Some Edge-magic Cubic Graphs*

W. C. Shiu†
Department of Mathematics
Hong Kong Baptist University
224 Waterloo Road, Kowloon Tong
Hong Kong, China.
and
Sin-Min Lee
Department of Mathematics and Computer Science
San José State University
One Washington Square,
San José, CA 95192-0103, U.S.A.

Abstract

It was conjectured by Lee that a cubic simple graph with $4k+2$ vertices is edge-magic [5]. In this paper we show that the conjecture is not true for multigraphs or disconnected simple graphs in general. Several new classes of cubic edge-magic graphs are exhibited.

Key words and phrases : Edge-magic, supermagic, cubic graph.
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1 Introduction, Notations and Basic Concepts

In this paper, the term "graph" means finite multigraph (not necessary connected) having no loop and no isolated vertex. All undefined symbols and concepts may be looked up from [1]. A graph $G=(V,E)$ is a (p,q)-graph if p and q are its order and size respectively, i.e. $|V|=p$ and $|E|=q$.

Let $G=(V,E)$ be a (p,q)-graph. Let $f: E \to \{d, d+1, \ldots, d+q-1\}$ be a bijection for some $d \in \mathbb{Z}$. The *induced mapping* $f^+ : V \to \mathbb{Z}_r$ of f is defined by $f^+(u) = \sum_{uv \in E} f(uv)$ for $u \in V$, the sum is taken in $\mathbb{Z}_r$ for some $r \geq 0$. Note that we denote $\mathbb{Z}$ by $\mathbb{Z}_0$. If f^+ is a constant mapping,

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then G is called *d-edge-magic over $\mathbb{Z}_r$*. If $d = 1$, then G is simply called *edge-magic over $\mathbb{Z}_r$*, f an *edge-magic labeling of G over $\mathbb{Z}_r$* and the value of f^+ an *edge-magic value of G over $\mathbb{Z}_r$.*. Moreover, G being edge-magic over $\mathbb{Z}_p$ or $\mathbb{Z}$ is called *edge-magic* or *supermagic*, the labeling f is called an *edge-magic labeling* or *supermagic labeling*, respectively. These concepts were introduced by Lee, Seah and Tan in 1992 [4] and Stewart in 1966 [9] respectively. Note that edge-magic value is not unique in general. A necessary condition of a (p, q)-graph being edge-magic is

$$q(q+1) \equiv 0 \pmod p \tag{1.1}$$

It is easy to see the following theorem.

Theorem 1.1: Suppose G is d-edge-magic over $\mathbb{Z}_r, r \geq 0$. Then G is d-edge-magic over $\mathbb{Z}_s$ if s is a factor of r.

Let G be a graph and let k be a positive integer. $G[k]$ is a graph which is made up of k copies of G with the same set of vertices. We call $G[k]$ the *k-fold* of G. kG denote the disjoint union of k copies of G, i.e., $kG = \underbrace{G + \cdots + G}_{k \text{ times}}$, it is called *$k$-duplicate graph* of G.

Let S be a set. We use $S \times n$ to denote the multiset of n-copies of S. Note that S may be a multiset itself. From now on, the term "set" means multiset. Set operations are viewed as multiset operations. Let S be a set containing mn elements. Let $\mathscr{P}$ be a partition of S. If each class of $\mathscr{P}$ contains n elements, then $\mathscr{P}$ is called an *(m, n)-partition* of S. Moreover, if S is a set of numbers and the sum of numbers in each class is a constant, then $\mathscr{P}$ is called an *(m, n)-balance partition* of S. We shall use $[r]$ to denote the set $\{1, 2, \ldots, r\}$ for $r \geq 1$. Some special cases of balance partitions were considered in [2].

In 1993 the second author proposed the following conjecture [5] :

Conjecture: Every cubic simple graph of order $p \equiv 2 \pmod 4$ is edge-magic (over $\mathbb{Z}_p$).

