ON THE INTEGER-MAGIC SPECTRA OF GRAPHS

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ABSTRACT. Let A be a non-trivial abelian group and $A^* = A - \{0\}$. A graph is A-magic if there exists an edge-labeling using elements of A^* which induces a constant vertex labeling of the graph. Although a fair amount of research has been done on A-magic labelings of graphs, there is much which is still unknown. In this paper, we construct a large collection of non-intuitive examples and counter-examples, which provide further insight into the integer-magic spectra of graphs. Particular attention is devoted to the integer-magic spectra of products of graphs.

1. INTRODUCTION

Let $G = (V, E)$ be a connected simple graph. For any non-trivial abelian group A (written additively), let $A^* = A - \{0\}$. A function $f : E \to A^*$ is called a *labeling* of G. Any such labeling induces a map $f^+ : V \to A$, defined by $f^+(v) = \Sigma f(u, v)$, where the sum is over all $(u, v) \in E$. If there exists a labeling f whose induced map on V is a constant map, we say that f is an *A-magic labeling* of G and that G is an *A-magic* graph. The corresponding constant is called an *A-magic value*. The *integer-magic spectrum* of a graph G is the set $\mathrm{IM}(G) = \{k : G \text{ is } \mathbb{Z}_k\text{-magic and } k \geq 2\}$. Note that the integer-magic spectrum of a graph is not to be confused with the set of achievable magic values. $\mathbb{Z}$-magic graphs were considered by Stanley [23, 24], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1, 2, 3] and others [7, 9, 15, 16, 21] have studied A-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [4, 6, 8, 10, 11, 12, 13, 14, 17, 18, 22].

2. SOME RESULTS

The main thrust of this paper is to provide non-intuitive examples and counter-examples regarding the integer-magic spectra of graphs. We first recall some important results from [16].

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Theorem A. *Every eulerian graph is $\mathbb{Z}_2$-magic.*

Theorem B. *Let G be an eulerian graph with an even number of edges. Then, G is A-magic, for every non-trivial abelian group A.*

Theorem C. *Let G be a $\mathbb{Z}_k$-magic graph, with $k|n$. Then, G is a $\mathbb{Z}_n$-magic graph.*

What more can be said about the integer-magic spectra of graphs? Let us look at the integer-magic spectra from some different points of view.

Definition. Suppose that graph G is A-magic, for some non-trivial abelian group A. If G is A_1-magic for all non-trivial subgroups A_1 of A, then G is an A-**divisible** magic graph.

We immediately note that if G is *fully-magic*, then G is an A-divisible magic graph, for any non-trivial A. The natural question to ask is the following: Are there ways to construct infinite classes of $\mathbb{Z}_k$-divisible magic graphs which are not fully-magic? Indeed, the answer to this question is yes. In a later section of the paper, some results are established which allow for this type of construction. For now, we leave the reader with the following example.

Example. In [18], Salehi and Bennett proved the following result: If $n \geq 3$ and $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the prime factorization of $n-1$, then $\mathrm{IM}(K_{1,n}) = \bigcup_{i=1}^{k} p_i\mathbb{N}$. Thus, this result gives us infinite classes of $\mathbb{Z}_{n-1}$-divisible magic graphs, where $n \geq 3$. For example, $K_{1,37}$ is a $\mathbb{Z}_{36}$-divisible magic graph.

Of course, one can view the integer-magic spectra in another way.

Definition. Suppose that graph G is A-magic, for some non-trivial abelian group A. If G is not A_1-magic for all non-trivial subgroups A_1 of A, then G is an A-**indivisible** magic graph.

To begin with, let us construct some infinite classes of $\mathbb{Z}_n$-indivisible magic graphs. The *derived graph* $D(G)$ of a graph G is the graph obtained from G by deleting all pendants of G. A *caterpillar* is a tree T such that $D(T) = P$ is a path. Namely, suppose $P = x_1 \cdots x_t$ and suppose n_i pendants are adjacent with x_i in T, where $n_1 \geq 1$, $n_t \geq 1$ and $n_i \geq 0$ for $2 \leq i \leq t-1$. Then, we denote T by $cat(n_1, \ldots, n_t)$. The path P is called the *central path*.

