Supermagicness of the composition of a cycle with a null graph*

Wai Chee Shiu, Peter Che Bor Lam
Department of Mathematics,
Hong Kong Baptist University
Kowloon, Hong Kong, China.

Sin-Min Lee
Department of Mathematics and Computer Science,
San Jose State University,
San Jose, CA 95192, U.S.A.

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Running Title

Supermagic Composition Graphs

All correspondence should be sent to:

Wai Chee SHIU
Department of Mathematics,
Hong Kong Baptist University,
224 Waterloo Road, Kowloon,
Hong Kong.

e-mail address: wcshiu@math.hkbu.edu.hk

Abstract

Given two graphs G and H. The composition of G with H is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. In this paper, we prove that the composition of a cycle with a null graph is supermagic.

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Key words and phrases : Supermagic, edge-magic, composition of graphs, orthogonal Latin square, magic square.

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1. Introduction

Let $G = (V, E)$ be a (p, q)-graph, i.e., $|V| = p$ and $|E| = q$. If there exists a bijection

$$f : E \to \{k,\ k+1,\ \cdots,\ q-1+k\}$$

for some $k \in \mathbb{Z}$ such that the map $f^+(u) = \sum_{v \in N(u)} f(uv)$ induces a constant map from V to $\mathbb{Z}_p$, then G is called *k-edge-magic* and f is called a *k-edge-magic labeling* of G. If $k = 1$, then G is simply called *edge-magic* and f an *edge-magic labeling* of G. This concept was initiated by Lee, Seah and Tan [1]. Stewart ([2], [3]) called G *supermagic* if $k = 1$ and f^+ is a constant map from V to $\mathbb{Z}$. We shall call such f a *supermagic labeling* of G. Clearly, a supermagic graph is edge-magic. However, there exists lots of edge-magic graphs which are not supermagic. Hartsfield and Ringel had also studied supermagic graphs [4].

Given two graphs G and H. The composition of G with H, denoted as $G \circ H$, is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. For example, $C_3 \circ K_2$ is shown in Figure 1.

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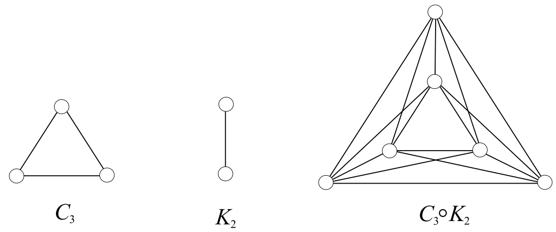

Figure 1

Recently, the authors [5] proved that for $m \geq 2$, $n \geq 2$ but $(m,n) \neq (2,2)$, $C_m \circ N_n$, where C_m is the m-cycle and N_n is the null graph on n vertices, is edge-magic. In this paper, we shall prove that $C_m \circ N_n$ is supermagic for those m and n.

2. Supermagicness of Regular Graphs

If $G = (V, E)$ is an r-regular (p, q)-graph, then $2q = pr$. Suppose $f : E \to \{1,\ 2,\ \cdots,\ q\}$ is a bijection. For any integer k, we can define a bijection $g : E \to \{k,\ k+1,\ \cdots,\ k+q-1\}$ by $g(e) = f(e) + k - 1$ for any $e \in E$. Then $g^+(u) = f^+(u) + r(k-1)$. Therefore f^+ is a constant mapping if and only if g^+ is a constant mapping.

In this paper we shall only consider simple regular graphs, and we shall label the edges of graphs by numbers $0,\ 1,\ \cdots,\ q-1$.

Definition: Let $G = (V, E)$ be a graph and S be a set. Suppose $f : E \to S$ is a mapping. A *labeling matrix* for a labeling f of G is a matrix whose rows and columns are named by the vertices of G and the (u, v)-entry is $f(uv)$ if $uv \in E$, and is $*$ otherwise. The label $f(uv)$ is sometimes written as $f(u, v)$.

A regular (p, q)-graph $G = (V, E)$ is supermagic if and only if there exists a bijection $f : E \to \{0,\ 1,\ \cdots,\ q-1\}$ such that the row sums and the column sums of the labeling matrix of G associated with f are all equal. For purposes of these sums, entries labeled with $*$ will be treated as 0.

