Some k-fold Edge-graceful Labelings of $(p, p-1)$-graphs*

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Abstract

An edge-graceful (p, q)-graph $G = (V, E)$ is a graph with p vertices and q edges for which there is a bijection $f : E \to \{1, 2, \ldots, q\}$ such that the induced mapping $f^+ : V \to \mathbb{Z}_p$ defined by $f^+(u) \equiv \sum\limits_{uv\in E} f(uv) \pmod{p}$, for $u \in V$, is a bijection. In this paper, some results on edge-gracefulness of trees are extended to k-fold graphs based on graphs with p vertices and $p-1$ edges. A k-fold multigraph $G[k]$ derived from a graph G is one in which each edge of G has been replaced by k parallel edges with the same vertices as the original edge. Certain classes of k-fold multigraphs derived from paths, combs, and spiders are shown to be edge-graceful, as well as other graphs constructed by combining these graphs in specified ways.

1 Introduction

In this paper, the term "graph" means finite multigraph (not necessary connected) having no loop and no isolated vertex. The term "set" means multiset. Set operations are viewed as multiset operations. Let A be a set and let n be a positive integer. $A \times n$ denotes the set which is n copies of A. All undefined symbols and concepts may be looked up in [1]. A graph $G = (V, E)$ is a (p, q)-graph if p and q are its order and size respectively, i.e., $|V| = p$ and $|E| = q$.

*Partially supported by FRG, Hong Kong Baptist University.

†Research is done while on sabbatical at San José State University in 1998.

Let $G = (V, E)$ be a (p, q)-graph. Let $f : E \to \{d, d+1, \ldots, d+q-1\}$ be a bijection for some $d \in \mathbb{Z}$. The *induced mapping* $f^+ : V \to \mathbb{Z}_p$ of f is defined by $f^+(u) \equiv \sum_{uv \in E} f(uv) \pmod{p}$ for $u \in V$. If f^+ is a bijection or a constant mapping, then G is called *d-edge-graceful* or *d-edge-magic* respectively. If $d = 1$, then G is simply called *edge-graceful* or *edge-magic*, f an *edge-graceful labeling* or *edge-magic labeling of* G respectively. A necessary condition for a (p, q)-graph being edge-graceful is

$$q(q+1) \equiv \tfrac{1}{2}p(p-1) \pmod{p} \tag{1.1}$$

Since edge-graceful labelings were introduced by Lo [10] in 1985. many researchers have investigated certain families of graphs. The interested reader is referred to [3, 5, 6, 7, 10, 11, 13, 14, 17, 18]. Recently, Riskin and Wilson [12, 19] investigated the disjoint union and product of cycles; some of their results were shown earlier by Schaffer [13]. A graph with $p \equiv 2 \pmod 4$ cannot be edge-graceful [3].

Let G be a graph and let k be a positive integer. $G[k]$ is a graph which is obtained by splitting each edge into k parallel edges. We call $G[k]$ the *k-fold multigraph* (or *k-fold graph*) of G. kG denotes the k copies of the graph, i.e., $kG = \underbrace{G + \cdots + G}_{k \text{ times}}$.

Let S be a set. We use $S \times n$ to denote the n-copies of S. Suppose S contains qk elements. If $\mathcal{P}$ is a partition of S such that each class of $\mathcal{P}$ contains k elements, then $\mathcal{P}$ is called a (q, k)*-partition* of S.

Suppose A is a set consisting of r integers. If the sum (in the ring $\mathbb{Z}_p$) of elements of A is s, then A is called an $(s; r)$*-set* and s is called the *sum* of A. If $r = 1, 2$, or 3, it is called an *s-singleton*, an *s-doubleton* or an *s-tripleton* respectively. We will frequently use the term *s-set* when the value r is clear.

We let $[\boldsymbol{r}] = \{1, 2, \ldots, r\}$ for a positive integer r and $[\mathbf{0}] = \varnothing$. A mapping f is called a *k-fold edge-graceful labeling* or *k-fold edge-magic labeling* of a (p, q)-graph G if there is a (q, k)-partition $\mathcal{P}$ of $[\boldsymbol{qk}]$ such that $f : E \to \mathcal{P}$ is a bijection and the induced mapping $f^+ : V \to \mathbb{Z}_p$ is a bijection or a constant mapping respectively, where

$$f^+(u) \equiv \sum_{uv \in E} \sum_{i \in f(uv)} i \pmod{p}.$$

Thus, $G[k]$ is edge-graceful (respectively, edge-magic) if and only if there is a k-fold edge-graceful labeling (respectively, k-fold edge-magic labeling) of G.

Consider $q = pQ + R$, where $0 \le R < p$. After taking modulo p, the set $[\boldsymbol{q}]$ is equal to the set $([\boldsymbol{p}] \times Q) \cup [\boldsymbol{R}]$. Thus, for a (p, q)-graph $G = (V, E)$, there is an edge-graceful labeling $f : E \to [\boldsymbol{q}]$ if and only if there is a bijection $g : E \to ([\boldsymbol{p}] \times Q) \cup [\boldsymbol{R}]$ such that $g^+ : V \to \mathbb{Z}_p$ is a bijection.

