INTEGER-ANTIMAGIC SPECTRA OF COMPLETE BIPARTITE GRAPHS AND COMPLETE BIPARTITE GRAPHS WITH A DELETED EDGE

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ABSTRACT. Let A be a non-trivial abelian group. A connected simple graph $G = (V, E)$ is A-antimagic if there exists an edge labeling $f : E(G) \to A \setminus \{0\}$ such that the induced vertex labeling $f^+ : V(G) \to A$, defined by $f^+(v) = \sum_{uv \in E(G)} f(uv)$, is injective. The integer-antimagic spectrum of a graph G is the set $\mathrm{IAM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$. In this article, we determine the integer-antimagic spectra of complete bipartite graphs and complete bipartite graphs with a deleted edge.

1. INTRODUCTION

Let G be a connected simple graph. For any non-trivial abelian group A (written additively), let $A^* = A \setminus \{0\}$, where 0 is the additive identity of A. Let a function $f : E(G) \to A^*$ be an edge labeling of G and $f^+ : V(G) \to A$ be its induced labeling, which is defined by $f^+(v) = \sum_{uv \in E(G)} f(uv)$. If there exists a labeling f whose induced map on $V(G)$ is a constant map, we say that f is an *A-magic labeling* of G and that G is an *A-magic* graph. The corresponding constant m is called an *A-magic value.*

If there exists an edge labeling f of G whose induced labeling f^+ on $V(G)$ is injective, then we say that f is an *A-antimagic labeling* of G and that G is an *A-antimagic* graph. The *integer-antimagic spectrum* of a graph G is the set

$$\mathrm{IAM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}.$$

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The concept of the A-antimagicness property for a graph G (introduced in [1]) naturally arises as a variation of the A-magic labeling problem (where the induced vertex labeling is a constant map). For some classes of graphs, the integer-antimagic spectra have been determined [1, 2, 7, 8]. An *antimagic labeling* of G (where the edges of G are labeled distinctly with $1, 2, \ldots, |E(G)|$) is a particular type of $\mathbb{Z}$-antimagic labeling. Readers interested in antimagic labelings can find many papers within the literature. In this paper, we determine the integer-antimagic spectra of $K_{m,n}$ and $K_{m,n} - \{e\}$.

A *labeling matrix* for a labeling f of a graph G is a matrix whose rows and columns are indexed by the vertices of G and the (u, v)-entry is $f(uv)$ if $uv \in E$, and is $*$ otherwise. In particular, if f is an A-magic labeling of G, then a labeling matrix of f is called an *A-magic labeling matrix* of G. Thus, G is A-magic if and only if there exists a labeling $f : E(G) \to A^*$ such that the row sums (as well as the column sums) of the labeling matrix for f are a constant value m, where entries with $*$ will be treated as 0. Similarly, if f is an A-antimagic labeling of G, then a labeling matrix of f is called an *A-antimagic labeling matrix* of G. Thus, G is A-antimagic if and only if there exists a labeling $f : E(G) \to A^*$ such that the row sums (as well as the column sums) of the labeling matrix for f are distinct. Note that if G is A-antimagic, then the order of A must be at least the order of G, i.e., $|A| \geq |G|$.

For the complete bipartite graph $K_{m,n}$ (where $n, m \geq 1$) and a suitable indexing of its vertices, a labeling matrix for any edge labeling is of the form

$$\begin{pmatrix} \bigstar_m & L \\ L^T & \bigstar_n \end{pmatrix},$$

where $\bigstar_r$ is a square matrix of order r (with all entries being $*$) and L is an $m \times n$ matrix whose entries are elements of A^*. So, in order to find an A-antimagic labeling of $K_{m,n}$, we need to find an $m \times n$ matrix L such that the row sums together with the column sums are distinct.

Notation. $[a, b]$ denotes the set of integers between a and b inclusive. Similarly, $[a, \infty)$ denotes the set of integers greater than or equal to a. For multi-sets S and T, we say that $S \equiv T \pmod{k}$ if and only if the sets S and T are equal, after reducing modulo k.

Example 1.1. Consider the graph $K_{3,4}$. Let

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}.$$

Then, the row sums of L are 10, 14, 18 and the column sums of L are 6, 9, 12, 15. Clearly,

$$\begin{pmatrix} \bigstar_3 & L \\ L^T & \bigstar_4 \end{pmatrix} = \left(\begin{array}{ccc|cccc} * & * & * & 1 & 2 & 3 & 4 \\ * & * & * & 2 & 3 & 4 & 5 \\ * & * & * & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 3 & * & * & * & * \\ 2 & 3 & 4 & * & * & * & * \\ 3 & 4 & 5 & * & * & * & * \\ 4 & 5 & 6 & * & * & * & * \end{array}\right)$$

So, $K_{3,4}$ is $\mathbb{Z}$-antimagic. Moreover, if all of the sums are taken in $\mathbb{Z}_7$, then we see that the row sums (as well as the column sums) of the above matrix are distinct. So, $K_{3,4}$ is also $\mathbb{Z}_7$-antimagic. Note that this matrix is not a $\mathbb{Z}_8$-antimagic labeling matrix of $K_{3,4}$. Later, we establish that $\mathrm{IAM}(G)(K_{3,4}) = [7, \infty)$.

