Characterization of 3-Regular Halin Graphs with Edge-face Total Chromatic Number Equal to Four*

W. H. Chan
Department of Mathematics and Information Technology, The Hong Kong Institute of Education, Hong Kong

Peter C. B. Lam and W. C. Shiu
Department of Mathematics, Hong Kong Baptist University, Hong Kong

Abstract

The edge-face total chromatic number of 3-regular Halin graphs was shown to be 4 or 5 in [5]. In this paper, we shall provide a necessary and sufficient condition to characterize 3-regular Halin graphs with edge-face total chromatic number equal to four.

Key words and phrases: Halin graphs, edge-face total chromatic number.

2000 AMS Subject Classifications: 05C15

1 Introduction

A *Halin* graph G is a plane graph embedding of a tree T with order at least 4 and whose interior vertices have degree at least 3, and a cycle C^* connecting all end vertices of T. The tree T is called the *characteristic tree* of G, and C^* is called the *adjoint cycle* of G. Vertices and edges on the cycle C^* are called the *outer vertices* and *outer edges* respectively. Other vertices and edges are called *inner vertices* and *inner edges* respectively. A path consisting of inner edges is called an *inner path.* The face incident with all outer vertices and outer edges is called the *outer face* and is denoted by f_0. All other faces are called *inner faces.* Faces of degree 3 are sometimes called *triangles.* Note that an inner face is bounded by one outer edge and an inner path. Two end vertices of the characteristic tree of a Halin graph are called *neighboring vertices* if they are linked by an edge of the adjoint cycle C^*. Two inner faces are *neighbors* of each other, or *neighboring faces*, if they are incident with a common outer vertex. A Halin graph is said to be *3-regular* if all the interior vertices of its characteristic tree are of degree 3.

*Partially supported by Faculty Research Grant (FRG/07-08/II-30), Hong Kong Baptist University; and Research Grant Council Grant (HKBU210207), Hong Kong.

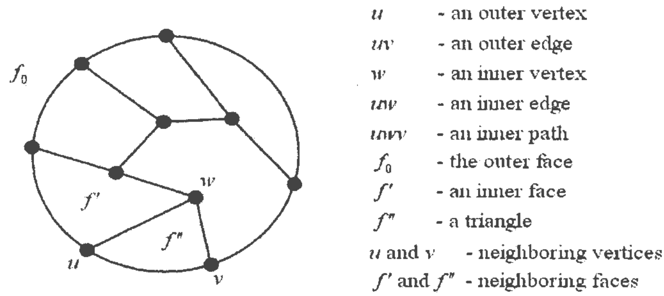

Figure 1: An example of 3-regular Halin graph

Definition 1.1 A *proper k-edge-face total* coloring, of a loopless plane graph G is an assignment of k colors $1, 2, \ldots, k$ to all edges and faces in $E \cup F$ such that no two adjacent or incident elements have the same color. A graph G is *k-edge-face total colorable* if there exists a *k-edge-face total coloring* on G. Moreover,

$$\chi_{ef}(G) = \min\{k |\ G \textit{ is } k\textit{-edge-face total colorable } \}$$

is called the *edge-face total chromatic number* of G.

The edge-face total chromatic number has been investigated as early as the conjecture that the edges and faces of each plane graph G may be colored with $\Delta(G) + 3$ colors so that any two adjacent or incident elements receive different colors, where $\Delta(G)$ is the maximum degree of G, was raised by Melnikov [8] in 1975. Since then, many researchers have been working on this problem [2, 3, 4, 6, 7, 9]. In 2000, it was shown in [5] that if G is a 3-regular Halin graph, then $\chi_{ef}(G)$ is either 4 or 5. In this paper, we provide a necessary and sufficient condition to characterize those graphs with $\chi_{ef}(G) = 4$. The reader is referred to [1] for standard terminology of graph theory not defined in this paper.

2 A necessary and sufficient condition

From the structure of Halin graphs, it can be observed that there is a one-to-one correspondence between inner faces and outer edges. Thus, the inner faces can also be regarded as cyclically ordered according to the order of the outer edges incident with them.

Theorem 2.1 *Suppose G is a 3-regular Halin graph. $\chi_{ef}(G) = 4$ if and only if for any two triangles T and T', the sequence of faces $f_1 f_2 \ldots f_m$ separating T and T' in the cyclic order contains at least one even face.*

Proof Suppose G is a 3-regular Halin graph and, between any two triangles of G, there is an inner face of even order. We shall construct a 4-edge-face total coloring of G by the following steps. The first two steps are from the 5-EFT coloring procedures in [5] to color all inner edges with colors c_2, c_3 and c_4, and the outer face with c_1.

