On Independent Domination Number of Regular Graphs[1]

by

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Abstract

Let G be a simple graph. The independent domination number $i(G)$ is the minimum cardinality among all maximal independent sets of G. Haviland (1995) conjectured that any connected regular graph G of order n and degree $\delta \leq n/2$ satisfies $i(G) \leq \lceil 2n/3\delta \rceil \delta/2$. In this paper, we will settle the conjecture of Haviland in the negative by constructing counterexamples. Therefore a larger upper bound is expected. We will also show that a connected cubic graph G of order $n \geq 8$ satisfies $i(G) \leq 2n/5$, providing a new upper bound for cubic graphs.

Key Words and phrases: Independent domination number, regular graph.

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1. Introduction

Let $G = (V, E)$ be a simple graph of order n and minimum degree δ. For a nonempty set $W \subset V$, its *neighborhood* $N(W)$ denote the set of all elements of V adjacent with at least one element of W. If $W = \{v\}$, then $N(W)$ is simply written as $N(v)$. An *independent set* is a set of pairwise non-adjacent vertices of G. A subset I of V is a *dominating set* if $N(I) \cup I = V$. The *independent domination number* $i(G)$ is the minimum cardinality among all independent dominating sets of G. An independent set is dominating if and only if it is maximal, so $i(G)$ is also the minimum cardinality of a maximal independent set in G.

The parameter $i(G)$ was introduced by Cockayne and Hedetniemi in [5] and some results on it can be found in [1-10]. Favaron [6] and Haviland [8] established upper bounds for $i(G)$ in terms of n and δ. For regular graphs of degree different from zero, we can prove that $i(G) \leq n/2$. However, for most values of δ this is far from best possible. In [6] it was shown that for any graph with $n/2 \leq \delta \leq n$, we have $i(G) \leq n - \delta$, and this bound could be attained only by complete multipartite graphs with vertex classes all of the same order. By adapting arguments from [8], the following result can readily be proved (see [9]).

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Proposition 1.1. *Let G be a regular graph. If $n/4 \leq \delta \leq (3-\sqrt{5})n/2$ then $i(G) \leq n - \sqrt{n\delta}$ and if $(3-\sqrt{5})n/2 \leq \delta \leq n/2$ then $i(G) \leq \delta$.*

If $n = 2m\delta$, then $i(mK_{\delta,\delta}) = n/2$ and $mK_{\delta,\delta}$ is disconnected for $m > 1$. Haviland [8] thought that if G was connected then the upper bound for $i(G)$ could be a function of n and δ. She also stated the following Conjecture in [9].

Conjecture 1.2. *If G is a connected r-regular graph with $r = \delta \leq n/2$, then $i(G) \leq \lceil 2n/3\delta \rceil \delta/2$.*

In section 2, we provide counterexamples to Conjecture 1.2. In section 3, we shall show that $i(G) \leq 2n/5$ for any connected cubic graphs, providing a new upper bound for $i(G)$ as a function of the number of vertices.

2. Counterexamples

Lemma 2.1 *Given positive integers $r \geq 2$ and $s \geq 3$, let $G(r,s)$ be the family of graphs such that $V = \bigcup_{j=1}^{r}(X_j \cup Y_j \cup Z_j)$, and $E = (E_1 \cup E_2 \cup E_3 \cup E_4)$, where*

1. $X_j = \{x_{j1}, x_{j2}, \cdots, x_{j(s-1)}\}$,
2. $Y_j = \{y_{j1}, y_{j2}, \cdots, y_{js}\}$,
3. $Z_j = \{z_{j1}, z_{j2}, \cdots, z_{js}\}$,
4. $E_1 = \bigcup_{j=1}^{r}\{x_{jk}y_{jl} |\ 1 \leq k \leq s-1,\ 1 \leq l \leq s\}$,
5. $E_2 = \bigcup_{j=1}^{r}\{y_{jk}z_{jk} |\ 1 \leq k \leq s\}$,
6. $E_3 = \bigcup_{j=1}^{r}[\{z_{jk}z_{jl} |\ 1 \leq k,\ l \leq s,\ k \neq l,\ \} \setminus \{z_{j1}z_{js}\}]$, *and*
7. $E_4 = \{z_{js}z_{(j+1)1} |\ 1 \leq j \leq r-1\} \ \cup\ \{z_{rs}z_{11}\}$.

