# On Strong Chromatic Index of Halin Graph

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**Abstract**

A strong $k$-edge-coloring of a graph $G$ is an assignment of $k$ colors to the edges of $G$ in such a way that any two edges meeting at a common vertex, or being adjacent to the same edge of $G$, are assigned different colors. The strong chromatic index of $G$ is the smallest number $k$ for which $G$ has a strong $k$-edge-coloring. A Halin graph is a planar graph consisting of a tree with no vertex of degree two and a cycle connecting the leaves of the tree. A caterpillar is a tree such that the removal of the leaves becomes a path. In this paper, we show that the strong chromatic index of cubic Halin graph is at most 9. That is, every cubic Halin graph is edge-decomposable into at most 9 induced matchings. Also we study the strong chromatic index of a cubic Halin graph whose characteristic tree is a caterpillar.

**Keywords** : Strong chromatic index, necklace, Halin graph, caterpillar
**AMS 2000 MSC** : 05C15

## 1 Introduction and notations

All graphs in this paper are finite and simple. All undefined symbols and concepts may be looked up from [2]

For $k$ being a positive integer, let $[\![k]\!] = \{1, 2, \dots, k\}$. A *strong $k$-edge-coloring* of a graph $G = (V, E)$ is a mapping $c : E \to [\![k]\!]$ in such a way that any two edges meeting at a common vertex, or being adjacent to the same edge of $G$, are assigned different values (colors). The *strong chromatic index* of $G$, denoted by $s\chi'(G)$, is the smallest number $k$ for which $G$ has a strong $k$-edge-coloring. A matching in a graph $G$ is *induced* if no two edges in the matching are joined by an edge in $G$. So $s\chi'(G) \le k$ if and only if $G$ is edge-decomposable into $k$ induced matchings.

A *Halin graph* $G = T \cup C$ is a plane graph that consists of a plane embedding of a tree $T$ and a cycle $C$ connecting the leaves (vertices of degree 1) of the tree such that $C$ is the boundary of the exterior face and the degree of each interior vertex (also called node) of $T$ is at least three. The tree $T$ and the cycle $C$ are called the *characteristic tree* and the *adjoint cycle* of $G$, respectively.

A tree is called a $(3,1)$-*tree* if the degree of each node is 3. A $(3,1)$-*caterpillar* $T$ is a $(3,1)$-tree if the removal of the leaves (together with their incident edges) becomes a path which is called the *spine* of $T$. In this paper, "caterpillar" means $(3,1)$-caterpillar.

Suppose $G$ is a Halin graph of order $2h+2$ with a caterpillar $T$ as its characteristic tree, $h \geq 1$. We name the vertices along the spine $P_h$ by 1, 2, ..., $h$. The vertices adjacent with 1 are named by 0 and $1'$. The vertices adjacent with $h$ are named by $h+1$ and $h'$. Other leaf adjacent with $i$ is named by $i'$, $2 \leq i \leq h-1$. Note that $0, 1', \ldots, h', h+1$ are vertices lying on the adjoint cycle $C_{h+2}$. We shall use this vertex labeling through this paper. Let $\mathcal{G}_h$ be the set of all cubic Halin graphs whose characteristic trees are caterpillars of order $2h+2$. Figure 1.1 shows all graphs in $\mathcal{G}_4$.

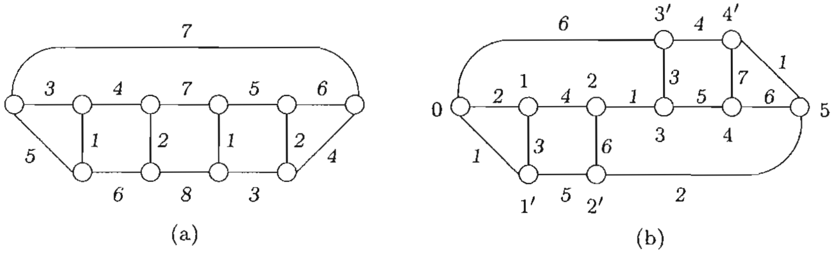


Figure 1.1. All two graphs in $\mathcal{G}_4$: (a) $Ne_4$, (b) another graph in $\mathcal{G}_4$.

