Journal of Combinatorics and Number Theory
Volume 5, Number 3, pp. 227–244

ISSN 1942-5600

FULL FRIENDLY INDEX SETS AND FULL PRODUCT-CORDIAL INDEX SETS OF SOME PERMUTATION PETERSEN GRAPHS*

Wai Chee Shiu† ***and Man-Ho Ho***‡
Department of Mathematics, Hong Kong Baptist University,
Kowloon Tong, Hong Kong, China

Abstract

Let $G = (V, E)$ be a connected graph without loops. A vertex labeling $g : V \to \mathbb{Z}_2$ induces two edge labelings $f^+, f^* : E \to \mathbb{Z}_2$, given by $f^+(uv) = f(u) + f(v)$ and $f^*(uv) = f(u)f(v)$ for each $uv \in E$ respectively. For $j \in \mathbb{Z}_2$, let $v_f(j) = |f^{-1}(j)|$, $e_{f^+}(j) = |(f^+)^{-1}(j)|$ and $e_{f^*}(j) = |(f^*)^{-1}(j)|$. A vertex labeling f is called friendly if $|v_f(1) - v_f(0)| \le 1$. For a friendly labeling f of G, the friendly index of G with respect to f is defined to be $i^+_f(G) = e_{f^+}(1) - e_{f^+}(0)$, and the product-cordial index is defined to be $i^*_f(G) = e_{f^*}(1) - e_{f^*}(0)$. The full friendly index set (FFI) and the full product-cordial index set (FPCI) of G contain precisely all the values $i^+_f(G)$ and $i^*_f(G)$ taken over all friendly labelings of G, respectively. In this paper, we study the FFI and the FPCI of odd twisted cylinder and two permutation Petersen graphs.

2000 Mathematics Subject Classification: Primary 05C78; Secondary 05C25

Keywords: Full friendly index sets, full product-cordial index sets, permutation Petersen graph.

1. Introduction

In this paper all graphs $G = (V, E)$ are assumed to be loopless and connected. A vertex labeling $f : V \to \mathbb{Z}_2$ induces two edge labelings $f^+, f^* : E \to \mathbb{Z}_2$, given by

$$f^+(uv) = f(u) + f(v),$$
$$f^*(uv) = f(u)f(v),$$

where $uv \in E$. For $j \in \mathbb{Z}_2$, let $v_f(j) = |f^{-1}(j)|$, $e_{f^+}(j) = |(f^+)^{-1}(j)|$ and $e_{f^*}(j) = |(f^*)^{-1}(j)|$, i.e., $v_f(i)$ is the number of vertices labeled by i, and $e_{f^+}(i)$, $e_{f^*}(i)$ are the

*Received: Oct. 23, 2013; Accepted: Dec. 5, 2013.
This work is partially supported by the Faculty Research Grant, Hong Kong Baptist University.
†E-mail address: wcshiu@hkbu.edu.hk (Corresponding author)
‡E-mail address: homanho@math.hkbu.edu.hk

numbers of edges labeled by i with respect to f^+ and f^* respectively. A vertex labeling f is said to be *friendly* if

$$|v_f(1) - v_f(0)| \leq 1.$$

The *friendly index* $i_f^+(G)$ of G with respect to a friendly labeling f is defined to be

$$i_f^+(G) := e_{f^+}(1) - e_{f^+}(0).$$

The *friendly index set* $\mathrm{FI}(G)$ of G [1] contains exactly the absolute value of friendly indices of all possible friendly labelings. In [11] Shiu–Kwong generalize the friendly index set to the *full friendly index set* $\mathrm{FFI}(G)$ of G:

$$\mathrm{FFI}(G) = \{i_f(G) \mid f \text{ is a friendly labeling of } G\}.$$

The *product-cordial index* $i_f^*(G)$ of G [6] with respect to a friendly labeling f is defined to be

$$i_f^*(G) := e_{f^*}(1) - e_{f^*}(0).$$

The *product-cordial index set* $\mathrm{PCI}(G)$ of G contains exactly the absolute value of the product-cordial indices of all possible friendly labelings. In [10] Shiu–Kwong generalize the product-cordial index set to the *full product-cordial index set* $\mathrm{FPCI}(G)$ of G:

$$\mathrm{FPCI}(G) = \{i_f^*(G) \mid f \text{ is a friendly labeling of } G\}.$$

Friendly index of some graphs are studied in [2,3,5,7]. Let $m \geq 3$ and $n \geq 2$. Denote by C_m an m-cycle and P_n an n-path. The full friendly index sets are studied in the case of a torus $C_m \times C_n$ [13,14], a cylinder $C_m \times P_n$ [9,15,16], a grid $P_2 \times P_n$ [11] and twisted cylinders [12].

Product-cordial index set is defined and studied for paths P_n, complete graphs K_n, $K_n - e$, bipartite graph $K_{m,n}$, double stars $DS(m, n)$, cycles C_n and wheels W_n in [6]. The product-cordial index sets of cylinders $C_m \times P_n$ is studied in [4]. In [10] Shiu–Kwong study the full product-cordial index sets of cycles C_n and torus $C_m \times C_n$, and establish the relationship between the friendly index and the product-cordial index of regular graphs. In [12] Shiu–Lee study the full product-cordial index sets of twisted cylinders.

In this paper we study the full friendly index set and the full product-cordial index set of odd twisted cylinder and two permutation Petersen graphs.

2. Notation and Preliminary Results

For any vertex labeling f, a vertex x is said to be a k-*vertex* if $f(x) = k$, and an edge is said to be an (a, b)-*edge* if it is incident with an a-vertex and a b-vertex. An edge e is said to be a k-*edge* if $f^*(e) = k$, where f^* is the edge labeling induced by the vertex labeling f. The number of (a, b)-edges is denoted by $E_f(a, b)$.

Lemma 2.1. [5, Corollary 2.3] *Let G be an r-regular graph of even order and of size q. If f is a friendly of G, then $i_f^*(G) = -(q + i_f^+(G))/2$.*

Since $e_f(1)$ is always positive for any connected graph G, $i_f^+(G) > -q$. Thus, the product-cordial index of friendly labeling of any regular connected graph G of even order is always negative, so $\mathrm{FPCI}(G) = -\,\mathrm{PCI}(G)$.

Lemma 2.2. [14, Corollary 2.3] *Let G be a regular graph of even order and of size q. If f is a friendly labeling of G, then $E_f(1,1) = E_f(0,0)$ and $i_f^+(G) = q - 4E_f(1,1)$.*

In this paper we consider cubic graphs of even order. By Lemma 2.2, to determine the full friendly index sets of cubic graphs it suffices to find the range and the non-existing values of $E_f(1,1)$. Define a set

$$E(G)(1,1) = \{E_f(1,1) \mid f \text{ is a friendly labeling of } G\}.$$

Then $\mathrm{FFI}(G) = q - 4E(G)(1,1)$. Lemma 2.1 enables us to determine the full product-cordial index set of the cubic graphs once its full friendly index set is known.

3. FFI and FPCI of Odd Twisted Cylinder

Let $m \geq 3$. A *permutation cubic graph* of order $2m$ is defined by taking two vertex-disjoint m-cycles and equip a perfect matching between the vertices of these two cycles. More precisely, let $C = u_1u_2\cdots u_m$ and $C^* = v_1v_2\cdots v_m$ be two m-cycles, and $\sigma \in S_m$, where S_m is the permutation group on the set $\{1,\ldots,m\}$. The permutation cubic graph of order $2m$, denoted by $\mathscr{P}(m;\sigma) = (V,E)$, is a simple graph with $V = \{u_1,\ldots,u_m,v_1,\ldots,v_m\}$ and $E = E(C) \cup E(C^*) \cup \{u_iv_{\sigma(i)} \mid 1 \leq i \leq m\}$.

A special case of permutation cubic graph is *twisted cylinder* $\mathscr{P}(2n;\sigma)$, where $\sigma \in S_{2n}$ is given by $\sigma = (1,2)(3,4)\cdots(2n-1,2n)$. The full friendly index set and the full product-cordial index set of $\mathscr{P}(2n;\sigma)$ are determined in [12].

In this section we study the full friendly index set of *odd twisted cylinder* $\mathscr{P}(2n+1;\sigma_1)$, where $\sigma_1 \in S_{2n+1}$ is given

$$\sigma_1 = (1,2)(3,4)\cdots(2n-1,2n).$$

Since the size of $\mathscr{P}(2n+1;\sigma_1)$ is $6n+3$, it follows from Lemma 2.2 that

$$i_f^+(\mathscr{P}(2n+1;\sigma_1)) = 6n+3-4E_f(1,1)$$

for any friendly labeling f of $\mathscr{P}(2n+1;\sigma_1)$.

