On Bandwidth and Cyclic Bandwidth of Graphs

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Abstract

Let $B(G)$ and $B_c(G)$ denote the bandwidth and cyclic bandwidth of graph G, respectively. In this paper, we shall give a sufficient condition for graphs to have equal bandwidth and cyclic bandwidth. This condition is satisfied by trees. Thus all theorems on bandwidth of graphs apply to cyclic bandwidth of graphs satisfying the sufficiency condition, and in particular, to trees. We shall also give a lower bound of $B_c(G)$ in terms of $B(G)$.

Key words and phrases: Bandwidth, cyclic bandwidth, tree, cycle.

AMS 1991 Subject Classifications: 05C05, 05C78.

1. Introduction

For a graph $G = (V, E)$ of order p, a one-to-one mapping from V onto $\{1, 2, \cdots, p\}$ is called a numbering of G. A one-to-one mapping from V into the set of integers $\mathbb{Z}$ is called an extended numbering of G.

Definition 1.1. Suppose f is a numbering of G. Let

$$B(G, f) = \max_{uv \in E} |f(u) - f(v)|.$$

The bandwidth of G, denoted by $B(G)$, is defined as

$$B(G) = \min\{B(G, f)|\ f \text{ is a numbering of } G\}.$$

Definition 1.2. Suppose f is a numbering of G. Let

$$B_c(G, f) = \max_{uv \in E} |f(u) - f(v)|_c,$$

where $|x|_c = \min\{|x|,\ p - |x|\}$ for $0 < |x| < p$. The cyclic bandwidth of G, denoted by $B_c(G)$, is defined as

$$B_c(G) = \min\{B_c(G, f)|\ f \text{ is a numbering of } G\}.$$

A numbering f of G satisfying $B_c(G) = B_c(G, f)$ is called a *cb*-optimal numbering of G.

The bandwidth problem of graphs has a wide range of applications including sparse matrix computation, data structure, coding theory and circuit layout of VLSI designs (see [1]). The problem became very important since the mid-sixties - see Chinn et al [2] or Chun and Seymour [3]. In its original formulation, the problem is to lay vertices of a graph on a path in such a way so that the maximum distance between any two vertices connected by an edge is minimized. Besides a path, other candidates are also available, and at times may even be more appropriate. In [1] and [4], laying vertices on a grid $P_m \times P_n$ and on a cycle C_n, respectively, are considered. When vertices are laid on a cycle, we get cyclic bandwidth as in Definition 1.2.

In this paper, we shall study cyclic bandwidth. In section 2, we introduce the concept of a *relative numbering* and in section 3, we use this concept to show that bandwidth is equal to cyclic bandwidth for a class of graphs, which includes trees. Finally, we give a lower bound for cyclic bandwidth in terms of bandwidth. Notation and terminology of graph theory are the same as described in the book of Bondy and Murty [5], and Grimaldi [6] unless otherwise defined in this paper.

2. Relative Numbering

In the rest of this paper, G shall denote a graph with vertex set V of order p and edge set E.

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Definition 2.1 Suppose f is a numbering of G. A one-to-one mapping g from V into the set of integers $\mathbb{Z}$ is called an extended numbering of G relative to f if

$$|g(v) - g(u)| \leq |f(v) - f(u)|_c \tag{2.1}$$

for all $uv \in E$.

Lemma 2.1 Suppose f is a numbering of a tree G. Then there exists an extended numbering of G relative to f.

Proof We shall use the following algorithm to construct an extended numbering of G relative to f.

1. Choose a vertex $v \in V$. Set $S = \{v\}$ and put $g(v) = f(v)$.
2. $G[S]$ is a tree. For any $v \in N(S)$, there exist $u \in S$ which is adjacent to v. This u is also unique, because two vertices in S cannot be both adjacent to v. Otherwise there will be a path in S connecting these two vertices, and this path together with the edges connecting v to its end-points will form a cycle in G. But that is impossible because G is a tree. Put $g(v) = g(u) + f(v) - f(u) + p\delta_{u,v}$, where

$$\delta_{u,v} = \begin{cases} 0 & |f(v) - f(u)| \leq \frac{p}{2}, \\ -1 & f(v) - f(u) > \frac{p}{2}, \\ 1 & f(v) - f(u) < -\frac{p}{2}. \end{cases} \tag{2.2}$$

