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[GTN LVIII:1] ORDERING TREES BY THEIR LARGEST LAPLACIAN EIGENVALUES

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Abstract

Let $\Delta(T)$ and $\mu_1(T)$, respectively, denote the maximum degree and largest Laplacian eigenvalue of a tree T. Let $\mathcal{T}_n$ be the set of trees of order n, and let $\mathcal{T}_n^{(\Delta)} = \{T \in \mathcal{T}_n : \Delta(T) = \Delta\}$. In this paper, among all trees in $\mathcal{T}_n^{(\Delta)}$, we characterize the tree that minimizes the largest Laplacian eigenvalue, as well the tree that maximizes the largest Laplacian eigenvalue when $n-1 \geq \Delta \geq \lceil n/2 \rceil$. Furthermore, we prove that, for two trees T_1 and T_2 in $\mathcal{T}_n$, if $\Delta(T_1) \geq \lceil 2n/3 \rceil - 1$ and $\Delta(T_1) > \Delta(T_2)$, then $\mu_1(T_1) > \mu_1(T_2)$. Using this result, we extend the order of trees in $\mathcal{T}_n$ by their largest Laplacian eigenvalues to the 13th tree when $n \geq 15$. This extends the results of Guo and Yu *et al.*

1. Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. For $v_i \in V(G)$, let $N_G(v_i)$ (or $N(v_i)$ if the context is clear) be the set of all *neighbors* of v_i and let $d_i = d(v_i) = |N(v_i)|$ be the degree of v_i. Let $A(G)$ be the *adjacency matrix* of G and let $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees of G, then the matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of the graph G. The characteristic polynomial of $L(G)$ is $\det(xI - L(G))$, which is denoted by $P(G, x)$. The eigenvalues of $L(G)$ are called the *Laplacian eigenvalues* of G. It is well known that $L(G)$ is positive semi-definite (symmetric) and singular. Moreover, since G is connected, $L(G)$ is irreducible. Denote the eigenvalues of $L(G)$ by

$$\mu_1(G) \geq \mu_2(G) \geq \ldots \geq \mu_n(G) = 0,$$

enumerated in non-increasing order and repeated according to their multiplicity. In particular, $\mu_1(G)$ is called the *largest Laplacian eigenvalue* (or *Laplacian spectral radius*) of G. Throughout this paper, let $\mathcal{T}_n$ denote the set of trees of order n, and let $\mathcal{T}_n^{(\Delta)} = \{T \in \mathcal{T}_n : \Delta(T) = \Delta\}$, where $\Delta(T)$ is the maximum degree of a tree T.

Laplacian eigenvalues of graphs are important structural invariants that have numerous applications in quantum chemistry and theoretical chemistry. Laplacian eigenvalues of graphs have been intensively investigated (see **[1]**–**[11]**). Most of the known results can be found in **[1]**–**[4]**. In 1981, Cvetković **[4]** indicated 12 directions for further investigations of graph spectra, one of which is “classifying and ordering graphs”. Hence, ordering graphs with various properties by their spectra, especially by their largest Laplacian eigenvalues, is an attractive topic (see **[8]**, **[9]**, and **[12]**). Guo **[8]** and Yu *et al.* **[12]** considered the order of trees in $\mathcal{T}_n$ by their largest Laplacian eigenvalues. The main results of these authors can be combined into the following theorem.

Theorem 1.1 ([8][12]): For $n \geq 15$, when the trees in $\mathcal{T}_n$ are ordered by their largest Laplacian eigenvalue, the trees $S_n^1, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7$, and S_n^8 take the first eight positions. S_n^1 is the star S_n and S_n^i $(2 \leq i \leq 8)$ are as shown in Figure 1. ∎