Lemma 1. *Let A be an abelian group and let $k_1, \ldots, k_s$ be s fixed positive integers. If $G = cat(k_1+1, \ldots, k_s+1, k+1)$ is A-magic with A-magic value*

m, then

$$km = m\sum_{j=1}^{s}(-1)^{s+j}k_j.$$

Proof. Let the edges of the central path are labeled by $a_1, \dots, a_s$ in the natural order. Then by considering each vertex of the central path, we get G is A-magic with A-magic value m if and only if

$$a_1 = -k_1 m;\ a_i = -k_i m - a_{i-1}, 2 \le i \le s;\ \text{and } km = -a_s$$

is solvable in A^*. Thus, we have

$$a_i = m\sum_{j=1}^{i}(-1)^{i+j-1}k_j \text{ for } 1 \le i \le s, \text{ and}$$

$$km = m\sum_{j=1}^{s}(-1)^{s+j}k_j.$$

□

Theorem 4. *Let $l_1, l_2, \dots, l_s$ be distinct positive integers. Suppose $n \ge 2$ is not a factor of l_i for all i. Let $k_1 = l_1$ and $k_i = l_i + l_{i-1}$ for $2 \le i \le s$. Let k be a positive integer such that $k \equiv \sum_{j=1}^{s}(-1)^{s+j}k_j = l_s \pmod{n}$. Then, $G = cat(k_1+1, \dots, k_s+1, k+1)$ is $\mathbb{Z}_n$-magic but not $\mathbb{Z}_{l_i}$-magic for each i $(1 \le i \le s)$, if $l_i \ge 2$.*

Proof. Let us keep the notation defined in Lemma 1. From the proof of Lemma 1, we choose $m = 1$. Then, we get $a_i = \sum_{j=1}^{i}(-1)^{i+j-1}k_j = -l_i \not\equiv 0 \pmod{n}$. Hence we have a $\mathbb{Z}_n$-magic labeling for G with $\mathbb{Z}_n$-magic value 1.

Suppose there is a $\mathbb{Z}_{l_i}$-magic labeling of G with $\mathbb{Z}_{l_i}$-magic value m. By the proof of Lemma 1, we have $a_i = -l_i m \equiv 0 \pmod{l_i}$ if $l_i \ge 2$. □

Example. Suppose $n = 6$. Using the notation introduced in Theorem 4, $l_1 = 3$ and $l_2 = 2$. Hence $k_1 = 3$, $k_2 = 5$ and $k = 2$. One can verify that $cat(4, 6, 3)$ is $\mathbb{Z}_6$-magic with $\mathbb{Z}_6$-magic value 1, but it is neither $\mathbb{Z}_3$-magic nor $\mathbb{Z}_2$-magic.

Example. Suppose $n = 12$. Using the notation introduced in Theorem 4, $l_1 = 6$, $l_2 = 3$ and $l_3 = 2$. Hence $k_1 = 6$, $k_2 = 9$, $k_3 = 5$ and $k = 2$. One can verify that $cat(7, 10, 6, 3)$ is $\mathbb{Z}_{12}$-magic with $\mathbb{Z}_{12}$-magic value 1, but it is neither $\mathbb{Z}_6$-magic, $\mathbb{Z}_3$-magic nor $\mathbb{Z}_2$-magic.

A natural question to ask is the following: Is there a way to construct a graph G, where $\mathrm{IM}(G) = \{k : k \geq n\}$ for a fixed n? The following corollary answers this in the affirmative.