Lemma 3.1. [11, Corollary 5] *Let f be a labeling of a graph G that contains a cycle C. If C contains an 1-edge, then the number of 1-edges in C is a positive even integer.*

Corollary 3.1. *The number of 0-edges of any odd cycle is odd with respect to any labeling. Thus, there is at least one 0-edge in any odd cycle.*

First of all we find the extreme friendly indices of $\mathscr{P}(2n+1;\sigma_1)$.

Theorem 3.1. *If f is friendly labeling of $\mathscr{P}(2n+1;\sigma_1)$, then the minimum value of $E_f(1,1)$ is 1 and the maximum value of $i_f^+(\mathscr{P}(2n+1;\sigma_1))$ is $6n-1$.*

Proof. Note that C and C^* are two odd cycles contained in $\mathscr{P}(2n+1;\sigma_1)$. Since the number of 0-edges is odd for any odd cycle, by Corollary 3.1, the number of 0-edges of $\mathscr{P}(2n+1;\sigma_1)$ is at least 2. Since $\mathscr{P}(2n+1;\sigma_1)$ is a 3-regular graph of even order, it follows from Lemma 2.2 that $E_f(1,1) = E_f(0,0) \geq 1$ for any friendly labeling f. Define a labeling $f_{\max}$ on $\mathscr{P}(2n+1;\sigma_1)$ by

$$f_{\max}(u) = \begin{cases} 1, & \text{for } u = u_{2n+1}, \text{ or } u_{2i}, v_{2i}, \text{ for } 1 \leq i \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $f_{\max}$ is friendly and $E_{f_{\max}}(1,1) = 1$. Therefore, the minimum value of $E_f(1,1) = 1$. It follows from Lemma 2.2 that the maximum value of $i_f^+(\mathscr{P}(2n+1;\sigma_1))$ is $i_{f_{\max}}^+(\mathscr{P}(2n+1;\sigma_1)) = 6n+3-4E_{f_{\max}}(1,1) = 6n+3-4 = 6n-1$. □

Theorem 3.2. *If f is a friendly labeling of $\mathscr{P}(2n+1;\sigma_1)$, then the maximum value of $E_f(1,1)$ is $3n-1$ and therefore the minimum value of $i_f^+(G)$ is $-6n+7$.*

Proof. Define a labeling $f_{\min}$ of $\mathscr{P}(2n+1;\sigma_1)$ by

$$f_{\min}(u) = \begin{cases} 1, & \text{for } u = u_i, \text{ for } 1 \leq i \leq n; \\ 1, & \text{for } u = v_i, \text{ for } 1 \leq i \leq n+1; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $f_{\min}$ is friendly, and $E_{f_{\min}}(1,1) = 3n-1$.

Let f be a friendly labeling of $\mathscr{P}(2n+1;\sigma_1)$. Let G be the subgraph of $\mathscr{P}(2n+1;\sigma_1)$ induced by all the 1-vertices. Thus the order of G is $2n+1$. For each $0 \leq j \leq 3$, denote by a_j the number of vertices of degree j in G. By the Handshaking Lemma, we have

$$\begin{aligned} 2E_f(1,1) &= a_1 + 2a_2 + 3a_3 \\ &= (a_1+a_2+a_3) + a_2 + 2a_3 \\ &= 2n+1-a_0+a_2+2a_3. \end{aligned}$$

If all the 1-vertices lie on either C or C^* but not both, then G is C or C^* and $E_f(1,1) = 2n+1 \leq E_{f_{\min}}(1,1)$; where equality holds only if $n=2$. Since $n \geq 3$ by assumption, to maximize $E_f(1,1)$ we can assume both C and C^* have at least one 1-vertex. Thus C or C^* contains at least two 1-vertices of degree less than 3 in G.

Without loss of generality, we assume C contains $2n$ 1-vertices and C^* contains one 1-vertex with respect to a friendly labeling h. Then $E_h(1,1) = 2n$ or $2n-1$, so $E_h(1,1) < E_{f_{\min}}(1,1)$. Therefore, to maximize $E_f(1,1)$ we can assume both C and C^* contain at least two 1-vertices of degree less than 3 in G. Thus $a_3 \leq 2n+1-4 = 2n-3$. Thus we have to maximize

$$E_f(1,1) = n + \frac{1-a_0+a_2}{2} + a_3$$

subject to

$$0 \leq a_0 + a_2 + a_3 \leq 2n+1 \text{ and } a_3 \leq 2n-3.$$

By the simplex method, the algebraic maximum value is $3n-1$ when $(a_0,a_1,a_2,a_3) = (0,0,5,2n-4)$ or $(a_0,a_1,a_2,a_3) = (0,1,3,2n-3)$. Combining with the friendly labeling

$f_{\min}$, the maximum value of $E_f(1,1)$ is $3n-1$. The minimum value of $i_f^+(\mathscr{P}(2n+1;\sigma_1))$ is $i_{f_{\min}}^+(\mathscr{P}(2n+1;\sigma_1)) = 6n+3-4E_{f_{\min}}(1,1) = 6n+3-4(3n-1) = -6n+7$. $\square$

We realize all the values lying in the extremely friendly indices of $\mathscr{P}(2n+1;\sigma_1)$ as friendly indices. The following two lemmas will be used repeatedly throughout this paper.

Lemma 3.2. [14, Lemma 2.7] *Let f be a labeling of a graph G such that $E_f(1,1) = k$. Suppose there are two non-adjacent vertices u, v such that $f(u) = 1$, $f(v) = 0$, the vertex u is adjacent to x 1-vertices, and the vertex v is adjacent to y 1-vertices. If g is a labeling on G defined by $g(u) = 0$ and $g(v) = 1$, and $g(w) = f(w)$ for all $w \in V(G) \setminus \{u, v\}$, then $E_g(1,1) = k-x+y$, and the numbers of 1-vertices and 0-vertices with respect to the labeling g are the same as those with respect to f.*

Lemma 3.3. [14, Lemma 2.8] *Let f be a labeling of a graph G such that $E_f(1,1) = k$. Suppose there are two adjacent vertices u, v such that $f(u) = 1$, $f(v) = 0$, the vertex u is adjacent to x 1-vertices, and the vertex v is adjacent to y 1-vertices. If g is a labeling on G defined by $g(u) = 0$ and $g(v) = 1$, and $g(w) = f(w)$ for all $w \in V(G) \setminus \{u, v\}$, then $E_g(1,1) = k-x+y-1$, and the numbers of 1-vertices and 0-vertices with respect to the labeling g are the same as those with respect to f.*

Let $[a,b] = \{i \in \mathbb{Z} \mid a \le i \le b\}$.

Theorem 3.3. $E(\mathscr{P}(2n+1;\sigma_1))(1,1) = [1, 3n-1]$. *Thus*

$$\mathrm{FFI}(\mathscr{P}(2n+1;\sigma_1)) = \{6n+3-4i \mid 1 \le i \le 3n-1\}.$$

Proof. Recall that $\mathscr{P}(2n+1;\sigma_1)$ consists of two odd cycles C and C^* whose vertices are labeled by u_i's and v_i's respectively, for $1 \le i \le 2n+1$.

Consider the friendly labeling $f_{\max}$ given in Theorem 3.1. Interchange the labelings of v_{2j-1} with v_{2j} consecutively for $1 \le j \le n$, and denote the resulting labeling by f_j with $f_0 = f_{\max}$. By Lemma 3.3 we have $E_{f_1}(1,1) = E_{f_0}(1,1) - 0 + 2 - 1 = E_{f_0}(1,1)+1$. It is easy to see that $E_{f_j}(1,1) = E_{f_{j-1}}(1,1) + 1$ for each $1 \le j \le n$. Thus $[1, n+1] \subseteq E(\mathscr{P}(2n+1;\sigma_1))(1,1)$.

After performing the above procedure, the labeling on $\mathscr{P}(2n+1;\sigma_1)$ is f_n and $E_{f_n}(1,1) = n+1$. Interchange the labelings of u_{2j-1} and v_{2j-1} consecutively for $1 \le j \le n$, and denote the resulting labeling by g_j with $g_0 = f_n$. By Lemma 3.2, $E_{g_1} = n+2$. It is easy to see that $E_{g_j}(1,1) = E_{g_{j-1}}(1,1)+1$ for each $1 \le j \le n$. Thus $[n+2, 2n+1] \subseteq E(\mathscr{P}(2n+1;\sigma_1))(1,1)$.

After performing the above procedure, the labeling on $\mathscr{P}(2n+1;\sigma_1)$ is g_n and $E_{g_n}(1,1) = 2n+1$. Interchange the labelings of u_{n+j} and v_j consecutively for $1 \le j \le n+1$, and denote the resulting labeling by h_j with $h_0 = g_n$. By Lemma 3.2, $E_{h_1} = 2n$. It is easy to see that $E_{h_j}(1,1) = E_{h_{j-1}}(1,1)+1$ for each $2 \le j \le n-1$. Note that $E_{h_{n-1}}(1,1) = 3n-2$. If n is even, then $E_{h_n}(1,1) = 3n-1$. If n is odd, then $E_{h_n}(1,1) = 3n-2$ and $E_{h_{n+1}}(1,1) = 3n-1$. Thus $[2n, 3n-1] \subseteq E(\mathscr{P}(2n+1;\sigma_1))(1,1)$.

