H), \ldots, \mu_{s-1}(H), 0)$. Then

$$t(G \circ H) = t(G) \prod_{i=1}^{s-1} (\mu_i(H)+1)^r. \tag{3.3}$$

Proof. Let

$$f(x) := \begin{vmatrix} -L(G) & -(x-s-1)I_r \\ xI_r & (x-1)I_r \end{vmatrix} = \sum_{i=0}^{2r} a_i x^i.$$

Then, by Lemma 2.2, we have

$$\Phi(G \circ H, x) = f(x) \left(\prod_{i=1}^{s-1} (x-1-\mu_i(H))^r \right).$$

Since $1 + \mu_i(H) > 0$ for $1 \le i \le s-1$, $f(x)$ can be written as

$$f(x) = x(x-b_1)(x-b_2)\cdots(x-b_{2r-1}),$$

where $b_i > 0$ is a root of equation $f(x) = 0$, $1 \le i \le 2r-1$.

Hence, (1.1) implies that

$$t(G \circ H) = \frac{\prod_{i=1}^{2r-1} b_i \prod_{i=1}^{s-1} (\mu_i(H)+1)^r}{(s+1)r} = \frac{(-a_1) \prod_{i=1}^{s-1} (\mu_i(H)+1)^r}{(s+1)r}.$$

In what follows, we will prove that $a_1 = -r(s+1)t(G)$.

For a matrix C, let $C(i)$ and $C(i,j)$ denote the submatrices obtained from C by deleting the ith row and column, and by deleting the ith row and jth column, respectively. Let $C^*(i)$ and $C^\dagger(i)$ denote the submatrices obtained from C by deleting the ith row and ith column, respectively. Let I_r and 0_r be the identity matrix and the zero matrix of order r, respectively.

Since

$$\begin{aligned} f(x) &= \begin{vmatrix} -L(G) & -(x-s-1)I_r \\ xI_r & (x-1)I_r \end{vmatrix} \\ &= \left| \begin{bmatrix} -L(G) & -(x-s-1)I_r \\ xI_r & (x-1)I_r \end{bmatrix} \begin{bmatrix} I_r & 0_r \\ -I_r & I_r \end{bmatrix} \begin{bmatrix} I_r & I_r \\ 0_r & I_r \end{bmatrix} \right| \\ &= \begin{vmatrix} (x-s-1)I_r - L(G) & -L(G) \\ I_r & xI_r \end{vmatrix}. \end{aligned}$$

Let $M := \begin{bmatrix} -(s+1)I_r - L(G) & -L(G) \\ I_r & 0_r \end{bmatrix}$. Note that $f(0) = \det M$.

Since the $(r+i)$th row of $M(i)$ has all zero entries when $1 \le i \le r$, $\det M(i) = 0$ for $1 \le i \le r$.

$$\begin{aligned} a_1 = \sum_{i=1}^{2r} \det M(i) &= \sum_{i=r+1}^{2r} \det M(i) \\ &= \sum_{i=1}^{r} \begin{vmatrix} -(s+1)I_r - L(G) & -L^\dagger(G)(i) \\ I_r^*(i) & 0_{r-1} \end{vmatrix} \end{aligned}$$

Let A_i be the $r \times r$ matrix whose (i,i)−entry is $s+1$ and other entries are all zero. By consecutively interchanging the ith column with the $(i+1)$th, $(i+2)$th, $\cdots$ and $(r+i-1)$th columns in the last determinant. We have

$$\begin{aligned} a_1 &= (-1)^r(-1)^{r-1} \sum_{i=1}^{r} \begin{vmatrix} (s+1)I_r^\dagger(i) + L^\dagger(G)(i) & L(G) + A_i \\ I_{r-1} & 0_r^*(i) \end{vmatrix} \\ &= (-1)(-1)^{2r-2} \sum_{i=1}^{r} \begin{vmatrix} L(G) + A_i & (s+1)I_r^\dagger(i) + L^\dagger(G)(i) \\ 0_r^*(i) & I_{r-1} \end{vmatrix} \\ &= -\sum_{i=1}^{r} \det(L(G) + A_i) = -\sum_{i=1}^{r} (\det L(G) + (s+1)\det L(G)(i)) \\ &= -(s+1) \sum_{i=1}^{r} t(G) = -r(s+1)t(G). \end{aligned}$$