Then

(1) $|V| = r(3s-1)$

(2) *$G(r,s)$ is both connected and s-regular, and*

(3) $i(G(r,s)) = rs$.

(Note that $G(r,s)$ contains r subgraphs, which we shall call *blocks*, isomorphic to each other. A typical block of $G(r,5)$, consisting of 14 vertices, is shown in Fig. 2.1.)

Proof of Lemma 2.1: (1) and (2) are trivial. Because $\bigcup_{j=1}^{r} Y_j$ is an independent dominating set, $i(G(r,s)) \le rs$. So (3) is also proved if we can show that $i(G(r,s)) \ge rs$.

We claim that for every $1 \le j \le r$, $|I \cap (X_j \cup Z_j)| \ge s$. Consider any such j between 1 and r. If $X_j \cap I \ne \emptyset$, then $Y_j \cap I = \emptyset$. Since I must dominate X_j, we have $X_j \subseteq I$. Now for any $1 < k < s$, $z_{j,k}$ is not adjacent to any vertex outside of $Y_j \cup Z_j$, and so in order for I to dominate $z_{j,k}$, it must be that $Z_j \cap I \ne \emptyset$. Thus $|I \cap (X_j \cup Z_j)| \ge s$.

On the other hand, if $X_j \cap I = \emptyset$, then for each $1 \le k \le s$, exactly one of $y_{j,k}$ or $z_{j,k}$ is in I. And so in this case it also follows that $|I \cap (Y_j \cup Z_j)| \ge s$.

Thus, it follows that $i(G(r,s)) \ge rs$, and hence that $i(G(r,s)) = rs$. ∎

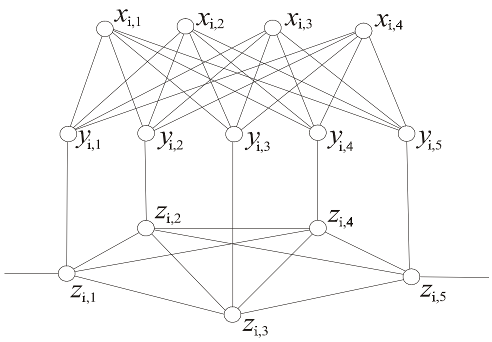


Fig. 2.1

Theorem 2.2 *If r is sufficiently large and $s \ge 3$, then $G = G(r,s)$ is a connected s-regular graph with $i(G) > \lceil 2n/3s \rceil s/2$, where n is the order of G.*

Proof: We have
$$\begin{aligned} \lceil 2r(3s-1)/3s \rceil s/2 &\le (2r - \lfloor 2r/3s \rfloor)s/2 \\ &= rs - (\lfloor 2r/3s \rfloor s/2) \\ &< i(G), \end{aligned}$$
provided r is sufficiently large. ∎

Theorem 2.2 settles Conjecture 1.2 in the negative for all $\delta \ge 3$. If $\delta = 3$, then by Theorem 2.2, the upper bound of $i(G)/n$ is at least $3/8$, as shown by the previous example with $s = 3$. However, if Conjecture 1.2 holds, then this upper bound would have been less than $(n+4)/3n$, which is strictly less than $3/8$ if $n > 32$.

Note that in the above, δ is fixed and n is large. In what follows, we shall construct connected regular graphs G with $\delta(G)$ small relative to n, but $i(G)/n$ is as close to $1/2$ as we wish.

