A note on weakly connected domination number in graphs

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Abstract: Let G be a connected graph. A weakly connected dominating set of G is a dominating set D such that the edges not incident to any vertex in D do not separate the graph G. In this paper, we first consider the relationship between weakly connected domination number $\gamma_w(G)$ and the irredundance number $ir(G)$. We prove that $\gamma_w(G) \leq \frac{5}{2}ir(G)-2$ and this bound is sharp. Furthermore, for a tree T, we give a sufficient and necessary condition for $\gamma_c(T) = \gamma_w(T) + k$, where $\gamma_c(G)$ is the connected domination number and $0 \leq k \leq \gamma_w(T) - 1$.

Keywords: domination number; weakly connected domination

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§1 Introduction

Throughout this paper $G = (V, E)$ will be an undirected connected graph. We begin by recalling some standard definitions from domination theory. For any vertex $v \in V$, the *open neighborhood of* v, denoted by $N_G(v)$, is $\{u \in V | uv \in E\}$. The *closed neighborhood of* v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. For $S \subseteq V$, the *open neighborhood of* S, denoted by $N_G(S)$, is $\bigcup_{v \in S} N_G(v)$, while the *closed neighborhood of* S, denoted by $N_G[S]$, is $\bigcup_{v \in S} N_G[v]$. The *private neighbor set of* v *with respect to* S is given by $PN_G[v, S] = N_G[v] - N_G[S - \{v\}]$. The vertex v is a *leaf* if $|N_G(v)| = 1$. The vertex v is a *support vertex* if it is adjacent to a leaf. Let $L(G)$ denote the set of leaves of G. The subscripts G will be omitted when the context is clear. Let $\langle S \rangle$ denote the subgraph of G induced by S.

A set $D \subseteq V$ is a *dominating set* of G if $N[D] = V$. The *domination number* of G, denoted by $\gamma(G)$, is the size of its smallest dominating set. D is a *connected dominating set* if D is a dominating set and $\langle D \rangle$ is connected. The *connected domination number* of G is the size of its smallest connected dominating set, and is denoted by $\gamma_c(G)$. Results related to the connected domination number may be found in [1, 2].

A set $D \subseteq V$ is an *irredundant set* if for every $x \in D$, $N[x] \not\subseteq \bigcup_{y \in D - \{x\}} N[y]$. The *irredundance number*, denoted by $ir(G)$, is the minimum size of a maximal irredundant set of vertices. A set $D \subseteq V$ is an independent set if no two vertices of D are adjacent. The *independence number* of G, denoted by $\beta(G)$, is the maximum size of an independent set.

For a set $D \subseteq V$, $|D|$ denotes the cardinality of D. We denote a set D as an ir-set if D is a maximal irredundant set with $|D| = ir(G)$.

In [3], Dunbar et al. introduced the concept of a *weakly connected dominating set*. A *weakly connected dominating set* for a connected graph is a dominating set D of vertices of the graph such that the edges not incident to any vertex in D do not separate the graph. For a set $D \subseteq V$, the *subgraph weakly induced by* D is the graph $\langle D\rangle_w = (N[D], E \cap (D \times N[D]))$. Notice that a set D is a weakly connected dominating set of G if D is dominating set and $\langle D\rangle_w$ is connected. Clearly a connected dominating set must be weakly connected, but the converse is not true. The *weakly connected domination number* of G, denoted by $\gamma_w(G)$, is the size of a smallest weakly connected dominating set for G. We then have $\gamma(G) \leq \gamma_w(G) \leq \gamma_c(G)$.

The inequality $\gamma(G) \leq 2ir(G) - 1$ was obtained independently in [4, 5]. Bo and Liu in [1] proved that $\gamma_c(G) \leq 3ir(G) - 2$ for a connected graph G and this result is best possible.

In this paper, we first consider the relationship between weakly connected domination number $\gamma_w(G)$ and the irredundance number $ir(G)$. We prove that $\gamma_w(G) \leq \frac{5}{2}ir(G) - 2$ and this bound is sharp. Furthermore, for a tree T, we give a sufficient and necessary condition for $\gamma_c(T) = \gamma_w(T) + k$, where $\gamma_c(G)$ is the connected domination number and $0 \leq k \leq \gamma_w(T) - 1$.

§2 Main results

First, we have the following two lemmas.