Example 1.2. Let $L = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$. The row and column sums of L are 2, 3, 4 and 5. Hence, $K_{2,2}$ is $\mathbb{Z}_k$-antimagic for $k \geq 4$ (and also $\mathbb{Z}$-antimagic).

2. Some useful results

In this section, we list a few known results and new lemmas. These will be used to establish the integer-antimagic spectra of $K_{m,n}$ and $K_{m,n} - \{e\}$.

Lemma 2.1 ([1, Lemma 1]). *Let $m \geq 0$. A graph G of order $4m + 2$ is not $\mathbb{Z}_{4m+2}$-antimagic.*

Theorem 2.2 ([1, Theorem 6]). *For odd $n \geq 5$, $K_{1,n-1}$ is $\mathbb{Z}_k$-antimagic for all $k \geq n$. For even $n \geq 4$, $K_{1,n-1}$ is $\mathbb{Z}_k$-antimagic for all $k \geq n + 2$; but not $\mathbb{Z}_n$-antimagic nor $\mathbb{Z}_{n+1}$-antimagic.*

Clearly, $K_{1,1} \cong P_2$ is not $\mathbb{Z}_k$-antimagic, for all $k \geq 2$. Also, $K_{1,2} \cong P_3$ is $\mathbb{Z}_k$-antimagic, for all $k \geq 3$, since one merely labels the two edges of $K_{1,2}$ with 1 and 2. Thus, we only consider $K_{m,n}$, for $m, n \geq 2$.

Theorem 2.3 ([5, Theorem 1]). *Suppose that A is a non-trivial abelian group, and m, n are even (≥ 2). Then, $K_{m,n}$ has an A-magic labeling with magic value* 0.

Theorem 2.4 ([5, Theorem 3]). *Suppose m is odd, with $m \geq 3$ and $n \geq 2$. For any abelian group A where $|A| \geq 3$, $K_{m,n}$ has an A-magic labeling with magic value* 0.

Corollary 2.5. *Suppose A is an abelian group of order at least 3. There is an A-magic labeling for $K_{m,n}$ with magic value 0, where $m \geq 2$ and $n \geq 2$.*

Proof. This follows immediately from Theorems 2.3 and 2.4. □

We now focus our attention on $\mathbb{Z}_k$-antimagic labelings of $K_{m,n}$. For a fixed non-trivial abelian group A, let $M_{m,n}$ be an $m \times n$ matrix such that

$$\begin{pmatrix} \bigstar_m & M_{m,n} \\ M_{m,n}^T & \bigstar_n \end{pmatrix}$$

is an A-magic labeling matrix (for $K_{m,n}$) with magic value 0. That is, the row and column sums of $M_{m,n}$ are zero, under the operation of A.

For $n, m \geq 3$ and a fixed $k \geq m + n$, we want to find an $m \times n$ matrix L (whose entries are in $\mathbb{Z}_k^*$) with distinct row and column sums, modulo k. Consider a matrix of the form

$$L = \left(\begin{array}{ccc|c} & & & r_1 \\ & & & r_2 \\ & M_{m-1,n-1} & & \vdots \\ & & & r_{m-1} \\ \hline c_1 & c_2 & \cdots \; c_{n-1} & b \end{array}\right). \tag{2.1}$$

Then, the row and column sums of L are $r_1, \ldots, r_{m-1}, b + \sum_{j=1}^{n-1} c_j$ and $c_1, \ldots, c_{n-1}, b + \sum_{i=1}^{m-1} r_i$, respectively. Thus, we need to find elements $r_1, \ldots,$ $r_{m-1}, c_1, \ldots, c_{n-1}, b$ in $\mathbb{Z}_k^*$ such that $r_1, \ldots, r_{m-1}, b + \sum_{j=1}^{n-1} c_j; c_1, \ldots, c_{n-1}, b + \sum_{i=1}^{m-1} r_i$ are distinct.

Lemma 2.6. *Let $x, y \in \mathbb{Z}_k$ and $x \neq 0$. If $x = y + a$, then there exists an element $b \in \mathbb{Z}_k^*$ such that $x + b = 0$ and $y + b = -a$.*

Proof. Let $b = -x$. Since $x \neq 0$, we have that $b \in \mathbb{Z}_k^*$. Clearly, $x + b = 0$ and $y + b = -a$. □

Lemma 2.7. *Suppose $m, n \geq 3$ and $k \geq m + n$. If there exists a set of $m + n - 2$ integers $\{r_1, \ldots, r_{m-1}, c_1, \ldots, c_{n-1}\} \equiv [1, m + n - 2] \pmod{k}$ such that $\sum_{j=1}^{n-1} c_j \not\equiv 0 \pmod{k}$ and $\sum_{j=1}^{n-1} c_j - \sum_{i=1}^{m-1} r_i \equiv 1 \pmod{k}$, then $K_{m,n}$ is $\mathbb{Z}_k$-antimagic.*

