On Some Three-color Ramsey Numbers*

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Abstract

In this paper we study three-color Ramsey numbers. Let $K_{i,j}$ denote a complete i by j bipartite graph. We shall show that (i) for any connected graphs G_1, G_2 and G_3, if $r(G_1, G_2) \geq s(G_3)$, then $r(G_1, G_2, G_3) \geq (r(G_1, G_2) - 1)(\chi(G_3) - 1) + s(G_3)$, where $s(G_3)$ is the chromatic surplus of G_3; (ii)$(k + m - 2)(n - 1) + 1 \leq r(K_{1,k}, K_{1,m}, K_n) \leq (k + m - 1)(n - 1) + 1$, and if k or m is odd, the second inequality becomes an equality; (iii) for any fixed $m \geq k \geq 2$, there is a constant c such that $r(K_{k,m}, K_{k,m}, K_n) \leq c(n/\log n)^k$, and $r(C_{2m}, C_{2m}, K_n) \leq c(n/\log n)^{m/(m-1)}$ for sufficiently large n.

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1 Introduction

In this paper, all graphs are simple. Undefined symbols and concepts may be looked up from Bondy and Murty [2]. Suppose $S \subseteq E$ is a subset of edges of the graph $G = (V, E)$. Then $G'[S]$ denotes the graph (V, S). Let $G_1, G_2, \ldots, G_h$ be graphs. The h-color Ramsey number $r(G_1, G_2, \ldots, G_h)$ is the smallest integer N such that if we color the edges of $G = K_N$ by the color-set $\{c_1, \cdots, c_h\}$, then there exists some i, $1 \leq i \leq h$, such that the graph $G_i \subseteq G'[E_i]$, where E_i is the set of edges colored in c_i. We write $r(i, j, \ldots, h)$ for $r(K_i, K_j, \ldots, K_h)$. It is clear that in this notation, G_i and G_j are inter-changeable. In this paper, we shall only study 3-color Ramsey numbers. Throughout this paper, we shall use E_r, E_b and E_y to denote the set of edges colored in red, blue and yellow respectively.

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In Section 2 we give a general lower bound for 3-color Ramsey numbers. In Section 3, we give an upper bound and a lower bound for $r(K_{1,k}, K_{1,m}, K_n)$ and that for $r(T_{1+k}, T_{1+m}, K_n)$. In Section 4, we obtain two asymptotic bounds on $r(K_{k,m}, K_{k,m}, K_n)$ and $r(C_{2m}, C_{2m}, K_n)$, where C_{2m} is a cycle of length $2m$.

2 A general lower bound

The following definition is due to Burr [3].

Definition: Let G be a connected graph. The *chromatic surplus* of G denoted by $s(G)$ is the minimum number of vertices in any vertex-color class, taken over all proper $\chi(G)$-coloring of the vertices of G.

It follows that $s(K_N) = s(C_{2m+1}) = 1$ and $s(C_{2m}) = m$. In [3], Burr also proved that if G_1 and G_2 are connected graphs and $|V(G_1)| \geq s(G_2)$, then,

$$r(G_1, G_2) \geq (|V(G_1)| - 1)(\chi(G_2) - 1) + s(G_2) \tag{1}$$

We shall now generalize this result as follows.

Theorem 2.1 *If G_1, G_2 and G_3 are connected graphs and $r(G_1, G_2) \geq s(G_3)$, then*

$$r(G_1, G_2, G_3) \geq (r(G_1, G_2) - 1)(\chi(G_3) - 1) + s(G_3).$$

Proof: Put $N_1 = r(G_1, G_2) - 1$ and $N = N_1(k-1) + s(G_3) - 1$, where $k = \chi(G_3)$. We shall color the edges of $G = K_N$ in red, blue and yellow. The theorem follows if we can color G in such a way that G_1, G_2 and G_3 are not contained in $G'[E_r]$, $G'[E_b]$ and $G'[E_y]$ respectively.