Let $G \in \mathcal{G}_h$. If $\{0,1'\}$, $\{1',2'\}$, ..., $\{(h-1)',h'\}$, $\{h',h+1\}$, and $\{h+1,0\}$ are edges of the adjoint cycle of $G$ (i.e., vertices $0, 1' \cdots, h', h+1$ in $C_{h+2}$ are in order), then $G$ is called a *necklace*. It is denoted by $Ne_h$ (see Figure 1.2). It is easy to see that, $\mathcal{G}_h = \{Ne_h\}$ for $h = 1, 2, 3$.

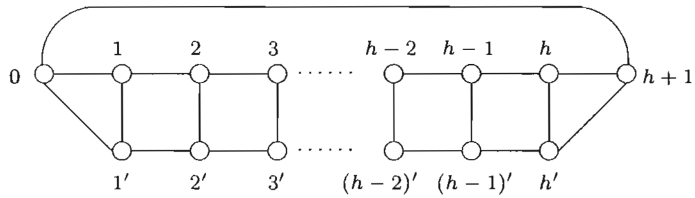


Figure 1.2. Necklace $Ne_h$.

In this paper, we investigate the strong chromatic index of a cubic Halin graph whose characteristic tree is a caterpillar.

# 2 Conjectures and Known Results

We use $\Delta$ to denote the maximum degree of a graph $G$. J. L. Fouquet and J. L. Jolivet [7, 8] first studied the strong edge-coloring of cubic planar graphs. In 1988, P. Erdős and J. Nešetřil [4, 5] posed the following conjecture.

**Conjecture 1 [4, 5]:** For any simple graph $G$,

$$s\chi'(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even;} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

Faudree, Gyárfás, Schelp and Tuza [6] asked in 1990 whether $s\chi'(G) \leq 9$ if $G$ is cubic and planar. The upper bound is attained by the complement of $C_6$. So if the upper bound is valid, it would be the best possible. The problem is still open. The following theorem can be found in [6, 8].

**Theorem 2.1** [6, 8] $s\chi'(G) \leq 2\Delta(\Delta - 1)$.

On the other hand, a trivial upper bound for the strong chromatic index of $G$ is given by $s\chi'(G) \leq 2\Delta^2 - 2\Delta + 1$ (see [9]). This inequality only shows that the conjecture of Erdős and Nešetřil is true for $\Delta \leq 2$.

If $\Delta = 3$, then $s\chi'(G) \leq 10$ by Conjecture 1. This result was proved by L. Andersen [1], and independently, Horák, Qing and Trotter [9]. For $\Delta = 4$, by Conjecture 1 that $s\chi'(G) \leq 20$. Recently, Cranston [3] obtained that $s\chi'(G) \leq 21$ for $\Delta = 4$.

In [6], an obvious lower bound for the strong chromatic index of $G$ is given by the inequality $s\chi'(G) \geq \max\limits_{uv \in E} \{\deg(u) + \deg(v) - 1\}$. The equality holds for trees.

**Theorem 2.2** [6] *If $G$ is a tree, then* $s\chi'(G) = \max\limits_{uv \in E} \{\deg(u) + \deg(v) - 1\}$.

# 3 Main Results

In the following, we consider the strong chromatic index of a cubic Halin graph whose characteristic tree is a caterpillar. Also we find sharp bounds for the strong chromatic index of cubic Halin graphs.

It is easy to check that $s\chi'(Ne_1) = 6$ (see Figure 3.2). It is straightforward to see that all edges of $Ne_2$ must be assigned distinct colors. Hence, $s\chi'(Ne_2) = 9$ (see Figure 3.3). Also we shall show in Theorem 3.3 that

$s\chi'(Ne_3) = 6$ (see Figure 3.1). It suffices to consider the strong chromatic index graphs in $\mathcal{G}_h$ for $h \geq 4$.

