INTEGER-ANTIMAGIC SPECTRA OF TADPOLE AND LOLLIPOP GRAPHS

WAI CHEE SHIU, PAK KIU SUN, AND RICHARD M. LOW

ABSTRACT. Let A be a non-trival abelian group. A connected simple graph $G = (V, E)$ is A-antimagic if there exists an edge labeling $f : E(G) \to A \setminus \{0\}$ such that the induced vertex labeling $f^+ : V(G) \to A$, defined by $f^+(v) = \sum_{uv \in E(G)} f(uv)$, is injective. The integer-antimagic spectrum of a graph G is the set $\mathrm{IAM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$. In this article, we determine the integer-antimagic spectra of tadpole and lollipop graphs.

1. INTRODUCTION

Let G be a connected simple graphs. For any abelian group A (written additively), let $A^* = A \setminus \{0\}$, where 0 is the additive identity of A. Let a function $f : E(G) \to A^*$ be an edge labeling of G and $f^+ : V(G) \to A$ be its induced labeling, which is defined by $f^+(v) = \sum_{uv \in E(G)} f(uv)$. If there exists an edge labeling f whose induced labeling f^+ on $V(G)$ is injective, then we say that f is an *A-antimagic labeling* and that G is an *A-antimagic* graph. The *integer-antimagic spectrum* of a graph G is the set $\mathrm{IAM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$.

The concept of A-antimagicness property for a graph G (introduced in [1]) naturally arises as a variation of the A-magic labeling problem (where the induced vertex labeling is a constant map). $\mathbb{Z}$-magic (or $\mathbb{Z}_1$-magic) graphs were considered by Stanley in [25, 26], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [2, 3, 4] and others [7, 9, 15, 16,

Date: November 2, 2015.

2010 Mathematics Subject Classification. 05C78.

This work is partially supported by Faculty Research Grant, Hong Kong Baptist University.

21] have studied A-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [5, 6, 8, 10, 11, 12, 13, 14, 17, 18, 19, 20, 22, 23].

2. Some Known Results

The following three lemmas will be used throughout this article.

Lemma 2.1 ([1, Lemma 1]). *For $m \geq 1$, a graph of order $4m+2$ is not $\mathbb{Z}_{4m+2}$-antimagic.*

Lemma 2.2 ([1, Theorem 3]). *The path P_3 is $\mathbb{Z}_k$-antimagic for all $k \geq 3$, and the cycle C_3 is $\mathbb{Z}_k$-antimagic for all $k \geq 4$ but not for $k=3$.*

Lemma 2.3 ([1, Theorem 4]). *For $m \geq 1$, C_{4m+r} and P_{4m+r} are $\mathbb{Z}_k$-antimagic, for all $k \geq 4m+r$ if $r = 0,1,3$; C_{4m+2} and P_{4m+2} are $\mathbb{Z}_k$-antimagic, for all $k \geq 4m+3$.*

Also, we will use the following $\mathbb{Z}_k$-antimagic labelings g and f for paths and cycles, respectively, found in [1]. For integers $a \leq b$, let $[a, b]$ denote the set of integers from a to b, inclusive.

Remark 2.1. In this paper, we let $P_n = v_1v_2\cdots v_n$, and $e_1 = v_1v_2, e_2 = v_2v_3, \ldots, e_{n-1} = v_{n-1}v_n$ be its edges.

Case 1. $n = 4m, m \geq 1$:

$$g(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd;} \\ \frac{i}{2} & \text{if } i \text{ is even and } 2 \leq i \leq 2m-2; \\ \frac{i+2}{2} & \text{if } i \text{ is even and } 2m \leq i \leq 4m-2. \end{cases}$$

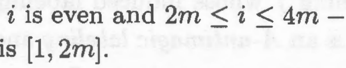
The range of g is $[1, 2m]$.

Case 2. $n = 4m+1$ with $m \geq 2$:

$$g(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even;} \\ \frac{i+3}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq 2m-3; \\ \frac{i+5}{2} & \text{if } i \text{ is odd and } 2m-1 \leq i \leq 4m-1. \end{cases}$$

The range of g is $[1, 2m+2]$.

The labeling for P_5 in [1] is not valid and we correct it as follows:



Case 3. $n = 4m+2, m \geq 1$:

$$g(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd;} \\ \frac{i+2}{2} & \text{if } i \text{ is even and } 2 \leq i \leq 2m-2; \\ \frac{i+4}{2} & \text{if } i \text{ is even and } 2m \leq i \leq 4m. \end{cases}$$

The range of g is $[1, 2m+2]$.

Case 4. $n = 4m+3, m \geq 1$:

$$g(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even;} \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq 2m-1; \\ \frac{i+3}{2} & \text{if } i \text{ is odd and } 2m+1 \leq i \leq 4m+1. \end{cases}$$

The range of g is $[1, 2m+2]$.

Case 5. $n = 2$: Even thought P_2 is not antimagic, we still define $g(v_1v_2) = 1$ in this article.

Case 6. $n = 3$: We define $g(v_1v_2) = 1$ and $g(v_2v_3) = 2$.

Remark 2.2. In this paper, we let $C_p = u_1u_2 \cdots u_nu_1$ and $e_1 = u_1u_2$, $e_2 = u_2u_3, \ldots, e_p = u_pu_1$ be its edges.

Case 1. $p = 4n, n \geq 1$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n; \\ 3 + 2(2n - \lceil \frac{i}{2} \rceil) & \text{if } 2n+1 \leq i \leq 4n. \end{cases}$$

Case 2. $p = 4n+1, n \geq 1$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n; \\ 3 + 2(2n - \lceil \frac{i}{2} \rceil) & \text{if } 2n+1 \leq i \leq 4n+1. \end{cases}$$

Case 3. $p = 4n+2, n \geq 1$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n+3; \\ 3 + 2(2n - \lceil \frac{i-2}{2} \rceil) & \text{if } 2n+4 \leq i \leq 4n+2. \end{cases}$$

Case 4. $p = 4n-1, n \geq 2$:

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n+1; \\ 3 + 2(2n - \lceil \frac{i+1}{2} \rceil) & \text{if } 2n+2 \leq i \leq 4n-1. \end{cases}$$

Case 5. $p = 3$: We label the edges of C_3 by 1, 2 and 3, hence $I_f(C_3) = [3, 5]$ and so C_3 is $\mathbb{Z}_k$-antimagic for $k \geq 4$.

