Characterization of Graphs with Equal Bandwidth and Cyclic Bandwidth[1]

Peter C. B. Lam,[2] W. C. Shiu[2] and W. H. Chan[2]

Abstract

$B(G)$ and $B_c(G)$ denote the bandwidth and cyclic bandwidth of graph G, respectively. In this paper, we shall give a characterization of graphs with equal bandwidth and cyclic bandwidth. Those graphs include any plane graph G with $B(G) < \frac{p}{m}$, where p and m are the number of vertices and the maximum degree of bounded faces of G, respectively. Hence convex triangulation meshes $T_{m,n,l}$ with $\min\{m,n,l\} \geq 4$ and grids $P_m \times P_n$ with $m \geq 3$ also fall in this class.

Key words and phrases: Bandwidth, cyclic bandwidth, convex triangulation meshes.

AMS 1991 Subject Classifications: $\bar{0}$5C78

1 Introduction

In this paper, $G = (V, E)$ shall be a graph of order p. A one-to-one mapping from V onto $\{1, 2, ..., p\}$ is called a *numbering* of G.

Definition 1.1 *Suppose f is a numbering of G. Let $B(G, f) = \max_{uv \in E} |f(u) - f(v)|$. The* bandwidth *of G, denoted by $B(G)$, is*

$$\min_f \{B(G, f) : \ f \text{ is a numbering of } G\}.$$

A numbering f of G satisfying $B(G) = B(G, f)$ is called an optimal numbering *of G.*

Definition 1.2 *Suppose f is a numbering of G. Let $B_c(G, f) = \max_{uv \in E} ||f(u) - f(v)||_c$, where $||x||_c = \min\{|x|, \ p - |x|\}$ for $0 < |x| < p$. The* cyclic bandwidth *of G, denoted by $B_c(G)$, is defined as*

$$B_c(G) = \min_f \{B_c(G, f) : \ f \text{ is a numbering of } G\}.$$

A numbering f of G satisfying $B_c(G) = B_c(G, f)$ is called a cb-optimal numbering *of G.*

The bandwidth problem of graphs has a wide range of applications including sparse matrix computation, data structure, coding theory and circuit layout of VLSI designs (see [6]). The problem became very important since the mid-sixties - see Chinn et al [2]

[1]Partially supported by FRG, Hong Kong Baptist University; and RGC Grant, Hong Kong
[2]Department of Mathematics, Hong Kong Baptist University, Hong Kong.

or Chung and Seymour [3]. In its original formulation, the problem is to lay vertices of a graph on a path in such a way so that the maximum distance between any two vertices connected by an edge is minimized. Besides a path, other candidates are also available, and at times may even be more appropriate. In [6] and [12], laying vertices on grids $P_m \times P_n$ (product of two paths) and on a cycle C_n, respectively, are considered. When vertices are laid on a cycle, we get cyclic bandwidth (Definition 1.2), which we shall study in this paper.

For a graph G in general, $B_c(G) \leq B(G) \leq 2B_c(G)$, and both bounds are sharp. In [10], we obtained a sufficient condition for a graph to have equal bandwidth and cyclic bandwidth, namely graphs without *long cycles*. However, many graphs possess long cycles and yet their bandwidth and cyclic bandwidth are equal. For example, let G be a graph consisting of a cycle C and a vertex v which does not belong to the cycle, but is adjacent to one of the vertices of the cycle. In this paper, we shall give a necessary and sufficient conditons for a graph to have equal bandwidth and cyclic bandwidth.

In Section 2 and 3, we introduce the concept of *zero/non-zero cycles* and *proper realignment* respectively. In Section 4, we use these concepts to show that bandwidth is equal to cyclic bandwidth for a graph with a *cb*-optimal numbering containing no non-zero cycles. Finally, we show also that convex triangulation meshes $T_{m,n,l}$ with $\min\{m, n, l\} \geq 6$ and grids $P_m \times P_n$ with $m \geq 5$ fall in this class. For notation and terminology of graph theory, please refer to the book of Bondy and Murty [1] and Grimaldi [5] unless defined otherwise.