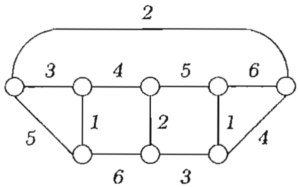


Figure 3.1. $Ne_3$.

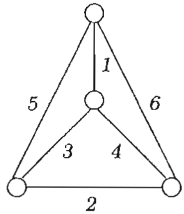


Figure 3.2. $Ne_1$.

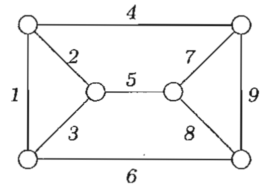


Figure 3.3. $Ne_2$.

**Theorem 3.1** *For $h \geq 4$ and $G \in \mathcal{G}_h$, we have $6 \leq s\chi'(G) \leq 8$.*

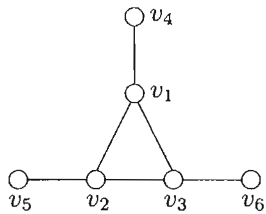


Figure 3.4. The graph $W$.

**Proof:** Suppose $G \in \mathcal{G}_h$. It is easy to see that there are at least two triangles contained in $G$ when $h > 1$ (readers may also find the proof from Theorem 2.2 of [10]). So $G$ contains a subgraph isomorphic to the graph $W$ (see Figure 3.4). In fact, $G$ contains only two such subgraphs. It is easy to see that $s\chi'(W) = 6$. Hence $s\chi'(G) \geq 6$.

Now we are going to give a strong 8-edge-coloring of $G$. For convenience, we let $0' = 0$ and $(h+1)' = h+1$. Consider the subgraph $W$ that contains the triangle $011'$ first. Let the third vertex adjacent with 0 be $r'$ ($r \neq 1$), and the third vertex adjacent with $h+1$ be $m'$ ($m \neq h$). If $v_1 = 1$, $v_2 = 0$ and $v_3 = 1'$, then $v_4 = 2$ and is adjacent with either $v_5$ or $v_6$. If $v_6 = 2'$, then $v_5 = r'$ ($r \neq 2$). If $v_5 = 2'$, then rename $v_3$ as 0 and $v_2$ as $1'$ such that $v_6 = r'$ ($r \neq 2$). By a similar argument, the other subgraph which is isomorphic to $W$ can be named by $v_1 = h$, $v_2 = h+1$, $v_3 = h'$, $v_4 = h-1$, $v_5 = m'$ ($m \neq h-1$) and $v_6 = (h-1)'$. So $G$ is described as either Figure 3.5 or Figure 3.6. If $G$ is the graph in Figure 3.5, then either $r = m = 0$ (or $h+1$) or $3 \leq r \leq m \leq h-2$. For the latter case, it implies that $h \geq 5$. If $G$ is the graph in Figure 3.6, then $3 \leq r \leq h-1$ and $2 \leq m \leq h-2$. First we are going to color the edges of

$$\widetilde{G} = G - \Big\{\{h-1, h\}, \{h, h+1\}, \{h-1, (h-1)'\}, \{h, h'\}, \{(h-1)', h'\}, \{h', h+1\}\Big\}$$

by using the color set $[\![7]\!]$. First we use 6 colors to color the edges $\{0,1\}$, $\{1,1'\}$, $\{0,1'\}$, $\{1,2\}$, $\{1',2'\}$, $\{0,r'\}$ arbitrarily. By the construction of $G$, $G$ contains a subgraph $H$ (Figure 3.7) for $0 \leq j < i < k \leq h+1$.

