The neighbor expanded sum distinguishing index of Halin graphs*

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Abstract

Let $G = (V,E)$ be a graph. A total k-weighting c of G is a function $c : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$. For $x \in V(G)$, define $w(x) = \sum_{y\in N(x)}(c(xy)+c(y))$. A total k-weighting c of G is called neighbor expanded sum distinguishing (nesd for short) if $w(u) \neq w(v)$ for every $uv \in E(G)$. The smallest value of k for which such a nesd total k-weighting of G exists is called the neighbor expanded sum distinguishing index of G, denoted by $\mathrm{nesd}_t(G)$. In this paper, we prove that $\mathrm{nesd}_t(G) \leq 3$ for any Halin graph G. Furthermore, $\mathrm{nesd}_t(G) = 2$ for the cubic Halin graphs G.

Key words: Halin graphs, total weighting, neighbor expanded sum distinguishing index.
AMS Subject Classifications: 05C15.

1 Introduction

Graphs considered in this paper are finite and simple. The terminologies and notation used but undefined in this paper can be found in [1]. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively.

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We call $T(G) = V(G) \cup E(G)$ the total set of G. Let $N(v)$ denote the set of neighbors of v in G and let $d(v) = |N(v)|$ denote the degree of v in G. Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of a vertex in G, respectively. A 3-regular graph is called a *cubic* graph. For $n \geq 3$, an n-cycle is a cycle of length n. An n-cycle is odd (or even) if n is odd (or even). For every $n \geq 3$, a *wheel* W_n is the graph obtained from an n-cycle by adding a new vertex w and joining it to every vertex of the n-cycle. Such vertex w is called the *center* of the wheel. A wheel W_n is odd (or even) if n is odd (or even).

A Halin graph G is a plane graph constructed as follows. Let T be a tree on at least 4 vertices. All vertices of T are either of degree 1, called leaves, or of degree at least 3. Let C be a cycle connecting the leaves of T in such a way that C forms the boundary of the unbounded face. The tree T and the cycle C are called the *characteristic tree* and the *adjoint cycle* of G, respectively. For simplicity, we usually write $G = T \cup C$. The vertices in $V(C)$ and in $V(G) \setminus V(C)$ are called *outer vertices* and *inner vertices* of G, respectively. Let $V_{in}(G)$ denote the set of inner vertices in a Halin graph G. If $|V_{in}(G)| = 1$, then G is just a wheel.

Let $[k]$ denote the set $\{1, 2, \ldots, k\}$. A *vertex k-weighting* c of G is a function $c : V(G) \to [k]$. A vertex k-weighting c of G is called proper if $c(u) \neq c(v)$ for every $uv \in E(G)$. A graph G is k-colorable if there exists a proper vertex k-weighting. The chromatic number $\chi(G)$ of G is the smallest value of k such that a graph G is k-colorable.

An *edge k-weighting* c of G is a function $c : E(G) \to [k]$. For a vertex $x \in V(G)$, we define $\sigma(x) = \sum\limits_{y \in N(x)} c(xy)$. An edge k-weighting c of G is called *neighbor sum distinguishing* if $\sigma(u) \neq \sigma(v)$ for every $uv \in E(G)$. Let $\mathrm{nsd}_e(G)$ be the smallest value of k so that G admits a neighbor sum distinguishing edge k-weighting.

A graph G is *nice* if no component is isomorphic to K_2. In [5], Karoński, Łuczak and Thomason proposed the following "1-2-3 Conjecture".

Conjecture 1 *If G is a nice graph, then* $\mathrm{nsd}_e(G) \leq 3$.

The best known upper bound on $\mathrm{nsd}_e(G)$ is due to Kalkowski et al., who showed that $\mathrm{nsd}_e(G) \leq 5$ if G is nice (see [4]).

A *total k-weighting* c of G is a function $c : V(G) \cup E(G) \to [k]$. For a vertex $x \in V(G)$, we define $t(x) = c(x) + \sigma(x) = c(x) + \sum\limits_{y \in N(x)} c(xy)$. A total k-weighting c of G is called *neighbor sum distinguishing* if $t(u) \neq t(v)$ for every $uv \in E(G)$. Let $\mathrm{nsd}_t(G)$ be the smallest value of k so that G admits a neighbor sum distinguishing total k-weighting.

In [6], Przybyło and Woźniak posed the following "1-2 conjecture".

