On a Construction of Supermagic Graphs

Wai Chee Shiu*, Peter Che Bor Lam

Department of Mathematics,
Hong Kong Baptist University
Kowloon, Hong Kong.

Sin-Min Lee

Department of Mathematics and Computer Science,
San José State University,
San José, CA 95192, U.S.A.

Abstract

Given two graphs G and H. The composition of G with H is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. In this paper, we prove that the composition of regular supermagic graph with a null graph is supermagic. With the help of this result we show that the composition of a cycle with a null graph is always supermagic.

Key words and phrases : Supermagic, edge-magic, composition of graphs, orthogonal Latin square, magic square.

AMS 2000 MSC : 05C78, 05B15

1. Introduction

Many combinatorial problems are very difficult to solve, but once a solution is known, it may seem easy. To prove a graph being supermagic is one such problem.

Let $G = (V, E)$ be a (p, q)-graph, i.e., $|V| = p$ and $|E| = q$. If there exists a bijection

$$f : E \to \{k,\ k+1,\ \cdots,\ k+(q-1)\}$$

for some $k \in \mathbb{Z}$ such that the map $f^+(u) = \sum_{uv \in E} f(uv)$ induces a constant map from V to $\mathbb{Z}_p$, then G is called *k-edge-magic* and f is called a *k-edge-*

*Partially supported by Research Grant Council Grant, Hong Kong; and Faculty Research Grant, Hong Kong Baptist University

magic labeling of G. If $k = 1$, then G is simply called *edge-magic graph* and f an *edge-magic labeling* of G. This concept was initiated by Lee, Seah and Tan [6]. Moreover, if f^+ is a constant map from V to $\mathbb{Z}$, then G is called *k-supermagic* and f is called a *k-supermagic labeling.* Similarly G is called *supermagic* and f a *supermagic labeling* of G if $k = 1$ [10, 11]. Clearly, a supermagic graph is edge-magic. However, there exists lots of edge-magic graphs which are not supermagic. Hartsfield and Ringel had also studied supermagic graphs [4]. Only a few graphs were shown to be supermagic [4, 9, 10, 11]. In this paper, a construction of supermagic graphs is given.

2. Supermagicness of Regular Graphs

If $G = (V, E)$ is an r-regular (p, q)-graph, then $2q = pr$. Suppose $f : E \to \{1,\ 2,\ \cdots,\ q\}$ is a bijection. For any integer k, we can define a bijection $g : E \to \{k,\ k+1,\ \cdots,\ k+(q-1)\}$ by $g(e) = f(e) + k - 1$ for any $e \in E$. Then $g^+(u) = f^+(u) + r(k-1)$. Therefore f^+ is a constant mapping if and only if g^+ is a constant mapping. Thus, from now on we simply call f is a supermagic or edge-magic labeling if f is a k-supermagic or k-edge-magic labeling for some k, respectively.

Definition: Let $G = (V, E)$ be a simple graph and S be a set. Suppose $f : E \to S$ is a mapping. A *labeling matrix* for a labeling f of G is a matrix whose rows and columns are named by the vertices of G and the (u, v)-entry is $f(uv)$ if $uv \in E$, and is $*$ otherwise. Sometimes, we call this matrix to be a *labeling matrix of* G. In other words, suppose A is an adjacency matrix of G and f is a labeling of G. Then a labeling matrix for f is obtained from $A = (a_{u,v})$ by replacing $a_{u,v}$ by $f(uv)$ if $a_{u,v} = 1$ and by $*$ if $a_{u,v} = 0$. Moreover, if f is a supermagic (edge-magic) labeling, then a labeling matrix of f is called a *supermagic* (*edge-magic*) *labeling matrix* of G.

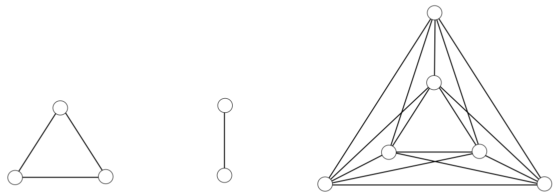
In the following we shall only consider simple regular graphs, and we shall label the edges of graphs by numbers $0,\ 1,\ \cdots,\ q-1$.

Thus, a regular (p, q)-graph $G = (V, E)$ is supermagic if and only if there exists a bijection $f : E \to \{0,\ 1,\ \cdots,\ q-1\}$ such that the row sums and the column sums of the labeling matrix for f are the same. For purposes of these sums, entries labeled with $*$ will be treated as 0.

3. Main Result

In this section, we shall obtain a useful construction to construct a class of supermagic graphs.