Lemma 1(Hedetniemi [7]) *If S is an ir-set of graph G,* *and S is independent, then $ir = \gamma$.*

Lemma 2(Dunbar et al. [3]) *If G is a connected graph, then $\gamma(G) \leq \gamma_w(G) \leq 2\gamma(G) - 1$.*

Theorem 1 *If a graph G is connected, then $\gamma_w(G) \leq \frac{5}{2}ir(G) - 2$.*

Proof Let G be a connected graph and let $S = \{v_1, v_2, \cdots, v_l\}$ be an ir-set of G. All components of $\langle S \rangle$ are denoted by $S_1, S_2, \cdots, S_n$ for $1 \leq n \leq l = ir$. Suppose that there are t isolated vertices $v_1, v_2, \cdots, v_t$ in $\langle S \rangle$, where $v_1, v_2, \cdots, v_t$ belong to the components $S_1, S_2, \cdots, S_t$, respectively. Then each of the other $n - t$ components contain at least two vertices. Hence,

$$2(n-t) + t \leq ir \text{ i.e., } 2n - t \leq ir. \tag{1}$$

First we prove that $\gamma_w(G) \leq \frac{5}{2}ir(G) - 1$.

If $t = n$, then S is independent set. By Lemmas 1 and 2, it follows that $\gamma_w(G) \leq 2\gamma(G) - 1 = 2ir(G) - 1 \leq \frac{5}{2}ir(G) - 1$. Without loss of generality, we can assume that $t < n$. Since S is an irredundant set, $N[v_i] \not\subseteq \bigcup_{j \neq i} N[v_j]$ for any $v_i \in S$. Assume that $N_i = N[v_i] - \bigcup_{j \neq i} N[v_j]$ for $i = 1, 2, \cdots, l$. Since $N_i \neq \emptyset$, we may choose one vertex $u_i \in N_i$ for $i = t+1, t+2, \cdots, l$. Let $S_1' = S \cup \{u_{t+1}, \cdots, u_l\}$. It is clear that

$$|S_1'| = ir + ir - t = 2ir - t. \tag{2}$$

Since S is an ir-set of G, it follows that S_1' is a dominating set of G.

If $G_1 = \langle S_1' \rangle_w$ is connected, then $\gamma_w(G) \leq |S_1'| = 2ir - t \leq \frac{5}{2}ir - 1$. Suppose that G_1 has $q \geq 2$ components. Note that $q \leq$

n. Let w_1 be an arbitrary vertex of S_1', let W_1 be the vertex set of the component of G_1 that contains w_1, and let $T_1 = V(G) - W_1$. Let $t_1 \in T_1$ be chosen so that $d(w_1, t_1) = min\{d(w_1, x) | x \in T_1\}$, and let $P = y_{11}, y_{12}, \cdots, y_{1k}$ be the shortest t_1w_1-path, where $y_{11} = t_1$ and $y_{1k} = w_1$. Then $y_{1i} \in W_1$ for $2 \leq i \leq k$. Furthermore, $y_{12} \notin S_1'$ and $t_1 \in T_1 - S_1'$. Let $S_2' = S_1' \cup \{y_{12}\}$. Then $G_2 = \langle S_2' \rangle_w$ has at most $q-1$ components.

If G_2 is connected, then $\gamma_w(G) \leq |S_2'| = 2ir - t + 1 \leq \frac{5}{2}ir - 1$. Suppose that G_2 has at most $q-1$ components. Let w_2 be an arbitrary vertex of S_2' and W_2 be the vertex set of the component of G_2 that contains w_2. Let $T_2 = V(G) - W_2$. Let $t_2 \in T_2$ be chosen so that $d(w_2, t_2) = min\{d(w_2, x) | x \in T_2\}$, and let $P = y_{21}, y_{22}, \cdots, y_{2l}$ be the shortest t_2w_2-path, where $y_{21} = t_2$ and $y_{2l} = w_2$. Then $y_{2i} \in W_2$ for $2 \leq i \leq l$. Furthermore $y_{22} \notin S_2'$ and $t_2 \in T_2 - S_2'$. Thus if we let $S_3' = S_2' \cup \{y_{22}\}$, then $G_3 = \langle S_3' \rangle_w$ has at most $q-2$ components, and so on. We will make a set $Y = \{y_{12}, y_{22}, \cdots, y_{(s-1)2}\}$, where $s \leq q \leq n$. It is clear that $S_1' \cup Y$ is a weakly connected dominating set of G. By (1), it follows that $n - \frac{t}{2} \leq \frac{ir}{2}$. Hence,

