Orthogonal (g,f)-Factorizations in Networks[1]

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Abstract

Let $G=(V,E)$ be a graph and let g and f be two integer-valued functions defined on V such that $k \le g(x) \le f(x)$ for all $x \in V$. Let $H_1, H_2, \cdots, H_k$ be subgraphs of G such that $|E(H_i)| = m, 1 \le i \le k$, and $V(H_i) \cap V(H_j) = \varnothing$ when $i \ne j$. In this paper it is proved that every $(mg+m-1, mf-m+1)$-graph G has a (g,f)-factorization orthogonal to H_i for $i = 1, 2, \cdots, k$ and shown that there are polynomial-time algorithms to find the desired (g,f)-factorizations.

Key words and phrases: Network, graph, (g,f)-factorization, orthogonal factorization

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1 Introduction

Many physical structures can conveniently be modeled by networks. Examples include a communication network with the nodes and links modeling cities and communication channels respectively; or a railroad network with nodes and links representing railroad stations and railways between two stations respectively. Orthogonal factorizations in networks are very useful in combinatorial design, network design, circuit layout and so on [1]. It is well-known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes respectively. Henceforth we use the term "graph" instead of "network".

Graphs considered in this paper will be finite undirected graphs with neither multiple edges nor loops. Let $G=(V,E)$ be a graph, g and f be two integer-valued functions defined on V such that $g(x) \le f(x)$ for all $x \in V$. Then a (g,f)-*factor* of G is a spanning subgraph F of G with $g(x) \le d_F(x) \le f(x)$ for all $x \in V(F)$. In particular, if G itself is a (g,f)-factor, then G is called a (g,f)-*graph.* A (g,f)-*factorization* $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ of G is a partition of G into edge-disjoint (g,f)-factors $F_1, F_2, \cdots, F_m$. Let a and b be two non-negative integers with $a \le b$. If $g(x) = a$ and $f(x) = b$ for all $x \in V$, then a (g,f)-factor is called an $[a,b]$-*factor.*

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An $[a,b]$-*factorization* can be defined similarly. Let H be a subgraph of G. A factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ of G is *orthogonal* to H if $|E(H) \cap E(F_i)| = 1, 1 \leq i \leq m$. A subgraph with m edges is called an *m-subgraph*. Undefined notations and definitions can be found in [3].

Recently Xu, Liu and Tokuda studied the connected factors in $K_{1,n}$-free graphs containing a (g,f)-factor [12]. Niessen gave a characterization of graphs having all (g,f)-factors [10]. Kano obtained some sufficient conditions for a graph to have $[a,b]$-factorizations [6]. Anstee, Hell and Kirkpatrick discussed algorithms for (g,f)-factors [2, 5]. The interested reader may find many relevant results about factors and factorizations in [1] and [11]. Alspach *et al* [1] posed the following problem: Given a subgraph H of G, does there exist a factorization $\mathcal{F}$ of G of certain type orthogonal to H? Liu proved that every $(mg+m-1, mf-m+1)$-graph has a (g,f)-factorization orthogonal to a star or a matching [7, 8]. We consider the more general problem: Given subgraphs $H_1, H_2, \cdots, H_k$ of G, does there exist a factorization $\mathcal{F}$ of G orthogonal to every $H_i, 1 \leq i \leq k$? The purpose of this paper is to prove that for any vertex-disjoint m-subgraphs $H_1, H_2, \cdots, H_k$ of an $(mg+m-1, mf-m+1)$-graph G, there exists a (g,f)-factorization of G orthogonal to every $H_i, 1 \leq i \leq k$, where $k \leq g(x) \leq f(x)$ for every $x \in V$. We shall use various technique from [7] and [8]. Furthermore, we shall show that polynomial-time algorithms for finding the particular orthogonal (g,f)-factorizations can be deduced.