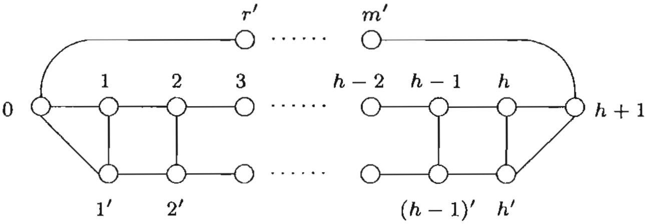


Figure 3.5.

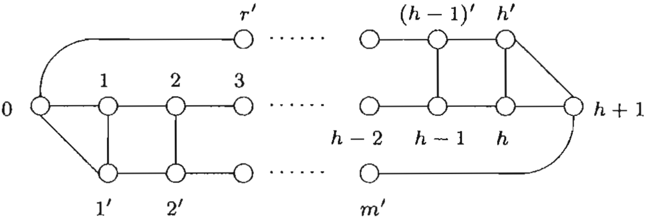


Figure 3.6.

For $2 \leq i \leq h-2$, suppose $\{i-1, i\}$ and $\{j', i'\}$ of $H$ have been colored. We color the remaining edges of $H$ in such a way that $\{i, i'\}$ is the first edge to be colored, follow by the edges $\{i, i+1\}$ and then $\{i', k'\}$. Then at most six colors are forbidden for each of the edges $\{i, i'\}$, $\{i, i+1\}$ and $\{i', k'\}$. So we have a strong 7-edge-coloring for $\widetilde{G}$.

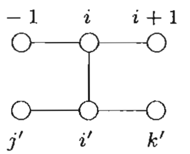


Figure 3.7. The graph $H$.

To color the remaining edges of $G$, we look at three cases. We consider $G$ as in Figure 3.5 first. Let $c$ be the strong edge-coloring of $\widetilde{G}$ defined above.

Case 1: If $m = h-2$ (Figure 3.8), then we color the edges in the following order: $\{h-1, (h-1)'\}$, $\{h-1, h\}$, $\{(h-1)', h'\}$, $\{h, h+1\}$, $\{h, h'\}$, $\{h', h+1\}$. Then at most six colors are forbidden for both $\{h-1, (h-1)'\}$ and $\{h-1, h\}$; at most seven colors are forbidden for $\{(h-1)', h'\}$, $\{h, h+1\}$ and $\{h, h'\}$. Finally, assign $c(\{h-2, h-1\})$ to $\{h', h+1\}$. Hence, all edges of $G$ require at most eight colors.

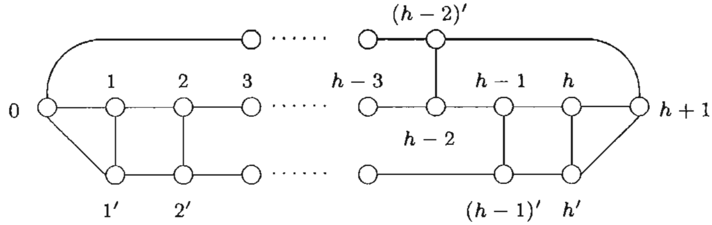


Figure 3.8.

Case 2: Suppose $0 < m \leq h-3$ (Figure 3.9). By the above assignment we have used $1, 2, \ldots, 7$ to color the edges of $\widetilde{G}$. Let $l'$ be the vertex adjacent with $m'$, $0 \leq l \leq m-1$. Without loss of generality, we may assume $c(\{h-2, h-1\}) = 1$, $c(\{h-2, (h-2)'\}) = 2$, $c(\{(h-2)', (h-1)'\}) = 3$, $c(\{p', (h-2)'\}) = 4$, where $2 \leq p \leq h-3$, $c(\{h-3, h-2\}) = 5$. Let $c(\{m', h+1\}) = x$, $c(\{l', m'\}) = y$ and $c(\{m, m'\}) = z$, where $x, y, z \in [\![\mathbf{7}]\!]$.