2 General Edge-graceful Properties of k-fold Graphs

Let T be a $(p, p-1)$-graph (not necessarily connected). Note that if p is even, then T is never edge-graceful, by condition (1.1), though $T[k]$ may be edge-graceful. However, if $T[k]$ is edge-graceful, then an application of the congruence in (1.1) gives

$$p \not\equiv 2 \pmod 4 \text{ and } k(k-1) \equiv \begin{cases} 0 \pmod p & \text{if } p \text{ is odd,} \\ \frac{p}{2} \pmod p & \text{if } p \text{ is even.} \end{cases} \tag{2.1}$$

Lemma 2.1 [2, 8]: *Let G be a (p, q)-graph.*

(a) There is an edge-magic labeling f of $G[2p]$ such that f^+ is a zero mapping;

(b) if p is odd, then there is an edge-magic labeling f of $G[p]$ such that f^+ is a zero mapping;

(c) if p is even and all vertex degrees are odd (respectively even), then there is an edge-magic labeling f of $G[p]$ such that the value of f^+ is $\frac{p}{2}$ (respectively 0).

Lemma 2.2: *Let G be a (p, q)-graph. For $k \geq 1$*

(a) $G[2p + k]$ is edge-graceful if $G[k]$ is edge-graceful;

(b) if p is odd, then $G[p + k]$ is edge-graceful if $G[k]$ is edge-graceful;

(c) if p is even and all vertex degrees are odd (or even), then $G[p + k]$ is edge-graceful if $G[k]$ is edge-graceful.

Proof: Suppose $G[k]$ is edge-graceful. There is a k-fold edge-graceful labeling g of $G = (V, E)$. Let f be a $2p$-fold edge-magic labeling of G. Now the set of integers $[\boldsymbol{q(2p + k)}] \pmod p$ is equal to $([\boldsymbol{p}] \times 2q) \cup [\boldsymbol{qk}]$ and $G[2p + k] \cong G[2p] \cup G[k]$, where $G[2p]$ and $G[k]$ are edge-disjoint. Define $\phi(e) = g(e) \cup f(e)$ for $e \in E$. Clearly, ϕ is a $(2p + k)$-fold edge-graceful labeling of G. We have proved part (a). By a similar argument we have parts (b) and (c). ∎

Corollary 2.3: *Let G be a (p, q)-graph. Suppose $G[n]$ is edge-magic and p is a factor of nq. For $k \geq 1$, if $G[k]$ is edge-graceful then $G[n + k]$ is edge-graceful.*

Proof: Let $qn = cp$ for some c. Since $[\boldsymbol{q(n + k)}] \pmod p$ is equal to $([\boldsymbol{p}] \times c) \cup [\boldsymbol{qk}]$, by a similar proof of Lemma 2.2(a) we have the corollary. ∎

Corollary 2.4: *Let G be a (p, q)-graph. Suppose $G[p]$ is edge-magic. For $k \geq 1$, if $G[k]$ is edge-graceful then $G[p + k]$ is edge-graceful.*

If G is a (p, q)-graph. By Lemma 2.2, we can reduce the problem for proving that $G[k]$ is edge-graceful to $k \leq 2p$, or $k \leq p$ if G satisfies Corollary 2.4.

3 Edge-gracefulness of k-fold $(p, p-1)$-graphs

In this section we shall consider the edge-gracefulness of k-fold $(p, p-1)$-graphs. First we show a useful lemma.

Lemma 3.1: *For any integer* $1 \le k \le 2m+1$, *if* $k(k-1) \equiv 0 \pmod{2m+1}$, *then* $[\boldsymbol{2mk}]$ *has a* $(2m, k)$*-partition such that the sums of the classes are nonzero and distinct taken in* $\mathbb{Z}_{2m+1}$.

Proof: In this proof, arithmetic is taken in $\mathbb{Z}_{2m+1}$. We observe that $[\boldsymbol{2mk}] \pmod{2m+1}$ is equal to $\big([\boldsymbol{2m+1}] \times (k-1)\big) \cup [\boldsymbol{2m+1-k}]$. Each $[\boldsymbol{2m+1}]$ may be grouped into m zero-doubletons and 1 zero-singleton as follows:

—	1	2	$\cdots$	$m-1$	m
$2m+1$	$2m$	$2m-1$	$\cdots$	$m+2$	$m+1$

(3.1)

We shall deal with $[\boldsymbol{2m+1-k}]$. $[\boldsymbol{2m+1}]$ is clearly a 0-set, and the condition $k(k-1) \equiv 0 \pmod{2m+1}$ implies that $[\boldsymbol{2mk}]$ is a 0-set, so $[\boldsymbol{2m+1-k}]$ is also a 0-set. If $2m+1-k \ge m+1$, then the set $\{k, k+1, \dots, 2m+1-k\}$ may be grouped into $m-k+1$ zero-doubletons. So we only have to deal with $[\boldsymbol{k-1}]$. If $2m+1-k \le m$, then $2m+1-k \le k-1$. Both of these cases may be reduced to the case of handling the 0-set $[\boldsymbol{r}]$, where $0 \le r \le k-1$.

Case 1: k is even.

Since $k = 2$ does not satisfy $k(k-1) \equiv 0 \pmod{2m+1}$ we may assume $k \ge 4$. First we choose 2 copies of $[\boldsymbol{2m+1}]$ and arrange them as follows:

	—	1	2	3	$\cdots$	m	$m+1$	$m+2$	$\cdots$	$2m$	$2m+1$
	1	2	3	4	$\cdots$	$m+1$	$m+2$	$m+3$	$\cdots$	$2m+1$	—
Sum	1	3	5	7	$\cdots$	$2m+1$	2	4	$\cdots$	$2m$	0

We have $2m$ doubletons which are 2-, 3-, …, $(2m)$-, $(2m+1)$-sets, one 1-singleton and one 0-singleton. Let A_i denote the i-doubleton obtained above, $2 \le i \le 2m$. The 0-doubleton $\{m, m+1\}$ will be handled together with other 0-doubletons.