(1) Choose any inner vertex of G and assign colors c_2, c_3 and c_4 to the three edges incident with that vertex in the clockwise direction. An inner vertex whose three incident edges have been assigned colors is marked as *labelled*. An inner vertex which has not yet been marked as labelled is called *unlabelled*.

(2) If there are unlabelled vertices remaining, then choose an unlabelled vertex v adjacent to a labelled vertex u. Without loss of generality, we may assume that c_2 has been assigned to the edge uv, and that colors have been assigned to the three edges incident to u in the clockwise direction. We then assign the remaining two colors, c_3 and c_4, to edges incident to v in the anti-clockwise direction and mark v as a labelled vertex. This process will continue until all inner vertices have been marked as labelled.

In [5], it was also shown that every face of G is surrounded by an outer edge and an inner path, in which the inner path is colored alternately by any two of colors c_2, c_3 and c_4.

(3) Put the color in $\{c_2, c_3, c_4\} \setminus \{c_i, c_j\}$ to f, where c_i and c_j are the colors on its inner path.

If two inner faces f_x and f_y are adjacent, the pairs of colors on the inner paths of the two faces cannot be identical. The colors of f_x and f_y are thus distinct.

(4) Suppose T_1 and T_2 are any two triangles of G and there is no other triangles between T_1 and T_2 in the clockwise direction from T_1 to T_2. Let $T_1, f_1, f_2, \ldots, f_n, T_2$ be a sequence of inner faces in the clockwise direction, and $v_{i-1}v_i$ be the outer edge incident with f_i, $1 \le i \le n$. Change the color of the inner edges incident with v_1, v_2, ... and v_{n-1} to c_1. From the assumption of the theorem, there is an inner face of even order in the above sequence. Let f_j be a face of even order and put the color of f_i to $v_i v_{i+1}$ for $i = 1, \ldots, j-1$ and to $v_{i-2}v_{i-1}$ for $i = j+2, \ldots, n$, and put the color of T_1 to $v_0 v_1$ and the color of T_2 to $v_{n-1}v_n$.

Clearly, every outer edge $v_{i-1}v_i$ incident with f_i, $1 \le i \le n$, is colored with a color different from the color of f_i. Since the colors on the adjacent inner faces of f_j should be distinct from the color of f_j and the color of the outer face, there are only two colors of the adjacent faces of f_j and they should be arranged alternately. Moreover, the order of f_j is even. Therefore the colors of f_{j-1} and f_{j+1} must be the same. Thus, the colors of v_0v_1, v_1v_2, ..., $v_{j-1}v_j$ are respectively identical to the colors of T_1, f_1, ..., $f_j - 1$ and the colors of $v_{j-1}v_j$, v_jv_{j+1}, ..., $v_{n-1}v_n$ are respectively identical to the colors of f_{j+1}, f_{j+2}, ..., T_2.

(5) For any triangle $T = uvw$ of G, where u and v are outer vertices, let f' be the neighboring face of T incident with uw. From (4), the colors of the outer edges incident with u and v (distinct from uv) are the same as that of T, so we can simply put the color of f' to the outer edge of T and change the color of vw to c_1 as shown in Figure 2.

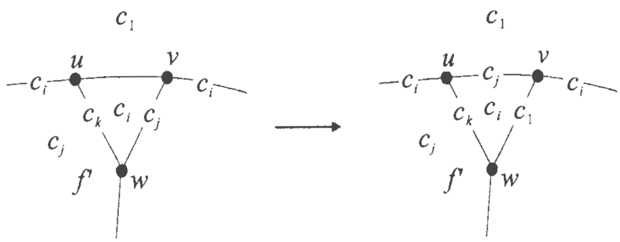

Figure 2: Coloring of the outer edge of a triangle

We can see that (i) at each vertex, all incident edges have distinct colors, (ii) the color of each face is distinct from those of its incident edges, and (iii) adjacent faces received different colors.

Hence, the construction of the 4-edge face total coloring of G is completed.