Let ϕ be an edge labeling of G and ϕ^+ be its induced vertex labeling. Let

$$I_\phi(G) = \{\phi^+(v) \mid v \in V(G)\},$$

where G is the graph being considered.

For multi-sets S and T, we denote by $S \equiv T \pmod{k}$ if the sets S and T are equal after taking modulo k.

The following are some important properties of the labelings g and f.

Proposition 2.4. *All elements in $[a, b]$ are distinct after taking modulo k for $k \geq b - a + 1$.*

Corollary 2.5. *For $m \geq 1$ and labeling g for paths provided in Remark 2.1, we have $I_g(P_{4m}) = [1, 4m]$, $I_g(P_{4m+1}) = [2, 4m+2]$, $I_g(P_{4m+2}) = [1, 4m+3] \setminus \{2\}$, $I_g(P_{4m-1}) = [1, 4m-1]$ and $I_g(P_2) = \{1, 1\}$ (a multiset). Moreover,*

$$\begin{cases} g^+(v_1) = 1, & g^+(v_{4m}) = 2m, & \text{for } P_{4m}; \\ g^+(v_1) = 2, & g^+(v_{4m+1}) = 2m, & \text{for } P_{4m+1} \text{ and } m \geq 2; \\ g^+(v_1) = 1, & g^+(v_{4m+2}) = 2m+1, & \text{for } P_{4m+2}; \\ g^+(v_1) = 1, & g^+(v_{4m+3}) = 2m+1, & \text{for } P_{4m+3}; \\ g^+(v_1) = 2, & g^+(v_5) = 3, & \text{for } P_5; \\ g^+(v_1) = 1, & g^+(v_3) = 2, & \text{for } P_3; \\ g^+(v_1) = 1, & g^+(v_2) = 1, & \text{for } P_2. \end{cases}$$

Corollary 2.6. *For $n \geq 1$ and labeling f for cycles provided in Remark 2.2, we have $I_f(C_{4n-1}) = [3, 4n+1]$, $I_f(C_{4n}) = [3, 4n+2]$, $I_f(C_{4n+1}) = [2, 4n+2]$ and $I_f(C_{4n+2}) = [3, 4n+5] \setminus \{4n+2\}$.*

Theorem 2.7 ([24]). *Suppose $h : E(G) \to [1, p]$ is a labeling of a graph G of order p such that $h^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $p \equiv 2 \pmod 4$ and $a < b$. Then, $b - a$ is odd.*

Theorem 2.8 ([24]). *Suppose $h : E(G) \to [1, p]$ is a labeling of a graph G of order p such that $h^+ : V(G) \to [b-p+1, b]$ is bijective, where $p \equiv 1 \pmod 4$. Then, b must be even.*

Theorem 2.9 ([24]). *Suppose $h : E(G) \to [1, p]$ is a labeling of a graph G of order p such that $h^+ : V(G) \to [b-p+1, b]$ is bijective, where $p \equiv 3 \pmod 4$. Then, b must be odd.*

3. Some Useful Lemmas

For $S \subset \mathbb{Z}$ and $a \in \mathbb{Z}$, we define the set $a + S = \{a + s \mid s \in S\}$.

Lemma 3.1. *Let $g : E(P_{2n}) \to \mathbb{Z}$ be a labeling and $c \in \mathbb{Z}$. There exists a labeling h such that $I_h(P_{2n}) = c + I_g(P_{2n})$, $h^+(v_1) = c + 1$ and $h^+(v_{2n}) = n + c$. Note that the range of h is a subset of $[1, n+1] \cup [c+1, c+n]$.*

Proof. Relabel the edge v_iv_{i+1} by $g(v_iv_{i+1}) + c$, for odd i, and leave the other labels unchanged. This yields h. □

Lemma 3.2. *Suppose $n \geq 2$. For $d \in \{2\ell \mid 1 \leq \ell \leq n\} \cup \{2\ell + 1 \mid n \leq \ell \leq 2n\}$, there is a vertex $v \in V(P_{4n+1})$ and a labeling h such that the multiset $I_h(P_{4n+1}) = \{d\} \cup [c, 4n+c-1]$ and $h^+(v) = d$, for any integer c.*

Proof. According to the labeling g defined in Remark 2.1, $g^+(v_{2\ell-1}) = 2\ell$ for $1 \le \ell \le n-1$; $g^+(v_{2\ell-1}) = 2\ell+1$ for $n \le \ell \le 2n$; and $g^+(v_{4n+1}) = 2n$. Hence there is a vertex $v_j \in V(P_{4n+1})$, where j is odd, such that $g^+(v_j) = d$. Consider the graph $P_{4n+1} - v_j$, which is either a path of even order or two paths of even order. As in the proof of Lemma 3.1, there is a labeling $\widetilde{h}$ such that $I_{\widetilde{h}}(P_{4n+1} - v_j) = c + I_g(P_{4n+1} - v_j)$. After inserting back the removed edge(s) with the original label(s), we obtain the desired labeling h for P_{4n+1}. □

Similarly, we have the following lemma.