3. Labeling Matrix of $C_m \circ N_n$

From now on, we identify $C_m \circ N_n$ as a Cayley graph which we shall describe. Let $\mathfrak{C}_m = \langle g \rangle$ be the (multiplicative) cyclic group of order m (≥ 2) generated by g. Let $H = \{h_0 = e,\ h_1,\ \cdots,\ h_{n-1}\}$ be any group of order n, where $n \geq 2$ and e is the identity of H. Throughout this paper we shall use e to denote the identity of a group. Let $\mathfrak{C}_m\{H\}$ denote the Cayley graph of $\mathfrak{C}_m \times H$ generated by $\{g,\ g^{-1}\} \times H$.

For $m \geq 3$, $\mathfrak{C}_m\{H\}$ is an (mn, mn^2)-graph; and $\mathfrak{C}_2\{H\}$ is an $(2n, n^2)$-graph. Moreover, $\mathfrak{C}_m\{H\}$ is isomorphic to $C_m \circ N_n$ for $m \geq 2$. Note that we may view $\mathfrak{C}_m\{H\}$ as a (simple) graph. For simplicity, we identify $(g^i, x) \in \mathfrak{C}_m \times H$ with $g^i x$ and choose $H = \mathfrak{C}_n = \langle h \rangle$.

When $m = 2$ and $n \geq 2$, $\mathfrak{C}_2\{\mathfrak{C}_n\} \cong K_{n,n}$. We can verify that $K_{2,2}$ is not supermagic. Since magic square of any order higher than 2 always exists (see [6] or [7]), $K_{n,n}$ is supermagic for $n \geq 3$. So we may assume that $m \geq 3$ and $n \geq 2$.

We list the elements of $\mathfrak{C}_n$ in the following order $\{e = h^0,\ h^1,\ h^2,\ \cdots,\ h^{n-1}\}$, and list the elements of $\mathfrak{C}_m \times \mathfrak{C}_n$ in the following order: $\{e\mathfrak{C}_n = g^0\mathfrak{C}_n,\ g^1\mathfrak{C}_n,\ \cdots,\ g^{m-1}\mathfrak{C}_n\}$. If $f : E \to S$ is a mapping, then the labeling matrix for f is

$$\begin{pmatrix} * & A_0 & * & \iddots & * & A_{m-1}^T \\ A_0^T & * & A_1 & \iddots & \ddots & \ddots \\ * & A_1^T & * & \iddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \iddots & \ddots & \ddots \\ * & * & * & \iddots & * & A_{m-2} \\ A_{m-1} & * & * & \iddots & A_{m-2}^T & * \end{pmatrix}. \tag{3.1}$$

This matrix is visualized as an $m \times m$ matrix $X = (x_{ij})$, each entry of which is occupied by an $n \times n$ matrix. For $i = 1,\ 2,\ \cdots, m$, the entry $x_{i,i+1}$ is the $n \times n$ matrix A_{i-1} and the entry $x_{i+1,i}$ is A_{i-1}^T, where $i+1$ is taken to be 1 if $i = m$ and $A_i = \left(a_{i'j'}^{(i)}\right)$. For $1 \leq i',\ j' \leq n$, the entry $a_{i'j'}^{(i)}$ of A_i corresponds to the $(g^i h^{i'-1}, g^i h^{j'-1})$-entry of the labeling matrix, and has the label $f(g^i h^{i'-1}, g^i h^{j'-1})$. Each remaining entry of X is occupied by an $n \times n$ matrix of $*$'s, meaning that no edge connects the corresponding vertices.

Thus $f : E \to \{0,\ 1, \cdots,\ mn^2 - 1\}$ is a supermagic labeling of $\mathfrak{C}_m\{\mathfrak{C}_n\}$ if and only if row sums and column sums of the above matrix are constant, and the problem is reduced to determining whether we can assign $\{0,\ 1, \cdots,\ mn^2 - 1\}$ into entries of m matrices A_i such that row sums and column sums of the above matrix are constant.

Let S be a set of mn integers, where $m,\ n \geq 2$. If there is a partition of S containing m classes such that each class has n elements and whose sum in each class is the same, then we call S has an (m, n)-*balance partition.*

Lemma 3.1 [5]: *If n is even, or if both n and m are odd, then $\{0,\ 1, \cdots,\ mn-1\}$ has an (m, n)-balance partition.*

Suppose there is a partition of S into m classes with n elements in each class. If the sums of elements in $\frac{m}{2}$ of the classes are all equal to one value, and the sums of elements in the remaining classes are all equal to another value, then we call S has an (m, n)-*semi-balance partition.*

Lemma 3.2 [5]: *If n is odd and m is even, then $\{0,\ 1,\ \cdots,\ mn-1\}$ has an (m, n)-semi-balance partition.*

4. Supermagic Labeling of $C_m \circ N_n$

In this section, we shall prove that $C_m \circ N_n$, where C_2 will be replaced by P_2 if necessary, is supermagic. To do that, we have to make use of Latin square.