Given two graphs G and H. The *composition of G with H*, denoted by $G \circ H$, is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. For example, $C_3 \circ K_2$ is shown in the figure below.



C_3 K_2 $C_3 \circ K_2$

Let $G = (V, E)$ be a simple graph and N_n be the null graph of order n. Suppose $A = (a_{u,v})$ is an adjacency matrix of G, where $u, v \in V$. Let J_n be the $n \times n$ matrix whose entries are 1. Then under the lexicographic order the adjacency matrix of $G \circ N_n$ is $A \otimes J_n$, the Kroneck product of A and J_n.

Example 3.1: A labeling matrix of $C_m \circ N_n$ is of the form

$$\begin{pmatrix} * & A_0 & * & \ddots & * & A_{m-1}^T \\ A_0^T & * & A_1 & \ddots & \ddots & \ddots \\ * & A_1^T & * & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ * & * & * & \ddots & * & A_{m-2} \\ A_{m-1} & * & * & \ddots & A_{m-2}^T & * \end{pmatrix}, \tag{3.1}$$

where A_i is an $n \times n$ matrix, $0 \le i \le m-1$ and $*$ denotes the $n \times n$ matrix whose entries are $*$.

Theorem 3.1: *If G is an r-regular supermagic graph, then $G \circ N_n$ is an rn-regular supermagic graph for $n \ge 3$.*

Proof: Let G be a (p, q)-graph. By definition, $G \circ N_n$ is rn-regular with qn^2 edges. Let A be an adjacency matrix of G. Since $A \otimes J_n$ is an adjacency matrix of $G \circ N_n$, to find a labeling matrix of $G \circ N_n$, we would replace 0's by $*$'s and each J_n by a suitable numeral $n \times n$ matrix from the adjacency matrix of $G \circ N_n$.

Let $L = (l_{i,j})$ be a supermagic labeling matrix of G and let k be the row sum of L. Let Φ be a matrix obtained from L by replacing $*$ by the $n \times n$ matrix whose entries are $*$, $l_{i,j}$ by $n^2 l_{i,j} J_n + M_{i,j}$ and $l_{j,i}$ by $n^2 l_{i,j} J_n + M_{i,j}^T$ if $l_{i,j} \neq *$ and $i < j$, where $M_{i,j}$ is a magic square of order n on $\{0,\ 1,\ \cdots,\ n^2-1\}$. Since for any integer $a \in \{0, 1, \ldots, qn^2-1\}$ there exist $0 \leq b \leq q-1$ and $0 \leq c \leq n^2-1$ uniquely such that $a = n^2 b + c$, Φ is a labeling matrix for a bijective labeling. It is easy to see that Φ is a supermagic labeling matrix of $G \circ N_n$ with row (column) sum $kn^3 + rm$, where m is the magic sum of the magic squares $M_{i,j}$ for all $i < j$. ∎

Corollary 3.2: *If G is an r-regular edge-magic graph, then $G \circ N_n$ is an rn-regular edge-magic graph for $n \geq 3$.*

We use the following example to illustrate the proof above. Note that when $m, n \geq 2$, $K_{m,m} \circ N_n \cong K_{mn,mn}$ which is supermagic by the existence of magic square.

Example 3.2: Consider $K_{3,3}$ which is supermagic with the following supermagic labeling matrix

$$\left(\begin{array}{ccc|ccc} * & * & * & 3 & 8 & 1 \\ * & * & * & 2 & 4 & 6 \\ * & * & * & 7 & 0 & 5 \\ \hline 3 & 2 & 7 & * & * & * \\ 8 & 4 & 0 & * & * & * \\ 1 & 6 & 5 & * & * & * \end{array}\right).$$

We choose $M = M_{i,j} = \begin{pmatrix} 3 & 8 & 1 \\ 2 & 4 & 6 \\ 7 & 0 & 5 \end{pmatrix}$ for all $1 \leq i < j \leq 3$. Then the matrix

$$\left(\begin{array}{ccc|ccc} * & * & * & 27J_3+M & 72J_3+M & 9J_3+M \\ * & * & * & 18J_3+M & 36J_3+M & 54J_3+M \\ * & * & * & 63J_3+M & M & 45J_3+M \\ \hline 27J_3+M^T & 18J_3+M^T & 63J_3+M^T & * & * & * \\ 72J_3+M^T & 36J_3+M^T & M^T & * & * & * \\ 9J_3+M^T & 54J_3+M^T & 45J_3+M^T & * & * & * \end{array}\right)$$

is a supermagic labeling matrix for $K_{3,3} \circ N_3$, where the upper right corner block is