2 Some Lemmas

Let $G = (V, E)$ be a graph and $S \subseteq V$. For any function f defined on V, we put $f(S) = \sum_{x \in S} f(x)$ and $f(\varnothing) = 0$. If $S \subset V$, we denote by $G - S$ the subgraph obtained from G by deleting the vertices in S together with the edges incident with vertices in S. If $G' = (V', E')$ is a subgraph of G and $E^* \subset E$, then $G' - E^*$ denotes the graph $(V', E' \setminus E^*)$, whihc is a sub-graph of G. For a vertex x of G, the degree of x in G is denoted by $d_G(x)$. Let S and T be disjoint subsets of V. We write $E(S,T) = \{xy : xy \in E, x \in S \text{ and } y \in T\}$ and $e(S,T) = |E(S,T)|$. Let g and f be two integer-valued functions defined on V. If C is a component of $G - (S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$, then we say that C is *odd* or *even* if $e(T, V(C)) + f(V(C))$ is odd or even respectively. Let $h(S,T)$ denote the number of odd components of $G - (S \cup T)$. Set

$$\delta_G(S,T) = d_{G-S}(T) - g(T) - h(S,T) + f(S).$$

The following result is proved by Lovász in 1970.

Lemma 1 [9] *Let G be a graph, and let g and f be two integer-valued functions defined on V such that $g(x) \le f(x)$ for all $x \in V$. Then G has a (g, f)-factor if and only if $\delta_G(S, T) \ge 0$ for any two disjoint subsets S and T of V.*

Note that when $f(x) \ne g(x)$ for all $x \in V$, $h(S, T) = 0$. Let S and T be two disjoint subsets of V, E_1 and E_2 be two disjoint subsets of E. Let

$$U = V \setminus (S \cup T), E(S) = \{xy \in E : x, y \in S\},$$

and

$$E(T) = \{xy \in E : x, y \in T\}.$$

Write

$$E_1' = E_1 \cap E(S), \ \ E_1'' = E_1 \cap E(S, U);$$

$$E_2' = E_2 \cap E(T), \ \ E_2'' = E_2 \cap E(T, U);$$

$$\alpha(S, T; E_1, E_2) = 2|E_1'| + |E_1''|,$$

$$\beta(S, T; E_1, E_2) = 2|E_2'| + |E_2''|.$$

and

$$\Delta(S, T; E_1, E_2) = \alpha(S, T; E_1, E_2) + \beta(S, T; E_1, E_2).$$

If there is no ambiguity, we substitute $\alpha(S, T; E_1, E_2), \beta(S, T; E_1, E_2)$ and $\Delta(S, T; E_1, E_2)$ for α, β and Δ, respectively.

Lemma 2 *Let $G = (V, E)$ be a graph, and let g and f be two integer-valued functions defined on V and $0 \le g(x) < f(x) \le d_G(x)$ for all $x \in V$. Let E_1 and E_2 be two disjoint subsets of E. Then G has a (g, f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \varnothing$ if and only if for any two disjoint subsets S and T of V,*

$$\delta_G(S, T) = d_{G-S}(T) - g(T) + f(S) \ge \Delta(S, T; E_1, E_2).$$

Proof. Let $G' = G - E_2$. Then G has a (g, f)-factor F such that $E(F) \cap E_2 = \varnothing$ if and only if G' has a (g, f)-factor. By Lemma 1, this is true if and only if for all disjoint subsets S and T of V,

$$\delta_{G'}(S, T) = d_{G'-S}(T) - g(T) + f(S) \ge 0.$$

It is easy to see that $\delta_{G'}(S,T) = \delta_G(S,T) - \beta$. Therefore $\delta_{G'}(S,T) \geq 0$ if and only if $\delta_G(S,T) \geq \beta$. Let $g'(x) = d_G(x) - f(x)$ and $f'(x) = d_G(x) - g(x)$. It is obvious that G has a (g,f)-factor containing all edges of E_1 if and only if G has a (g',f')-factor excluding all edges of E_1. By the above argument, this is equivalent to

$$\delta_G(S,T;g',f') = d_{G-S}(T) - g'(T) + f'(S) \geq 2|E_1 \cap E(T)| + |E_1 \cap E(T,U)|.$$