The theorem follows by combining all the above cases. $\square$

Corollary 3.2. $\mathrm{FI}(\mathscr{P}(2n+1;\sigma_1)) = \{2i+1 \mid 1 \le i \le 3n-1\} \setminus \{6n-3\}$.

By Lemma 2.1 we have the following corollary.

Corollary 3.3. *For* $n \geq 1$,

$$\mathrm{FPCI}(\mathscr{P}(2n+1;\sigma_1)) = \{2i-6n-3 \mid 1 \leq i \leq 3n-1\} = \{-2j-1 \mid j \in [2,3n]\}.$$

Thus $\mathrm{PCI}(\mathscr{P}(2n+1;\sigma_1)) = \{2j+1 \mid j \in [2,3n]\}$.

4. FFI and FPCI of Two Permutation Petersen Graphs

Let $n \geq 3$. A *permutation Petersen graph* $P(n;\sigma)$ of order $2n$, where $\sigma \in S_n$ has no fixed point, is the graph with vertex set $\{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\}$ and edge set $\{x_i x_{i+1}, x_i y_i, y_i y_{\sigma(i)} \mid 1 \leq i \leq n\}$, where the addition is taken modulo n. The cycle $C = x_1 x_2 \cdots x_n x_1$ is called the *outer cycle* of $P(n;\sigma)$. If $\sigma = \sigma_1\sigma_2\cdots\sigma_r$ is the cycle decomposition of σ and $\sigma_i = (i_1, i_2, \ldots, i_s)$, we write $C^i = y_{i_1} y_{i_2} \cdots y_{i_s} y_{i_1}$, and let H be the disjoint union of all C^i's. This is a generalization of permutation cubic graph defined in [8], namely, $\mathscr{P}(m;\sigma) = P(m;\sigma^{-1}\tau\sigma)$, where τ is the m-cycle $(1, 2, \ldots, m)$.

We study the full friendly index set of two permutation Petersen graphs: $P(3n;\sigma_2)$, where

$$\sigma_2 = (1,2,3)(4,5,6)\cdots(3n-2,3n-1,3n),$$

and $P(4n;\sigma_3)$, where

$$\sigma_3 = (1,2,3,4)(5,6,7,8)\cdots(4n-3,4n-2,4n-1,4n).$$

The size of $P(3n;\sigma_2)$ and $P(4n;\sigma_3)$ are $9n$ and $12n$, respectively. It follows from Lemma 2.2 that

$$\begin{aligned} i_f^+(P(3n;\sigma_2)) &= 9n - 4E_f(1,1), \\ i_f^+(P(4n;\sigma_3)) &= 12n - 4E_f(1,1), \end{aligned}$$

where f is a friendly labeling.

4.1. Full Friendly Index Set of $P(3n;\sigma_2)$

First of all we find the extreme friendly indices of $P(3n;\sigma_2)$.

Theorem 4.1. *Let f be a friendly labeling of $P(3n;\sigma_2)$. The minimum value of $E_f(1,1)$ is $n/2$ if n is even and $(n+1)/2$ if n is odd. Thus the maximum value of $i_f^+(P(3n;\sigma_2))$ is $7n$ if n is even and $7n-2$ if n is odd.*

Proof. Note that all the C^i's are C_3 and $P(3n;\sigma_2)$ contains n C_3's. Since C_3 is an odd cycle, it follows from Corollary 3.1 that there are at least n 0-edges in $P(3n;\sigma_2)$. Since $E_f(1,1) = E_f(0,0)$ by Lemma 2.2, it follows that, if n is even, then $E_f(1,1) \geq \frac{n}{2}$; if n is odd, $E_f(1,1) \geq \frac{n+1}{2}$.

For even n, define a labeling $f_{\max}$ on $P(3n;\sigma_2)$ by

$$f_{\max}(v) = \begin{cases} 1, & \text{if } v = x_{2i} \text{ or } y_{2i-1} \text{ for } 1 \leq i \leq \frac{3n}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $f_{\max}$ is friendly. Since only $\frac{n}{2}$ of the cycles of σ_2 contain two odd integers, it follows that $E_{f_{\max}}(1,1) = \frac{n}{2}$. Thus the maximum value of $i_f^+(P(3n;\sigma_2))$ is $i_{f_{\max}}^+(P(3n;\sigma_2)) = 9n - 4(\frac{n}{2}) = 7n$ by Lemma 2.2.

For odd n, define a labeling $g_{\max}$ on $P(3n;\sigma_2)$ by

$$g_{\max}(v) = \begin{cases} 1, & \text{if } v = x_{2i} \text{ or } y_{2i-1} \text{ for } 1 \le i \le \frac{3n+1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $g_{\max}$ is friendly. Since only $\frac{n+1}{2}$ of the cycles of σ_2 contain two odd integers, it follows that $E_{f_{\max}}(1,1) = \frac{n+1}{2}$. Thus the maximum value of $i_f^+(P(3n;\sigma_2))$ is $i_{g_{\max}}^+(P(3n;\sigma_2)) = 9n - 4(\frac{n+1}{2}) = 7n - 2$ by Lemma 2.2. □

Theorem 4.2. *Let f be a friendly labeling of $P(3n;\sigma_2)$. The maximum value of $E_f(1,1)$ is $4n-1+n/2$ if n is even and $4n-2+(n-1)/2$ if n is odd. Thus the minimum value of $i_f^+(P(3n;\sigma_2))$ is $-9n+4$ if n is even and $-9n+10$ if n is odd.*

Proof. For even n, define a labeling $f_{\min}$ on $P(3n;\sigma_2)$ by

$$f_{\min}(v) = \begin{cases} 1, & \text{if } v = x_i \text{ or } y_i \text{ for } 1 \le i \le \frac{3n}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $f_{\min}$ is friendly. There are $\frac{3n}{2} - 1$ $(1,1)$-edges on the outer circle, $\frac{3n}{2}$ $(1,1)$-edges on the edges x_iy_i and $\frac{3n}{2}$ $(1,1)$-edges on the C^i's, so $E_{f_{\min}}(1,1) = \frac{3n}{2} - 1 + \frac{3n}{2} + \frac{3n}{2} = 4n - 1 + \frac{n}{2}$.

For odd n, define a labeling $g_{\min}$ on $P(3n;\sigma_2)$ by

$$g_{\min}(v) = \begin{cases} 1, & \text{if } v = x_i \text{ for } 1 \le i \le \frac{3(n-1)}{2}; \\ 1, & \text{if } v = y_i \text{ for } 1 \le i \le \frac{3(n-1)}{2} \text{ or } 3n-2 \le i \le 3n; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $g_{\min}$ is friendly. There are $\frac{3(n-1)}{2} - 1$ $(1,1)$-edges on the outer circle, $\frac{3(n-1)}{2}$ $(1,1)$-edges on the x_iy_i edges and $\frac{3(n-1)}{2} + 3$ $(1,1)$-edges on the C^i's, so $E_{g_{\min}}(1,1) = 4n - 2 + \frac{n-1}{2}$.

Let f be a friendly labeling of $P(3n;\sigma_2)$, and G the subgraph of $P(3n;\sigma_2)$ induced by all the 1-vertices. Then G is of order $3n$. For each $0 \le j \le 3$, denote by a_j the number of vertices of degree j in G. By Handshaking Lemma, we have

$$\begin{aligned} 2E_f(1,1) &= a_1 + 2a_2 + 3a_3 \\ &= (a_1 + a_2 + a_3) + a_2 + 2a_3 \\ &= 3n - a_0 + a_2 + 2a_3. \end{aligned}$$

If all the 1-vertices lie on the outer cycle C, then $G = C_{3n}$ and therefore $E_f(1,1) = 3n < E_{g_{\min}}(1,1)$. Thus, to maximize $E_f(1,1)$ we can assume the outer cycle C contains at least one 0-vertex. Let k be the number of 1-vertices contained in C.

Case 1: If $k \ge 2$, then $G \cap C$ is a forest of order greater than 1. Then G contains at least two vertices of degree less than 3.

Case 2: If $k = 1$, then H contains only one 0-vertex, which is adjacent to two 1-vertices in H. Thus G contains at least three vertices of degree less than 3.