Conjecture 2 *If G is a connected graph, then* $\mathrm{nsd}_t(G) \leq 2$.

It is known that for every graph G, $\mathrm{nsd}_t(G) \leq 3$ (see [3]).

Recently, Flandrin et al. [2] introduced the neighbor expanded sum distinguishing index. For a vertex $x \in V(G)$, define

$$w(x) = \sum_{y \in N(x)} (c(xy) + c(y)),$$

where c is a total k-weighting of G. The value $w(x)$ is called an *expanded sum at x*. A total k-weighting c of G is called *neighbor expanded sum distinguishing* (nesd for short) if $w(u) \neq w(v)$ for every $uv \in E(G)$. The smallest value of k so that G admits a nesd total k-weighting of G is called the *neighbor expanded sum distinguishing index* of G, and denoted it by $\mathrm{nesd}_t(G)$.

In the same paper, they completely determined the neighbor expanded sum distinguishing indices for paths, cycles, trees, complete graphs and complete bipartite graphs. Furthermore, they proposed the following conjecture.

Conjecture 3 *For every graph G,* $\mathrm{nesd}_t(G) \leq 2$.

In this paper, we investigate the neighbor expanded sum distinguishing index of Halin graphs. To prove our main results, we need some useful results as follows.

Lemma 1.1 ([2]) *Let k be an odd integer and let G be a connected k-colorable graph. Then* $\mathrm{nesd}_t(G) \leq k$.

Lemma 1.2 ([7]) *If G is a Halin graph, then $\chi(G) \leq 4$ with equality if and only if G is isomorphic to an odd wheel.*

2 Halin graphs

In the section, we give an upper bound on the neighbor expanded sum distinguishing index of Halin graphs.

Theorem 2.1 *For $n \geq 3$,* $\mathrm{nesd}_t(W_n) = 2$.

Proof. Let w be the center of W_n and $C = v_1v_2\cdots v_n$ be the adjoint cycle of W_n. For convenience, we let $e_i = v_iv_{i+1}$, for $1 \leq i \leq n-1$ and $e_n = v_nv_1$. The proof is split into the following cases:

Case 1: $n = 3m$.

For $m = 1$, we weight the center w with 2 and all the edges of T with 1. Then weight the sequence of $v_1, e_1, v_2, e_2, v_3, e_3$ by $1, 2, 2, 1, 1, 2$ in order. Then the expanded sum at w is 7 and the expanded sums of v_1, v_2, v_3 are 10,8,9, respectively.

For $m \geq 2$, we weight the center w and all edges of T with 1. Then weight the vertex-edge alternating sequence of v_1, e_1, v_2, e_2, ...,

v_{3m}, e_{3m} by 1, 2, 2, 1, 1, 2 cyclically. Then the expanded sums of the vertices $v_1, v_2, \ldots, v_{3m}$ are 9, 7, 8 cyclically, and the expanded sum at w is $3m + 4m = 7m \geq 14$.

Case 2: $n = 3m + 1$.

We weight w and wv_{3m+1} with 1, and all other edges of T with 2. Then weight the vertex-edge alternating sequence of v_1, e_1, v_2, e_2, $\ldots$, v_{3m+1}, e_{3m+1} by 1, 2, 2, 1, 1, 2 cyclically. Then the expanded sums of the vertices $v_1, v_2, \ldots, v_{3m}$ are 10, 8, 9 cyclically, the expanded sum at v_{3m+1} is 8, and the expanded sum at w is $6m + 1 + 4m + 1 = 10m + 2 \geq 12$.

Case 3: $n = 3m + 2$.

We weight w and all edges of T with 1. Then weight the vertex-edge alternating sequence of $v_1, e_1, v_2, e_2, \ldots, v_{3m+2}, e_{3m+2}$ by 1, 2, 2, 1, 1, 2 cyclically. Then the expanded sums of the vertices $v_2, v_3, \ldots, v_{3m+1}$ are 7, 8, 9 cyclically, the expanded sum at v_1 and v_{3m+2} are 9 and 7, respectively, and the expanded sum at w is $3m + 2 + 4m + 3 = 7m + 5 \geq 12$.

This completes the proof. ■

Theorem 2.2 *For any Halin graph G,* $\mathrm{nesd}_t(G) \leq 3$.