Note that

$$\begin{aligned}\delta_G(S,T;g',f') &= d_{G-S}(T) - g'(T) + f'(S) = d_{G-S}(T) - d_G(T) + f(T) + d_G(S) - g(S)\\ &= d_{G-T}(S) - g(S) + f(T) = \delta_G(T,S;g,f) = \delta_G(T,S).\end{aligned}$$

Hence G has a (g,f)-factor containing all edges of E_1 if and only if

$$\delta_G(T,S) \geq 2|E_1 \cap E(T)| + |E_1 \cap E(T,U)|,$$

that is

$$\delta_G(S,T) \geq 2|E_1 \cap E(S)| + |E_1 \cap E(S,U)| = \alpha.$$

Since G has a (g,f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \varnothing$ if and only if G' has a (g,f)-factor F such that $E_1 \subseteq E(F)$. By the above discussion, this is equivalent to $\delta_{G'}(S,T) \geq \alpha$. Note that $\delta_{G'}(S,T) = \delta_G(S,T) - \beta$. It follows that G has a (g,f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \varnothing$ if and only if $\delta_G(S,T) \geq \alpha + \beta$. ∎

In the following we always assume that G is an $(mg+m-1, mf-m+1)$-graph where $m \geq 1$ is an integer. Define

$$p(x) = \max\{g(x), d_G(x) - (m-1)f(x) + m - 2\},$$

and

$$q(x) = \min\{f(x), d_G(x) - (m-1)g(x) - m + 2\}.$$

It follows that

$$g(x) \leq p(x) < q(x) \leq f(x).$$

Let

$$\Delta_1(x) = \frac{1}{m}d_G(x) - p(x), \qquad \Delta_2(x) = q(x) - \frac{1}{m}d_G(x).$$

We have the following lemma.

Lemma 3 *For every $x \in V$ and $m \geq 2$*

$$\Delta_1(x) \geq \begin{cases} \frac{1}{m}, & \textit{if } p(x) > g(x) \textit{ and } d_G(x) = mf(x) - m + 1, \\ \frac{m-1}{m}, & \textit{otherwise.} \end{cases}$$

and

$$\Delta_2(x) \geq \begin{cases} \frac{1}{m}, & \textit{if } q(x) < f(x) \textit{ and } d_G(x) = mg(x) + m - 1, \\ \frac{m-1}{m}, & \textit{otherwise.} \end{cases}$$

Proof. If $p(x) = g(x)$, then

$$\Delta_1(x) \geq \frac{mg(x) + m - 1}{m} - g(x) = \frac{m-1}{m}.$$

Otherwise, by the definition of $p(x)$, we have $p(x) = d_G(x) - (m-1)f(x) + m - 2$.

If $d_G(x) \leq mf(x) - m$, then

$$\begin{aligned} \Delta_1(x) &= \frac{1}{m} d_G(x) - p(x) \\ &\geq \frac{1}{m} d_G(x) - d_G(x) + (m-1)f(x) - m + 2 \\ &\geq \frac{1-m}{m}(mf(x) - m) + (m-1)f(x) - m + 2 = 1 \geq \frac{m-1}{m}. \end{aligned}$$

Otherwise, we have $d_G(x) = mf(x) - m + 1$, and then

$$\Delta_1(x) \geq \frac{1-m}{m}(mf(x) - m + 1) + (m-1)f(x) - m + 2 = \frac{1}{m}.$$

So the first inequality is proved. Similarly, we can prove the second inequality. ∎

Lemma 4 *For any two disjoint subsets S and T of V*

$$\delta_G(S, T; p, q) = \Delta_1(T) + \Delta_2(S) + \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S).$$