Therefore, $t(G \circ H) = \frac{(-a_1)\prod_{i=1}^{s-1}(\mu_i(H)+1)^r}{(s+1)r} = t(G)\prod_{i=1}^{s-1}(\mu_i(H)+1)^r$, which completes the proof. $\square$

Note that $S(H \vee K_1) = (s+1,\, \mu_1(H)+1,\, \ldots,\, \mu_{s-1}(H)+1,\, 0)$ and $t(H \vee K_1) = \prod_{i=1}^{s-1}(\mu_i(H)+1)$. Then (3.3) can be rewritten as follows.

Theorem 3.3 *Let G be a connected graph of order r. Then $t(G \circ H) = t(G)t^r(H \vee K_1)$.*

3.2 $G \vee H$

Theorem 3.4 *Let G and H be two graphs of order r and s, respectively. If*

$S(G) = (\mu_1(G), \ldots, \mu_{r-1}(G), 0)$ *and* $S(H) = (\mu_1(H), \ldots, \mu_{s-1}(H), 0)$,

then

$$t(G \vee H) = \prod_{i=1}^{r-1}(s + \mu_i(G)) \prod_{i=1}^{s-1}(r + \mu_i(H)).$$

Proof. By Lemma 2.3 and Eq.(1.1), the result follows. $\square$

Let K_n and $\overline{K_n}$ denote the complete graph and empty graph of order n, respectively, $K_{m,n}$ denote the complete bipartite graph such that one part has n vertices and the other has m vertices. It is interesting that $t(G \vee H)$ not only can be determined by the Laplacian spectra of G and H, but also can be expressed as the following form.

Theorem 3.5 *Let G and H be two graphs of order r and s, respectively. Then*

$$t(G \vee H) = \frac{t(G \vee \overline{K_s})t(H \vee \overline{K_r})}{t(K_{r,\,s})}.$$

Proof. By Lemma 2.3, the Laplacian spectra of $G \vee \overline{K_s}$ and $H \vee \overline{K_r}$ are

$$S(G \vee \overline{K_s}) = (r+s,\, \underbrace{r,\, \ldots,\, r}_{s-1},\, \mu_1(G)+s,\, \ldots,\, \mu_{r-1}(G)+s,\, 0),$$

$$S(H \vee \overline{K_r}) = (r+s,\, \underbrace{s,\, \ldots,\, s}_{r-1},\, \mu_1(H)+r,\, \ldots,\, \mu_{s-1}(H)+r,\, 0).$$

Note that the Laplacian spectrum of $K_{r,\,s}$ is

$$S(K_{r,s}) = (r+s, \underbrace{r, \ldots, r}_{s-1}, \underbrace{s, \ldots, s}_{r-1}, 0).$$

Therefore we have $t(G \vee \overline{K_s}) = r^{s-1} \prod_{i=1}^{r-1} (\mu_i(G) + s)$, $t(H \vee \overline{K_r}) = s^{r-1} \prod_{i=1}^{s-1} (\mu_i(H) + r)$ and $t(K_{r,s}) = r^{s-1}s^{r-1}$.

Hence, by Theorem 3.4, the result follows. $\square$

Theorem 3.6 *Let G and H be two connected graphs of order r and s, respectively. Then*

$$\frac{t(G \vee H)}{t(G)t(H)} \geq (r+s)^2 (1 + \frac{r}{s})^{s-2} (1 + \frac{s}{r})^{r-2},$$

and equality holds if and only if $G \vee H$ is complete.