In either case, we have $a_3 \leq 3n - 2$. Thus we have to maximize

$$E_f(1,1) = \frac{3n - a_0 + a_2}{2} + a_3$$

subject to

$$0 \leq a_0 + a_2 + a_3 \leq 3n \text{ and } a_3 \leq 3n - 2.$$

By the simplex method, the algebraic maximum value is $4n + \frac{n}{2} - 1$ when $(a_0, a_1, a_2, a_3) = (0, 0, 2, 3n - 2)$. Suppose n is even. By considering the friendly labeling $f_{\min}$, the maximum value of $E_f(1,1)$ is $4n + \frac{n}{2} - 1$, so the minimum value of $i_f^+(P(3n;\sigma_2))$ is

$$i_{g_{\min}}^+(P(3n;\sigma_2)) = 12n - 4E_{g_{\min}}(1,1) = 9n - 4\left(4n + \frac{n}{2} - 1\right) = -9n + 4.$$

Suppose n is odd. Since the algebraic maximum value is $4n + \frac{n}{2} - 1$, we have $E_f(1,1) \leq 4n - 1 + \frac{n-1}{2}$. Suppose $E_f(1,1) = 4n - 1 + \frac{n-1}{2}$. Then either $(a_0, a_1, a_2, a_3) = (0, 1, 1, 3n - 2)$ or $(a_0, a_1, a_2, a_3) = (0, 0, 3, 3n - 3)$.

Case 1: Suppose $(a_0, a_1, a_2, a_3) = (0, 1, 1, 3n - 2)$. From the above argument, the two vertices of degree less than 3 are in the outer cycle C. All the 1-vertices in H, considered as a subgraph of G, are of degree 3 and consequently there are $3k$ 1-vertices in H for some $k \in \mathbb{N}$, as H is the disjoint union of n C_3's. Otherwise, not all of them are of degree 3. Each of these $3k$ 1-vertices in H is adjacent to a 1-vertex in C of degree 3 except the one of degree 2. For the vertex of degree 1, it is adjacent to a 1-vertex in C. Therefore, $3n - 2 = a_3 = 3k + 3k - 1 = 6k - 1$, which is impossible.

Case 2: Suppose $(a_0, a_1, a_2, a_3) = (0, 0, 3, 3n - 3)$. Since each vertex in $G \cap C$ is adjacent to at most one vertex in H, $G \cap C$ does not contain any isolated vertex. Since $G \cap C$ is a forest and $a_2 = 3$, $G \cap C$ must be a path.

Suppose one of the vertices of degree 2 in G lies in H, say y_i. Since each vertex in $G \cap C$ is adjacent to a vertex in H, y_i is adjacent to a 0-vertex contained in C; otherwise there is a 0-vertex in H, which will produce one more vertex of degree less than 3. Thus there are $(3k - 1)$ 1-vertices of degree 3 in H for some $k \in \mathbb{N}$. Each of these $(3k - 1)$ 1-vertices is adjacent to a 1-vertex in C of degree is 3 except two 1-vertices of degree 2. Therefore $3n - 3 = a_3 = 3k - 1 + 3k - 1 - 2 = 6k - 4$, which is impossible.

Suppose one of the vertices of degree 2 in G lies in C, say x_i. Since each vertex in $G \cap C$ except x_i is adjacent to a vertex in H, and all the vertices in H are of degree 3 in G, there are $3k$ such vertices for some $k \in \mathbb{N}$, and each of them is adjacent to a vertex in $G \cap C$. Therefore, $3n - 3 = a_3 = 3k + 3k - 2 = 6k - 2$, which is impossible.

Thus $E_f(1,1) \neq 4n - 1 + \frac{n-1}{2}$, and therefore $E_f(1,1) \leq 4n - 2 + \frac{n-1}{2}$ when n is odd. By considering the friendly labeling $g_{\min}$, the maximum value of $E_f(1,1)$ is $E_{g_{\min}}(1,1) = 4n - 2 + \frac{n-1}{2}$, and therefore the minimum value of $i_f^+(P(3n;\sigma_2))$ is

$$i_{g_{\min}}^+(P(3n;\sigma_2)) = 9n - 4E_{g_{\min}}(1,1) = 12n - 4\left(4n - 2 + \frac{n-1}{2}\right) = -9n + 10$$

by Lemma 2.2. $\square$

Theorem 4.3. *For $n \geq 1$,*

$$E(P(3n;\sigma_2)) = \begin{cases} [n/2, 4n-1+n/2], & \text{if } n \text{ is even;} \\ [n+1/2, 4n-2+(n-1)/2], & \text{if } n \text{ is odd.} \end{cases}$$

Thus

$$\mathrm{FFI}(P(3n;\sigma_2)) = \begin{cases} \{9n-4i \mid i \in [n/2, 4n-1+n/2]\}, & \text{if } n \text{ is even;} \\ \{9n-4i \mid i \in [(n+1)/2, 4n-2+(n-1)/2]\}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose n is even. Consider the friendly labeling $f_{\max}$ defined in the proof of Theorem 4.1. Then $E_{f_{\max}}(1,1) = \frac{n}{2}$. Interchange the labelings of y_{6j-1} with y_{6j} consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by f_j^1 with $f_0^1 = f_{\max}$. By Lemma 3.3, we have $E_{f_1} = E_{f_0} + 1$. It is easy to see that $E_{f_j^1}(1,1) = E_{f_{j-1}^1}(1,1)+1$ for each $1 \leq j \leq n/2$. Thus $[\frac{n}{2}, n] \subseteq E(P(3n;\sigma_2))$.

The current labeling on $P(3n;\sigma_2)$ is $f_{n/2}^1$, and $E_{f_{n/2}^1}(1,1) = n$. Interchange the labelings of y_{6j-5} with y_{6j-4} consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by f_j^2 with $f_0^2 = f_{n/2}^1$. By Lemma 3.3, we have $E_{f_1^2}(1,1) = E_{f_0^2} + 1 = n+1$. It is easy to see that $E_{f_j^2}(1,1) = E_{f_{j-1}^2}(1,1) + 1$ for each $1 \leq j \leq n/2$. Thus $[n, n+\frac{n}{2}] \subseteq E(P(3n;\sigma_2))$.

The current labeling on $P(3n;\sigma_2)$ is $f_{n/2}^2$, and $E_{f_{n/2}^2}(1,1) = n+\frac{n}{2}$. Interchange the labelings of y_{6j} with y_{6j-5} consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by f_j^3 with $f_0^3 = f_{n/2}^2$. By Lemma 3.2, we have $E_{f_1^3}(1,1) = E_{f_0^3} + 1 = n + \frac{n}{2} + 1$. It is easy to see that $E_{f_j^3}(1,1) = E_{f_{j-1}^3}(1,1) + 1$ for each $1 \leq j \leq n/2$. Thus $[n+\frac{n}{2}, 2n] \subseteq E(P(3n;\sigma_2))$.

The current labeling on $P(3n;\sigma_2)$ is $f_{n/2}^3$, and $E_{f_{n/2}^3}(1,1) = 2n$. Interchange the labelings of y_{6j-1} with y_{6j-2} consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by f_j^4 with $f_0^4 = f_{n/2}^3$. By Lemma 3.3, we have $E_{f_1^4}(1,1) = E_{f_0^4} + 1 = 2n+1$. It is easy to see that $E_{f_j^4}(1,1) = E_{f_{j-1}^4}(1,1)+1$ for each $1 \leq j \leq n/2$. Thus $[2n, 2n+\frac{n}{2}] \subseteq E(P(3n;\sigma_2))$.

The current labeling on $P(3n;\sigma_2)$ is $f_{n/2}^4$, and $E_{f_{n/2}^4}(1,1) = 2n+\frac{n}{2}$. Interchange the labelings of y_{6j} with y_{6j-3} consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by f_j^5 with $f_0^5 = f_{n/2}^4$. By Lemma 3.2, we have $E_{f_1^5}(1,1) = E_{f_0^5} + 1 = 2n + \frac{n}{2} + 1$. It is easy to see that $E_{f_j^5}(1,1) = E_{f_{j-1}^5}(1,1) + 1$ for each $1 \leq j \leq n/2$. Thus $[2n+\frac{n}{2}, 3n] \subseteq E(P(3n;\sigma_2))$.

The current labeling on $P(3n;\sigma_2)$ is $f_{n/2}^5$, and $E_{f_{n/2}^5}(1,1) = 3n$. Interchange the labelings of y_{6j-1} with y_{6j-2} consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by f_j^6 with $f_0^6 = f_{n/2}^5$. By Lemma 3.3, we have $E_{f_1^6}(1,1) = E_{f_0^6} + 1 = 3n+1$. It is easy to see that $E_{f_j^6}(1,1) = E_{f_{j-1}^6}(1,1) + 1$ for each $1 \leq j \leq \frac{n}{2}$. Thus $[3n, 3n+\frac{n}{2}] \subseteq E(P(3n;\sigma_2))$.

The current labeling on $P(3n;\sigma_2)$ is $f_{n/2}^6$, and $E_{f_{n/2}^6}(1,1) = 3n+\frac{n}{2}$. Interchange the labelings of y_{6j-2} with y_{6j-5} consecutively for $1 \leq j \leq n/2$, and denote the resulting labeling by f_j^7 with $f_0^7 = f_{n/2}^6$. By Lemma 3.2, we have $E_{f_1^7}(1,1) = E_{f_0^7}+1 = 3n+\frac{n}{2}+1$.