Lemma 3.3. *Suppose $n \ge 2$. For $d \in \{2\ell - 1 \mid 1 \le \ell \le n\} \cup \{2\ell \mid n \le \ell \le 2n-1\}$, there is a vertex $v \in V(P_{4n-1})$ and a labeling h such that the multiset $I_h(P_{4n-1}) = \{d\} \cup [c, 4n+c-3]$ and $h^+(v) = d$, for any integer c.*

Corollary 3.4. *For any integer c, there is a labeling h such that the multiset $I_h(P_{4n+1}) = \{2\} \cup [c, 4n+c-1]$ and $h^+(v_1) = 2$. Moreover, the image of h is a subset of $[2, 2n+2] \cup [c-2, 2n+c-3]$ if $n \ge 2$; and equal to $\{2, 3, c-1, c\}$ if $n = 1$.*

Proof. For $n \ge 2$, the result follows from Lemma 3.2 by choosing $d = 2$. For $n = 1$, we relabel v_2v_3 and v_4v_5 by $c-1$ and c, respectively, and obtain the result. □

Corollary 3.5. *For any integer c, there is a labeling h such that the multiset $I_h(P_{4n-1}) = \{1\} \cup [c, 4n+c-3]$ and $h^+(v_1) = 1$. Moreover, the image of h is a subset of $[1, 2n] \cup [c-1, 2n+c-3]$ if $n \ge 2$; and equal to $\{1, c\}$ if $n = 1$.*

4. $\mathbb{Z}_k$-ANTIMAGICNESS OF $G^{uv}P_s$

Let G and H be connected simple graphs. Let $u \in V(G)$ and $v \in V(H)$. The graph $G^{uv}H$ is obtained from G and H by add a new edge (bridge) uv. In this section, we construct some group-antimagic graphs from other group-antimagic graphs.

Let G be a simple connected graph of order $p \ge 3$ and assume that $f : E(G) \to [1, p]$. Since all values of f are positive and G is connected, the induced labeling f^+ is a positive mapping. In addition, we assume that $f^+ : V(G) \to [b-p+1, b]$ is bijective when $p \not\equiv 2 \pmod 4$, where

$b-p \geq 0$; and $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ when $p \equiv 2 \pmod 4$, where $1 \leq b-p < a < b$.

We use the following construction in this article. First, we relabel some edges of $P_s = v_1 v_2 \ldots v_s$ based on g to obtain a new labeling h. Then, we choose a suitable vertex u from G and a suitable vertex v from P_s to construct the graph $G^{uv}P_s$. Lastly, we label this bridge uv by a suitable label to construct a $\mathbb{Z}_k$-antimagic labeling ϕ of $G^{uv}P_s$.

4.1. $\mathbb{Z}_k$-antimagic Labelings of $G^{uv}P_{4m}$.

Theorem 4.1. *Suppose $f : E(G) \to [1,p]$ is a labeling of a graph G of order p such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $p \not\equiv 2 \pmod 4$ and $b - p \leq 2m+1$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_{4m}$ is $\mathbb{Z}_k$-antimagic for $k \geq p + 4m$.*

Proof. By setting $c = b-2$ into Lemma 3.1, we have a labeling h such that $I_h(P_{4m}) = [b-1, 4m+b-2]$. Choose the vertex $u \in V(G)$ with $f^+(u) = b$ and assign $\phi(uv_1) = -p$. Then $I_\phi(G^{uv_1}P_{4m}) = [b-p-1, 4m+b-2]$ with ϕ equals to f on G. After taking modulo k for $k \geq p+4m$, all labels are non-zero and the induced labels are distinct, hence $G^{uv_1}P_{4m}$ is $\mathbb{Z}_k$-antimagic. □

Theorem 4.2. *Suppose $f : E(G) \to [1,p]$ is a labeling of a graph G of order p such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $p \equiv 2 \pmod 4$ and $b-p \leq 2m+2$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_{4m}$ is $\mathbb{Z}_k$-antimagic for $k \geq p + 4m + 1$.*

Proof. By setting $c = b-2$ into Lemma 3.1, we have a labeling h such that $I_h(P_{4m}) = [b-1, 4m+b-2]$. Choose the vertex $u \in V(G)$ with $f^+(u) = b$ and assign $\phi(uv_1) = -p-1$. Then $I_\phi(G^{uv_1}P_{4m}) = [b-p-2, 4m+b-2] \setminus \{a\}$ with ϕ equals to f on G. After taking modulo k for $k \geq p+4m+1$, all labels are non-zero and the induced labels are distinct, hence $G^{uv_1}P_{4m}$ is $\mathbb{Z}_k$-antimagic. □

4.2. $\mathbb{Z}_k$-antimagic Labelings of $G^{uv}P_{4m+2}$.

Theorem 4.3. *Let $m \geq 1$. Suppose $f : E(G) \to [1,p]$ is a labeling of a graph G of order p such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $p \equiv 1$ or $3 \pmod 4$ and $b - p \leq 2m+2$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_{4m+2}$ is $\mathbb{Z}_k$-antimagic for $k \geq p + 4m + 2$.*

Proof. Recall that $I_g(P_{4m+2}) = [1, 4m+3] \setminus \{2\}$. By setting $c = b-2$ into Lemma 3.1, we have a labeling h such that $I_h(P_{4m+2}) = [b-1, 4m+b+1] \setminus \{b\}$ and $h^+(v_1) = b-1$. Choose $u \in V(G)$ with $f^+(u) = b-(p+1)/2$ (it is valid since $p \geq 3$) and assign $\phi(uv_1) = (-p+1)/2$. Then $I_\phi(G^{uv_1}P_{4m+2}) = [b-p, 4m+b+1]$. Hence ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_{4m+2}$ for $k \geq p+4m+2$. □

Theorem 4.4. *Let $m \geq 1$. Suppose $f : E(G) \to [1,p]$ is a labeling of a graph G of order p such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $p \equiv 2 \pmod 4$, $b-p \leq 2m+2$ and $3 \leq b-a \leq 2p-1$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_{4m+2}$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+2$.*

Proof. Note that $b-a$ is odd by Theorem 2.7. Setting $c = b-2$ in Lemma 3.1, there is a labeling h of P_{4m+2} such that $I_h(P_{4m+2}) = [b-1, 4m+b+1] \setminus \{b\}$ and $h^+(v_1) = b-1$. Choose $u \in V(G)$ with $f^+(u) = (a+b-1)/2$ and assign $\phi(uv_1) = (a-b+1)/2$. Then $I_\phi(G^{uv_1}P_{4m+2}) = [b-p, 4m+b+1]$ and hence $G^{uv_1}P_{4m+2}$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+2$. □

