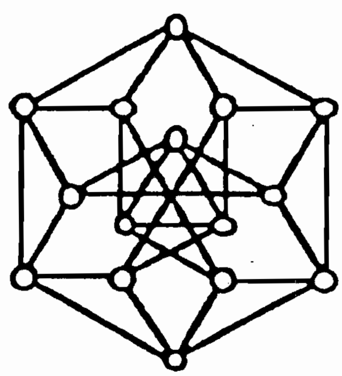
COMBINATORICS, GRAPH THEORY, AND ALGORITHMS

Volume II

Proceedings of the Eighth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms, and Applications

Western Michigan University

Edited by
Y. Alavi, D. R. Lick, and A. Schwenk



NEW ISSUES PRESS
WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan
1999

Entire Chromatic Number of Maximal Outerplanar Graphs with Maximum Degree at most 4[1]

by

Wai Chee Shiu[2], Peter Che Bor Lam[2] and Zhongfu Zhang[3]

Abstract

Let $\chi_e(G)$ be the vertex-edge-face entire chromatic number of a plane graph G. Kronk and Mitchem (1973) conjectured that if G is a simple graph with maximum degree $\Delta(G)$, then $\Delta(G) \leq \Delta(G)+4$. Lin, Zhang, Han et al. (1993) proved that if G is a maximal outerplanar graph with $\Delta(G) \geq 6$, then $\chi_e(G) \leq \Delta(G)+1$. In this paper, we find the value of $\chi_e(G)$ when G is a maximal outerplanar graph with maximum degree at most 4.

Key Words and phrases: Entire coloring, entiree chromatic number, maximal outerplanar graph.

AMS 1991 Subject Classification: 05C15

1. Introduction

We first recall the following definition.

Definition: Let G be a simple plane graph. A *proper k-vertex-edge-face coloring* or a *k-entire coloring* of G is an assignment of k colors to all the vertices, edges and faces of G such that no two adjacent or incident elements have the same color. The entire chromatic number of G, denoted by $\chi_e(G)$, is the smallest number k such that G has a k-entire coloring.

In [2], the following conjecture was given:

If G is a simple plane graph with maximum degree $\Delta(G)$, then $\chi_e(G) \leq \Delta(G)+4$.

Definition: A (simple) plane graph is said to be an *outerplanar* graph if all of its vertices are at the boundary of a given face f_0. The face f_0 is called the *outer face*, other faces are called *inner faces*.

In [3], [4] and [5], Lin, Zhang, Han, Wang et al. proved the following theorem.

[1]Partially supported by FRG, Hong Kong Baptist University; and RGC grant, Hong Kong.

[2]Department of Mathematics, Hong Kong Baptist University, Kowloon, Hong Kong.

[3]Department of Mathematics, Lanzhou Railway Institute, Lanzhou 730070, China. Research was done while visiting Hong Kong Baptist University.

$\chi_c(G) \geq 6$, and hence $\chi_c(G) = 6$.

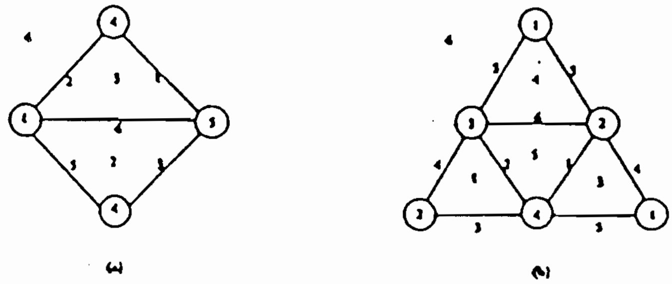


Figure 1

Suppose each inner face of G has at least one edge in common with the boundary of the outer face f_0. By Lemma 2.1, there is a vertex u with $deg(u) = 2$. Let uvw and vwy be two triangles of G. Suppose $\chi_c(G) \leq 5$ and τ is a 5-entire coloring with $\tau(f_0) = 5$, $\tau(u) = 1$, $\tau(v) = 2$, $\tau(w) = 3$, and $\tau(uvw) = 4$. Then $\tau(uv) = 3$ and $\tau(uw) = 2$. We get a contradiction when we consider the τ-values of vw, vwy, vy and wy which implies $\chi_c(G) \geq 6$. In the following we shall prove that there exists a 6-entire coloring of G and therefore $\chi_c(G) = 6$.

For now on we assume the color of the outer face to be 6. By applying mathematical induction on $|V(G)|$, the order of G, we have the following theorem.

Theorem 2.3: *If G is a 2-connected maximal outerplanar graph with $\Delta(G) = 4$, then $\chi_c(G) \leq 6$.*

Proof: If G has 5 vertices then G is a fan and $\chi_c(G) \leq 6$ - Figure 2(a). Now we assume that G contains n vertices, $n \geq 6$: Suppose that there are u, v, $b \in V(G)$ such that $deg(u) = deg(v) = 2$, $deg(b) = 4$ and bu, $bv \in E(G)$. In this case G only contains 6 vertices and we can find a 6-entire coloring for G - Figure 2(b).