On the other hand, we shall prove that if there exist two triangles T' and T'' of G such that all inner faces $f_1, f_2, \ldots, f_m$ between T' to T'' are odd, then G is not 4-edge-face total colorable. Suppose G is edge-face colorable by the color set $\{c_1, c_2, c_3, c_4\}$. Without loss of generality, we assume that f_0, T' and f_1 are colored with c_1, c_2 and c_3 respectively. The color of f^*, which is the other neighboring face of T', must be the fourth color, c_4. There are 2 possible ways to color the edges incident with T' as shown in Figure 3. Obviously, in both cases, the color of the outer edge incident with f_1 must be c_2. Because the colors of the faces adjacent to f_1 should not be c_1 (the color of the outer face) nor c_3 (the color of f_1), each face adjacent to f_1 should receive either c_2 or c_4. Moreover, the two colors of the faces adjacent to f_1 should be arranged alternately. Since f_2

is one of the adjacent faces of f_1 and the order of f_1 is odd, the color that appears on f_2 must be c_4 and hence the outer edge of f_2 must be colored with c_3. Since f_3 is one of the adjacent faces of f_2 and the order of f_2 is odd, the color of f_3 must be c_2 and that on the outer edge of f_3 must be c_4. Similarly, we got c_3 and c_2 on f_4 and the outer edge of f_4 respectively as shown in Figure 3.

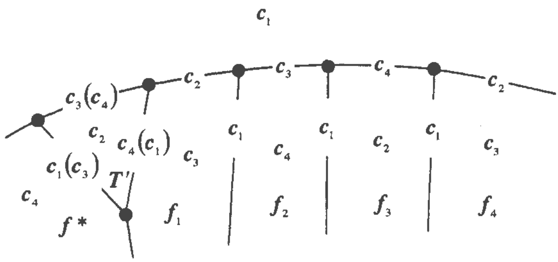

Figure 3: The color pattern on the odd faces and their outer edges

It can be easily observed that the colors c_3, c_4 and c_2 appear cyclically on faces $f_1, f_2, \ldots, f_m$. Moreover, the colors of the outer edges are c_2, c_3 and c_4 when the color of their incident inner face is respectively c_3, c_4 and c_2.

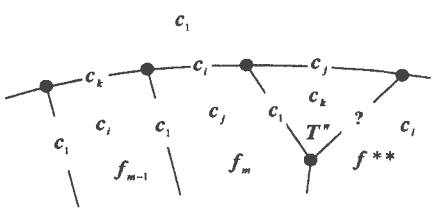

Figure 4: No proper 4-edge-face total coloring of G

Finally, suppose the colors of f_{m-1}, f_m and T'' are c_i, c_j and c_k respectively, where none of i, j, k is equal to 1. Then the colors of the outer edges of f_{m-1}, f_m and T'' must be, respectively, c_k, c_i and c_j (see Figure 4). Hence the color of f^{**} which is the other neighboring face of T'' should be c_i, and the two inner edges of the triangle T'' should be colored with c_1, a contradiction. ▮

Acknowledgment

The authors wish to thank the anonymous reviewer for the constructive suggestions and insightful comments that have helped improve the presentation of this paper.

References

[1] J. A. BONDY and U. S. R. MURTY, *Graph theory with applications*, The Macmillan Press Ltd, London, (1976).

[2] O. V. BORODIN, Simultaneous colorings of graphs on the plane, *Metody Diskretn. Anal.*, **45** 21-27 (1987).

[3] C. F. CHANG, J. X. CHANG, X. C. LU, P. C. B. LAM, J. F. WANG, Edge-face total chromatic number of outerplanar graphs with $\Delta(G) = 6$. In *Lecture Notes in Computer Science 959 - Proceedings of the First Annual International Conference on Computing and Combinatorics*, (Edited by D.F. Du and M. Li), pp. 396-399, Springer, New York, (1995).

[4] G. Z. HU and Z. F. ZHANG, On the edge-face total colorings of planar graphs, *J. Tsinghua Univ.*, **32**(3) 18-23 (1992).

[5] P. C. B. LAM, W. C. SHIU and W. H. CHAN, Edge-face total chromatic number of 3-regular Halin graphs, *Congressus Numerantium*, **145** 161-165 (2000).

[6] C. LIN, G. Z. HU, and Z. F. ZHANG, A six-color theorem for the edge-face coloring of plane graphs, *Discrete Mathematics*, **141** 291-297 (1995).

[7] R. LUO and C. Q. ZHANG, Edge-face chromatic number and edge chromatic number of simple plane graphs, *Journal of Graph Theory*, **49**(3) 234-256 (2005).

[8] L. S. MELNIKOV, Problem 9, Recent advances in graph theory, *Academic Praha.* **543** (1975).

[9] W. WANG and K. W. LIH, A new proof of Melnikov's conjecture on the edge-face coloring of plane graphs, *Discrete Mathematics*, **253** 87-95 (2002).