More on the Generalized Fibonacci Numbers and Associated Bipartite Graphs

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Abstract

For a positive integer $k \geq 2$, the k-Fibonacci sequence $\{f_n^{(k)}\}$ is defined by $f_n^{(k)} = f_{n-1}^{(k)} + f_{n-2}^{(k)} + \cdots + f_{n-k}^{(k)}$, for $n \geq k$, with initial value $f_0^{(k)} = f_1^{(k)} = \cdots = f_{k-2}^{(k)} = 0$, $f_{k-1}^{(k)} = 1$. For a fixed $\alpha = (a_1, a_2, \ldots, a_m)$, the (k, α)-sequence is defined by $s(\alpha)_n^{(k)} = \sum_{i=1}^{m} a_i f_{n-1+k-i}^{(k)}$ for $k \geq 2$, $m \geq 1$ and $n \geq 1$. In this paper, we consider the relationship between $s(\alpha)_n^{(k)}$ and perfect matchings of a bipartite graph.

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1. Introduction

Let $A = (a_{i,j})$ be a square matrix of order n over a ring R. The *permanent* of A is defined by

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where S_n denotes the symmetric group on n letters. It is easy to see that for any square matrix A and any permutation matrices P and Q, $\mathrm{per}(A) = \mathrm{per}(PAQ)$. Let

$A_{i,j}$ be the matrix obtained from a square matrix $A = (a_{i,j})$ by deleting the i-th row and the j-th column. Then it is also easy to see that $\text{per}(A) = \sum_{k=1}^{n} a_{i,k}\text{per}(A_{i,k}) = \sum_{k=1}^{n} a_{k,j}\text{per}(A_{k,j})$ for any i, j.

In this paper, all undefined terminologies and symbols of graph can be found in [1]. Let G be a bipartite graph with bipartition (X, Y). If G contains a perfect matching, then $|X| = |Y|$. Let A be an adjacency matrix of G. It is known [7] that the number of perfect matchings (or 1-factor) of G is $\sqrt{\text{per}(A)}$. Namely, if $|X| = |Y| = n$ then $A = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix}$ for some square matrix B of order n, where O is the zero matrix of order n. Such matrix B is called a *bipartite adjacency matrix*. We shall denote the graph G as $G(B)$. Note that the matrix B is not unique. The number of perfect matchings of $G(B)$ is $\text{per}(B)$, see [7].

Let $\{F_n\}$ be the Fibonacci sequence, i.e., $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

The *k-Fibonacci sequence* $\{f_n^{(k)}\}$ for positive integer $k \geq 2$ is defined recursively by

$$f_n^{(k)} = f_{n-1}^{(k)} + f_{n-2}^{(k)} + \cdots + f_{n-k}^{(k)}, \text{ for } n \geq k,$$

with initial value $f_0^{(k)} = f_1^{(k)} = \cdots = f_{k-2}^{(k)} = 0$, $f_{k-1}^{(k)} = 1$. The number $f_n^{(k)}$ is called the *n-th k-Fibonacci number*. It is known that [6]

$$f_j^{(k)} = 2^{j-k}, \text{ for } k \leq j \leq 2k-1.$$

Note that $\{f_n^{(2)}\}$ is the Fibonacci sequence.

The *k-Lucas sequence* $\{l_n^{(k)}\}$ is defined by $l_n^{(k)} = f_{n-1}^{(k)} + f_{n+k-1}^{(k)}$ and $l_n^{(k)}$ is called the *n-th k-Lucas number*. It is known that $l_j^{(k)} = 2^{j-1}$, $1 \leq j \leq k-1$, and $l_k^{(k)} = 1 + 2^{k-1}$, see [6]. More about Lucas sequence can be found in [3]. Note that $\{l_n^{(2)}\}$ is the Lucas sequence.

A matrix is said to be a *$(0,1)$-matrix* if each of its entries is either 0 or 1. Suppose n and k are positive integers. Let $T_n = (t_{i,j})$ be an $n \times n$ tridiagonal $(0,1)$-matrix, where $t_{i,j} = 1$ if and only if $|j-i| \leq 1$. Let $U_n^{(k)} = (u_{i,j})$ be an $n \times n$ upper triangular $(0,1)$-matrix, where $u_{i,j} = 1$ if and only if $2 \leq j-i \leq k-1$ if $k \leq n$ and $U_n^{(k)} = U_n^{(n)}$ if $k > n$. Let $\mathscr{F}^{(n,k)} = T_n + U_n^{(k)}$ and let $\mathscr{C}^{(n,k)} = \mathscr{F}^{(n,k)} + E_{1,k+1} - \sum_{j=2}^{k} E_{1,j}$ for $n \geq 3$, where $E_{i,j}$ denotes the $n \times n$ matrix with 1 at the (i,j)-th entry and zeros elsewhere.