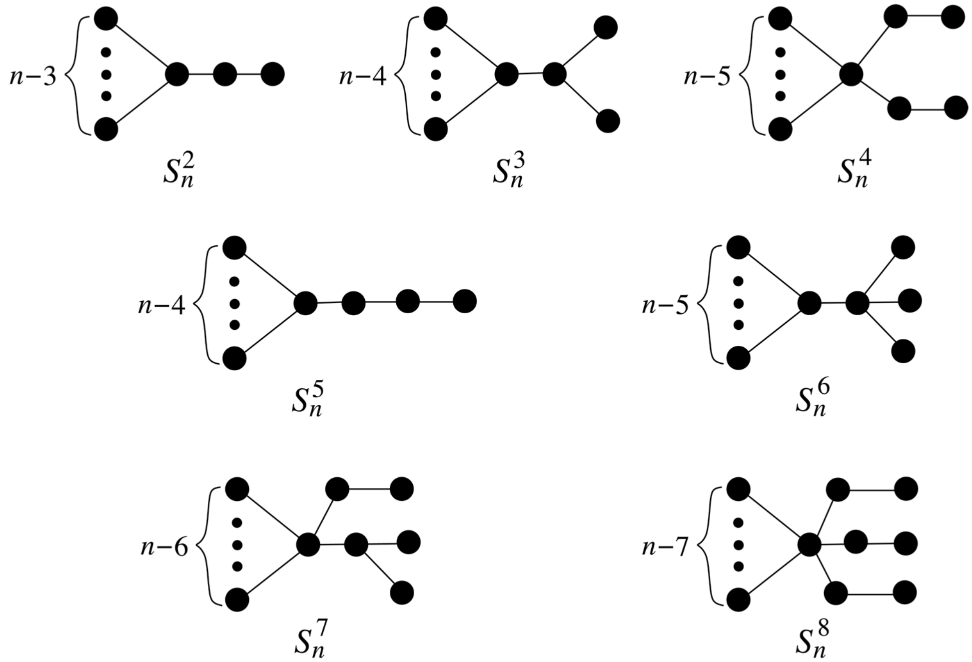

Figure 1: Trees S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, and S_n^8.

In this paper, we investigate the relation between the largest Laplacian eigenvalue and the maximum degree of a tree in $\mathcal{T}_n$, and ordering trees by their largest Laplacian eigenvalues. Let $B_{n,\,n-\Delta+1}$ $(2 \le \Delta \le n-1)$ be a *broom* (see **[1]**) and let $T^1_{i,\,j}$ $(i + j = n-2)$ be a *double star*, as shown in Figure 2. In Section 3, we show that among all trees in $\mathcal{T}_n$, $B_{n,\,n-\Delta+1}$ minimizes the largest Laplacian eigenvalue and $T^1_{\Delta-1,\,n-\Delta-1}$ maximizes the largest Laplacian eigenvalue when $n-1 \ge \Delta \ge \lceil n/2 \rceil$ (see Theorem 3.1). Furthermore, we prove that, for two trees T_1 and T_2 in $\mathcal{T}_n$, if $\Delta(T_1) \ge \lceil 2n/3 \rceil - 1$ and $\Delta(T_1) > \Delta(T_2)$, then $\mu_1(T_1) > \mu_1(T_2)$. Using this result, we extend the order of trees in $\mathcal{T}_n$ by their largest Laplacian eigenvalues to the 13th tree when $n \ge 15$ (see Theorem 4.1).

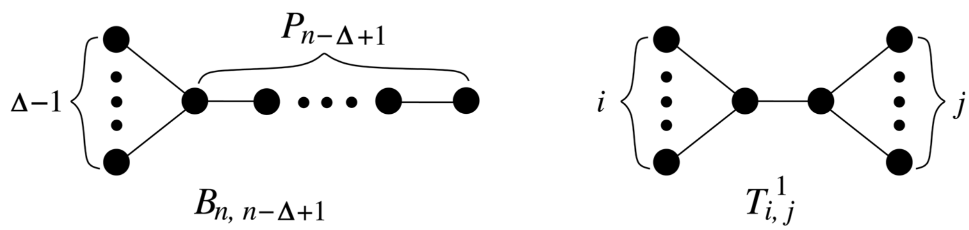

Figure 2: Trees $B_{n,\,n-\Delta+1}$ and $T^1_{i,\,j}$.

2. Preliminaries

Let $K(G) = D(G) + A(G)$ and let $\rho(K(G))$ denote the *spectral radius* of $K(G)$ (i.e., the largest eigenvalue of $K(G)$). The following description of the spectral radius of $K(G)$, or any symmetric matrix, is well known (see for example, **[5]** page 49):

(1) $$\rho(K(G)) = \sup_{\|x\| = 1} x^T K(G) x \quad (x \in \mathbb{R}^n).$$

We note here that the maximum is attained in **(1)** if and only if x is an eigenvector (for the spectral radius) of $K(G)$. Moreover, for a connected graph G, $K(G)$ is non-negative (i.e., all entries are non-negative) and irreducible, and by the Perron–Frobenius theorem for non-negative matrices, $\rho(K(G))$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho(K(G))$. We shall refer to such an eigenvector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ as the *Perron eigenvector* of $K(G)$, where the positive real number x_i corresponds to the vertex v_i ($i = 1, 2, \ldots, n$). Then, for all simple graphs, we have

(2) $$\rho(K(G))x_i = d_i x_i + \sum_{v_j \in N(v_i)} x_j,\ i = 1, 2, \ldots, n.$$

We use this notation in the following two theorems and lemma. In fact, **(2)** is an eigenvalue equation for the ith vertex (corresponding to the spectral radius of $K(G)$).

