Extreme Friendly Indices of $C_m \times P_n$*

Wai Chee Shiu, Fook Sun Wong
Department of Mathematics, Hong Kong Baptist University,
224 Waterloo Road, Kowloon Tong,
Hong Kong, China.

Abstract

Let $G = (V, E)$ be a connected simple graph. A labeling $f : V \to \mathbb{Z}_2$ induces an edge labeling $f^* : E \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |f^{*-1}(i)|$. If $|v_f(1) - v_f(0)| \leq 1$, then f is called a friendly labeling of G. For a friendly labeling f of a graph G, we define the friendly index of G under f by $i_f(G) = e_f(1) - e_f(0)$. The set $\{i_f(G) \mid f$ is a friendly labeling of $G\}$ is called the full friendly index set of G. In this paper, we will present the extreme friendly indices, i.e., the maximum and minimum friendly indices of Cartesian product of a cycle and a path.

Keywords: Vertex labeling, friendly labeling, friendly index set, Cartesian product of a cycle and a path.
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1 Introduction and Notations

In this paper, all graphs are simple and connected. All undefined symbols and concepts can be referred to [1]. Let G be a graph. A labeling $f : V \to \mathbb{Z}_2$ induces an edge labeling $f^* : E \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, define $v_f(i) = |f^{-1}(i)|$ and $e_f(i) = |f^{*-1}(i)|$. A labeling f is called *friendly* if $|v_f(1) - v_f(0)| \leq 1$. For a friendly labeling f of a graph G, we define the *friendly index of G under f* by $i_f(G) = e_f(1) - e_f(0)$. The set

$$\{|i_f(G) \mid f \text{ is a friendly labeling of } G\}$$

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is called the *friendly index set of* G, which was first introduced by Chartrand *et al.* [2]. The set

$$\{i_f(G) \mid f \text{ is a friendly labeling of } G\}$$

is called the *full friendly index set of* G, which was first introduced by Shiu and Kwong [4].

The full friendly indices of the graphs $P_2 \times P_n$ and $C_m \times C_n$ were found [3–5]. In this paper, we are interested on the bounds of the full friendly index set of $C_m \times P_n$.

Given cycle C_m and path P_n with vertex sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively, the Cartesian product $C_m \times P_n$ is a simple graph with vertex sets consisting of mn vertices labeled (i, j), where $1 \le i \le m$ and $1 \le j \le n$. Two vertices (i, j) and (h, k) are adjacent in $C_m \times P_n$ if either $i = h$ and v_j is adjacent to v_k in graph P_n, or $j = k$ and v_i is adjacent to v_h in graph C_m. Note that $C_m \times P_n$ is a graph of order mn and size $2mn - m$. In this paper, the vertices (i, j) are denoted as u_{ij}, where $1 \le i \le m$ and $1 \le j \le n$.

2 The upper bounds

For a fixed labeling f, a vertex v is called a *k-vertex* if $f(v) = k$ and an edge e is called a *k-edge* if $f^*(e) = k$. A graph G is called a *(p, q)-graph* if the order and the size of G are p and q, respectively. It is easy to get the following natural upper bound of the friendly index.

Lemma 2.1 *If f is a friendly labeling of a (p, q)-graph G, then $i_f(G) \le q$.*

Corollary 2.2 *If f is a friendly labeling of the graph $C_m \times P_n$, then $i_f(C_m \times P_n) \le 2mn - m$.*

Lemma 2.3 *An odd cycle C in a graph with f contains at least one 0-edge.*

Proof: Since $\sum_{e \in E(C)} f^*(e) = 2 \sum_{v \in V(C)} f(v) \equiv 0 \pmod 2$, there exist at least one 0-edge in the odd cycle C. □

Theorem 2.4 *If f is a friendly labeling of the graph $C_m \times P_n$, then $i_f(C_m \times P_n) \leq 2mn - m - 2n$ when m is odd.*

Proof: The graph $C_m \times P_n$ contains at least n disjoint odd cycles. So we have $e_f(0) \geq n$ and $e_f(1) \leq 2mn - m - n$. Hence, $i_f(C_m \times P_n) \leq 2mn - m - 2n$ $\square$

From the above theorem, the upper bounds of friendly indices of $C_m \times P_n$ are $2mn - m$ and $2mn - m - 2n$ according to m is even and odd, respectively.

