Elementary Blocks of Plane Bipartite Graphs*

Peter C.B. Lam,† W.C. Shiu†, Heping Zhang‡

Abstract

Let G be a 2-connected plane bipartite graph with perfect matchings, with more than one cycle and with minimum degree of interior vertices larger than 2. An edge of G is called non-fixed if it belongs to some, but not to all, perfect matchings of G . Every component of the subgraph formed by all non-fixed edges of G is called an elementary component. If each interior face of an elementary component is a face of G, then this elementary component is called an elementary block. It is known that each elementary component of a hexagonal system is an elementary block. In this paper, the concept of maximal fixed alternating path, is introduced and will be used conveniently to simplify the discussion of elementary components. In particular, we give a sharp lower bound for the number of elementary blocks of G whenever G has a cycle as its elementary block. As an application, a stronger result on the number of normal components of hexagonal systems is obtained.

Keywords: Perfect matching, alternating path, elementary component, plane bipartite graph, non-crossing partition

AMS 1991 Subject Classification: 05C70, 05C90, 92E10

1. Introduction

In this article we consider plane bipartite graph $G = (V, E)$ with perfect matchings, or *Kekulé structures* in chemical context. If G is a bipartite graph we color the vertices of G with black and white such that adjacent vertices receive different colors. A finite face f of a plane graph G is called a *cell* if the boundary is a cycle. Denote by ∂f the boundary of the face f. The boundary of the infinite face is called the boundary of G, and is denoted by ∂G. The vertices and edges not on ∂G are called *interior* vertices and edges of G. For undefined terms in graph theory, we shall refer to [1].

An edge of G is called a *fixed single* or *fixed double* edge if it belongs to no or all perfect matchings of G respectively. An edge of G is called *fixed* if it is either a fixed single or a fixed double edge of G.

*This work is partially supported by Research Grant Council Grant of Hong Kong, Faculty Research Grant of Hong Kong Baptist University and National Natural Science Foundation of China.

†Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong, China, e-mail address: cblam@hkbu.edu.hk, wcshiu@hkbu.edu.hk

‡Department of Mathematics, Lanzhou University, Lanzhou, Gansu 730000, China, e-mail address: zhanghp@lzu.edu.cn

A connected graph is *elementary* if the union of all perfect matchings forms a connected subgraph. It is well-known that a connected bipartite graph with more than one edge is elementary if and only if each edge is not fixed; or if and only if the deletion of any two distinct colored vertices results in a subgraph with perfect matchings. Also, any elementary bipartite graphs with more than one edge is 2-connected. Other important properties on elementary bipartite graphs can be found in [16,17]. Let G be a plane bipartite graph with a perfect matching M. A cycle C or a path P of G is called *M-alternating* if the edges of C or P, respectively, appear alternately in M and $E(G)\backslash M$. All non-fixed edges of G form a subgraph H, each component of which is elementary and is therefore called an *elementary component*. Moreover, H is called an *elementary block* if each interior face of H is also a face of the original graph G.

A hexagonal system with fixed edge is also called *essentially disconnected*. It has been proved [12] that all elementary components of a hexagonal system are elementary blocks, also called *normal components* in chemical context. A hexagonal system, also called a benzenoid system, or a honeycomb, or a benzenoid, is a finite connected plane bipartite graph with no cut-vertices in which every interior region is bounded by a regular hexagon of unit side [19]. Since a hexagonal system with at least one perfect matching may be viewed as the carbon-skeleton of a benzenoid hydrocarbon, various topological properties of hexagonal systems have been extensively studied [6,9,11,22]. Normal components of hexagonal systems play a key role in resonance theory of condensed aromatic hydrocarbons. The decomposition of benzenoids with fixed bonds into normal components can simplify some calculation appeared in Kekulé structure number [6], Clar aromatic sextet theory [2] and Randić conjugated model [18]. Much contribution has been devoted to essentially disconnected benzenoids; for example, see [4–6,12,20]. A basic result is that an essentially disconnected benzenoid has at least two normal components. Further, Hansen and Zheng [12] obtained the following result, which was used conveniently in some small normal hexagonal systems determined uniquely by Clar covering polynomial [25,29].