Proof. Let $S(G) = (\mu_1(G), \ldots, \mu_{r-1}(G), 0)$ and $S(H) = (\mu_1(H), \ldots, \mu_{s-1}(H), 0)$. Then by Theorem 3.4, we have

$$\frac{t(G \vee H)}{t(G)t(H)} = rs \prod_{i=1}^{r-1} (1 + \frac{s}{\mu_i(G)}) \prod_{i=1}^{s-1} (1 + \frac{r}{\mu_i(H)}).$$

And

$$(r+s)^2 (1 + \frac{s}{r})^{r-2} (1 + \frac{r}{s})^{s-2} = rs(1 + \frac{s}{r})^{r-1} (1 + \frac{r}{s})^{s-1}.$$

Hence,

$$\frac{\frac{t(G \vee H)}{t(G)t(H)}}{(r+s)^2 (1 + \frac{s}{r})^{r-2} (1 + \frac{r}{s})^{s-2}} = \prod_{i=1}^{r-1} \frac{1 + \frac{s}{\mu_i(G)}}{1 + \frac{s}{r}} \prod_{i=1}^{s-1} \frac{1 + \frac{r}{\mu_i(H)}}{1 + \frac{r}{s}}.$$

Note that $\mu_i(G) \leq r$ for all $1 \leq i \leq r-1$ and $\mu_i(H) \leq s$ for all $1 \leq i \leq s-1$. Hence the desired inequality holds, and the equality holds if and only if $\mu_i(G) = r$ for all $1 \leq i \leq r-1$ and $\mu_i(H) = s$ for all $1 \leq i \leq s-1$. Therefore G and H are complete graphs. Thus $G \vee H$ is also complete. $\square$

3.3 Examples

The Laplacian spectra of P_n, C_n and K_n [1] are

$$S(P_n) = \left(4\sin^2 \frac{(n-1)\pi}{2n}, 4\sin^2 \frac{(n-2)\pi}{2n}, \ldots, 4\sin^2 \frac{\pi}{2n}, 0\right),$$

$$S(C_n) = \left(4\sin^2 \frac{(n-1)\pi}{n}, 4\sin^2 \frac{(n-2)\pi}{n}, \ldots, 4\sin^2 \frac{\pi}{n}, 0\right) \quad \text{and}$$

$$S(K_n) = (n, n, \ldots, n, 0).$$

And it is well known that $t(P_n) = 1$, $t(C_n) = n$ and $t(K_n) = n^{n-2}$.

The fan graph $F_{r,s}$ and cone graph $C_{r,s}$ are defined as $P_r \vee \overline{K_s}$ and $C_r \vee \overline{K_s}$, respectively. Hence by Theorem 3.4, we have

$$t(F_{r,s}) = r^{s-1} \prod_{i=1}^{r-1} \left(s + 4\sin^2 \frac{i\pi}{2r} \right) \quad \text{and} \quad t(C_{r,s}) = r^{s-1} \prod_{i=1}^{r-1} \left(s + 4\sin^2 \frac{i\pi}{r} \right).$$

In particular, for the fan F_{r+1} and wheel graph W_{r+1} which are defined as $F_{r,1}$ and $C_{r,1}$, we have

$$t(F_{r+1}) = \prod_{i=1}^{r-1} \left(1 + 4\sin^2 \frac{i\pi}{2r} \right) \quad \text{and} \quad t(C_{r+1}) = \prod_{i=1}^{r-1} \left(1 + 4\sin^2 \frac{i\pi}{r} \right).$$

The r-corona graph of a graph G, denoted by $I_r(G)$, is defined as $G \circ \overline{K_r}$. Since $\overline{K_r} \vee K_1 = S_{r+1}$, and $t(S_{r+1}) = 1$, by Theorem 3.3, we have

$$t(I_r(G)) = t(G).$$

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