We first partition $V(G)$ into mutually disjoint subsets $V_1, V_2, \cdots, V_k$ with $|V_i| = N_1$ for $i = 1, \cdots, k-1$ and $|V_k| = s(G_3) - 1$. Since $|V_i| \leq r(G_1, G_2) - 1$ for all i, we may color each $G[V_i]$ in red and blue so that G_1 and G_2 are not contained in $G[V_i] \cap G'[E_r]$ and $G[V_i] \cap G'[E_b]$ respectively. For all edges of G not colored in red or blue as above, we color them yellow. Clearly G_1 and G_2 are not subgraphs of $G'[E_r]$ and $G'[E_b]$ respectively. Moreover, $G'[E_y]$ contains k independent sets $V_1, V_2, \cdots, V_k$ with $|V_k| < s(G_3)$. It follows that if H is a subgraph of $G'[E_y]$, then $\chi(H) \leq k$ and $s(H) < s(G_3)$. Therefore H cannot be isomorphic to G_3. ∎

The following result can be proved similarly by putting $N = N_1(k-1) + r(G_1, G_2) - 1$.

Theorem 2.2 *If G_1, G_2 and G_3 are connected graphs and $r(G_1, G_2) \le s(G_3)$, then*

$$r(G_1, G_2, G_3) \ge (r(G_1, G_2) - 1)(\chi(G_3) - 1) + r(G_1, G_2).$$

Theorem 2.1 is truly a generalization of Burr's lower bound because (1) follows from it and the fact that $r(G_1, G_2) = r(G_1, K_2, G_2)$.

The simple result in Theorem 2.1 generally does not give good lower bounds. However, it may give reasonable lower bounds for some small Ramsey numbers which are usually obtained by use of computers. We list some of them in the following table in which the second row contains the values of $r(G_1, G_2, G_3)$ together with their references. For values of the relevant 2-color Ramsey numbers $r(G_1, G_2)$, we refer to the survey by Radziszowski [20]. In the table, $N_1 = r(G_1, G_2) - 1$, $k = \chi(G_3)$, $s = s(G_3)$, and $B = K_4 - e$ is the graph obtained from K_4 by deleting an edge.

G_1, G_2, G_3	C_5, C_5, C_5	C_6, C_6, C_6	C_7, C_7, C_7	C_4, C_4, K_3	$K_{1,3}, C_4, K_4$	$B, B, K_{1,2}$
r/ Ref.	17/[26]	12/[27]	25/[10]	12/[22]	16/[15]	11/[9]
$N_1(k-1)+s$	17	10	25	11	16	10

Table 1: Some small $r(G_1, G_2, G_3)$ and $(r(G_1, G_2) - 1)(\chi(G_3) - 1) + s(G_3)$.

Note that for odd integer $k \ge 5$, $r(C_k, C_k) = 2k - 1$ hence our result yields a lower bound $r(C_k, C_k, C_k) \ge 4k - 3$, and Bondy and Erdős [5] conjectured that the equality holds.

3 Tree-tree-complete graph

In [4], Chvátal determined the 2-color Tree-Complete Ramsey number.

$$r(T_{1+k}, K_n) = k(n-1) + 1, \tag{2}$$

where T_{1+k} is a tree on $1 + k$ vertices. In this section, we shall study the 3-color Tree-Tree-Complete Ramsey number. We shall need Turán's inequality: If G is a graph of order N, and its average degree and independence number are d and $\alpha(G)$ respectively, then

$$\alpha(G) \ge \frac{N}{1+d}.$$

We need the following lemma for the main result of this section.

Lemma 3.1 *For any stars $K_{1,k}$ and $K_{1,m}$, and for any complete graph K_n, we have*

$$r(K_{1,k}, K_{1,m}, K_n) \leq (k+m-1)(n-1)+1.$$

Proof: Suppose we color the edges of $G = K_N$, where $N = (k+m-1)(n-1)+1$, in red, blue and yellow. Let $G^* = G'[E_b \cup E_r]$. If G^* contains neither a red $K_{1,k}$ nor a blue $K_{1,m}$, then the degree of each vertex of G^* is at most $k+m-2$. By Turán's inequality, we have

$$\alpha(G^*) \geq \left\lceil \frac{N}{k+m-1} \right\rceil = \left\lceil n-1+\frac{1}{k+m-1} \right\rceil = n.$$

Since an independent set of G^* induces a complete subgraph in $G'[E_y]$, the theorem follows. ▮

Harary [12] proved

$$r(K_{1,k}, K_{1,m}) = \begin{cases} k+m & \text{if } k \text{ or } m \text{ is odd,} \\ k+m-1 & \text{otherwise.} \end{cases}$$

Combining this result with Theorem 2.1 and Lemma 3.1, we have

Theorem 3.1 *For any positive integers k, m, and n,*

$$(k+m-2)(n-1)+1 \leq r(K_{1,k}, K_{1,m}, K_n) \leq (k+m-1)(n-1)+1.$$

If k or m is odd, the second inequality becomes an equality.