Corollary 1. *For any positive integer $n \geq 2$, there is a caterpillar T with $\mathrm{IM}(T) = \{k : k \geq n\}$. Hence, T is an $\mathbb{Z}_n$-indivisible magic graph.*

Proof. Choose $\{l_1, l_2, \ldots, l_{n-2}\} = \{2, \ldots, n-1\}$ in Theorem 4. From this, the corollary follows. □

Example. Let $G = cat(3, 6, 8, 5)$. Then, $\mathrm{IM}(G) = \{5, 6, 7, \ldots\}$.

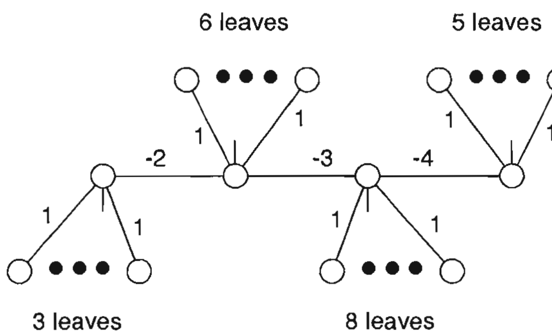

FIGURE 1. $\mathrm{IM}(cat(3, 6, 8, 5)) = \{5, 6, 7, \ldots\}$.

Of course, an A-magic graph might be A_1-magic for some (but not all) non-trivial subgroups A_1 of A. In particular, the converse of Theorem C is not true in general.

Example. Graph H in Figure 2 is eulerian and hence, is $\mathbb{Z}_2$-magic. By Theorem C, H is $\mathbb{Z}_6$-magic. However, it is straight-forward to show that H is not $\mathbb{Z}_3$-magic.

3. Integer-magic spectra of graph products

Now, we focus on the integer-magic spectra of certain types of graph products. In [15], the following theorems were established.

Theorem E. *Let A be an abelian group. If G_1 and G_2 are A-magic graphs, then $G_1 \times G_2$ is an A-magic graph.*

Theorem F. *Let A be an abelian group. Then, the lexicographic product of two A-magic graphs is A-magic.*

From Theorems E and F, we immediately get the following two corollaries.

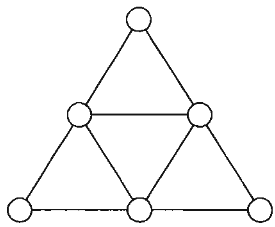

FIGURE 2. Graph H.

Corollary 2. *The Cartesian product of two A-divisible magic graphs is an A-divisible magic graph.*

Proof. Let G_1 and G_2 be two A-divisible magic graphs. In particular, G_1 and G_2 are A-magic which implies that $G_1 \times G_2$ is A-magic. Now, let A_1 be a non-trivial subgroup of A. Since G_1 and G_2 are A_1-magic graphs, this implies that $G_1 \times G_2$ is A_1-magic. Thus, $G_1 \times G_2$ is A-divisible magic. □

Corollary 3. *The lexicographic product of two A-divisible magic graphs is an A-divisible magic graph.*

Proof. The result follows from an analogous argument, as found in the proof of Corollary 2. □

Also, one should note that the converses of Theorems E and F are not true in general.

Example. The following result was established in [14]: For $n \geq 2$ and $m \geq 3$, $\mathrm{IM}(P_n \times P_m) = \{3, 4, 5, \ldots\}$. However, clearly $\mathrm{IM}(P_k) = \{2, 3, 4, 5, \ldots\}$ for $k = 2$, and $\mathrm{IM}(P_k) = \varnothing$ for $k \geq 3$.

Let $S_k = K_{1,k}$ be the star graph. The vertex of maximum degree in S_k is called the *center* of S_k.