Proof. If G is a wheel, then by Theorem 2.1, we obtain that $\mathrm{nesd}_t(G) = 2$. Otherwise, we can get $\mathrm{nesd}_t(G) \leq 3$ by Lemmas 1.1 and 1.2. Hence the theorem holds. ■

3 Cubic Halin graphs

In the section, we determine exactly the neighbor expanded sum distinguishing index of cubic Halin graphs.

Lemma 3.1 *Let $G = T \cup C$ be a cubic Halin graph with $|V_{in}(G)| \geq 5$. Then G contains H_1 or H_2 of Fig. 1 as a subgraph.*

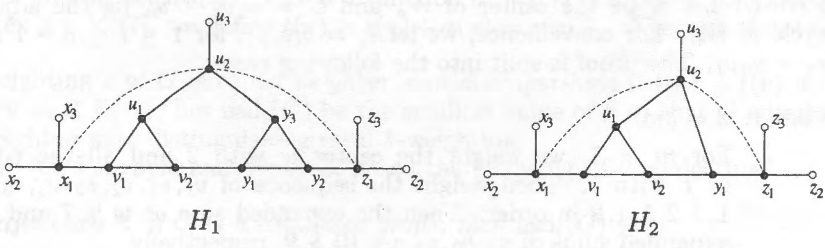

Figure 1: Two subgraphs of a Halin graph.

Proof. Let $P = u_0u_1 \cdots u_l$ be a longest path of T. Then $l \geq 4$. Let $N(u_1) = \{u_2, v_1, v_2\}$. Then v_1 and v_2 are leaves of T. Since $d(u_2) = 3$. Let $N(u_2) = \{u_1, u_3, x\}$. Suppose x is not a leaf of T. By means of the longest path P, x is adjacent to two leaves, say y_1 and y_2. After renaming suitably, we obtain the subgraph H_1 described in Fig. 1. Suppose x is a leaf of T. We rename it by y_1. After renaming suitably, we get the subgraph H_2 described in Fig. 1. ∎

Suppose c is a total 2-weighting of a cubic graph G. Let $w^c(x)$ denote the expanded sum at x under c, for $x \in V(G)$. Let $c^* = 3 - c$. That means that c^* swaps the weights 1 to 2 and 2 to 1 assigned by c, respectively. Clearly $w^{c^*}(x) = 18 - w^c(x)$. Thus c is a nesd total 2-weighting of G if and only if c^* is a nesd total 2-weighting of G. We call c^* the *dual weighting* of c.

Theorem 3.2 *For any cubic Halin graph* $G = T \cup C$, $\text{nesd}_t(G) = 2$.

Proof. We prove the theorem by induction on the number of the inner vertices. If $|V_{in}(G)| = 1$, then G is a wheel and we have done by Theorem 2.1. If $2 \leq |V_{in}(G)| \leq 4$, then there are only five Halin graphs G, depicted in Fig. 2, in which a nesd total 2-weighting is established.

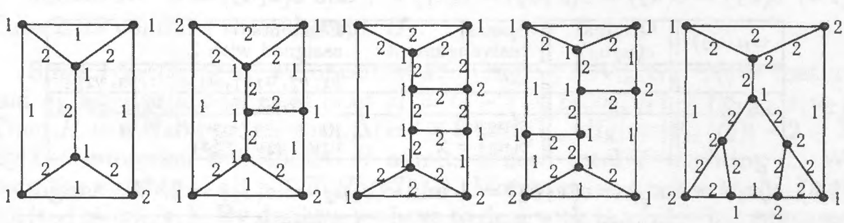

Figure 2: Five cubic Halin graphs of small order.

Now assume that $|V_{in}(G)| \geq 5$. By Lemma 3.1, G contains subgraph H_1 or H_2 described in Fig. 1. We shall keep the notation depicted in Fig. 1.

Case 1: G contains the subgraph H_1.

Since $|V_{in}(G)| \geq 5$, $u_1, u_2, u_3, y_3 \in V_{in}(G)$. Note that x_1 and z_1 are distinct vertices of C. Let $H = G - \{u_1, v_1, v_2, y_1, y_2, y_3\} + \{x_1u_2, z_1u_2\}$, Then H is a Halin graph with $\Delta(H) = 3$ and $|V_{in}(H)| = |V_{in}(G)| - 3 \geq 2$. By the induction hypothesis, H admits a nesd total 2-weighting c. We keep the weights of all $x \in T(H) \cap T(G)$. Moreover, $w_H(u_2) \neq w_H(x_1)$ and $w_H(u_2) \neq w_H(z_1)$, where $w_H(x)$ is the expanded sum in H at x. For each case, $w_H(u_2)$ has 3 possible cases which will be shown at the following tables. By symmetry and duality, we have to deal with the following subcases.

