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ALGEBRAIC STRUCTURE OF SCHUR RINGS

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ABSTRACT. Schur ring (S–ring) is known to have applications on group theory and combinatorial design theory (see [1],[5]). In this paper, we study the general structure of Schur rings. Schur subrings, normal Schur subrings, quotient S–rings, S–ring homomorphism and direct product of two S–rings are introduced. As in group theory, three isomorphism theorems of S–rings are shown. We also show that the set of all normal Schur subrings of a given S–ring ordered by inclusion is a modular lattice. Hence it satisfies the Jordan–Hölder–Dedekind theorem. However some of our results are found different from the structure of groups. For instance, the kernel of an S–ring homomorphism may not be normal; the direct product of two Schur subrings of an S–ring may not be its Schur subring.

0. PRELIMINARIES

Let G be a finite multiplicative group (in this paper, all groups are nontrivial and finite) and let e be the identity of G. For any $D \subseteq G$, $t \in \mathbb{Z}$, we define $D^{(t)} = \{d^t | d \in D\}$ and $\bar{D} = \sum\limits_{d \in D} d \in \mathbb{C}[G]$ (if $D = \phi$, we set $\bar{D} = 0$) where $\mathbb{C}[G]$ denotes the group algebra of G over $\mathbb{C}$. All the algebraic structures hold when we replace the group algebra $\mathbb{C}[G]$ by the group ring $\mathbb{Z}[G]$.

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Let $\varphi : G \to H$ be a group homomorphism (or antihomomorphism) from group G into group H. If we define $\varphi^* : \mathbb{C}[G] \to \mathbb{C}[H]$ by

$$\varphi^*\Big[\sum_{g\in G} a_g g\Big] = \sum_{g\in G} a_g \varphi(g),$$

then φ^* becomes an algebra homomorphism (or antihomomorphism) from $\mathbb{C}[G]$ into $\mathbb{C}[H]$. (See [3]:§2.2)

In this paper, we shall use the notation φ^* to denote the algebra homomorphism (antihomomorphism) induced from φ.

For $t \in \mathbb{Z}$, if the group homomorphism $\varphi : G \to G$ is defined by $\varphi(g) = g^t$, $g \in G$, we often use the notation $X^{(t)}$ to denote $\varphi^*(X)$ for any $X \in \mathbb{C}[G]$.

1. SCHUR RINGS

Let $\mathcal{P} = \{D_0 = \{e\}, D_1, \cdots, D_d\}$ be a family of nonempty subsets of G satisfying the following conditions:

[S1] $\mathcal{P}$ is a partition of G;

[S2] for each $D_i \in \mathcal{P}$, $D_i^{(-1)} = D_{i^*}$ for some $i^* \in \{0, 1, \cdots, d\}$;

[S3] $\bar{D}_i\bar{D}_j = \sum_{k=0}^{d} p_{ij}^k \bar{D}_k$ for all i, j where $p_{ij}^k \in \{0, 1, 2, \cdots\} = I\!N$.

The subalgebra, denoted by $\mathfrak{S} = (G; \mathcal{P})$, of $\mathbb{C}[G]$ generated by $\bar{D}_0, \bar{D}_1, \cdots, \bar{D}_d$ is called a Schur ring (for short, S–ring) of dimension $d+1$ over G. The integers p_{ij}^k are called the intersection numbers of the S–ring $\mathfrak{S}$. Each $\bar{D}_i$ is called a principal basis element of $\mathfrak{S}$ and each D_i is called an $\mathfrak{S}$–principal subset of G.

An S–ring $\mathfrak{S} = (G; \mathcal{P})$ (of dimension $d+1$) is called commutative if $\bar{D}_i\bar{D}_j = \bar{D}_j\bar{D}_i$ for all i, j, $0 \le i, j \le d$. $\mathfrak{S}$ is called symmetric if $D_i^{(-1)} = D_{i^*} = D_i$ for all i, $0 \le i \le d$.

A Schur ring can be used to construct an association scheme (See [1]:§II.6) and some properties of association schemes can be applied on Schur rings. But in this paper, the author try to prove the properties of Schur rings by elementary proofs as long as it applies.

Proposition 1.1. *Symmetric S-rings over G are commutative.* (*See* [4])

Proposition 1.2. *Let p_{ij}^k be intersection numbers of an S-ring $\mathfrak{S}$ of dimension $d+1$. Then we have*

(i) $v_i = |D_i| = |D_{i^*}| = v_{i^*}$; (ii) $p_{0j}^k = \delta_{jk}$; (iii) $p_{j0}^k = \delta_{jk}$;

(iv) $p_{ij}^0 = v_i\delta_{ij^*}$; (v) $p_{ij}^k = p_{j^*i^*}^{k^*}$; (vi) $\sum_{j=0}^{d} p_{ij}^k = v_i$;

(vii) $v_k p_{ij}^k = v_i p_{kj^*}^i = v_j p_{i^*k}^j$.

Proof. (i) is obvious. (ii) and (iii) follow from $\bar{D}_0\bar{D}_j = \bar{D}_j = \bar{D}_j\bar{D}_0$. (iv) follows from $e \in D_iD_j$ if and only if $j = i^*$.

From the proof of proposition 1.1, we have

$$\sum_{k^*} p_{j^*i^*}^{k^*}\bar{D}_{k^*} = \bar{D}_{j^*}\bar{D}_{i^*} = \sum_k p_{ij}^k\bar{D}_{k^*} = \sum_{k^*} p_{ij}^k\bar{D}_{k^*}$$

Hence (v) follows.

$\sum_j \bar{D}_i\bar{D}_j = \sum_k\Big[\sum_j p_{ij}^k\Big]\bar{D}_k$. On the other hand,

$\sum_j \bar{D}_i\bar{D}_j = \bar{D}_i\Big[\sum_j \bar{D}_j\Big] = \bar{D}_i\bar{G} = v_i\bar{G} = v_i\Big[\sum_k \bar{D}_k\Big]$. Since $\{D_0, D_1, \cdots, D_d\}$ is a basis of $\mathfrak{S}$, (vi) follows.

By comparing the coefficients of $\bar{D}_0$ in $(\bar{D}_i\bar{D}_j)\bar{D}_{k^*}$ and $\bar{D}_i(\bar{D}_j\bar{D}_{k^*})$ we have

$$v_k p_{ij}^k = v_i p_{jk^*}^{i^*}$$

Since $p_{ij}^k = p_{j^*i^*}^{k^*}$ and $i^{**} = i$ for any $i = 0, 1, \cdots, d$, $v_k p_{ij}^k = v_i p_{kj^*}^i$ and $v_i p_{kj^*}^i = v_{i^*} p_{jk^*}^{i^*} = v_j p_{ik^*}^j$. □

Examples.

(1) Let $\mathcal{P} = \{\{g\} | g \in G\}$. Clearly $(G; \mathcal{P}) = \mathbb{C}[G]$ is an S-ring. Thus Schur ring is a generalization of group algebra (or group ring).

(2) Let G be a group and let $D_0 = \{e\}$, $D_1 = G\backslash\{e\}$. Then $(G; \{D_0, D_1\})$ is an S-ring of dimension 2. Note that this is the unique S-ring of dimension 2 over a given group and is called the trivial Schur ring.

(3) Let G be a group and let $\mathcal{P} = \{D_0 = \{e\}, D_1, \cdots, D_d\}$ be the conjugacy classes of G. Then $(G;\mathcal{P})$ is a commutative S–ring. The S–ring is symmetric if and only if g and g^{-1} are conjugate in G for all $g \in G$.