Several classes of edge-magic cubic graphs were exhibited in [5]. In this paper we want to show that the conjecture is not true for multigraphs or disconnected simple graphs in general. Some new classes of edge-magic cubic multigraphs are introduced in this paper.

2 Some Non-edge-magic Cubic Multigraphs

In this section, we shall show some connected cubic multigraphs which are not edge-magic.

Before showing some examples, we would like to make some conventions. Let S and T be sets of integers. $S \equiv T \pmod{r}$ means that two sets are equal after their elements are taken modulo r, where $r \geq 2$. Since all arithmetic will be taken in $\mathbb{Z}_r$ for finding an edge-magic labeling of a graph G over $\mathbb{Z}_r$, the labels may be taken modulo r before labeling the edges. Therefore, the requirement of edge-magic over $\mathbb{Z}_r$ is equivalent to the following.

Let $G = (V, E)$ be a (p, q)-graph and let $S \equiv \{1, 2, \ldots, q-1\} \pmod{r}$. There exists a bijection $f : V \to S$ such that its induced mapping $f^+ : V \to \mathbb{Z}_r$ is a constant mapping.

Example 2.1: The graph described in Figure 2.1 is not edge-magic.

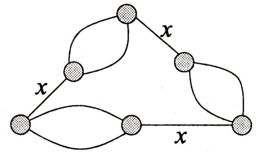


Figure 2.1

If the above graph is edge-magic then three single edges must be labeled by x for some x. But there are at most 2 labels can be the same. It is a contradiction. ∎

Example 2.2: The graph G described in Figure 2.2 is not edge-magic over $\mathbb{Z}_5$.

Suppose the graph $G = (V, E)$ has an edge-magic labeling over $\mathbb{Z}_5$. Then the labels must be assigned as in Figure 2.2, for some x, y, z, a and b in $\mathbb{Z}_5$.

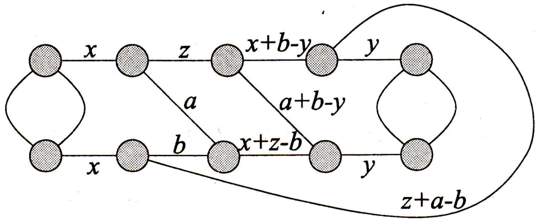


Figure 2.2

In this example, the arithmetic is taken in $\mathbb{Z}_5$. Then the mapping $g : E \to [\mathbf{15}] = [\mathbf{5}] \times 3$, defined by $g(e) = f(e) - x$ for any $e \in E$, is a $(1-x)$-edge-magic labeling over $\mathbb{Z}_5$. Since $\{g(e) \big| e \in E\} = \{k - x \big| k \in [\mathbf{15}]\} = [\mathbf{5}] \times 3$, g is also an edge-magic labeling of G over $\mathbb{Z}_5$. Thus we may assume $x = 0$.

Then $z + (a + b - y) + (b - y) = z + a$ implies $2(b - y) = 0$ and hence $b = y$. Thus Figure 2.2 becomes

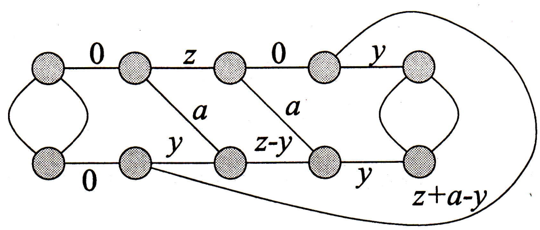


Figure 2.3

Since $z, z - y, a + z - y, y, 0$ are distinct, i.e., $\{z, z - y, a + z - y, y, 0\} = \mathbb{Z}_5$. Since $a \in \mathbb{Z}_5$ and $a \neq 0, a \neq y$ or $a \neq a + z - y, a = z$ or $a = z - y$.