Proof. Since for every $x \in V$ $p(x) < q(x)$, we have

$$\begin{aligned} \delta_G(S, T) &= d_{G-S}(T) - p(T) + q(S) \\ &= d_G(T) - e(S, T) - p(T) + q(S) \\ &= (\frac{1}{m} d_G(T) - p(T)) + (q(S) - \frac{1}{m} d_G(S)) + \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \\ &= \Delta_1(T) + \Delta_2(S) + \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S). \end{aligned}$$

∎

3 The Proofs of Main Results

Let G be a graph and let g and f be two integer-valued functions defined on V such that $k \le g(x) < f(x)$ for all $x \in V$. Let $H_1, H_2, \cdots, H_k$ be mutually vertex-disjoint m-subgraphs of G. For $i = 1, 2, \cdots, k$, we put

$$A_{i1} = \{xy \in E(H_i) : p(x) \ge g(x) + 1 \text{ and } p(y) \ge g(y) + 1\},$$

$$A_{i2} = \{xy \in E(H_i) : p(x) \ge g(x) + 1 \text{ or } p(y) \ge g(y) + 1\},$$

and

$$A_i = \begin{cases} A_{i1}, & \text{if } A_{i1} \ne \varnothing, \\ A_{i2}, & \text{if } A_{i1} = \varnothing \text{ and } A_{i2} \ne \varnothing, \\ E(H_i), & \text{otherwise.} \end{cases}$$

Choose $u_i v_i \in A_i$ for $i = 1, 2, \cdots, k$. Let $E_1 = \{u_i v_i : 1 \le i \le k\}$ and $E_2 = \left(\bigcup_{i=1}^{k} E(H_i) \right) \setminus E_1$. We have $|E_1| = k$ and $|E_2| = (m-1)k$. For any two disjoint subsets S and T of V, we define $E_1', E_1'', E_2', E_2'', \alpha, \beta$ and Δ as the same as in Section 2. By the definition of α and β, we have

$$\alpha \le \min\{2k, |S|\}$$

and

$$\beta \le \min\{(m-1)|T|, 2(m-1)k\}.$$

Define $p(x)$ and $q(x)$ as the same as in Section 2. To prove the main theorem we first prove the following essential lemma.

Lemma 5 *Let G be an $(mg + m - 1, mf - m + 1)$-graph, where $m \ge 1$. Then G has a (p, q)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \varnothing$.*

Proof. By Lemma 2, it suffices to show that for any two disjoint subsets S and T

$$\delta_G(S, T) = \delta_G(S, T; p, q) \ge \Delta = \alpha + \beta.$$

Suppose to the contrary that there exist disjoint subsets S and T of V such that $\delta_G(S, T) < \alpha + \beta$. If $S = \varnothing$, then by Lemma 4,

$$\begin{aligned} \delta_G(S, T) &\ge \frac{m-1}{m} d_G(T) + \Delta_1(T) \\ &\ge \frac{m-1}{m}(mg(T) + m|T| - |T|) + \frac{|T|}{m} \\ &\ge (m-1)|T| \ge \beta = \alpha + \beta, \end{aligned}$$

which is a contradiction. If $T = \varnothing$, then by Lemma 4,

$$\begin{aligned}
\delta_G(S,T) &\geq \frac{1}{m}d_G(S) + \frac{|S|}{m} \\
&\geq \frac{1}{m}(mg(S) + m|S| - |S|) + \frac{|S|}{m} \\
&\geq |S| \geq \alpha = \alpha + \beta,
\end{aligned}$$

which is a contradiction also. Therefore we may assume that $S \neq \varnothing$ and $T \neq \varnothing$. Set

$$S_0 = \{x \in S : q(x) = f(x) \text{ or } d_G(x) > mg(x) + m - 1\}, \ \ S_1 = S \setminus S_0$$

$$T_0 = \{x \in T : p(x) = g(x) \text{ or } d_G(x) < mf(x) - m + 1\}, \ \text{ and } \ T_1 = T \setminus T_0.$$

We shall show that $p(x) = g(x)$ for all $x \in S_1$. Suppose $x \in S_1$, then $q(x) < f(x)$ and $d_G(x) = mg(x)+m-1$. By the definition of $q(x)$, we have $q(x) = mg(x)+m-1-(m-1)g(x)-m+2 = g(x)+1$. Since $g(x)+1 = q(x) \geq p(x)+1$ and $p(x) \geq g(x)$, we have $p(x) = g(x)$. It also follows from the definition of T_0 that $p(x) \geq g(x)+1$ for all $x \in T_1$.