3. Put $S = S \cup N(S)$. If $S \neq V$, then go to (2). Otherwise stop.

We shall show that g is an extended numbering of G relative to f. Suppose u and v are two vertices of G. Because G is a tree, u and v are connected by a unique path in G. Let this path be $s_1 s_2 \ldots s_n$, where $s_1 = u$ and $s_n = v$. We have

$$\begin{aligned} g(v) - g(u) &= \sum_1^{n-1} \{g(s_{i+1}) - g(s_i)\} \\ &= \sum_1^{n-1} \{f(s_{i+1}) - f(s_i) + p\delta_{s_i, s_{i+1}}\} \\ &= f(v) - f(u) + p\delta, \end{aligned} \tag{2.3}$$

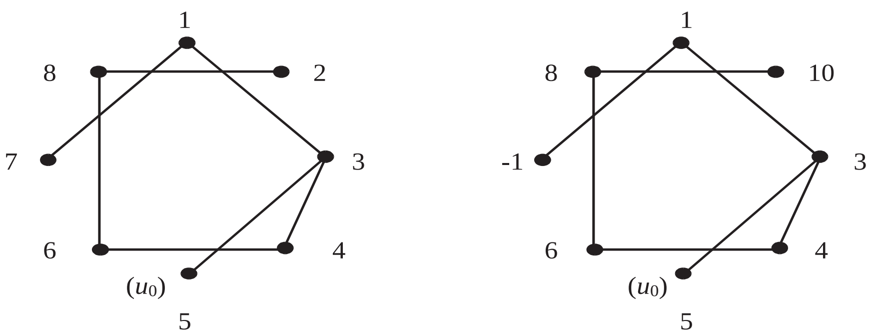
where $\delta = \sum_1^{n-1} \delta_{s_i, s_{i+1}}$. Since p does not divide $f(v) - f(w)$, therefore $g(v) \neq g(u)$.

The definition of $\delta_{u,v}$ ensures that if $uv \in E$, then

$$|g(v) - g(u)| = |f(v) - f(u) + p\delta_{u,v}| = |f(v) - f(u)|_c. \tag{2.4}$$

This shows that g is an extended numbering of G relative to f. ∎

Figure 1 illustrates a tree with numbering f. Also shown is the numbering g which is an extended numbering relative to f. Vertex u_0 is the initial vertex in the algorithm for obtaining g.



A graph G with a numbering f

g is an extended numbering relative to f

Figure 1

3. Graphs with Cyclic Bandwidth Equal to Bandwidth

In this section, we shall describe a class of graphs with cyclic bandwidth equal to bandwidth, and this class of graphs includes trees.

Theorem 3.1 Suppose G is a tree, then $B(G) = B_c(G)$.

Proof It is sufficient to show that $B(G) \leq B_c(G)$. Let f be a *cb*-optimal numbering of G, and let g be the entended numbering of G relative to f as obtained in Lemma 2.1. Arrange the vertices of G in ascending order of values under g, so that $g(u_1) < g(u_2) < ... < g(u_p)$. Define a numbering h of G by letting $h(u_i) = i$ for $i = 1,\ 2,\ ...,\ p$. We can see that h is a numbering of G. It also follows from (2.4) that if $u_i u_j \in E$, then

$$|h(u_i) - h(u_j)| \leq |g(u_i) - g(u_j)| = |f(u_i) - f(u_j)|_c. \tag{3.1}$$

Therefore $B(G) \leq B(G, h) \leq B_c(G, f) = B_c(G)$. ▮

Theorem 3.2 Let $G = (V, E)$ be a graph. If the length of the longest cycle is r and $rB_c(G) < p$, then $B_c(G) = B(G)$.

Proof Let $G_T = (V, E_T)$ be a spanning tree of G and f a numbering of G with $B_c(G, f) = B_c(G)$. Then f is also a numbering of G_T. As in lemma 2.1, we can obtain a numbering g of G_T relative to f.