(1.1) $c(x_1) = c(u_2) = c(z_1) = c(x_1u_2) = c(z_1u_2) = 1$. Hence $6 \leq w_H(u_2) \leq 8$. We assign:

Conditions		Original	Assignments	
$w_H(u_2)$	$w_H(z_1)$	$c(u_2u_3)$	Special rearrangement	Elements assigned with 2
6	$\neq 9$			$y_1, u_1v_2, y_1y_2, y_2y_3$
	$= 9$			$y_1, u_1v_2, y_1y_3, y_2y_3$
7	$\neq 9$			$y_1, v_1v_2, y_1y_2, y_2y_3$
	$= 9$			$y_1, v_1v_2, y_1y_3, y_2y_3$
8		2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$v_2, u_1v_1, u_1v_2, u_2y_3$, y_1y_2, y_1y_3, y_2y_3

(1.2) $c(x_1) = c(u_2) = c(x_1u_2) = c(z_1u_2) = 1$ and $c(z_1) = 2$. We assign:

Condition	Assignment
$w_H(u_2)$	Elements assigned with 2
7	$y_1, y_3, v_1v_2, v_2y_1, y_1y_2, y_1y_3, y_2y_3$
8	$v_2, y_3, u_1v_1, u_1v_2, y_1y_3$
9	$v_2, y_3, u_1v_1, v_1v_2, y_2y_3$

(1.3) $c(x_1) = c(z_1) = c(x_1u_2) = c(z_1u_2) = 1$ and $c(u_2) = 2$. We assign:

$w_H(u_2)$	Elements assigned with 2
6	v_1, y_2, y_1y_3
7	$v_1, y_2, u_1v_1, v_2y_1, y_1y_3, y_2y_3$
8	$v_1, v_2, y_1, y_2, u_1v_2, v_1v_2, v_2y_1, y_2y_3$

(1.4) $c(x_1) = c(u_2) = c(x_1u_2) = c(z_1) = 1$ and $c(z_1u_2) = 2$. We assign:

$w_H(u_2)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
7			$y_1, y_2, u_1v_1, u_1v_2, u_2y_3, v_2y_1$
8			$v_2, y_2, y_3, u_1v_1, u_1v_2, y_1y_2$
9	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$u_1, y_1, u_1v_1, u_2y_3, v_1v_2$, v_2y_1, y_2y_3, y_2z_1

(1.5) $c(x_1) = c(u_2) = c(x_1u_2) = 1$ and $c(z_1u_2) = c(z_1) = 2$. We assign:

$w_H(u_2)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
8			$u_1, y_1, y_2, u_2y_3, v_1v_2, v_2y_1, y_1y_3$
9			$u_1, v_2, u_1v_1, u_2y_3, y_1y_2, y_2z_1$
10	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$u_1, v_2, u_1u_2, u_1v_1, u_1v_2, u_2y_3$, $v_1v_2, y_1y_2, y_2y_3, y_2z_1$

(1.6) $c(x_1) = c(u_2) = c(z_1u_2) = 1$ and $c(x_1u_2) = c(z_1) = 2$. We assign:

$w_H(u_2)$	Elements assigned with 2
8	$u_1, v_1, v_1v_2, u_2y_3, y_1y_3, y_2y_3$
9	$u_1, y_1, u_1v_1, u_1v_2, u_2y_3, v_1x_1, v_2y_1, y_1y_3, y_2y_3$

$w_H(u_2)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
10	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$u_1, v_2, y_3, x_1v_1, v_1v_2$, u_1u_2, y_1y_2, y_2y_3

(1.7) $c(x_1) = c(x_1u_2) = c(z_1) = 1$ and $c(z_1u_2) = c(u_2) = 2$. We assign:

$w_H(u_2)$	Elements assigned with 2
7	$v_1, y_2, u_1v_1, v_2y_1, y_1y_3, u_2y_3, y_2z_1$
8	$v_1, v_2, y_2, u_1v_1, u_1v_2, v_2y_1, y_1y_2, y_1y_3, y_2z_1, u_2y_3$
9	$v_2, y_2, y_3, x_1v_1, u_1v_1, u_1v_2, v_2y_1, y_1y_3, y_2y_3, y_2z_1$