$$\left(\begin{array}{ccc|ccc|ccc}
30 & 35 & 28 & 75 & 80 & 73 & 12 & 17 & 10 \\
29 & 31 & 33 & 74 & 76 & 78 & 11 & 13 & 15 \\
34 & 27 & 32 & 79 & 72 & 77 & 16 & 9 & 14 \\ \hline
21 & 26 & 19 & 39 & 44 & 37 & 57 & 62 & 55 \\
20 & 22 & 24 & 38 & 40 & 42 & 56 & 58 & 60 \\
25 & 18 & 23 & 43 & 36 & 41 & 61 & 54 & 59 \\ \hline
66 & 71 & 64 & 3 & 8 & 1 & 48 & 53 & 46 \\
65 & 67 & 69 & 2 & 4 & 6 & 47 & 49 & 51 \\
70 & 63 & 68 & 7 & 0 & 5 & 52 & 45 & 50
\end{array}\right).$$

∎

4. Applications

In this section, we shall apply Theorem 3.1 to complete m-partite graphs. Stewart [11] proved the following theorem.

Theorem 4.1: *K_m is supermagic if and only if $m > 5$ and $m \not\equiv 0 \pmod 4$.*

Applying Theorem 3.1, we have

Theorem 4.2: *$K_m \circ N_n \cong K_{\underbrace{n, n, \ldots, n}_{m \text{ times}}}$ is supermagic if $n \geq 3$, $m > 5$ and $m \not\equiv 0 \pmod 4$.*

Even though there is no 2 by 2 magic square, by a similar idea of the proof of Theorem 3.1, we have the following theorem.

Theorem 4.3: *$K_m \circ N_2$ is supermagic if $m > 5$ and $m \equiv 1 \pmod 4$.*

Proof: Note that the labeling matrix of $K_m \circ N_2$ is of the form

$$\begin{pmatrix}
* & A_{1,2} & A_{1,3} & \cdots & \cdots & A_{1,m} \\
A_{2,1} & * & A_{2,3} & \cdots & \cdots & A_{2,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & * & A_{m-1,m} \\
A_{m,1} & A_{m,2} & A_{m,3} & \cdots & A_{m,m-1} & *
\end{pmatrix}, \qquad (4.1)$$

where $A_{i,j}$ is a 2×2 matrix and $A_{i,j} = A_{j,i}^T$. Let

$$A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}.$$

Define

$$M = \begin{pmatrix} * & A & B & A & B & \cdots & \cdots & A & B \\ A^T & * & B & A & B & \cdots & \cdots & A & B^T \\ B^T & B^T & * & A & B & \cdots & \cdots & A & B \\ A^T & A^T & A^T & * & B & \cdots & \cdots & A & A^T \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A^T & A^T & A^T & A^T & A^T & \cdots & \cdots & * & A^T \\ B^T & B & B^T & A & B^T & \cdots & \cdots & A & * \end{pmatrix},$$

i.e., according to the notations in (4.1), for $i < j$,

$$A_{i,j} = \begin{cases} A & \text{if } j \text{ is even} \\ B & \text{if } j \text{ is odd and } i \neq m, \text{ or } j = m \text{ and } i \text{ is odd} \\ B^T & \text{if } j = m \text{ and } i \equiv 2 \pmod 4 \\ A^T & \text{if } j = m \text{ and } i \equiv 0 \pmod 4. \end{cases}$$

It is easy to check that the row sums of M are equal to $3(m-1)$.

Let L be a supermagic labeling matrix of K_m. Since for each $a \in \{0, 1, \cdots, 4q-1\}$ with $q = \frac{1}{2}m(m-1)$ there exist b and c with $0 \leq b \leq q-1$ and $0 \leq c \leq 3$ uniquely such that $a = 4b + c$, $\Omega = 4L \otimes J_2 + M$ is a supermagic labeling matrix of $K_m \circ N_2$. Note that for convenience we define $*J_2 = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$. ▮

Note that there is a supermagic labeling of $K_5 \circ N_2$ shown by Ho and Lee in [5, Example 3].

Example 4.1: The matrix

$$L = \begin{pmatrix} * & 18 & 16 & 30 & 6 & 9 & 26 & 23 & 12 \\ 18 & * & 19 & 7 & 29 & 4 & 34 & 2 & 27 \\ 16 & 19 & * & 15 & 20 & 11 & 24 & 35 & 0 \\ 30 & 7 & 15 & * & 25 & 17 & 31 & 5 & 10 \\ 6 & 29 & 20 & 25 & * & 21 & 3 & 14 & 22 \\ 9 & 4 & 11 & 17 & 21 & * & 13 & 32 & 33 \\ 26 & 34 & 24 & 31 & 3 & 13 & * & 1 & 8 \\ 23 & 2 & 35 & 5 & 14 & 32 & 1 & * & 28 \\ 12 & 27 & 0 & 10 & 22 & 33 & 8 & 28 & * \end{pmatrix}$$