If $a = z$, then we have to label two copies of $a - y$ and $2a - y$ to the unlabeled edges such that edge-magic value over $\mathbb{Z}_5$ is $2a$. We list all combinations as follows:

$+$	$a - y$	$2a - y$
$a - y$	$2a - 2y$	$3a - 2y$
$2a - y$	$3a - 2y$	$4a - 2y$

We need one of the three values equals to $2a$. Because $a \neq y$ and $y \neq 0$, the possible case is $3a - 2y = 2a$, i.e., $a = 2y$. In this case, consider the upper right corner vertex of the graph, we have $(3a - 2y) + y \neq 2a$. Thus this is not an edge-magic labeling.

If $a = z - y$, then we have to label two copies of $a + y$ and $2a$ to the unlabeled edges such that edge-magic value over $\mathbb{Z}_5$ is $2a + y$. We list all combinations as follows:

$+$	$a+y$	$2a$
$a+y$	$2a+2y$	$3a+y$
$2a$	$3a+y$	$4a$

We need one of the three values equals to $2a + y$. But this is impossible because $a \neq 0$, $y \neq 0$ and $2a \neq y$. ∎

3 A Class of Supermagic Duplicate Graphs

In this section we shall show that $mK_2[n]$ is supermagic when n is even or both m and n are odd. In particular, the duplicate cubic graph $mK_2[3]$ is supermagic when m is odd.

Lemma 3.1 ([6, Lemmas 3.1 and 3.3]): *Suppose $m, n \geq 2$. If n is even or both m and n are odd, then $[\mathbf{mn}]$ has an (m, n)-balance partition.*

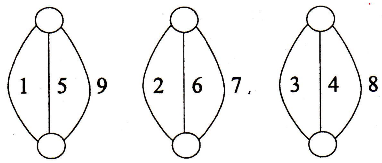
Theorem 3.2: For m, $n \geq 2$, $mK_2[n]$ *is supermagic if and only if n is even or both m and n are odd.*

Proof: If $mK_2[n]$ is supermagic, from (1.1) $mn(mn + 1) \equiv 0 \pmod{2m}$ or equivalently $n(mn + 1) \equiv 0 \pmod{2}$. Then n is even or both m and n are odd.

Conversely, by Lemma 3.1 we have an (m, n)-balance partition $\mathscr{P}$ of $[\mathbf{mn}]$. Elements of each class of $\mathscr{P}$ are labeled to edges of a copy of $K_2[n]$. ∎

Corollary 3.3: If m is odd, then $mK_2[3]$ *is supermagic.*

Example 3.1: $\Big\{\{1, 5, 9\}, \{2, 6, 7\}, \{3, 4, 8\}\Big\}$ is a (3,3)-balance partition of $[\mathbf{9}]$. We label $3K_2[3]$ as follows :



More classes of supermagic graphs were exhibited in [3, 7, 8, 9, 10].

4 A Class of Edge-magic Connected Graphs

In this section we shall give a class of edge-magic connected cubic graph. For $n \geq 1$, let $L_n = (V, E)$, a *ladder graph*, where

$$V = \{u_0, u_1, \ldots, u_n, v_0, v_1, \ldots, v_n\}$$

and

$$E = \{u_{i-1}u_i \big| 1 \leq i \leq n\} \cup \{v_{i-1}v_i \big| 1 \leq i \leq n\} \cup \{u_j v_j \big| 0 \leq j \leq n\} \cup \{e_0, e_n\},$$

e_0 and e_n are parallel edges of u_0v_0 and u_nv_n respectively.
Note that there are two parallel edges incident with u_0, v_0 and u_n, v_n respectively.