A *Latin square* is a square matrix in which each row and each column consists of the same set of entries without repetition [8]. Two Latin squares $A = (a_{i,j})$ and $B = (b_{i,j})$ of order n are *orthogonal* if the n^2 pairs $(a_{i,j}, b_{i,j})$ are all distinct. It is easy to see that there is no pair of orthogonal Latin squares of order 2. In 1900, G. Tarry examined all cases and proved that there is no pair of orthogonal Latin squares of order 6. In 1960, R.C. Bose, S.S. Shrikhande and E.T. Parker proved the following theorem in [9].

Theorem 4.1: *There exist pairs of orthogonal Latin squares of order n if $n \geq 3$ and $n \neq 6$.*

There is a proof written in the book by van Lint and Wilson ([10], 251-260). The nonexistence proof for the case $n = 6$ is long. In 1984, D.R. Stinson gave a short proof ([11]). Because of Theorem 4.1, we have the following theorem.

Theorem 4.2: *$\mathfrak{C}_2\{\mathfrak{C}_n\}$ is supermagic if $n \geq 3$, and $\mathfrak{C}_m\{\mathfrak{C}_n\}$ is supermagic if $m \geq 3$, $n \geq 3$ and $n \neq 6$.*

Proof: It was shown in section 3 that $\mathfrak{C}_2\{\mathfrak{C}_n\} \cong K_{n,n}$ is supermagic if $n \geq 3$. So we only have to consider $m \geq 3$, $n \geq 3$ and $n \neq 6$. Let X and Y be a pair of orthogonal Latin squares of order n.

Case 1: Suppose n is odd and m is even. By Lemma 3.2, we have an (m, n)-semi-balance partition of $Q = \{0,\ 1,\ \cdots, mn-1\}$. Let $\{P_0,\ P_1,\ \cdots,\ P_{m-1}\}$ be this partition such that the sum of elements of P_i, where i is odd, is equal to one value and the sum of elements of P_i, where i is even, is equal to another value. Using the format of X we obtain a Latin square A_j with entries consisting of elements of P_j, $0 \leq j \leq m-1$, and substitute these $A_j's$ into (3.1) to obtain a labeling matrix of $\mathfrak{C}_m\{\mathfrak{C}_n\}$, denoted by Ω.

Case 2: Suppose n is even or both n and m are odd. By Lemma 3.1, we have an (m, n)-balance partition of Q. As in Case 1, we obtain a labeling matrix Ω of $\mathfrak{C}_m\{\mathfrak{C}_n\}$.

Note that the matrix Ω, obtained from each of the above cases, is a labeling matrix for an edge-labeling of $\mathfrak{C}_m\{\mathfrak{C}_n\}$ (see [5]).

In the same way, we may use the format of Y to obtain a Latin square B with entries $0,\ mn,\ 2mn,\ \cdots,\ (n-1)mn$. Substituting B for A_j of (3.1), $0 \le j \le m-1$, we have a matrix, say Ψ. Because of the orthogonality of A_j's (obtained from case 1 or case 2) and B, $\Omega + \Psi$ is a labeling matrix for a supermagic labeling of $\mathfrak{C}_m\{\mathfrak{C}_n\}$. ∎

Example 4.1: Consider $\mathfrak{C}_4\{\mathfrak{C}_3\}$. A $(4,3)$-semi-balance partition of $\{0,\ 1,\ \cdots,\ 11\}$ is $P_0 = \{0,7,11\}$, $P_1 = \{1,5,9\}$, $P_2 = \{2,6,10\}$, and $P_3 = \{3,4,8\}$. Choose