is a supermagic labeling matrix of K_9 by using $\{0, 1, \ldots, 35\}$. The original labeling was shown in [11]. Let

$$M = \begin{pmatrix} * & A & B & A & B & A & B & A & B \\ A^T & * & B & A & B & A & B & A & B^T \\ B^T & B^T & * & A & B & A & B & A & B \\ A^T & A^T & A^T & * & B & A & B & A & A^T \\ B^T & B^T & B^T & B^T & * & A & B & A & B \\ A^T & A^T & A^T & A^T & A^T & * & B & A & B^T \\ B^T & B^T & B^T & B^T & B^T & B^T & * & A & B \\ A^T & A^T & A^T & A^T & A^T & A^T & A^T & * & A^T \\ B^T & B & B^T & A & B^T & B & B^T & A & * \end{pmatrix},$$

where A and B are defined in the proof of Theorem 4.3. Then

$$\Omega = \left(\begin{array}{cc|cc|cc|cc|cc|cc|cc|cc|cc}
* & * & 72 & 74 & 67 & 65 & 123 & 121 & 24 & 26 & 36 & 38 & 107 & 105 & 92 & 94 & 51 & 49 \\
* & * & 75 & 73 & 64 & 66 & 120 & 124 & 27 & 25 & 39 & 37 & 104 & 106 & 95 & 93 & 48 & 50 \\ \hline
72 & 75 & * & * & 79 & 77 & 28 & 30 & 119 & 117 & 16 & 18 & 139 & 137 & 8 & 10 & 111 & 108 \\
74 & 73 & * & * & 76 & 78 & 31 & 29 & 116 & 118 & 19 & 17 & 136 & 138 & 11 & 9 & 109 & 110 \\ \hline
67 & 64 & 79 & 76 & * & * & 60 & 62 & 83 & 81 & 44 & 46 & 99 & 97 & 140 & 142 & 3 & 1 \\
65 & 66 & 77 & 78 & * & * & 63 & 61 & 80 & 82 & 47 & 45 & 96 & 98 & 143 & 141 & 0 & 2 \\ \hline
123 & 120 & 28 & 31 & 60 & 63 & * & * & 103 & 101 & 68 & 70 & 127 & 125 & 20 & 22 & 40 & 43 \\
121 & 122 & 30 & 29 & 62 & 61 & * & * & 100 & 102 & 71 & 69 & 124 & 126 & 23 & 21 & 42 & 41 \\ \hline
24 & 27 & 119 & 116 & 83 & 80 & 103 & 100 & * & * & 84 & 86 & 15 & 13 & 56 & 58 & 91 & 89 \\
26 & 25 & 117 & 118 & 81 & 82 & 101 & 102 & * & * & 87 & 85 & 12 & 14 & 59 & 57 & 88 & 90 \\ \hline
36 & 39 & 16 & 19 & 44 & 47 & 68 & 71 & 84 & 87 & * & * & 55 & 53 & 128 & 130 & 135 & 132 \\
38 & 37 & 18 & 17 & 46 & 45 & 70 & 69 & 86 & 85 & * & * & 52 & 54 & 131 & 129 & 133 & 134 \\ \hline
107 & 104 & 139 & 136 & 99 & 96 & 127 & 124 & 15 & 12 & 55 & 52 & * & * & 4 & 6 & 35 & 33 \\
105 & 106 & 137 & 138 & 97 & 98 & 125 & 126 & 13 & 14 & 53 & 54 & * & * & 7 & 5 & 32 & 34 \\ \hline
92 & 95 & 8 & 11 & 140 & 143 & 20 & 23 & 56 & 59 & 128 & 131 & 4 & 7 & * & * & 112 & 115 \\
94 & 93 & 10 & 9 & 142 & 141 & 22 & 21 & 58 & 57 & 130 & 129 & 6 & 5 & * & * & 114 & 113 \\ \hline
51 & 48 & 111 & 109 & 3 & 0 & 40 & 42 & 91 & 88 & 135 & 133 & 35 & 32 & 112 & 114 & * & * \\
49 & 50 & 108 & 110 & 1 & 2 & 43 & 41 & 89 & 90 & 132 & 134 & 33 & 34 & 115 & 113 & * & *
\end{array}\right)$$

is a supermagic labeling matrix of $K_9 \circ N_2$. ∎

More about the supermagicness of regular complete m-partite graphs, the reader is referred to [5].

Theorem 3.1 holds when G is a regular supermagic graph. We shall show you that $G \circ N_n$ can be supermagic even though G is not supermagic.

