

Edge Game Coloring of Graphs¹

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Abstract

Corresponding to the game chromatic number of graphs, we consider in this paper the game chromatic index χ'_g of graphs, which is defined similarly, except that edges, instead of vertices of graphs are colored. Upper bounds for trees and wheels are given.

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1. Introduction

In 1991, Bodlaender [2] introduced the game-coloring problem of graphs. Let G be a graph and $X = \{1, \dots, k\}$ be a set of colors. Consider a two-person game on G as following: Player 1 and Player 2 make moves alternatively with Player 1 moving first. Each *feasible* move consists of choosing an uncolored vertex, and coloring it with a color from X , so that in the subgraph H of G induced by the colored vertices, adjacent vertices get distinct colors. The game ends as soon as one of the two players can no longer execute any feasible move. Player 1 wins if all the vertices of G are colored, otherwise Player 2 wins. A graph G is called *k-game-colorable* if Player 1 has a winning strategy for $|X| = k$, and the *game-chromatic number* $\chi_g(G)$ of G is the least integer k such that G is *k-game-colorable*. Upper bounds for the game chromatic number of some classes of graphs are given in [3, 4, 5].

In this paper, we consider the edge-version of the game-coloring of graphs. It is defined similarly except that the two players will be coloring the edges of a graph G instead of its vertices. In this case, a *feasible* move consists of choosing an uncolored edge, and choosing a color from X , so that in the subgraph H of G induced by the colored edges, incident edges get distinct colors. A color that can be assigned to an edge to constitute a feasible move is called a *feasible* color. The smallest number k such that Player 1 has a winning strategy with k colors in edge game-coloring

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G is called the *edge game-chromatic index* of G , denoted by $\chi'_g(G)$. An obvious lower bound of $\chi'_g(G)$ is $\chi'(G)$. This lower bound can be reached because $\chi'_g(K_{1,r}) = \chi'(K_{1,r}) = r$, and $\chi'_g(G)$ can be strictly greater than $\chi'(G)$ because $\chi'_g(P_n) = 3$ but $\chi'(P_n) = 2$ when $n \geq 5$, where $\chi'(G)$, $K_{1,r}$ and P_n denote the chromatic index of graph G , a star of order $r + 1$ and a path of order n , respectively. Note that each edge is adjacent to at most $2\Delta(G) - 2$ distinct edges, where $\Delta(G)$ denotes the maximum degree of G . So $2\Delta(G) - 1$ is a trivial upper bound of $\chi'_g(G)$. We can also see that $\chi'_g(C) = 3$.

All graphs considered in this paper are finite and simple. Undefined symbols and concepts can be found in [1]. In section 2, we shall establish an upper bound for trees. In Section 3, we shall obtain the game chromatic number of wheels.

2. Game chromatic number of trees

First we give an upper bound to χ'_g for trees using the method given by Faigle, Kern, Kierstead and Trotter [3].

Theorem 1 $\chi'_g(T) \leq \Delta(T) + 2$ for each tree T .

Proof : We give a winning strategy for Player 1 using $\Delta(T) + 2$ colors.

Initially, Player 1 chooses an arbitrary edge $e = v_0v_1$ of T , where $\deg_T(v_0) = 1$, and assigns a color to it. Let $T^* = \{e\}$. Henceforth, T is regarded as a digraph with v_0 as its root.

Suppose that Player 2 has just moved by coloring an edge e_1 . Let P be the directed path from v_0 to e_1 in T , and let e^* be the last edge P has in common with T^* . We update T^* to $T^* \cup \{P\}$, i.e., $T^* := T^* \cup \{P\}$.

If e^* is uncolored, then we assign a feasible color to e^* . If e^* is colored and T^* contains an uncolored edge f , then assign a feasible color to f . Otherwise, we color any edge f adjacent to some edges of T^* and let $T^* := T^* \cup \{f\}$.

Suppose $f = \overrightarrow{uv}$ is the last edge of the directed path in T from v_0 to v . If at most one arc out of v has been colored, then the total number of colored arcs incident with f is at most $\Delta(G)$. As soon as a second outgoing arc of v has been colored, Player 1 will color f unless it has already been colored beforehand. At this moment, at most $\Delta(G) + 1$ colored arcs are incident with f . With $\Delta(G) + 2$ available colors, Player 1 can always find a feasible color for f . ■

3. Game chromatic number of wheels

A *wheel* W_n is a graph obtained from a cycle C_n by adding a new vertex and joining it to each of the vertices of C_n . The added new vertex is called the *center* of the wheel. It is clear that $\chi'_g(W_n) \leq n + 2$.

Theorem 2 $\chi'_g(W_3) = 5$ and $\chi'_g(W_n) = n + 1$ when $n \geq 4$.

Proof : $\chi'_g(W_3) = 5$ can be verified directly.