Theorem 4.5. *Let $m \geq 1$. Suppose $f : E(G) \to [1,p]$ is a labeling of a graph G of order p such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $p \equiv 0 \pmod 4$ and $b-p \leq 2m+3$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_{4m+2}$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+3$.*

Proof. By setting $c = b-2$ into Lemma 3.1, we have a labeling h such that $I_h(P_{4m+2}) = [b-1, 4m+b+1] \setminus \{b\}$ and $h^+(v_1) = b-1$. Choose $u \in V(G)$ with $f^+(u) = b-1-p/2$ (it is valid since $p \geq 4$) and assign $\phi(uv_1) = -p/2$. Then $I_\phi(G^{uv_1}P_{4m+2}) = [b-p-1, 4m+b+1] \setminus \{b-p\}$ and hence ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_{4m+2}$ for $k \geq p+4m+3$. □

Theorem 4.6. *Suppose $f : E(G) \to [1,p]$ is a labeling of a graph G of order p such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $p \not\equiv 2 \pmod 4$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_2$ is $\mathbb{Z}_k$-antimagic for $k \geq p+3$ and $k \neq b-p-1$.*

Proof. Let h be an edge labeling for P_2 defined by $h(v_1v_2) = b-p-1$. Choose $u \in V(G)$ with $f^+(u) = b-p+1$ and assign $\phi(uv_1) = -1$. Then $I_\phi(G^{uv_1}P_2) = [b-p-2, b] \setminus \{b-p+1\}$ and so ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_2$ for $k \geq p+3$ and $k \neq b-p-1$. □

Theorem 4.7. *Suppose $f : E(G) \to [1, p]$ is a labeling of a graph G of order p such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $p \equiv 2 \pmod 4$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_2$ is $\mathbb{Z}_k$-antimagic for $k \geq p+3$ and $k \neq a$.*

Proof. Let h be an edge labeling for P_2 defined by $h(v_1v_2) = a$. Choose $u \in V(G)$ with $f^+(u) = a+1$ and assign $\phi(uv_1) = b-a+1$. Then, $I_\phi(G^{uv_1}P_2) = [b-p, b+2] \setminus \{a+1\}$ and so ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_2$ for $k \geq p+3$ and $k \neq a$. □

4.3. $\mathbb{Z}_k$-antimagic Labelings of $G^{uv}P_{4m+1}$.

Theorem 4.8. *Suppose $f : E(G) \to [1, p]$ is a labeling of a graph G of order p such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $p \equiv 0$ or $3 \pmod 4$ with b is odd and $b - p \leq 2m$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_{4m+1}$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+1$.*

Proof. By setting $c = b+2$ into Corollary 3.4, we have $I_h(P_{4m+1}) = \{2\} \cup [b+2, 4m+b+1]$. Note that $b-p \leq 2m$ implies that the label of P_{4m+1} under h are positive and less than $p+4m+1$ for $m \geq 1$. Choose $u \in V(G)$ with $f^+(u) = (b+3)/2$ and assign $\phi(uv_1) = (b-1)/2$. Then $I_\phi(G^{uv_1}P_{4m+1}) = [b-p+1, 4m+b+1]$ and so ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_{4m+1}$ for $k \geq p+4m+1$. □

Theorem 4.9. *Suppose $f : E(G) \to [1, p]$ is a labeling of a graph G of order p such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $p \equiv 0$ or $1 \pmod 4$ with b is even and $b-p \leq 2m$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_{4m+1}$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+2$.*

Proof. By setting $c = b+3$ into Corollary 3.4, we have $I_h(P_{4m+1}) = \{2\} \cup [b+3, 4m+b+2]$. Note that $b-p \leq 2m$ implies that the label of P_{4m+1} under h are positive and less than $p+4m+2$ for $m \geq 1$. Choose $u \in V(G)$ with $f^+(u) = 2+b/2$. and assign $\phi(uv_1) = b/2$. Then $I_\phi(G^{uv_1}P_{4m+1}) = [b-p+1, 4m+b+2] \setminus \{b+1\}$ and thus ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_{4m+1}$, for $k \geq p+4m+2$. □

Theorem 4.10. *Suppose $f : E(G) \to [1, p]$ is a labeling of a graph G of order $p \equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective with* *a is even and*

$$\begin{cases} b-p \le 2m-1 & \text{if } a=2; \\ b-p \le 2m+1 & \text{if } a \ge 4 \text{ and } a/2 \ge b-p-1. \end{cases}$$

Then, there is a vertex $u \in V(G)$ *such that* $G^{uv_1}P_{4m+1}$ *is* $\mathbb{Z}_k$*-antimagic for* $k \ge 4m+p+1$.

Proof. Suppose $a=2$, then $b-p=1$. By setting $c=b+3$ into Corollary 3.4, we have $I_h(P_{4m+1}) = \{2\} \cup [b+3, 4m+b+2]$. Choose $u \in V(G)$ with $f^+(u)=1$ and assign $\phi(uv_1)=b$. Then $I_\phi(G^{uv_1}P_{4m+1}) = [3, 4m+p+3]$ and so ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_{4m+1}$ for $k \ge p+4m+1$.

Suppose $a \ge 4$. By setting $c=b+1$ into Corollary 3.4, we have $I_h(P_{4m+1}) = \{2\} \cup [b+1, 4m+b]$. Choose $u \in V(G)$ with $f^+(u)=a/2+1$ (note that $a/2+1 \in [b-p, b] \setminus \{a\}$) and assign $\phi(uv_1)=a/2-1$. Then $I_\phi(G^{uv_1}P_{4m+1}) = [b-p, 4m+b]$ and hence ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_{4m+1}$ for $k \ge p+4m+1$. □

Theorem 4.11. *Suppose* $f: E(G) \to [1,p]$ *is a labeling of a graph* G *of order* $p \equiv 2 \pmod 4$ *such that* $f^+: V(G) \to [b-p, b] \setminus \{a\}$ *is bijective, where* $b-p \le 2m-1$, *a is odd and* $(a+1)/2 \ge b-p$. *Then, there is a vertex* $u \in V(G)$ *such that* $G^{uv_1}P_{4m+1}$ *is* $\mathbb{Z}_k$*-antimagic for* $k \ge 4m+p+2$.