$$\begin{aligned}
\gamma_w(G) &\leq |S_1' \cup Y| \leq 2ir - t + s - 1 \\
&\leq 2ir - t + n - 1 \leq 2ir - 1 + (n - \tfrac{t}{2}) - \tfrac{t}{2} \\
&\leq \tfrac{5}{2}ir - 1 - \tfrac{t}{2} \\
&\leq \tfrac{5}{2}ir - 1
\end{aligned} \tag{3}$$

Suppose that $\gamma_w(G) = \frac{5}{2}ir(G) - 1$. Then $t = 0$, $s = q = n = \frac{ir}{2}$ and $|Y| = n - 1$. So, $|S_1'| = 2ir$ and $|S_i| = 2$ for $i = 1, \cdots, n$. Without loss of generality, we can assume that $S_i = \{v_{2i-1}, v_{2i}\}$ for $i = 1, \cdots, n$. Furthermore, $G_1 = \langle S_1' \rangle_w$ has n components. Let $G_{11}, \cdots, G_{1n}$ denote the components of G_1. For $u_1 \in S_1'$, there exists $v_i \in S$ such that u_1 is adjacent to each vertex of

$PN[v_i, S]$. If $v_i \in S - \{v_1, v_2\}$, then the components number of G_1 is less than n, which is a contradiction. If $v_i \in \{v_1, v_2\}$, then $S_1' \cup Y - \{v_i\}$ is a weakly connected dominating set of G with cardinality less than $S_1' \cup Y$, which is a contradiction. Hence, $\gamma_w(G) \leq \frac{5}{2}ir(G) - 2$.

Theorem 2 *Let G be a connected graph. If $\gamma_w(G) = \frac{5}{2}ir(G) - 2$, then $ir(G) = 2$.*

Proof Let S, S_1', Y be defined as above. Since $\gamma_w(G) = \frac{5}{2}ir(G) - 2$, it follows that $ir(G)$ is even. We consider the following two cases.

Case 1 G_1 is connected. If $t \geq 2$, then $\gamma_w(G) \leq |S_1'| \leq 2ir(G) - t < \frac{5}{2}ir(G) - 2$. So, we only consider the case $t \leq 1$. If $t = 1$ and $ir(G) \geq 4$, then $\gamma_w(G) \leq |S_1'| \leq 2ir(G) - t < \frac{5}{2}ir(G) - 2$. It is obvious that it is impossible for $t = 1$ and $ir(G) = 2$. If $t = 0$ and $ir(G) \geq 6$, then $\gamma_w(G) \leq |S_1'| \leq 2ir(G) - t < \frac{5}{2}ir(G) - 2$. If $t = 0$ and $ir(G) = 4$, then for $u_1 \in S_1'$ there exists $v_i \in S$ such that u_1 is adjacent to each vertex of $PN[v_i, S]$. Hence $S_1' - \{v_i\}$ is a weakly connected dominating set of G with cardinality less than 8, which is a contradiction.Hence, $t = 0$ and $ir(G) = 2$.

Case 2 G_1 has $q \geq 2$ components. Then $q = n$. Otherwise, if $q \leq n - 2$, then

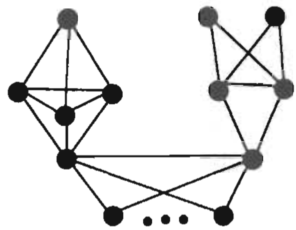
$$\begin{aligned}
\gamma_w(G) &\leq |S_1' \cup Y| \leq 2ir - t + s - 1 \\
&\leq 2ir - t + q - 1 \leq 2ir - t + n - 3 \\
&\leq 2ir - 3 + (n - \tfrac{t}{2}) - \tfrac{t}{2} \\
&\leq \tfrac{5}{2}ir - 3 - \tfrac{t}{2} \\
&\leq \tfrac{5}{2}ir - 3
\end{aligned}$$

If $q = n - 1$, then $s = q$, $t = 0$ and $n = \frac{ir(G)}{2}$. Let $G_{11}, G_{12}, \cdots, G_{1(n-1)}$ denote the components of G_1 such that $|G_{11} \cap S| = 4$. Without loss of generality, we can assume that $G_{11} \cap S = \{v_1, v_2, v_3, v_4\}$. For $u_5 \in S_1'$, there exists $v_i \in \{v_5, v_6\}$ such that u_5 is adjacent to each vertex of $PN[v_i, S]$. Then $(S_1' \cup Y) - \{v_i\}$ is a weakly connected dominating set of G with cardinality less than $\frac{5}{2}ir - 2$, which is a contradiction.