Note that if k or m is odd, then by Theorem 2.1, the equality in Theorem 3.1 becomes

$$r(K_{1,k}, K_{1,m}, K_n) = (r(K_{1,k}, K_{1,m}) - 1)(n-1)+1. \tag{3}$$

By using Theorem 3.2 described below, it is easy to see that (3) also holds when $k = m = 2$.

It would be interesting to find other pairs of trees to replace the stars in (3). However, we have a weaker result. We need the following lemma which is due to Chvátal.

Lemma 3.2 *Let G be a graph with $\delta(G) > k-1$. Then G contains every tree of order $k+1$.*

The following result is well-known:

Lemma 3.3 *Let G be a graph with average degree d. Then there exists a subgraph of G with minimum degree at least $d/2$.*

Proof: If the minimum degree of G is at least $d/2$, we are done. Otherwise, choosing a vertex $v \in V(G)$ with $\deg(v) < d/2$, the (induced) subgraph $G - v$ has average degree at least d. Continuing this procedure if necessary, we will eventually get the desired subgraph. ∎

Theorem 3.2 *For any positive integers k, m, n, we have*

$$(r(T_{1+k}, T_{1+m}) - 1)(n-1) + 1 \leq r(T_{1+k}, T_{1+m}, K_n) \leq 2(k+m-1)(n-1) + 1. \tag{4}$$

Proof: The lower bound follows from Theorem 2.1. We may assume that $k,\ m \geq 2$, otherwise (4) is trivial. Color the edges of K_N, where $N = 2(k+m-1)(n-1)+1$, in red, blue, and yellow. Let $G^* = G'[E_r \cup E_b]$. If K_n is not contained in $G'[E_y]$, then by Turán's inequality we have

$$n - 1 \geq \alpha(G^*) \geq \frac{N}{1 + d(G^*)} > \frac{2(k+m-1)(n-1)}{1+d(G^*)},$$

where $d(G^*)$ is the average degree of G^*. Therefore $d(G^*) \geq 2(k+m-1)$, and consequently either $d(G'[E_r]) \geq 2k-1$ or $d(G'[E_b]) \geq 2m-1$. Without loss of generality, we may assume that $d(G'[E_r]) \leq 2k-1$.

By Lemma 3.3, $G_r = G'[E_r]$ has a subgraph H with $\delta(H) \geq k - \frac{1}{2} > k-1$. The theorem follows from Lemma 3.2. ∎

Corollary 3.1 *For any positive integers k, m, n, we have*

$$(r(T_{1+k}, K_{1,m}) - 1)(n-1) + 1 \leq r(T_{1+k}, K_{1,m}, K_n) \leq (2k+m-1)(n-1) + 1. \tag{5}$$

Proof: Following the proof of Theorem 3.2, we let $N = (2k+m-1)(n-1)+1$. Then we have $d(G^*) \geq 2k+m-1$ and consequently either $d(G'[E_r]) \geq 2k-1$ or $d(G'[E_b]) \geq m$. Therefore, either $G'[E_r]$ contains a tree on $1+k$ vertices or $G'[E_b]$ contains a star $K_{1,m}$. ∎

Erdős and Sós conjectured that a graph with average degree more than $k-1$ contains every tree on $k+1$ vertices. If it is true then the same argument of the proof of Theorem 3.2 will yield the bound

$$r(T_{1+k}, T_{1+m}, K_n) \leq (k+m-1)(n-1) + 1.$$

4 Complete bipartite-bipartite-large complete graph

In this section, we study the case in which G_1 and G_2 are fixed and G is large. We shall first show that

$$r(G_1, G_2, T_n) \leq (r(G_1, G_2) - 1)(n-1) + 1,$$

where T_n is a tree on n vertices. Suppose the edges of $G = K_N$, where $N = (k-1)(n-1)+1$ and $k = r(G_1, G_2)$, are colored in red, blue and yellow. By Chvátal's result (2), if T_n is not contained in $G'[E_y]$, then K_k is contained in $G'[E_r \cup E_b]$. Since $k = r(G_1, G_2)$, either G_1 is contained in $G'[E_r]$ or G_2 is contained in $G'[E_b]$.