(4) Let G be a group and $H_0 = \{e\} < H_1 < \cdots < H_d = G$ be a chain of subgroups of G. Let $D_0 = H_0$ and $D_i = H_i \backslash H_{i-1}$ for $i = 1, 2, \cdots, d$. Then $(G; \{D_0, D_1, \cdots, D_d\})$ is a symmetric S–ring.

2. SCHUR SUBRINGS AND QUOTIENT SCHUR RINGS

Suppose $H \leq G$ and let $\mathfrak{S} = (G;\mathcal{P})$ $\mathfrak{S}' = (H;\mathcal{P}')$ are S–rings over G and H, respectively. If $\mathcal{P}' \subseteq \mathcal{P}$, then we call $\mathfrak{S}'$ a Schur subring (for short, S–subring) of $\mathfrak{S}$ and denote $\mathfrak{S}' \leq \mathfrak{S}$; call $\mathfrak{S}'$ normal if $H \triangleleft G$ and denote $\mathfrak{S}' \triangleleft \mathfrak{S}$. $\mathfrak{S}$ is called simple if it contains only two normal S–subrings $\mathfrak{S}$ and $\mathfrak{S}_0 = (\{e\}; \{\{e\}\})$.

Lemma 2.1. *Let $\mathfrak{S} = (G;\mathcal{P})$ be an S-ring. Let $\phi \neq \mathcal{P}' \subseteq \mathcal{P}$ satisfy*

(i) *$D_{i^*} \in \mathcal{P}'$ for each $D_i \in \mathcal{P}'$.*

(ii) *$\bar{D}_i\bar{D}_j$ is a linear combination of $\bar{D}_k$ with $D_k \in \mathcal{P}'$ all $D_i, D_j \in \mathcal{P}'$.*

Then $H = \bigcup_{D \in \mathcal{P}'} D \leq G$ and $\mathfrak{S}' = (H;\mathcal{P}') \leq \mathfrak{S}$.

An S–ring $\mathfrak{S} = (G;\mathcal{P})$ (of dimension $d+1$) over G is called primitive if $< D_i >= G$ for each $D_i \in \mathcal{P}$, $1 \leq i \leq d$. Otherwise it is called imprimitive. Clearly, $\mathfrak{S}$ is primitive if and only if no $D_i \in \mathcal{P}$, $1 \leq i \leq d$, is contained in a proper subgroup of G.

Proposition 2.2. *$\mathfrak{S} = (G;\mathcal{P})$ is an imprimitive S-ring if and only if $\mathfrak{S}$ contains a proper S-subring $\mathfrak{S}' \neq (\{e\};\{D_0\})$. Hence primitive S-rings are simple. (See [4])*

Let $\mathfrak{S}' = (H;\mathcal{P}')$ be an S–subring of $\mathfrak{S} = (G;\mathcal{P})$, where $\mathcal{P} = \{D_0, D_1, \cdots, D_d\}$. Without loss of generality, we may assume $\mathcal{P}' = \{D_i | 0 \leq i \leq s\}$, $0 \leq s \leq d$. Let $\mathcal{D} = \{0, 1, \cdots, d\}$. We define a relation $\mathcal{R}$ on $\mathcal{D}$ by $(i,j) \in \mathcal{R}$ if and only if the intersection number p_{it}^{j} of $\mathfrak{G}$ is not zero for some t, $0 \le t \le s$. It is equivalent to $(i,j) \in \mathcal{R}$ if and only if $D_i^{(-1)} D_j \cap H \ne \phi$ (since from Proposition 1.2 (vii) $p_{i^*j}^{t} \ne 0 \Leftrightarrow p_{it}^{j} \ne 0$). Clearly $\mathcal{R}$ is an equivalence relation.

Proposition 2.3. *Let $[i]$ be an equivalence class of $\mathcal{R}$ containing i. For $g \in G$, set $S(g) = \{j | gH \cap D_j \ne \phi\}$. Then $[i] = S(g)$ for any $g \in D_i$.*

Proof. For any fixed $g \in D_i$, suppose $j \in S(g)$ $\exists h \in H$ such that $gh \in D_j$ for some $h \in H$. So we have $h \in g^{-1} D_j$ and $D_i^{-1} D_j \cap H \ne \phi$. Hence $(i,j) \in \mathcal{R}$ and $S(g) \subseteq [i]$.

Suppose $(i,j) \in \mathcal{R}$. Since $p_{jt}^{i} \ne 0$ for some $0 \le t \le s$,

$$\bar{D}_j \bar{D}_t = p_{jt}^{i} \bar{D}_i + \cdots.$$

Therefore there exist $w \in D_j$ and $h \in D_t \subseteq H$ such that $wh = g$, i.e., $w = gh^{-1} \in gH$. Hence $[i] = S(g)$. □

Corollary 2.4. *If $g \in D_i$ and $j \in [i]$ then there exists $w \in D_j$ such that $wH = gH$.*

Corollary 2.5. *If $H \lhd G$ and let $\nu : G \to G/H$ be the natural group epimorphism. Then $(i,j) \in \mathcal{R}$ if and only if $\nu(D_i) = \nu(D_j)$.*

Corollary 2.6. *If $H \lhd G$, then either $\nu(D_i)$ and $\nu(D_j)$ are disjoint or identical.*

Proof. Suppose $\nu(D_i) \cap \nu(D_j) \ne \phi$, then $D_i^{(-1)} D_j \cap H \ne \phi$. Thus $(i,j) \in \mathcal{R}$. The corollary follows from Corollary 2.5. □

Suppose $H \le G$. Let $T_0 = [0] = \{0, 1, \cdots, s\}$, $T_1, \cdots, T_t$ be the equivalence classes of $\mathcal{R}$ on $\mathcal{D}$. If $T_a = [i]$ we set $E_a = \nu(D_i)$. Let $B = \{e = g_1, g_2, \cdots, g_q\} \subseteq G$ be the set of all representatives of the left cosets of H in G. If $H \lhd G$ then for each E_a there is $B_a \subseteq B$ such that $E_a = \nu(B_a)$. Moreover if $T_a = [i]$ then $D_i = \bigcup_{g \in B_a} gA_g$ where A_g are nonempty subsets of H. It is clear that by Corollaries 2.5 and 2.6 we have

$$\bigcup_{i \in T_a} D_i = B_a H.$$

We denote $\bigcup_{i \in T_a} D_i$ by F_a. If fact, the above result is also true when H is not normal in G.

Lemma 2.7. *Keep the notations defined above. Suppose $H \le G$ and let $C = \sum_{k=1}^{q} b_k g_k \in \mathbb{C}[G]$. If $C\bar{H} = \sum_{r=1}^{d} a_r \bar{D}_r$ for some $a_r \in \mathbb{C}$ then $a_i = a_j$ if $(i,j) \in \mathcal{R}$.*

Proof. $C\bar{H} = \sum_{k=1}^{q} b_k \overline{g_k H}$. By Corollary 2.4 that for any $d_i \in D_i$ $\exists d_j \in D_j$ such that $d_i H = d_j H$ if $(i,j) \in \mathcal{R}$. Since $d_i \in g_k H$ for some k, $a_i = b_k = a_j$. □

Note that it is easy to show that $b_k = b_{k'}$ if $S(g_k) = S(g_{k'})$.