Theorem 4.1: L_{2t} is *edge-magic* for $t \geq 1$.
Proof: In this proof, the arithmetic is taken in $\mathbb{Z}_{4t+2}$. Define $f : E \to$ $[\mathbf{4t+2}] \cup [\mathbf{2t+1}]$ by

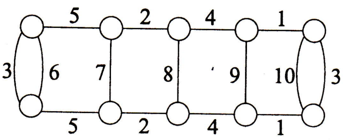
$$\begin{aligned} f(u_{i-1}u_i) = f(v_{i-1}v_i) &= \begin{cases} 2t+1-\frac{i-1}{2} & \text{if } i \text{ is odd} \\ t+1-\frac{i}{2} & \text{if } i \text{ is even} \end{cases}, 1 \leq i \leq 2t; \\ f(u_j v_j) &= 2t+2+j,\ 0 \leq j \leq 2t; \end{aligned}$$

and

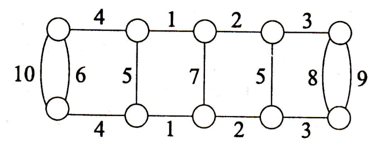
$$f(e_0) = f(e_{2t}) = t+1.$$

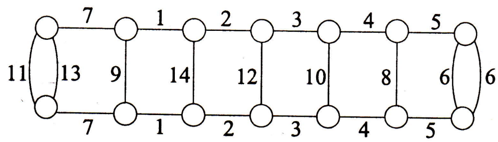
It is easy to see that f is a bijection and $f^+(w) = t+2$ for each $w \in V$. ▮

Example 4.1: Consider L_4. An edge-magic labeling is



There are other edge-magic labelings for L_4 and L_6 with edge-magic values 0 and 3 described below, respectively.





5 Some Edge-magic Disconnected Cubic Graphs

In this section, we shall consider the edge-magicness of two classes of cubic graphs $mK_4 + nK_2[3]$ and $mK_4 + nK_{3,3}$ for $m \geq 0$ and $n \geq 0$. They are $(4m+2n, 6m+3n)$- and $(4m+6n, 6m+9n)$-graphs respectively. If they are edge-magic, then from (1.1), n must be odd. For $m = 0$ we have shown that $nK_2[3]$ is supermagic. $nK_{3,3}$ is also supermagic. In [8], there is a general result on supermagicness of $sK_{n,n}$. We shall provide a construction of an edge-magic labeling of $nK_{3,3}$ over $\mathbb{Z}$ below. Thus we assume that n is odd and $m \geq 1$.

Theorem 5.1: *Suppose $m, n \geq 1$ and n is odd. If $n < m$, then $H = mK_4 + nK_2[3]$ is not edge-magic over $\mathbb{Z}_{2m+n}$. The graph $n(K_4 + K_2[3])$ is edge-magic over $\mathbb{Z}_{3n}$.*

Proof: In this proof, the arithmetic is taken in $\mathbb{Z}_{2m+n}$. Let $H_1 = mK_4$ and $H_2 = nK_2[3]$. Suppose there is an edge-magic labeling f of H over $\mathbb{Z}_{2m+n}$. Let the labels be assigned on a component of H_1, which is isomorphic to K_4, as Figure 5.1 (a):

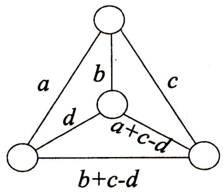


Figure 5.1 (a)

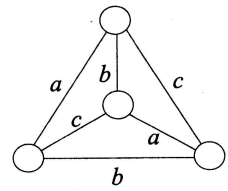


Figure 5.1 (b)

Then $(a + c - d) + (b + c - d) = a + b$, and hence $c = d$. Since $[\mathbf{6m + 3n}] = [\mathbf{2m + n}] \times 3$, a, b and c are distinct. Thus two copies of three distinct numbers are labeled in each component of H_1 and numbers labeled in difference components of H_1 are distinct. Therefore, there are two copies of $3m$ distinct numbers labeled on components of H_1, hence $n \geq m$.