Let $e = v_r v_s$ be an edge of a graph G and assume that the vertex v_r is not adjacent to v_t. A *rotation* $\mathcal{R}(s, t)$ (around v_r) consists of the deletion of edge e followed by the addition of edge $e' = v_r v_t$.

Theorem 2.1: Let G' be a graph obtained from a connected graph G of order n by rotation $\mathcal{R}(s, t)$. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T$ be the Perron eigenvectors for $K(G)$ and $K(G')$, respectively. Then $\boldsymbol{x} \neq \boldsymbol{y}$.

Proof: If $\boldsymbol{x} = \boldsymbol{y}$, then

$$\rho(K(G))x_s = d_s x_s + \sum_{v_j \in N_G(v_s)} x_j, \text{ and}$$

$$\rho(K(G'))x_s = (d_s - 1)x_s + \sum_{v_j \in N_{G'}(v_s)} x_j.$$

Then, $\rho(K(G))x_s - \rho(K(G'))x_s = x_s + x_r > 0$. Hence, $\rho(K(G)) > \rho(K(G'))$. Interchanging the roles of v_s and v_t we obtain that $\rho(K(G)) < \rho(K(G'))$. This is a contradiction. ■

Theorem 2.2: Let G' be a graph obtained from a connected graph G of order n by the rotation $\mathcal{R}(s, t)$ (around v_r). Let x and y be defined as in Theorem 2.1. If $x_t \geq x_s$, then $\rho(K(G')) > \rho(K(G))$.

Proof: Recall that $\|x\| = 1$. By Theorem 2.1 and the uniqueness of Perron eigenvector,

$$\begin{aligned}\rho(K(G')) - \rho(K(G)) &= \sup_{\|x\| = 1} z^T K(G')z - x^T K(G)x \\ &> x^T K(G')x - x^T K(G)x \\ &= x^T (K(G') - K(G))x \\ &= (2x_r + x_t + x_s)(x_t - x_s) \geq 0.\end{aligned}$$

Hence, $\rho(K(G')) > \rho(K(G))$. ■

Lemma 2.3: Let G be a connected graph with a vertex v_r adjacent to v_s but not adjacent to v_t. Let G', obtained from G by the rotation $\mathcal{R}(s, t)$ (around v_r), also be a connected graph. Let $\boldsymbol{x}$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ be the Perron eigenvectors of $K(G)$ and $K(G')$, respectively. Then $x_t \geq x_s$ implies $y_t > y_s$.

Proof: Since $x_t \geq x_s$, by Theorem 2.2, then $\rho(K(G)) > \rho(K(G'))$. Interchanging the roles of G and G', by the contrapositive of Theorem 2.2, we obtain $y_t > y_s$. ■

From Theorem 2.2 and Lemma 2.3 it is easy to obtain the following corollary.

Corollary 2.4: Let u and v be two vertices in a non-trivial connected graph G and suppose that s edges $uu_1, uu_2, \ldots, uu_s$ are attached to G at u and t edges $vv_1, vv_2, \ldots, vv_t$ are attached to G at v to form $G_{s,t}$. Then either $\rho(K(G_{s+i,t-i})) > \rho(K(G_{s,t}))$ ($1 \leq i \leq t$), or $\rho(K(G_{s-i,t+i})) > \rho(K(G_{s,t}))$ ($1 \leq i \leq s$). ■

Grone *et al.* **[7]** showed that if G is a bipartite graph then $K(G)$ and $L(G)$ are unitarily similar; that is, there is a invertible matrix U such that $U^* K(G) U = L(G)$, where U^* is the conjugate transpose of U and $U^* U$ is the identity matrix. This implies the following lemma.

Lemma 2.5: If G is a bipartite graph, then $K(G)$ and $L(G)$ have the same spectrum. ■

Pan **[11]** proved that $\mu_1(G) \leq \rho(K(G))$. Moreover, the equality holds if and only if G is a bipartite graph. When G is a tree, it follows from Lemma 2.5 that investigation of the Laplacian spectrum of G may be reduced to investigation of the spectrum of $K(G)$.