In [4, 5], Lee et al. found a class of bipartite graphs whose number of perfect matchings is $f_n^{(k)}$ and prove the following result.

Theorem 1.1: *For $n \geq 2$, the number of perfect matchings of $G(\mathscr{F}^{(n,k)})$ is $f_{n-1+k}^{(k)}$.*

In [2], Brualdi proved the following result:

Theorem 1.2: *For $n \geq 2$, let $A^{(n)} = I + U^{(n)}$. Then* $\operatorname{per}(A^{(n)}) = 2^{n-1}$.

Making use of Theorem 1.2, Lee [6] proved the following result.

Theorem 1.3: *For $n \geq 3$, the number of perfect matchings of $G(\mathscr{C}^{(n,k)})$ is $l_{n-1}^{(k)}$.*

In this paper, we shall show in Section 2 that the permanent of a special matrix is a linear combination of k-Fibonacci numbers. By making use of this result we obtain the number of perfect matchings of a larger class of bipartite graphs. Theorems 1.1 to 1.3 are special cases of this result. Moreover, in Section 3 we shall use this permanent to obtain the number of perfect matchings of certain bipartite graph which is not isomorphic to the graphs studied in [6].

2. Main results

For a fixed $\alpha = (a_1, a_2, \ldots, a_m) \in R^m$, where R is a ring. We define the (k, α)-*sequence* by

$$s(\alpha)_n^{(k)} = a_1 f_{n+k-2}^{(k)} + a_2 f_{n+k-3}^{(k)} + \cdots + a_m f_{n+k-m-1}^{(k)} = \sum_{i=1}^{m} a_i f_{n-1+k-i}^{(k)}, \ k \geq 2, \ n \geq 1.$$

The number $s(\alpha)_n^{(k)}$ is called the *n-th (k,α)-number*. Note that, if $\alpha = (1, \cdots, 1) \in \mathbb{Z}^k$, then $s(\alpha)_n^{(k)}$ is the $(n-1+k)$-th k-Fibonacci number $f_{n-1+k}^{(k)}$; if $\alpha = (1, 0, \cdots, 0, 1) \in \mathbb{Z}^{k+1}$, then $s(\alpha)_n^{(k)}$ is the $(n-1)$-st k-Lucas number $l_{n-1}^{(k)}$.

Theorem 2.1: *Suppose $n, k \geq 2$. Let*

$$B_n = \left(\begin{array}{c|ccc} a_1 & a_2 & \cdots & a_n \\ \hline 1 & & & \\ 0 & & \mathscr{F}^{(n-1,k)} & \\ \vdots & & & \\ 0 & & & \end{array} \right),$$

for some elements $a_1, a_2, \ldots, a_n$ in a ring R. Then $\operatorname{per}(B_n) = \sum_{i=1}^{n} a_i f_{n-1+k-i}^{(k)}$.

Proof: We shall prove the theorem by mathematical induction on n. Since

$$\operatorname{per}(B_2) = a_1 + a_2 = a_1 f_k^{(k)} + a_2 f_{k-1}^{(k)},$$

the theorem is true for $n = 2$.

Assume that the theorem is true for some $n \geq 2$. Expanding the permanent by the first column and by Theorem 1.1 and the induction assumption, we have

$$\mathrm{per}(B_{n+1}) = a_1\mathrm{per}(\mathscr{F}^{(n,k)}) + \mathrm{per}\begin{pmatrix} \begin{array}{c|ccc} a_2 & a_3 & \cdots & a_{n+1} \\ \hline 1 & & & \\ 0 & & \mathscr{F}^{(n-1,k)} & \\ \vdots & & & \\ 0 & & & \end{array} \end{pmatrix}$$

$$= a_1 f_{n+k-1} + \sum_{i=1}^{n} a_{i+1} f^{(k)}_{n-1+k-i}$$

$$= a_1 f_{n+k-1} + \sum_{i=2}^{n+1} a_i f^{(k)}_{n+k-i} = \sum_{i=1}^{n+1} a_i f^{(k)}_{n+k-i}.$$

