On the ℓ-distance face coloring of regular plane graphs*

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Abstract

The ℓ-distance face chromatic number of a connected plane graph is the minimum number of colors in a coloring of its faces so that whenever two different faces are at distance ℓ or less, they receive different colors. In this paper, we estimate the ℓ-distance face chromatic numbers for connected 6-regular plane graphs. Also, we have a general result on n-regular plane graphs with $n \geq 6$.

Key words and phrases: Plane graph, ℓ-distance face chromatic number

2000 AMS MSC: 05C15

1 Introduction

In this paper, all graphs $G = (V, E, F)$ are connected plane graphs with at least two vertices, loops and multiple edges are allowed, where V, E and F are the sets of vertices, edges and faces of G respectively. We denote the numbers of its vertices, edges and faces by ν, ε and ϕ respectively.

Let $G = (V, E, F)$. The *degree* of a face f of G, denoted by $d_G(f)$ (or simply $d(f)$), is the number of edges incident with f (edges incident with exactly one face are counted twice). Suppose $g_1, g_2 \in F$ and $u \in V$. The

*Partially supported by Faculty Research Grant, Hong Kong Baptist University.

distance between u and g_2, denoted by $d_G(u, g_2)$ (or simply $d(u, g_2)$), is the minimum distance $d_G(u, x_2)$ of all vertices x_2 incident with g_2. The *distance between g_1 and g_2*, denoted by $d_G(g_1, g_2)$ (or simply $d(g_1, g_2)$), is the minimum distance $d_G(x_1, g_2)$ of all vertices x_1 incident with g_1.

For $\ell \geq 0$, an *ℓ-distance face k-coloring* of a graph $G = (V, E, F)$ is a mapping $\varphi : F \to \{1, 2, \ldots, k\}$ such that if $d(g_1, g_2) \leq \ell$ for $g_1 \neq g_2$, then $\varphi(g_1) \neq \varphi(g_2)$. The *$\ell$-distance face chromatic number* $\chi^{\ell}_{df}(G)$ is the minimum k such that there is an ℓ-distance face k-coloring of G.

The special face coloring was originally studied for cubic plane graphs by Bouchet *et al.* [3] and Bordin [2] as the Heawood face coloring.

An Heawood-coloring (or h-coloring for short) of F is a mapping $h : F \to \{1, 2, \ldots, k\}$ such that for each edge e, the faces incident with the ends of e have pairwise different colors, which in fact is 1-distance face coloring. In [4], the h-coloring was generalized and studied by Hornak and Jendrol for 4-regular plane graphs and prove that $\chi^1_{df}(G) \leq 21$ for any 4-regular plane graph G. In this paper, we shall study the ℓ-distance face coloring for connected n-regular plane graphs G with $n \geq 6$ and give the upper bound and lower bound of $\chi^{\ell}_{df}(G)$. For simplicity proof, we first proof the result on 6-regular connected graphs.

2 Lemmas

We need several auxiliary lemmas for our main theorem.

Lemma 1: *If f is a face of a 6-regular graph G, then the number of faces of G at distance at most ℓ from f is at most* $1 + 4d(f)5^{\ell}$.

Proof: Let $v_i(f)$ be the number of vertices and $\phi_i(f)$ the number of faces at distance i from f. Any face of G at distance i from f is incident with a vertex at distance i from f. If x and y are vertices of G at distance i and $i-1$ from f, respectively, where $i \geq 1$, and if xy is an edge of G, then faces of G incident with xy are at distance at most $i-1$ from f. Hence at most four among the faces incident with x are at distance i from f and we have $\phi_i(f) \leq 4v_i(f)$ for every $i \geq 1$. Similarly, we have $v_i(f) \leq 5v_{i-1}(f)$ for $i \geq 2$ and $v_1(f) \leq 4v_0(f) = 4d(f)$. Since $\phi_0(f) \leq 1 + 4d(f)$ (note that f is at distance 0 from itself), the number of faces at distance at most ℓ

from f is

$$\begin{aligned}
\sum_{i=0}^{\ell} \phi_i(f) &\leq 1 + 4d(f) + \sum_{i=1}^{\ell} 4v_i(f) \\
&= 1 + 4d(f) + 4\sum_{i=1}^{\ell} v_i(f) \\
&\leq 1 + 4d(f) + 4\sum_{i=1}^{\ell} 4d(f)5^{i-1} \\
&= 1 + 4d(f) + 4d(f)(5^{\ell} - 1) \\
&= 1 + 4d(f)5^{\ell}. \qquad \blacksquare
\end{aligned}$$

By a similar proof, we obtain the following general result.