Theorem 7. *There is a $\mathbb{Z}_{k+1}$-magic labeling with $\mathbb{Z}_{k+1}$-magic value* 0 *of the Cartesian product $G = S_k \times S_k$.*

Proof. Denote the vertices of the first S_k by $c_1, x_1, \ldots, x_k$ and those of the second S_k by $c_2, y_1, \ldots, y_k$, where c_i is the center of S_k, $i = 1, 2$. Define $f : E(G) \to \mathbb{Z}_{k+1}$ by

$$\begin{cases} f((c_1, c_2)(x_i, c_2)) = -1, & 1 \leq i \leq k; \\ f((c_1, c_2)(c_1, y_j)) = 1, & 1 \leq j \leq k; \\ f((c_1, y_j)(x_i, y_j)) = 1, & 1 \leq i \leq k; \\ f((x_i, c_2)(x_i, y_j)) = -1, & 1 \leq j \leq k. \end{cases}$$

Then $f^+(c_1,c_2) = \sum_{i=1}^{k} f((c_1,c_2)(x_i,c_2)) + \sum_{j=1}^{k} f((c_1,c_2)(c_1,y_j)) = -k+k = 0$; $f^+(c_1,y_j) = f((c_1,c_2)(c_1,y_j)) + \sum_{i=1}^{k} f((c_1,y_j)(x_i,y_j)) = 1+k \equiv 0 \pmod{k+1}$; $f^+(x_i,c_2) = f((c_1,c_2)(x_i,c_2)) + \sum_{j=1}^{k} f((x_i,c_2)(x_i,y_j)) = -1-k \equiv 0 \pmod{k+1}$; $f^+(x_i,y_j) = f((c_1,y_j)(x_i,y_j)) + f((x_i,c_2)(x_i,y_j)) = 1-1=0$.

□

Example. Note that S_k is not $\mathbb{Z}_{k+1}$-magic for $k \geq 2$ and that Theorem 7 provides another counter-example to the converse of Theorem E.

Example. Consider the lexicographic product $G = P_4 \circ N_2$, where N_2 is the null graph of order two (see Figure 3). Since G is an eulerian graph with an even number of edges, Theorem B tells us that G is A-magic. Clearly, P_4 and N_2 are not A-magic. This provides a counter-example for the converse of Theorem F.

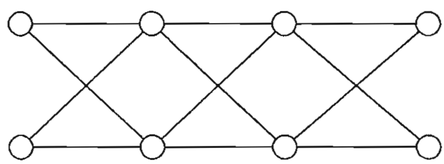

FIGURE 3. The lexicographic product of P_4 and N_2.

However, in the case where $A = \mathbb{Z}_2$, the converse of Theorem E is true.

Theorem 8. *If $G_1 \times G_2$ is $\mathbb{Z}_2$-magic, then G_1 and G_2 are $\mathbb{Z}_2$-magic.*

Proof. Suppose $G_1 \times G_2$ is $\mathbb{Z}_2$-magic. Then, $\deg_{G_1\times G_2}(u,v) = \deg_{G_1}(u) + \deg_{G_2}(v)$ are of the same parity for all $(u,v) \in V(G_1 \times G_2)$. For a fixed u, $\deg_{G_2}(v)$ are of the same parity for all $v \in V(G_2)$. Similarly, $\deg_{G_1}(u)$ are of the same parity for all $u \in V(G_1)$. Hence, G_1 and G_2 are $\mathbb{Z}_2$-magic. □

We conclude with a few additional results on the integer-magic spectra of $G_1 \times G_2$.

Theorem 9. *Let G_1 and G_2 be $\mathbb{Z}_m$-magic and $\mathbb{Z}_n$-magic graphs, respectively. Then, $\{kl : k \in \mathbb{N}$ and $l = \mathrm{lcm}(m,n)\} \subseteq \mathrm{IM}(G_1 \times G_2)$.*

Proof. Let $l = \mathrm{lcm}(m,n)$. Since G_1 is $\mathbb{Z}_m$-magic, Theorem C implies that G_1 is $\mathbb{Z}_l$-magic. Similarly, G_2 is $\mathbb{Z}_l$-magic. By Theorem E, $G_1 \times G_2$ is

$\mathbb{Z}_l$-magic. Since $l|2l$, $l|3l$, etc., the claim is established by using Theorem C. □

Theorem 10. *Let l be even, and $k_i \geq 3$. Then,* $\mathrm{IM}(P_{k_1} \times P_{k_2} \times \cdots \times P_{k_l}) = \{3, 4, 5, \ldots\}$.