Consider the graph $C_m \circ N_n$, $m, n \geq 2$. We view C_2 as P_2. For $m \geq 3$, $C_m \circ N_n$ is an (mn, mn^2)-graph; and $C_2 \circ N_n$ is an $(2n, n^2)$-graph. When $m = 2$, $C_2 \circ N_n \cong K_{n,n}$. We can verify that $K_{2,2}$ is not supermagic. Since magic square of any order higher than 2 always exists (see [1] or [2]), $K_{n,n}$ is supermagic for $n \geq 3$. So we may assume that $m \geq 3$ and $n \geq 2$.

Thus $f : E \to \{0,\ 1, \cdots,\ mn^2 - 1\}$ is a supermagic labeling of $C_m \circ N_n$ if and only if row sums and column sums of the labeling matrix for f are the same. The problem is reduced to determining whether we can assign $\{0,\ 1, \cdots,\ mn^2-1\}$ to the entries of m matrices A_i in (3.1) such that row sums and column sums of the matrix are the same.

To reach the purpose, we introduce some concept defined in [8] first. Let S be a set of mn integers, where $m,\ n \geq 2$. If there is a partition of S containing m classes such that each class has n elements and whose sum in each class is the same, then we call S has an (m,n)-*balance partition.*

Lemma 4.4 [8]: *If n is even, or both n and m are odd, then $\{0,\ 1, \cdots,\ mn-1\}$ has an (m,n)-balance partition.*

Suppose there is a partition of S into m classes with n elements in each class. If the sums of elements in $\frac{m}{2}$ of the classes are all equal to one value, and the sums of elements in the remaining classes are all equal to another value, then we call S has an (m,n)-*semi-balance partition* [8].

Lemma 4.5 [8]: *If n is odd and m is even, then $\{0,\ 1,\ \cdots,\ mn-1\}$ has an (m,n)-semi-balance partition.*

Recently, the authors [8] proved that for $m \geq 2$, $n \geq 2$ but $(m,n) \neq (2,2)$, $C_m \circ N_n$ is edge-magic. Now, we are going to prove that $C_m \circ N_n$ is supermagic for those m and n. To do that, we have to make use of Latin squares.

A *Latin square* is a square matrix in which each row and each column consists of the same set of entries without repetition. Two Latin squares $A = (a_{i,j})$ and $B = (b_{i,j})$ of order n are *orthogonal* if the n^2 pairs $(a_{i,j}, b_{i,j})$ are all distinct. It is easy to see that there is no pair of orthogonal Latin squares of order 2. In 1900, G. Tarry examined all cases and proved that there is no pair of orthogonal Latin squares of order 6. In 1960, R.C. Bose, S.S. Shrikhande and E.T. Parker proved the following theorem in [3].

Theorem 4.6: *There exist pairs of orthogonal Latin squares of order n if $n \geq 3$ and $n \neq 6$.*

There is a proof written in the book by van Lint and Wilson ([7], 251-260). The nonexistence proof for the case $n = 6$ is long. In 1984, D.R. Stinson [12] gave a short proof. Because of Theorem 4.6, we have the following theorem.

Theorem 4.7: *$C_2 \circ N_n$ is supermagic if $n \geq 3$, and $C_m \circ N_n$ is supermagic if $m \geq 3$, $n \geq 3$ and $n \neq 6$.*

Proof: It was shown earlier that $C_2 \circ N_n \cong K_{n,n}$ is supermagic if $n \geq 3$. So we only have to consider $m \geq 3$, $n \geq 3$ and $n \neq 6$. Let X and Y be a pair of orthogonal Latin squares of order n.

Case 1: Suppose n is odd and m is even. By Lemma 4.5, we have an (m, n)-semi-balance partition of $Q = \{0,\ 1,\ \cdots, mn-1\}$. Let $\{P_0,\ P_1,\ \cdots,\ P_{m-1}\}$ be this partition such that the sum of elements of P_i, where i is odd, is equal to one value and the sum of elements of P_i, where i is even, is equal to another value.

Using the format of X we obtain a Latin square A_j with entries consisting of elements of P_j, $0 \leq j \leq m-1$, and substitute these $A_j's$ into (3.1) to obtain a labeling matrix of $C_m \circ N_n$, denoted by Ω.

Case 2: Suppose n is even or both n and m are odd. By Lemma 4.4, we have an (m, n)-balance partition of Q. As in Case 1, we obtain a labeling matrix Ω of $C_m \circ N_n$.

Note that the matrix Ω, obtained from each of the above cases, is an edge-magic labeling matrix of $C_m \circ N_n$ (see [8]).