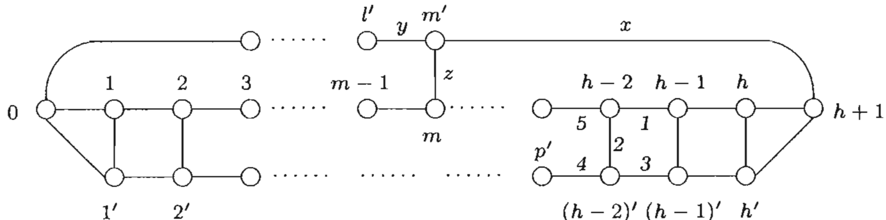


Figure 3.9.

Case 2-1: Suppose $x = 7$. Then $y, z \in [\![\mathbf{6}]\!]$. Define $c(\{h-1, (h-1)'\}) = 7$, $c(\{(h-1)', h'\}) = 5$, $c(\{h, h'\}) = 2$ and $c(\{h-1, h\}) = 4$.

(a) If $1 \notin \{y, z\}$, then assign $2$ to $\{h', h+1\}$ and $8$ to $\{h, h+1\}$.

(b) If $1 \in \{y, z\}$, then assign $8$ to $\{h', h+1\}$ and either $3$ or $6$ to $\{h, h+1\}$ depending on the values of $y$ and $z$.

Case 2-2: Suppose $x \neq 7$.

(a) If $1 \notin \{y, z\}$, then define $c(\{h-1, (h-1)'\}) = 6$, $c(\{(h-1)', h'\}) = 7$, $c(\{h', h+1\}) = 1$ and $c(\{h-1, h\}) = 8$. At most 7 colors are forbidden for $\{h, h+1\}$. After coloring $\{h, h+1\}$, at most 6 colors are forbidden for $\{h, h'\}$. So we can color $G$ by 8 colors.

(b) If $1 \in \{y, z\}$, then recolor $\{h-1, h-2\}$ by $8$. Define $c(\{h-1, (h-1)'\}) = 6$, $c(\{(h-1)', h'\}) = 1$, $c(\{h', h+1\}) = 8$ and $c(\{h-1, h\}) = 7$. Now the edge $\{h, h'\}$ may be colored by $2$, $4$ or $5$ depending on the values of $x, y, z$ (note that $1 \in \{y, z\}$). Similarly the edge $\{h, h+1\}$ may be colored by at least one color from $\{2, 3, 4, 5\}$.

So, at most eight colors are used to color all the edges of $G$ in this case.

Case 3: If $m = 0$, then $G \cong Ne_h$. To finish the coloring, follow Case 2 and replace $\{l', m'\}$, $\{m, m'\}$ and $p'$ by $\{0, 1'\}$, $\{0, 1\}$ and 0 respectively in Case 2. Actually, we can find the strong chromatic index of $Ne_h$. Please see Theorem 3.3.

Similarly, we have a strong 8-edge-coloring in both Cases 1 and 2 for $G$ in Figure 3.6.

Consequently, we find a strong 8-edge-coloring for $G$. Therefore, $6 \leq s\chi'(G) \leq 8$. □

In fact, Theorem 3.1 gives a strong edge-coloring of necklace. In the following, we will provide another coloring for necklace and determine the strong chromatic index of it.

**Lemma 3.2** *Suppose $G$ is a graph with $s\chi'(G) \geq 6$. Let two adjacent vertices $A$ and $B$ of $G$ be of degree 3. Let $X, Y$ and $Z, W$ be the other two neighbors of $A$ and $B$, respectively ($X, Y, Z, W$ are not necessarily distinct). Let $\widetilde{G}$ be a graph obtained from $G$ by replacing the edge-induced subgraph $G[\{AB\}]$ by a ladder graph of length 4 (see Figure 3.10). Then $s\chi'(\widetilde{G}) \leq s\chi'(G)$.*

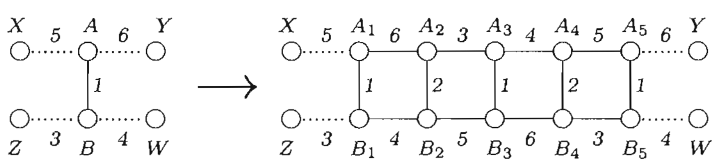


Figure 3.10.