It is easy to see that $E_{f_j^7}(1,1) = E_{f_{j-1}^7}(1,1)+1$ for each $1 \leq j \leq n/2$. Thus $[3n+\frac{n}{2}, 4n] \subseteq E(P(3n;\sigma_2))$.

The current labeling on $P(3n;\sigma_2)$ is $f_{n/2}^7$, and $E_{f_{n/2}^6}(1,1) = 4n$. For each $1 \leq i \leq n$, define a set

$$V_i = \{x_{3i-2}, x_{3i-1}, x_{3i}, y_{3i-2}, y_{3i-1}, y_{3i}\}.$$

Note that all the 1-vertices are contained in V_{2i-1} for $1 \leq i \leq n/2$. For each $1 \leq i \leq n/2-1$, interchange the labelings of V_{i+1} with V_{2i+1} consecutively, i.e., interchange the labelings of $x_{3(i+1)-k}$ with $x_{3(2i+1)-k}$ and $y_{3(i+1)-k}$ with $y_{3(2i+1)-k}$ for each $0 \leq k \leq 2$ simultaneously. Denote the resulting labeling by f_j^8 with $f_0^8 = f_{n/2}^7$. It is easy to see that $E_{f_1^8}(1,1) = 4n+1$ and $E_{f_j^8}(1,1) = E_{f_{j-1}^8}(1,1)+1$ for each $1 \leq j \leq n/2-1$. Thus $[4n, 4n-1+\frac{n}{2}] \subseteq E(P(3n;\sigma_2))$.

Combining all the above cases we get the desired result for $E(P(3n;\sigma_2))$ for even n.

Now suppose n is odd. Perform similar procedures as in the case where n is even, except we start with the friendly labeling $g_{\max}$ defined in the proof of Theorem 4.1, and after the 7-th step we interchange the labelings of x_{3n-1} with y_{3n-2}, and denote by $\widetilde{g}$ the resulting labeling. Since $E_{g^7_{(n-1)/2}}(1,1) = 4n-2$, it follows from Lemma 3.2 that $E_{\widetilde{g}}(1,1) = E_{g^7_{(n-1)/2}}(1,1)+1 = 4n-1$. Now perform the last procedure in the case where n is even to get the desired result for $E(P(3n;\sigma_2))$ for odd n. $\square$

Corollary 4.1. *For* $n \geq 1$,

$$\mathrm{FI}(P(3n;\sigma_2)) = \begin{cases} \{9n-4i \mid n/2 \leq i \leq 9n/4\}, & \textit{if } n \textit{ is even;} \\ \{9n-4i \mid (n+1)/2 \leq i \leq (9n+1)/4\}, & \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 4.2. *For* $n \geq 1$,

$$\mathrm{FPCI}(P(3n;\sigma_2)) = \begin{cases} \{-2j \mid j \in [1,4n]\}, & \textit{if } n \textit{ is even;} \\ \{1-2j \mid j \in [3,4n]\}, & \textit{if } n \textit{ is odd.} \end{cases}$$

4.2. Full Friendly Index Set of $P(4n;\sigma_3)$

First of all we find the extreme friendly indices of $P(4n;\sigma_3)$.

Theorem 4.4. *Let f be a friendly labeling of $P(4n;\sigma_3)$. The minimum value of $E_f(1,1)$ is 0. Thus the maximum value of $i_f^+(P(4n;\sigma_3))$ is $12n$.*

Proof. For any graph, $E_f(1,1) \geq 0$. Define a friendly labeling $f_{\max}$ on $P(4n;\sigma_3)$ by

$$f_{\max}(v) = \begin{cases} 1, & \text{if } v = x_{2i} \text{ or } v = y_{2i-1} \text{ for } 1 \leq i \leq 2n; \\ 0, & \text{otherwise.} \end{cases}$$

Then $E_{f_{\max}}(1,1) = 0$. Therefore the maximum value of $i_f^+(P(4n;\sigma_3))$ is $12n$ by Lemma 2.2. $\square$

Theorem 4.5. *Let f be a friendly labeling of $P(4n;\sigma_3)$. The maximum value of $E_f(1,1)$ is $6n-1$ for even n, and is $6n-2$ for odd n. Thus the minimum value of $i_f^+(P(4n;\sigma_3))$ is $-12n+4$ for even n and is $-12n+8$ for odd n.*

Proof. Define a labeling $f_{\min}$ on $P(4n;\sigma_3)$ as follows. If n is even, then $f_{\min}$ is defined by

$$f_{\min}(v) = \begin{cases} 1, & \text{if } v = x_i \text{ or } v = y_i \text{ for } 1 \le i \le 2n; \\ 0, & \text{otherwise,} \end{cases}$$

if n is odd, then $f_{\min}$ is defined by

$$f_{\min}(v) = \begin{cases} 1, & \text{if } v = x_i \text{ or } v = y_i \text{ for } i \in [1, 2n-2] \cup \{4n-1, 4n\}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $f_{\min}$ is friendly in both cases and $E_{f_{\min}}(1,1) = 12(\frac{n}{2}) - 1 = 6n-1$ if n is even, and $E_{f_{\min}}(1,1) = 12(\frac{n-1}{2}) + 4 = 6n-2$ if n is odd.

Let f be a friendly labeling of $P(4n;\sigma_3)$, and G the subgraph of $P(4n;\sigma_3)$ induced by all the 1-vertices. Then G is of order $4n$. For each $0 \le j \le 3$, denote by a_j the number of vertices of degree j in G. By Handshaking Lemma, we have

$$\begin{aligned} 2E_f(1,1) &= a_1 + 2a_2 = a_3 \\ &= (a_1 + a_2 + a_3) + a_2 + 2a_3 \\ &= 4n - a_0 + a_2 + 2a_3. \end{aligned}$$

If all the 1-vertices lie on the outer cycle C, then G is C_{4n} and $E_f(1,1) = 4n < E_{f_{\min}}(1,1)$. Thus, to maximize $E_f(1,1)$ we can assume C contains at least one 0-vertices. By a similar argument in the proof of Theorem 4.2 we may assume $a_3 \le 4n-2$. We maximize

$$E_f(1,1) = 2n + \frac{-a_0 + a_2}{2} + a_3$$

subject to

$$0 \le a_0 + a_2 + a_3 \le 2n+1 \text{ and } a_3 \le 4n-2.$$

By the simplex method, the algebraic maximum value is $6n-1$ when $(a_0, a_1, a_2, a_3) = (0, 0, 2, 4n-2)$. For even n, by considering the friendly labeling $f_{\min}$, the maximum of $E_f(1,1)$ is $E_{f_{\min}}(1,1) = 6n-1$. Thus the minimum value of $i_f^+(P(4n;\sigma_3))$ is

$$i_{f_{\min}}^+(P(4n;\sigma_3)) = 12n - 4E_{f_{\min}}(1,1) = 12n - 4(6n-1) = -12n+4.$$

For odd n, suppose $E_f(1,1) = 6n-1$. Then we have $(a_0, a_1, a_2, a_3) = (0, 0, 2, 4n-2)$. Similar to the proof of Theorem 4.2, we know that $G \cap C$ is a path. Since $a_2 = 2$, all the 1-vertices in H are of degree 3, so there are $4k$ 1-vertices in H, where $k \in \mathbb{N}$, as H is the disjoint union of n C_4's. Each of these $4k$ 1-vertices in H is adjacent to one 1-vertex in C of degree 3 except two 1-vertices. Thus $4n-2 = a_3 = 4k + 4k - 2 = 8k - 2$. It implies $n = 2k$, which contradicts n being odd. Thus for odd n, $E_f(1,1) \ne 6n-1$, and therefore $E_f(1,1) \le 6n-2$. By considering the friendly labeling $f_{\min}$ in the case of odd n, the maximum value of $E_f(1,1)$ is $E_{f_{\min}}(1,1) = 6n-2$. So the minimum value of $i_f^+(P(4n;\sigma_3))$ is $i_{f_{\min}}^+(P(4n;\sigma_3) = 12n - 4E_{f_{\min}}(1,1) = 12n - 4(6n-2) = -12n+8$ by Lemma 2.2. $\square$

Theorem 4.6.

$$E(P(4n;\sigma_3)) = \begin{cases} [0, 6n-1], & \textit{if } n \textit{ is even;} \\ [0, 6n-2], & \textit{if } n \textit{ is odd.} \end{cases}$$

Thus

$$\mathrm{FFI}(P(4n;\sigma_3)) = \begin{cases} \{12n-4i \mid 0 \le i \le 6n-1\}, & \textit{if } n \textit{ is even;} \\ \{12n-4i \mid 0 \le i \le 6n-2\}, & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. For each $1 \le i \le n$, define a set

$$V_i = \{x_{4i-3}, x_{4i-2}, x_{4i-1}, x_{4i}, y_{4i-3}, y_{4i-2}, y_{4i-1}, y_{4i}\}.$$

Suppose n is even. Consider the friendly labeling $f_{\max}$ defined in the proof of Theorem 4.4. Then $E_{f_{\max}}(1,1) = 0$. Define a labeling f_1^1 on $P(4n;\sigma_3)$ by

$$f_1^1(v) = \begin{cases} f_{\max}(v) + 1, & \text{if } v \in V_2; \\ f_{\max}(v), & \text{otherwise.} \end{cases}$$

For each $2 \le i \le n/2$, define a labeling f_i^1 on $P(4n;\sigma_3)$ by

$$f_i^1(v) = \begin{cases} f_{\max}(v) + 1, & \text{if } v \in V_{2i}; \\ f_{i-1}^1(v), & \text{otherwise.} \end{cases}$$

Note that f_i^1 is friendly for all $1 \le i \le n/2$. It is easy to see that $E_{f_1^1}(1,1) = 1$ and $E_{f_i^1}(1,1) = E_{f_{i-1}}(1,1) + 1$ for each $2 \le i \le n/2$. Thus $[0, \frac{n}{2}] \subseteq E(P(3n;\sigma_2))$.