If G is an n-regular plane graph with $n \geq 2$, then the number of faces of G at distance at most ℓ from f is at most $1 + (n-2)d(f)(n-1)^{\ell}$.

A subset H of F is said to be an (ℓ, k)-*colorable* of a graph $G = (V, E, F)$ if its elements can be denoted $h_1, h_2, \ldots, h_n$ in such a way that the number of the faces in the set $F \setminus \{h_{i+1}, \ldots, h_n\}$ which are at distance at most ℓ from the face h_i is at most k for any $i = 1, 2, \ldots, n$.

Lemma 2 [4]: *If φ is a partial ℓ-distance face k-coloring of a connected plane graph G such that the set H of the uncolored faces is (ℓ, k)-colorable, then φ can be extended to an ℓ-distance face k-coloring of G.*

Lemma 3 [1]: *If G is a simple connected plane graph, then $\varepsilon \leq 3\nu - 6$.*

3 The main result

Theorem 4: *Let $\ell \geq 1$. Suppose $G = (V, E, F)$ is a 6-regular graph. Then $6 \leq \chi_{df}^{\ell}(G) \leq \max\{1 + 8 \times 5^{\ell}, 2(\nu - 2)\}$.*

Proof: Let $F_k(G)$ be the set of all faces of degree k in G and let $f_k(G)$ be its cardinality. Let $r = \max\{1 + 8 \times 5^{\ell}, \sum_{i=3}^{\infty} f_k(G)\}$. It is clear that there is a partial ℓ-distance face r-coloring for faces of G of degree at least 3 such that each color occurs at most once. Because $r \geq 1 + 8 \times 5^{\ell} \geq 1 + 4d(f)5^{\ell}$

for any face $f \in H$, and because of Lemma 1, the set $H = \bigcup_{i=1}^{2} F_i(G)$ is (ℓ, r)-colorable. By Lemma 2, we have $\chi_{df}^{\ell}(G) \leq r$.

By Lemma 3 and because $\varepsilon = 3\nu$, there exist at least 6 loops or multiple edges, so $|H| \geq 6$. By Euler's formula, $\phi = \varepsilon - \nu + 2 = 3\nu - \nu + 2 = 2\nu + 2$, we have $\sum_{i=3}^{\infty} f_k(G) \leq 2\nu + 2 - 6 = 2(\nu - 2)$, which implies that $\chi_{df}^{\ell}(G) \leq \max\{1 + 8 \times 5^{\ell}, 2(\nu - 2)\}$.

Let $xy \in E$. Let F' be the set of faces incident with either x or y. Let $K = (V', E', F')$ be the subgraph of G induced by F' (i.e., K is the subgraph induced by the edges incident with faces in F'). Since there are no vertices of degree 1, the degree of a face f is equal to the number of vertices occurred in its boundary. Since each vertex is incident with at least two faces, we have

$$2|E'| = \sum_{f \in F'} d_K(f) \geq 2(|V'| - 2) + d_K(x) + d_K(y) = 2(|V'| - 2) + 12.$$

This implies $|E'| \geq |V'| + 4$. By Euler's formula, $|F'| = |E'| - |V'| + 2 \geq 6$. So $\chi_{df}^{\ell}(G) \geq 6$. This completes the proof of the theorem. ∎

Remark: We conjecture that $\chi_{df}^{1}(G) \leq 41$ for any 6-regular graph G, but the method of transforming graphs in [4] does not work.

By a proof similar to that of Theorem 4, we have the following theorem.

Theorem 5: *Let $\ell \geq 1$. Suppose $G = (V, E, F)$ is an n-regular graph with $n \geq 6$. Then $n \leq \chi_{df}^{\ell}(G) \leq \max\{1 + 2(n-2)(n-1)^{\ell}, 2(\nu - 2)\}$.*

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