**Proof:** Figure 3.10 shows that $s\chi'(\widetilde{G})$ does not exceed $s\chi'(G)$. □

**Theorem 3.3** *Suppose $h \geq 1$.*

$$s\chi'(Ne_h) = \begin{cases} 6 & \text{if } h \text{ is odd}, \\ 7 & \text{if } h \geq 6 \text{ and is even}, \\ 8 & \text{if } h = 4, \\ 9 & \text{if } h = 2. \end{cases}$$

**Proof:** For $h$ being odd, it suffices to give a strong 6-edge-coloring for $Ne_h$. It is shown in Figures 3.2 and 3.1 that $s\chi'(Ne_h) = 6$ for $h = 1$ and 3, respectively. Applying Lemma 3.2 repeatedly we get $s\chi'(Ne_h) = 6$ for all odd positive integers $h$.

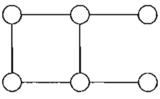

Figure 3.11. The graph $H'$.

For $h$ being even, we have seen that $s\chi'(Ne_2) = 9$. So we may assume $h \geq 4$. The edges $\{1, 1'\}$, $\{h+1, 0\}$, $\{0, 1\}$, $\{1, 2\}$, $\{0, 1'\}$ and $\{1', 2'\}$ must be colored in different colors. Without loss of generality, we may assume they are colored by *1*, *2*, *3*, *4*, *5* and *6*, respectively. Since the edges of a subgraph $H'$ (see Figure 3.11) of $Ne_h$ require all six colors, the edges $\{2, 3\}$, $\{2', 3'\}$ and $\{2, 2'\}$ must receive colors *5*, *3* and *2* respectively. Continuing in this fashion, we see that the edge $\{j, j'\}$, where $1 \leq j \leq h - 1$ is *1* or *2*, according to whether $j$ is odd or even, respectively. In particular $\{h-1, (h-1)'\}$ is colored by *1*. We can also see that $\{h-1, h\}$ and $\{(h-1)', h'\}$ are either colored by *4* and *6* respectively, or the other way round. So the remaining three edges $\{h, h+1\}$, $\{h', h+1\}$ and $\{h, h'\}$ cannot be properly colored by six colors. Thus, $s\chi'(Ne_h) \geq 7$ for $h \geq 4$.

To color $Ne_4$, we first note that it has 15 edges. It is also straightforward to verify that no color may be used for three times. Therefore $s\chi'(Ne_4) \geq 8$. A strong 8-edge-coloring of $Ne_4$ is given in Figure 1.1.

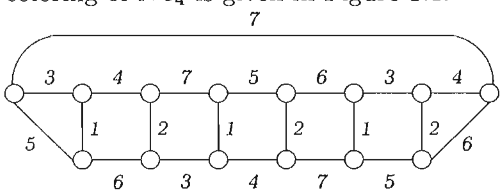


Figure 3.12. A strong 7-edge-coloring for $Ne_6$.

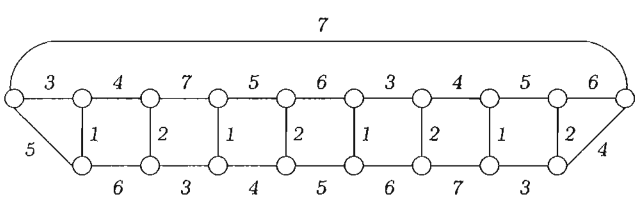


Figure 3.13. A strong 7-edge-coloring for $Ne_8$.

To prove $s\chi'(Ne_h) = 7$ for even $h \geq 6$, it suffices to find a 7-edge-coloring for $Ne_h$. It is shown in Figures 3.12 and 3.13 that $s\chi'(Ne_h) = 7$ for $h = 6$ and 8, respectively. Applying Lemma 3.2 repeatedly we get $s\chi'(Ne_h) = 7$ for all even positive integers $h$ which are greater than 4. $\square$

We illustrate a strong 6-edge-coloring of $Ne_5$ in Figure 3.14.