Now let

$$S_0' = \{x \in S_0 : p(x) \geq g(x) + 1\}, \ \ S_1' = S \setminus S_0',$$

$$T_0' = \{x \in T_0 : p(x) = g(x)\}, \ \text{ and } \ T_1' = T \setminus T_0'.$$

Thus $p(x) = g(x)$ for every $x \in S_1' \cup T_0'$ and $p(x) \geq g(x) + 1$ for every $x \in T_1' \cup S_0'$. Note that when $m = 1$, Lemma 5 is trivial. So we may assume that $m \geq 2$.

Claim 1 $|S_0'| + |T_0'| < 2k - 2$.

Proof of Claim 1 Suppose to the contrary that $|S_0'| + |T_0'| \geq 2k - 2$. By Lemma 3 and Lemma 4, we have

$$\begin{aligned}
\delta_G(S,T) &\geq \frac{|T_1'| + (m-1)|T_0'|}{m} + \frac{|S_1'| + (m-1)|S_0'|}{m} + \frac{m-1}{m}d_{G-S}(T) + \frac{1}{m}d_{G-T}(S) \\
&= \frac{|T| + (m-2)|T_0'|}{m} + \frac{|S| + (m-2)|S_0'|}{m} + \frac{m-1}{m}d_{G-S}(T) + \frac{1}{m}d_{G-T}(S) \\
&= \frac{(m-2)|T_0'|}{m} + \frac{(m-2)|S_0'|}{m} + \frac{|T| + d_{G-T}(S)}{m} + \frac{|S| + d_{G-S}(T)}{m} + \frac{(m-2)d_{G-S}(T)}{m} \\
&\geq \frac{(m-2)(|S_0'| + |T_0'|)}{m} + \frac{mg(x) + m - 1 + \alpha - 1}{m} \\
&\quad + \frac{mg(y) + m - 1 + \beta - (m-1)}{m} + \frac{(m-2)\beta}{m}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{(m-2)(|S_0'|+|T_0'|)+2mk+m+\alpha+(m-1)\beta-2}{m}\\
&= \frac{(m-2)(|S_0'|+|T_0'|)+(m-1)2k+\alpha+(m-1)\beta+m+2k-2}{m}\\
&\geq \frac{(m-2)(|S_0'|+|T_0'|)+m\alpha+(m-1)\beta+2k+m-2}{m}\\
&\geq \frac{(m-2)(2k-2)+m\alpha+(m-1)\beta+2k-2+m}{m}\\
&= \frac{(m-1)(2k-2)+m\alpha+(m-1)\beta+m}{m}\\
&= \frac{2(m-1)k+m\alpha+(m-1)\beta-m+2}{m}\\
&\geq \frac{m\alpha+m\beta-m+2}{m}=\alpha+\beta-\frac{m-2}{m}.
\end{aligned}
$$

Because $\delta_G(S,T)$ is an integer, we have $\delta_G(S,T)\geq\alpha+\beta$, which is a contradiction. ∎