In the same way, we may use the format of Y to obtain a Latin square B with entries 0, mn, $2mn$, $\cdots$, $(n-1)mn$. Substituting B for A_j of (3.1), $0 \leq j \leq m-1$, we have a matrix, say Ψ. Because of the orthogonality of A_j's (obtained from case 1 or case 2) and B, $\Omega + \Psi$ is a supermagic labeling matrix of $C_m \circ N_n$. ∎

Example 4.2: Consider $C_4 \circ N_3$. A $(4, 3)$-semi-balance partition of $\{0,\ 1,\ \cdots,\ 11\}$ is $P_0 = \{0, 7, 11\}$, $P_1 = \{1, 5, 9\}$, $P_2 = \{2, 6, 10\}$, and $P_3 = \{3, 4, 8\}$. Choose

$$X = \begin{pmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{pmatrix} \text{ and } Y = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix},$$

which are orthogonal Latin squares. Then $B = \begin{pmatrix} 0 & 12 & 24 \\ 12 & 24 & 0 \\ 24 & 0 & 12 \end{pmatrix}$ and

$$\Omega = \left(\begin{array}{ccc|ccc|ccc|ccc}
* & * & * & 0 & 7 & 11 & * & * & * & 3 & 8 & 4 \\
* & * & * & 11 & 0 & 7 & * & * & * & 4 & 3 & 8 \\
* & * & * & 7 & 11 & 0 & * & * & * & 8 & 4 & 3 \\
\hline
0 & 11 & 7 & * & * & * & 1 & 5 & 9 & * & * & * \\
7 & 0 & 11 & * & * & * & 9 & 1 & 5 & * & * & * \\
11 & 7 & 0 & * & * & * & 5 & 9 & 1 & * & * & * \\
\hline
* & * & * & 1 & 9 & 5 & * & * & * & 2 & 6 & 10 \\
* & * & * & 5 & 1 & 9 & * & * & * & 10 & 2 & 6 \\
* & * & * & 9 & 5 & 1 & * & * & * & 6 & 10 & 2 \\
\hline
3 & 4 & 8 & * & * & * & 2 & 10 & 6 & * & * & * \\
8 & 3 & 4 & * & * & * & 6 & 2 & 10 & * & * & * \\
4 & 8 & 3 & * & * & * & 10 & 6 & 2 & * & * & *
\end{array}\right).$$

We have

$$\Omega + \Psi = \left(\begin{array}{ccc|ccc|ccc|ccc}
* & * & * & 0 & 19 & 35 & * & * & * & 3 & 20 & 28 \\
* & * & * & 23 & 24 & 7 & * & * & * & 16 & 27 & 8 \\
* & * & * & 31 & 11 & 12 & * & * & * & 32 & 4 & 15 \\
\hline
0 & 23 & 31 & * & * & * & 1 & 17 & 33 & * & * & * \\
19 & 24 & 11 & * & * & * & 21 & 25 & 5 & * & * & * \\
35 & 7 & 12 & * & * & * & 29 & 9 & 13 & * & * & * \\
\hline
* & * & * & 1 & 21 & 29 & * & * & * & 2 & 18 & 34 \\
* & * & * & 17 & 25 & 9 & * & * & * & 22 & 26 & 6 \\
* & * & * & 33 & 5 & 13 & * & * & * & 30 & 10 & 14 \\
\hline
3 & 16 & 32 & * & * & * & 2 & 22 & 30 & * & * & * \\
20 & 27 & 4 & * & * & * & 18 & 26 & 10 & * & * & * \\
28 & 8 & 15 & * & * & * & 34 & 6 & 14 & * & * & *
\end{array}\right),$$

which is a supermagic labeling matrix of $C_4 \circ N_3$. ∎

Theorem 4.8: *If $m \geq 4$ and is even, then $C_m \circ N_n$ is supermagic.*

Proof: Let Ω be an edge-magic labeling matrix of $C_m \circ N_n$ constructed in the proof of Theorem 4.7. Let $\vec{\mathbf{1}}^T$ be the transpose of $\vec{\mathbf{1}} = (1, 1, \cdots, 1)$. Let A be the $n \times n$ matrix whose i-th column is $(i-1)mn\vec{\mathbf{1}}^T$ and B be the $n \times n$ matrix whose i-th row is $(n-i)mn\vec{\mathbf{1}}$. Then A and B are orthogonal to each numeral block matrix of Ω, which are Latin squares. Substituting A and B for A_j of (3.1) if j is even and odd, respectively, we have a matrix Ψ. Then each row of Ψ contains two copies of $\{0,\ mn,\ \cdots,\ (n-1)mn\}$ and $(m-2)n$ $*$'s or n copies of $\{imn,\ (n-i)mn\}$ and $(m-2)n$ $*$'s for some i, $0 \leq i \leq n-1$. Thus the row sums and the column sums are the same, namely it is equal to mn^2. Then $\Omega + \Psi$ is a required supermagic labeling matrix of $C_m \circ N_n$. ▮