The current labeling on $P(4n;\sigma_3)$ is $f_{n/2}^1$, and $E_{f_{n/2}^1}(1,1) = \frac{n}{2}$. Interchange the labelings of x_{8j-7} with y_{8j-7} consecutively for $1 \le j \le n/2$, and denote the resulting labeling by f_j^2 with $f_0^2 = f_{n/2}^1$. By Lemma 3.3, we have $E_{f_1^2}(1,1) = E_{f_0^2} + 1 = \frac{n}{2} + 1$. It is easy to see that $E_{f_j^2}(1,1) = E_{f_{j-1}^2}(1,1) + 1$ for each $1 \le j \le n/2$. Thus $[\frac{n}{2}, n] \subseteq E(P(4n;\sigma_3))$.

The current labeling on $P(4n;\sigma_3)$ is $f_{n/2}^2$, and $E_{f_{n/2}^2}(1,1) = n$. Interchange the labelings of x_{8j-7} with y_{8j-6} consecutively for $1 \le j \le n/2$, and denote the resulting labeling by f_j^3 with $f_0^3 = f_{n/2}^2$. By Lemma 3.2, we have $E_{f_1^3}(1,1) = E_{f_0^3} + 1 = n+1$. It is easy to see that $E_{f_j^3}(1,1) = E_{f_{j-1}^3}(1,1) + 1$ for each $1 \le j \le n/2$. Thus $[n, n+\frac{n}{2}] \subseteq E(P(4n;\sigma_3))$.

The current labeling on $P(4n;\sigma_3)$ is $f_{n/2}^3$, and $E_{f_{n/2}^3}(1,1) = n + \frac{n}{2}$. Interchange the labelings of x_{8j} with y_{8j} consecutively for $1 \le j \le n/2$, and denote the resulting labeling by f_j^4 with $f_0^4 = f_{n/2}^3$. By Lemma 3.3, we have $E_{f_1^4}(1,1) = E_{f_0^4} + 1 = n + \frac{n}{2} + 1$. It is easy to see that $E_{f_j^4}(1,1) = E_{f_{j-1}^4}(1,1) + 1$ for each $1 \le j \le n/2$. Thus $[n+\frac{n}{2}, 2n] \subseteq E(P(4n;\sigma_3))$.

The current labeling on $P(4n;\sigma_3)$ is $f_{n/2}^4$, and $E_{f_{n/2}^4}(1,1) = 2n$. Interchange the labelings of x_{8j} with y_{8j-1} consecutively for $1 \le j \le n/2$, and denote the resulting labeling by f_j^5 with $f_0^5 = f_{n/2}^4$. By Lemma 3.2, we have $E_{f_1^5}(1,1) = E_{f_0^5} + 1 = 2n+1$. It is easy to see that $E_{f_j^5}(1,1) = E_{f_{j-1}^5}(1,1) + 1$ for each $1 \le j \le n/2$. Thus $[2n, 2n+\frac{n}{2}] \subseteq E(P(4n;\sigma_3))$.

The current labeling on $P(4n;\sigma_3)$ is $f^5_{n/2}$, and $E_{f^5_{n/2}}(1,1) = 2n + \frac{n}{2}$. Interchange the labelings of x_{8j-2} with y_{8j-6} consecutively for $1 \le j \le n/2$, and denote the resulting labeling by f^6_j with $f^6_0 = f^5_{n/2}$. By Lemma 3.2, we have $E_{f^6_1}(1,1) = E_{f^6_0}+1 = 2n+\frac{n}{2}+1$. It is easy to see that $E_{f^6_j}(1,1) = E_{f^6_{j-1}}(1,1)+1$ for each $1 \le j \le n/2$. Thus $[2n+\frac{n}{2}, 3n] \subseteq E(P(4n;\sigma_3))$.

The current labeling on $P(4n;\sigma_3)$ is $f^6_{\frac{n}{2}}$, and $E_{f^6_{n/2}}(1,1) = 3n$. Interchange the labelings of x_{8j-6} with y_{8j-6} consecutively for $1 \le j \le n/2$, and denote the resulting labeling by f^7_j with $f^7_0 = f^6_{n/2}$. By Lemma 3.3, we have $E_{f^7_1}(1,1) = E_{f^7_0} + 1 = 3n + 1$. It is easy to see that $E_{f^7_j}(1,1) = E_{f^7_{j-1}}(1,1) + 1$ for each $1 \le j \le n/2$. Thus $[3n, 3n + \frac{n}{2}] \subseteq E(P(4n;\sigma_3))$.

The current labeling on $P(4n;\sigma_3)$ is $f^7_{n/2}$, and $E_{f^7_{n/2}}(1,1) = 3n + \frac{n}{2}$. Interchange the labelings of y_{8j-3} with y_{8j-6} consecutively for $1 \le j \le n/2$, and denote the resulting labeling by f^8_j with $f^8_0 = f^7_{n/2}$. By Lemma 3.2, we have $E_{f^8_1}(1,1) = E_{f^8_0}+1 = 3n+\frac{n}{2}+1$. It is easy to see that $E_{f^8_j}(1,1) = E_{f^8_{j-1}}(1,1)+1$ for each $1 \le j \le n/2$. Thus $[3n+\frac{n}{2}, 4n] \subseteq E(P(4n;\sigma_3))$.

The current labeling on $P(4n;\sigma_3)$ is $f^8_{n/2}$, and $E_{f^8_{n/2}}(1,1) = 4n$. Interchange the labelings of x_{8j-5} with y_{8j-5} consecutively for $1 \le j \le n/2$, and denote the resulting labeling by f^9_j with $f^9_0 = f^8_{n/2}$. By Lemma 3.3, we have $E_{f^9_1}(1,1) = E_{f^9_0} + 1 = 4n+1$. It is easy to see that $E_{f^9_j}(1,1) = E_{f^9_{j-1}}(1,1) + 1$ for each $1 \le j \le n/2$. Thus $[4n, 4n + \frac{n}{2}] \subseteq E(P(4n;\sigma_3))$.

The current labeling on $P(4n;\sigma_3)$ is $f^9_{n/2}$, and $E_{f^9_{n/2}}(1,1) = 4n + \frac{n}{2}$. Interchange the labelings of x_{8j-5} with y_{8j} consecutively for $1 \le j \le n/2$, and denote the resulting labeling by f^{10}_j with $f^{10}_0 = f^9_{n/2}$. By Lemma 3.2, we have $E_{f^{10}_1}(1,1) = E_{f^{10}_0} + 1 = 4n + \frac{n}{2} + 1$. It is easy to see that $E_{f^{10}_j}(1,1) = E_{f^{10}_{j-1}}(1,1) + 1$ for each $1 \le j \le n/2$. Thus $[4n + \frac{n}{2}, 5n] \subseteq E(P(4n;\sigma_3))$.

The current labeling on $P(4n;\sigma_3)$ is $f^{10}_{\frac{n}{2}}$, and $E_{f^{10}_{n/2}}(1,1) = 5n$. Interchange the labelings of x_{8j-4} with y_{8j} consecutively for $1 \le j \le n/2$, and denote the resulting labeling by f^{11}_j with $f^{11}_0 = f^{10}_{n/2}$. By Lemma 3.2, we have $E_{f^{11}_1}(1,1) = E_{f^{11}_0} + 1 = 5n + 1$. It is easy to see that $E_{f^{11}_j}(1,1) = E_{f^{11}_{j-1}}(1,1) + 1$ for each $1 \le j \le n/2$. Thus $[5n, 5n + \frac{n}{2}] \subseteq E(P(4n;\sigma_3))$.