Corollary 2.8. *For $i \in T_a$ and $D_i = \bigcup_{g \in B_a} gA_g$ where $\phi \ne A_g \subseteq H$. Then $\bar{D}_i\bar{H} = c_{ai}\bar{F}_a$ where $c_{ai} = \sum_{j=0}^{s} p_{ij}^k$ for any $k \in T_a$. Moreover, $|A_g| = c_{ai}$ for any $g \in B_a$.*

Proof. Since $(i,k) \in \mathcal{R} \Leftrightarrow p_{ij}^k \ne 0$,

$$\begin{aligned}\bar{D}_i\bar{H} &= \bar{D}_i \sum_{j=0}^{s} \bar{D}_j = \sum_{j=0}^{s}\sum_{k=0}^{d} p_{ij}^k \bar{D}_k = \sum_{j=0}^{s}\sum_{k \in T_a} p_{ij}^k \bar{D}_k \\ &= \sum_{k \in T_a}\left[\sum_{j=0}^{s} p_{ij}^k\right]\bar{D}_k = \sum_{k \in T_a} q_{ki}\bar{D}_k,\end{aligned}$$

where $q_{ki} = \sum_{j=0}^{s} p_{ij}^k$.

Since $D_i = \bigcup_{g \in B_a} gA_g$, $\bar{D}_i\bar{H} = \sum_{g \in B_a} |A_g| g\bar{H} = C\bar{H}$ where $C = \sum_{g \in B_a} |A_g| g$. By Lemma 2.7 that q_{ki} are equal, say c_{ai}. Hence $\bar{D}_i\bar{H} = c_{ai}\sum_{k \in T_a}\bar{D}_k = c_{ai}\bar{F}_a$. Clearly, $|A_g| = c_{ai}$ for any $g \in B_a$. □

Corollary 2.9. *Suppose $H \triangleleft G$. Let ν be the natural epimorphism from G onto G/H. If $i \in T_a$ then $\nu^*(\bar{D}_i) = c_{ai}\bar{E}_a$.*

Proof. Follows from Corollary 2.8. □

Theorem 2.10. *Let $\mathfrak{S}' = (H; \{D_0, D_1, \cdots, D_s\})$ be a normal S–subring of $\mathfrak{S} = (G; \{D_0, D_1, \cdots, D_d\})$ and let ν be the natural group epimorphism from G onto G/H. Keep the notations $\mathcal{R}$ and $E_0, E_1, \cdots, E_t$ which were defined below Corollary 2.6. Then $\bar{\mathfrak{S}} = (G/H; \{E_0, E_1, \cdots, E_t\})$ is an S–ring over G/H. Moreover $\nu^*(\mathfrak{S}) = \bar{\mathfrak{S}}$. Such S–ring will be called the quotient S–ring of $\mathfrak{S}$ related to $\mathfrak{S}'$ and denoted as $\mathfrak{S}/\mathfrak{S}'$.*

Proof. Obviously $E_0 = \{H\}$. [S1] follows by Corollary 2.6.

Since $\nu(D_i)^{(-1)} = \nu(D_i^{(-1)}) = \nu(D_{i^*})$, $E_a^{(-1)} = E_{a^*}$ for $i \in T_a$ and $i^* \in T_{a^*}$ for some $0 \le a^* \le t$. Hence **[S2]** holds.

Let $h = |H|$. Since $F_a = B_a H = \bigcup_{i \in T_a} D_i$,

$$\begin{aligned}\bar{F}_a\bar{F}_b &= \bar{B}_a\bar{H}\bar{B}_b\bar{H} = \bar{B}_a\bar{B}_b\bar{H}\bar{H} \text{ (since } H \text{ is normal in } G).\\ &= h\bar{B}_a\bar{B}_b\bar{H} = hC\bar{H} \quad \text{where} \quad C = \sum_{k=1}^{q} b_k g_k \quad \text{for some } b_k \in I\!N.\end{aligned}$$

By Lemma 2.7 that $\bar{F}_a\bar{F}_b = h\sum_{c=1}^{t} \mathcal{P}_{ab}^c\bar{F}_c$ for some $\mathcal{P}_{ab}^c \in I\!N$.

Since $F_a = B_a H = \bigcup_{i \in T_a} D_i$, $\nu^*(\bar{F}_a) = h\bar{E}_a$ and

$$h\bar{E}_a h\bar{E}_b = \nu^*(\bar{F}_a)\nu^*(\bar{F}_b) = h\sum_{c=1}^{t}\mathcal{P}_{ab}^c\nu^*(\bar{F}_c) = h^2\sum_{c=1}^{t}\mathcal{P}_{ab}\bar{E}_c.$$

Hence $\bar{E}_a\bar{E}_b = \sum_{c=1}^{t}\mathcal{P}_{ab}^c\bar{E}_c$. □

Example. Let $G = Q_8 = \{e, e', i, i', j, j', k, k'\}$ be the quaternion group, where

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} \zeta & 0 \\ 0 & -\zeta \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & \zeta \\ \zeta & 0 \end{bmatrix},$$

$$e' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad i' = \begin{bmatrix} -\zeta & 0 \\ 0 & \zeta \end{bmatrix}, \quad j' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad k' = \begin{bmatrix} 0 & -\zeta \\ -\zeta & 0 \end{bmatrix}$$

and $\zeta = \sqrt{-1}$.

Let $D_0 = \{e\}$, $D_1 = \{e'\}$, $D_2 = \{i, i'\}$, $D_3 = \{j, j'\}$, $D_4 = \{k, k'\}$. Then $\mathcal{G} = (Q_8; \{D_0, D_1, D_2, D_3, D_4\})$ is a symmetric S–ring. The multiplication table is

	d_0	d_1	d_2	d_3	d_4
d_0	d_0	d_1	d_2	d_3	d_4
d_1	d_1	d_0	d_2	d_3	d_4
d_2	d_2	d_2	$2d_0 + 2d_1$	$2d_4$	$2d_3$
d_3	d_3	d_3	d_4	$2d_0 + 2d_1$	$2d_2$
d_4	d_4	d_4	$2d_3$	$2d_2$	$2d_0 + 2d_1$

where $d_i = \bar{D}_i$, $i = 0, 1, 2, 3, 4$.

Clearly, $\mathfrak{S}_1 = (H; \{D_0, D_1\})$ and $\mathfrak{S}_2 = (K; \{D_0, D_1, D_2\})$ are S–subrings, where $H = \{e, e'\}$ and $K = \{e, e', i, i'\}$. Moreover $\mathfrak{S}_1 \lhd \mathfrak{S}_2 \lhd \mathfrak{S}$ and $\mathfrak{S}_1 \lhd \mathfrak{S}$.

Consider the case $\mathfrak{S}_1 \lhd \mathfrak{S}$. $T_0 = \{0, 1\}$, $T_1 = \{2\}$, $T_2 = \{3\}, T_3 = \{4\}$ and $F_0 = H$, $F_1 = D_2$, $F_2 = D_3$, $F_3 = D_4$. We have

	f_0	f_1	f_2	f_3
f_0	$2f_0$	$2f_1$	$2f_2$	$2f_3$
f_1	$2f_1$	$2f_0$	$2f_3$	$2f_2$
f_2	$2f_2$	$2f_3$	$2f_0$	$2f_1$
f_3	$2f_3$	$2f_2$	$2f_1$	$2f_0$

where $f_i = \bar{F}_i$, $i = 0, 1, 2, 3$. Thus $E_0 = \{H\}$, $E_1 = \{iH\}$, $E_2 = \{jH\}$, $E_3 = \{kH\}$ and $\mathfrak{S}/\mathfrak{S}_1 = (Q_8/H; \{E_0, E_1, E_2, E_3\})$ which is isomorphic to $\mathbb{C}[\mathbb{C}_2 \times \mathbb{C}_2]$, where $\mathbb{C}_2$ is the cyclic group of order 2.