Theorem 1.1 : [12] *Let H be a hexagonal system with more than one hexagon. If H has a single hexagon as a normal component, then H has at least three normal components.*

I. Gutman [7,8] and P. John [13] defined the so-called cell polynomial or resonant ring polynomial for polycyclic unsaturated alternant hydrocarbons and thus extend the concept for sextet polynomial of hexagonal systems to plane bipartite graphs. Certain correspondence between the resonant or Clar patterns and perfect matchings was revealed [7,8,14,26]. Elementary plane bipartite graphs were considered in [27,28]. The above results on elementary blocks have also been extended to plane *weakly elementary bipartite graphs*, in which all elementary components are elementary blocks. In general, an elementary component of a plane bipartite graph is not necessarily an elementary block. However, the following two results were obtained in [21]:

Theorem 1.2 : *Let G be a connected plane bipartite graph with perfect matchings. If all vertices with degree one of G are of the same color and lie on the boundary of G or if $\delta(G) \geq 2$, then G has at least one elementary block.*

Corollary 1.3: *Let G be a plane bipartite graph with perfect matchings. If all vertices with degree one of G are of the same color and lie on the boundary of G or if $\delta(G) \geq 2$, then for any perfect matching M of G, G contains a cell whose boundary is M-alternating.*

In this paper we attempt to estimate the number of elementary blocks of plane bipartite graphs having a single cycle as an elementary block. In Section 2, a concept for maximal fixed alternating path is introduced to simplify the discussion of elementary components. In particular, we obtain the following unexpected result, which is a complete extension of the result in Theorem 1.1 to plane bipartite graphs. The proof will be given in Section 3.

Theorem 1.4: *Suppose that a 2-connected plane bipartite graph G has a perfect matching and more than one cycle. Moreover, all interior vertices are of degree ≥ 3. If G has a cycle as an elementary block, then G has at least three elementary blocks.*

In Theorem 1.5, we obtain a sharp lower bound for the number of elementary blocks, which depends on the number of vertices of degree at least 3 lying on the single cycle.

Theorem 1.5: *Let G be a connected plane bipartite graph with perfect matchings, more than one cycle, $\delta(G) \geq 2$ and degrees of interior vertices ≥ 3. Suppose that G has a cycle C as an elementary block and let $\delta_3(C)$ be the number of vertices of C with degree ≥ 3 in G. Then the number of elementary blocks of G has the following sharp lower bound*:

$$\min\{\lceil \tfrac{1}{2}(\delta_3(C)+1)\rceil + 1, \tfrac{1}{2}|V(C)| + 1\}.$$

Theorem 1.5 is an extension of Theorem 1.4 since $\delta_3(C) \geq 2$ if G is 2-connected. For hexagonal systems, the following stronger result is obtained.

Theorem 1.6: *Let H be a hexagonal system with more than one hexagon. Suppose that H has a single hexagon h as a normal component and let $\delta_3(h)$, where $2 \leq \delta_3(h) \leq 6$, denote the number of vertices of hexagon h with degree 3 in H. Then H has at least $\delta_3(h)+1$ normal components.*

Proofs of Theorems 1.5 and 1.6 will be given in Section 4 and 5 respectively. The lower bounds for the number of elementary blocks in these two theorems can be achieved. For example, see Figures 1.1 and 1.2.

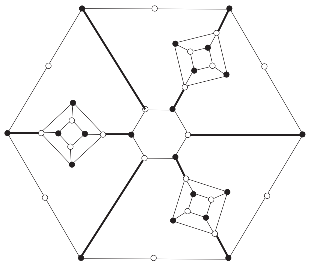

Figure 1.1: A plane bipartite graph with four elementary blocks.
(Heavy lines indicate fixed single edges).

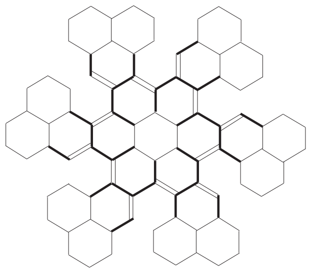

Figure 1.2: A hexagonal system with seven normal components.
(Heavy lines indicate fixed single edges and double lines indicate fixed double edges).

2. Maximal fixed alternating path and some Lemmas

In this section we only consider bipartite graphs with fixed edges. Proof of the following basic result can be found in [12].

Lemma 2.1: *Let G be a bipartite graph with perfect matchings. An edge of G is not fixed if and only if it belongs to an M-alternating cycle for any perfect matching M of G.*

Definition 2.1: Let G be a bipartite graph with fixed edges, implying that G has perfect matchings. A path P of G is called a *fixed alternating path* (FAP), if the edges of P are alternately fixed single and fixed double edges of G. A fixed alternating path P of G is *maximal* if it is not a proper sub-path of any other FAP of G.