The current labeling on $P(4n;\sigma_3)$ is $f^{11}_{\frac{n}{2}}$, and $E_{f^{11}_{\frac{n}{2}}}(1,1) = 5n + \frac{n}{2}$. For each $1 \le i \le n/2$, all the vertices in V_{2i} are 1-vertices. For each $1 \le i \le n/2 - 1$, switch the labelings of V_{2i} with V_i, and denote the resulting labeling by f^{12}_j with $f^{12}_0 = f^{11}_{n/2}$. Then $E_{f^{12}_1}(1,1) = E_{f^{11}_1}(1,1)+1 = 5n+\frac{n}{2}+1$. It is easy to see that $E_{f^{12}_j}(1,1) = E_{f^{12}_{j-1}}(1,1)+1$ for each $1 \le j \le n/2$. Thus $[5n + \frac{n}{2}, 6n - 1] \subseteq E(P(4n;\sigma_3))$.

Combining all the above cases we get the desired result for $E(P(3n;\sigma_2))$ for even n.

Now suppose n is odd. Perform similar procedures as in the case where n is even, except the followings. After the first step, define a labeling $\widetilde{f}$ on $P(4n;\sigma_3)$ by

$$f^1_i(v) = \begin{cases} f_{\max}(v) + 1, & \text{if } v \in V_n; \\ f_{\max}(v), & \text{otherwise.} \end{cases}$$

$\widetilde{f}^1$ is friendly. Note that $E_{\widetilde{f}}(1,1) = (n-1)/2$. Then, after the fourth step, we interchange the labelings of x_{4n} and y_{4n}, and denote by $\widetilde{f}_1^4$ the resulting labeling. By Lemma 3.3, $E_{\widetilde{f}_1^4}(1,1) = n+(n-1)/2$. Then interchange the labelings of x_{4n-3} and y_{4n}, and denote by $\widetilde{f}_2^4$ the resulting labeling. By Lemma 3.2, $E_{\widetilde{f}_2^4}(1,1) = E_{\widetilde{f}_1^4}(1,1)+1 = n+(n-1)/2+1$. After the last step, interchange the labelings of x_{4n-2} and y_{4n-1}, and denote by $\widetilde{f}^{12}$ the resulting labeling. By Lemma 3.2, $E_{\widetilde{f}^{12}}(1,1) = 6n-2$. Thus we get the desired result for $E(P(3n;\sigma_2))$ for odd n. □

Corollary 4.3. $\mathrm{FI}(P(4n;\sigma_3)) = \{48-4i \mid 0 \le i \le 3n\}$.

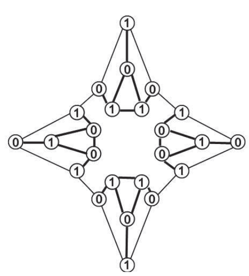
Corollary 4.4.

$$\mathrm{FPCI}(P(4n;\sigma_3)) = \begin{cases} \{-2j \mid j \in [1,6n]\}, & \textit{if } n \textit{ is even;} \\ \{-2j \mid j \in [2,6n]\}, & \textit{if } n \textit{ is odd.} \end{cases}$$

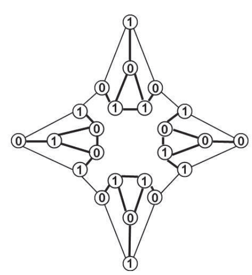
Appendix

In this appendix we illustrate the labeling procedure of Theorem 4.3 for $P(12;\sigma_2)$ and the one of Theorem 4.6 for $P(16,\sigma_3)$.