Proof. Using Lemma 2.6, let $a = 1$, $x = \sum_{j=1}^{n-1} c_j \not\equiv 0 \pmod{k}$, and $y = \sum_{i=1}^{m-1} r_i \equiv 1 \pmod{k}$. Then, there is an element $b \in \mathbb{Z}_k^*$ such that

$$\left\{ b + \sum_{j=1}^{n-1} c_j, b + \sum_{i=1}^{m-1} r_i \right\} = \{-1, 0\} \text{ in } \mathbb{Z}_k.$$

Thus,

$\{r_1, \ldots, r_{m-1}, b + \sum_{j=1}^{n-1} c_j, c_1, \ldots, c_{n-1}, b + \sum_{i=1}^{m-1} r_i\} \equiv [-1, m + n - 2]$ in $\mathbb{Z}_k$.

Hence, $K_{m,n}$ is $\mathbb{Z}_k$-antimagic. □

3. Integer-antimagic spectrum of $K_{m,n}$

In this section, we prove the following theorem.

Theorem 3.1. *For $m, n \geq 2$,*

$$\mathrm{IAM}(K_{m,n}) = \begin{cases} [m+n, \infty), & \text{if } m+n \not\equiv 2 \pmod{4}; \\ [m+n+1, \infty), & \text{if } m+n \equiv 2 \pmod{4}. \end{cases}$$

From Example 1.2 or [1], we know that $K_{2,2} \cong C_4$ is $\mathbb{Z}_k$-antimagic, for $k \geq 4$. So without loss of generality, we assume that $n \geq 3$.

3.1. $m = 2$.

In this subsection, we do not construct L as described in Eq. (2.1).

1. Suppose $n = 4r$, where $r \geq 1$. Consider the $2 \times 4r$ matrix L_0 in Figure 1. We swap the entries in the second column of L_0 to obtain the matrix L in Figure 2. Hence, the set of row and column sums of L is $[-2r - 1, 2r + 1] \setminus \{0\}$. This implies that $K_{2,4r}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 4r + 3$. Note that $K_{2,4r}$ is not $\mathbb{Z}_{4r+2}$-antimagic (by Lemma 2.1).

Column nos.	1	2	$\cdots$	$2r-1$	$2r$	$2r+1$	$2r+2$	$\cdots$	$4r-1$	$4r$	Row sum
$L_0 =$	1	1	$\cdots$	1	1	-1	-1	$\cdots$	-1	-1	0
	$-2r-1$	$-2r$	$\cdots$	-3	-2	2	3	$\cdots$	$2r$	$2r+1$	0
Column sum	$-2r$	$-2r+1$	$\cdots$	-2	-1	1	2	$\cdots$	$2r-1$	$2r$	

FIGURE 1

Column nos.	1	2	$\cdots$	$2r-1$	$2r$	$2r+1$	$2r+2$	$\cdots$	$4r-1$	$4r$	Row sum
$L =$	1	$-2r$		1	1	-1	-1		-1	-1	$-2r-1$
	$-2r-1$	1	$\cdots$	-3	-2	2	3	$\cdots$	$2r$	$2r+1$	$2r+1$
Column sum	$-2r$	$-2r+1$	$\cdots$	-2	-1	1	2	$\cdots$	$2r-1$	$2r$	

FIGURE 2

Note. To save space, we will omit the "Column nos.", "Column sum" and "Row sum" headings on subsequent matrices within this paper.

2. Suppose $n = 4r + 2$, where $r \geq 1$. Consider the $2 \times (4r + 2)$ matrix L_0 in Figure 3.

	1	$\cdots$	r	$r+1$	$r+2$	$\cdots$	$2r+1$	$2r+2$	$2r+3$	$\cdots$	$4r+1$	$4r+2$	
$L_0 =$	1	$\cdots$	1	1	1	$\cdots$	1	-1	-1	$\cdots$	-1	-1	0
	$-2r-2$	$\cdots$	$-r-3$	$-r-2$	$-r-1$	$\cdots$	-2	2	3	$\cdots$	$2r+1$	$2r+2$	0
	$-2r-1$	$\cdots$	$-r-2$	$-r-1$	$-r$	$\cdots$	-1	1	2	$\cdots$	$2r$	$2r+1$	

FIGURE 3

We change the first entry of the $(r+1)$-st column from 1 to $-r$ to obtain the matrix L in Figure 4.

	1	$\cdots$	r	$r+1$	$r+2$	$\cdots$	$2r+1$	$2r+2$	$2r+3$	$\cdots$	$4r+1$	$4r+2$	
$L =$	1	$\cdots$	1	$-r$	1	$\cdots$	1	-1	-1	$\cdots$	-1	-1	$-r-1$
	$-2r-2$	$\cdots$	$-r-3$	$-r-2$	$-r-1$	$\cdots$	-2	2	3	$\cdots$	$2r+1$	$2r+2$	0
	$-2r-1$	$\cdots$	$-r-2$	$-2r-2$	$-r$	$\cdots$	-1	1	2	$\cdots$	$2r$	$2r+1$	

FIGURE 4

Hence, the set of row and column sums of L is $[-2r-2, 2r+1]$. This implies that $K_{2,4r+2}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 4r+4$.