Lemma 2.3 *Given positive integers $r \geq 1$ and $s \geq 2$, let $G^*(r,s)$ be the graph (V,E) with $V = U \bigcup \left[\bigcup_{j=1}^{2r+1}(V_j \cup W_j)\right]$, and $E = (E_1 \cup E_2 \cup E_3 \cup E_4)$, where*

1. $U = \{u_1, u_2, \cdots, u_{2r+1}\}$,
2. $V_j = \{v_{j,1}, v_{j,2}, \cdots, v_{j,s+2r}\}$,
3. $W_j = \{w_{j,1}, w_{j,2}, \cdots, w_{j,s+2r-1}\}$,
4. $E_1 = \{u_j u_k \mid 1 \leq j < k \leq 2r+1\}$,
5. $E_2 = \bigcup_{j=1}^{2r+1}\{u_j v_{j,k} \mid 1 \leq k \leq s\}$,
6. $E_3 = \bigcup_{j=1}^{2r+1}[\{v_{j,s+2k-1} v_{j,s+2k} \mid 1 \leq k \leq r\}]$ *and*
7. $E_4 = \bigcup_{j=1}^{2r+1}[\{v_{j,k} w_{j,l} \mid 1 \leq k \leq s+2r,\ 1 \leq l \leq s+2r-1\}]$.

Then

(1) $|V| = 2(2r+1)(s+2r)$,

(2) $G^*(r,s)$ *is both connected and* $(s+2r)$*-regular, and*

(3) $i(G^*(r,s)) = 2r(s+r) + r + 1$.

Proof: (1) and (2) are trivial. Because $S =$

$$\left[\bigcup_{j=1}^{2r}[\{v_{j,k} \mid 1 \leq k \leq s\} \cup \{v_{j,s+2k} \mid 1 \leq k \leq r\}]\right] \bigcup [\{u_{2r+1}\} \cup \{v_{2r+1,s+2k} \mid 1 \leq k \leq r\}]$$

is a maximal independent set of $G^*(r,s)$, and $|S| = 2r(s+r) + r + 1$, we have $i(G^*(r,s)) \leq 2r(s+r) + r + 1$.

Suppose I is a maximal independent set of order $i(G^*(r,s))$ and $I_j = I \cap [V_j \cup W_j \cup \{u_j\}]$ for $1 \leq j \leq 2r+1$. Clearly, $I = \bigcup_{j=1}^{2r+1} I_j$ and $|I| = \sum_{j=1}^{2r+1} |I_j|$. If $u_j \notin I$, then $|I \cap (V_j \cup W_j)| \geq s + r$, and if $u_j \in I$, then $|I \cap (V_j \cup W_j)| \geq r$. Because I is independent and the induced subgraph on U is complete, there is at most one j with $u_j \in I$. It follows that $i(G^*(r,s)) \geq 2r(s+r) + r + 1$ and (3) follows. ∎

Theorem 2.4 *Suppose $0 < \epsilon < 1$ and $N \geq 2$. Then there exists a connected δ-regular graph of order n with $\delta < n/N$ and $i(G) > n/(2+\epsilon)$.*

Proof: Let r_1 be the smallest integer such that $2(2r_1+1) > N$. Because $\lim_{r\to\infty} \frac{r}{2r+1} = \frac{1}{2}$, we can find r_2 such that if $r \geq r_2$, then $\frac{r}{2r+1} > \frac{1}{2} - \frac{\epsilon}{12}$. Put $r = \max\{r_1,\ r_2\}$. Also, for fixed r, we have $\lim_{s\to\infty} \frac{2r(s+r)+r+1}{2(2r+1)(s+2r)} = \frac{r}{2r+1}$, so we can find s such that $\frac{2r(s+r)+r+1}{2(2r+1)(s+2r)} > \frac{r}{2r+1} - \frac{\epsilon}{12}$. Let $G = G^*(r,s)$ and $n = |G|$. Then G is a δ-regular graph with $\delta = s + 2r$.

By Lemma 2.3 and the definition of r, $\delta/n = 1/2(2r+1) < 1/N$. Moreover, by Lemma 2.3 again and the definition of r and s,

$$\frac{i(G)}{n} = \frac{2r(s+r)+r+1}{2(2r+1)(s+2r)} > \frac{r}{2r+1} - \frac{\epsilon}{12} > \frac{1}{2} - \frac{\epsilon}{6} > \frac{1}{2+\epsilon}$$

provided $0 < \epsilon < 1$. ∎

3. Regular Cubic Graphs

In this section, we obtain an upper bound for the independent domination number of a connected cubic graph.