Let T be a tree in $\mathcal{T}_n$ with $n \geq 4$. Let $e = uv$ be a non-pendant edge of T, and let X and Y be two components of $T - e$ such that $u \in X$ and $v \in Y$. Define T_0 to be the graph obtained from T in the following way:

Contract the edge $e = uv$ and add a pendant edge to the vertex $u(v)$, where $u(v)$ is the new vertex created by identifying u and v. This procedure is called the *edge-growing transformation* of T (on edge e) (see Figure 3).

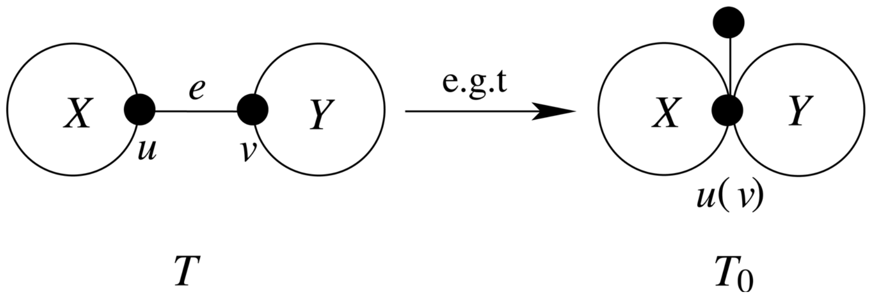

Figure 3: Transformation of T to T_0.

Lemma 2.6: Let T be a tree in $\mathcal{T}_n$ with at least one non-pendant edge $e = uv$. If T' is obtained from T by an edge-growing transformation, then $\mu_1(T) < \mu_1(T')$.

Proof: Assume x_u and x_v to be the components of the Perron vector of $K(T)$ corresponding to u and v, respectively. Let $N(u) = \{v, v_1, v_2, \ldots, v_s\}$ and $N(v) = \{u, u_1, u_2, \ldots, u_t\}$. Since $e = uv$ is a non-pendant edge, then $s, t \geq 1$.

If $x_u \geq x_v$, let $T_1 = T - \{vu_1, vu_2, \ldots, vu_t\} + \{uu_1, uu_2, \ldots, uu_t\}$.

If $x_u < x_v$, let $T_2 = T - \{uv_1, uv_2, \ldots, uv_s\} + \{vv_1, vv_2, \ldots, vv_s\}$.

Obviously, for each of the above cases, $T' = T_1 = T_2$. By Theorem 2.2 and Lemma 2.3, it follows that $\rho(K(T)) < \rho(K(T'))$. Since T is a tree, by Lemma 2.5, $\mu_1(T) < \mu_1(T')$. ■

For each $T \in \mathcal{T}_n^{(\Delta)}$ let $u \in V(T)$ with $d(u) = \Delta$. If $E(T)\backslash\{uv : v \in N(u)\}$ contains at least one non-pendant edge, then T can be transformed to T^* (as shown in Figure 4) by carrying out edge-growing transformations repeatedly on each non-pendent edge, such that $\mu_1(T) < \mu_1(T^*)$; otherwise, let $T^* = T$. Then, if $T^* \neq T^1_{\Delta-1, n-\Delta-1}$, by Corollary 2.4, T^* can be further transformed to $T^1_{\Delta-1, n-\Delta-1}$ (as shown in Figure 4) such that $\mu_1(T^*) < \mu_1(T^1_{\Delta-1, n-\Delta-1})$.

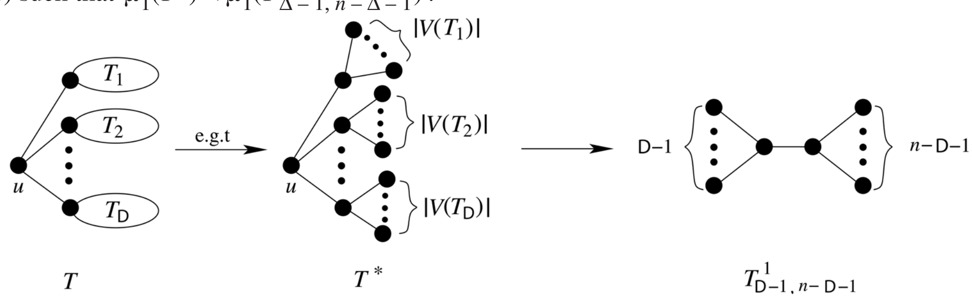

Figure 4: Transformation of T to T^* and T^* to $T^1_{\Delta-1, n-\Delta-1}$.