Similarly,

$$\begin{aligned} \mathfrak{S}/\mathfrak{S}_2 &= (Q_8/K; \{\{K\}, \{jK\}\}) \cong \mathbb{C}[\mathbb{C}_2] \quad \text{and} \\ \mathfrak{S}_2/\mathfrak{S}_1 &= (K/H; \{\{H\}, \{iH\}\}) \cong \mathbb{C}[\mathbb{C}_2]. \end{aligned}$$

Let $\mathfrak{S} = (G; \mathcal{P} = \{D_0, D_1, \cdots, D_d\})$ be an S–ring. Let $\phi \neq \mathcal{P}' \subseteq \mathcal{P}$ and let

$$\begin{aligned} \mathcal{Q} &= \left\{ D \in \mathcal{P} \;\middle|\; \begin{array}{l} \bar{D} \text{ appears in the expression of } \bar{D}_i\bar{D}_j \text{ as a linear} \\ \text{combination of } \bar{D}_0, \bar{D}_1, \cdots, \bar{D}_d \text{ for some } D_i, D_j \in \mathcal{P}' \end{array} \right\} \\ &= \{D \in \mathcal{P} | D \subseteq D_iD_j \text{ for some } D_i, D_j \in \mathcal{P}'\}. \end{aligned}$$

Let $H = \bigcup_{D \in \mathcal{Q}} D$, then by Lemma 2.1 that $(H; \mathcal{Q})$ is an S–subring. We call it the S–subring generated by $\mathcal{P}'$ and denote it by $< \mathcal{P}' >$.

Note that $< \mathcal{P}' >$ is the smallest S–ring having all $\mathfrak{S}$–principal subsets in $\mathcal{P}'$.

Theorem 2.11. *Suppose $\mathfrak{S}_H = (H; \mathcal{P}_H)$ and $\mathfrak{S}_K = (K; \mathcal{P}_K)$ are S-subrings of $\mathfrak{S} = (G; \mathcal{P})$. Then $< \mathcal{P}_H \cap \mathcal{P}_K > = (H \cap K; \mathcal{P}_H \cap \mathcal{P}_K)$, and $< \mathcal{P}_H \cup \mathcal{P}_K >$ is S-ring over $< H \cup K >$.*

Proof. Since $H = \bigcup_{D \in \mathcal{P}_H} D$, $K = \bigcup_{E \in \mathcal{P}_K} E$, $H \cap K = \bigcup_{\substack{D \in \mathcal{P}_H \\ E \in \mathcal{P}_K}} (D \cap E)$. Since $D \cap E = \phi$ or $D = E$ for $D \in \mathcal{P}_H$ and $E \in \mathcal{P}_K$, $H \cap K = \bigcup_{D \in \mathcal{P}_H \cap \mathcal{P}_K} D$. Clearly $\mathcal{P}_H \cap \mathcal{P}_K$ satisfies the conditions of the Lemma 2.1. Thus $(H \cap K; \mathcal{P}_H \cap \mathcal{P}_K)$ is an S–ring which contains all $\mathfrak{S}$–principal subsets in $\mathcal{P}_H \cap \mathcal{P}_K$. So $< \mathcal{P}_H \cap \mathcal{P}_K > = (H \cup K; \mathcal{P}_H \cap \mathcal{P}_K)$.

Let $< \mathcal{P}_H \cup \mathcal{P}_K >= (L; \mathcal{Q})$ for some subgroup L of G and some subset $\mathcal{Q}$ of $\mathcal{P}$. Clearly $H \cup K \subseteq \bigcup_{D \in \mathcal{Q}} D = L$. So $< H \cup K > \subseteq L$. $\forall g \in L$, $g \in D$ for some $D \in \mathcal{Q}$. Since $D \subseteq D_i D_j$; for some $D_i, D_j \in \mathcal{P}_H \cup \mathcal{P}_K$, $g \in < H \cup K >$. Hence $L =< H \cup K >$. □

Theorem 2.12. *Suppose $\mathfrak{S}_H = (H; \mathcal{P}_H)$ and $\mathfrak{S}_K = (K; \mathcal{P}_K)$ are S–subrings of $\mathfrak{S} = (G; \mathcal{P})$. Then*

(i) $\mathfrak{S}_H \cap \mathfrak{S}_K = (H \cap K; \mathcal{P}_H \cap \mathcal{P}_K)$.

(ii) *If $\mathfrak{S}_H \triangleleft \mathfrak{S}$ then $\mathfrak{S}_K \mathfrak{S}_H = \mathfrak{S}_H \mathfrak{S}_K = \{xy | x \in \mathfrak{S}_H,\ y \in \mathfrak{S}_K\} < \mathcal{P}_H \cup \mathcal{P}_K >$ is an S–ring over HK.*

Proof. Clearly $< \mathcal{P}_H \cap \mathcal{P}_K > \subseteq \mathfrak{S}_H \cap \mathfrak{S}_K$. For any $x \in \mathfrak{S}_H \cap \mathfrak{S}_K$, $x = \sum_{D \in \mathcal{P}_H} a_D \bar{D} = \sum_{E \in \mathcal{P}_K} b_E \bar{E}$. Since $\{\bar{D} | D \in \mathcal{P}_H\}$ and $\{\bar{E} | E \in \mathcal{P}_K\}$ are parts of basis for the vector space $\mathfrak{S}$ over $\mathbb{C}$, $a_D = 0 = b_E$ except $D = E \in \mathcal{P}_H \cap \mathcal{P}_K$. In that case, $a_D = b_D$ and hence by Corollary 1.3 that $x \in < \mathcal{P}_H \cap \mathcal{P}_K >$. Thus $\mathfrak{S}_H \cap \mathfrak{S}_K =< \mathcal{P}_H \cap \mathcal{P}_K >$. By Theorem 2.11 that $\mathfrak{S}_H \cap \mathfrak{S}_K = (H \cap K; \mathcal{P}_H \cap \mathcal{P}_K)$.

Since $H \triangleleft G$, $\bar{D}\bar{E} = \bar{E}D^{(-1)}$ for $D \in \mathcal{P}_H$ and $E \in \mathcal{P}_K$. So that $\mathfrak{S}_K \mathfrak{S}_H = \mathfrak{S}_H \mathfrak{S}_K$ and $\mathfrak{S}_H \mathfrak{S}_K$ is an subalgebra of $\mathfrak{S}$. Hence $< \mathcal{P}_H \cup \mathcal{P}_K > \subseteq \mathfrak{S}_H \mathfrak{S}_K$. For any $xy \in \mathfrak{S}_H \mathfrak{S}_K$, $x = \sum_{D \in \mathcal{P}_H} a_D \bar{D}$ and $y = \sum_{E \in \mathcal{P}_K} b_E \bar{E}$. Then $xy = \sum_{\substack{D \in \mathcal{P}_H \\ E \in \mathcal{P}_K}} a_D b_E \bar{D}\bar{E}$. Since $\bar{D}\bar{E} \in < \mathcal{P}_H \cup \mathcal{P}_K >$, $xy \in < \mathcal{P}_H \cup \mathcal{P}_K >$. Hence $\mathfrak{S}_H \mathfrak{S}_K =< \mathcal{P}_H \cup \mathcal{P}_K >$.