Theorem 3.1 *If G is a connected cubic graph of order n, where $n \geq 8$, then*

$$i(G) \leq \frac{2n}{5}.$$

Proof: Let I be an independent dominating set (IDS) of cardinality $i(G)$. Also let $J = V \backslash I$ and $B = (I, J)$ be the bipartite graph induced by edges of G joining a vertex in I to a vertex in J. Among all such choices of I, choose one so that B contains the smallest number of $K_{2,3}$'s. If $v \in J$ is connected to $u \in I$ by an edge in B, we say that v is *guarded* by u and that u is a *guardian* of v. For each $t = 1,\ 2$ and 3, let $J_t = \{v \in J : v \text{ has } t \text{ guardians}\}$. Since I is a dominating set, J is the disjoint union of $J_1,\ J_2$ and J_3. If $|J_3| \leq |J_1|$, then

$$\begin{aligned} 3n = \sum_{v\in V} d_G(v) &= 2\sum_{v\in I} d_G(v) + 2|J_1| + |J_2| \\ &\geq 6i(G) + |J_1| + |J_2| + |J_3| \\ &= 6i(G) + (n - i(G)), \end{aligned}$$

and therefore $i(G) \leq 2n/5$. So the theorem is proved if we can construct an injective map $f : J_3 \to J_1$. A vertex $v \in J$ is *guarded* by $I' \subset I$ if it is guarded by at least one vertex $u \in I'$. The set of guardians of a vertex $v_0 \in J_3$ shall be denoted by $I_0 = N(v_0) = \{u_1,\ u_2,\ u_3\}$. A vertex v that is guarded only by vertices of I_0 is called *exclusive* (with respect to v_0), otherwise *not exclusive*. V_{ex}

shall denote the set of exclusive vertices. Note that v_0 is not adjacent to any vertex in $V_{ex}\backslash\{v_0\}$. If $|V_{ex}| \leq 2$, then $[I \cup V_{ex}]\backslash I_0$ is a subminimal IDS. Henceforth, we suppose $|V_{ex}| \geq 3$. We have three possible cases.

Case 1: I_0 does not guard any J_1-vertex.

In this case, $V_{ex} \subset J_2 \cup J_3$. If $|V_{ex} \cap J_3| = 3$, then $n = 6$. So besides v_0, there is at most one J_3-vertex in V_{ex}, and thus $J_2 \cap V_{ex} \neq \emptyset$. If $|V_{ex} \cap J_3| = 2$ and if w_1 and w_2 are two exclusive vertices in J_2 and $J_3\backslash\{v_0\}$ respectively, then $[I \cup \{w_1\}]\backslash[N(w_1) \cap I]$ is a subminimal IDS. Hence $V_{ex} \cap J_3 = \{v_0\}$ and there are at least two J_2-vertices in V_{ex}, say v_1 and v_2. Suppose the guardian sets of v_1 and v_2 are not identical (see H_1 of Fig 3.1). The third vertex guarded by u_3 must be guarded by a vertex in $I\backslash\{u_2,\ u_3\}$ and therefore $[I\backslash\{u_2,\ u_3\}] \cup \{v_2\}$ is a subminimal IDS. So we assume that v_1 and v_2 have the same guardian set (see H_2 of Fig 3.1). Moreover, $V_{ex} = \{v_0,\ v_1,\ v_2\}$.