Remark 1:

1. If $\left\lceil \frac{n}{2} \right\rceil \leq \Delta \leq n-1$, then $\Delta - 1 \geq n - \Delta - 1$ and $T^1_{\Delta-1, n-\Delta-1} \in \mathcal{T}_n^{(\Delta)}$.

2. If $\left\lceil \frac{n}{3} \right\rceil + 1 \leq \Delta \leq \left\lceil \frac{2n}{3} \right\rceil - 1$, then neither $\Delta - 1$ nor $n - \Delta - 1$ are less than $\left\lfloor \frac{n}{3} \right\rfloor$. In this case,

   $T^1_{\Delta-1, n-\Delta-1}$ can be further transformed to $T^1_{\lfloor n/3 \rfloor, \lceil 2n/3 \rceil - 2} \cong T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor}$

   by Corollary 2.4 such that $\mu_1(T) < \mu_1(T^*) < \mu_1(T^1_{\Delta-1, n-\Delta-1}) \leq \mu_1(T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor})$.

Lemma 2.7 ([13]): Let u be a vertex of a tree T in $\mathcal{T}_n$ and let $T^0_{k,l}$ denote the tree obtained from T by adding pendent paths of length k and l at u. If $k \ge l \ge 1$, then $\mu_1(T^0_{k,l}) > \mu_1(T^0_{k+1,l-1})$. ■

We call the transformation in Lemma 2.7, from $T^0_{k,l}$ to $T^0_{k+1,l-1}$ ($k \ge l \ge 1$), an α *transformation* of $T^0_{k,l}$.

Remark 2: For each $T \in \mathcal{T}^{(\Delta)}_n$, if $T \ne B_{n,n-\Delta+1}$, T can be transformed to T^{**}, and then to $B_{n,n-\Delta+1}$ by performing a series of α transformations. Hence, $\mu_1(T) > \mu_1(B_{n,n-\Delta+1})$ (see Figure 5).

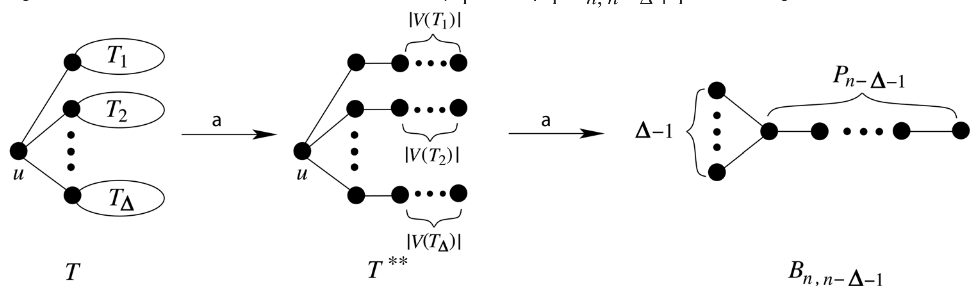

Figure 5: Transformation of T to T^{**} and T^{**} to $B_{n,n-\Delta+1}$.

Lemma 2.8 ([6]): Let G have at least one edge. Then $\mu_1(G) \ge \Delta(G) + 1$. For a connected graph G on $n > 1$ vertices, the equality holds if and only if $\Delta(G) = n - 1$. ■

Lemma 2.9 ([14]): For a graph G, $\mu_1(G) \le \max\{d(u) + d(v) : uv \in E(G)\}$. ■

3. Main Results

Applying α transformations and edge-growing transformations, it is easy to determine the trees in $\mathcal{T}^{(\Delta)}_n$ that have extreme largest Laplacian eigenvalues.

Remark 1(1) and Remark 2 suggest the following theorem.

Theorem 3.1: Let T be a tree in $\mathcal{T}^{(\Delta)}_n$ with order $n \ge 4$. Then $\mu_1(T) \ge \mu_1(B_{n,n-\Delta-1})$. Moreover, equality holds if and only if $T \cong B_{n,n-\Delta-1}$. If $\lceil n/2 \rceil \le \Delta \le n-1$, then $T^1_{\Delta-1,n-\Delta-1} \in \mathcal{T}^{(\Delta)}_n$ and $\mu_1(T) \le \mu_1(T^1_{\Delta-1,n-\Delta-1})$, where equality holds if and only if $T \cong T^1_{\Delta-1,n-\Delta-1}$.

Lemma 3.2: Let $T^{(\Delta)}$ be a tree in $\mathcal{T}^{(\Delta)}_n$, where $\lceil n/2 \rceil \le \Delta \le n-1$ and $n \ge 4$. Then $\mu_1(T^{(n-1)}) > \mu_1(T^{(n-2)}) > \ldots > \mu_1(T^{(\lceil n/2 \rceil)})$.