Proof. Note that b is even in this case and thus, $b-p \ge 2$.

By Corollary 3.4, we have $I_h(P_{4m+1}) = \{2\} \cup [b+2, 4m+b+1]$. Relabel the edge v_1v_2 of the path P_{4m+1} by $a-b$ and still denote this new labeling by h. Hence $I_h(P_{4m+1}) = \{a-b, a\} \cup [b+3, 4m+b+1]$ and $h^+(v_1) = a-b$. Choose $u \in V(G)$ with $f^+(u) = (a+1)/2$ and assign $\phi(uv_1) = b-(a-1)/2$ (nonzero). Then $I_\phi(G^{uv_1}P_{4m+1}) = [b-p, 4m+b+1] \setminus \{b+2\}$and so ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_{4m+1}$ for $k \ge p+4m+2$. □

4.4. $\mathbb{Z}_k$-antimagic Labelings of $G^{uv}P_{4m-1}$.

Theorem 4.12. *Suppose* $f: E(G) \to [1,p]$ *is a labeling of a graph* G *of order* $p \equiv 0$ *or* $1 \pmod 4$ *such that* $f^+: V(G) \to [b-p+1, b]$ *is bijective, where* $b-p \le 2m-2$, $b \le 2p$ *and b is even. Then, there is a vertex* $u \in V(G)$ *such that* $G^{uv_1}P_{4m-1}$ *is* $\mathbb{Z}_k$*-antimagic for* $k \ge p+4m-1$.

Proof. By setting $c = b+2$ into Corollary 3.5, there is a labeling h such that $I_h(P_{4m-1}) = \{1\} \cup [b+2, 4m+b-1]$. Note that the maximum label of h is $2m+b-1$ when $m \geq 2$ and $b+2$ when $m = 1$, respectively. When $b-p \leq 2m-2$, the maximum label of h is less than $p+4m-1$. Choose $u \in V(G)$ with $f^+(u) = b/2+1$ and assign $\phi(uv_1) = b/2$. Then $I_\phi(G^{uv}P_{4m-1}) = [b-p+1, 4m+b-1]$ and hence $G^{uv_1}P_{4m-1}$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m-1$. □

Theorem 4.13. *Suppose $f : E(G) \to [1,p]$ is a labeling of a graph G of order $p \equiv 0$ or $3 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $b-p \leq 2m-2$, $b \leq 2p+1$ and b is odd. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_{4m-1}$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m$.*

Proof. By Corollary 3.5 there is a labeling h such that $I_h(P_{4m-1}) = \{1\} \cup [b+3, 4m+b]$. Choose $u \in V(G)$ with $f^+(u) = (b+3)/2$ and let $\phi(uv_1) = (b+1)/2$. Then $I_\phi(G^{uv}P_{4m-1}) = [b-p+1, 4m+b] \setminus \{b+1\}$ and thus $G^{uv_1}P_{4m-1}$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m$. □

Theorem 4.14. *Suppose $f : E(G) \to [1,p]$ is a labeling of a graph G of order $p \equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $b-p \leq 2m$, a is odd and $(a+1)/2 \geq b-p$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_{4m-1}$ is $\mathbb{Z}_k$-antimagic for $k \geq 4m+p-1$.*

Proof. By Corollary 3.5, we have $I_h(P_{4m+1}) = \{1\} \cup [b+1, 4m+b-2]$. Choose $u \in V(G)$ with $f^+(u) = (a+1)/2$ and assign $\phi(uv_1) = (a-1)/2$. Then $I_\phi(G^{uv_1}P_{4m-1}) = [b-p, 4m+b-2]$ and hence ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_{4m-1}$ for $k \geq p+4m-1$. □

Theorem 4.15. *Let $m \geq 2$. Suppose $f : E(G) \to [1,p]$ is a labeling of a graph G of order $p \equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $b-p \leq 2m$, a is even and $a/2 \geq b-p$. Then there is a vertex $u \in V(G)$ such that $G^{uv_1}P_{4m-1}$ is $\mathbb{Z}_k$-antimagic for $k \geq 4m+p$.*

Proof. By Corollary 3.5, we have $I_h(P_{4m-1}) = \{1\} \cup [b+2, 4m+b-1]$. Relabel the edge v_1v_2 of the path P_{4m-1} by $a-b-1$ and still denote this new labeling by h. Hence $I_h(P_{4m-1}) = \{a-b-1, a\} \cup [b+3, 4m+b-1]$ and $h^+(v_1) = a-b-1$. Choose $u \in V(G)$ with $f^+(u) = a/2$ and assign $\phi(uv_1) = b+1-a/2$. Then $I_\phi(G^{uv_1}P_{4m-1}) = [b-p, 4m+b-1] \setminus \{b+2\}$. Hence ϕ is a $\mathbb{Z}_k$-antimagic labeling of $G^{uv_1}P_{4m-1}$ for $k \geq p+4m$. □

Theorem 4.16. *Suppose $f : E(G) \to [1,p]$ is a labeling of a graph G of order $p \equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where a is even, $3a \neq 2b - p$ and $a \leq p$. Then, there is a vertex $u \in V(G)$ such that $G^{uv_1}P_3$ is $\mathbb{Z}_k$-antimagic for $k \geq p+3$.*

Proof. There is a vertex $u \in V(G)$ such that $f^+(u) = b - (p+a)/2$, which is not equal to a. Define $\phi(v_1v_2) = b - a + 1$, $\phi(v_2v_3) = a$ and $\phi(v_1u) = -1 - (p-a)/2$. Then, $I_\phi(G^{uv_1}P_3) = [b-p-1, b+1]$. Hence $G^{uv_1}P_3$ is $\mathbb{Z}_k$-antimagic for $k \geq p+3$. □

5. Application to Tadpole Graphs

The *tadpole* graph $T(r,s)$ is obtained by joining a cycle C_r and a path P_s by a bridge, where $r \geq 3$ and $s \geq 1$. The idea for finding an antimagic labeling of $T(r,s)$ is to modify the labeling of the path that provided in Remark 2.1, and then joining a suitable labeled vertex from the cycle by a bridge with the end vertex v_1 or v_s of the path.