Spencer [24] proved that for fixed $m \geq 3$,

$$r(C_m, K_n) \geq c \left(\frac{n}{\log n} \right)^{(m-1)/(m-2)}.$$

Combining this with Chvátal's result (2), we know that for a fixed connected graph G, $r(G, K_n)$ grows linearly in n if and only if G is a tree. From the upper bound for 3-color Ramsey number given by Theorem 3.2, for fixed connected graphs G_1 and G_2, $r(G_1, G_2, K_n)$ grows linearly in n if and only if both G_1 and G_2 are trees. Let us consider $r(G_1, G_2, K_n)$ when n is large and neither G_1 nor G_2 is a tree.

We first list some known facts on small Ramsey numbers of the form $r(3, n)$ and $r(3, 3, n)$ in the following table, in which the third row contains the corresponding references. They seem to be quite different. The order of magnitude for $r(3, n)$ is exactly known by Kim's result [14]. Erdős and Sós [7, 23] conjectured $r(3, 3, n)/r(3, n) \to \infty$ and $r(3, n+1) - r(3, n) \to \infty$.

n	3	4	5	6	7	8	9
$r(3,n)$	6	9	14	18	23	28	36
$r(3,3,n)$	17	30 - 31	45 - 57	≥ 60	≥ 72	-	≥ 110
Ref.	[11]	[13, 19]	[8, 16]	[21]	[25]	-	[25]

Table 2 Some small $r(3, n)$ and $r(3, 3, n)$.

A simple use of Turan's inequality and a recursive argument for Ramsey numbers yields

$$r(3, 3, n) \leq (n-1)(2r(3, n) - 1) + 1 \leq 2nr(3, n).$$

This upper bound may be far from the truth. We shall do better for $r(K_{k,m}, K_{k,m}, K_n)$. In the following asymptotic upper bounds, although they are stated for all K_n, but we are only interested in the case in which n is sufficiently large.

Theorem 4.1 *For any fixed integers $m \geq k \geq 2$, there exists constant $c = c(k,m) > 0$ such that for all n,*

$$r(K_{k,m}, K_{k,m}, K_n) \leq c\left(\frac{n}{\log n}\right)^k.$$

We need the following result from [18]. A slight weaker form of it is in [17].

Lemma 4.1 *Let G be a graph with N vertices and average degree d. If for any vertex v of G, the average degree of subgraph induced by $N(v)$ is at most a, then $\alpha(G) \geq N f_{a+1}(d)$, where for $b \geq 1$ and $x > 0$, the function*

$$f_b(x) = \int_0^1 \frac{(1-t)^{1/b}}{b+(x-b)t}dt \geq \frac{\log(x/b)-1}{x}.$$

Moreover, $f_b(x)$ is decreasing in x.

Proof of Theorem 4.1: We first color $G = K_N$, where $N = r(K_{k,m}, K_{k,m}, K_n) - 1$, in red, blue and yellow so that $K_{k,m}$ is not contained in $G'[E_r]$ or $G'[E_b]$; and K_n is not contained in $G'[E_y]$. Let $G^* = G'[E_r \cup E_b]$. Then $n - 1 \geq \alpha(G^*)$. To apply Lemma 4.1, we need to obtain an upper for the average degree of G^* and the average degree of subgraphs induced by any neighborhood.

The Turán number of a graph F, denoted by $ex(F;N)$, is the maximum number of edges in a graph of order N not containing F. It is well known that

$$ex(K_{k,m}; N) \leq \frac{1}{2}[(m-1)^{\frac{1}{k}} N^{2-\frac{1}{k}} + (k-1)N].$$

Since $K_{k,m}$ is not contained in $G'[E_r]$, the average degree of $G'[E_r]$, and similarly that of $G'[E_b]$, is at most $(m-1)^{\frac{1}{k}} N^{1-\frac{1}{k}} + (k-1)$. Therefore

$$d(G^*) \leq 2(m-1)^{\frac{1}{k}} N^{1-\frac{1}{k}} + 2(k-1) \leq c_0 N^{1-\frac{1}{k}}, \tag{6}$$

for large N, where c_0 is a constant independent of N and hence of n. Henceforth, other similar constants, c_1, c_2, etc., will be chosen in due course.