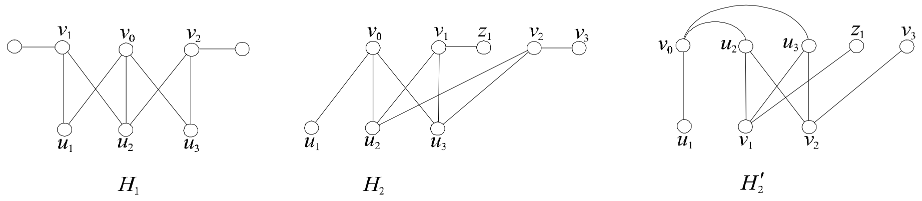


Fig. 3.1

If $z_1 = v_2$, then $(I \cup \{v_1\})\backslash\{u_2, u_3\}$ is a subminimal IDS. Hence $z_1 \neq v_2$. If $z_1 \neq v_3$, then $I' = I \cup \{v_1,\ v_2\}\backslash\{u_2,\ u_3\}$ is an IDS with $|I| = |I'|$, but the bipartite graph $(I', V\backslash I')$ contains a smaller number of $K_{2,3}$'s (compare H_2 and H_2' in Fig. 3.1). Therefore $z_1 = v_3 \in J_1$ and G contains the subgraph H_3 in Fig. 3.2. We let $f(v_0) = z_1$.

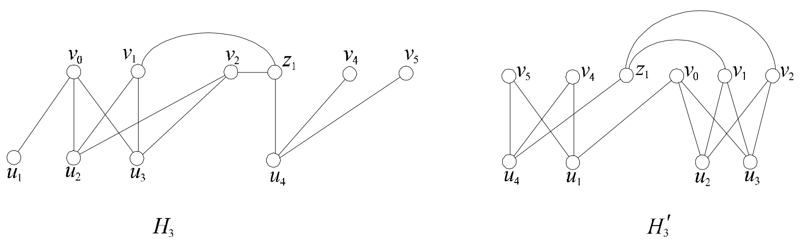


Fig. 3.2

Suppose $N^*(u_4)$ is the set of vertices which are guarded by u_4 but not by any vertex in $I\backslash[I_0 \cup \{u_4\}]$. If either $v_4 \notin N^*(u_4)$ or $v_5 \notin N^*(u_4)$, then $[I \cup \{v_0\} \cup N^*(u_4)]\backslash\{u_1,\ u_2,\ u_3,\ u_4\}$ is a subminimal IDS. Therefore $|N^*(u_4)| = 3$. It follows that neither v_4 nor v_5 is in J_3. Moreover, if

v_4 is in J_2, then it must be guarded by u_1. The same is true for v_5. If both v_4 and v_5 are in J_2 then G contains the subgraph H_3' in Fig. 3.2.

Case 2: I_0 guards exactly one J_1-vertex v', which is guarded by $u_3 \in I_0$.

Besides v_0 and v', there is an exclusive vertex in $J_2 \cup J_3$, because $|V_{ex}| \geq 3$. We have the following sub-cases.

Sub-case 2.1: $[V_{ex} \backslash \{v_0,\ v'\}] \subset J_2$ and there exists $v_2 \in V_{ex} \backslash \{v_0,\ v'\}$ guarded by u_3.

In this sub-case, H_4 appears (Fig. 3.3). Relabeling v' as z_2, we let $f(v_0) = z_2$.