Since $q = n$, by inequality (3), it follows that $t \leq 2$. Let $G_{11}, \cdots, G_{1n}$ denote the components of G_1, where $S_i \subseteq G_{1i}$ for $i = 1, \cdots, n$. Suppose that $n - t \geq 2$. For $u_{t+1} \in S_1' \cap G_{1(t+1)}$, there exists $v_i \in S \cap G_{1(t+1)}$ such that u_{t+1} is adjacent to each vertex of $PN[v_i, S]$. For $u_n \in S_1' \cap G_{1n}$, there exists $v_j \in S \cap G_{1n}$ such that u_n is adjacent to each vertex of $PN[v_j, S]$. Then $S_1' \cup Y - \{v_i, v_j\}$ is a weakly connected dominating set of G with cardinality less than $\frac{5}{2}ir(G) - 2$, which is a contradiction. Hence $n - t \leq 1$. Since $n \geq 2$, it follows that $t \geq 1$.

If $t = 2$, then $s = q = n$ and $n - 1 = \frac{ir(G)}{2}$. If $n = 2$, then $ir(G) = 2$. If $n = 3$, suppose that $|S_1| = |S_2| = 1$ and $|S_3| \geq 2$. For $u_3 \in S_1'$, there exists $v_i \in \{v_3, v_4\}$ such that u_3 is adjacent to each vertex of $PN[v_i, S]$. Then $(S_1' \cup Y) - \{v_i\}$ is a weakly connected dominating set of G with cardinality less than $\frac{5}{2}ir - 2$, which is a contradiction.

If $t = 1$, then $s = q = n = 2$ and $n = \frac{ir(G)}{2}$. By a similar way as above, then there exists a weakly connected dominating set of G with cardinality less than $\frac{5}{2}ir - 2$, which is a contradiction. So $ir(G) = 2$.



Graphs with $\gamma_w(G) = 3$ and $ir(G) = 2$

Lemma 3 (Dunbar et al. [3]) *If a graph G is connected, then $\gamma_w(G) \leq \gamma_c(G) \leq 2\gamma_w(G) - 1$.*

Lemma 4 (Domke et al.[6]) *If T is a tree of order p, then $\gamma_w(T) = p - \beta(T)$.*

Theorem 3 *Let T denote a tree of order p, then $\gamma_c(T) = \gamma_w(T) + k$ if and only if $\beta(T') = k$, where $0 \leq k \leq \gamma_w(T) - 1$ and $T' = T - N[L]$.*

Proof Let S be an independent set of T such that $|S \cap L|$ is maximum. Then $S \cap L = L$. Otherwise, if there exists a vertex $v \in L$ such that $v \notin S$, then $N(v) \in S$. So $S' = (S - \{N(v))\} \cup \{v\}$ is an independent set of T. Furthermore, $|S' \cap L|$ is more than $|S \cap L|$, which is a contradiction. So $S - L$ is an independent set of T' and $\beta(T') \geq |S - L| = \beta(T) - |L|$.

Let D be an independent set of T'. Then $D \cup L$ is an independent set of T, Hence, $\beta(T) \geq |D \cup L|$. That is $\beta(T) \geq \beta(T') + |L|$. Therefore, $\beta(T) = \beta(T') + |L|$.

Suppose $\gamma_c(T) = \gamma_w(T) + k$. Since $\gamma_c(T) = p - |L|$ and

$\gamma_w(T) = p - \beta(T)$, it follows that $\beta(T) = |L| + k$. Hence, $\beta(T') = k$.

Conversely, if $\beta(T') = k$, then $\beta(T) = |L| + k$. Hence, $\gamma_c(T) = \gamma_w(T) + k$.

Corollary 1 *Let T denote a tree of order p, then $\gamma_c(T) = \gamma_w(T)$ if and only if every vertex of T is a leaf or a support vertex.*

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