In this section, we will use some results proved in [24].

Lemma 5.1 ([24]). *For $d \in [2, 4n+2]$ and any integer c, there is a labeling h such that $I_h(C_{4n+1})$ is the multiset $([c, 4n+c] \setminus \{c+d-2\}) \cup \{d\}$. Note that the range of h is a subset of $[1, 2n+1] \cup [c-1, c-1+2n]$.*

Lemma 5.2 ([24]). *For $d \in [3, 4n+1]$ and any integer c, there is a labeling h such that $I_h(C_{4n-1})$ is the multiset $([c, 4n+c-2] \setminus \{c+d-3\}) \cup \{d\}$. Note that the range of h is a subset of $[1, 2n+1] \cup [c-2, c-2+2n]$.*

Lemma 5.3 ([24]). *Let $g : V(C_{2n}) \to \mathbb{Z}$ and $c \in \mathbb{Z}$. There is a labeling h such that $I_h(C_{2n}) = c + I_g(C_{2n})$. Note that the range of h is a subset of $[1, n+2] \cup [c+2, c+n+1]$.*

Theorem 5.4. *The graph $T(p, 4m)$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m$ if $p \not\equiv 2 \pmod 4$, and is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+1$ if $p \equiv 2 \pmod 4$.*

Proof. From Corollary 2.6, we have $1 \leq b-p \leq 3$ and the results follow from Theorems 4.1 and 4.2. □

Theorem 5.5. *Let $m \geq 1$. The graph $T(p, 4m+2)$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+2$ if $p \not\equiv 0 \pmod 4$, and is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+3$ if $p \equiv 0 \pmod 4$.*

Proof. From Corollary 2.6, we have $1 \leq b-p \leq 3$; and $b-a=3$ when $p \equiv 2 \pmod 4$. The results follow from Theorems 4.3, 4.4 and 4.5. □

Theorem 5.6. *The graph $T(p,2)$ is $\mathbb{Z}_k$-antimagic for $k \geq p+2$ if $p \not\equiv 0 \pmod 4$, and is $\mathbb{Z}_k$-antimagic for $k \geq p+3$ if $p \equiv 0 \pmod 4$.*

Proof.

Case 1: Suppose $p = 4n+1$ for some $n \geq 1$. By setting $d=2$ and $c=3$ into Lemma 5.1, there is a labeling h such that $I_h(C_p) = [4, p+2] \cup \{2\}$ and $h^+(u_1) = 2$. We label the edge v_1v_2 of P_2 by 1 and label the bridge u_1v_1 by 1 and denote the new labeling by ϕ. Then $\phi^+(u_1) = 3$, $\phi^+(v_1) = 2$ and $\phi^+(v_2) = 1$. Hence $I_\phi(T(p,2)) = [1, p+2]$ and so ϕ is an $\mathbb{Z}_k$-antimagic labeling for $k \geq p+2$.

Case 2: Suppose $p = 4n+2$ for some $n \geq 1$. In this case $I_f(C_p) = [3, p+3] \setminus \{p\}$. We choose $u \in V(C_p)$ such that $f^+(u) = p+2$ (indeed $u = u_{2n+4}$). We label the edge v_1v_2 of P_2 by p and label the bridge uv_1 by 2 and denote the new labeling by ϕ. Then $\phi^+(u) = p+4$, $\phi^+(v_1) = p+2$ and $\phi^+(v_2) = p$. Hence $I_\phi(T(p,2)) = [3, p+4]$ and thus, ϕ is an $\mathbb{Z}_k$-antimagic labeling for $k \geq p+2$.

Case 3: Suppose $p = 4n-1$ for some $n \geq 1$. By setting $d=3$ and $c=4$ into Lemma 5.2, there is a labeling h such that $I_h(C_p) = [5, p+3] \cup \{3\}$ and $h^+(u_2) = 3$. We label the edge v_1v_2 of P_2 by 4 and label the bridge u_2v_1 by -1 and denote the new labeling by ϕ. Then $\phi^+(u_2) = 2$, $\phi^+(v_1) = 3$ and $\phi^+(v_2) = 4$. Hence $I_\phi(T(p,2)) = [2, p+3]$ and thus, ϕ is an $\mathbb{Z}_k$-antimagic labeling for $k \geq p+2$.

Case 4: Suppose $p = 4n$ for some $n \geq 1$. Then, the result follows from Theorem 4.6. □

Theorem 5.7. *For $m \geq 1$, $T(p, 4m+1)$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+1$ if $p \not\equiv 1 \pmod 4$, and is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+2$ if $p \equiv 1 \pmod 4$.*

Proof.

Case 1: Suppose $p = 4n-1$ for some $n \geq 1$. In this case, $b = 4n+1$ and $b-p=2$. By Theorem 4.8, we obtain the result.

Case 2: Suppose $p = 4n+2$ for some $n \geq 1$. In this case, $b-p=3$ and $a = 4n+2 \neq 2$. The result follows from Theorem 4.10.

Case 3: Suppose $p = 4n$ for some $n \geq 1$. In this case, $b = 4n+2$ and $b-p=2$. By Theorem 4.9, $T(p, 4m+1)$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+2$.