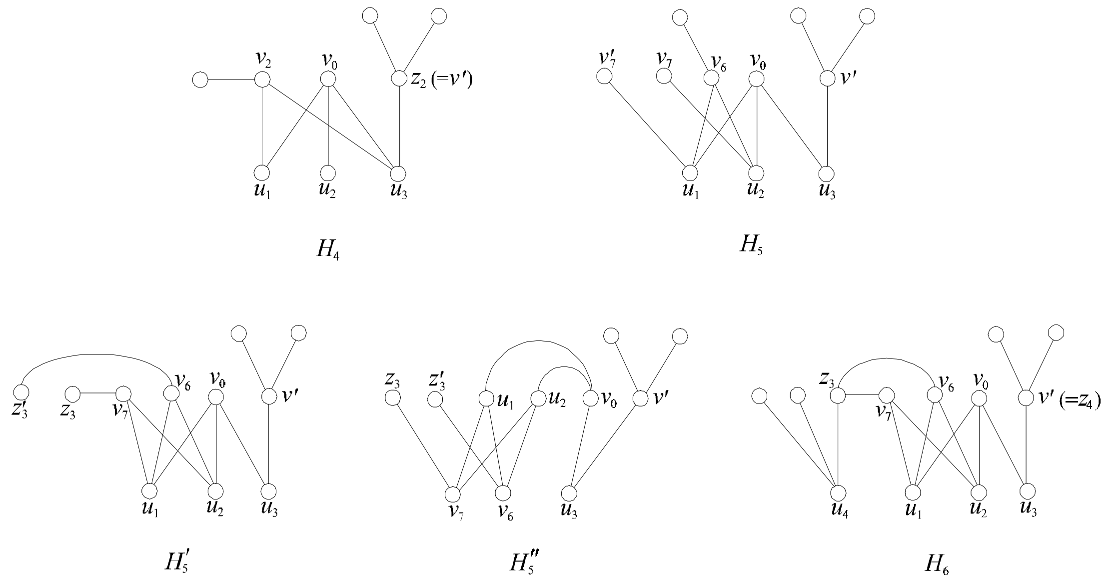


Fig. 3.3

Sub-case 2.2: $[V_{ex} \backslash \{v_0,\ v'\}] \subset J_2$ and no vertex in $V_{ex} \backslash \{v_0,\ v'\}$ is guarded by u_3.

Suppose $v_6 \in V_{ex} \backslash \{v_0,\ v'\}$ is guarded by u_1 and u_2. Because G is cubic, u_1 guards a remaining vertex besides v_0 and v_6. The same is true for u_2. If these two remaining vertices v_7 and v_7' are distinct, i.e. G contains H_5 (Fig. 3.3), then since they are not in J_1, $I \cup \{v_6\} \backslash \{u_1,\ u_2\}$ is a subminimal IDS. Therefore both u_1 and u_2 guard the same remaining vertex, and G contains H_5' (Fig. 3.3). We have v_6 not adjacent to v_7, otherwise $I \cup \{v_6\} \backslash \{u_1,\ u_2\}$ is a subminimal IDS. We also have $z_3 = z_3'$, otherwise $I' = I \cup \{v_6,\ v_7\} \backslash \{u_1,\ u_2\}$, is an IDS with $|I| = |I'|$, but $(I',\ V \backslash I')$ contains less $K_{2,3}$'s than B (compare H_5' with H_5'' in Fig. 3.3). The vertex z_3 is different from v'

for otherwise $(I \cup \{v_o, v'\})\backslash\{u_1, u_2, u_3\}$ is a subminimal IDS. Therefore the guardian of z_3 is also different from the guardian of v', i.e. $u_4 \neq u_3$, and G contains the subgraph H_6.

If u_3 also guards a J_3-vertex besides v_0, then all vertices in $N(u_4)\backslash J_1$ would be guarded by at least one vertex in $I\backslash(I_0 \cup \{u_4\})$. Moreover $(I \cup \{v_0,\ v'\} \cup [N(u_4) \cap J_1])\backslash\{u_1,\ u_2,\ u_3,\ u_4\}$ would be a subminimal IDS if $|N(u_4) \cap J_1| = 1$. Therefore $|N(u_4) \cap J_1| \geq 2$ and u_4 guards another J_1-vertex besides z_3. In this case, we let $f(v_0) = z_3$. If u_3 does not guard another J_3-vertex besides v_0, i.e. the third vertex it guards is in J_2, then we relabel v' as z_4 and let $f(v_0) = z_4$.

Subcase 2.3 $[V_{ex}\backslash\{v_0,\ v'\}] \cap J_3 \neq \emptyset$.

Suppose $v_2 \in V_{ex} \cap J_3$ and so G contains H_7. The set $I' = I \cup \{v'\}\backslash\{u_3\}$ is an IDS with $|I'| = |I|$ but $(I',\ V\backslash I')$ contains a less $K_{2,3}$'s unless there are vertices $u_4 \in I$ and $u_5 \in I$, both of which guards v_3 as well as v_4 (compare H_7 with H_7' in Fig. 3.4). Because G is cubic, u_1, and similarly u_2, cannot be u_4 or u_5. Therefore G must contain H_7'' (Fig. 3.4).