3. Suppose $n = 2s+1$, where $s \geq 1$. Consider the $2 \times (2s+1)$ matrix L in Figure 5. The set of row and column sums of L is $[-s-1, s+1]$. This implies that $K_{2,2s+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 2s+3$ (where $s \geq 1$).

	1	2	$\cdots$	$s-1$	s	$s+1$	$s+2$	$\cdots$	$2s-1$	$2s$	$2s+1$	
$L=$	1	1	$\cdots$	1	1	-1	-1	$\cdots$	-1	-1	$-s-1$	$-s-1$
	$-s-1$	$-s$	$\cdots$	-3	-2	2	3	$\cdots$	s	$s+1$	$s+1$	$s+1$
	$-s$	$-s+1$	$\cdots$	-2	-1	1	2	$\cdots$	$s-1$	s	0	

FIGURE 5

3.2. $m \geq 3$.

In this subsection, we construct L as described in Eq. (2.1).

1. One of m and n is odd and the other is even. Without loss of generality, we assume $m = 2i+1$ and $n = 2j$, where $i \geq 1$ and $j \geq 2$. Let $b = -1$ and let the sequences $(c_1, c_2, \ldots, c_{2j-5}, c_{2j-4}, c_{2j-3}, c_{2j-2}; c_{2j-1}) = (2, -2, \ldots, j-1, -j+1, j, -j; 1)$ and $(r_1, r_2, \ldots, r_{2i-3}, r_{2i-2}, r_{2i-1}, r_{2i}) = (j+1, -j-1, \ldots, j+i-1, -j-i+1, j+i, -j-i)$. The set of row and column sums is $[-i-j, i+j]$. Hence, $\mathrm{IAM}(K_{2i+1,2j}) = [2i+2j+1, \infty)$.

2. Both m and n are even and $m+n \equiv 2 \pmod 4$. That is, $m = 2i$ and $n = 2j$ with $i, j \geq 2$ and $i+j$ is odd. Let $b = -(i+j)$ and let the sequences $(c_1, c_2, \ldots, c_{2j-3}, c_{2j-2}; c_{2j-1}) = (2, -2, \ldots, j, -j; j+i+1)$ and $(r_1, r_2, \ldots, r_{2i-3}, r_{2i-2}; r_{2i-1}) = (j+1, -j-1, \ldots, j+i-1, -j-i+1; j+i)$. The set of row and column sums is $[-i-j+1, i+j+1] \setminus \{-1\}$. Hence, $K_{2i,2j}$ is $\mathbb{Z}_k$-antimagic for $k \geq 2i+2j+1$. Note that in this case, $K_{m,n}$ is not $\mathbb{Z}_{m+n}$-antimagic (by Lemma 2.1).

3. Both m and n are even and $m+n \equiv 0 \pmod 4$. That is, $m = 2i$ and $n = 2j$ with $i, j \geq 2$ and $i+j$ is even. When $k \geq 2i+2j+1$, the labeling defined above (in Case 2) shows that $K_{2i,2j}$ is $\mathbb{Z}_k$-antimagic.

So, the remaining case is when $k = 2i+2j$. First, note that if $i+j = 4$, then $i = j = 2$. It is easy to see that

$$L = \left(\begin{array}{ccc|c} & & & 3 \\ & M_{3,3} & & -2 \\ & & & 1 \\ \hline -3 & 2 & -1 & 2 \end{array}\right)$$

can be used to construct a $\mathbb{Z}_8$-antimagic labeling matrix of $K_{4,4}$. Now, let $i+j \neq 4$. In particular, $i+j \geq 6$. Our aim is to choose distinct $c_1, \ldots, c_{2j-1}, r_1, \ldots, r_{2i-1}$ from $[1, i+j] \cup [-i-j+1, -2] \equiv [1, 2i+2j-2]$ (mod $2i+2j$) such that they satisfy the hypothesis of Lemma 2.7. We let $c_1 = 1$, $c_2 = i+j-1$, $c_3 = (i+j)/2+1$, $r_1 = i+j$, $r_2 = -i-j+1$, $r_3 = -(i+j)/2-1$. They are distinct, since $i+j \neq 4$. We choose the remaining c_s (an even number of them) and r_t (an even number of them) from $[2, i+j-2] \cup [-i-j+2, -2] \setminus \{(i+j)/2+1, -(i+j)/2-1\}$ so that each c_s, c'_s (and each r_t, r'_t) pair add up to 0, (mod $2i+2j$). Thus, $\sum_{p=1}^{2j-1} c_p = c_1 + c_2 + c_3 = 3(i+j)/2 + 1 \not\equiv 0$, (mod $2i+2j$), and $\sum_{q=1}^{2i-1} r_q = r_1 + r_2 + r_3 = -(i+j)/2$, (mod $2i+2j$). Then,

$$\sum_{p=1}^{2j-1} c_p - \sum_{q=1}^{2i-1} r_q = 2(i+j) + 1 \equiv 1, \text{ (mod } 2i+2j).$$

Hence by Lemma 2.7, $K_{2i,2j}$ is $\mathbb{Z}_{2i+2j}$-antimagic, for $i+j \neq 4$.