Suppose $n \geq 4$. We shall call an edge joining the center to a vertex of the circle a *spoke*. The end vertex of a spoke lying on C_n is called its *end*. An edge joining the ends of two spokes is called a *rim*. A *bare* spoke is uncolored and have no colored rim incident with it.

It is enough to show that Player 1 can color all spokes with $n + 1$ colors when $n \geq 4$. Any rim is incident with two spokes and two other rims. Hence $n + 1 \geq 5$ assures that there will be a feasible color for any rim after all spokes are colored. Let c_k denote the k -th color introduced during the game. Player 1 plans to color r spokes with r distinct colors, for $1 \leq r \leq n - 2$. Initially, he colors an arbitrarily chosen spoke with color c_1 . Suppose $r - 1$ spokes have been colored with $r - 1$ colors when it is Player 2's turn, where $2 \leq r \leq n - 2$, so there are at least 3 uncolored spokes.

If Player 2 colors a spoke, he would be helping Player 1 to accomplish his goal. If Player 2 colors a rim, he can at his best prevent at most 2 of the uncolored spokes from being colored with c_r by Player 1, but there are at least 3 uncolored spokes.

When Player 1 colors the $(n - 2)$ -th spoke, he should ensure that no bare spokes are next to each other afterwards. This can be done by choosing to color a bare spoke if there are 2 next to each other, and by choosing to color the middle one if there are three of them next to each other. With this precaution, at best, Player 2 can color the rim incident with one of the 2 remaining uncolored spokes with c_{n-1} . Nevertheless, Player 1 can color the other one with c_{n-1} . So no matter what Player 2 does next, Player 1 can color the last uncolored spoke with c_n or c_{n+1} .

If Player 2 chooses to color the $(n - 2)$ -th spoke with c_{n-2} , Player 1 can simply color one of the 2 uncolored spokes with c_{n-1} . Any move by Player 2 cannot prevent Player 1 from coloring the last uncolored spoke with c_n or c_{n+1} . ■

4. Discussion

For each graph G , $E(G)$ can be partitioned into $k = \chi'(G)$ matchings E_1, E_2, \dots, E_k . We shall call each of them as a color-class of G . It is obvious that in the edge game-coloring of graph G , Player 1 always want to color the edges in a color-class with the same color, and Player 2 will do the opposite, i.e., try to color each edge in a matching with distinct colors. Let us consider the game chromatic index of K_5 . It is known that $\chi'(K_5) = 5$ and each matching of K_5 contains at most two edges. Let $X = \{1, 2, \dots, 6\}$. Initially, Player 1 color an arbitrary edge. Suppose Player 2 has just colored an edge e with color $j \in X$. If there is still uncolored edge, then Player 1 choose an edge e' which has no common vertex with e and color it with j also. This guarantees that there are 8 edges of G are colored with 4 colors of X . Therefore, Player 1 has a winning strategy using 6 colors.

It is known that there is no upper bound for $\chi_g(G)$ as a function of $\chi(G)$ [3] because there exists bipartite graphs of arbitrarily large game chromatic numbers. This may not be true when we consider the relation between $\chi'_g(G)$ and $\chi'(G)$. Maybe it is true that there is a constant c such that $\chi'_g(G) \leq \Delta(G) + c$ for any graphs. In addition, $\chi'_g(G) \geq \chi'(G)$ and this lower bound can be reached. How many graphs are there which satisfy $\chi'_g(G) = \chi'(G)$? Is it true that almost all graphs satisfy $\chi'_g(G) > \chi'(G)$? Finally, we conclude this paper with some open questions:

Question 1: Is there a constant $c \geq 2$ such that $\chi'_g(G) \leq \Delta(G) + c$ for each graph G ? If it is true, is $c = 2$ enough?

Question 2: Let \mathcal{G}_n be the set of graphs of order n and let $\mathcal{G}_n^* = \{G \in \mathcal{G}_n | \chi'_g(G) > \chi'(G)\}$. Is it true that

$$\lim_{n \rightarrow \infty} \frac{|\mathcal{G}_n^*|}{|\mathcal{G}_n|} = 1?$$

References

- [1] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, New York: Macmillan Ltd. Press, 1976.
- [2] H. L. Bodlaender, On the complexity of some coloring games, *Lecture Notes in Computer Science*, **484**, 30-40, Springer, 1991.

- [3] U. Faigle, U. Kern, H. Kierstead and W. T. Trotter, On the game chromatic number of some classes of graphs, *Ars Combin.*, **35**, 143-150, 1993.
- [4] H. A. Kierstead and W. T. Trotter, Planar graph coloring with an uncooperative partner, *J. Graph Theory*, **18**, No.6, 569-584, 1994
- [5] Thomas Dinski and Xuding Zhu, Game chromatic number of graphs, to appear in *Discrete Math.*