Now we are in the position to complete the proof of Lemma 5. Let $r=|V(E_1)\cap T_1'|$. Then $|V(E_1)\cap(S_0'\cup T_1')|\leq|S_0'|+r$. Let $u_iv_i\in E_1\cap E(H_i)$, where $u_i,\ v_i\in S\cup T$. By the choice of E_1, if $\{u_i,v_i\}\cap(S_0'\cup T_1')=\varnothing$, then $V(H_i)\cap T_1'=\varnothing$. If $|\{u_i,v_i\}\cap(S_0'\cup T_1')|=1$, then $|\{x,y\}\cap T_1'|\leq 1$ for any $xy\in E(H_i)$. If $\{u_i,v_i\}\subseteq S_0'\cup T_1'$, then $|\{x,y\}\cap T_1'|\leq 2$ for any $xy\in E(H_i)$. Let $r'=|V(E_1)\cap U|$, where $U=V\setminus(S\cup T)$. It is easy to see that $\beta(S,T_1')\leq|V(E_1)\cap(S_0'\cup T_1')|(m-1)+r'(m-1)\leq(|S_0'|+r+r')(m-1)$. Subsequently, we get

$$
\begin{aligned}
\beta(S,T) &= \beta(S,T_0')+\beta(S,T_1')\leq|T_0'|(m-1)+(|S_0'|+r+r')(m-1)\\
&= (m-1)(|S_0'|+|T_0'|+r+r').
\end{aligned}
$$

and $\alpha(S,T)\leq 2k-r-r'$, i.e. $2k\geq\alpha+r+r'$. By Lemma 4 and based on the proof of Claim 1, we obtain

$$
\begin{aligned}
\delta_G(S,T) &\geq \frac{(m-2)(|S_0'|+|T_0'|)+2mk+m+\alpha+(m-1)\beta-2}{m}\\
&\geq \frac{(m-2)(|S_0'|+|T_0'|)+(m-1)2k+m+\alpha+(m-1)\beta+2k-2}{m}\\
&> \frac{(m-1)(|S_0'|+|T_0'|)+(m-1)2k+m+\alpha+(m-1)\beta}{m}\\
&\geq \frac{(m-1)(|S_0'|+|T_0'|)+(m-1)(\alpha+r+r')+\alpha+(m-1)\beta+m}{m}\\
&\geq \frac{(m-1)(|S_0'|+|T_0'|+r+r')+m\alpha+(m-1)\beta+m}{m}\\
&\geq \frac{m\alpha+m\beta+m}{m}>\alpha+\beta.
\end{aligned}
$$

This contradiction completes the proof of Lemma 5. ∎

Now we are ready to prove the following main theorem.

Theorem 1 *Let G be an $(mg(x)+m-1, mf(x)-m+1)$-graph, where g and f are two integer-valued functions defined on V such that $k \leq g(x) \leq f(x)$. Let $H_1, H_2, \cdots, H_k$ be mutually vertex-disjoint m-subgraphs of G. Then G has a (g, f)-factorization orthogonal to every H_i, $1 \leq i \leq k$.*

Proof. We apply induction on m. The theorem is true for $m = 1$. Suppose the theorem holds for $m - 1$, where $m \geq 2$. By Lemma 5, G has a (p, q)-factor F_1 such that $E_1 \subseteq F_1$ and $E_2 \cap F_1 = \varnothing$. Clearly F_1 is also a (g, f)-factor of G. Set $G' = G - E(F_1)$. By the definition of $p(x)$ and $q(x)$

$$\begin{aligned} d_{G'}(x) &= d_G(x) - d_{F_1}(x) \geq d_G(x) - q(x) \\ &\geq d_G(x) - (d_G(x) - (m-1)g(x) - m + 2) \\ &= (m-1)g(x) + m - 2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} d_{G'}(x) &= d_G(x) - d_{F_1}(x) \leq d_G(x) - p(x) \\ &\leq (m-1)f(x) - m + 2. \end{aligned}$$

Hence G' is an $((m-1)g + m - 2, (m-1)f - m + 2)$-graph. Let $H_i' = H_i - E_1, 1 \leq i \leq k$. By the induction assumption G' has a (g, f)-factorization $\mathcal{F}' = \{F_2, \cdots, F_m\}$ orthogonal to every $H_i', 1 \leq i \leq k$. Thus G has a (g, f)-factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ orthogonal to every $H_i, 1 \leq i \leq k$. ∎

Substituting g and f by $g - 1$ and $f + 1$ in Theorem 1, respectively, we obtain an $(mg - 1, mf + 1)$-graph having a $(g - 1, f + 1)$-factorization orthogonal to any given k mutually vertex-disjoint m-subgraphs $H_1, H_2, \cdots, H_k$ where $g(x) \geq k + 1$. Therefore the following corollary holds.