4. Both m and n are odd and $m+n \equiv 2$ (mod 4). That is, $m = 2i+1$ and $n = 2j+1$ with $i, j \geq 1$ and $i+j$ is even. Without loss of generality, we assume $j \geq i \geq 1$.

Suppose $j = 1 = i$. Then,

$$L = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 2 & -3 \\ -1 & 3 & -2 \end{pmatrix}$$

can be used to construct a $\mathbb{Z}_k$-antimagic labeling matrix of $K_{3,3}$ for $k \geq 7$. Note that $K_{3,3}$ is not $\mathbb{Z}_6$-antimagic (by Lemma 2.1).

Now, suppose $j \geq 2$. Let $b = -2$ and let the sequences $(c_1, c_2, c_3, c_4; c_5, c_6 \ldots, c_{2j-1}, c_{2j}) = (1, 2, i+j, -j-i-1; 3, -3, \ldots, j, -j)$

and
$(r_1, r_2; r_3, r_4 \ldots, r_{2i-1}, r_{2i}) = (j+i+1, -j-i; j+1, -j-1, \ldots, j+i-1, -j-i+1)$. The set of row and column sums is $[-i-j-1, i+j+1] \setminus \{-2\}$. Hence, $K_{2i+1,2j+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 2i+2j+3$. Note that in this case, $K_{m,n}$ is not $\mathbb{Z}_{m+n}$-antimagic (by Lemma 2.1).

5. Both m and n are odd and $m+n \equiv 0 \pmod 4$. That is, $m = 2i+1$ and $n = 2j+1$ with $i, j \geq 1$ and $i+j$ is odd. Without loss of generality, we assume $j > i \geq 1$. When $k \geq 2i+2j+3$, the labeling defined above (in Case 4) shows that $K_{2i+1,2j+1}$ is $\mathbb{Z}_k$-antimagic.

   So, the remaining case is when $k = 2i+2j+2$. First, note that if $i+j=3$, then $i=1$ and $j=2$. Then,

$$L = \left(\begin{array}{cccc|c} & & & & 2 \\ & M_{2,4} & & & -3 \\ \hline -4 & -2 & 3 & 1 & 1 \end{array}\right)$$

   can be used to construct a $\mathbb{Z}_8$-antimagic labeling matrix of $K_{3,5}$. Now, let $i+j \neq 3$. In particular, $i+j \geq 5$. Our aim is to choose distinct $c_1, \ldots, c_{2j}, r_1, \ldots, r_{2i}$ from $[1, i+j+1] \cup [-i-j, -2] \equiv [1, 2i+2j] \pmod{2i+2j+2}$ such that they satisfy the hypothesis of Lemma 2.7. We let $c_1 = 1$, $c_2 = (i+j-1)/2$, $c_3 = i+j$, $c_4 = -i-j+1$, $r_1 = -(i+j-1)/2$, $r_2 = i+j-1$, $r_3 = -i-j$, $r_4 = i+j+1$. They are distinct, since $i+j \neq 3$. We choose the remaining c_s (an even number of them) and r_t (an even number of them) from $[2, i+j-2] \cup [-i-j+2, -2] \setminus \{(i+j-1)/2, -(i+j-1)/2\}$ so that each c_s, c_s' (and each r_t, r_t') pair add up to 0, $\pmod{2i+2j+2}$. Thus, $\sum_{p=1}^{2j} c_p = c_1+c_2+c_3+c_4 = 2+(i+j-1)/2 \not\equiv 0 \pmod{2i+2j+2}$ and $\sum_{q=1}^{2i} r_q = r_1+r_2+r_3+r_4 = i+j-(i+j-1)/2 \pmod{2i+2j+2}$. Then,

$$\sum_{p=1}^{2j} c_p - \sum_{q=1}^{2i} r_q = 1, \quad \pmod{2i+2j+2}.$$

Hence, by Lemma 2.7, $K_{2i+1,2j+1}$ is $\mathbb{Z}_{2i+2j+2}$-antimagic, for $i+j \neq 3$.

Therefore, we have established Theorem 3.1.

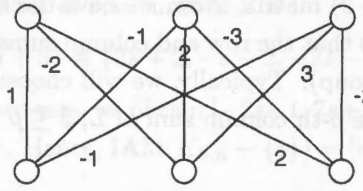

FIGURE 6. $K_{3,3}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 7$.

4. INTEGER-ANTIMAGIC SPECTRUM OF $K_{m,n} - \{e\}$

Let $K_{m,n} - \{e\}$ denote the graph obtained from $K_{m,n}$ by deleting a single edge. The *null set* of a graph G, denoted by $N(G)$, is the set of numbers $k \in \mathbb{N}$, where G has a $\mathbb{Z}_k$-magic labeling with magic-value 0 (by convention, $\mathbb{Z}_1 = \mathbb{Z}$). The null sets of some graphs have been established in [3–6]. The following lemmas were proved in [6] and will be used for this section of the paper.