Proof. Let $G = P_{k_1} \times P_{k_2} \times \cdots \times P_{k_l}$. Since G has vertices of even and odd degree, G is not $\mathbb{Z}_2$-magic. Because of the result mentioned in an earlier example, $\mathrm{IM}(P_{k_i} \times P_{k_{i+1}}) = \{3, 4, 5, \ldots\}$, for $i = 1, 3, 5, \ldots, l-1$. Thus by Theorem E, $\mathrm{IM}(P_{k_1} \times P_{k_2} \times \cdots \times P_{k_l}) = \{3, 4, 5, \ldots\}$. □

Theorem 11. *Let G be a graph with degree set $\mathcal{D}_G = \{d_1, d_2, \ldots, d_l\}$, T a non-trivial tree and $k \geq 4$. If $k \nmid d_i$ for all i, then there exists a $\mathbb{Z}_k$-magic labeling of $G \times T$.*

Proof. We proceed by induction on n, the number of vertices in T. Let the induction hypothesis be the following: If T is a tree of order n, then there exists a $\mathbb{Z}_k$-magic labeling of $G \times T$ (with magic-value 0), where the edges in each copy G_c of G are labeled with $r_c \in \{-1, -2, 1, 2\}$.

For the base case ($n = 2$), $T = P_2$. Let the vertices of T be v_1 and v_2. In $G \times T$, label all of the edges in the two copies of G with -1. Label each edge joining vertices (u_j, v_1) and (u_j, v_2) in $G \times T$, with the value $\deg_G(u_j)$. Since $k \nmid \deg_G(u_j)$, this ensures that $\deg_G(u_j) \not\equiv 0 \pmod{k}$. Thus, we have a $\mathbb{Z}_k$-magic labeling of $G \times P_2$, with magic-value 0.

Now, assume that the induction hypothesis holds for $m = 2, 3, \ldots, n-1$ and let T be a tree of order n. Suppose that v is a leaf of T and v' is its neighbor. Then, $T' = T - \{v\}$ is a tree of order $n - 1$. By the induction hypothesis, $G \times T'$ has a $\mathbb{Z}_k$-magic labeling with magic-value 0 (see Figure 4). There, we see that $f^+((u_j, v')) = (a_2 + \cdots + a_d) + r \cdot \deg_G(u_j) \equiv 0 \pmod{k}$.

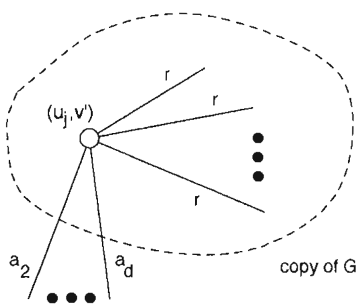

FIGURE 4. $\mathbb{Z}_k$-magic labeling of $G \times (T - \{v\})$, where $r = -1, -2, 1$, or 2.

We now use the labeling in Figure 4 to obtain a $\mathbb{Z}_k$-magic labeling of $G \times T$ with magic-value 0. There are two cases to consider.

CASE 1: Suppose that $r = -1$ or $r = 2$. Then, Figure 5 gives a $\mathbb{Z}_k$-magic labeling of $G \times T$ satisfying the induction hypothesis. Note that in Figure 5,

$$\begin{aligned} f^+((u_j, v)) &= \deg_G(u_j) + (-1) \cdot \deg_G(u_j) \\ &\equiv 0 \pmod{k}, \\ f^+((u_j, v')) &= \deg_G(u_j) + (r-1) \cdot \deg_G(u_j) + \sum_{i=2}^{d} a_i \\ &= r \cdot \deg_G(u_j) + \sum_{i=2}^{d} a_i \\ &\equiv 0 \pmod{k}. \end{aligned}$$