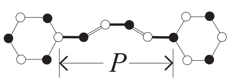

Figure 2.1: A maximal fixed alternating path P of a plane bipartite graph.

The following lemma follows from definition.

Lemma 2.2: *Let G be a bipartite graph. If e is a fixed edge of G, then G has a maximal FAP containing e.*

Lemma 2.3: *Let G be a bipartite graph with fixed edges. Then each end-vertex of any maximal FAP of G is either of degree one or in some elementary component of G.*

Proof: Let P be a maximal FAP of G. Let x be an end-vertex and xu an end-edge of P. Suppose $d(x) \geq 2$ and x belongs to none of elementary components of G. Then all edges incident with x are fixed and one must be a fixed double edge. If xu is a fixed single edge, let xy be such a fixed double edge of G. Then y does not belong to P. Otherwise $P(x, y)$ and xy form an alternating cycle with respect to a perfect matching of G, where $P(x, y)$ denotes the sub-path of P from x to y, which contradicts Lemma 2.1. Thus $P + xy$ is also a FAP, which contradicts the maximality of P. If xu is a fixed double edge, let xy be any fixed single edge. By the same reasoning, we can show that $P + xy$ is a FAP, which is also a contradiction. ▮

Lemma 2.4: *Let G be a bipartite graph with perfect matchings. Then G is elementary if and only if, for any perfect matching M and any two distinct colored vertices u and v of G, there exists an M-alternating path P connecting u and v with both end-edges belong to M.*

Proof: A bipartite graph is elementary if and only if the deletion of any two distinct colored vertices results in a subgraph with perfect matchings [17]. Let M be a perfect matching of G. Suppose that G is elementary. Let u and v be any two distinctly colored vertices of G and $G' =: G - \{u, v\}$. Then G' has a perfect matching M'. The symmetry difference $M \oplus M' =: (M \cup M')\backslash(M \cap M')$ forms a subgraph of G consisting of M- and M'-alternating paths and cycles; exactly one M-alternating path exists, as required. Conversely, suppose that for any two distinctly colored vertices u and v, G has an M-alternating path P connecting u and v with both end-edges belong to M. Then $M \oplus E(P)$ is a perfect matching of $G - \{u, v\}$. ▮

Lemma 2.5: *Let G be a bipartite graph with fixed edges. Then the end-vertices of any maximal FAP of G do not belong to the same elementary component of G.*

Proof: Let P be a maximal FAP of G. Suppose that the end-vertices x and y of P belong to the same elementary component G_1 of G. Since both end-edges of P are fixed single edges, P is of odd length and the end-vertices x and y of P are of different colors. By Lemma 2.4, for any perfect matching M of G, there exists an M-alternating path P' connecting x and y with end-edges belonging to M. Thus P and P' induce an M-alternating cycle in G_1, which is a contradiction. ▮

Corollary 2.6: *Let G be a plane bipartite graph with fixed edges and $\delta(G) \geq 2$. Then any maximal FAP P of G is of odd length, the end-edges are fixed single and the end-vertices lie on two different elementary components of G. Moreover, any internal vertex of P does not lie on any elementary component of G.*

3. Proof of Theorem 1.4

Using the concept of maximal FAP and the related results obtained in Section 2, we shall now prove Theorem 1.4. Let C be a cycle which is an elementary block of G; that is, all edges outside C incident with the vertices of C are fixed single edges of G and there is no vertices and edges in the interior of C. For any $u \in V(C)$ with $d_G(u) \geq 3$, let e_u be a fixed single edge incident with u. Then $e_u \notin E(C)$. By Lemma 2.2, G has a maximal FAP, denoted by P_u, containing e_u. By Corollary 2.6 let G_u denote the elementary component of G to which the end-vertex of P_u different from u belongs. An elementary component G_u for such a vertex u in C is called a *concentrated component*. We shall first establish the following claims.

Claim 1. *If u and v are different colored vertices in C with degrees ≥ 3, then $P_u \cap P_v = \emptyset$.*

Suppose that $P_u \cap P_v \neq \emptyset$. Let w be the first vertex of P_u meeting P_v when one traverses P_u from u. By Corollary 2.6 w is neither u nor v. Let $P_u(u, w)$ and $P_v(v, w)$ denote the sub-paths of P_u and P_v from u and v to w, respectively. Let $w'w$ be an end-edge of $P_u(u, w)$. If $w'w$ is a fixed double edge of G, then $w'w$ must be an edge of P_v, which contradicts the choice for w. Thus it follows that both end-edges of the FAP $P_u(u, w)$ are fixed single edges and thus the length is odd. Since w and u are of different colors, v and w are of the same color. So an end-edge of FAP $P_v(v, w)$ incident with w is a fixed double edge, and consequently $P' = P_u(u, w) \cup P_v(v, w)$ is a FAP connecting two vertices of the same elementary component C of G. Since both u and v are end vertices of maximal FAPs, P' is also maximal. This is a contradiction to Lemma 2.5.