It is clear that $HK =< H \cup K >$ and then $\mathfrak{S}_H \mathfrak{S}_K$ is an S–ring over HK. □

3. HOMOMORPHISMS

Let $\mathfrak{S} = (G; \{D_i | 0 \le i \le d\})$ and $\mathfrak{S}' = (G'; \{D'_i | 0 \le i \le d'\})$ be two S–rings. Suppose the algebra homomorphism $\Phi : \mathfrak{S} \to \mathfrak{S}'$ has the following property:

For each i, $\quad 0 \le i \le d, \quad \Phi(\bar{D}_i)\mu_i \bar{D}'_j$ for some $\mu_i > 0$ and $0 \le j \le d'$.

We call Φ a Schur ring homomorphism (for short, S–homomorphism). In addition, if Φ is an algebra isomorphism, we call it a Schur ring isomorphism (S–isomorphism).

Remark 3.1.

(1) It is easy to see that $\Phi(\bar{D}_0) = \bar{D}'_0$. (If $\Phi(\bar{D}_0) = \mu_0 \bar{D}'_j$ then $D'_j D'_j \subseteq D'_j$. Since D'_j is a finite set, $e \in D'_j$ and hence $D'_j = D'_0$.)

(2) If Φ is an S–isomorphism, then $d = d'$. Moreover, let p_{ij}^k and q_{ij}^k be intersection numbers of $\mathfrak{S}$ and $\mathfrak{S}'$, respectively. Then, by a suitable renumbering on the indices, $p_{ij}^k = [\frac{\mu_i \mu_j}{\mu_k}] q_{ij}^k$.

Set $T_j = \{i | \Phi(\bar{D}_i) = \mu_i \bar{D}'_j\}$. Let Ker$\Phi$ be the subalgebra of $\mathfrak{S}$ generated by $\{\bar{D}_i | i \in T_0\}$ and call it the kernel of Φ.

Lemma 3.2. *Keep all above notations. Then* $\Phi(\bar{D}_{i^*}) = \frac{\mu_{i^*}}{\mu_i}\Phi(\bar{D}_i)^{(-1)}$. *Hence if* $\bar{D}_i \in \mathrm{Ker}\Phi$ *then* $\bar{D}_{i^*} \in \mathrm{Ker}\Phi$.

Proof. Let p_{ij}^k and q_{ij}^k be the intersection numbers of $\mathfrak{S}$ and $\mathfrak{S}'$, respectively. Suppose $i \in T_j$ and $i^* \in T_h$.

$$\begin{aligned}\Phi(\bar{D}_i \bar{D}_{i^*}) &= \Big(\sum_{k=0}^{d} p_{ii^*}^k \bar{D}_k\Big) = \Phi\Big(\Big[\sum_{k\in T_0} p_{ii^*}^k \bar{D}_k\Big] + \cdots\Big) \\ &= \Big[\sum_{k\in T_0} p_{ii^*}^k \mu_k\Big] \bar{D}'_0 + \cdots .\end{aligned}$$

Since $p_{ii^*}^0 = v_i > 0$, $\mu_k > 0$ and $p_{ij}^k \geq 0$, $\sum\limits_{k\in T_0} p_{ii^*}^k \mu_k > 0$.

On the other hand,

$$\Phi(\bar{D}_i \bar{D}_{i^*} = \Phi(\bar{D}_i)\Phi(\bar{D}_{i^*}) = \mu_i \mu_{i^*} \bar{D}'_j \bar{D}'_h = \mu_i \mu_{i^*}(q_{jh}^0 \bar{D}'_0 + \cdots)$$

Thus $|D'_j|\delta_{jh^*} = q_{jh}^0 > 0$ and hence $h = j^*$ and $\Phi(\bar{D}_{i^*}) = \mu_{i^*}\bar{D}'_{j^*} = \mu_{i^*}\bar{D}_{j'}^{(-1)} = \frac{\mu_{i^*}}{\mu_i}\Phi(\bar{D}_i)^{(-1)}$. Thus $\bar{D}_{i^*} \in \ker\Phi$ if $\bar{D}_i \in \ker\Phi$. □

Lemma 3.3. KerΦ *is an S–subring of* $\mathfrak{S}$ *and* $\Phi(\mathfrak{S})$ *is an S–subring of* $\mathfrak{S}'$. *Moreover, the set of principal basis elements of* $\Phi(\mathfrak{S})$ *is* $\{\bar{D}'_j | T_j \neq \phi\}$.

Proof. It follows from Lemma 3.2 and Lemma 2.1. □

Note that, generally $\mathrm{Ker}\Phi$ is not normal. For example, let G be the symmetric group S_3 of degree 3 and G' be the alternating group A_3 of degree 3. Let $\mathfrak{S}=(G;\{D_0=\{e\},D_1=\{(1\,2)\},D_2=\{(1\,3),(2\,3)\},D_3=\{(1\,2\,3),(1\,3\,2)\}\})$ and $\mathcal{G}'=(G';\{D_0'=\{e\},D_1'=\{(1\,2\,3),(1\,3\,2)\}\})$ be two S-rings. Then we have the following multiplication tables.

	d_0	d_1	d_2	d_3
d_0	d_0	d_1	d_2	d_3
d_1	d_1	d_0	d_3	d_2
d_2	d_2	d_3	$2d_0+d_3$	$2d_1+d_2$
d_3	d_3	d_2	$2d_1+d_2$	$2d_0+d_3$

	d_0'	d_1'
d_0'	d_0'	d_1'
d_1'	d_i'	$2d_0'+d_1'$

where $d_i=\bar{D}_i$, $i=0,1,2,3$ and $d_j'=\bar{D}_j'$, $j=0,1$. We can define an S-homomorphism $\Phi:\mathfrak{S}\to\mathfrak{S}'$ by $\Phi(\bar{D}_0)=\bar{D}_0'$, $\Phi(\bar{D}_1)=\bar{D}_0'$, $\Phi(\bar{D}_2)=\bar{D}_1'$, $\Phi(\bar{D}_3)=\bar{D}_1'$. Then $\mathrm{Ker}\Phi=(\{e,(1\,2)\};\{D_0,D_1\})$ is not a normal S-ring of $\mathfrak{S}$.

Lemma 3.4. *Let $\Phi:\mathfrak{S}\to\mathfrak{S}'$ be an S-homomorphism. Φ is a monomorphism if and only if* $\mathrm{Ker}\Phi=<\bar{D}_0>$.

Proof. It is clear that $\mathrm{Ker}\Phi=<\bar{D}_0>$ if Φ is a monomorphism.