$$X = \begin{pmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix},$$

which are orthogonal Latin squares. Then

$$\Omega = \left(\begin{array}{ccc|ccc|ccc|ccc}
* & * & * & 0 & 7 & 11 & * & * & * & 3 & 8 & 4 \\
* & * & * & 11 & 0 & 7 & * & * & * & 4 & 3 & 8 \\
* & * & * & 7 & 11 & 0 & * & * & * & 8 & 4 & 3 \\
\hline
0 & 11 & 7 & * & * & * & 1 & 5 & 9 & * & * & * \\
7 & 0 & 11 & * & * & * & 9 & 1 & 5 & * & * & * \\
11 & 7 & 0 & * & * & * & 5 & 9 & 1 & * & * & * \\
\hline
* & * & * & 1 & 9 & 5 & * & * & * & 2 & 6 & 10 \\
* & * & * & 5 & 1 & 9 & * & * & * & 10 & 2 & 6 \\
* & * & * & 9 & 5 & 1 & * & * & * & 6 & 10 & 2 \\
\hline
3 & 4 & 8 & * & * & * & 2 & 10 & 6 & * & * & * \\
8 & 3 & 4 & * & * & * & 6 & 2 & 10 & * & * & * \\
4 & 8 & 3 & * & * & * & 10 & 6 & 2 & * & * & *
\end{array}\right),$$

and

$$B = \begin{pmatrix} 0 & 12 & 24 \\ 12 & 24 & 0 \\ 24 & 0 & 12 \end{pmatrix}.$$

We have

$$\Omega + \Psi = \left(\begin{array}{ccc|ccc|ccc|ccc}
* & * & * & 0 & 19 & 35 & * & * & * & 3 & 20 & 28 \\
* & * & * & 23 & 24 & 7 & * & * & * & 16 & 27 & 8 \\
* & * & * & 31 & 11 & 12 & * & * & * & 32 & 4 & 15 \\
\hline
0 & 23 & 31 & * & * & * & 1 & 17 & 33 & * & * & * \\
19 & 24 & 11 & * & * & * & 21 & 25 & 5 & * & * & * \\
35 & 7 & 12 & * & * & * & 29 & 9 & 13 & * & * & * \\
\hline
* & * & * & 1 & 21 & 29 & * & * & * & 2 & 18 & 34 \\
* & * & * & 17 & 25 & 9 & * & * & * & 22 & 26 & 6 \\
* & * & * & 33 & 5 & 13 & * & * & * & 30 & 10 & 14 \\
\hline
3 & 16 & 32 & * & * & * & 2 & 22 & 30 & * & * & * \\
20 & 27 & 4 & * & * & * & 18 & 26 & 10 & * & * & * \\
28 & 8 & 15 & * & * & * & 34 & 6 & 14 & * & * & *
\end{array}\right),$$

which is a labeling matrix for a supermagic labeling of $\mathfrak{C}_4\{\mathfrak{C}_3\}$. ∎

Theorem 4.3: *If $m \geq 4$ and is even, then $\mathfrak{C}_m\{\mathfrak{C}_n\}$ is supermagic.*

Proof: Let Ω be the labeling matrix for the edge-magic labeling of $\mathfrak{C}_m\{\mathfrak{C}_n\}$ constructed in the proof of Theorem 4.2. Let $\vec{\mathbf{1}}^T$ be the transpose of $\vec{\mathbf{1}} = (1, 1, \cdots, 1)$. Let A be the $n \times n$ matrix whose i-th column is $(i-1)mn\vec{\mathbf{1}}^T$ and B be the $n \times n$ matrix whose i-th row is $(n-i)mn\vec{\mathbf{1}}$. Then A and B are orthogonal to each numeral block matrix of Ω, which are Latin squares. Substituting A and B for A_j of (3.1) if j is even and odd, respectively, we have a matrix Ψ. Then each row of Ψ contains two copies of $\{0,\ mn,\ \cdots,\ (n-1)mn\}$ and $(m-2)n$ $*$'s or n copies of $\{imn,\ (n-i)mn\}$ and $(m-2)n$ $*$'s for some i, $0 \leq i \leq n-1$. Thus the row sums and the column sums are the same, namely it is equal to mn^2. Then $\Omega + \Psi$ is a required labeling matrix for a supermagic labeling of $\mathfrak{C}_m\{\mathfrak{C}_n\}$. ∎