(1.8) $c(x_1) = c(u_2) = c(z_1) = 1$ and $c(x_1u_2) = c(z_1u_2) = 2$. We assign:

$w_H(u_2)$	Elements assigned with 2
8	$u_1, v_1, y_3, v_1v_2, v_2y_1, y_1y_3, y_2z_1$
9	$u_1, y_1, u_1v_1, u_1v_2, u_2y_3, v_1x_1, v_2y_1, y_1y_3, y_2y_3, y_2z_1$

$w_H(u_2)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
10	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$u_1, v_2, y_3, u_1u_2, v_1v_2, v_1x_1, y_1y_2, y_2y_3, y_2z_1$

(1.9) $c(x_1) = c(x_1u_2) = c(z_1u_2) = 1$ and $c(u_2) = c(z_1) = 2$. We assign:

$w_H(u_2)$	Elements assigned with 2
7	$v_1, y_1, y_3, u_1v_2, v_1v_2, y_1y_2, y_2y_3, y_2z_1$
8	$v_2, y_2, u_1v_2, u_2y_3, v_1x_1, y_1y_3, y_2y_3$
9	$v_2, y_2, y_3, v_1v_2, v_1x_1, y_1y_2$

(1.10) $c(u_2) = c(x_1u_2) = c(z_1u_2) = 1$ and $c(x_1) = c(z_1) = 2$. We assign:

$w_H(u_2)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
8			$v_2, y_3, u_1u_2, u_1v_2, y_1y_3$
9			$u_1, v_2, y_3, v_2y_1, y_1y_2, y_1y_3$
10	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$u_1, v_2, y_3, u_1u_2, v_1v_2, y_1y_2, y_2y_3$

One can check that each total weighting c is a nesd total 2-weighting.

Case 2: G contains the subgraph H_2.

Since $|V_{in}(G)| \geq 5$, we obtain that $u_1, u_2, u_3 \in V_{in}(G)$. Note that x_1 and z_1 are distinct vertices. Let $H = G - \{u_1, v_1, v_2, y_1\} + \{x_1u_2, z_1u_2\}$. Then H is a Halin graph with $\Delta(H) = 3$ and $|V_{in}(H)| = |V_{in}(G)| - 2 \geq 3$. By the induction hypothesis, H admits a nesd total 2-weighting c. We keep the weights of all $x \in T(H) \cap T(G)$. Moreover, $w_H(u_2) \neq w_H(x_1)$ and $w_H(u_2) \neq w_H(z_1)$. By duality, we have to deal with the following subcases.

(2.1) $c(x_1) = c(u_2) = c(z_1) = c(x_1u_2) = c(z_1u_2) = 1$. We assign:

Conditions				Assignments
$w_H(u_2)$	$w_H(z_1)$	$w_H(u_3)$	$w_H(x_1)$	Elements assigned with 2
6	$\neq 7$			u_1v_2, v_2y_1
	$= 7$	$\neq 7$		u_1u_2, u_1v_2
	$= 7$	$= 7$	$\neq 8$	$u_1, u_1u_2, u_2y_1, v_1v_2, v_2y_1$
	$= 7$	$= 7$	$= 8$	$v_2, u_1u_2, u_1v_2, u_2y_1, v_2y_1$

Conditions		Original	Assignments	
$w_H(u_2)$	$w_H(u_3)$	$c(u_2u_3)$	Special rearrangement	Elements assigned with 2
7	$\neq 8$			$u_1u_2, u_1v_1, u_1v_2, v_2y_1$
	$= 8$			$u_1, u_1u_2, u_1v_2, v_2y_1$
8	$\neq 10$			$v_2, u_1u_2, u_1v_1, u_2y_1$
	$= 10$	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	v_2, u_1v_1

(2.2) $c(u_2) = c(x_1u_2) = c(z_1u_2) = c(z_1) = 1$ and $c(x_1) = 2$. We assign:

$w_H(u_2)$	$w_H(z_1)$	$w_H(x_1)$	Elements assigned with 2
7	$\neq 8$		$u_1, v_2, u_1v_1, v_1v_2, u_1v_2, v_2y_1$
	$= 8$	$\neq 10$	$u_1, u_1v_1, u_1v_2, v_1v_2$
	$= 8$	$= 10$	u_1v_1, u_1u_2, v_1v_2
8	$\neq 7$		u_1, u_1v_2, v_2y_1
	$= 7$		u_1u_2, u_1v_1, u_1v_2

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
9	$\neq 8$	2	$c(u_2u_3) = 1, c(u_2) = 2$	$v_2, u_1v_1, u_1v_2, u_2y_1$
	$= 8$			$v_2, u_1v_1, u_1v_2, v_2y_1$

(2.3) $c(x_1) = c(z_1) = c(x_1u_2) = c(z_1u_2) = 1$ and $c(u_2) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	$w_H(z_1)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
6	$\neq 8$				v_1, y_1
	$= 8$	$\neq 8$	1	$c(u_2u_3) = 2$, $c(u_2) = 1$	$v_1, u_1u_2, u_2y_1, y_1z_1$
	$= 8$	$= 8$			v_1, y_1, u_2y_1

$w_H(u_2)$	$w_H(u_3)$	Elements assigned with 2
7	$\neq 8$	v_1, y_1, u_1v_1
	$= 8$	v_1, y_1, u_1u_2, u_1v_1
8	$\neq 10$	$u_1, v_1, y_1, u_1v_1, v_2y_1$
	$= 10$	$u_1, v_1, u_1v_1, u_1v_2, y_1z_1$

(2.4) $c(x_1) = c(u_2) = c(x_1u_2) = c(z_1u_2) = 1$ and $c(z_1) = 2$. We assign:

$w_H(u_2)$	$w_H(z_1)$	Elements assigned with 2
7	$\neq 9$	u_2y_1, v_1v_2, v_2y_1
	$= 9$	$u_1u_2, u_1v_2, v_1v_2, v_2y_1$
8	$\neq 9$	$v_2, u_1u_2, u_1v_1, v_2y_1$
	$= 9$	u_1, u_1v_2, v_1v_2

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
9	$\neq 10$			$u_1, u_1v_1, u_1v_2,$ u_2y_1, v_1v_2, v_2y_1
	$= 10$	2	$c(u_2u_3) = 1, c(u_2) = 2$	$v_2, u_1u_2, u_1v_1,$ u_1v_2, v_1v_2

(2.5) $c(x_1) = c(u_2) = c(z_1u_2) = c(z_1) = 1$ and $c(x_1u_2) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
7	$\neq 6$			$v_1, u_1v_2, v_1v_2, v_2y_1$
	$= 6$			$v_1, u_1u_2, u_1v_1, u_1v_2, u_2y_1$
8	$\neq 7$			$v_1, v_2, u_1v_1, u_1v_2, v_2y_1$
	$= 7$			$u_1, v_1, , u_2y_1, v_1v_2, v_2y_1$
9	$\neq 11$			$u_1, v_1, v_2, u_1u_2, u_1v_2,$ u_2y_1, v_1v_2, v_2y_1
	$= 11$	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$v_2, u_1v_1, u_1v_2,$ v_1x_1, v_2y_1

(2.6) $c(x_1) = c(u_2) = c(x_1u_2) = c(z_1) = 1$ and $c(z_1u_2) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
7	$\neq 8$			y_1, u_2y_1, v_1v_2
	$= 8$			u_1, y_1, u_2y_1
8	$\neq 9$			$u_1, u_1u_2, v_1v_2, v_2y_1, y_1z_1$
	$= 9$			$v_2, u_1v_1, u_1v_2, y_1z_1$
9	$\neq 10$			$u_1, u_1v_1, u_1v_2, v_1v_2, v_2y_1, u_2y_1, y_1z_1$
	$= 10$	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$v_2, u_1u_2, u_1v_1, u_1v_2, v_1v_2, y_1z_1$

(2.7) $c(x_1u_2) = c(z_1u_2) = c(z_1) = 1$ and $c(u_2) = c(x_1) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
7	$\neq 8$			v_1, y_1, u_1u_2
	$= 8$	1	$c(u_2u_3) = 2$, $c(u_2) = 1$	v_1, y_1, u_1v_2, u_2y_1
8	$\neq 10$			$u_1, v_1, y_1, u_1u_2, v_2y_1$
	$= 10$			v_1, u_1v_1, y_1z_1
9	$\neq 12$			$u_1, v_1, v_2, y_1, u_1u_2, u_1v_2, u_2y_1$
	$= 12$			v_1, v_2, u_1v_1, y_1z_1