After grouping the remaining $[\boldsymbol{2m+1}]$'s as (3.1), we have a 0-set $[\boldsymbol{r}]$ with $0 \le r \le k-1$; A_i, where $2 \le i \le 2m$; one 1-singleton; some number of 0-doubletons and some number of 0-singletons. Note that r is odd.

We combine $[\boldsymbol{r}]$ with the 1-singleton and an appropriate number of 0-doubletons, if necessary, to form a $(1; k)$-set. Group the remaining 0-singletons into 0-doubletons. Then combining an appropriate number of 0-doubletons with A_i we obtain $2m$ sets whos e sums are $2, 3, \dots, 2m$. Hence we obtain a required partition.

Case 2: k is odd.

We partition a copy of $[\boldsymbol{2m+1}]$ into $2m+1$ singletons, namely $A_i = \{i\}$ for $1 \le i \le 2m$ and a 0-singleton $\{2m+1\}$. The other $[\boldsymbol{2m+1}]$'s, if any, are grouped as (3.1). Now we have a 0-set $[\boldsymbol{r}]$ with $0 \le r \le k-1$; A_i, where $1 \le i \le 2m$; some number of 0-doubletons and some number of 0-singletons. Note that r is even. Similar to Case 1, we combine $[\boldsymbol{r}]$ with A_1 and certain appropriate number of 0-doubletons, if necessary, to form a $(1;k)$-set. The rest is the same as Case 1. ∎

A technique similar to the above proof was used in some other papers, for example [2, 15, 16].

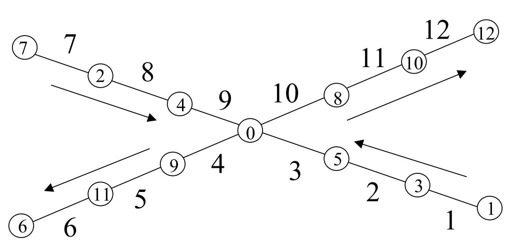
Theorem 3.2: *Let G be a $(2m+1, 2m)$-graph. If G is edge-graceful and $k(k-1) \equiv 0 \pmod{2m+1}$, then $G[k]$ is edge-graceful.*

Proof: We may assume that $1 \le k \le 2m+1$. From Lemma 3.1 we have a $(2m,k)$-partition $\{C_i \mid 1 \le i \le 2m\}$ of $[\boldsymbol{2mk}]$, where C_i is an i-set (taken in $\mathbb{Z}_{2m+1}$). Let g be an edge-graceful labeling of G. Define $f(e) = C_i$ if $g(e) = i$. Then f is a k-fold edge-graceful labeling of G. ∎

Let P_n and S_Δ denote the path with n vertices and the star with maximum degree Δ, respectively. It is known that P_{2m+1} and S_{2m} are edge-graceful for $m \ge 1$ [4, 10].

Let G be a graph. A vertex u of degree one in G is called a *pendant*. Let v be a vertex of G and let P be a path originating from v to a pendant of G. If all internal vertices of P are of degree 2 in G, then $V(P) \cup E(P) \setminus \{v\}$ is called a *leg* of v. The number of edges ℓ is called the *length* of the leg. The leg will also be called an *ℓ-leg*. A graph G is called a *spider with Δ legs* (called *superstar* in some articles) if it is obtained from the union of Δ paths with one of the end vertices of each path identified. The identified vertex is called the *center* of the spider graph. If all legs are ℓ-legs, then G is called a *regular spider graph with Δ legs of length ℓ*, and denoted by $S_{\Delta,\ell}$. Note that $S_{\Delta,1} = S_\Delta$. Lee [9] proved that $S_{2m,\ell}$ is edge- graceful. Small [18] also proved the same result. We use Example 3.1 below to illustrate the proof of Lee in [9].

Example 3.1: Consider the regular spider graph $S_{4,3}$. Following is an edge-graceful labeling f. Numbers labeled in vertices are the values of f^+.



This labeling method can be generalized to an edge-graceful labeling for $S_{2m,\ell}$. ∎

Then we have

Corollary 3.3: *Let $m \geq 1$. If $k(k-1) \equiv 0 \pmod{2m+1}$, then $P_{2m+1}[k]$ is edge-graceful; and if $k(k-1) \equiv 0 \pmod{2m\ell+1}$, then $S_{2m,\ell}[k]$ is edge-graceful.*

Some more families of trees such as odd caterpillars with at most one vertex of degree 2, complete $(2k+1)$-ary trees with $2s+1$ layers, are edge-graceful. The interested reader is referred to [4].

Let $G = (V, E)$ be a (p, q)-graph. A mapping (not necessary bijective) $f : E \to \mathbb{Z}_p$ is called a $\mathbb{Z}_p$-*edge-graceful labeling* of G if the *induced mapping* $f^+ : V \to \mathbb{Z}_p$ is a bijection, where $f^+(u) \equiv \sum_{uv \in E} f(uv) \pmod{p}$. A similar concept called $\mathbb{Z}_p$-edge-magic labeling was introduced in [15, 16].