Proof: Since $T^{(n-1)} \cong S^1_n$ and obviously $\mu_1(T^{(n-1)}) > \mu_1(T^{(n-2)})$, then by Theorem 3.1, it suffices to show that $\mu_1(B_{n,n-\Delta}) > \mu_1(T^1_{\Delta-1,n-\Delta-1})$ for $\lceil n/2 \rceil \le \Delta \le n-3$.

It is clear that the characteristic polynomial of $L(T^1_{\Delta-1,n-\Delta-1})$ is

$$P(T^1_{\Delta-1,n-\Delta-1}, x) = x(x-1)^{n-4}\{x^3 - (n+2)x^2 + [2n + (\Delta-1)(n-\Delta-1)+1]x - n\}.$$

Hence, $\mu_1(T^1_{\Delta-1,n-\Delta-1})$ is the largest root of the equation

$$f(x) = x^3 - (n+2)x^2 + [2n + (\Delta-1)(n-\Delta-1)+1]x - n = 0.$$

By differentiating $f(x)$ we see that $f(x)$ is a strictly increasing function in the interval

$$\left(\frac{2+n+\sqrt{n^2-(3\Delta-1)n+3\Delta^2-2}}{3}, \infty\right).$$

Moreover, by taking $x = \Delta + 2$

$$f(\Delta+2) = 2\Delta^2 - (n-6)\Delta - (3n-4).$$

Hence, $f(\Delta+2) > 0$ whenever $\Delta > \lceil n/2 \rceil > \frac{n-6+\sqrt{n^2+12n+4}}{4}$.

We can also check that when $\Delta > \lceil n/2 \rceil$,

$$\Delta + 2 > \frac{2 + n + \sqrt{n^2 - (3\Delta - 1)n + 3\Delta^2 - 2}}{3}.$$

Therefore, $\mu_1(T^1_{\Delta - 1, n - \Delta - 1}) < \Delta + 2$ when $\Delta > \lceil n/2 \rceil$. By Lemma 2.8, $\Delta + 2 < \mu_1(B_{n, n - \Delta})$. Hence the lemma. ■

Lemma 3.3: Let T be a tree in $\mathcal{T}_n^{(\Delta)}$, where $2 \le \Delta \le \lceil 2n/3 \rceil - 1$. Then $\mu_1(T) \le \mu_1(T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor})$. Moreover, equality holds if and only if $T \cong T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor}$.

Proof: We distinguish the following two cases.

<u>Case 1</u>: $\lfloor n/3 \rfloor + 1 \le \Delta \le \lceil 2n/3 \rceil - 1$.

Then from Remark 1(2), T can be transformed to $T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor}$ with $\mu_1(T) \le \mu_1(T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor})$ and $\mu_1(T) = \mu_1(T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor})$ if and only if $T \cong T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor}$.

<u>Case 2</u>: $2 \le \Delta \le \lfloor n/3 \rfloor$.

Then by Lemma 2.9, $\mu_1(T) \le 2\Delta \le \lceil 2n/3 \rceil$. By Lemma 2.8,

$$\mu_1(T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor}) \ge \left\lceil \frac{2n}{3} \right\rceil - 1 + 1 = \left\lceil \frac{2n}{3} \right\rceil.$$

Hence,

$$\mu_1(T) \le \left\lceil \frac{2n}{3} \right\rceil \le \mu_1(T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor}).$$

The proof is completed. ■

By combining Lemmas 3.2 and 3.3, we obtain the following main result.

Theorem 3.4: Let $T^{(\Delta)}$ be a tree in $\mathcal{T}_n^{(\Delta)}$ with $2 \le \Delta \le n - 1$ and $n \ge 4$. Then $\mu_1(T^{(n-1)}) > \mu_1(T^{(n-2)}) > \ldots > \mu_1(T^{(\lceil 2n/3 \rceil)}) > \mu_1(T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor}) \ge \mu_1(T^{(k)})$, where $2 \le k \le \lceil 2n/3 \rceil - 1$. Moreover, the final equality holds if and only if $T^{(k)} \cong T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor}$. ■