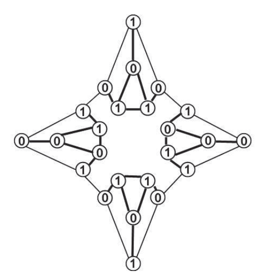
For $P(12;\sigma_2)$, we have



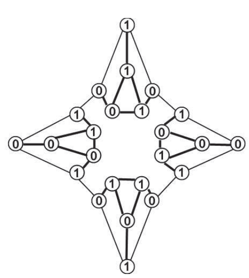
$E_{f_{\max}}(1,1) = 2$



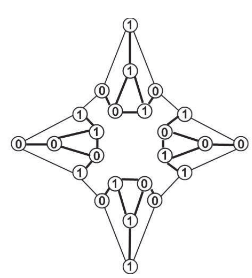
$E_{f_1^1}(1,1) = 3$



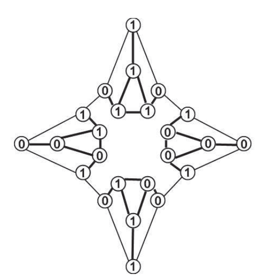
$E_{f_2^1}(1,1) = 4$



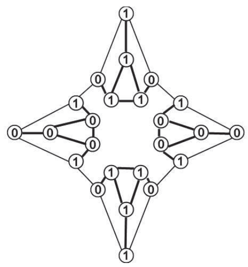
$E_{f_1^2}(1,1) = 5$



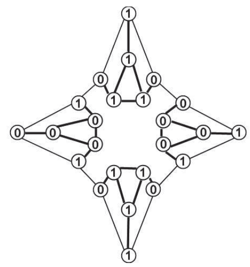
$E_{f_2^2}(1,1) = 6$



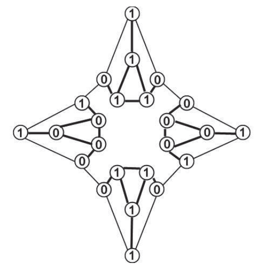
$E_{f_1^3}(1,1) = 7$



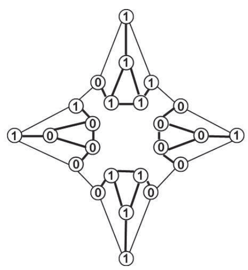
$E_{f_2^3}(1,1) = 8$



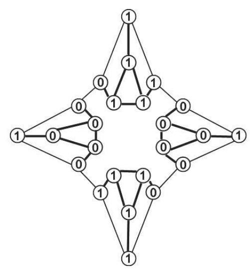
$E_{f_1^4}(1,1) = 9$



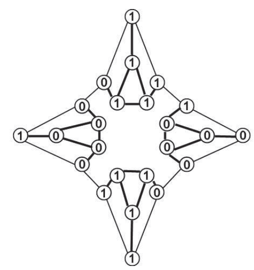
$E_{f_2^4}(1,1) = 10$



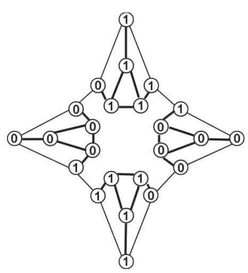
$E_{f_1^5}(1,1) = 11$



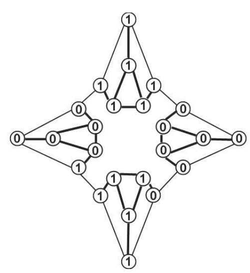
$E_{f_2^5}(1,1) = 12$



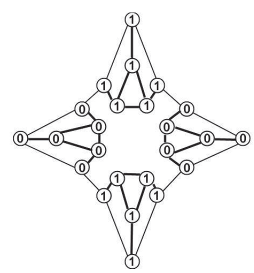
$E_{f_1^6}(1,1) = 13$



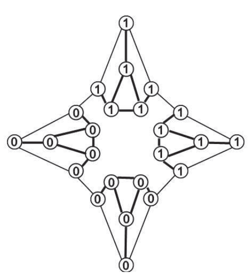
$E_{f_2^6}(1,1) = 14$



$E_{f_1^7}(1,1) = 15$

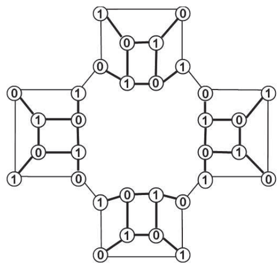


$E_{f_2^7}(1,1) = 16$

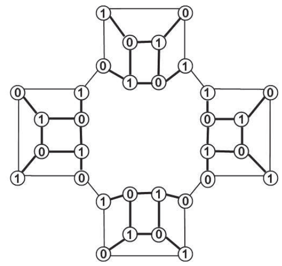


$E_{f_1^8}(1,1) = 17$

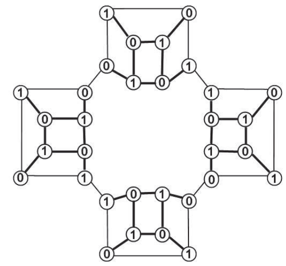
For $P(16;\sigma_3)$ we have



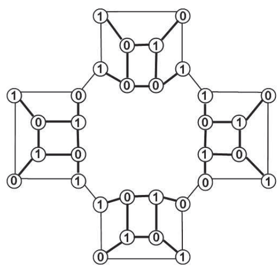
$E_{f_{\max}}(1,1)=0$



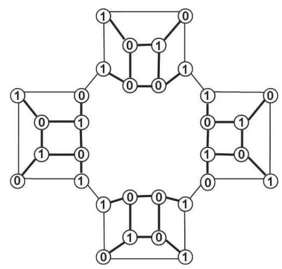
$E_{f_1^1}(1,1)=1$



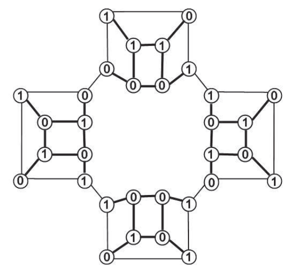
$E_{f_2^1}(1,1)=2$



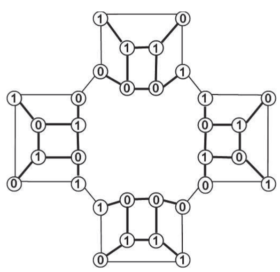
$E_{g_1^2}(1,1)=3$



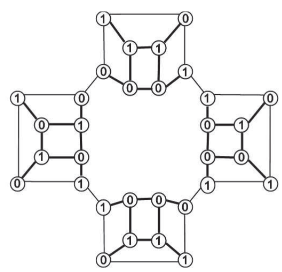
$E_{f_2^2}(1,1)=4$



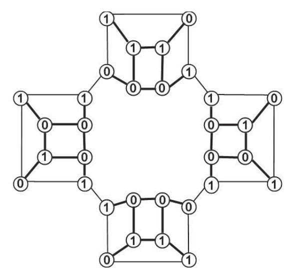
$E_{f_1^3}(1,1)=5$



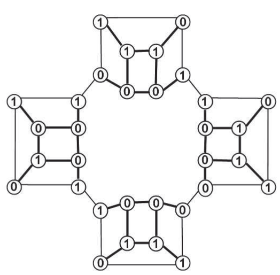
$E_{f_2^3}(1,1)=6$



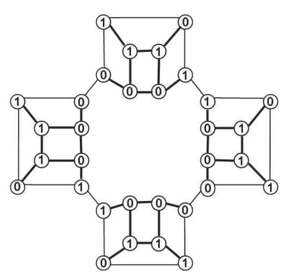
$E_{f_1^4}(1,1)=7$



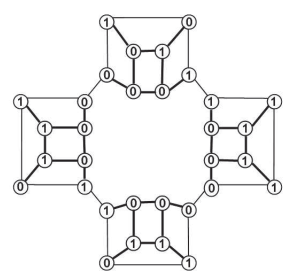
$E_{f_2^4}(1,1)=8$



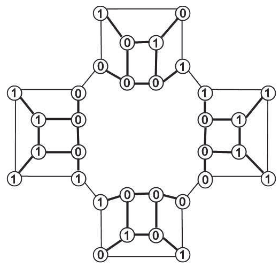
$E_{f_1^5}(1,1)=9$



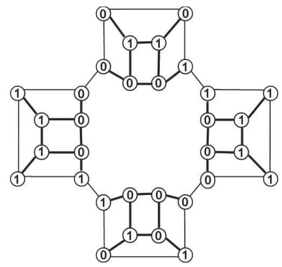
$E_{f_2^5}(1,1)=10$



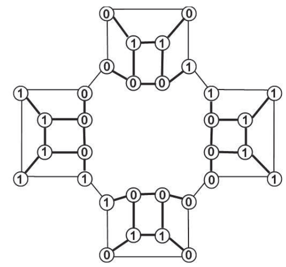
$E_{f_1^6}(1,1)=11$



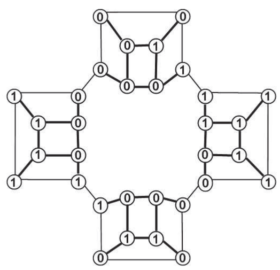
$E_{f_2^6}(1,1) = 12$



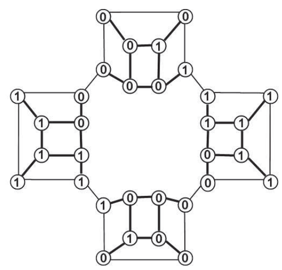
$E_{f_1^7}(1,1) = 13$



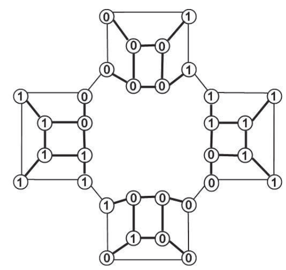
$E_{f_2^7}(1,1) = 14$



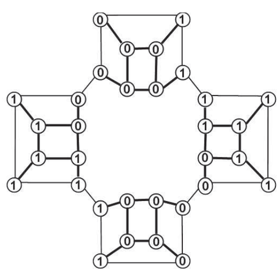
$E_{f_1^8}(1,1) = 15$



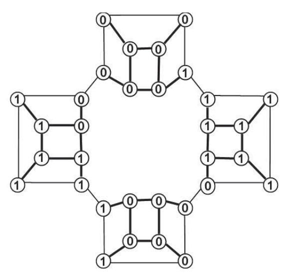
$E_{f_2^8}(1,1) = 16$



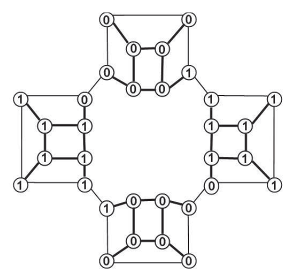
$E_{f_1^9}(1,1) = 17$



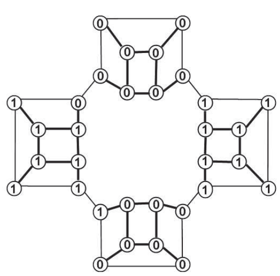
$E_{f_2^9}(1,1) = 18$



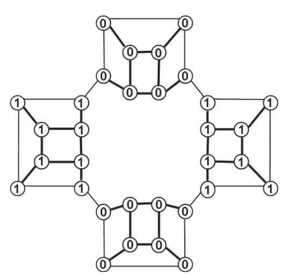
$E_{f_1^{10}}(1,1) = 19$



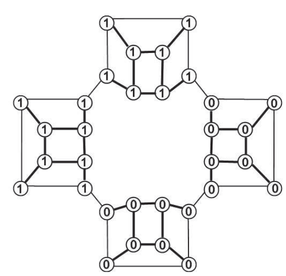
$E_{f_2^{10}}(1,1) = 20$



$E_{f_1^{11}}(1,1) = 21$



$E_{f_2^{11}}(1,1) = 22$



$E_{f_1^{12}}(1,1) = 23$

References

[1] G. Chartrand and S.-M. Lee and P. Zhang, Uniformly cordial graphs, *Discrete Math.*, 306 (2006), 726–737.

[2] H. Kwong and S.-M. Lee, On friendly index sets of generalized books, *J. Combin. Math. Combin. Comput.*, 66 (2008), 43–58.

[3] H. Kwong and S.-M. Lee and H.K. Ng, On friendly index sets of 2-regular graphs, *Discrete Math.*, 308 (2008), 5522–5532.

[4] H. Kwong, S.-M. Lee and H.K. Ng, On product-cordial index sets of cylinders, *Congr. Numer.*, 206 (2010), 139–150.

[5] S.-M. Lee and H.K. Ng, On friendly index sets of bipartite graphs, *Ars Combin.*, 86 (2008), 257–271.

[6] E. Salehi, PC-labeling of a graph and its PC-set, *Bull. Inst. Combin. Appl.*, 58 (2010), 112–121.

[7] E. Salehi and S.-M. Lee, On friendly index sets of trees, *Congr. Numer.*, 178 (2006), 173–183.

[8] W.C. Shiu, Super-edge-graceful labeling of some cubic graphs, *Acta Math. Sin. (Engl. Ser.)*, 22 (2006), 1621–1628.

[9] W.C. Shiu, M.-H. Ho, Full friendly index sets of slender and flat cylinder graphs, *Trans. Combin.*, 2(4) (2013), 63–80.

[10] W.C. Shiu and H. Kwong, Product–cordial index and friendly index of regular graphs, *Trans. Combin.*, 1(1) (2012), 15–20.

[11] W.C. Shiu and H. Kwong, Full friendly index sets of $P_2 \times P_n$, *Discrete Math.*, 308 (2008), 3688–3693.

[12] W.C. Shiu and S.-M. Lee, Full friendly index sets and full product-cordial index sets of twisted cylinders, *J. Comb. Number Theory*, 3 (2012), 209–216.

[13] W.C. Shiu and M.H. Ling, Extreme friendly indices of $C_m \times C_n$, *Congr. Numer.*, 188 (2007), 175–182.

[14] W.C. Shiu and M.H. Ling, Full friendly index sets of Cartesian products of two cycles, *Acta Math. Sin. (Engl. Ser.)*, 26 (2010), 1233–1244.

[15] W.C. Shiu and F.S. Wong, Extreme friendly indices of $C_m \times P_n$, *Congr. Numer.*, 197 (2009), 65–75.

[16] W.C. Shiu and F.S. Wong, Full friendly index sets of cylinder graphs, *Australas. J. Combin.*, 52 (2012), 141–162.