2 Zero and Non-zero Cycles

Definition 2.1 *Let f be a numbering of G. For any $u, v \in V$ such that $uv \in E$, the* cyclic displacement *of the numbering f from u to v, denoted by $d_f(u,v)$, is $f(v) - f(u) + p\delta_{v,u}$, where*

$$\delta_{v,u} = \begin{cases} 0 & \text{if } |f(v) - f(u)| \leq \frac{p}{2} \\ 1 & \text{if } f(v) - f(u) < -\frac{p}{2} \\ -1 & \text{if } f(v) - f(u) > \frac{p}{2} \end{cases}.$$

Note that $||f(v) - f(u)||_c = |d_f(u,v)|$.

Definition 2.2 *Let f be a numbering of G and $C: v_1v_2...v_kv_{k+1} = v_1$ a cycle in G. The* total cyclic displacement *of the numbering f on C, denoted by S_C, is the sum of cyclic displacements of edges in C.*

It is easy to see that $S_C = \lambda p$, where λ is an integer. We call the cycle C a *zero cycle* of f if $\lambda = 0$; otherwise, we call C a *non-zero cycle* of f. For examples, the 6-cycle is a zero cycle of the numberings indicated in Figures 1(a) and 1(b), and is a non-zero cycle of the numberings indicated in Figures 1(c) and 1(d).

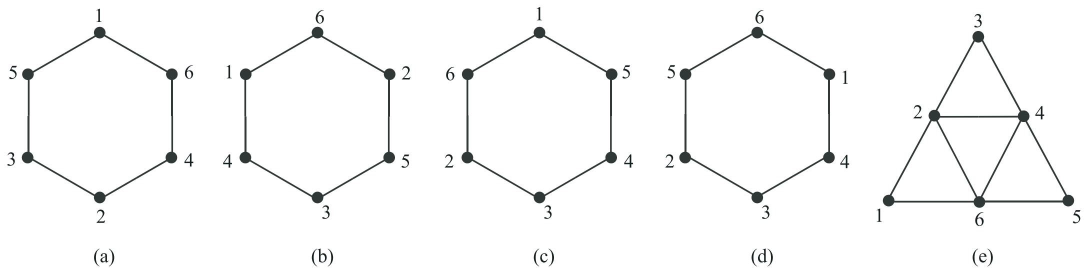

Figure 1

3 Proper Realignment

Definition 3.1 Suppose f is a numbering of G. A one-to-one mapping g from V into $\mathbb{N}$ is called a *proper realignment* of f if

$$|g(v) - g(u)| \leq ||f(v) - f(u)||_c, \text{ for any } uv \in E.$$

The following Lemma on proper realignment can be found in [10].

Lemma 3.2 *Suppose f is a numbering of a tree T. Then there exists a proper realignment of f.*

We can construct a proper realignment of f by the following steps:

1. Choose a vertex $v \in V$. Set $S = \{v\}$ and put $g(v) = f(v)$.

2. $T[S]$ is a tree. For any $v \in N(S)$, there exists $u \in S$ which is adjacent to v. This u is also unique, because T, being a tree, contains no cycles and two vertices in S cannot be both adjacent to v. Put $g(v) = g(u) + d_f(u, v)$.

3. Put $S = S \cup \{v\}$. If $S \neq V$, then go to (2). Otherwise stop.

Remarks follow Lemma 3.2:

1. If u and v are two vertices in $V[T]$, then $g(u) \neq g(v)$.

2. If $vu \in E[T]$, then $g(v) - g(u) = d_f(u, v)$.

4 Characterization of Graphs with Equal Bandwidth and Cyclic Bandwidth

Theorem 4.1 *Suppose G is a graph. There exists a cb-optimal numbering f of G containing no non-zero cycles if and only if $B_c(G) = B(G)$.*

Proof Suppose G is a graph and f is a cb-numbering of G containing no non-zero cycles. We take an arbitrary spanning tree T from G and then construct a proper realignment of f by the Proper Realignment Algorithm. For any $vu \in E[T]$, it is clear that $|g(v)-g(u)| \leq B_c(T,f) \leq B_c(G,f) = B_c(G)$. If we can also show that $|g(v) - g(u)| \leq B_c(G)$ for any $vu \in E[G] \setminus E[T]$, then we have $B_c(G) = B(G)$.