Claim 2. *If u and v are two different colored vertices of C with degree ≥ 3. Then G_u and G_v are two different elementary components of G, i.e., $G_u \cap G_v = \emptyset$.*

Suppose that $G_u = G_v$. Let u' and v' be the other end-vertices of maximal FAPs P_u and P_v different from end-vertices u and v, respectively. Thus both u' and v' belong to G_u. By Corollary 2.6 both P_u and P_v are of odd length and colors of u and v are different from that of u' and v' respectively. Since u and v are of different colors, u' and v' are of different colors also. Let M be any perfect matching of G. Since both G_u and C are elementary components of G, so by

Lemma 2.4, G_u and C have M-alternating paths $P_{u'v'}$ and P_{uv} with end-vertices u', v' and u, v respectively, and with end-edges belonging to M. In addition, by Claim 1, P_u and P_v are disjoint. Then $P_{u'v'} \cup P_{uv} \cup P_u \cup P_v$ forms an M-alternating cycle, which contradicts that both P_u and P_v are FAPs.

Claim 3. *G has at most one concentrated component G_u such that C lies in the interior of a finite face of G_u, where $u \in V(C)$ and $d_G(u) \geq 3$.*

Suppose the G has two distinct concentrated component G_u and $G_{u'}$, where $u, u' \in V(C)$, $d_G(u), d_G(u') \geq 3$, and C lies in the interior of finite faces f and f' of G_u and $G_{u'}$ respectively. Since $\partial f \cap \partial f' = \emptyset$, we may assume without loss of generality that f lies in the interior of f'. Thus G_u lies in the interior of f'. Let $P_{u'}$ be a maximal FAP connecting u' and a vertex of $\partial f'$. By Corollary 2.6, each internal vertex of $P_{u'}$ does not lie on G_u. But this contradicts the Jordan Curve Theorem [1].

Claim 4. *Suppose G has a concentrated component G_u with $u \in V(C)$, $d_G(u) \geq 3$ and C lies in the exterior of G_u. Then G has an elementary block which is an elementary block of $I[\partial G_u]$, the subgraph of G formed by the cycle ∂G_u together with its interior.*

Suppose that G_u is a concentrated component such that C lies in the exterior of G_u. Then $G' = I[\partial G_u]$ is a plane bipartite graph with perfect matchings and with $\delta(G') \geq 2$. Moreover, each finite face of G' is also a finite face of G. By Theorem 1.2, G' has an elementary block, which is an elementary block of G.

We shall now prove the theorem. Since G is 2-connected and contains more than one cycle, there are at least two vertices x and y in C with degree ≥ 3. Then there exists a path P from x to y in C the edges of which does not belong to the boundary of G. Thus C has at least two different colored vertices, say u and v, with degrees ≥ 3. By Claim 2, G has at least two distinct concentrated components G_u and G_v.

If G has at least three concentrated components, by Claim 3 G has two concentrated components, say G_u and G_v, such that C lies in the exteriors of G_u and G_v. By Claim 4, there are two distinct elementary blocks of G which are those of $I[\partial G_u]$ and $I[\partial G_v]$, respectively. Since C is an elementary block, G contains at least three elementary blocks.

So we may suppose that G has exactly two concentrated components G_u and G_v and C lies in the interior of a finite face f of G_v and lies in the exterior of G_u. Since P_u connects u and a vertex in G_u, by Jordan Curve Theorem, G_u lies in the interior of f. Since all vertices of C are interior vertices of G, the degrees of them are ≥ 3. Thus if $x, y \in V(C)$ are of the same color, then $G_x = G_y$.