For $1 \leq i \leq m$, $1 \leq j \leq n$, let $f(u_{ij}) = i + j \pmod 2$. It is easy to see that f is a friendly labeling of $C_m \times P_n$. For each edge $u_{ab}u_{cd} \in E(C_m \times P_n)$, either $a = c$ and $b = d \pm 1$ with $1 \leq b,\ d \leq n$, or $b = d$ and $a \equiv c \pm 1 \pmod m$. Thus,

$$\begin{aligned} f^*(u_{ab}u_{cd}) &= f(u_{ab}) + f(u_{cd}) = a + b + c + d \\ &\equiv \begin{cases} 0 \pmod 2 & \text{if } b = d,\ a = 1 \text{ and } c = m \text{ is odd,} \\ 1 \pmod 2 & \text{otherwise.} \end{cases} \end{aligned}$$

Then

$$e_f(0) = \begin{cases} n & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Hence

$$i_f(C_m \times P_n) = \begin{cases} 2mn - m - 2n & \text{if } m \text{ is odd,} \\ 2mn - m & \text{if } m \text{ is even.} \end{cases}$$

Therefore, the maximum friendly indices of $C_m \times P_n$ are $2mm - m - 2n$ when m is odd and $2mn - m$ when m is even, respectively. Hence, the bounds of Corollary 2.2 and Theorem 2.4 are sharp.

Labelings f of $C_6 \times P_3$, $C_6 \times P_4$, $C_7 \times P_3$ and $C_7 \times P_4$ in Fig. 1 and Fig. 2 illustrate the proof of Corollary 2.2 and Theorem 2.4.

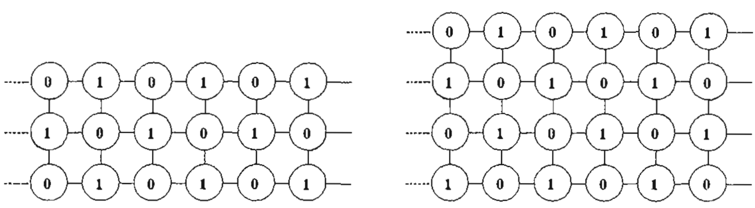

Figure 1: $i_f(C_6 \times P_3) = 30$ and $i_f(C_6 \times P_4) = 42$.

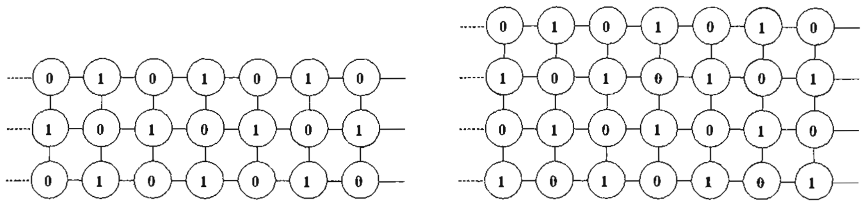

Figure 2: $i_f(C_7 \times P_3) = 29$ and $i_f(C_7 \times P_4) = 41$.

3 The lower bounds

Let f be any labeling of a graph containing a cycle C as its subgraph. The cycle C is called *mixed* (under f), if there is two vertices $u, v \in V(C)$ such that $f(u) = 1$ and $f(v) = 0$. Let f be any labeling of a graph containing a path P as its subgraph. The path P is called *mixed* (under f), if there is two vertices $u, v \in V(P)$ such that $f(u) = 1$ and $f(v) = 0$. Clearly, a mixed cycle and mixed path contains at least one 1-edge. The cycle C is called 1-*pure cycle* (under f), if $f(u) = 1$ for any vertex $u \in V(C)$. The cycle C is called 0-*pure cycle* (under f), if $f(u) = 0$ for any vertex $u \in V(C)$. The definitions of 0-*pure path* and 1-*pure path* are similarly.