Now let $e = vu \in E[G] \setminus E[T]$ and $C = u_1u_2...u_mu_{m+1}$, where $u_1 = u_{m+1} = v$ and $u_m = u$, be a cycle in $E[T] + e$. Then from Remark 2 in Section 3, we have

$$S_C = \sum_{i=1}^{m} d_f(u_i, u_{i+1}) = d_f(u, v) + g(u) - g(v).$$

Since all cycles in G are zero cycles, therefore $S_C = 0$ and

$$|g(u) - g(v)| = |-d_f(u, v)| \leq B_c(G)$$

Conversely, suppose h is an optimal numbering of G and $B(G) = B_c(G)$. Since $B_c(G) \leq \frac{p}{2}$, we have $|h(v) - h(u)| \leq \frac{p}{2}$ and hence $d_h(u, v) = h(v) - h(u)$ for any $uv \in E$. Moreover, $||h(v) - h(u)||_c = |d_h(u, v)| = |h(v) - h(u)| \leq B(G) = B_c(G)$. Therefore h is also a cb-optimal numbering of G. Also, for any cycle $C : v_1v_2...v_nv_{n+1}$ in G, where $v_{n+1} = v_1$, we have

$$S_C = \sum_{i=1}^{n} d_h(v_i, v_{i+1}) = \sum_{i=1}^{n} h(v_{i+1}) - h(v_i) = 0.$$

So h is a cb-optimal numbering of G containing no non-zero cycles. ∎

Because trees are acyclic, we obtain the following result of [10] from Theorem 4.1 as a Corollary.

Corollary 4.2 *If T is a tree, then $B_c(T) = B(T)$.*

It is known that the problem of determining the bandwidth of a graph is NP-complete even when it is restricted to trees with maximim degree three [6]. Therefore the following Corollary, a main result of [12], holds.

Corollary 4.3 *The problem of determining the cyclic bandwidth of a graph is NP-complete.*

5 Graphs with Equal Bandwidth and Cyclic Bandwidth

Because the problem of determining the cyclic bandwidth of a graph is NP-complete, it is in general very difficult to obtain a cb-optimal numbering of a given graph G, not to mention the requirement of containing no non-zero cycles. However, in this section, we demonstrate that in some graphs, in addition to trees, a cb-optimal numbering containing no non-zero cycles exists. So Theorem 4.1 is applicable to some graphs containing cycles.

Lemma 5.1 *Suppose G is a graph and f is a numbering of G. If there exists a non-zero n-cycle of f in G, then $nB_c(G,f) \geq p$.*

Proof Let $C : v_1v_2...v_nv_{n+1}$, where $v_{n+1} = v_1$, be a non-zero cycle in G of f. Then

$$p \leq |S_C| \leq \sum_{i=1}^{n} |d_f(v_i, v_{i+1})| = \sum_{i=1}^{n} ||f(v_i) - f(v_{i+1})||_c \leq nB_c(G,f). \qquad \blacksquare$$

Given a cycle C of a plane graph G, an edge is called an *internal edge* of C if it lies inside C. A path is called an *internal path* if it consists of internal edges of C solely.

Lemma 5.2 *Suppose G is a plane graph and f is a numbering of G. If the maximum degree of bounded faces of G is not greater than m, then either all cycles are zero cycles of f, or there exists a non-zero cycle of f with length m or less.*

Proof Suppose $C : u_1u_2\cdots u_lu_{l+1}$, where $u_{l+1} = u_1$, is a non-zero cycle of f enclosing k faces. If $k = 1$, or if $k \geq 2$ and there is no internal path joining any two vertices of C, then clearly $l \leq m$.

Suppose $k \geq 2$ and there is an internal path $u_1v_2\cdots v_mu_i$ joining u_1 to u_i, where $2 \leq i \leq l$. Consider the two cycles $C' : u_1v_2\cdots v_mu_iu_{i+1}\cdots u_lu_{l+1}$ and $C^* : u_1u_2\cdots u_iv_mv_{m-1}\cdots v_2u_1$. Noting that $d_f(u,v) = -d_f(v,u)$, we can show that

$$S_C = S_{C'} + S_{C^*}.$$

Since $S_C \neq 0$, therefore either $S_{C'} \neq 0$ or $S_{C^*} \neq 0$. In either case, we get a non-zero cycle of f enclosing at most $k-1$ faces. This process can continue until we get a non-zero cycle of f enclosing 1 face or having no internal paths joining any two vertices of the cycle. $\blacksquare$

Theorem 5.3 *Suppose G is a plane graph and the maximum degree of bounded faces is not greater than m. If $B(G) \leq \lceil \frac{p}{m} \rceil$, then $B_c(G) = B(G)$.*