Now we consider $H = n(K_4 + K_2[3])$. We also let $H_1 = nK_4$ and $H_2 = nK_2[3]$. By Lemma 3.1, $[\mathbf{3n}]$ has an $(n, 3)$-balance partition $\mathscr{P}$. Since $[\mathbf{9n}] = [\mathbf{3n}] \times \mathbf{3}$, we have 3 copies of $\mathscr{P}$. For each class of $\mathscr{P}$, say $C = \{a, b, c\}$, we use elements of two copies of C to label a component of H_1 as Figure 5.1 (b) and one copy to label a component of H_2 arbitrarily. Then this is an edge-magic labeling of H over $\mathbb{Z}_{3n}$. ▮

Corollary 5.2: *If $n \geq 1$ and n is odd, then $n(K_4 + K_2[3])$ is edge-magic.*
Proof : We keep the notations of the proof of Theorem 5.1. It suffices to extend the edge-magic labeling over $\mathbb{Z}_{3n}$ obtained in the proof of Theorem 5.1 to an edge-magic labeling of H. For each component of H_1, we add the numbers which are labeled on the exterior triangle by $3n$. Then this is an edge-magic labeling of H. ▮

Example 5.1: Consider $H = K_4 + K_2[3]$. Figure 5.2 (a) is an edge-magic labeling f of H over $\mathbb{Z}_3$ and Figure 5.2 (b) is an edge-magic labeling of H extended from f. Note that this labelings are unique up to isomorphism.

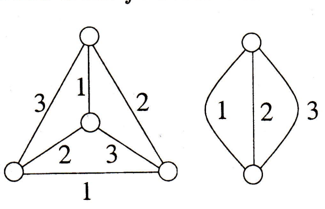


Figure 5.2 (a)

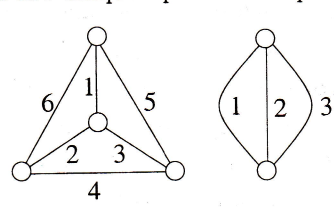
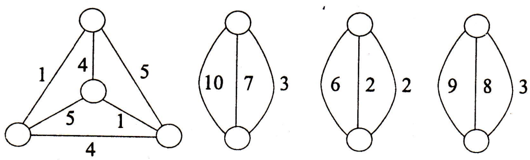


Figure 5.2 (b)

Example 5.2 : One can show that $K_4 + 2K_2[3]$ is not edge-magic over $\mathbb{Z}_4$. But $K_4 + 3K_2[3]$ is edge-magic. Here is an edge-magic labeling of it:



Lemma 5.3 [8]: *If n is odd then $nK_{3,3}$ is supermagic.*

Proof : To show $nK_{3,3}$ being supermagic is equivalent to find n 3×3 matrices $A_i, 1 \le i \le n$, such that the set of all entries of these matrices is $[\mathbf{9n}]$ and row sums and columns sums of these matrices are the same. For $n = 1$, it is known that $K_{3,3}$ is supermagic [3, 7, 9]. Thus we may assume $n \ge 3$.

By Lemma 3.1, $[\mathbf{3n}]$ has an $(n, 3)$-balance partition $\mathscr{P}$. Let $C_1, C_2, \dots, C_n$ be its classes. Let

$$X = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

Clearly they are two orthogonal Latin squares. Using elements of C_i and the format of X we obtain a Latin square $L_i, 1 \le i \le n$. Then $A_i = 3nY + L_i$ are required matrices. ∎

Theorem 5.4: *Suppose $m, n \ge 1$ and n is odd. If $3n < m$, then $H = mK_4 + nK_{3,3}$ is not edge-magic over $\mathbb{Z}_{2m+3n}$. The graph $3nK_4 + nK_{3,3}$ is edge-magic over $\mathbb{Z}_{9n}$.*

Proof: Let $H_1 = mK_4$ and $H_2 = nK_{3,3}$. The first part of the theorem follows from the same argument of the first part of the proof of Theorem 5.1.