(2.8) $c(u_2) = c(x_1u_2) = c(z_1u_2) = 1$ and $c(x_1) = c(z_1) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
8	$\neq 6$			u_1v_2, v_1v_2, v_2y_1
	$= 6$			u_2y_1, v_1v_2
9	$\neq 10$			$u_1, u_1u_2, u_1v_2, u_2y_1, v_1v_2, v_2y_1$
	$= 10$			$v_2, u_1u_2, u_1v_1, u_1v_2, v_2y_1$
10	$\neq 11$			$u_1, v_2, u_1u_2, u_2y_1, v_1v_2, v_2y_1$
	$= 11$	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$v_2, u_1v_1, v_1v_2, v_2y_1$

(2.9) $c(x_1) = c(x_1u_2) = c(z_1u_2) = 1$ and $c(z_1) = c(u_2) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
7	$\neq 9$	1	$c(u_2u_3) = 2$, $c(u_2) = 1$	$v_1, y_1, u_1u_2, u_1v_1, u_1v_2$
	$= 9$			u_1, v_1, y_1, u_1u_2
8	$\neq 9$			$v_1, y_1, u_1u_2, u_1v_1, u_1v_2, v_1v_2$
	$= 9$			$u_1, v_1, y_1, u_1u_2, u_1v_1, u_1v_2$
10	$\neq 10$			$u_1, y_1, u_1v_2, v_1v_2, v_1x_1, v_2y_1$
	$= 10$			$u_1, v_1, y_1, u_1u_2, u_1v_1, v_1v_2, v_2y_1$

(2.10) $c(u_2) = c(z_1u_2) = c(z_1) = 1$ and $c(x_1) = c(x_1u_2) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	$w_H(x_1)$	Elements assigned with 2
8	$\neq 7$		u_2y_1, v_1x_1, v_2y_1
	$= 7$	$\neq 10$	$v_2, u_1v_1, v_1x_1, v_2y_1$
	$= 7$	$= 10$	$u_1, v_1, v_2, u_1v_1, u_1u_2, u_2y_1, v_1v_2$

$w_H(u_2)$	$w_H(z_1)$	Elements assigned with 2
9	$\neq 8$	$v_1, v_2, u_1u_2, u_1v_1, u_2y_1$
	$= 8$	$u_1, u_1u_2, v_1x_1, v_2y_1$

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
10	$\neq 8$	2	$c(u_2u_3)=1$, $c(u_2)=2$	$v_2, u_1v_1, u_2y_1, v_1x_1, v_2y_1$
	$=8$			$v_1, v_2, u_1u_2, u_1v_1, u_2y_1, v_1v_2, v_2y_1$

(2.11) $c(u_2) = c(x_1u_2) = c(z_1) = 1$ and $c(x_1) = c(z_1u_2) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
8	$\neq 7$			u_2y_1, v_1v_2, y_1z_1
	$=7$			$y_1, u_1u_2, u_2y_1, v_1v_2, v_2y_1$
9	$\neq 8$			$v_2, u_1u_2, u_1v_1, u_1v_2, v_2y_1, y_1z_1$
	$=8$			$u_1, v_2, y_1, u_1v_2, u_2y_1, v_2y_1$
10	$\neq 11$			$u_1, v_2, u_1u_2, u_1v_1, u_2y_1, v_2y_1, y_1z_1$
	$=11$	2	$c(u_2u_3)=1$, $c(u_2)=2$	$v_2, u_1v_1, v_1v_2, v_2y_1, y_1z_1$

(2.12) $c(x_1) = c(z_1u_2) = c(z_1) = 1$ and $c(x_1u_2) = c(u_2) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
7	$\neq 8$	1	$c(u_2u_3)=2$, $c(u_2)=1$	v_1, y_1, u_1u_2, v_1x_1
	$=8$			$v_1, y_1, u_1u_2, v_1x_1, v_2y_1$