Let α, β, γ and τ be four consecutive vertices of C. We may assume that $G_u = G_\alpha = G_\gamma$ and $G_v = G_\beta = G_\tau$. A maximal FAP P_τ connects a vertex τ of C and a vertex of ∂f. Maximal FAPs P_α and P_γ connect vertices α and γ of C with vertices of ∂G_u, respectively, which are disjoint with P_τ. Then $\alpha\beta\gamma$, P_α, P_γ and a sub-path ∂G_u form a cycle C' the interior of which lies in the exteriors of both C and ∂G_u. On the other hand, from Claim 1 P_β and any one of P_α and P_γ and ∂G_u are disjoint. Moreover β is the unique common vertex of P_β and C. Then $P_\beta - \beta$ and C' are disjoint. By Jordan Curve Theorem it is impossible. The proof is completed. ▌

4. Proof of Theorem 1.5

For positive integers $a \leq n$, let $[a, n] =: \{x : x \text{ is an integer and } a \leq x \leq n\}$ be the totally ordered set with the natural order; and $[\boldsymbol{n}]$ be an abbreviation for $[1, n]$. Let $\mathcal{F} = \{I_1, I_2, \ldots, I_m\}$ be a partition of $[\boldsymbol{n}]$. Each I_i is called a *part* of $\mathcal{F}$. Two parts I_1 and I_2 of $\mathcal{F}$ are called *crossing parts* if there exists $\{a, c\} \subseteq I_1$ and $\{b, d\} \subseteq I_2$ such that $a < b < c < d$ or $b < a < c < d$. The partition $\mathcal{F}$ is called *non-crossing* if $\mathcal{F}$ has no crossing parts. Moreover, if each part of $\mathcal{F}$ consists of elements with the same parity, then $\mathcal{F}$ is called a *non-crossing partity partition* or for simplicity, an *NPP*.

Let $p_n =: \min\{|\mathcal{F}| : \mathcal{F} \text{ is an NPP of } [\boldsymbol{n}]\}$. Note that parity partition of $[\boldsymbol{n}]$ nad p_n can be extended to arbitrary $[a, n]$ in a natural way so that the number p_{n-a+1} only depends on the length $n - a + 1$ of $[a, n]$. We have the following result.

Lemma 4.1: *For any positive integer n, $p_n = \left\lceil \frac{1}{2}(n+1) \right\rceil$.*

Proof: It is clear that $p_1 = 1$, $p_2 = p_2 = 2$ and the Lemma holds for $n = 1,\ 2$ and 3. For $n \geq 4$, we put $I_0 = \{x :\ x \leq n \text{ and } x \text{ is odd}\}$ and $I_r = \{2r\}$ for $r = 1, \ \cdots,\ \lfloor \frac{n}{2} \rfloor$ to obtain an NPP of $[1, n]$ consisting of $\left\lceil \frac{1}{2}(n+1) \right\rceil$ parts. So $p_n \leq \left\lceil \frac{1}{2}(n+1) \right\rceil$. It is also clear that $p_{n-1} \leq p_n$ for all $n \geq 2$. Therefore if we can show that $p_{n-2} \leq p_n - 1$ for $n \geq 4$, where n is even, then we can deduce that $p_n \geq \left\lceil \frac{1}{2}(n+1) \right\rceil$ and the Lemma is proved.

Now suppose $n \geq 4$ is even and $p_n = r$. Let $\mathcal{F} = \{I_1, I_2, \cdots, I_r\}$ be an NPP of $[\boldsymbol{n}]$. Without loss of generality, we assume that $n \in I_r$. If $I_r \cap [\boldsymbol{n-1}] = \emptyset$, then $\mathcal{F}' = \mathcal{F} \setminus \{I_r\}$ is an NPP of $[\boldsymbol{n-1}]$ with $r - 1$ parts and so $p_{n-1} \leq r - 1 = p_n - 1$. If $I_r \cap [\boldsymbol{n-1}] \neq \emptyset$, then let $a = \min\{x : x \in I_r\}$. Because $a \in I_r \setminus \{n\}$, both $a - 1$ and $a + 1$ are odd; and they belong to two distinct parts of $\mathcal{F}$ other than I_r. Assume that $a - 1$ and $a + 1$ belong I_1 and I_2 respectively. We construct $\mathcal{F}' = \{I'_1, I'_2, \cdots, I'_r\}$ as follows: $I'_1 = I_1 \cup I_2$, $I'_2 = \{a\}$, $I'_r = I_r \setminus \{a\}$, and $I'_k = I_k$ for $k = 3, \cdots, r-1$. Then $\mathcal{F}'$ is also an NPP of $[\boldsymbol{n}]$ with r parts, $n \in I'_r$ and $I'_r < I_r$. This procedure can be continued until $I'_r \cap [n-1] = \emptyset$. The proof is completed. ∎

Proof of Theorem 1.5: Let $V_3(C) = \{u \in V(C) : d_G(u) \geq 3\}$. Each vertex u of $V_3(C)$ is connected to some elementary components, denoted by G_u, of G by maximal FAPs. Note that one vertex u of $V_3(C)$ may correspond to more than one elementary components G_u of G in the above way. Let $\mathcal{G} = \{G_u : u \in V_3(C)\}$. We assert that $|\mathcal{G}| \geq \lceil \frac{1}{2}(\delta_3(C)+1) \rceil$.