Lemma 4.1 ([6, Corollary 3.4]). *$N(K_{r,s} - \{e\}) = \mathbb{N} \setminus \{2\}$ for $r \geq s \geq 5$.*

Lemma 4.2 ([6, Corollary 3.8]). *$N(K_{r,3} - \{e\}) = \mathbb{N} \setminus \{2, 3\}$, for $r \geq 3$.*

Lemma 4.3 ([6, Theorem 3.10]). *$N(K_{r,4} - \{e\}) = \mathbb{N} \setminus \{2\}$ for $r \geq 4$.*

Lemma 4.4 ([6, Corollary 3.6]). *For $r \geq 3$, $N(K_{r,2} - \{e\}) = \varnothing$.*

For the sake of consistency in this paper, we will apply Lemmas 4.1–4.4 for $s \geq r \geq 3$. By using the above lemmas and applying the constructions of Section 3 to $K_{m,n} - \{e\}$ for $n \geq m \geq 4$, we see that $K_{m,n} - \{e\}$ is $\mathbb{Z}_k$-antimagic for $k \geq m + n$ if $m + n \not\equiv 2 \pmod 4$; and for $k \geq m + n + 1$ if $m + n \equiv 2 \pmod 4$. However, Lemma 4.4 indicates that new constructions (different from those found in Section 3) must be used, when $m = 3$ and 2.

4.1. $m = 3$.

In this subsection, we do not construct L as described in Eq. (2.1). For $K_{3,n} - \{e\}$, consider a labeling matrix L of the form

$$L = \left(\begin{array}{cc|cccccc} * & b & c_3 & c_4 & c_5 & \cdots & c_{n-1} & c_n \\ \hline l_{21} & l_{22} & & & & & & \\ & & & & & B & & \\ l_{31} & l_{32} & & & & & & \end{array} \right),$$

where B is a $2 \times (n-2)$ matrix. Now, we have to choose B and the values of the other entries so that the row and column sums of L are distinct (in a non-trivial abelian group). Typically, we will choose $B = M_{2,n-2}$. In this case, c_p represents the p-th column sum of $L, 3 \leq p \leq n$.

1. $n = 2j$, where $j \geq 2$. Let $\{c_p \mid 5 \leq p \leq 2j\} = [-j-1, j+1] \setminus \{0, 1, -1, 2, -2, 3, -3\}$, $b = -2$, $c_3 = 1$, $c_4 = 3$, $l_{21} = 2$, $l_{22} = -2$, $l_{31} = -3$ and $l_{32} = 1$. The first and second column sums are -1 and -3, respectively. The row sums are 2, 0 and -2. Hence, the set of row and column sums of L is $[-j-1, j+1]$. Thus, $K_{3,2j} - \{e\}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 2j + 3 = n + 3$.

2. $n = 2j + 1$, where $j \geq 1$ and j is odd. Then, $n + 3 \equiv 2 \pmod 4$. Let $\{c_p \mid 4 \leq p \leq 2j+1\} = [-j-2, j+2] \setminus \{0, 1, -1, 2, -2, 3, -3\}$, $b = 1$, $c_3 = -2$, $l_{21} = 1$, $l_{22} = 2$, $l_{31} = -1$ and $l_{32} = -2$. The first and second column sums are 0 and 1, respectively. The row sums are -1, 3 and -3. Hence, the set of row and column sums of L is $[-j-2, j+2] \setminus \{2\}$. Thus, $K_{3,2j+1} - \{e\}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 2j + 5 = n + 4$. By Lemma 2.1, $K_{3,n} - \{e\}$ is not $\mathbb{Z}_{n+3}$-antimagic. Hence, $\mathrm{IAM}(K_{3,n} - \{e\}) = [n+4, \infty)$.

3. $n = 4s + 1$, where $s \geq 1$. This corresponds to Case 2, where $j \geq 1$ and j is even. Note that the labeling described in Case 2 provides a $\mathbb{Z}_k$-antimagic labeling of $K_{3,4s+1}$, for $k \geq 4s + 5$. Thus, we need to construct a $\mathbb{Z}_{4s+4}$-antimagic labeling of $K_{3,4s+1}$. Let $\{c_p \mid 6 \leq p \leq 4s+1\} = [-2s, 2s] \setminus \{0, \pm(s+1), \pm(s+2)\}$ and let $b = 2s+2$, $c_3 = 2s+1$, $c_4 = s+1$, $c_5 = s+2$, $l_{21} = 2s+2$, $l_{22} = s$, $l_{31} = s+1$ and $l_{32} = s+2$. Namely,

$$L = \left(\begin{array}{cc|cccccc} * & 2s+2 & 2s+1 & s+1 & s+2 & c_6 \cdots & c_{4s} & c_{4s+1} \\ \hline 2s+2 & s & & & & & & \\ & & & & M_{2,4s-1} & & & \\ s+1 & s+2 & & & & & & \end{array} \right).$$

The set of column sums of L is

$$[-2s, 2s] \setminus \{0, \pm(s+1),$$

$$\pm(s+2)\} \cup \{3s+3, 4s+4, 2s+1, s+1, s+2\} \equiv [-2s, 2s+1] \setminus \{-s-2\},$$

(mod $4s+4$). The set of row sums of L is

$$\{6s+6, 3s+2, 2s+3\} \equiv \{2s+2, -s-2, -2s-1\} \pmod{4s+4}.$$

Combining these two sets, we obtain $[-2s-1, 2s+2]$. Thus, $K_{3,4s+1}-\{e\}$ is $\mathbb{Z}_{4s+4}$-antimagic. Hence, $\mathrm{IAM}(K_{3,n}-\{e\}) = [n+3, \infty)$.