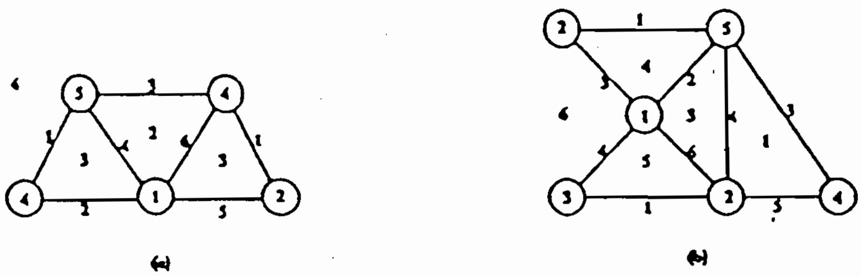


Figure 2

Suppose that there are $u, v \in V(G)$ such that $uv \in E(G)$, $deg(u) = 2$ and $deg(v) = 3$. Let buv and abv be triangles in G. Then $deg(b) = 4$ and $H = G - u$ is also a maximal outerplanar graph. By induction there is a 6-entire coloring τ for H. Let c be the other vertex adjacent to b. Without loss of generality, we may assume that $\tau(a) = 2$, $\tau(b) = 3$, $\tau(abv) = 4$ and $\tau(abc) = 5$. We shall extend τ to σ, a 6-entire coloring for G which agrees

Theorem 1.1: *If G is a maximal outerplanar graph with $\Delta(G) \geq 6$, then $\chi_e(G) = \Delta(G) + 1$.*

In [6], Wang obtained the following stronger results.

Theorem 1.2: *If G is a 2-edge-connected outerplanar graph with $\Delta(G) \geq 6$, then $\chi_e(G) = \Delta(G) + 1$.*

In this paper, we shall find the entire chromatic number of maximal outerplanar graph G with $\Delta(G) \leq 4$. Unless otherwise stated, we shall adopt the notation and terminology given in Bondy and Murty [1].

2. Entire Chromatic Number of G with $\Delta(G) \leq 4$

We first recall the following useful lemma. See for example [6].

Lemma 2.1: *Let G be a maximal outerplanar graph with $\Delta(G) \geq 4$. Then at least one of the following statements is true:*

1. *There are two adjacent vertices whose degrees are 2 and 3, respectively.*
2. *There are two vertices of degree 2 which have a common neighbor of degree 4.*

Let G be a maximal outerplanar graph. If $\Delta(G) = 2$, then G is isomorphic to K_3. Clearly $\chi_e(K_3) = 5$. If $\Delta(G) = 3$, then G is isomorphic a fan with 4 vertices and $\chi_e(G) = 6$ - see Figure 1(a).

By the definition of maximal outerplanar graphs, it is easy to see the following lemma:

Lemma 2.2: *Let G be a connected outerplanar graph. G is maximal if and only if all inner faces of G are triangles. If s is a 6-entire coloring of G, then each inner face of G has at least one pair of elements (vertex and edge) with the same color under s. If this inner face has one edge on the boundary of the outer face of G and the colors of its other edges are different from that of the outer face, then it has at least two pairs of elements with the same color.*

Suppose $\Delta(G) = 4$, where G is a maximal outerplaner graph. If there is an inner face with no edge on the boundary of the outer face, then G must be isomorphic to the Figure 1(b) and has a 6-entire coloring. Thus $\chi_e(G) \leq 6$. Let $S = V(G) \cup E(G) \cup F(G)$. Clearly $|S| = 20$. Let the color of the outer face be 6. Then there are at most two elements in S whose colors are 6. Since the order of the maximum independent set is 4, with τ on H. By considering possible choices for $\tau(ab)$, we get two cases.

1 $\tau(ab) = 6$. By considering the possible choices for $\tau(av)$, we have the following 2 subcases.

1.1 $\tau(av) = 3$. Suppose $\tau(bc) \neq 2$. Consider the triangle abc. Since $\tau(c) \neq 6$ and $\tau(ac) \neq 3$, Lemma 2.2 requires that $\tau(bc) = 2$. We define the colors $\sigma(bv) = 1$, $\sigma(v) = 5$, $\sigma(u) = 1$, $\sigma(bu) = 5$, $\sigma(buv) = 2$, $\sigma(uv) = 4$ and define σ to be the same as τ on the other elements of G. Then σ is a 6-entire coloring of G - Figure 3(a).