Proof Suppose $B_c(G) < B(G)$ and f is a cb-optimal numbering of G. By Theorem 4.1 and Lemma 5.2, f contains a non-zero cycle of length m or less. By Lemma 5.1, $mB_c(G) \geq p$. It follows that $\frac{p}{m} \leq B_c(G)$ and consequently $\lceil \frac{p}{m} \rceil < B(G)$. The contradiction shows that $B_c(G) = B(G)$. ▮

Definition 5.4 *A plane graph G whose bounded faces are all of degree m is called an* m-gonal *graph. If $m = 3$, G is called a* triangulated *graph.*

Theorem 5.5 *Suppose G is an m-gonal graph with $B(G) \leq \lceil \frac{p}{m} \rceil$. Then $B_c(G) = B(G)$.*

Corollary 5.6 *If G is a triangulated graph and $B(G) \leq \frac{p}{3}$, then $B_c(G) = B(G)$.*

The product of two paths P_m and P_n is called an *mn-grid*. Because the bandwidth of an mn-grid is $\min\{m, n\}$ by [4], the following corollary holds.

Corollary 5.7 *If G is an mn-grid with $n \geq m \geq 3$, then $B_c(G) = B(G)$.*

The definition of a convex triangulation mesh $T_{m,n,l}$ was given in [11]. Because the bandwidth of a convex triangulation mesh $T_{m,n,l}$ is $\min\{m, n, l\}$ by [7] and [11], the next corollary follows.

Corollary 5.8 *For all convex triangulation meshes $T_{m,n,l}$ with $\min\{m, n, l\} \geq 4$, we have $B_c(T_{m,n,l}) = B(T_{m,n,l})$.*

Note that $B_c(T_{m,n,l}) \neq B(T_{m,n,l})$ if $m = n = l = 3$. The numbering of $G^* = T_{3,3,3}$ indicated in Figure 1(e) shows that $B_c(G^*) \leq 2$, whereas $B(G^*) = 3$ by [7].

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, The Macmillan Press LTD, London, 1976.

[2] P. Z. Chinn, J. Chvátalová, A. A. Dewdney and N. E. Gibbs, The Bandwidth Problem for Graphs and Matrices - a Survey, *J. Graph Theory*, **6(3)** (1982), 223-254.

[3] F. R. K. Chung and P. D. Seymour, Graph Theory with Small Bandwidth and Cutwidth, *Discrete Mathematics*, **75** (1989), 113-119.

[4] J. Chvátalová, Optimal Labeling of A Product of Two Paths, *Discrete Mathematics*, **11** (1975), 249-253.

[5] R. P. GRIMALDI, *Discrete and Combinatorial Mathematics, an Applied Introduction*, Addison-Wesley Publishing Company, New York, 1989.

[6] M. R. GAREY, R. L. GRAHAM, D. S. JOHNSON and D. E. KNUTH, Complexity Results for Bandwidth Minimization, *SIAM J. Appl. Math.*, **34** (1978), 477-495.

[7] R. HOCHBERG, C. McDIARMID and M. SAKS, On the Bandwidth of Triangulated Triangles, *Discrete Mathematics*, **138** (1995), 261-265.

[8] P. C. B. LAM, On Number of Leaves and Bandwidth of Trees, *Acta Mathematicae Applicatae Sinica*, **14**(**2**) (1998), 193-196.

[9] P. C. B. LAM, On Bandwidth of M-ary Trees, *Technical Reports*, **93**, Department of Mathematics, Hong Kong Baptist University, 1995.

[10] P. C. B. LAM, W. C. SHIU and W. H. CHAN, Bandwidth and Cyclic Bandwidth of Graphs, *Ars Combinatoria*, **47** (1997), 87-92.

[11] P. C. B. LAM, W. C. SHIU, W. H. CHAN, and Y. X. LIN, On the Bandwidth of Convex Triangulation Meshes, *Discrete Mathematics*, **173** (1997), 285-289.

[12] Y. X. LIN, The Cyclic Bandwidth Problem, *Syst. Sci. Math. Sci.*, **7** No. **3** (1994), 282-288.

[13] J. WANG and B. YAO, On Upper Bounds of Bandwidths of Trees, *Acta Mathematicae Applicatae Sinica*, **11**(**2**) (1995), 1-8.