4.2. $\boldsymbol{m = 2.}$

In this subsection, we do not construct L as described in Eq. (2.1).

1. Suppose $n = 2r$, where $r \geq 1$.

When $r = 1$, we see that $K_{2,2}-\{e\} \cong P_4$, which is $\mathbb{Z}_k$-antimagic, for $k \geq 4$ (see [1]).

When $r = 2$, let

$$L = \begin{pmatrix} * & 2 & 2 & -2 \\ 1 & -2 & 1 & -1 \end{pmatrix}.$$

Hence $K_{2,4}-\{e\}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 7$. Note that $K_{2,4}-\{e\}$ is not $\mathbb{Z}_6$-antimagic (by Lemma 2.1).

When $r \geq 3$, let L be as described in Figure 7.

$$\begin{array}{c||cccccccccccc||c} & 1 & 2 & 3 & \cdots & r-2 & r-1 & r & r+1 & r+2 & \cdots & 2r-1 & 2r & \\ \hline L = & * & 2 & 3 & \cdots & r-2 & r-1 & r & r+1 & -r-1 & \cdots & -4 & -3 & 2 \\ & 1 & 1 & 1 & \cdots & 1 & -r+1 & 1 & 1 & 1 & \cdots & 1 & 1 & r \\ \hline & 1 & 3 & 4 & \cdots & r-1 & 0 & r+1 & r+2 & -r & \cdots & -3 & -2 & \end{array}$$

FIGURE 7

The set of row and column sums is $[-r, r+2] \setminus \{-1\}$. Hence, $K_{2,2r}-\{e\}$ is $\mathbb{Z}_k$-antimagic for $k \geq 2r+3$.

When $r = 2s$ is even, the discussion above (and Lemma 2.1) show that $\mathrm{IAM}(K_{2,4s}) = [4s+3, \infty)$, for $s \geq 1$. Hence, the remaining case which needs to be resolved is when $r = 2s+1$ and $k = 4s+4$, where $s \geq 1$.

When $s = 1$, let

$$\begin{array}{c||cccccc||c} L = & * & 1 & 2 & 3 & 4 & -3 & 7 \\ & 2 & -1 & -1 & 2 & 2 & -1 & 3 \\ \hline & 2 & 0 & 1 & 5 & 6 & -4 & \end{array}$$

Hence, $K_{2,6}-\{e\}$ is $\mathbb{Z}_8$-antimagic.

When $s = 2$, let

$$L = \left\|\begin{array}{cccccccccc||c}
* & 1 & 2 & 3 & 4 & 5 & -6 & -5 & -4 & -3 & -3 \\
-1 & -3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -12 \\
\hline
-1 & -2 & 1 & 2 & 3 & 4 & -7 & -6 & -5 & -4 &
\end{array}\right.$$

Hence, $K_{2,10} - \{e\}$ is $\mathbb{Z}_{12}$-antimagic.

When $s = 3$, let L be as described in Figure 8.

$$L = \left\|\begin{array}{cccccccccccccc||c}
* & 1 & 2 & 3 & 4 & 5 & 6 & 7 & -8 & -7 & -6 & -5 & -4 & -2 & -4 \\
-1 & -3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -16 \\
\hline
-1 & -2 & 1 & 2 & 3 & 4 & 5 & 6 & -9 & -8 & -7 & -6 & -5 & -3 &
\end{array}\right.$$

FIGURE 8

Hence, $K_{2,14} - \{e\}$ is $\mathbb{Z}_{16}$-antimagic.

When $s \geq 4$, let L_0 be as described in Figure 9.

$$L_0 = \left\|\begin{array}{cccccccccccccc||c}
1 & 2 & 3 & \cdots & 2s+2 & 2s+3 & 2s+4 & \cdots & 3s+4 & 3s+5 & 3s+6 & \cdots & 4s+2 & & \\
\hline
* & 1 & 2 & \cdots & 2s+1 & -2s-2 & -2s-1 & \cdots & -s-1 & -s & -s+1 & \cdots & -3 & & -2s+1 \\
-1 & -3 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & & -4s-4 \\
\hline
-1 & -2 & 1 & \cdots & 2s & -2s-3 & -2s-2 & \cdots & -s-2 & -s-1 & -s & \cdots & -4 & &
\end{array}\right.$$

FIGURE 9

Changing the $(1, 3s+5)$-th entry from $-s$ to -2, we obtain L as described in Figure 10.