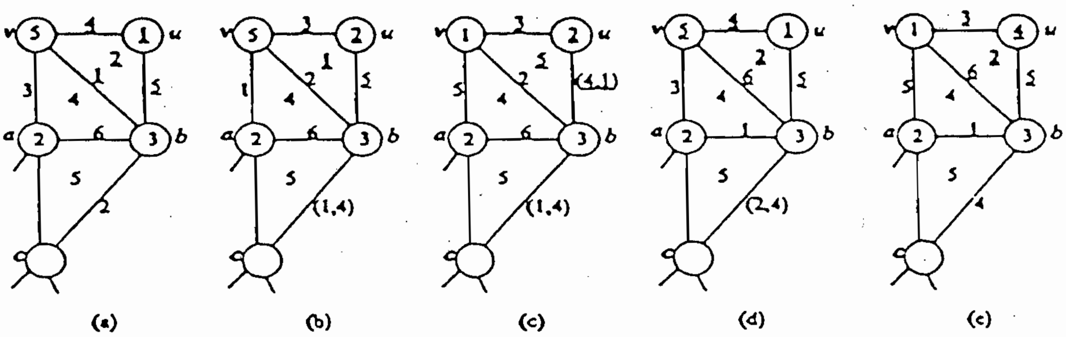


Figure 3

1.2 $\tau(av) = 1$. Applying Lemma 2.2 to the triangle abv, we get $\tau(bv) = 2$ and $\tau(v) = 5$. It follows that $\tau(bc) = 1$ or 4. We define the colors $\sigma(u) = 2$, $\sigma(bu) = 5$, $\sigma(buv) = 1$, $\sigma(uv) = 3$ and define σ to be the same as τ on the other elements of G. Then σ is a 6-entire coloring of G - Figure 3(b).

1.3 $\tau(av) = 5$. Similar to subcase 1.2, we get $\tau(bv) = 2$ and $\tau(v) = 1$ and $\tau(bc) = 1$ or 4. We define the colors $\sigma(bu) = 4$ or 1, avoiding conflict with $\tau(bc)$, $\sigma(u) = 2$, $\sigma(buv) = 5$, $\sigma(uv) = 3$ and define σ to be the same as τ on the other elements of G. Then σ is a 6-entire coloring of G - Figure 3(c).

2 $\tau(ab) = 1$. By considering the possible choices for $\tau(av)$, we have the following 2 subcases.

2.1 $\tau(av) = 3$. In this subcase, $\tau(bc) = 2$ or 4. We define the colors $\sigma(bv) = 6$, $\sigma(v) = 5$, $\sigma(u) = 1$, $\sigma(bu) = 5$, $\sigma(buv) = 2$, $\sigma(uv) = 4$ and define σ to be the same as τ on the other elements of G. Then σ is a 6-entire coloring of G - Figure 3(d).

2.2 $\tau(av) = 5$. Applying Lemma 2.2 to triangle abv, we have $\tau(v) = 1$ and $\tau(bv) = 2$, and this forces $\tau(bc) = 4$. We define the colors $\sigma(bv) = 6$, $\sigma(vu) = 3$, $\sigma(bu) = 5$, $\sigma(buv) = 2$, $\sigma(u) = 4$ and define σ to be the same as τ on the other elements of G. Then σ is a 6-entire coloring of G - Figure 3(e).

Theorem 2.4: *Let G be a maximal outerplanar graph. If $\Delta(G) = 2$ then $\chi_e(G) = 5$. If $3 \leq \chi_e(G) \leq 4$. then $\chi_e(G) = 6$. Moreover, in each optimal entire coloring, the color of each inner face is not equal to that of the outer face.*

Unsolved Case

All of the above results do not cover the case when the maximum degree of the outerplanar graph is 5. In an unsuccessful attempt of the authors of this note to find the entire chromatic number of maximal outerplanar graph G with $\Delta(G) = 5$ ([7]), they show that $\chi_e(G) \leq 7$. However, there are strong indication that the upper bound is not sharp. Therefore we conjecture that

The entire chromatic number of any simple maximal outerplanar with maximum degree 5 *is* 6.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, 1976, London, Macmillan.

[2] H. V. Kronk and J. A. Mitchem, A seven-color theorem on the sphere, *Discrete Math.*, 5, 253-260, 1973.

[3] W. F. Wang and Z. F. Zhang, On the complete chromatic number of outerplanar graphs (in Chinese), *J. Lanzhou Railway Institute*, **3**, 27-34, 1992.

[4] Z.F. Zhang, J. F. Wang, W. F. Wang and L. X. Wang, The entire chromatic number of some graphs, *Science in China, Series A*, **23**, No. **4**, 363-368, 1993.

[5] K. Lin, Z. Zhang and J. Han, On the complete chromatic number of maximal of outerplanar graphs with maximum degree $\Delta(G) = 6$, to appear.

[6] W. F. Wang, On coloring of outerplanar graphs, *Discrete Math.*, **147**, 257-269, 1995.

[7] W. C. Shiu, P. C. B. Lam and Z. F. Zhang, Complete Chromatic Number of 2-Connected Maximal Outerplanar with Small Maximum Degree, *Technical Report*, **116**, Department of Mathematics, Hong Kong Baptist University, 1996.