Theorem 3.4: *Let $G = (V, E)$ be a (p, q)-graph. If $G[k]$ is edge-graceful then there is a $\mathbb{Z}_p$-edge-graceful labeling f of G such that $\sum_{e \in E} f(e) \equiv \frac{1}{2}kq(kq+1) \pmod{p}$.*

Proof: $G[k]$ is edge-graceful if and only if G has a k-fold edge-graceful labeling g. For $e \in E$ if $g(e) = C_e$, then $\{C_e \mid e \in E\}$ is a (q, k)-partition of $[\boldsymbol{qk}]$. Hence $\sum_{e \in E} \|g(e)\| = \frac{1}{2}kq(kq+1)$, where $\|g(e)\| = \sum_{j \in C_e} j$. Define $f(e) \equiv \|g(e)\| \pmod{p}$. Since g is a k-fold edge-graceful labeling, f is a $\mathbb{Z}_p$-edge-graceful labeling of G and $\sum_{e \in E} f(e) \equiv \frac{1}{2}kq(kq+1) \pmod{p}$. ∎

Corollary 3.5: *Let $G = (V, E)$ be a $(p, p-1)$-graph. If $G[k]$ is edge-graceful then there is a $\mathbb{Z}_p$-edge-graceful labeling f of G such that if p is odd, then $\sum_{e \in E} f(e) \equiv 0 \pmod{p}$ and if p is even, then*

$$\sum_{e \in E} f(e) \equiv \begin{cases} \frac{1}{2}k(k-1) \pmod{p} & \text{if } k \text{ is even,} \\ -\frac{1}{2}k[k(p-1)+1] \pmod{p} & \text{if } k \text{ is odd.} \end{cases} \tag{3.2}$$

Proof: For p odd, since $k(p-1)[k(p-1)+1] \equiv 0 \pmod{p}$, $\sum_{e \in E} f(e) \equiv 0 \pmod{p}$. For p even,

$$\sum_{e \in E} f(e) \equiv \frac{1}{2}k(p-1)[k(p-1)-1] \equiv$$

$$\begin{cases} \frac{1}{2}k(k-1) \pmod{p} & \text{if } k \text{ is even,} \\ -\frac{1}{2}k[k(p-1)+1] \pmod{p} & \text{if } k \text{ is odd.} \end{cases}$$ ∎

Note that if p is even, from (3.2) we have

$$2\sum_{e \in E} f(e) \equiv k(k-1) \equiv \tfrac{p}{2} \pmod{p} \tag{3.3}$$

or equivalently

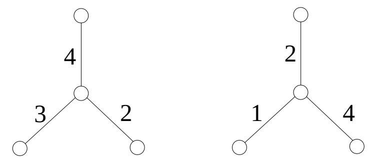
$$\sum_{e \in E} f(e) \equiv \tfrac{p}{4} \text{ or } \tfrac{3p}{4} \pmod{p}. \tag{3.4}$$

4 Examples

Example 4.1: If G is a $(4,3)$-graph. Then $G \cong P_4$ or S_3 if G is connected and $G \cong K_2[2] + K_2$ if G is disconnected.

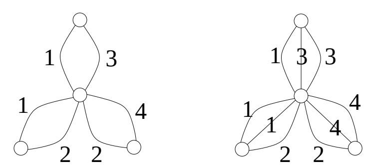
(a) Suppose $G \cong K_2[2] + K_2$. It is easy to see that there is no $\mathbb{Z}_4$-edge-graceful labeling.

(b) Suppose $G \cong S_3$ and f is a $\mathbb{Z}_4$-edge-graceful labeling of S_3. Then $f(e)$ are distinct for all $e \in E$ and $\sum\limits_{e\in E} f(e) \equiv 1$ or $3 \pmod 4$. The only possible cases are $\{f(e) \mid e \in E\} = \{2,3,4\}$ or $\{f(e) \mid e \in E\} = \{1,2,4\}$ (see the following figures).

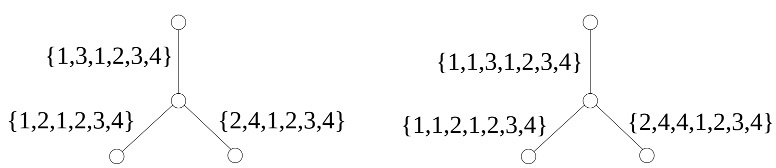


From (3.2), $k \equiv 2,\ 3,\ 6$ or $7 \pmod 8$. From (3.4), if $k \equiv 2$ or $3 \pmod 8$, then $\{f(e) \mid e \in E\} = \{2,3,4\}$; if $k \equiv 6$ or $7 \pmod 8$, then $\{f(e) \mid e \in E\} = \{1,2,4\}$.

For $k = 2$ or 3, it suffices to find a $(3,k)$-partition of $[\boldsymbol{3k}]$ such that the sums of the classes are 2, 3, and 4. For $k = 2$, $[\mathbf{6}]$ is partitioned as $\big\{\{2,4\},\{1,2\},\{1,3\}\big\}$. Following are edge-graceful labelings of $S_3[k]$ for $k = 2, 3$.

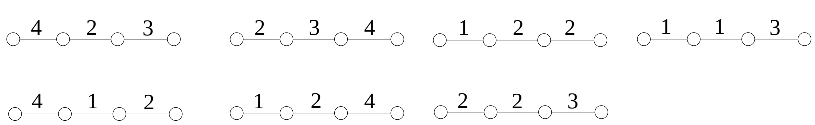


For $k = 6$ or 7, following are k-fold edge-graceful labelings of S_3.



In fact by Lemma 2.2(c) the edge-gracefulness of $S_3[2]$ and $S_3[3]$ imply that $S_3[6]$ and $S_3[7]$ are edge-graceful.