The following lemma is a particular case of Corollary 2 in [4].

Lemma 3.1 *For any labeling, the number of* 1*-edge in a mixed cycle is a positive even integer.*

Now we consider the graph $C_m \times P_n$. For $1 \leq i \leq m$, the path $u_{i1}u_{i2}\cdots u_{in}$ is called a *vertical path* and for $1 \leq i \leq n$, the cycle $u_{1j}u_{2j}\cdots u_{mj}u_{1j}$ is called a *horizontal cycle.*

Theorem 3.2 *Let f be a friendly labeling of the graph $C_m \times P_n$. If n is even with $m \leq 2n$, then $i_f(C_m \times P_n) \geq 3m - 2mn$.*

Proof: Let r be the number of horizontal 1-pure cycles and s be the number of horizontal 0-pure cycles. By the property of friendly labeling, we have $0 \leq r, s \leq \frac{n}{2}$.

If $r = s = 0$, then all horizontal cycles are mixed and hence the number of edge disjoint mixed cycles in $C_m \times P_n$ is at least n. Thus $e_f(1) \geq 2n \geq m$.

If $r = 0$ or $s = 0$, then, without loss of generality, we may assume $r \neq 0$ and $s = 0$. In this case, the number of horizontal mixed cycles is $n - r$. Hence there exist at least $\lceil \frac{mn/2}{n-r} \rceil$ vertical mixed paths since there are totally $\frac{mn}{2}$ 0-vertices lying in $n - r$ horizontal cycles. Therefore, there are at least $2(n-r) + \lceil \frac{mn/2}{n-r} \rceil$ 1-edges. Note that

$$\begin{aligned} 2(n-r) + \left\lceil \frac{mn/2}{n-r} \right\rceil &= 2n - 2r + \left\lceil \frac{mn/2}{n-r} \right\rceil \geq 2n - 2(\frac{n}{2}) + \left\lceil \frac{mn/2}{n} \right\rceil \\ &= n + \frac{m}{2} \geq \frac{m}{2} + \frac{m}{2} = m. \end{aligned}$$

If $r \neq 0$ and $s \neq 0$, then there exist m vertical mixed paths. Thus there are at least m 1-edges.

For each case, we have $e_f(1) \geq m$ and hence $e_f(0) \leq 2mn - m - m$. Therefore, $i_f(C_m \times P_n) \geq m - (2mn - 2m) = 3m - 2mn$. $\square$

Suppose n is even. Let $f(u_{ij}) = 0$ for $1 \leq i \leq m$, $1 \leq j \leq \frac{n}{2}$ and $f(u_{ij}) = 1$ for $1 \leq i \leq m$, $\frac{n}{2} + 1 \leq j \leq n$. It is easy to see that f is a friendly labeling of $C_m \times P_n$. For each edge $u_{ab}u_{cd} \in E(C_m \times P_n)$, either $a = c$ and $b = d \pm 1$ with $1 \leq b, d \leq n$, or $b = d$ and $a \equiv \pm 1 \pmod{m}$. Then

$$\begin{aligned} f^*(u_{ab}u_{cd}) &= f(u_{ab}) + f(u_{cd}) = a + b + c + d \\ &\equiv \begin{cases} 1 \pmod 2 & \text{if } a = c,\ b = d - 1 = \frac{n}{2} \\ 0 \pmod 2 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, $e_f(1) = m$ and the minimum friendly index of $C_m \times P_n$ is $m - (2mn - m - m) = 3m - 2mn$. That is, the bound of Theorem 3.2 is sharp.

Labelings f of $C_6 \times P_4$ and $C_7 \times P_4$ in Fig. 3 illustrate the proof of the Theorem 3.2.

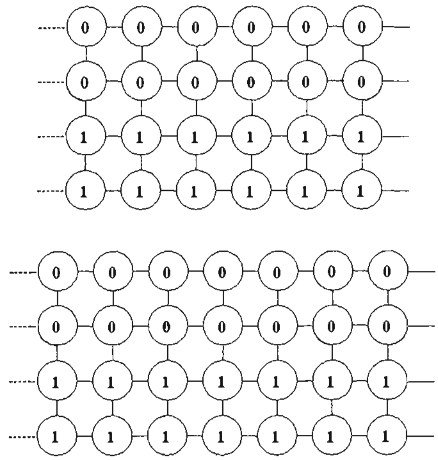

Figure 3: $i_f(C_6 \times P_4) = -30$ and $i_f(C_7 \times P_4) = -35$.