Example 4.2: Consider $\mathfrak{C}_4\{\mathfrak{C}_3\}$ again. Let Ω be that of Example 4.1, and

$$\Psi = \left(\begin{array}{ccc|ccc|ccc|ccc}
* & * & * & 0 & 12 & 24 & * & * & * & 24 & 12 & 0 \\
* & * & * & 0 & 12 & 24 & * & * & * & 24 & 12 & 0 \\
* & * & * & 0 & 12 & 24 & * & * & * & 24 & 12 & 0 \\
\hline
0 & 0 & 0 & * & * & * & 24 & 24 & 24 & * & * & * \\
12 & 12 & 12 & * & * & * & 12 & 12 & 12 & * & * & * \\
24 & 24 & 24 & * & * & * & 0 & 0 & 0 & * & * & * \\
\hline
* & * & * & 24 & 12 & 0 & * & * & * & 0 & 12 & 24 \\
* & * & * & 24 & 12 & 0 & * & * & * & 0 & 12 & 24 \\
* & * & * & 24 & 12 & 0 & * & * & * & 0 & 12 & 24 \\
\hline
24 & 24 & 24 & * & * & * & 0 & 0 & 0 & * & * & * \\
12 & 12 & 12 & * & * & * & 12 & 12 & 12 & * & * & * \\
0 & 0 & 0 & * & * & * & 24 & 24 & 24 & * & * & *
\end{array}\right).$$

Then

$$\Omega+\Psi=\left(\begin{array}{ccc|ccc|ccc|ccc}
* & * & * & 0 & 19 & 35 & * & * & * & 27 & 20 & 4 \\
* & * & * & 11 & 12 & 31 & * & * & * & 28 & 15 & 8 \\
* & * & * & 7 & 23 & 24 & * & * & * & 32 & 16 & 3 \\
\hline
0 & 11 & 7 & * & * & * & 25 & 29 & 33 & * & * & * \\
19 & 12 & 23 & * & * & * & 21 & 13 & 17 & * & * & * \\
35 & 31 & 24 & * & * & * & 5 & 9 & 1 & * & * & * \\
\hline
* & * & * & 25 & 21 & 5 & * & * & * & 2 & 18 & 34 \\
* & * & * & 29 & 13 & 9 & * & * & * & 10 & 14 & 30 \\
* & * & * & 33 & 17 & 1 & * & * & * & 6 & 22 & 26 \\
\hline
27 & 28 & 32 & * & * & * & 2 & 10 & 6 & * & * & * \\
20 & 15 & 16 & * & * & * & 18 & 14 & 22 & * & * & * \\
4 & 8 & 3 & * & * & * & 34 & 30 & 26 & * & * & *
\end{array}\right),$$

which is a labeling matrix for a supermagic labeling of $\mathfrak{C}_4\{\mathfrak{C}_3\}$.

Now we shall prove the remaining cases, i.e., $\mathfrak{C}_m\{\mathfrak{C}_2\}$ and $\mathfrak{C}_m\{\mathfrak{C}_6\}$ are supermagic for m is odd and $m \geq 3$.

Theorem 4.4: *If G is an r-regular supermagic graph, then $G \circ N_n$ is an rn-regular supermagic graph for $n \geq 3$.*

Proof: By definition, $G \circ N_n$ is rn-regular. Let A be an adjacency matrix of G. Then an adjacency matrix of $G \circ N_n$ is obtained from A by replacing the $0'$s and $1'$s by the matrices O and J respectively, where O and J are $n \times n$ matrices whose entries are all 0 and all 1 respectively. So to find a labeling matrix of $G \circ N_n$, we would replace $0'$s by $*'$s and J by a suitable numeral matrix from the adjacency matrix of $G \circ N_n$.

Let $L = (l_{i,j})$ be a labeling matrix for a supermagic labeling of G and let k be the row sums of L. Let Φ be a matrix obtained from L by replacing $*$ by the $n \times n$ matrix whose entries are $*$, $l_{i,j}$ by $n^2 l_{i,j} + M_{i,j}$ and $l_{j,i}$ by $n^2 l_{i,j} + M_{i,j}^T$ if $l_{i,j} \neq *$ and $i < j$, where $M_{i,j}$ is a magic square of order n on $\{0,\ 1,\ \cdots,\ n^2-1\}$. Let m be the magic sum of the magic squares $M_{i,j}$. It is easy to see that Φ is a labeling matrix for a supermagic labeling of $G \circ N_n$ with row (column) sum $kn^3 + rm$. ▮

Corollary 4.5: *If $\mathfrak{C}_m\{\mathfrak{C}_k\}$ is supermagic, then so is $\mathfrak{C}_m\{\mathfrak{C}_{kn}\}$ for $n \geq 3$.*

Proof: The conclusion follows from $(C_m \circ N_k) \circ N_n \cong C_m \circ N_{kn}$. ▮