When $k = p+4m+1$, we have $I_g(P_{4m+1}) = [2, 4m+2]$. By setting $c = 4m+1$ in Lemma 5.3, we have a labeling h such that $I_h(C_{4n}) = [4m+4, 4n+4m+3]$. Note that the image of h is a subset of $[1, 2n+2] \cup [4m+3, 4m+2n+2]$. Choose $u \in V(C_{4n})$ with $h^+(u) = 2n+4m+3$ and assign $\phi(uv_1) = -2n$. Then $\phi^+(u) = 4m+3$ and $\phi^+(v_1) = 2-2n \equiv 4m+2n+3 \pmod{4n+4m+1}$. Hence $I_\phi(T(4n, 4m+1)) \equiv [3, 4n+4m+3] \pmod{4n+4m+1}$ and so ϕ is an $\mathbb{Z}_{4n+4m+1}$-antimagic labeling for $T(4n, 4m+1)$.

Case 4: Suppose $p = 4n+1$ for some $n \geq 1$. In this case, $b = 4n+2$ and $b-p=1$. The result follows from Theorem 4.9. □

Theorem 5.8. *For $m \geq 1$, $T(p,1)$ is $\mathbb{Z}_k$-antimagic for $k \geq p+1$ if $p \not\equiv 1 \pmod 4$, and is $\mathbb{Z}_k$-antimagic for $k \geq p+2$ if $p \equiv 1 \pmod 4$.*

Proof. Let $P_1 = v$. From Corollary 2.6, we have $I_f(C_{4n-1}) = [3, 4n+1]$, $I_f(C_{4n}) = [3, 4n+2]$, $I_f(C_{4n+1}) = [2, 4n+2]$ and $I_f(C_{4n+2}) = [3, 4n+5] \setminus \{4n+2\}$.

Case 1: Suppose $p = 4n$. By setting $c = 1$ into Lemma 5.3, we have $I_h(C_{4n}) = [4, 4n+3]$. Choose $u \in V(C_{4n})$ with $h^+(u) = 2n+2$ and assign $\phi(uv) = 2n+2$. Then $\phi^+(u) = 4n+4$ and $\phi^+(v) = 2n+2$. Hence $I_\phi(T(4n,1)) = [4, 4n+4]$ and so ϕ is an $\mathbb{Z}_k$-antimagic labeling for $T(4n,1)$ for $k \geq 4n+1$.

Case 2: Suppose $p = 4n-1$. Since $I_f(C_{4n-1}) = [3, 4n+1]$, choose $u \in V(C_{4n-1})$ with $f^+(u) = 2n+1$ and assign $\phi(uv) = 2n+1$. Then $\phi^+(u) = 4n+2$ and $\phi^+(v) = 2n+1$. Hence $I_\phi(T(4n-1,1)) = [3, 4n+2]$ and so ϕ is an $\mathbb{Z}_k$-antimagic labeling for $T(4n-1,1)$ for $k \geq 4n$.

Case 3: Suppose $p = 4n+2$. Since $I_f(C_{4n+2}) = [3, 4n+5] \setminus \{4n+2\}$, choose $u \in V(C_{4n+2})$ with $f^+(u) = 2n+1$ and assign $\phi(uv) = 2n+1$. Then $\phi^+(u) = 4n+2$ and $\phi^+(v) = 2n+1$. Hence $I_\phi(T(4n+2,1)) = [3, 4n+5]$ and so ϕ is an $\mathbb{Z}_k$-antimagic labeling for $T(4n+2,1)$ for $k \geq 4n+3$.

Case 4: Suppose $p = 4n+1$. Since $I_f(C_{4n+1}) = [2, 4n+2]$, choose $u \in V(C_{4n+1})$ with $f^+(u) = 2n+2$ and assign $\phi(uv) = 2n+2$. Then $\phi^+(u) = 4n+4$ and $\phi^+(v) = 2n+2$. Hence $I_\phi(T(4n+1,1)) = [2, 4n+4] \setminus \{4n+3\}$ and so ϕ is an $\mathbb{Z}_k$-antimagic labeling for $T(4n+1, 1)$ for $k \geq 4n+3$. □

Theorem 5.9. *For $m \geq 1$, $T(p, 4m-1)$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m-1$ if $p \not\equiv 3 \pmod 4$, and is $\mathbb{Z}_k$-antimagic for $k \geq p+4m$ if $p \equiv 3 \pmod 4$.*

Proof.

Case 1: Suppose $p = 4n$. In this case, $b = 4n+2$ and $b - p = 2$. The result follows from Theorem 4.12.

Case 2: Suppose $p = 4n+1$. In this case, $b = 4n+2$ and $b - p = 1$. The result follows from Theorem 4.12.

Case 3: Suppose $p = 4n+2$. In this case, $b = 4n+5$, $b - p = 3$ and $a = 4n+2 = p$.

When $m = 1$, by Theorem 4.16, $T(p, 3)$ is $\mathbb{Z}_k$-antimagic for $k \geq p+3$.

When $m \geq 2$, by Theorem 4.15, $T(p, 4m-1)$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m$.

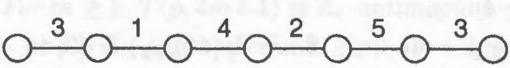
Therefore, we have to only deal with the case when $k = 4n + 4m + 1$. By setting $c = 4m-2$ in Lemma 5.3, we have a labeling h such that $I_h(C_{4n+2}) = [4m+1, 4m+4n+3] \setminus \{4m+4n\}$.

For $m \geq 3$, we define a new labeling for P_{4m-1} (based on g) by

$$\widetilde{g}(e_i) = \begin{cases} 3 & \text{if } i = 1; \\ g(e_i) + 1 & \text{if } i \text{ is even and } i \geq 4; \\ g(e_i) & \text{otherwise,} \end{cases}$$

where e_i are defined in Remark 2.1.

For $m = 2$, we define $\widetilde{g}$ as the following way:



Hence $I_{\widetilde{g}}(P_{4m-1}) = [3, 4m] \cup \{3\}$. Now we choose $u \in V(C_{4n+2})$ with $h^+(u) = 4m+4n+2$ and assign $\phi(uv_1) = -2$. Note that $\phi^+(v_1) = 1 \equiv 4m+4n+2 \pmod{4m+4n+1}$. Then $I_\phi(T(4n+$

2, $4m-1$)) $= [3, 4m+4n+3]$. Hence it is an $\mathbb{Z}_{4m+4n+1}$-antimagic labeling of $T(4n+2, 4m-1)$.