Corollary 1 *Let G be an (mg, mf)-graph and $k + 1 \leq g(x) \leq f(x)$. Then for any mutually vertex-disjoint m-subgraphs $H_1, H_2, \cdots, H_k$ of G, there exists a $(g - 1, f + 1)$-factorization of G orthogonal to every H_i, $1 \leq i \leq k$.*

By Theorem 1, the following corollary holds.

Corollary 2 *Let G be an $(mg+m-1, mf-m+1)$-graph and $k \le g(x) < f(x)$. Then for any km-matching M of G there is a (g,f)-factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ of G such that $|F_i \cap M| = k$ for all $i, 1 \le i \le m$.*

Remark 1 *The bounds $mg+m-1$ and $mf-m+1$ in* Theorem 1 *are sharp in the following sense: If the lower bound $mg+m-1$ decreases or the upper bound $mf-m+1$ increases just by one,* Theorem 1 *will not hold. In fact, the graph G may have no any (g,f)-factorizations. Some examples can be found in* [8]. *In the proof of* Lemma 5, *it is required that $g(x) \ge k$ for all $x \in V$. Nevertheless, we conjecture that $g(x) \ge k$ can be improved, i.e., in the main theorem the condition $g(x) \ge k$ can be substituted by $g(x) \ge k-1$. We have only known that the above conjecture is true in some cases when $k=1$* [8]. *Of course, it is easy to see that the condition $g(x) \ge k$ can be substituted by $d_G(x) \ge mk+m-k$ in* Lemma 5 *and* Theorem 1.

Remark 2 *From the proofs in this paper polynomial-time algorithms for finding the orthogonal (g,f)-factors of an $(mg+m-1, mf-m+1)$-graph G in* Theorem 1 *can be deduced. Using the theory on network flows, Anstee gave a polynomial-time algorithm which either finds a (g,f)-factor or shows that one does not exist in* $\mathrm{O}(|V|^3)$ *operations* [2]. *Hell and Kirkpatrick gave* $\mathrm{O}(\sqrt{(g(V)}|E|)$ *algorithms for the general (g,f)-factor problems* [5]. *In particular, when $g(x) \ne f(x)$ for every $x \in V$, it is shown that there is a very simple (g,f)-factor algorithm of time complexity* $\mathrm{O}(g(V)|E|)$ *by finding alternating paths in* [4]. *Clearly, we can find $p(x)$, $q(x)$ and E_1 by* $\mathrm{O}(|V|)$ *operations. Let $G_1 = G - E_2$. Set $p'(x) = p(x) - 1$ and $q'(x) = q(x) - 1$ when $x = u_i, v_i,\ 1 \le i \le k$. Otherwise, set $p'(x) = p(x)$ and $q'(x) = q(x)$. Then we can find a (p', q')-factor F_1 in G_1 by the algorithms in* [5] *or in* [2]. *It is easy to see that F_1 is a (g,f)-factor of G containing E_1 and excluding E_2. It follows from the proof of* Theorem 1, *$G' = G - F_1$ is an $((m-1)g+m-2, (m-1)f-m+2)$-graph. Repeating the above procedure, we can find after at most $m-1$ operations a (g,f)-factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ orthogonal to mutually vertex-disjoint subgraphs $H_1, H_2, \cdots, H_k$ in G.*

Finally we ask the following question: If $H_1, H_2, \cdots, H_k$ are mutually edge-disjoint m-subgraphs of G, will Theorem 1 still hold? Does there exist a polynomial-time algorithm to verify that?

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