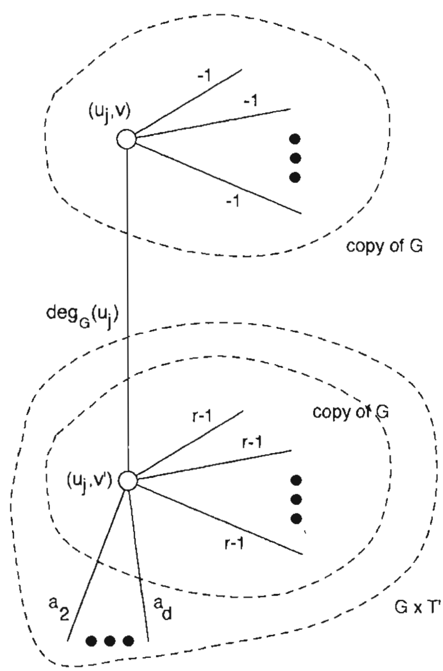

FIGURE 5. $\mathbb{Z}_k$-magic labeling of $G \times T$, for CASE 1.

CASE 2: Suppose that $r = -2$ or $r = 1$. Then, Figure 6 gives a $\mathbb{Z}_k$-magic labeling of $G \times T$ satisfying the induction hypothesis. Note that in

Figure 6,

$$\begin{aligned}
f^+((u_j, v)) &= -\deg_G(u_j) + (1) \cdot \deg_G(u_j) \\
&\equiv 0 \pmod{k}, \\
f^+((u_j, v')) &= -\deg_G(u_j) + (r+1) \cdot \deg_G(u_j) + \sum_{i=2}^{d} a_i \\
&= r \cdot \deg_G(u_j) + \sum_{i=2}^{d} a_i \\
&\equiv 0 \pmod{k}.
\end{aligned}$$

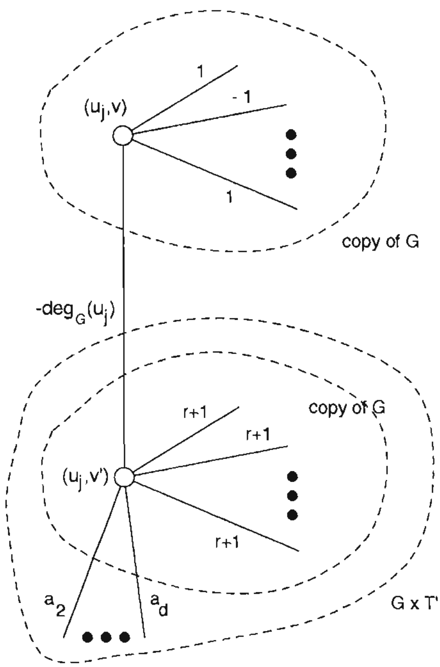

FIGURE 6. $\mathbb{Z}_k$-magic labeling of $G \times T$, for CASE 2.

In both cases, all of the labels on the edges of $G \times T$ are non-zero. Thus by induction, the theorem is established. □

Example. Figure 7 illustrates Theorem 11 for the graph $C_4 \times P_3$.

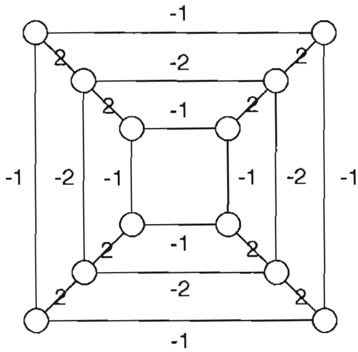

FIGURE 7. $\mathbb{Z}_k$-magic labeling of $C_4 \times P_3$, for $k \geq 3$.

Example. Let $H = Q_n \times K_m \times T_1 \times T_2$, where T_1 and T_2 are non-trivial trees and Q_n is the n-dimensional hypercube. Suppose $k \geq 4$ where $k \nmid n$ and $k \nmid m-1$. Then, it follows from Theorem 11 that $Q_n \times T_1$ and $K_m \times T_2$ are $\mathbb{Z}_k$-magic. Thus by Theorem E, H is $\mathbb{Z}_k$-magic.