Conversely, suppose $\Phi(\bar{D}_i)=\Phi(\bar{D}_k)$. Then

$$\Phi(\bar{D}_i\bar{D}_{k^*})=\Phi(\bar{D}_i)\frac{\mu_{i^*}}{\mu_i}\Phi(\bar{D}_k)^{(-1)}=c\bar{D}_0'+\cdots\quad\text{for some } c\neq 0.$$

Since $\bar{D}_i\bar{D}_{k^*}=p^0_{ik^*}\bar{D}_0+\sum_{j>0}p^j_{ik^*}\bar{D}_j$ and $\mathrm{Ker}\Phi=<\bar{D}_0>$, $p^0_{ik^*}=c\neq 0$. Hence $i=k$ and Φ is a monomorphism. □

Lemma 3.5. *Let $\mathfrak{S}_H=(H;\mathcal{P}_H)$ and $\mathfrak{S}_K=(K;\mathcal{P}_K)$ be S-subrings of $\mathfrak{S}=(G;\mathcal{P})$. If $D\in\mathcal{P}_K$ and $D\cap H\neq\phi$, then $D\in\mathcal{P}_H\cap\mathcal{P}_K$.*

Proof. It is because $\mathcal{P}$, $\mathcal{P}_H$ and $\mathcal{P}_K$ are partitions of G, H and K, respectively, and $\mathcal{P}_H$ and $\mathcal{P}_K$ are subsets of $\mathcal{P}$. □

Theorem 3.6. (*First Isomorphism Theorem of Schur rings*)

Let $\mathfrak{S}=(G;\{D_i|0\le i\le d\})$ and $\mathfrak{S}'=(G';\{D_i'|0\le i\le d'\})$ be two S-rings. If $\Phi:\mathfrak{S}\to\mathfrak{S}'$ is an S-homomorphism and $\mathrm{Ker}\Phi$ *is normal, then the S-ring $\Phi(\mathfrak{S})$ is isomorphic to the S-ring $\mathfrak{S}/\mathrm{Ker}\Phi$. We often denote $\Phi(\mathfrak{S})\cong\mathfrak{S}/\mathrm{Ker}\Phi$.*

Proof. Without loss of generality, we may assume Φ is onto, i.e., $\Phi(\mathfrak{S}) = \mathfrak{S}'$. Suppose Ker$\Phi$ is an S–ring over H, for some $H \lhd G$. We write $\mathfrak{S}/\text{Ker}\Phi$ by $\bar{\mathfrak{S}}$. Then $\bar{\mathfrak{S}} = (G/H; \{E_0, E_1, \cdots, E_t\})$ where E_a were defined as in section 2.

We define an algebra homomorphism $\Omega : \bar{\mathfrak{S}} \to \mathfrak{S}'$ as follows:

For $\bar{E}_a = \nu^*(\bar{D}_i)$ for some i, here ν is the natural epimorphism from G onto G/H, define

$$\Omega(\bar{E}_a) = \bar{D}'_j \quad \text{if and only if} \quad \Phi(\bar{D}_i) = \mu_i \bar{D}'_j.$$

Suppose $\bar{D}_i\bar{D}_{k^*} = \sum_{D_j \subseteq H} p^j_{ik^*}\bar{D}_j + \sum_{D_j \cap H = \phi} p^j_{ik^*}\bar{D}_j$, then

$$\nu^*(\bar{D}_i\bar{D}_{k^*}) = \Big(\sum_{D_j \subseteq H} p^j_{ik^*}v_j\Big)\bar{E}_0 + \sum_{D_j \cap H = \phi} p^j_{ik^*}\nu^*(\bar{D}_j), \quad \text{where} \quad v_j = |D_j|.$$

If $\nu^*(\bar{D}_i) = \nu^*(\bar{D}_k)$ then $\sum_{D_j \subseteq H} p^j_{ik^*}v_j \neq 0$ and $\sum_{D_j \subseteq H} p^j_{ik^*} \neq 0$. Hence $\sum_{D_j \subseteq H} p^j_{ik^*}\mu_j \neq 0$. So we have

$$\Phi(\bar{D}_i)\Phi(\bar{D}_{k^*}) = \Phi(\bar{D}_i\bar{D}_{k^*}) = \sum_{D_j \subseteq H} p^j_{ik^*}\mu_j\bar{D}'_0 + \sum_{D_j \cap H = \phi} p^j_{ik^*}\Phi(\bar{D}_j),$$

hence if $\Phi(\bar{D}_i) = \mu_i\bar{D}'_r$ and $\Phi(\bar{D}_k) = \mu_k\bar{D}'_h$ then $r = h$. So that, Ω is well-defined.

Obviously, Ω is onto.

Suppose $\Omega(\bar{E}_a) = \bar{D}'_0$. Let $\bar{E}_a = \nu^*(\bar{D}_i)$ then $\bar{D}_i \in \text{Ker}\Phi$, i.e., $D_i \subseteq H$. Thus $\bar{E}_a = \bar{E}_0$ is the identity of $\bar{\mathfrak{S}}$. Ω is an S–isomorphism by Lemma 3.4. □

Examples.

(1) Suppose $\varphi : G \to G'$ is a group momomorphism. If $\mathcal{G}$ is an S–ring over G, then $\mathfrak{S}' = \varphi^*(\mathfrak{S})$ is an S–ring over $\varphi(G)$ and $\varphi^*|_{\mathfrak{S}}$ is an S–momomorphism.

(2) Let $\mathfrak{S} = (G; \{D_i | 0 \le i \le d\})$ and $\mathfrak{S}' = (G'; \{D'_i | 0 \le i \le d'\})$ be two S–rings. Let $\varphi : G \to G'$ be a group homomorphism. If $\Phi = \varphi^*|_{\mathfrak{S}}$ is an S–homomorphism then $\Phi(\mathfrak{S}) \cong /\text{Ker}\Phi$, since $\bigcup_{\bar{D} \in \text{Ker}\Phi} D = \text{Ker}\varphi \lhd G$.

Theorem 3.7. (*Second Isomorphism Theorem of Schur rings*) *Suppose* $\mathfrak{S}_H = (H; \mathcal{P}_H)$ *and* $\mathfrak{S}_K = (K; \mathcal{P}_K)$ *are S-subrings of* $\mathfrak{S} = (G; \mathcal{P})$. *If* $\mathfrak{S}_H \lhd \mathfrak{S}$, *then* $\mathfrak{S}_K/\mathfrak{S}_H \cap \mathfrak{S}_K \cong \mathfrak{S}_H\mathfrak{S}_K/\mathfrak{S}_H$.

Proof. Let $\iota : K \to HK$ be the imbedding mapping and $\nu : HK \to HK/H$ be the natural epimorphism. Clearly $\iota^* : \mathfrak{S}_K \to \mathfrak{S}_H\mathfrak{S}_K$ is an S–momomorphism. By Theorem 2.10 that $\nu^* : \mathfrak{S}_H\mathfrak{S}_K \to \mathfrak{S}_H\mathfrak{S}_K/\mathfrak{S}_H$ is an S–epimorphism. So $\nu^*\iota^*$ is an S–homomorphism from $\mathfrak{S}_K$ into $\mathfrak{S}_H\mathfrak{S}_K/\mathfrak{S}_H$. Since $\nu\iota$ is onto, $\nu^*\iota^*$ is onto.