Lemma 3.2 indicates that, for $T_1, T_2 \in \mathcal{T}_n$, if $\Delta(T_2) \ge \lceil 2n/3 \rceil - 1$, then $\mu_1(T_1) > \mu_1(T_2)$. If all the trees of order n are arranged in descending order of their largest eigenvalues, by Theorem 3.4, the trees in $\mathcal{T}_n^{(n-1)}$, $\mathcal{T}_n^{(n-2)}$, …, $\mathcal{T}_n^{(\lceil 2n/3 \rceil)}$, and $T^1_{\lceil 2n/3 \rceil - 2, \lfloor n/3 \rfloor}$ take the first several positions in that order. In particular, if $n \ge 15$, trees in $\mathcal{T}_n^{(n-1)}$, $\mathcal{T}_n^{(n-2)}$, $\mathcal{T}_n^{(n-3)}$, $\mathcal{T}_n^{(n-4)}$, and $T^1_{n-6,4}$ take the first several positions in the order. It is easy to see that $\mathcal{T}_n^{(n-1)} = \{S_n^1\}$, $\mathcal{T}_n^{(n-2)} = \{S_n^2\}$, $\mathcal{T}_n^{(n-3)} = \{S_n^3, S_n^4, S_n^5\}$, and $\mathcal{T}_n^{(n-4)} = \{S_n^6, S_n^7, S_n^8, S_n^9, S_n^{10}, S_n^{11}, S_n^{12}\}$, where $S_n^9, S_n^{10}, S_n^{11}$, and S_n^{12} are as shown in Figure 6. Thus, $S_n^9, S_n^{10}, S_n^{11}, S_n^{12}$, and $T^1_{n-6,4}$ must take the next five positions.

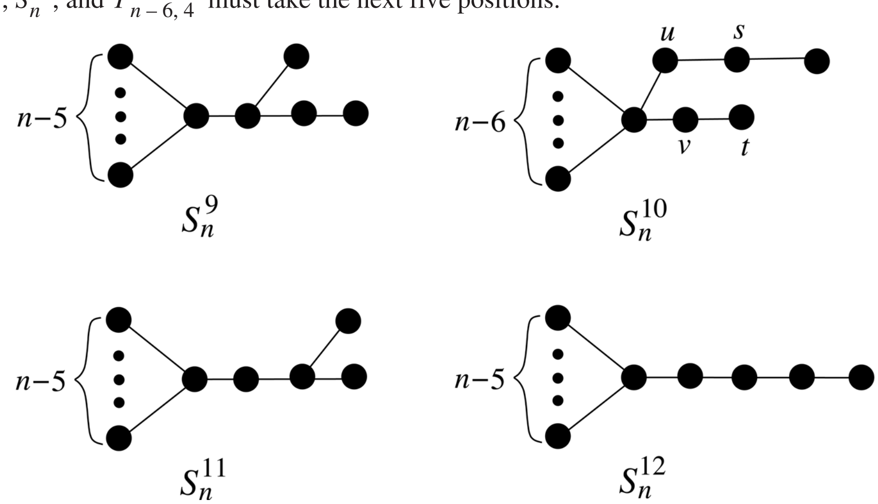

Figure 6: Trees $S_n^9, S_n^{10}, S_n^{11}$, and S_n^{12}.

Since the roots of the characteristic polynomial of $L(G)$ are real, we only consider polynomials with real roots in this paper. If $f(x)$ is a polynomial in x, we denote the degree of $f(x)$ by $\partial(f)$ and the largest root of the equation $f(x) = 0$ by $\mu_1(f)$. Many of the discussions in the rest of this paper involve comparing the largest root of a polynomial with that of another polynomial. The next result provides an effective method for this.

Lemma 3.5 ([2]): Let $f(x)$ and $g(x)$ be two monic polynomials with real roots, with $\partial(f) \geq \partial(g)$. If $f(x) = q(x)g(x) + r(x)$, where $q(x)$ is a monic polynomial, $\partial(r) < \partial(g)$ and $\mu_1(g) > \mu_1(q)$, then
1. when $r(x) = 0$, then $\mu_1(f) = \mu_1(g)$;
2. when $r(x) > 0$ for any x satisfying $x \geq \mu_1(g)$, then $\mu_1(f) < \mu_1(g)$;
3. when $r(\mu_1(g)) < 0$, then $\mu_1(f) > \mu_1(g)$. ■

Lemma 3.6: When $n \geq 15$, then $\mu_1(S_n^9) > \mu_1(S_n^{10}) > \mu_1(S_n^{11}) > \mu_1(S_n^{12})$.

Proof: Assume x_u and x_v to be the components of the Perron vector of $K(S_n^{10})$ corresponding to u and v, respectively, (see Figure 6).

If $x_u \geq x_v$, let $S_n^9 = S_n^{10} - \{vt\} + \{ut\}$. If $x_u < x_v$, let $S_n^9 = S_n^{10} - \{us\} + \{vs\}$. Then by Theorem 2.1 and Lemma 2.5, $\mu_1(S_n^9) > \mu_1(S_n^{10})$.