Theorem 3.3 *Let f be a friendly labeling of the graph $C_m \times P_n$. If n is even with $m \geq 2n$, then $i_f(C_m \times P_n) \geq 4n + m + 2 - 2mn$ for m is odd and $i_f(C_m \times P_n) \geq 4n + m - 2mn$ for m is even.*

Proof: We adopt the notations defined in the proof of Theorem 3.2. If $r = s = 0$, then all n horizontal cycles are mixed. Thus, $e_f(1) \geq 2n$. Suppose m is odd. By the property of friendly labeling, there is at least one mixed path. So $e_f(1) \geq 2n + 1$.

Suppose either $r = 0$ or $s = 0$. By using the same argument of the proof of Theorem 3.2, there are at least $2(n-r) + \lceil \frac{mn/2}{n-r} \rceil$ 1-edges. Note that $2(n-r) + \lceil \frac{mn/2}{n-r} \rceil = 2n - 2r + \lceil \frac{mn/2}{n-r} \rceil \geq 2n - 2(\frac{n}{2}) + \lceil \frac{mn/2}{n} \rceil \geq 2n - n + \lceil \frac{m}{2} \rceil$. If m is odd, then $2n - n + \lceil \frac{m}{2} \rceil = n + \frac{m+1}{2} \geq n + \frac{2n+1+1}{2} = 2n + 1$ and if m is even, then $2n - n + \lceil \frac{m}{2} \rceil = n + \frac{m}{2} \geq n + \frac{2n}{2} = 2n$.

Suppose $r \neq 0$ and $s \neq 0$. Then there exist m vertical mixed paths. Thus, there are at least m 1-edges. Since $m \geq 2n + 1$ for m is odd and $m \geq 2n$ for m is even, there are at least $2n + 1$ 1-edges for m is odd and $2n$ 1-edges for m is even.

For each case, we have $e_f(1) \geq 2n + 1$ for m is odd and and $e_f(1) \geq 2n$ for m is even. Thus, $e_f(0) \leq 2mn - m - 2n - 1$ for m is odd and $e_f(0) \leq 2mn - m - 2n$ for m is even, and hence $i_f(C_m \times P_n) \geq 4n + m + 2 - 2mn$ for m is odd and $i_f(C_m \times P_n) \geq 4n + m - 2mn$ for m is even. The proof is complete. $\square$

Suppose n is even with $m \geq 2n$. Let

$$f(u_{ij}) = \begin{cases} 0 & \text{if } 1 \leq i \leq \lfloor \frac{m}{2} \rfloor, 1 \leq j \leq n, \\ 1 & \text{if } \lfloor \frac{m}{2} \rfloor + 1 \leq i \leq 2\lfloor \frac{m}{2} \rfloor, 1 \leq j \leq n, \\ 0 & \text{if } i = m \text{ is odd}, 1 \leq j \leq \frac{n}{2}, \\ 1 & \text{if } i = m \text{ is odd}, \frac{n}{2} + 1 \leq j \leq n. \end{cases}$$

It is easy to check that f is friendly. Then

$$f^*(u_{ab}u_{cd}) \equiv \begin{cases} 1 \pmod 2 & \text{if } b = d,\ a = c - 1 = \lfloor \frac{m}{2} \rfloor, \\ 1 \pmod 2 & \text{if } b = d,\ a = 1 \text{ and } c = m \text{ is even}, \\ 1 \pmod 2 & \text{if } \frac{n}{2} + 1 \leq b = d \leq n,\ a = 1 \text{ and } c = m \text{ is odd}, \\ 1 \pmod 2 & \text{if } 1 \leq b = d \leq \frac{n}{2},\ a + 1 = c = m \text{ is odd}, \\ 1 \pmod 2 & \text{if } b = d - 1 = \frac{n}{2},\ a = c = m \text{ is odd}, \\ 0 \pmod 2 & \text{otherwise}. \end{cases}$$

Hence, $e_f(1) = 2n$ and the minimum friendly index of $C_m \times P_n$ is $4n + m - 2mn$.