Thus, the theorem is true for each $n \geq 2$. ■

Corollary 2.2: *For a fixed $m \geq 1$, suppose $n, k \geq 2$ and $n \geq m$. Let*

$$\mathscr{S}^{(n,k)}_{\alpha} = \begin{pmatrix} \begin{array}{c|cccccc} a_1 & a_2 & \cdots & a_m & 0 & \cdots 0 \\ \hline 1 & & & & & \\ 0 & & & \mathscr{F}^{(n-1,k)} & & \\ \vdots & & & & & \\ 0 & & & & & \end{array} \end{pmatrix}.$$

Then the number of perfect matching of $G(\mathscr{S}^{(n,k)}_{\alpha})$ is the n-th (k,α)-number with $\alpha = (a_1, a_2, \ldots, a_m)$.

Applying Corollary 2.2 and Theorem 2.1 by choosing $\alpha = (1, 1, \cdots, 1) \in \mathbb{Z}^k$ for $n > k$ and by choosing $a_i = 1$ for all $i = 1, 2, \ldots, n$ for $n \leq k$ respectively, we get Theorem 1.1. Applying Corollary 2.2 and Theorem 2.1 by choosing $\alpha = (1, 0, \cdots, 0, 1) \in \mathbb{Z}^{k+1}$ for $n > k$ and by choosing $a_1 = 1$ and $a_i = 0$ for all $i = 2, \ldots, n$ for $n \leq k$ respectively, we get Theorem 1.3.

3. Other results

From Theorem 1.3, the number of perfect matchings of $G(\mathscr{C}^{(n,2)})$ is $l^{(2)}_{n-1}$. In [6], there is a bipartite graph G, which is not isomorphic to $G(\mathscr{C}^{(n,2)})$ and whose number of perfect matchings is also $l^{(2)}_{n-1}$. Namely $G = G(B^{(n)})$, where $B^{(n)} = T_n + E_{1,3} - E_{2,3} + E_{2,4} - E_{3,4}$ for $n \geq 4$. Now we shall show another bipartite graph whose number of perfect matchings is $l^{(2)}_{n-1}$ too.

Let $C^{(n)} = T_n - E_{2,3} + E_{1,5}$ for $n \geq 5$. It is easy to see that both $G(C^{(n)})$ and $G(\mathscr{C}^{(n,2)})$ contain exactly one vertex of degree 4 when $n \geq 6$. It is easy to show that $G(C^{(6)})$ is not isomorphic to $G(\mathscr{C}^{(6,2)})$. Let a and b be the vertices of degree 4 in $G(C^{(n)})$ and $G(\mathscr{C}^{(n,2)})$ respectively. For $n \geq 7$, since b is adjacent to a vertex of degree 2 but a is not, $G(C^{(n)})$ is not isomorphic to $G(\mathscr{C}^{(n,2)})$. Since $G(B^{(n)})$ does not contain any vertex of degree 4 when $n \geq 6$, $G(C^{(n)})$ is not isomorphic to $G(B^{(n)})$. Note that $G(C^{(5)})$ is isomorphic to $G(B^{(5)})$.

Expanding the permanent by the first column and by Theorem 2.1 we have

$$\begin{aligned}
\operatorname{per}(C^{(n)}) =& \operatorname{per}\left(\begin{array}{ccccc|ccc}
1 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & \cdots \\
\hline
0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 0 & & \mathscr{F}^{(n-5,k)} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{array}\right) \\
=& \operatorname{per}\left(\begin{array}{c|ccc}
1 & 0 & 0 & \cdots \\
\hline
1 & & & \\
0 & & \mathscr{F}^{(n-2,k)} & \\
\vdots & & &
\end{array}\right) + \operatorname{per}\left(\begin{array}{c|cccccc}
1 & 0 & 0 & 1 & 0 & \cdots \\
\hline
1 & & & & & \\
0 & & & \mathscr{F}^{(n-2,k)} & & \\
\vdots & & & & &
\end{array}\right) \\
=& f_{n-1}^{(2)} + f_{n-1}^{(2)} + f_{n-4}^{(2)} = f_{n-1}^{(2)} + f_{n-2}^{(2)} + f_{n-3}^{(2)} + f_{n-4}^{(2)} = f_{n}^{(2)} + f_{n-2}^{(2)} = l_{n-1}^{(2)}
\end{aligned}$$

Thus the number of perfect matchings of the bipartite graph $G(C^{(n)})$ is $l_{n-1}^{(2)}$.

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