(c) Suppose $G \cong P_4$ and f is a $\mathbb{Z}_4$-edge-graceful labeling of P_4. Then the possible cases are $\{f(e) | e \in E\} = \{2,3,4\}$, $\{1,2,2\}$, $\{1,1,3\}$, $\{1,2,4\}$ or $\{2,2,3\}$. Following are all the $\mathbb{Z}_4$-edge-graceful labelings of P_4.

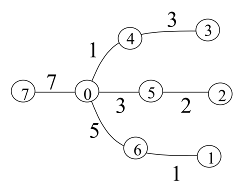


Similar to (b) if $k \equiv 2$ or $3 \pmod 8$, then $\{f(e) \mid e \in E\} = \{2,3,4\}$, $\{1,2,2\}$ or $\{1,1,3\}$; if $k \equiv 6$ or $7 \pmod 8$, then $\{f(e) \mid e \in E\} = \{1,2,4\}$ or $\{2,2,3\}$.

For $k = 2$ or 3, if $\{f(e) \mid e \in E\} = \{2,3,4\}$, then a partition of $[\mathbf{3k}]$ is the same as in part (b); if $\{f(e) \mid e \in E\} = \{1,2,2\}$, then $\big\{\{2,3\},\{1,1\},\{2,4\}\big\}$ is a required partition of $[\mathbf{6}] \equiv [\mathbf{4}] \cup [\mathbf{2}] \pmod 4$ and $\big\{\{1,4,4\},\ \{2,1,3\},$
$\{2,1,3\}\big\}$ is a required partition of $[\mathbf{9}] \equiv ([\mathbf{4}] \times 2) \cup [\mathbf{1}] \pmod 4$; if $\{f(e) \mid e \in E\} = \{1,1,3\}$, then $\big\{\{2,3\},\ \{1,4\},\ \{1,2\}\big\}$ is a required partition of $[\mathbf{6}]$ and $\big\{\{1,4,4\},\ \{1,2,2\},\ \{1,3,3\}\big\}$ is a required partition of $[\mathbf{9}]$.

For $k = 6$ or 7, if $\{f(e) \mid e \in E\} = \{1,2,4\}$, then a partition of $[\mathbf{3k}]$ is the same as in (b); if $\{f(e) \mid e \in E\} = \{2,2,3\}$, then $\big\{\{1,3\} \cup [\mathbf{4}],\ \{2,2\} \cup [\mathbf{4}],\ \{1,4\} \cup [\mathbf{4}]\big\}$ and $\big\{\{1,1,2\} \cup [\mathbf{4}],\ \{1,3,4\} \cup [\mathbf{4}],\ \{2,3,4\} \cup [\mathbf{4}]\big\}$ are required partitions of $[\mathbf{18}] \equiv ([\mathbf{4}] \times 4) \cup [\mathbf{2}] \pmod 4$ and $[\mathbf{21}] \equiv ([\mathbf{4}] \times 5) \cup [\mathbf{1}] \pmod 4$ respectively. ∎

Example 4.2: We consider a spider graph Ψ described in the following figure. In this example, arithmetic is taken in $\mathbb{Z}_8$. A $\mathbb{Z}_8$-edge-graceful labeling, say f, of Ψ is also described in the figure. Numbers labeled on the vertices are values of f^+.



To show $\Psi[4]$ edge-graceful, it suffices to partition $[\mathbf{28}] = ([\mathbf{8}] \times 3) \cup [\mathbf{4}]$ into a $(7,4)$-partition which consists of two 1-sets, two 3-sets, one 2-set, one 5-set and one 7-set. A possible partition is:

	3	2	1	1	3	1	7
	4	4	2	1	5	2	8
	4	5	3	2	5	3	8
	6	6	4	7	6	7	8
Sum	1	1	2	3	3	5	7

Each column forms a class of the partition. ∎

5 Labelings of k-fold Combs and Paths

In this section we consider a class of trees called the combs. Let E_n be a $(2n, 2n-1)$-graph whose vertex set is $A \cup B$, where $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, such that the induced subgraph $E_n[A] \cong P_n$ and b_i is only adjacent to a_i, $1 \le i \le n$. Then E_n is called a *comb*. By the remark preceding Lemma 2.1, E_n is never edge-graceful, and if $E_n[k]$ is edge-graceful, then $n = 2t$

for some $t \geq 1$. From Corollary 3.5, a necessary condition of $E_{2t}[k]$ being edge-graceful is that there exists a $\mathbb{Z}_{4t}$-edge-graceful labeling f such that

$$\sum_{e\in E} f(e) \equiv t \text{ or } 3t \pmod{4t},$$

where E is the edge set of E_{2t}. From now on, unless stated otherwise arithmetic is taken in $\mathbb{Z}_{4t}$.

Lemma 5.1: *There are two $\mathbb{Z}_{4t}$-edge-graceful labelings g_1 and g_3 of $E_{2t} = (V, E)$ such that $\sum\limits_{e\in E} g_a(e) \equiv at \pmod{4t}$, $a = 1, 3$.*

Proof: Keep all the notations defined as above. Define $g_1 : E \to \mathbb{Z}_{4t}$ by

$$g_1(a_ib_i) = i,\ 1 \leq i \leq 2t \text{ and } g_1(a_ja_{j+1}) = \begin{cases} 2t & \text{if } j \text{ is odd} \\ 0 & \text{if } j \text{ is even} \end{cases},\ 1 \leq j \leq 2t-1.$$

Then $\sum\limits_{e\in E} g_1(e) = t$.