$$L = \left\|\begin{array}{cccccccccccccc||c}
1 & 2 & 3 & \cdots & 2s+2 & 2s+3 & 2s+4 & \cdots & 3s+4 & 3s+5 & 3s+6 & \cdots & 4s+2 & & \\
\hline
* & 1 & 2 & \cdots & 2s+1 & -2s-2 & -2s-1 & \cdots & -s-1 & -2 & -s+1 & \cdots & -3 & & -s-1 \\
-1 & -3 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & & -4s-4 \\
\hline
-1 & -2 & 1 & \cdots & 2s & -2s-3 & -2s-2 & \cdots & -s-2 & -3 & -s & \cdots & -4 & &
\end{array}\right.$$

FIGURE 10

The set of row and column sums of L is $[-2s - 3, 2s]$, mod $4s + 4$. Hence, $K_{2,4s+2} - \{e\}$ is $\mathbb{Z}_{4s+4}$-antimagic.

2. Suppose $n = 2r+1$, where $r \geq 1$.

When $r = 1$, let

$$L = \begin{pmatrix} * & 1 & 2 \\ 1 & 3 & -2 \end{pmatrix}.$$

Hence, $K_{2,3} - \{e\}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 5$.

When $r \geq 2$, let L_0 be as described in Figure 11.

$$L_0 = \begin{array}{c||cccccccccccc||c} & 1 & 2 & 3 & \cdots & r & r+1 & r+2 & r+3 & \cdots & 2r-1 & 2r & 2r+1 & \\ \hline & * & 1 & 2 & \cdots & r-1 & r & -r & -r+1 & \cdots & -3 & -2 & -1 & 0 \\ & -r & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & -r & -1 \\ \hline & -r & 2 & 3 & \cdots & r & r+1 & -r+1 & -r+2 & \cdots & -2 & -1 & -r-1 & \end{array}$$

FIGURE 11

Changing the $(1, 2r)$-th entry from -2 to -1, we obtain L as described in Figure 12.

$$L = \begin{array}{c||cccccccccccc||c} & 1 & 2 & 3 & \cdots & r & r+1 & r+2 & r+3 & \cdots & 2r-1 & 2r & 2r+1 & \\ \hline & * & 1 & 2 & \cdots & r-1 & r & -r & -r+1 & \cdots & -3 & -1 & -1 & 1 \\ & -r & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & -r & -1 \\ \hline & -r & 2 & 3 & \cdots & r & r+1 & -r+1 & -r+2 & \cdots & -2 & 0 & -r-1 & \end{array}$$

FIGURE 12

The set of row and column sums of L is $[-r-1, r+1]$, mod $2r+3$. Hence, $K_{2,2r+1} - \{e\}$ is $\mathbb{Z}_k$-antimagic for $k \geq 2r+3$.

Combining the discussion above, we have established the following theorem.

Theorem 4.5. *For $m, n \geq 2$,*

$$\mathrm{IAM}(K_{m,n} - \{e\}) = \begin{cases} [m+n, \infty), & \text{if } m+n \not\equiv 2 \pmod 4; \\ [m+n+1, \infty), & \text{if } m+n \equiv 2 \pmod 4. \end{cases}$$

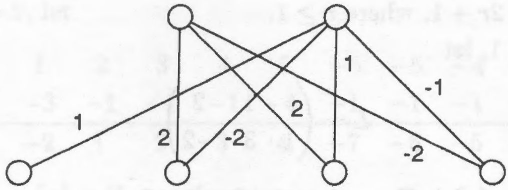

FIGURE 13. $K_{2,4} - \{e\}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 7$.

REFERENCES

[1] W.H. Chan, R.M. Low and W.C. Shiu, Group-antimagic labelings of graphs, *Congr. Numer.*, **217** (2013), 21–31.

[2] D. Roberts and R.M. Low, Group-antimagic labelings of multi-cyclic graphs, preprint, 2014.

[3] E. Salehi, Zero-sum magic graphs and their null sets, *Ars Combin.*, **82** (2007), 41–53.

[4] E. Salehi, On zero-sum magic graphs and their null sets, *Bulletin of the Institute of Mathematics, Academia Sinica*, **3** (2008), 255–264.

[5] W.C. Shiu and R.M. Low, Group magicness of complete n-partite graphs, *J. Combin. Math. Combin. Comput.*, **58** (2006), 129–134.

[6] W.C. Shiu and R.M. Low, Group-magic labelings of graphs with deleted edges, *Australas. J. Combin.*, **57** (2013), 3–19.

[7] W.C. Shiu and R.M. Low, The integer-antimagic spectra of dumbbell graphs, preprint, 2014.

[8] W.C. Shiu, P.K. Sun and R.M. Low, Integer-antimagic spectra of tadpole and lollipop graphs, preprint, 2015.

[9] T-M. Wang and C-C. Hsiao, On anti-magic labeling for graph products, *Discrete Math.*, **308** (2008), 3624–3633.

[10] T-M. Wang, C-C. Hsiao and S-M. Lee, A note on edge-graceful spectra for the square of paths, *Discrete Math.*, **308** (2008), 5878–5885.

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