If $V_3(C) = V(C)$, let $P_1 = C - e$ for an edge e of C. Otherwise, deleting all vertices of $V(C)$ with degree 2, we get a subgraph of C, the components of which are paths and denoted by $P_1, \cdots, P_r$. We have $V_3(C) = \cup_{i=1}^r V(P_i)$.

Suppose that $r = 1$. Let $P_1 = u_1 u_2 \cdots u_{\delta_3}$, where $\delta_3 = \delta_3(C)$. Note that $V(P_1)$ may be viewed as the order set $[\boldsymbol{\delta_3}]$ of subscripts of vertices. Let F be the union of all maximal FAPs each of which has an end-vertex in C. We now define a partition $\mathcal{F}$ of $V(P_1)$ as follows: u_i and u_j belong to the same part of $\mathcal{F}$ if and only if u_i and u_j belong to the same component of F. Then $|\mathcal{G}| \geq |\mathcal{F}|$. By Claim 1 in the proof of Theorem 1.4, distinctly colored vertices of $V_3(C)$ must belong to the different parts of $\mathcal{F}$. Further, by the planarity of graph G (Jordan Curve Theorem), it can be shown that $\mathcal{F}$ is an NPP of $V(P_1)$. Thus by Lemma 4.1 we have $|\mathcal{G}| \geq |\mathcal{F}| \geq \lceil \frac{1}{2}(\delta_3(C)+1) \rceil$.

Suppose that $r \geq 2$. Let $\mathcal{G}^i = \{G_u : u \in V(P_i)\}$, where $1 \leq i \leq r$. It follows that $\{\mathcal{G}^1, \cdots, \mathcal{G}^r\}$ is a partition of $\mathcal{G}$. For each P_i, the above discussion is still valid. Thus

$$|\mathcal{G}| = \sum_{i=1}^{r} |\mathcal{G}^i| \geq \sum_{i=1}^{r} \frac{|V(P_i)|+1}{2} > \frac{1}{2}(\delta_3(C)+1).$$

If C lies in the exterior of all members of $\mathcal{G}$, then by Claim 4 of Section 3 each $I[\partial G_u]$ has at least one elementary block which is also elementary block of G. Then G has at least $\lceil \frac{1}{2}(\delta_3(C)+1) \rceil + 1$ elementary blocks. Otherwise, by Claim 3, $\mathcal{G}$ has exactly one member G_v such that C lies in its interior, and thus the other members of $\mathcal{G}$ other than G_v are contained in the interior of a finite face f of G_v. Thus each vertex of C is an interior vertex of G and $V_3(C) = V(C)$. Further, we have that $|\mathcal{G}| \geq \frac{1}{2}|V(C)| + 1$. For other member G_x of $\mathcal{G}$ other than G_v, $I[\partial G_v]$ has an elementary block which is also an elementary block of G. Thus G has at least $\frac{1}{2}|V(C)| + 1$ elementary blocks. Therefore the theorem is proved. ∎

5. Proof of Theorem 1.6

Let H be a hexagonal system, e an edge of H and s a hexagon of H containing e. Throughout this section, we use $L_{s,e}$ to denote the segment perpendicular to e which starts from the midpoint of e, passes through the central point of s and ends at the boundary of H, and is totally contained in the interior region of H. By Lemma 3 of [24] we have the following:

Lemma 5.1: *Let H be a hexagonal system with a perfect matching M. Suppose that a hexagon s of H contains a fixed single edge e of H. If two edges of s adjacent to e belong to M, then all edges intersecting $L_{s,e}$ are fixed single edges.*

Proof of Theorem 1.6: We still adopt the notations used in Section 3. Let h be a hexagon of H which is also a normal component of H. It is sufficeint to prove that for any two vertices u and v of h with degree 3, $H_u \cap H_v = \emptyset$.