$w_H(u_2)$	$w_H(u_3)$	Elements assigned with 2
8	$\neq 9$	$v_1, v_2, y_1, u_1v_2, u_1u_2, v_1x_1$
	$=9$	$v_1, u_1v_1, v_1x_1, y_1z_1$
9	$\neq 10$	$v_1, v_2, y_1, u_1u_2, u_1v_1, u_1v_2, v_1x_1, v_2y_1$
	$=10$	$u_1, v_1, y_1, u_1u_2, u_1v_2, u_2y_1, v_1v_2, v_1x_1, v_2y_1$

(2.13) $c(x_1) = c(x_1u_2) = c(z_1) = 1$ and $c(z_1u_2) = c(u_2) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
7	$\neq 10$	1	$c(u_2u_3)=2$, $c(u_2)=1$	$u_1, v_1, y_1, u_1u_2, y_1z_1$
	$=10$			$v_1, y_1, u_1u_2, u_1v_1, u_1v_2, y_1z_1$

$w_H(u_2)$	$w_H(u_3)$	Elements assigned with 2
8	$\neq 10$	$u_1, v_1, y_1, u_1u_2, v_1v_2, y_1z_1$
	$=10$	$v_1, y_1, u_1u_2, u_1v_1, v_1v_2, y_1z_1$
9	$\neq 10$	$v_1, v_2, y_1, u_1u_2, u_1v_1, u_1v_2, v_1v_2, y_1z_1$
	$=10$	$u_1, y_1, u_1v_2, u_2y_1, v_1v_2, v_1x_1, y_1z_1$

(2.14) $c(x_1) = c(u_2) = c(z_1u_2) = 1$ and $c(x_1u_2) = c(z_1) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Elements assigned with 2
8	$\neq 7$	u_2y_1, v_1v_2, v_1x_1
	$=7$	$u_1v_2, v_1v_2, v_1x_1, v_2y_1$
9	$\neq 8$	$v_2, u_1u_2, u_1v_1, u_1v_2, v_1x_1, v_2y_1$
	$=8$	$u_1, u_1u_2, u_1v_2, u_2y_1, v_1v_2, v_1x_1, v_2y_1$

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
10	$\neq 11$			$u_1, v_2, u_1u_2, u_1v_1, u_2y_1, v_1x_1, v_2y_1$
	$=11$	2	$c(u_2u_3)=1$, $c(u_2)=2$	$v_2, u_1v_1, u_2y_1, v_1v_2, v_1x_1$

(2.15) $c(x_1) = c(u_2) = c(x_1u_2) = 1$ and $c(z_1u_2) = c(z_1) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Elements assigned with 2
8	$\neq 9$	$u_1, y_1, u_1v_2, u_2y_1, v_1v_2$
	$= 9$	$u_1, y_1, u_1u_2, u_2y_1, v_1v_2$
9	$\neq 10$	$u_1, y_1, u_1v_1, u_1v_2, u_2y_1, v_1v_2, v_2y_1$
	$= 10$	$v_2, u_1v_1, v_1v_2, y_1z_1$

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
10	$\neq 11$			$u_1, v_2, u_1u_2, u_1v_1, u_2y_1, v_1v_2, y_1z_1$
	$= 11$	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$u_1, v_2, u_1v_1, v_1v_2, y_1z_1$

(2.16) $c(x_1) = c(u_2) = c(z_1) = 1$ and $c(x_1u_2) = c(z_1u_2) = 2$. We assign:

$w_H(u_2)$	$w_H(u_3)$	Elements assigned with 2
8	$\neq 6$	$u_1v_2, v_1v_2, v_1x_1, v_2y_1, y_1z_1$
	$= 6$	$u_1, v_1, y_1, u_1u_2, u_1v_2, v_1v_2, u_2y_1, v_2y_1$
9	$\neq 8$	$u_1v_1, u_2y_1, v_1v_2, v_1x_1, v_2y_1, y_1z_1$
	$= 8$	$u_1, v_1, v_2, y_1, u_2y_1, v_1v_2, v_2y_1$

$w_H(u_2)$	$w_H(u_3)$	Original $c(u_2u_3)$	Special rearrangement	Elements assigned with 2
10	$\neq 9$	2	$c(u_2u_3) = 1$, $c(u_2) = 2$	$u_1, v_1, v_2, u_1u_2, u_1v_1, u_1v_2, v_1v_2, v_2y_1, y_1z_1$
	$= 9$			$v_2, u_1v_1, v_1v_2, v_1x_1, v_2y_1, y_1z_1$

One may check that each total weighting c defined above is a nesd total 2-weighting. This completes the proof. ∎

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