Example 4.3: Consider $C_4 \circ N_3$ again. Let Ω be that of Example 4.2, and

$$\Psi = \left(\begin{array}{ccc|ccc|ccc|ccc}
* & * & * & 0 & 12 & 24 & * & * & * & 24 & 12 & 0 \\
* & * & * & 0 & 12 & 24 & * & * & * & 24 & 12 & 0 \\
* & * & * & 0 & 12 & 24 & * & * & * & 24 & 12 & 0 \\
\hline
0 & 0 & 0 & * & * & * & 24 & 24 & 24 & * & * & * \\
12 & 12 & 12 & * & * & * & 12 & 12 & 12 & * & * & * \\
24 & 24 & 24 & * & * & * & 0 & 0 & 0 & * & * & * \\
\hline
* & * & * & 24 & 12 & 0 & * & * & * & 0 & 12 & 24 \\
* & * & * & 24 & 12 & 0 & * & * & * & 0 & 12 & 24 \\
* & * & * & 24 & 12 & 0 & * & * & * & 0 & 12 & 24 \\
\hline
24 & 24 & 24 & * & * & * & 0 & 0 & 0 & * & * & * \\
12 & 12 & 12 & * & * & * & 12 & 12 & 12 & * & * & * \\
0 & 0 & 0 & * & * & * & 24 & 24 & 24 & * & * & *
\end{array}\right).$$

Then

$$\Omega + \Psi = \left(\begin{array}{ccc|ccc|ccc|ccc}
* & * & * & 0 & 19 & 35 & * & * & * & 27 & 20 & 4 \\
* & * & * & 11 & 12 & 31 & * & * & * & 28 & 15 & 8 \\
* & * & * & 7 & 23 & 24 & * & * & * & 32 & 16 & 3 \\
\hline
0 & 11 & 7 & * & * & * & 25 & 29 & 33 & * & * & * \\
19 & 12 & 23 & * & * & * & 21 & 13 & 17 & * & * & * \\
35 & 31 & 24 & * & * & * & 5 & 9 & 1 & * & * & * \\
\hline
* & * & * & 25 & 21 & 5 & * & * & * & 2 & 18 & 34 \\
* & * & * & 29 & 13 & 9 & * & * & * & 10 & 14 & 30 \\
* & * & * & 33 & 17 & 1 & * & * & * & 6 & 22 & 26 \\
\hline
27 & 28 & 32 & * & * & * & 2 & 10 & 6 & * & * & * \\
20 & 15 & 16 & * & * & * & 18 & 14 & 22 & * & * & * \\
4 & 8 & 3 & * & * & * & 34 & 30 & 26 & * & * & *
\end{array}\right),$$

which is a supermagic labeling matrix of $C_4 \circ N_3$. ▮

Now we shall prove the remaining cases, i.e., $C_m \circ N_2$ and $C_m \circ N_6$ are supermagic for m is odd and $m \geq 3$.

By applying Theorem 3.1, we have

Corollary 4.9: *If $C_m \circ N_k$ is supermagic, then so is $C_m \circ N_{kn}$ for $n \geq 3$.*

Proof: The conclusion follows from $(C_m \circ N_k) \circ N_n \cong C_m \circ N_{kn}$. ▮

Theorem 4.10: *$C_m \circ N_2$ is supermagic if $m \geq 3$ and is odd.*

Proof: The following is a supermagic labeling matrix of $C_3 \circ N_2$:

$$\left(\begin{array}{cc|cc|cc} * & * & 0 & 10 & 4 & 8 \\ * & * & 11 & 1 & 3 & 7 \\ \hline 0 & 11 & * & * & 6 & 5 \\ 10 & 1 & * & * & 9 & 2 \\ \hline 4 & 3 & 6 & 9 & * & * \\ 8 & 7 & 5 & 2 & * & * \end{array}\right).$$

When $m \geq 5$, we let

$$A_0 = \begin{pmatrix} 0 & 4m-2 \\ 4m-1 & 1 \end{pmatrix}, \quad A_{m-1} = \begin{pmatrix} 2m-2 & 2m-3 \\ 2m+2 & 2m+1 \end{pmatrix},$$

$$A_{m-2} = \begin{pmatrix} 2m & 2m-1 \\ 2m+3 & 2m-4 \end{pmatrix}$$

and for $1 \leq j \leq m-3$,

$$A_j = \begin{cases} \begin{pmatrix} 2j+1 & 4m-2j-2 \\ 4m-2j-1 & 2j \end{pmatrix} & \text{if } j \text{ is odd.} \\[2ex] \begin{pmatrix} 2j & 4m-2j-2 \\ 4m-2j-1 & 2j+1 \end{pmatrix} & \text{if } j \text{ is even.} \end{cases}$$