For the second part, similar to the proof of the second part of Theorem 5.1, we have 3 copies of a $(3n, 3)$-balance partition $\mathscr{P}$ of $[\mathbf{9n}]$. Use two copies of $\mathscr{P}$ to label the components of H_1. Then the edge-magic value over $\mathbb{Z}_{9n}$ is $\frac{3(3n+1)}{2}$. Now we have to label the elements of $[\mathbf{9n}]$ to the edges of H_2. Since H_2 is supermagic, there is a supermagic labeling of H_2 and the edge-magic value over $\mathbb{Z}$ is $\frac{3(9n+1)}{2}$. Since $\frac{3(9n+1)}{2} \equiv \frac{3(3n+1)}{2} \pmod{9n}$, H

has an edge-magic labeling over $\mathbb{Z}_{9n}$. ∎

Corollary 5.5: *$3nK_4 + nK_{3,3}$ is edge-magic.*
Proof : Same proof as the proof of Corollary 5.2. ∎

Example 5.3: Consider $H = 3K_4 + K_{3,3}$. Use the $(3,3)$-balance partition of $[9]$ described in Example 3.1 to label three K_4 components. Let $M = \begin{pmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{pmatrix}$ be a magic square (for the case of $n = 1$, we use magic square to construct a supermagic labeling of $K_{3,3}$). Hence Figure 5.3 (a) is an edge-magic labeling f of H over $\mathbb{Z}_9$ and Figure 5.3 (b) is an edge-magic labeling of H extended from f.

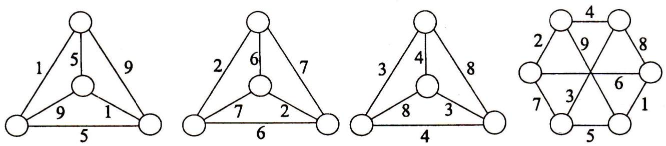


Figure 5.3 (a)

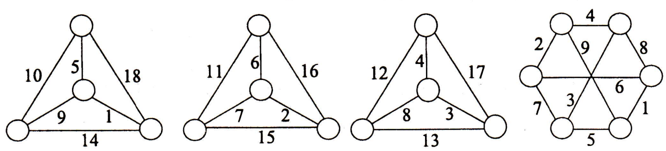


Figure 5.b (a)

Example 5.4: *$H = K_4 + K_{3,3}$ is not edge-magic over $\mathbb{Z}_5$.*

In this example the arithmetic is taken in $\mathbb{Z}_5$. If H has an edge-magic labeling f over $\mathbb{Z}_5$, then by the proof of Theorem 5.1 or 5.4 the K_4 component is labeled as Figure 5.1 (b) for some a, b and c in $\mathbb{Z}_5$. Let the other two numbers differing from a, b and c be x and y. We have to fill $\{a, b, c, x, x, x, y, y, y\}$ into a matrix A such that its row sums and column sums are the same. If two of a, b and c are in the same row (similarly column) then the other one must lie in the same row. If this is a case, since $x \neq y$, the unique assignment is $\begin{pmatrix} a & b & c \\ x & x & x \\ y & y & y \end{pmatrix}$. It is impossible because of a, b and c are distinct. Thus a, b and c must be filled in the diagonal (under an isomorphism). If two x lie in the same row, without loss of generality, we may assume (a, x, x) is the first row of A, then the first column must be $(a, y, y)^T$. Since $x \neq y$, it is impossible. Thus each row contains x and y. Since a, b and c are distinct, it is also impossible. ∎

Example 5.5: $H = 2K_4 + K_{3,3}$ *is edge-magic.*

Suppose H has an edge-magic labeling over $\mathbb{Z}_7$. Let a_1, b_1, c_1 and a_2, b_2, c_2 be numbers in $\mathbb{Z}_7$ that labeled on edges of two K_4 components and d be the rest number.