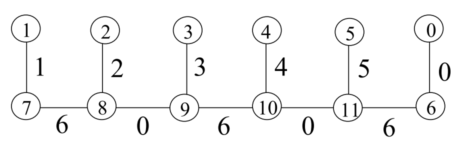
Define $g_3 : E \to \mathbb{Z}_{4t}$ by $g_3(a_{2t}b_{2t}) = g_1(a_{2t}b_{2t}) + 2t = 0$ and $g_3(e) = g_1(e)$ otherwise. Then $\sum\limits_{e\in E} g_3(e) = 3t$. It is easy to check that g_1 and g_3 are $\mathbb{Z}_{4t}$-edge-graceful labelings of E_{2t}. ∎

We shall show that $E_{2t}[k]$, where $k(k-1) = 2t$, is edge-graceful. By Lemma 2.2 we may assume that $1 \leq k \leq 8t$ and k satisfies the (congruence) equation $k(k-1) = 2t$. In fact, $k = 1$ is not a case. It suffices to partition $[\boldsymbol{(4t-1)k}]$ into a $(4t-1, k)$-partition $\mathcal{P}$ such that $\mathcal{P}$ consists of t $(2t)$-sets; $(t-1)$ zero-sets; one 1-, 2-, $\dots$, $(2t-1)$-set each; and either one additional $2t$-set or 0-set according to the $\mathbb{Z}_{4t}$-edge-graceful labeling g_1 or g_3 defined in Lemma 5.1, respectively. The choice of g_1 or g_3 depends on the value of k.

Remark: Let t be a fixed positive integer. If k_0 satisfies the (congruence) equation $k(k-1) = 2t$, then so does $k_0 + 4t$ and vice versa. If k_0 satisfies $\frac{1}{2}(4t-1)k[(4t-1)k+1] = t$ then $k_0 + 4t$ satisfies $\frac{1}{2}(4t-1)k[(4t-1)k+1] = 3t$.

If $k = 2$, then k satisfies $k(k-1) = 2t$ only when $t = 1$. Since $E_2 \cong P_4$, from Example 4.1 $E_2[2]$ is edge-graceful. If $k = 3$ and satisfies $k(k-1) = 2t$, then $t = 1$ or 3. Also from Example 4.1, $E_2[3]$ is edge-graceful. We shall show that $E_6[3]$ is edge-graceful in the following example.

Example 5.1: Since $k = 3$ and $t = 3$, we use g_3 as our frame (see the figure below). Thus we need to partition $[\mathbf{33}] = ([\mathbf{12}] \times 2) \cup [\mathbf{9}]$ into a (11, 3)-partition such that it consists of three 6-sets; three 0-sets, one 1-, 2-, 3-, 4-, 5-set each.



	1	2	3	4	5	6	7	8	9	10	11
	11	10	9	8	7	9	5	4	8	2	1
	1	2	3	4	5	3	6	6	7	12	12
Sum	1	2	3	4	5	6	6	6	0	0	0

Each column forms a class. $\blacksquare$

Lemma 5.2: *Let* $4 \le k \le 8t$ *and* $a = 1$ *or* 3. *If* $\frac{1}{2}(4t-1)k[(4t-1)k+1] \equiv at \pmod{4t}$ *then* $[\boldsymbol{(4t-1)k}]$ *has a* $(4t-1, k)$*-partition* $\mathcal{P}$ *such that* $\mathcal{P}$ *consists of* t $(2t)$*-sets;* $(t-1)$ 0*-sets; one* 1-,2-, $\ldots$, $(2t-1)$*-set each and one* $(t+at)$*-set.*

Proof: $[\boldsymbol{(4t-1)k}] = \big([\boldsymbol{4t}] \times (k-b)\big) \cup [\boldsymbol{4tb-k}]$, where $b = 1$ if $4 \le k \le 4t$ and $b = 2$ if $4t+1 \le k \le 8t$.

We shall deal with $[\boldsymbol{4tb-k}]$. Since $[\boldsymbol{(4t-1)k}]$ is an (at)-set and $[\boldsymbol{4t}]$ is a $(2t)$-set, $[\boldsymbol{4tb-k}]$ is a $(2bt+at)$-set if k is even and $(2bt+(a-2)t)$-set if k is odd.

If $4tb-k \ge 2t+1$, then the set $\{k, k+1, \ldots, 4tb-k\}$ may be grouped into some 0-doubletons and one $(2t)$-singleton $\{2t\}$. So we only have to deal with the set $R = [\boldsymbol{k-1}] \cup \{2t\}$. If $4tb-k \le 2t$, then $4tb-k \le k$. We let $R = [\boldsymbol{4tb-k}]$. Both of these cases may be reduced to the case of handling a residual set R with $|R| \le k$ which is a $(2bt+at)$-set if k is even and $(2bt+(a-2)t)$-set if k is odd. Note that if $|R| = k$ then $2t \in R$.

Each $[\boldsymbol{4t}]$ may be grouped into $(2t-1)$ 0-doubletons, one $(2t)$-singleton and one 0-singleton as follows:

$$\begin{array}{|c|}\hline - \\ 4t \\ \hline\end{array}\ \begin{array}{|cccc|}\hline 1 & 2 & \cdots & 2t-1 \\ 4t-1 & 4t-2 & \cdots & 2t+1 \\ \hline\end{array}\ \begin{array}{|c|}\hline 2t \\ - \\ \hline\end{array} \tag{5.1}$$

Case 1: Suppose $4 \le k \le 4t$, i.e., $b = 1$.