Case 4: Suppose $p = 4n-1$. In this case, $b = 4n+1$, $b-p = 2$. By Theorem 4.13, we have the result. □

Summarizing the results in this section, we have

Theorem 5.10. *For $r \geq 3$ and $s \geq 1$,*

$$\mathrm{IAM}(T(r,s)) = \begin{cases} [r+s, \infty) & \text{if } r+s \not\equiv 2 \pmod 4; \\ [r+s+1, \infty) & \text{if } r+s \equiv 2 \pmod 4. \end{cases}$$

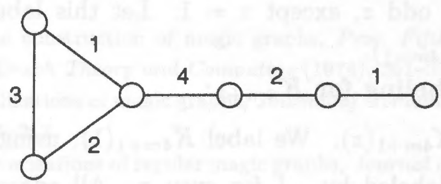


FIGURE 1. $T(3,3)$ is $\mathbb{Z}_k$-antimagic, for $k \geq 7$.

6. APPLICATION TO LOLLIPOP GRAPHS

The *lollipop* graph $L(r,s)$ is obtained by joining a complete graph K_r and a path P_s by a bridge, where $r \geq 3$ and $s \geq 1$. To make this paper self-contained, we provide the labeling of K_p (directly adopted from [24]). The image of the induced vertex labeling is the same as that of C_p (given by Lemma 2.3). The results in the preceding sections of this paper are then used to determine the integer-antimagic spectra of lollipop graphs.

Let the vertex set of K_p be $\{u_1, \ldots, u_p\}$. Let z be an integer with $1 \leq z \leq \lfloor p/2 \rfloor$. We construct a spanning subgraph $K_p(z)$ of K_p in which two vertices u_i and u_j are adjacent if $j \equiv i+z \pmod p$. Then, $K_p(z)$ is a union of $\gcd(z,p)$ cycles (each of order $p/\gcd(z,p)$). Note that if $z = p/2$, then $K_p(z)$ is a perfect matching. Also, observe that $K_p = \bigcup_{z=1}^{\lfloor p/2 \rfloor} K_p(z)$.

$\mathbb{Z}_k$-Antimagic labeling for K_{4m}:

$K_{4m} = \bigcup_{z=1}^{2m} K_{4m}(z)$. We label $K_{4m}(1)$, using $g+1$. All edges of $K_{4m}(z)$ are labeled by -1 for even z, except $z = 2m$. All edges of $K_{4m}(z)$ are labeled by 1 for odd z, except $z = 1$. All edges of $K_{4m}(2m)$ are labeled by -2. Let this labeling be f. Then, $I_f(K_{4m}) = I_g(C_{4m})$.

$\mathbb{Z}_k$-Antimagic labeling for K_{4m+2}:

$K_{4m+2} = \bigcup_{z=1}^{2m+1} K_{4m+2}(z)$. We label $K_{4m+2}(1)$, using g. All edges of $K_{4m+2}(z)$ are labeled by 1 for even z. All edges of $K_{4m+2}(z)$ are labeled by -1 for odd z, except $z = 1$ and $2m + 1$. All edges of $K_{4m+2}(2m+1)$ are labeled by -2. Let this labeling be f. Then, $I_f(K_{4m+2}) = I_g(C_{4m+2})$.

$\mathbb{Z}_k$-Antimagic labeling for K_{4m-1}:

$K_{4m-1} = \bigcup_{z=1}^{2m-1} K_{4m-1}(z)$ for $m \geq 2$. We label $K_{4m-1}(1)$, using g. All edges of $K_{4m-1}(z)$ are labeled by 1 for even z. All edges of $K_{4m-2}(z)$ are labeled by -1 for odd z, except $z = 1$. Let this labeling be f. Then, $I_f(K_{4m-1}) = I_g(C_{4m-1})$.

$\mathbb{Z}_k$-Antimagic labeling for K_{4m+1}:

$K_{4m+1} = \bigcup_{z=1}^{2m} K_{4m+1}(z)$. We label $K_{4m+1}(1)$, using $g + 1$. All edges of $K_{4m+1}(z)$ are labeled by -1 for even z. All edges of $K_{4m-2}(z)$ are labeled by 1 for odd z, except $z = 1$. Let this labeling be f. Then, $I_f(K_{4m+1}) = I_g(C_{4m+1})$.

Observe that in all of these cases, the domain of f is a subset of $[-2, p-1] \setminus \{0\}$, where p is the order of the graph under consideration.

If we change the domain of f (described in the lemmas and theorems in Sections 3 and 4) to $[-2, p-1] \setminus \{0\}$, then those results continue to hold. By substituting G by K_r and H by P_s and using these results and similar arguments as in Section 5, we see that

Theorem 6.1. *For $r \geq 3$ and $s \geq 1$,*

$$\mathrm{IAM}(L(r,s)) = \begin{cases} [r+s, \infty) & \text{if } r+s \not\equiv 2 \pmod 4; \\ [r+s+1, \infty) & \text{if } r+s \equiv 2 \pmod 4). \end{cases}$$

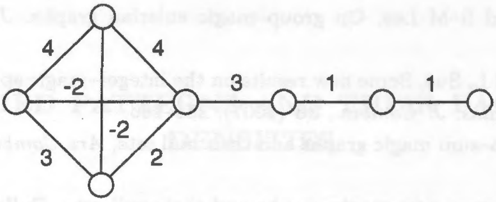


FIGURE 2. $L(4,3)$ is $\mathbb{Z}_k$-antimagic, for $k \geq 7$.

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Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Hong Kong, P.R. China

E-mail address: wcshiu@hkbu.edu.hk

Department of Mathematics, Hong Kong Baptist University, 224 Waterloo Road, Hong Kong, P.R. China

E-mail address: lionel@hkbu.edu.hk

Department of Mathematics, San Jose State University, San Jose, CA 95192, USA

E-mail address: richard.low@sjsu.edu