Labelings f of $C_6 \times P_2$ and $C_7 \times P_2$ in Fig. 4 illustrate the proof of Theorem 3.3

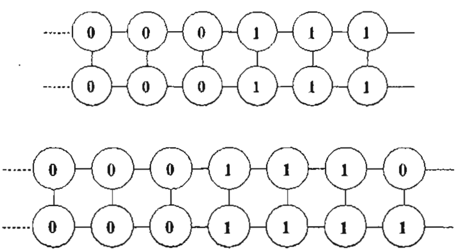

Figure 4: $i_f(C_6 \times P_2) = -10$ and $i_f(C_7 \times P_2) = -11$.

Theorem 3.4 *Let f be a friendly labeling of the graph $C_m \times P_n$. If n is odd with $m \leq 2n - 1$, then $i_f(C_m \times P_n) \geq 3m + 4 - 2mn$.*

Proof: We still adopt the notations defined in the proof of Theorem 3.2. If $r = s = 0$, then all horizontal cycles are mixed and hence the number of edge disjoint mixed cycles in $C_m \times P_n$ is at least n and $e_f(1) \geq 2n$. If m is odd, there is at least 1 mixed path. Hence $e_f(1) \geq 2n+1 \geq m+2$. If m is even, $e_f(1) \geq 2n \geq m+2$.

If $r = 0$ or $s = 0$, then, without loss of generality, we may assume $r \neq 0$ and $s = 0$. In this case, the number of horizontal mixed cycles is $n - r$. Since there are totally $\frac{mn}{2}$ 0-vertices lying in $n - r$ horizontal cycles, there exist at least $\lceil \frac{mn/2}{n-r} \rceil$ vertical mixed paths. Therefore, there are at least $2(n-r) + \lceil \frac{mn/2}{n-r} \rceil$ 1-edges. Note that $2(n-r) + \lceil \frac{mn/2}{n-r} \rceil \geq 2n - 2r + \lceil \frac{m}{2} \rceil \geq 2n - 2(\frac{n-1}{2}) + \lceil \frac{m}{2} \rceil = 2n - n + 1 + \lceil \frac{m}{2} \rceil = n + \lceil \frac{m}{2} \rceil + 1$.
If m is odd, then $n + \lceil \frac{m}{2} \rceil + 1 = n + \frac{m+1}{2} + 1 \geq \frac{m+1}{2} + \frac{m+1}{2} + 1 = m + 2$ and
if m is even, then $n + \lceil \frac{m}{2} \rceil + 1 = n + \frac{m}{2} + 1 \geq \frac{m+2}{2} + \frac{m}{2} + 1 = m + 2$.

If $r \neq 0$ and $s \neq 0$, then there exist m vertical mixed paths, there are at least m 1−edges, as n is odd, there are at least one mixed cycle, so there are at least $m + 2$ 1−edges.

For each case, we have $e_f(1) \geq m+2$ and $e_f(0) \leq 2mn - m - (m+2) = 2mn - 2m - 2$. Thus $i_f(C_m \times P_n) \geq m + 2 - (2mn - 2m - 2) = 3m + 4 - 2mn$. The proof is complete. □

Suppose n are odd with $m \leq 2n - 1$. Let

$$f(u_{ij}) = \begin{cases} 0 & \text{if } 1 \leq i \leq m,\ 1 \leq j \leq \frac{n-1}{2}, \\ 0 & \text{if } 1 \leq i \leq \lfloor \frac{m}{2} \rfloor,\ j = \frac{n+1}{2}, \\ 1 & \text{if } \lceil \frac{m+1}{2} \rceil \leq i \leq m,\ j = \frac{n+1}{2}, \\ 1 & \text{if } 1 \leq i \leq m,\ \frac{n+3}{2} \leq j \leq n. \end{cases}$$