Subcase 1.1: Suppose k is even. Since $k \ge 4$, we can choose two copies of $[\boldsymbol{4t}]$ and group them as follows:

$$\begin{array}{c|ccccc|c|cccc|cc|}
\cline{2-13}
 & 4t & 1 & 2 & \cdots & t-1 & t & t+1 & t+2 & \cdots & 2t-1 & 2t & - \\
 & 1 & 2 & 3 & \cdots & t & - & 3t-1 & 3t-2 & \cdots & 2t+1 & - & 2t \\
\cline{2-13}
\text{Sum} & 1 & 3 & 5 & \cdots & 2t-1 & & 0 & 0 & \cdots & 0 & & \\
\cline{2-13}
\end{array}$$

$$\begin{array}{c|cccc|cccc|c|}
\cline{2-10}
 & 2t+1 & 2t+2 & \cdots & 3t & 3t+1 & 3t+2 & \cdots & 4t-1 & - \\
 & 4t-1 & 4t-2 & \cdots & 3t & t+1 & t+2 & \cdots & 2t-1 & 4t \\
\cline{2-10}
\text{Sum} & 2t & 2t & \cdots & 2t & 2 & 4 & \cdots & 2t-2 & \\
\cline{2-10}
\end{array} \tag{5.2}$$

or

$$
\begin{gathered}
\begin{array}{r|ccccc|c|cccc|c|}
\hline
 & 4t & 1 & 2 & \cdots & t-1 & t & t+1 & t+2 & \cdots & 2t-1 & - \\
 & 1 & 2 & 3 & \cdots & t & 3t & 3t-1 & 3t-2 & \cdots & 2t+1 & 2t \\
\hline
\text{Sum} & 1 & 3 & 5 & \cdots & 2t-1 & 0 & 0 & 0 & \cdots & 0 & \\
\hline
\end{array}
\\
\begin{array}{r|ccccc|c|cccc|}
\hline
 & 2t & 2t+1 & 2t+2 & \cdots & 3t-1 & 3t & 3t+1 & 3t+2 & \cdots & 4t-1 \\
 & 4t & 4t-1 & 4t-2 & \cdots & 3t+1 & - & t+1 & t+2 & \cdots & 2t-1 \\
\hline
\text{Sum} & 2t & 2t & 2t & \cdots & 2t & & 2 & 4 & \cdots & 2t-2 \\
\hline
\end{array}
\end{gathered}
\tag{5.3}
$$

After grouping the other $[\boldsymbol{4t}]$'s as (5.1), we have t $(2t)$-doubletons; one $(2t+at)$-set R; some 0-singletons; some $(2t)$-singletons; some 0-doubletons, one 1- to $(2t-1)$-doubleton each and one t-singleton or $(3t)$-singleton according to (5.2) or (5.3). Note that $|R|$ is even.

For this subcase we use (5.2). If $|R| \le k-2$, then combine R with the t-singleton, a $(2t)$-singleton and an appropriate number of 0-doubletons, if necessary, to form a $(t+at)$-set. If $|R| = k$, then combine $R \setminus \{2t\}$ with the t-singleton and an appropriate number of 0-doubletons, if necessary, to form a $(t+at)$-set. Group the remaining 0-singletons and $(2t)$-singletons into 0-doubletons. Combine each of the other doubletons obtained from (5.2) with an appropriate number of 0-doubletons to obtain a required class. Thus we obtain a required partition.

Subcase 1.2: Suppose k is odd. Since $k \ge 4$, we can choose three copies of $[\boldsymbol{4t}]$ and group as follows:

$$
\begin{gathered}
\begin{array}{r|c|cccc|c|cccc|}
\hline
 & - & 1 & 2 & \cdots & t-1 & t & t+1 & t+2 & \cdots & 2t \\
 & 1 & 2 & 3 & \cdots & t & - & 3t-1 & 3t-2 & \cdots & 2t \\
 & 4t-1 & 4t-3 & 4t-5 & \cdots & 2t+1 & - & 1 & 3 & \cdots & 2t-1 \\
\hline
\text{Sum} & 0 & 0 & 0 & \cdots & 0 & & 1 & 3 & \cdots & 2t-1 \\
\hline
\end{array}
\\
\begin{array}{r|cccc|cccc|c|c|}
\hline
 & 2t+1 & 2t+2 & \cdots & 3t & 3t+1 & 3t+2 & \cdots & 4t-1 & 4t & - \\
 & 4t-1 & 4t-2 & \cdots & 3t & t+1 & t+2 & \cdots & 2t-1 & 4t & - \\
 & 2t+2 & 2t+4 & \cdots & 4t & 2t-2 & 2t-4 & \cdots & 2 & - & 2t \\
\hline
\text{Sum} & 2 & 4 & \cdots & 2t & 2t & 2t & \cdots & 2t & 0 & \\
\hline
\end{array}
\end{gathered}
\tag{5.4}
$$