S_n^{12} can be transformed to S_n^{11} by an edge-growing transformation. Thus, $\mu_1(S_n^{11}) > \mu_1(S_n^{12})$ by Lemma 2.6. It remains to show that $\mu_1(S_n^{10}) > \mu_1(S_n^{11})$. By an elementary calculation, the characteristic polynomials of $L(S_n^{10})$ and $L(S_n^{11})$ are

$$P(S_n^{10}, x) = x(x-1)^{n-7}g(x) \text{ and}$$
$$P(S_n^{11}, x) = x(x-1)^{n-5}h(x),$$

where

$$g(x) = x^6 - (n+5)x^5 + (8n-2)x^4 - (22n-40)x^3 + (24n-53)x^2 - (9n-13)x + n \text{ and}$$
$$h(x) = x^4 - (n+3)x^3 + (6n-10)x^2 - (8n-22)x + n.$$

Then $\mu_1(S_n^{10})$ and $\mu_1(S_n^{11})$ are the largest roots of the equations $g(x) = 0$ and $h(x) = 0$, respectively. Comparing $g(x)$ and $h(x)$, then

$$g(x) = (x^2 - 2x + 2)h(x) + [4x^3 - (5n-11)x^2 + (9n-31)x - n].$$

Let $r(x) = 4x^3 - (5n-11)x^2 + (9n-31)x - n$. By differentiating $r(x)$, we see that $r(x)$ is a strictly increasing function in the interval

$$\left(\frac{5n - 11 + \sqrt{25n^2 - 218n + 493}}{12}, \infty\right).$$

By Lemmas 2.8 and 2.9,

$$n - 3 = n - 4 + 1 < \mu_1(S_n^i), \text{ for } i = 10, 11.$$

It is easily verified that

$$\frac{5n - 11 + \sqrt{25n^2 - 218n + 493}}{12} < n - 3 \text{ when } n \geq 15.$$

Hence, $r(\mu_1(S_n^{11})) < r(n-2) = -(n^2 - 14n + 37)(n-2) - n < 0$, when $n \geq 15$.

By Lemma 3.5(3), $\mu_1(S_n^{10}) > \mu_1(S_n^{11})$. ■

4. Conclusion

Denote $T^1_{n-6,4}$ by S^{13}_n. By Theorems 3.1, 3.4, and Lemma 3.6, the following result is obvious.

Theorem 4.1: When $n \geq 15$ in the order of trees in $\mathcal{T}_n$ by their largest Laplacian eigenvalues, the trees $S^9_n, S^{10}_n, S^{11}_n, S^{12}_n$, and S^{13}_n take the 9th to 13th positions, respectively. ■

By calculating the largest Laplacian eigenvalues of $S^9_n, S^{10}_n, S^{11}_n, S^{12}_n$, and S^{13}_n, the 9th to 13th largest values of the largest Laplacian eigenvalues of trees in $\mathcal{T}_n$ can be obtained immediately. Furthermore, we can easily further extend this order when $n \geq 15$, since by Theorem 3.4, we only need to order a few trees (for example, the trees in $\mathcal{T}^{(n-5)}_n$).

From the table for the spectra of all trees with n vertices ($2 \leq n \leq 15$) in **[3]**, we can easily determine the order of trees in $\mathcal{T}_n$ by their largest Laplacian eigenvalues. When $n \geq 15$, combining Theorem 4.1 with the results in **[1][2]**, we show the distribution of the first thirteen trees in the order of trees in $\mathcal{T}_n$ by their largest Laplacian eigenvalues in terms of the partition

$$\mathcal{T}_n = \bigcup_{\Delta=2}^{n-1} \mathcal{T}^{(\Delta)}_n$$

in the following table.

$\mathcal{T}^{(n-1)}_n$	S^1_n	—	—	—	—	—	—	—
$\mathcal{T}^{(n-2)}_n$	S^2_n	—	—	—	—	—	—	—
$\mathcal{T}^{(n-3)}_n$	S^3_n	S^4_n	S^5_n	—	—	—	—	—
$\mathcal{T}^{(n-4)}_n$	S^6_n	S^7_n	S^8_n	S^9_n	S^{10}_n	S^{11}_n	S^{12}_n	—
$\mathcal{T}^{(n-5)}_n$	S^{13}_n	…	…	…	…	…	…	…
$\mathcal{T}^{(n-6)}_n$	…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…	…

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