It is easy to check that f is friendly. Then

$$f^*(u_{ab}u_{cd}) \equiv \begin{cases} 1 \pmod 2 & \text{if } b = d = \frac{n+1}{2},\ a = c - 1 = \lfloor \frac{m}{2} \rfloor, \\ 1 \pmod 2 & \text{if } b = d = \frac{n+1}{2},\ a = 1,\ c = m, \\ 1 \pmod 2 & \text{if } 1 \leq a = c \leq \lfloor \frac{m}{2} \rfloor,\ b = d - 1 = \frac{n+1}{2}, \\ 1 \pmod 2 & \text{if } \lceil \frac{m+1}{2} \rceil \leq a = c \leq m,\ b = d - 1 = \frac{n-1}{2}, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

Hence, $e_f(1) = m + 2$ and the minimum friendly index of $C_m \times P_n$ is

$3m + 4 - 2mn$.

Labelings f of $C_5 \times P_5$ and $C_6 \times P_5$ in Fig. 5 and Fig. 6 illustrate the proof of the Theorem 3.4.

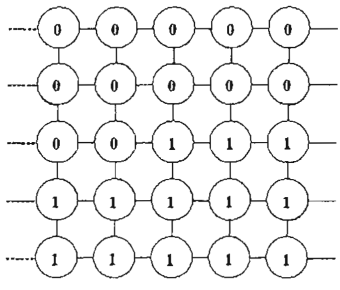

Figure 5: $i_f(C_5 \times P_5) = -31$.

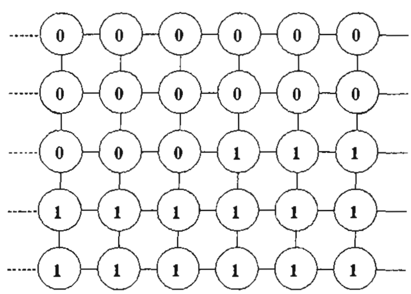

Figure 6: $i_f(C_6 \times P_5) = -38$.

Theorem 3.5 *Let f be a friendly labeling of the graph $C_m \times P_n$. If n is odd with $m \geq 2n - 2$, then $i_f(C_m \times P_n) \geq 4n + m + 2 - 2mn$ for m is odd and $i_f(C_m \times P_n) \geq 4n + m - 2mn$ for m is even.*

Proof: We still adopt the notations defined in the proof of Theorem 3.2. If $r = s = 0$, then all horizontal cycles are mixed and hence the number of edge disjoint mixed cycles in $C_m \times P_n$ is at least n and $e_f(1)$ is at least $2n$. Suppose m is odd. There is at least 1 mixed path. Then $e_f(1) \geq 2n + 1$.

If $r = 0$ or $s = 0$, then, without loss of generality, we may assume $r \neq 0$ and $s = 0$. In this case, the number of horizontal mixed cycles is $n - r$.

Since there are totally $\frac{mn}{2}$ 0-vertices lying in $n-r$ horizontal cycles, there exist at least $\lceil \frac{mn/2}{n-r} \rceil$ vertical mixed paths. Therefore, there are at least $2(n-r)+\lceil \frac{mn/2}{n-r} \rceil$ 1−edges. Note that $2(n-r)+\lceil \frac{mn/2}{n-r} \rceil \geq 2n-2r+\lceil \frac{m}{2} \rceil \geq 2n-2(\frac{n-1}{2})+\lceil \frac{m}{2} \rceil \geq 2n-n+1+\lceil \frac{m}{2} \rceil = n+1+\lceil \frac{m}{2} \rceil$.
If m is odd, then $n+1+\lceil \frac{m}{2} \rceil = n+1+\frac{m+1}{2} \geq n+1+\frac{2n-1+1}{2} \geq 2n+1$ and
if m is even, then $n+1+\lceil \frac{m}{2} \rceil = n+1+\frac{m}{2} \geq n+1+\frac{2n-2}{2} = n+1+n-1 = 2n$.