or

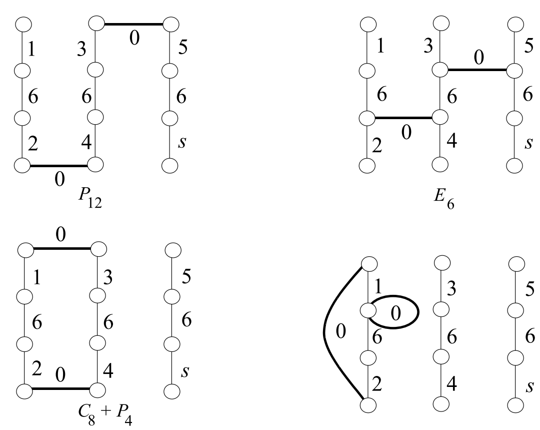
$$
\begin{gathered}
\begin{array}{r|c|cccc|c|cccc|}
\hline
 & - & 1 & 2 & \cdots & t-1 & t & t+1 & t+2 & \cdots & 2t \\
 & 1 & 2 & 3 & \cdots & t & 3t & 3t-1 & 3t-2 & \cdots & 2t \\
 & 4t-1 & 4t-3 & 4t-5 & \cdots & 2t+1 & - & 1 & 3 & \cdots & 2t-1 \\
\hline
\text{Sum} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 3 & \cdots & 2t-1 \\
\hline
\end{array}
\\
\begin{array}{r|cccc|c|cccc|c|c|}
\hline
 & 2t+1 & 2t+2 & \cdots & 3t-1 & 3t & 3t+1 & 3t+2 & \cdots & 4t-1 & 4t & - \\
 & 4t-1 & 4t-2 & \cdots & 3t+1 & - & t+1 & t+2 & \cdots & 2t-1 & 4t & - \\
 & 2t+2 & 2t+4 & \cdots & 4t-2 & - & 2t-2 & 2t-4 & \cdots & 2 & 2t & 4t \\
\hline
\text{Sum} & 2 & 4 & \cdots & 2t-2 & & 2t & 2t & \cdots & 2t & 2t & \\
\hline
\end{array}
\end{gathered}
\tag{5.5}
$$

After grouping the other [$\boldsymbol{4t}$]'s as (5.1), we have t $(2t)$-tripletons; one (at)-set R; $(t-1)$ 0-tripletons; some 0-singletons; some $(2t)$-singletons; some 0-doubletons, one 1- to $(2t-1)$-tripleton each and one t-singleton or $(3t)$-singleton according to (5.4) or (5.5). Note that $|R|$ is odd.

If $|R| \le k-2$, then we use (5.4) and combine R with the t-singleton, a 0-singleton and an appropriate number of 0-doubletons, if necessary, to form a $(t+at)$-set. If $|R| = k$, then we use (5.5) and combine $R \setminus \{2t\}$ with the $(3t)$-singleton and an appropriate number of 0-doubletons, if necessary, to form a $(t+at)$-set. Group the remaining 0-singletons and $(2t)$-singletons into 0-doubletons. Combine each of the other tripletons obtained from (5.4) or (5.5) with an appropriate number of 0-doubletons to obtain a required class. Thus we obtain a required partition.

Case 2: Suppose $4t+1 \le k \le 8t$, i.e., $b=2$. For k even, we use (5.3) and the combination is similar to Subcase 1.1. For k odd, we use (5.5) and (5.4) for the cases $|R| \le k-2$ and $|R| = k$ respectively, and the combination is similar to Subcase 1.2. ▮

The construction above may be used to give k-fold edge-graceful labelings of a number of graphs. Suppose a graph G is composed of t copies of P_4 together with $t-1$ *"linking edges"* K_2 such that each linking edge has its vertices identified with any (not necessarily distinct) vertices of the P_4's. The edges of the P_4's are labeled $(1, 2t, 2)$, $(3, 2t, 4)$, $\ldots$, $(2t-3, 2t, 2t-2)$ and $(2t-1, 2t, s)$ in order, where s is either 0 or $2t$ depending on the value of k. The edges K_2 are all labeled 0. This labeling is a $\mathbb{Z}_{4t}$-edge-graceful labeling of G. It may be given a k-fold edge-graceful labeling based on the partition obtained from Lemma 5.2. As the diagram below shows, many graphs may be composed in this way, including the combs E_{2t} and the paths P_{4t}.



In the above figure, s is 0 or 6 and heavy edges are linking edges.

By Lemma 5.2 we have the following theorem.

Theorem 5.3: *Suppose $k \ge 1$ and satisfies $k(k-1) \equiv 2t \pmod{4t}$. Then $E_{2t}[k]$ and $P_{4t}[k]$ are edge-graceful.*

As further examples of graphs which are not themselves edge-graceful, but which may have

k-fold edge-graceful labelings, define a *comb with n legs or "bristles" of length c*, $E_n(c)$, in the same way as E_n, except that we now subdivide each edge a_ib_i by $c-1$ additional vertices, thus replacing edge a_ib_i by a path of length c. Then $E_n(c)$, for $c \equiv 3 \pmod 4$, and $E_{2t}(c)$, for $c \equiv 1 \pmod 4$, are each easily decomposed into a number of P_4's and one fewer linking edges K_2, and so Lemma 5.2 provides k-fold edge-graceful labelings. Also, if $t = n_0 + n_1 + \cdots + n_d$ is any partition of t, then the disjoint union of a path and cycles $P_{4n_0} + n_1C_4 + n_2C_8 + \cdots + n_dC_{4d}$ may be similarly decomposed and k-fold edge-gracefully labeled.

In our experience, there are many ways for us to partition $[\boldsymbol{qk}]$ into a required (q,k)-partition, so we believe that the following conjecture is true.

Conjecture: Let $G = (V, E)$ be a (p, q)-graph. For $k \geq 2$, if there is a $\mathbb{Z}_p$-edge-graceful labeling f of G such that $\sum\limits_{e \in E} f(e) \equiv \frac{1}{2}kq(kq+1) \pmod p$, then $G[k]$ is edge-graceful.

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