Clearly $< \mathcal{P}_H \cap \mathcal{P}_K > \subseteq \mathrm{Ker}(\nu^*\iota^*)$. Suppose $D \in \mathcal{P}_K$ and $\nu^*\iota^*(\bar{D}) = ae$ for some $a \in I\!N$. Since ι is an imbedding, $\nu^*(\bar{D}) = ae$ and hence $D \subseteq H$. By Lemma 3.5 that $D \in \mathcal{P}_H \cap \mathcal{P}_K$. Thus $\mathrm{Ker}(\nu^*\iota^*) = < \mathcal{P}_H \cap \mathcal{P}_K > = \mathfrak{S}_H \cap \mathfrak{S}_K$. By Theorem 3.6 we get the conclusion of this theorem. □

Theorem 3.8. (*Third Isomorphism Theorem of Schur rings*) *Let* $\mathfrak{S}_K = (K; \mathcal{P}_K)$ *be a normal S–subring of* $\mathfrak{S} = (G; \mathcal{P})$, $\mathfrak{S}_H = (H; \mathcal{P}_H)$ *an S–subring of $\mathfrak{S}$ containing* $\mathfrak{S}_K$. *Then* $\bar{\mathfrak{S}}_H = \mathfrak{S}_H/\mathfrak{S}_K$ *is an S–subring of* $\bar{\mathfrak{S}} = \mathfrak{S}/\mathfrak{S}_K$. *If* $\mathfrak{S}_H$ *is normal then* $\bar{\mathfrak{S}}/\bar{\mathfrak{S}}_H \cong /\mathfrak{S}_H$.

Proof. Obviously, $\bar{\mathfrak{S}}_H < \bar{\mathfrak{S}}$ and $\bar{\mathfrak{S}}_H \lhd \bar{\mathfrak{S}}$ if $\mathfrak{S}_H \lhd \mathfrak{S}$.

Let $\varphi : G/K \to G/H$ be a homomorphism defined by $\varphi(gK) = gH$, where $g \in G$. Let $\nu_H : G \to H$ and $\nu_K : G \to K$ be the natural epimorphisms, respectively. Clearly $\varphi\nu_K = \nu_H$. Suppose E is an $\bar{\mathfrak{S}}$–principal subset of G/K. From Corollary 2.9 we know that $\nu_H^*(\bar{D}) = a\bar{E}$ for some $a \in I\!N \backslash \{0\}$ and $D \in \mathcal{P}$. Thus

$$(**) \qquad a\varphi^*(\bar{E}) = \varphi^*\nu_K^*(\bar{D}) = \nu_H^*(\bar{D}) = b\bar{F}$$

for some $b \in I\!N\backslash\{0\}$ and for some $\mathfrak{S}/\mathfrak{S}_H$–principal subset F of G/H. Hence φ^* is an S–homomorphism. Suppose $\bar{E}$ is a generator of $\mathrm{Ker}\varphi^*$, where E is an $\bar{\mathfrak{S}}$–principal subset of G/K. From (**) $E \subseteq H/K$, and then $\bar{E} \in \bar{\mathfrak{S}}_H$. Hence $\mathrm{Ker}\varphi^* \subseteq \bar{\mathfrak{S}}_H$.

Let $\bar{E}$ be a principal basis of $\bar{\mathfrak{S}}_H$ then $\bar{E} = \sum_{h\in S} hK$, for some subset S in H. Clearly, $\bar{E} \in \mathrm{Ker}\varphi^*$. Thus $\mathrm{Ker}\varphi^* = \bar{\mathfrak{S}}_H$. □

4. DIRECT PRODUCT

Given two groups G and H, it is known that we can construct a new group $G \times H$ which is called the direct product of G and H. Now when given two S–rings

$\mathfrak{S}_1$ and $\mathfrak{S}_2$ over G and H, respectively, we also want to construct a new S–ring over $G \times H$ analogously.

For convenience, we write all the identities of groups as e and we identify A with $A \times \{e\}$ and B with $\{e\} \times B$, where $A \subseteq G$ and $B \subseteq H$.

Now suppose $\mathfrak{S}_1 = (\ G; \{D_0, D_1, \cdots, D_d\}\ )$ and $\mathfrak{S}_2 = (\ H; \{E_0, E_1, \cdots, E_r\}\ )$ are S–rings over G and H, respectively. Consider the set $\mathcal{P} = \{D_i \times E_j | 0 \le i \le d,\ 0 \le j \le r\}$. Clearly, $\mathcal{P}$ satisfies the axiom **[S1]**.

For each $D_i \times E_j$, $(D_1 \times E_j)^{(-1)} = D_{i^*} \times E_{j^*}$. So that $\mathcal{P}$ satisfies **[S2]**.

Since

$$\begin{aligned}
(\overline{D_i \times E_j})(\overline{D_h \times E_k}) &= (\bar{D}_i \bar{D}_h)(\bar{E}_j \bar{E}_k) \\
&= \Big(\sum_\alpha p_{ih}^\alpha \bar{D}_\alpha\Big)\Big(\sum_\beta q_{jk}^\beta \bar{E}_\beta\Big) \\
&= \sum_\alpha \sum_\beta (p_{ih}^\alpha q_{jk}^\beta) \overline{D_\alpha \times E_\beta}
\end{aligned}$$

where p_{ih}^α and q_{jk}^β are intersection numbers of $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively, so that $\mathcal{P}$ satisfies **[S3]**. Hence $(G \times H; \mathcal{P})$ is an S–ring over $G \times H$. This S–ring is called the direct product of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ and is denoted by $\mathfrak{S}_1 \otimes \mathfrak{S}_2$.

In group theory, if H and K are normal subgroups of a group G then $HK \cong H \times K$ is a normal subgroup of G. For S–ring, there is no any analogous conclusion. That is, if $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are normal S–rings of an S–ring $\mathfrak{S}$ then $\mathfrak{S}_1 \otimes \mathfrak{S}_2$ may not be an S–subring of $\mathfrak{S}$. There is an example below:

Let $G = <\rho | \rho^{12} = e>$ then $\mathfrak{S} = (G; \mathcal{P})$ is an S–ring where $\mathcal{P} = \{\{e\}, \{\rho, \rho^{11}\}, \{\rho^5, \rho^7\}, \{\rho^4, \rho^8\}, \{\rho^2, \rho^{10}\}, \{\rho^3, \rho^9\}, \{\rho^6\}\}$. It contains two normal S–subrings $\mathfrak{S}_1 = (H; \{\{e\}, \{\rho^3, \rho^9\}, \{\rho^6\}\})$ and $\mathfrak{S}_2 = (K; \{\{e\}, \{\rho^4, \rho^8\}\})$, where $H = \{e, \rho^3, \rho^6, \rho^9\}$ and $K = \{e, \rho^4, \rho^8\}$. Then $\mathfrak{S}_1 \otimes \mathfrak{S}_2 = (G; \{\{e\}, \{\rho, \rho^{11}, \rho^5, \rho^7\}, \{\rho^4, \rho^8\}, \{\rho^2, \rho^{10}\}, \{\rho^3, \rho^9\}, \{\rho^6\}\})$ which is not an S–subring of $\mathfrak{S}$. Note that $\mathfrak{S}/\mathfrak{S}_1 \cong \mathfrak{S}_2$, which is simple, under the S–homomorphism which is induced by the group homomorphism $\varphi : G \to K$ defined by $\rho \mapsto \rho^4$.

In group theory, if H and K are normal subgroups of a group G and $|H||K| = |G|$ then $H \times K \cong G$. Analogously, for S–ring there is the following proposition.

Proposition 4.1. *If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are normal S–rings of an S-ring $\mathfrak{S}$ and the dimension of $\mathfrak{S}$ is equal to the product of the dimensions of the S–subrings. Then $\mathfrak{S}_1 \otimes \mathfrak{S}_2 = \mathfrak{S}$.*

Proof. Since $\mathfrak{S}_1 \otimes \mathfrak{S}_2$ is a subalgebra of $\mathfrak{S}$ and $\dim_{\mathbb{C}} \mathfrak{S}_1 \otimes \mathfrak{S}_2 = \dim_{\mathbb{C}} \mathfrak{S}_1 \times \dim_{\mathbb{C}} \mathfrak{S}_2$, $\mathfrak{S}_1 \otimes \mathfrak{S}_2 = \mathfrak{S}$. □

Note that it is easy to see that $\mathfrak{S}_1 \otimes \mathfrak{S}_2 \cong \mathfrak{S}_2 \otimes \mathfrak{S}_1$, where $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are S–rings.