If $r \neq 0$ and $s \neq 0$, then there exist m vertical mixed paths. Thus, there are at least m 1-edges. As n is odd, there are at least one mixed cycle. So there are at least $m+2$ 1-edges. Hence we have $m+2 \geq 2n-1+2 = 2n+1$ for m is odd and $m+2 \geq 2n-2+2 = 2n$ for m is even.

For each case, we have$e_f(1) \geq 2n+1$ for m is odd and $e_f(1) \geq 2n$ for m is even. Therefore, $e_f(0) \leq 2mn-m-(2n+1) = 2mn-m-2n-1$ for m is odd and $e_f(0) \leq 2mn-m-(2n) = 2mn-m-2n$ for m is even. Hence $i_f(C_m \times P_n) \geq 4n+m+2-2mn$ for m is odd and $i_f(C_m \times P_n) \geq 4n+m-2mn$ for m is even. The proof is complete. □

Suppose n is odd with $m \geq 2n-2$. Let

$$f(u_{ij}) = \begin{cases} 0 & \text{if } 1 \leq i \leq \lfloor \frac{m}{2} \rfloor,\ 1 \leq j \leq n, \\ 1 & \text{if } \lfloor \frac{m}{2} \rfloor + 1 \leq i \leq 2\lfloor \frac{m}{2} \rfloor,\ 1 \leq j \leq n, \\ 0 & \text{if } i = m \text{ is odd},\ 1 \leq j \leq \frac{n-1}{2}, \\ 1 & \text{if } i = m \text{ is odd},\ \frac{n-1}{2}+1 \leq j \leq n. \end{cases}$$

It is easy to check that f is friendly. Then

$$f^*(u_{ab}u_{cd}) \equiv \begin{cases} 1 \pmod 2 & \text{if } b = d,\ a = c-1 = \lfloor \frac{m}{2} \rfloor \\ 1 \pmod 2 & \text{if } b = d,\ a = 1 \text{ and } c = m \text{ is even}, \\ 1 \pmod 2 & \text{if } \frac{n-1}{2}+1 \leq b = d \leq n,\ a = 1 \text{ and } c = m \text{ is odd}, \\ 1 \pmod 2 & \text{if } 1 \leq b = d \leq \frac{n-1}{2},\ a+1 = c = m \text{ is odd}, \\ 1 \pmod 2 & \text{if } b = d-1 = \frac{n-1}{2},\ a = c = m \text{ is odd}, \\ 0 \pmod 2 & \text{otherwise}. \end{cases}$$

Hence, $e_f(1) = 2n+1$ for m is odd and $e_f(1) = 2n$ for m is even. The minimum friendly index of $C_m \times P_n$ is $4n+m+2-2mn$ for m is odd and $4n+m-2mn$ for m is even.

Labelings f of $C_7 \times P_3$ and $C_6 \times P_3$ in Fig. 7 and Fig. 8 illustrate the proof of the Theorem 3.5.

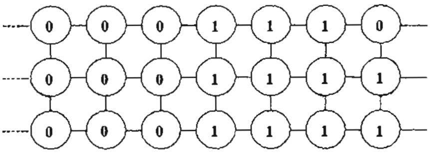

Figure 7: $i_f(C_7 \times P_3) = -21$.

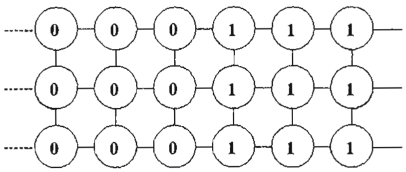

Figure 8: $i_f(C_6 \times P_3) = -18$.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, 1976.

[2] G. Chartrand, S.-M. Lee and P. Zhang, Uniformly cardial graphs, *Discrete Math.*, **306** (2006). 726-737.

[3] W.C. Shiu, M.H. Ling, Extreme friendly indices of $C_m \times C_n$, *Congressus Numerantium*, **188** (2007), 175-182.

[4] W.C. Shiu and H. Kwong, Full friendly index sets of $P_2 \times P_n$, *Discrete Math.*, **308** (2008). 3688-3693.

[5] W.C. Shiu, M.H. Ling, Full friendly fndex sets of Cartesian products of two cycles, preprint, 2008.