There is a one-to-one correspondence between entries of A_j, $0 \leq j \leq m-1$, and $\{0,\ 1,\ \cdots,\ 4m-1\}$. Substituting these matrices into (3.1), we obtain a labeling matrix L of $C_m \circ N_2$. We shall show that the row sums of this labeling matrix are the same.

The first two row sums of L are contributed by the matrices A_0 and A^T_{m-1}. These two row sums are both $8m-2$. Similarly, the last two row sums of L are contributed by the matrices A_{m-1} and A^T_{m-2}. These two row sums are also both $8m-2$. Sum of the $(2j+1)$-th and the $(2j+2)$-th rows, where $1 \leq j \leq m-2$, are contributed by the matrices A_j and A^T_{j-1}, which are also both $8m-2$. Therefore L is a supermagic labeling matrix of $C_m \circ N_2$. ∎

Corollary 4.11: *$C_m \circ N_6$ is supermagic if $m \geq 3$ and is odd.*

Example 4.4: The following is a supermagic labeling matrix of $C_5 \circ N_2$.

$$\left(\begin{array}{cc|cc|cc|cc|cc}
* & * & 0 & 18 & * & * & * & * & 8 & 12 \\
* & * & 19 & 1 & * & * & * & * & 7 & 11 \\ \hline
0 & 19 & * & * & 3 & 16 & * & * & * & * \\
18 & 1 & * & * & 17 & 2 & * & * & * & * \\ \hline
* & * & 3 & 17 & * & * & 4 & 14 & * & * \\
* & * & 16 & 2 & * & * & 15 & 5 & * & * \\ \hline
* & * & * & * & 4 & 15 & * & * & 10 & 9 \\
* & * & * & * & 14 & 5 & * & * & 13 & 6 \\ \hline
8 & 7 & * & * & * & * & 10 & 13 & * & * \\
12 & 11 & * & * & * & * & 9 & 6 & * & *
\end{array}\right).$$

According to the proof of Theorem 3.1, suppose we choose the magic square

$$M = \begin{pmatrix} 3 & 8 & 1 \\ 2 & 4 & 6 \\ 7 & 0 & 5 \end{pmatrix} = M_{i,j}, \qquad 1 \leq i < j \leq 10.$$

If we replace each numeral x above the diagonal, y below the diagonal and

$*$ in the labeling matrix of $C_5 \circ N_2$ by the 3×3 matrix $3^2 x J_3 + M$, the 3×3 matrix $3^2 y J_3 + M^T$ and the 3×3 matrix with $*$ as entries respectively, then we obtain a supermagic labeling matrix of $C_5 \circ N_6$. ▮

References

[1] W. W. R. Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, 13th ed., Dover, 1987.

[2] W. H. Benson and O. Jacoby, *New Recreations with Magic Squares*, Dover, 1976.

[3] R. C. Bose, S. S. Shrikhande and E. T. Parker, Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture, *Canad. J. Math.*, **12**, 189-203, 1960.

[4] N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *Journal of Recreational Mathematics*, **21**, 107-115, 1989.

[5] Yong-Song Ho and Sin-Min Lee, Some initial result of supermagicness of regular complete k-partite graphs, to appear in *J. of Comb. Math. and Comb. Comp.*

[6] Sin-Min Lee, Eric Seah and S. K. Tan, On edge-magic graphs, *Congressus Numerantium*, **86**, 179-191, 1992.

[7] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, Cambridge, 1992.

[8] W. C. Shiu, P. C. B. Lam and Sin-Min Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium*, **132**, 9-18, 1998.

[9] W. C. Shiu, P. C. B. Lam and H.L. Cheng, Supermagic Labeling of an s-duplicate of $K_{n,n}$, *Congressus Numerantium*, **146**, 119-124, 2000.

[10] B. M. Stewart, Magic graphs, *Canadian Journal of Mathematics*, **18**, 1031-1059, 1966.

[11] B. M. Stewart, Supermagic Complete Graphs, *Canadian Journal of Mathematics*, **19**, 427-438, 1967.

[12] D. R. Stinson, A Short Proof of the Nonexistence of a Pair of Orthogonal Latin Squares of Order Six, *Journal of Combinatorial Theory, Series A*, **36**, 373-376, 1984.