Proposition 4.2. *If an S–ring $\mathfrak{S} = \mathfrak{S}_1 \otimes \mathfrak{S}_2$, where $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are S–subrings of $\mathfrak{S}$, then $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are normal and $\mathfrak{S}/\mathfrak{S}_1 \cong \mathfrak{S}_2$.*

Proof. Suppose $\mathfrak{S}_i$ is an S–ring over G_i, $i = 1, 2$. Consider the natural group epimorphism $\nu : G_1 \times G_2 \to G_2$. Then $\nu^*|_{\mathfrak{S}}$ is an S–epimorphism with kernel $\mathfrak{S}_1$.□

The converse of proposition 4.2 is not true (see the example above).

5. LATTICE OF S–SUBRINGS OF AN S–RING

It is clear that the set of all S–subrings of an S–ring $\mathfrak{S}$ ordered by inclusion is a partially ordered set. For any S–subrings $\mathfrak{S}_H = (H; \mathcal{P}_H)$ and $\mathfrak{S}_K = (K; \mathcal{P}_K)$ of $\mathfrak{S}$, if we write $< \mathcal{P}_H \cup \mathcal{P}_K >$ and $\mathfrak{S}_H \cap \mathfrak{S}_K$ as $\mathfrak{S}_H \vee \mathfrak{S}_K$ and $\mathfrak{S}_H \wedge \mathfrak{S}_K$, respectively. Then $\mathfrak{S}_H \vee \mathfrak{S}_K$ and $\mathfrak{S}_H \wedge \mathfrak{S}_K$ are the least upper bound and the greatest lower bound of $\mathfrak{S}_H$ and $\mathfrak{S}_K$, respectively. Hence the set of all S–subrings of an S–ring $\mathfrak{S}$ ordered by inclusion is a complete lattice.†

Lemma 5.1. *For any S–subrings $\mathfrak{S}_H = (H; \mathcal{P}_H)$, $\mathfrak{S}_K = (K; \mathcal{P}_K)$ and $\mathfrak{S}_L = (H; \mathcal{P}_L)$ of $\mathfrak{S}$, $(\mathfrak{S}_H \wedge \mathfrak{S}_K) \vee (\mathfrak{S}_H \wedge \mathfrak{S}_L) \subseteq \mathfrak{S}_H \wedge (\mathfrak{S}_K \vee \mathfrak{S}_L)$, where "$\vee$" and "$\wedge$" are defined above.*

Proof. Suppose $x \in (\mathfrak{S}_H \wedge \mathfrak{S}_K) \vee (\mathfrak{S}_H \wedge \mathfrak{S}_L) = < (\mathcal{P}_H \cap \mathcal{P}_K) \cup (\mathcal{P}_H \cap \mathcal{P}_L) > = < \mathcal{P}_H \cap (\mathcal{P}_K \cup \mathcal{P}_L) >$. Clearly $x \in \mathfrak{S}_H \wedge (\mathfrak{S}_K \vee \mathfrak{S}_L)$. □

Theorem 5.2. *The lattice of all normal S–subrings of an S–ring ordered by inclusion is modular.*‡

† The general consideration of lattices can be found in [2].
‡ The definition can be found in [2].

Proof. By the Lemma 5.1 and Theorem 2.12 (ii), we suffice to show that if $\mathfrak{S}_K = (K;\mathcal{P}_K) \subseteq \mathfrak{S}_H = (H;\mathcal{P}_H)$ then $\mathfrak{S}_K(\mathfrak{S}_H \cap \mathfrak{S}_L) \supseteq \mathfrak{S}_H \cap (\mathfrak{S}_K\mathfrak{S}_L)$.

Suppose D is an $\mathfrak{S}_H \cap (\mathfrak{S}_K\mathfrak{S}_L)$-principal subset then $D \in \mathcal{P}_H$ and $\bar{D}$ appears in the expression of $\bar{E}\bar{F}$ where $E, F \in \mathcal{P}_K \cup \mathcal{P}_L$. If both E and F are in $\mathcal{P}_K$ then $\bar{D} \in < \mathcal{P}_K >$ and hence $\bar{D} \in < \mathcal{P}_K \cup (\mathcal{P}_H \cap \mathcal{P}_L) >$. If both E and F are in $\mathcal{P}_L$ then $\bar{D} \in < \mathcal{P}_L >$, i.e., $D \in \mathcal{P}_L$. Since $D \in \mathcal{P}_H$, $\bar{D} \in < \mathcal{P}_H \cap \mathcal{P}_L >$ and hence $\bar{D} \in < \mathcal{P}_K \cup (\mathcal{P}_H \cap \mathcal{P}_L) >$. If $E \in \mathcal{P}_K$ and $F \in \mathcal{P}_L$ then $D \subseteq EF$. Thus $E^{(-1)}D \cap F \neq \phi$. Since $E \subseteq H$, $F \cap H \neq \phi$. By Lemma 3.5 that $F \in \mathcal{P}_H \cap \mathcal{P}_L$. Hence $\bar{D} \in \mathfrak{S}_K(\mathfrak{S}_H \cap \mathfrak{S}_K)$. □

Thus the theorem of Jordan–Hölder–Dedekind (see [2]) holds for the lattice of all normal S–subrings of a given S–ring. That is, we have the following theorem which is analogue of the Jordan–Hölder theorem for finite groups (see [2]).

Theorem 5.3. *Let $\mathfrak{S}$ be an S–ring and let $\mathfrak{S} = \mathfrak{S}_0 \triangleright \mathfrak{S}_1 \triangleright \cdots \triangleright \mathfrak{S}_s = (\{e\};\{D_0\})$ and $\mathfrak{S} = \mathfrak{S}'_0 \triangleright \mathfrak{S}'_1 \triangleright \cdots \triangleright \mathfrak{S}'_t = (\{e\};\{D_0\})$ be two composition series for $\mathfrak{S}$ (i.e., $\mathfrak{S}_{i-1}/\mathfrak{S}_i$ and $\mathfrak{S}'_{j-1}/\mathfrak{S}'_j$ are simple for all $1 \leq i \leq s$ and $1 \leq j \leq t$). Then $s = t$ and there exists a permutation f on $\{0, 1, \cdots, s\}$ such that $\mathfrak{S}_{i-1}/\mathfrak{S}_i \cong \mathfrak{S}'_{f(i)-1}/\mathfrak{S}'_{f(i)}$.*

Now we have the Jordan–Hölder–Dedekind theorem. Naturally, the next question will be the extension problem. That is, given a normal S–subring $\mathfrak{S}_1$ of an unknown S–ring $\mathfrak{S}$ and also given the quotient S–ring $\mathfrak{S}/\mathfrak{S}_1$, may one recapture $\mathfrak{S}$? The group extension problem is a difficult problem. The S–ring extension problem is, of course, more difficult then group extension problem. It is because that S–rings are generalization of groups in some sense. For example, given a subgroup K of a cyclic group G and G/K then G is determined uniquely. But in S–ring this may not hold. Please see the example described in section 4.

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