Let N_i be the number of vertices of G^* with degree i, then $\sum_{i \geq 0} iN_i = Nd$. For any $\eta > 0$,

$$\sum_{i \geq (1+\eta)d} N_i \leq \frac{1}{(1+\eta)d} \sum_{i \geq (1+\eta)d} iN_i \leq \frac{Nd}{(1+\eta)d} = \frac{N}{1+\eta}.$$

Denote by H the subgraph of G^* induced by all vertices with degree less than $(1+\eta)d$. Then the order of H is at least $N - \frac{N}{1+\eta} = \frac{\eta}{1+\eta}N$. If $\eta = 1$, then the order of H is at least $N/2$ and the maximum degree $\Delta(H)$ of H satisfies

$$\Delta(H) < 2d \leq 2c_0 N^{1-\frac{1}{k}}$$

for all large N. It is clear that H contains neither red nor blue $K_{k,m}$. For any vertex u of H, its neighbors induces a subgraph of H which contains neither red nor blue $K_{k,m}$. Applying the same argument and inequality (6) to this subgraph implies that its average degree a satisfies

$$a \leq c_0(\Delta(H))^{1-\frac{1}{k}} \leq c_1 N^{\left(1-\frac{1}{k}\right)^2}.$$

Since H is an induced subgraph of G^*, so its independence number is at most that of G^*, and therefore $n-1 \geq \alpha(H)$. Using Lemma 4.1 and letting $d(H) \leq \Delta(H)$ be the average degree of H,

$$\begin{aligned}
n-1 &\geq \alpha(H) \geq \frac{N}{2} f_{a+1}(d(H)) \\
&\geq (1-o(1))\frac{N}{2}\frac{\log(d(H)/a)}{d(H)} \\
&\geq (1-o(1))\frac{N}{2}\frac{\log(N^{1-\frac{1}{k}}/N^{(1-\frac{1}{k})^2})}{2c_0 N^{1-\frac{1}{k}}} \\
&\geq (1-o(1))c_3 N^{\frac{1}{k}} \log N.
\end{aligned}$$

$$\text{i.e.} \qquad n-1 \geq c_4 N^{\frac{1}{k}} \log N. \tag{7}$$

We will show that

$$N \leq (1+o(1))\left(\frac{n}{c_4 k \log n}\right)^k,$$

for all large n. Suppose to the contrary, there exist $\delta > 0$ and infinitely many n such that $N \geq (1+\delta)(\frac{n}{c_4 k \log n})^k$. Then the right side of (7) would be at least $(1-o(1))(1+\delta)^{\frac{1}{k}} n$ which yields a contradiction as n is sufficiently large. Therefore

$$r(K_{k,m}, K_{k,m}, K_n) \leq (c+o(1))\left(\frac{n}{\log n}\right)^k,$$

where $c = (1/c_4 k)^k$. ∎

Theorem 4.2 *Let $m \geq 2$ be a fixed integer. Then there exists constant $c = c(m)$ such that for all n*

$$r(C_{2m}, C_{2m}, K_n) \leq c\left(\frac{n}{\log n}\right)^{m/(m-1)}.$$

Proof: The proof is similar to that for Theorem 4.1. The only exception is that the role of the upper bound for $ex(K_{k,m}; N)$ is replaced by the known fact $ex(C_{2m}; N) \leq 90mN^{1+\frac{1}{m}}$, see Bollobás [1, pp. 158-161]. ∎

Erdős, Faudree, Rousseau and Schelp [6] proved that for a fixed graph G with p vertices and q edges,

$$r(G, K_n) \geq c \left(\frac{n}{\log n}\right)^{(q-1)/(p-2)}$$

as $n \to \infty$. Since $r(G_1, G_2, K_2) = r(G_1, G_2)$, we have

$$r(K_{k,m}, K_{k,m}, K_n) \geq r(K_2, K_{k,m}, K_n) = r(K_{k,m}, K_n) \geq c_1 \left(\frac{n}{\log n}\right)^{(km-1)/(k+m-2)},$$

which shows that for any fixed k, if m is moderately large, the exponent $(km-1)/(k+m-2)$ of $n/\log n$ can be arbitrarily close to the upper bound of Theorem 4.1. This is also the case for the lower bound

$$r(C_{2m}, C_{2m}, K_n) \geq c_2 \left(\frac{n}{\log n}\right)^{(2m-1)/(2m-2)}$$

and the upper bound in Theorem 4.2.

Note: It is clear that the methods and results in this section can be generalized for estimating the bounds for h-color Ramsey numbers such as $r(K_{k,m}, \ldots, K_{k,m}, K_n)$ and $r(C_{2m}, \ldots, C_{2m}, K_n)$ when $h > 3$.

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