Theorem 4.6: *$\mathfrak{C}_m\{\mathfrak{C}_2\}$ is supermagic if $m \geq 3$ and is odd.*

Proof: The following is a labeling matrix for a supermagic labeling of $\mathfrak{C}_3\{\mathfrak{C}_2\}$:

$$\left(\begin{array}{cc|cc|cc}
* & * & 0 & 10 & 4 & 8 \\
* & * & 11 & 1 & 3 & 7 \\
\hline
0 & 11 & * & * & 6 & 5 \\
10 & 1 & * & * & 9 & 2 \\
\hline
4 & 3 & 6 & 9 & * & * \\
8 & 7 & 5 & 2 & * & *
\end{array}\right).$$

When $m \geq 5$, we let

$$A_0 = \begin{pmatrix} 0 & 4m-2 \\ 4m-1 & 1 \end{pmatrix}, \quad A_{m-1} = \begin{pmatrix} 2m-2 & 2m-3 \\ 2m+2 & 2m+1 \end{pmatrix}, \quad A_{m-2} = \begin{pmatrix} 2m & 2m-1 \\ 2m+3 & 2m-4 \end{pmatrix}$$

and for $1 \leq j \leq m-3$,

$$A_j = \begin{cases} \begin{pmatrix} 2j+1 & 4m-2j-2 \\ 4m-2j-1 & 2j \end{pmatrix} & \text{if } j \text{ is odd.} \\[2ex] \begin{pmatrix} 2j & 4m-2j-2 \\ 4m-2j-1 & 2j+1 \end{pmatrix} & \text{if } j \text{ is even.} \end{cases}$$

There is a one-to-one correspondence between entries of A_j, $0 \leq j \leq m-1$, and $\{0, 1, \cdots, 4m-1\}$. Substituting these matrices into (3.1), we obtain a labeling matrix L of $\mathfrak{C}_m\{\mathfrak{C}_2\}$. We shall show that the row sums of this labeling matrix are all the same.

The first two row sums of L are contributed by the matrices A_0 and A_{m-1}^T. These two row sums are both $8m-2$. Similarly, the last two row sums of L are contributed by the matrices A_{m-1} and A_{m-2}^T. These two row sums are also both $8m-2$. Sum of the $(2j+1)$-th and the $(2j+2)$-th rows, where $1 \leq j \leq m-2$, are contributed by the matrices A_j and A_{j-1}^T, which are also both $8m-2$. Therefore L is a labeling matrix for a supermagic labeling of $\mathfrak{C}_m\{\mathfrak{C}_2\}$. ∎

Corollary 4.7: *$\mathfrak{C}_m\{\mathfrak{C}_6\}$ is supermagic if $m \geq 3$ and is odd.*

Example 4.3: The following is a labeling matrix for a supermagic labeling of $\mathfrak{C}_5\{\mathfrak{C}_2\}$.

$$\left(\begin{array}{cc|cc|cc|cc|cc}
* & * & 0 & 18 & * & * & * & * & 8 & 12 \\
* & * & 19 & 1 & * & * & * & * & 7 & 11 \\
\hline
0 & 19 & * & * & 3 & 16 & * & * & * & * \\
18 & 1 & * & * & 17 & 2 & * & * & * & * \\
\hline
* & * & 3 & 17 & * & * & 4 & 14 & * & * \\
* & * & 16 & 2 & * & * & 15 & 5 & * & * \\
\hline
* & * & * & * & 4 & 15 & * & * & 10 & 9 \\
* & * & * & * & 14 & 5 & * & * & 13 & 6 \\
\hline
8 & 7 & * & * & * & * & 10 & 13 & * & * \\
12 & 11 & * & * & * & * & 9 & 6 & * & *
\end{array}\right).$$

Suppose we choose the magic square

$$M = \begin{pmatrix} 3 & 8 & 1 \\ 2 & 4 & 6 \\ 7 & 0 & 5 \end{pmatrix} = M_{i,j}, \qquad 1 \le i < j \le 10.$$

If we replace each numeral x above the diagonal, y below the diagonal and $*$ in the labeling matrix of $\mathfrak{C}_5\{\mathfrak{C}_2\}$ by the 3×3 matrix $3^2xJ + M$, the 3×3 matrix $3^2yJ + M^T$ and the 3×3 matrix with $*$ as entries respectively, then we obtain a labeling matrix for a supermagic labeling of $\mathfrak{C}_5\{\mathfrak{C}_6\}$.

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