Suppose u and v are two vertices of h with degree 3. If they are of different colors, then from Claim 2 of the proof of Theorem 1.4, we have $H_u \cap H_v = \emptyset$. Suppose they are of the same color, and that $H_u = H_v$. Then H has maximal FAP's P_u and P_v, connecting u to a vertex $u' \in \partial H_u$ and $v' \in \partial H_u$ respectively. Note that $u' = v'$ is possible. A path Q from u to v in h, P_u, P_v and a path of ∂H_u between u' and v' whenever $P_u \cap P_v = \emptyset$, forms a cycle C', the interior of which lies in the exteriors of both h and ∂H_u.

Let $Q = uxw\cdots v$. Note that $v = w$ may be allowed. Thus x lies in the interior of the cycle $C' \oplus h$, which implies that x is an interior vertex of H with $d_G(x) = 3$. Let s and s' denote two hexagons of H other than h which contains edges xu and xw, respectively. Let $e = xy \in E(s) \cap E(s')$. Obviously $d_G(y) = 3$. Let y' and y'' denote the vertices, other than x, adjacent to y such that yy' and yy'' are edges of s and s' respectively - Figure 5.1.

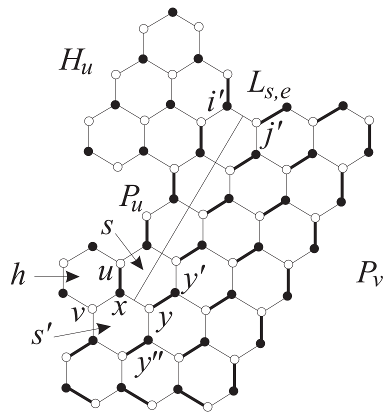

Figure 5.1: Illustration of proof of Theorem 1.6.
(Heavy lines indicate some matched edges).

Since h is a normal component of H, xy is a fixed single edge and h is an alternating hexagon with respect to any perfect matching of H. Let M be a perfect matching of H. Suppose $yy' \in M$

(The proof is similar if $yy'' \in M$). We may also assume that $ux \in M$, otherwise consider a new perfect matching $M' = M \oplus h$). Denote by $\mathcal{L}$ the set of edges of H intersecting $L_{s,e}$. Then $L_{s,e}$ intersects the cycle C' and thus $\mathcal{L} \cap E(C') \neq \emptyset$. Since $\mathcal{L}$ does not intersect h, $\mathcal{L}$ must intersect one of P_u, P_v and ∂H_u. Since xy is a fixed single edge and $ux, yy' \in M$, by Lemma 5.1 all the edges of $\mathcal{L}$ are fixed single edges. Since each edge of H_u is non-fixed, $\mathcal{L} \cap \partial H_u = \emptyset$. Further, the edges belonging to M incident with end-vertices of edges in $\mathcal{L}$ are determined uniquely and illustrated in Figure 5.1.

We now assert that $\mathcal{L} \cap P_u = \emptyset$. Otherwise, let ij denote the first edge of P_u entering at $\mathcal{L}$ when one traverses P_u from u such that u is nearer to i than j. Thus $ij \in \mathcal{L}$ is a fixed single edge of H. Then u and j are of the same color, whereas u and i are of different colors. Since the sub-path of P_u from u to j is an FAP with even length and the end-edge incident with u is a fixed single edge, the other end-edge ij must be a fixed double edge, which is a contradiction. So the assertion is proved.

Thus $\mathcal{L} \cap P_v \neq \emptyset$. Let $i'j'$ denote the first edge of P_v entering at $\mathcal{L}$ when one traverses P_v from v such that v is nearer to j' than i'. The sub-path $P_v(v, i')$ of P_v from v to i' is a FAP. Since the end-edges of $P_v(v, i')$ are fixed single edges, its length is odd. It follows that $P_v(v, j') = P_v(v, i') - i'$ is a FAP of even length and both v and j' are of the same color. Let $P'(x, j')$ denote the M-alternating path from x to j' formed by xy and some edges adjacent to the edges of $\mathcal{L}$. Then $P'(x, j')$ are of odd length and the end-edges do not belong to M. Thus $P'(x, j') \cup P_v(v, j')$ is an M-alternating path connecting two different colored vertices v and x such that its end-edges do not belong to M. There is an M-alternating path $P(x, v)$ in h which connects x and v such that the end-edges belong to M. Thus $P'(x, j') \cup P_v(v, j') \cup P(x, v)$ is an M-alternating cycle, each edge of which is non-fixed, which contradicts that both P_u and P_v are FAPs. ▮

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan Press LTD, 1976.