Then

$$a_1 + b_1 + c_1 \equiv a_2 + b_2 + c_2 \equiv 3d \pmod 7.$$

We list all the possible cases below (up to isomorphism):

$d \pmod 7$	1	2	3	4	5	6	7
$\{a_1, b_1, c_1\} \pmod 7$	$\{2,3,5\}$	$\{1,5,7\}$	$\{1,2,6\}$	$\{1,5,6\}$	$\{1,3,4\}$	$\{1,3,7\}$	$\{1,2,4\}$
$\text{Sum}_1 \pmod{14}$	10	13	9	12	8	11	7
$\{a_2, b_2, c_2\} \pmod 7$	$\{4,6,7\}$	$\{3,4,6\}$	$\{4,5,7\}$	$\{2,3,7\}$	$\{2,6,7\}$	$\{2,4,5\}$	$\{3,5,6\}$
$\text{Sum}_2 \pmod{14}$	3	13	2	12	1	11	0
$3d+7 \pmod{14}$	11	13	2	5	8	11	0
$\text{Sum}_1 + 21 \pmod{14}$	3	6	2	5	1	4	0
$\text{Sum}_2 + 21 \pmod{14}$	10	6	9	5	8	4	7

Table 5.1

We have to fill $\{a_1, b_1, c_1, a_2, b_2, c_2, d, d, d\}$ into a 3×3 matrix A such that its row sums and columns are the same. By similar argument in Example 5.4, we can show that there is the unique assignment (up to isomorphism) which is $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ d & d & d \end{pmatrix}$, where $a_1 + a_2 \equiv b_1 + b_2 \equiv c_1 + c_2 \equiv 2d \pmod 7$.

For example, $\begin{pmatrix} 1 & 2 & 4 \\ 6 & 5 & 3 \\ 7 & 7 & 7 \end{pmatrix}$. In each possible case an edge-magic labeling of H over $\mathbb{Z}_7$ can be constructed.

Now if H has an edge-magic labeling. Then this labeling must be extended from one of the above labelings. If it is possible, then seven distinct numbers among all the labels must be increased by 7. We shall call these numbers to be *added numbers*. Then each row sum of A is $3d+7 \pmod{14}$.

Therefore, the edge-magic value of H (over $\mathbb{Z}_{14}$) is $3d+7$.

If all added numbers are labeled in the $K_{3,3}$ component, then Sum_1, Sum_2, (the edge-magic values come from two K_4 components), Sum_1+21 (the edge-magic values come from the $K_{3,3}$ component) and $3d+7$ are the same modulo 14. Table 5.1 shows that it is impossible. Thus there is at least one added number, without loss of generality say a_1, which is labeled in a K_4 component. Then one of b_1 and one of c_1 in that component must be increased by 7. Thus the first row sum of A is $a_1+b_1+c_1 \equiv \mathrm{Sum}_1 \pmod{14}$. Only the 2nd and the 5th columns of Table 5.1 will be possible. By considering the edge-magic value come from the other K_4 component, the 5th column is not a possible case. Fortunately, an edge-magic labeling can be extended from the case when $d=2$. Namely,

$$A = \begin{pmatrix} 1 & 5 & 7 \\ 3 & 6 & 4 \\ 9 & 2 & 2 \end{pmatrix}.$$

∎

We invite the reader to consider the following unsolved problems.

Question 1: Is $K_4+nK_2[3]$ edge-magic for odd $n \geq 5$?

Question 1′ : Is $mK_4+nK_2[3]$ edge-magic for odd $n \geq 5$ and $1 \leq m < n$?

Question 1″ : Is $mK_4+nK_2[3]$ edge-magic over $\mathbb{Z}_{2m+n}$ for even $n \geq 2$ and $m \geq 1$?

Question 2 : Is $K_4+nK_{3,3}$ edge-magic for odd $n \geq 3$?

Question 2′ : Is $mK_4+nK_{3,3}$ edge-magic for odd $n \geq 3$ and $1 \leq m \leq 3n$?

Question 2″ : Is $mK_4+nK_{3,3}$ edge-magic over $\mathbb{Z}_{2m+3n}$ for even $n \geq 2$ and $m \geq 1$?

Finally we propose a modified conjecture below

Conjecture: Every connected cubic simple graph of order $n \equiv 2 \pmod 4$ is edge-magic.

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