Suppose $v_5 = v_6 = w$. If $N(w) \cap I \subset U = \{u_1,\ u_2,\ u_3,\ u_4,\ u_5\}$, then $I \cup \{v',\ w\}\backslash[N(w) \cup \{u_3\}]$ is a subminimal IDS. If $[N(w) \cap I]\backslash U \neq \emptyset$, then since u_1 and u_2 does not guard any J_1-vertex, $I \cup \{v_0,\ v_2,\ v_3,\ v_4\}\backslash U$ is a subminimal IDS. Therefore $v_5 \neq v_6$.

If both $[N(v_5) \cap I]\backslash U$ and $[N(v_6) \cap I]\backslash U$ are non-empty, then $I \cup \{v_0,\ v_2,\ v_3,\ v_4\}\backslash U$ is a subminimal IDS. Therefore one of $[N(v_5) \cap I]\backslash U$ and $[N(v_6) \cap I]\backslash U$ must be empty. Without loss of generality, we may assume that $[N(v_5) \cap I]\backslash U = \emptyset$. Then $[I \cup \{v_5\}]\backslash[N(v_5) \cap \{u_1,\ u_2,\ u_4\}]$ is a subminimal IDS unless $|N(v_5) \cap \{u_1,\ u_2,\ u_4\}| = 1$. So v_5 is not guarded by u_1 or by u_2. Therefore v_5 is a J_1-vertex. Relabeling v' and v_5 as z_5 and z_6 respectively, we put $f(v_0) = z_5$ and $f(v_2) = z_6$.

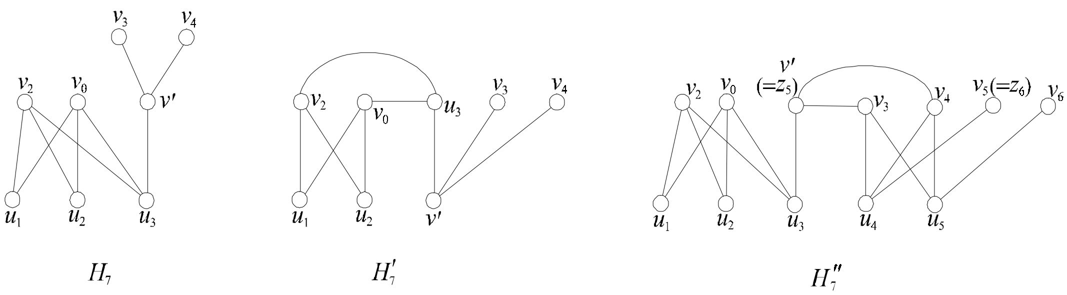
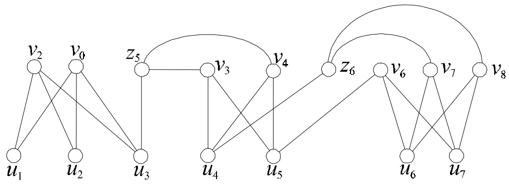


Fig. 3.4

So far, the mapping is injective. Vertex z_1 is guarded by a vertex which guards only J_1- or J_2-vertices. Vertex z_2 is guarded by a vertex which also guards one J_2- and one J_3-vertex. The latter is the pre-image of z_2. Similar argument can be applied to z_4. Vertex z_3 is guarded by a vertex which guards at least one other J_1-vertex. Vertex z_5 is guarded by a vertex which guards two other J_3-vertices. Vertex z_6 is guarded by a vertex which guards two other J_2-vertices. If z_6 were the image of another J_3-vertex as z_1, then G would contain H_8 in Fig. 3.5 (see also H_3' of Fig. 3.2), and $I \cup \{z_5,\ z_6,\ v_6\} \backslash \{u_i |\ 3 \le i \le 7\}$ would be a subminimal IDS. Therefore z_6 will not be mapped as z_1.



H_8

Fig. 3.5

Case 3: I_0 guards two or more J_1-vertices.