REFERENCES

[1] M. Doob, *On the construction of magic graphs*, Proc. Fifth S.E. Conference on Combinatorics, Graph Theory and Computing (1974), 361–374.

[2] M. Doob, *Generalizations of magic graphs*, Journal of Combinatorial Theory, Series B, **17** (1974), 205–217.

[3] M. Doob, *Characterizations of regular magic graphs*, Journal of Combinatorial Theory, Series B, **25** (1978), 94–104.

[4] M.C. Kong, S-M Lee, and H. Sun, *On magic strength of graphs*, Ars Combinatoria, **45** (1997), 193–200.

[5] A. Kotzig and A. Rosa, *Magic valuations of finite graphs*, Canad. Math. Bull., **13** (1970), 451–461.

[6] S-M Lee, Yong-Song Ho and R.M. Low, *On the integer-magic spectra of maximal planar and maximal outerplanar graphs*, Congressus Numerantium, **168** (2004), 83–90.

[7] S-M Lee, A. Lee, Hugo Sun, and Ixin Wen, *On group-magic graphs*, Journal of Combinatorial Mathematics and Combinatorial Computing, **38** (2001), 197–207.

[8] S-M Lee and F. Saba, *On the integer-magic spectra of two-vertex sum of paths*, Congressus Numerantium, **170** (2004), 3–15.

[9] S-M Lee, F. Saba, E. Salehi, and H. Sun, *On the V_4-group magic graphs*, Congressus Numerantium, **156** (2002), 59–67.

[10] S-M Lee, F. Saba, and G. C. Sun, *Magic strength of the k-th power of paths*, Congressus Numerantium, **92** (1993), 177–184.

[11] S-M Lee and E. Salehi, *Integer-magic spectra of amalgamations of stars and cycles*, Ars Combinatoria, **67** (2003), 199–212.

[12] S-M Lee, E. Salehi and H. Sun, *Integer-magic spectra of trees with diameters at most four*, JCMCC., **50** (2004), 3–15.

[13] S-M Lee, L. Valdes, and Yong-Song Ho, *On group-magic spectra of trees, double trees and abbreviated double trees*, JCMCC., **46** (2003), 85–95.
[14] R.M. Low and S-M Lee, *On the integer-magic spectra of tessellation graphs*, Australas. J. Combin., **34** (2006), 195–210.
[15] R.M. Low and S-M Lee, *On the products of group-magic graphs*, Australas. J. Combin., **34** (2006), 41–48.
[16] R.M. Low and S-M Lee, *On group-magic eulerian graphs*, JCMCC., **50** (2004), 141–148.
[17] R.M. Low and L. Sue, *Some new results on the integer-magic spectra of tessellation graphs*, Australas. J. Combin., **38** (2007), 255–266.
[18] E. Salehi and P. Bennett, *On integer-magic spectra of caterpillars*, JCMCC., **61** (2007), 65–71.
[19] J. Sedlácek, *On magic graphs*, Math. Slov, **26** (1976), 329–335.
[20] J. Sedlácek, *Some properties of magic graphs*, in Graphs, Hypergraph, and Bloc Syst. 1976, Proc. Symp. Comb. Anal., Zielona Gora (1976), 247–253.
[21] W.C. Shiu and R.M. Low, *Group-magicness of complete n-partite graphs*, JCMCC., **58** (2006), 129–134.
[22] W.C. Shiu and R.M. Low, *Integer-magic spectra of sun graphs*, J. Comb. Optim., **14** (2007), 309–321.
[23] R.P. Stanley, *Linear homogeneous diophantine equations and magic labelings of graphs*, Duke Math. J., **40** (1973), 607–632.
[24] R.P. Stanley, *Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings*, Duke Math. J., **40** (1976), 511–531.
[25] W.D. Wallis, *Magic Graphs*, Birkhauser Boston, (2001).

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