Let $uv \in E \backslash E_T$. Because G_T is a tree, there is a unique path in G_T joining u and v. Letting this path be $s_1 s_2 ... s_n$, where $s_1 = u$ and $s_n = v$, we have (2.3). Because $uv \in E$, we can determine the integer $\delta_{u,v}$ so that $|f(v) - f(u)|_c = |f(v) - f(u) + p\delta_{u,v}|$. Now the last two expressions of (2.3) may be re-stated as

$$\sum_{1}^{n-1} \{f(s_{i+1}) - f(s_i) + p\delta_{s_i, s_{i+1}}\} = f(v) - f(u) + p\delta_{u,v} + p(\delta - \delta_{u,v}). \tag{3.2}$$

Since the path P together with the edge uv form a cycle of length n in G, we have $nB_c(G) \leq rB_c(G) < p$. Rearranging (3.2), we get

$$\begin{aligned} |p(\delta - \delta_{u,v})| &\leq \sum_{1}^{n-1} |f(s_{i+1}) - f(s_i) + p\delta_{s_i, s_{i+1}}| + |f(v) - f(u) + p\delta_{u,v}| \\ &\leq nB_c(G) < p. \end{aligned} \tag{3.3}$$

Inequality (3.3) is possible only if $\delta - \delta_{u,v} = 0$, and (2.3) now becomes

$$\begin{aligned} |g(v) - g(u)| &= |f(v) - f(u) + p\delta_{u,v}| \\ &= |f(v) - f(u)|_c \leq B_c(G). \end{aligned} \tag{3.4}$$

Clearly $|g(v) - g(u)| = |f(v) - f(u)|_c$ if $uv \in E_T$, therefore (3.4) actually holds for any $uv \in E$.

Now we define a numbering h of G as in theorem 3.1. Because of (3.1) and (3.4), we see that $B(G) \leq B(G, h) \leq B_c(G, f) = B_c(G)$. ▮

Theorem 3.3 Suppose $B(G) = B$ and $B_c(G) = B_c$. Then G may be imbedded into another graph H such that $B_c(H) = B(H) = B$.

Proof There is nothing to prove if G is a tree. Let r be the length of the longest cycle in G. If $rB < p$, then the theorem follows from theorem 3.2. So we assume that $rB \geq p$. Now let f be a numbering of G with $B(G, f) = B$, and u be the vertex with $f(u) = p$. Construct a graph H by adding a path $u_1\ u_2\ ...\ u_k$ to u, where $u_j \notin V(G)$ and $k = rB + 1 - p$. Obviously, H is of order $rB + 1$ and $B(H) = B$. Since $rB_c(H) \leq rB(H) = rB < |H|$, we can apply theorem 3.2 to get $B_c(H) = B(H) = B$.

Since the bandwidth and cyclic bandwidth of trees are identical, all results on bandwidth of trees ([7] - [9]) will be applicable to cyclic bandwidth of trees. We shall conclude this paper with the following theorem, which establishes a bound relationship between bandwidth and cyclic bandwidth of a general graph.

Theorem 3.4 For any graph G, we have $\frac{B(G)}{2} \leq B_c(G) \leq B(G)$.

Proof It is sufficient to show that $B(G) \leq 2B_c(G)$. Let f be given. We define g as follows

$$g(u) = \begin{cases} 2f(u) & \text{if } |f(u)| \leq \lfloor \frac{p}{2} \rfloor, \\ 2[p - f(u)] + 1 & \text{if } f(u) > \lfloor \frac{p}{2} \rfloor. \end{cases} \tag{3.5}$$

Clearly g is a numbering of G. Let $uv \in E$.

If $f(u),\ f(v) \leq \lfloor \frac{p}{2} \rfloor$; or if $f(u),\ f(v) > \lfloor \frac{p}{2} \rfloor$; then

$$|g(u) - g(v)| \leq 2|f(u) - f(v)| = 2|f(u) - f(v)|_c. \tag{3.6}$$

If $f(u) \leq \lfloor \frac{p}{2} \rfloor$ and $f(v) > \lfloor \frac{p}{2} \rfloor$, then

$$\begin{aligned} |g(u) - g(v)| &= |2f(u) - 2[p - f(v)] - 1| \\ &= |2[f(v) - f(u) + 2f(u) - p] - 1| \\ &\leq 2|f(u) - f(v)|. \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} |g(v) - g(u)| &= |2[p - f(v)] + 1 - 2f(u)| \\ &= |2[p - f(v) + f(u)] - 4f(u)] + 1| \\ &\leq 2|p - f(v) + f(u)|. \end{aligned} \tag{3.8}$$

Inequalities (3.6) to (3.8) imply that $|g(u) - g(v)| \leq 2|f(u) - f(v)|_c$. So we have proved that $B(G) \leq B(G, g) \leq 2B_c(G)$. ▮

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