Suppose u_1, one of the three guardians of v_0, guards two J_1-vertices v_8 and v_8'. By examining the type of vertices guarded by u_1, we can conclude that neither v_8 nor v_8' can possibly be mapped as z_i unless $i = 3$. If only one of v_8 and v_8' (say v_8) have been mapped as z_3, then we rename v_8' as z_7 and define $f(v_0) = z_7$. If both v_8 and v_8' have been mapped as z_3 according to Sub-case 2.2, then G contains the sub-graph in Fig 3.6. If v_0 is not guarded by both u_4 and u_4', then $I \cup \{v_1, v_2, v_8, v_1', v_2', v_8'\} \backslash \{u_1, u_2, u_3, u_4, u_2', u_3', u_4'\}$ is a subminimal IDS. Therefore v_0 is guarded by both u_4 and u_4' and we relabel v_2 as z_8 and let $f(v_0) = z_8$. We know that z_7 has not been mapped as z_3. Because z_7 is guarded by a vertex which guards two J_1-vertices and one J_3-vertex, its pre-image, it cannot be z_i for $i = 1, \cdots, 6$ and it cannot be the image of two distinct J_3-vertices. The guardian of z_8 guards two J_3-vertices and the J_1-vertex z_8, but the guardian set of one of these two J_3-vertices guards at least two J_1-vertices. Among z_i, $i = 1, \cdots, 7$, only z_5 is guarded by a vertex which guards two J_3-vertices, but both of these two J_3-vertices has the same guardian set which guards exactly one J_1-vertex.

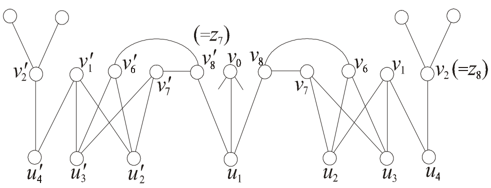


Fig. 3.6

Suppose v' is a J_3-vertex whose guardian set guards two or more J_1-vertices, but each guardian guards at most one J_1-vertex. Let v be one of these J_1-vertices. Because the guardian of v guards exactly one J_1-vertex and at least one J_3-vertex, v cannot have been mapped as z_1, z_3, z_6 and z_7. The guardian of z_2 guards exactly one J_3-vertex whose guardian set guards exactly one J_1-vertex. Because the guardian set of v guards one J_3-vertex whose guardian set guards at least two J_1-vertices, v cannot have been mapped as z_2. For the same reason, it cannot have been mapped as z_4 or z_5. The guardian of z_8 guards two J_3-vertices v_0 and v_1, of Fig 3.6. The guardian set of v_1 guards exactly one J_1-vertex, so v' cannot be v_1. The guardian of v_0 guards one J_3-vertex and two J_1-vertices, so v' cannot be v_0. Since the guardian of v guards the J_3-vertex v', v cannot have been mapped as z_8.

Let $W = \{w_1, w_2, \cdots, w_k\}$ be the set of J_3-vertices whose guardian set guards two or more J_1-vertices, but each guardian guards at most one J_1-vertex; V_i be the set of J_1-vertices guarded by the guardian set of w_i, $i = 1, \cdots, k$; and $V^* = \bigcup_{i=1}^{k} V_i$. If the vertex v belongs to three distinct sets V_{i_1}, V_{i_2} and V_{i_3}, then the guardian of v will guard w_{i_1}, w_{i_2} and w_{i_3}. This is impossible because the graph is cubic. Therefore a vertex may belong to at most two distinct sets V_{i_1} and V_{i_2}, and $|V^*| \geq \frac{1}{2}\sum_{i=1}^{k} |V_i| \geq k$. We may now finish defining the injective map f from J_3 into J_1. ▮

Note that the graph G' in Fig. 3.7 has 10 vertices and $i(G') = 4 = 2n/5$. For $n \geq 12$, we do not know if there exists a graph G'' such that $i(G'') = 2n/5$, but we suspect that such graph does not exist. Moreover, we do not know how close this upper bound is to being the best possible.

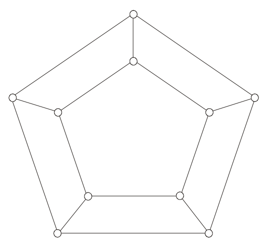


Fig. 3.7

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