[2] E. Clar, *The Aromatic Sextet*, Wiley, London, 1972.

[3] R. Chen and F. Zhang, Coronoid systems with fixed bonds, *J. Mol. Struct.*, **403** (1997), 237-243.

[4] R. Chen, S.J. Cyvin and B.N. Cyvin, Recognition of essentially disconnected benzenoids, *Match*, **25** (1990), 71-81.

[5] R. Chen, S.J. Cyvin and B.N. Cyvin, Recognition of essentially disconnected benzenoids, *Acta Math. Appl. Sinica*, **8(4)** (1992), 377-382.

[6] S.J. Cyvin and I. Gutman, *Kekulé Structures in Benzenoid Hydrocarbons*, Lectures Notes in Chemistry 46, Springer, Berlin, 1988.

[7] I. Gutman, The R-polynomial. A new combinatorial technique in resonance theory, *Bull. Soc. Chim. Beograd*, **46** (1981), 17-22.

[8] I. Gutman, Some combinatorial consequences of Clar' resonance sextet theory, *Match*, **11** (1981), 127-143.

[9] I. Gutman and S.J. Cyvin, *Advances in the Theory of Benzenoid Hydrocarbons*, Springer, Berlin, 1990.

[10] P. Hansen and M. Zheng, A linear algorithm for fixed bonds of hexagonal systems, *J. Mol. Strct. (Theochem)*, **257** (1992), 75-83.

[11] P. Hansen and M. Zheng, A linear algorithm for perfect matchings of benzenoid systems, *Discrete Math.*, **122** (1993), 297-304.

[12] P. Hansen and M. Zheng, Normal components of benzenoid systems, *Theor. Chim. Acta*, **85** (1993), 335-344.

[13] P.E. John, Calculating the cell polynomial of catacondensed polycyclic hydrocarbons, *J. Chem. Inf. Comput. Sci.*, **34** (1994), 357-359.

[14] P.E. John, H. Sachs and M. Zheng, Kekulé patterns and Clar patterns in bipartite plane graphs, *J. Chem. Inf. Comput. Sci.*, **35** (1995), 1019-1021.

[15] G. Kreweras, Sur les partitions noncroisées d'un cycle, *Discrete Math.*, **1** (1972), 333-350.

[16] L. Lovász and M.D. Plummer, On minimal elementary bipartite graphs, *J. Combin. Theory Ser. B*, **23** (1977), 127-138.

[17] L. Lovász and M.D. Plummer, Matching Theory, *Annals of Discrete Mathematics*, **29**, North-Holland, Amsterdam, 1986.

[18] M. Randić, Conjugated circuits and resonance energies of benzenoid hydrocarbons, *Chem. Phys. Lett.*, **38** (19767), 63-70.

[19] H. Sachs, Perfect matchings in hexagonal systems, *Combinatorica*, **4(1)** (1980), 89-99.

[20] R. Sheng, S.J. Cyvin and I. Gutman, "Essentially disconnected benzenoids" are essentially disconnected, *Match*, **26** (1991), 191.

[21] W.C. Shiu, P.C.B. Lam, F. Zhang and H. Zhang, Normal components, Kekué patterns, and Clar patterns in plane bipartite graphs, submitted.

[22] N. Trinajastić, *Chemical Graph Theory*, Second edition, CRC Press, Boca Raton. Florida, 1992.

[23] F. Zhang and H. Zhang, A note on the number of perfect matchings of bipartite graphs, *Discrete Appl. Math.*, **73** (1997), 163-173.

[24] F. Zhang and M. Zheng, Generalized hexagonal systems with each hexagon being resonant, *Discrete Appl. Math.*, **36** (1992), 67-73.

[25] H. Zhang, The Clar covering polynomial of hexagonal systems with an application to chromatic polynomials, *Discrete Math.*, **172** (1997), 163-173.

[26] H. Zhang and F. Zhang, The rotation graphs of perfect matchings of plane bipartite graphs, *Discrete Appl. Math.*, **73** (1997), 5-12.

[27] H. Zhang, and F. Zhang, Perfect matchings of polyomino graphs, *Graphs Combin.*, **13** (1997), 295-304.

[28] H. Zhang and F. Zhang, Plane elementary bipartite graphs, *Discrete Appl. Math.*, **105** (2000), 291-311.

[29] H. Zhang and F. Zhang, The Clar covering polynomial of hexagonal systems III, *Discrete Math.*, **212** (2000),261-269.