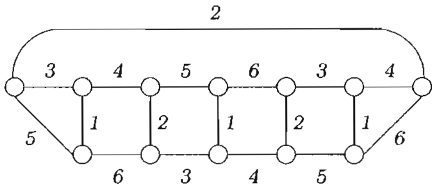


Figure 3.14. A strong 6-edge-coloring for $Ne_5$.

We are now going to find general bounds for the strong chromatic index of cubic Halin graphs. We have mentioned that any cubic Halin graphs $G$ contains at least two triangles. It is easy to see that $G \in \mathcal{G}_h$ for some $h \geq 2$ if and only if $G$ contains only two triangles.

**Theorem 3.4** *If $G$ is a cubic Halin graph, then $6 \leq s\chi'(G) \leq 9$ and the bounds are sharp.*

**Proof:** As mentioned before, every cubic Halin graph $G$ contains a subgraph isomorphic to the subgraph $W$ (Figure 3.4). Since the edges of $W$ must be assigned distinct colors, we have $s\chi'(G) \geq s\chi'(W) = 6$

Let $G = T \cup C$, where $T$ is a $(3,1)$-tree and $C = C_n$. Let $v_1, v_2, \ldots, v_n$ be vertices lying in $C_n$ clockwisely. Let $e_i$ be edge in $T$ which is incident with $v_i$, $1 \leq i \leq n$.

If $G$ contains two triangles sharing an edge, then $G \cong K_4$. Hence $s\chi'(G) = 6$. If $G$ contains only two triangles, $G \in \mathcal{G}_h$ for some $h \geq 2$. By Theorem 3.1 $s\chi'(G) \leq 9$. Thus from now on, we assume that $G$ contains at least three triangles. Then $n \geq 6$. When $n = 6$, the characteristic tree of $G$ is a complete cubic tree of height 2 (Figure 3.15). Then $s\chi'(G) \leq 7$ (actually $s\chi'(G) = 7$, see [11]). So we assume $n \geq 7$. The number of leaves is two more than the number of nodes in a $(3,1)$-tree $T$. So $|V(T)| \geq 12$ and even. By Theorem 2.2 $s\chi'(T) = 5$. Let $c_0$ be a strong 5-edge-coloring for $T$. We shall extend $c_0$ to be a strong 9-edge-coloring of $G$.

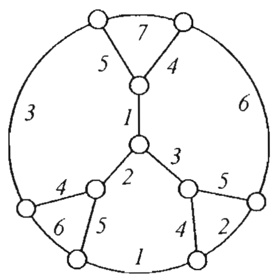


Figure 3.15.

Since $n \geq 7$, either there is an edge $e_t$, for some $t$, that is not an edge of any triangle or $G$ contains at least four triangles. For the first case, after renumbering the vertices in $C_n$ we may assume $t = n$ and $\{v_1, v_2\}$ is an edge of a triangle. We shall use notation $\triangle_s$ to denote the triangle containing the edge $\{v_s, v_{s+1}\}$, $1 \leq s \leq n-1$.

Suppose $\triangle_s$ is a triangle in $G$. If $1 \leq s \leq n-2$, then by exchanging the colors of $e_s$ and $e_{s+1}$ if necessary, we may assume $c_0(e_{s-1}) \neq c_0(e_s)$ (where $e_0 = e_n$). Note that, $c_0(e_{n-2})$ may equal to $c_0(e_{n-1})$ if $e_{n-1}$ and $e_n$ are edges of the triangle $\triangle_{n-1}$.

First we perform the change colors procedure below (we shall call this procedure CCP):

$$\text{Starting from } j = 2 \text{ to } j = n-1,$$
$$\text{if } c_0(e_j) = c_0(e_{j+1}), \text{ then we redefine } c_0(e_{j+1}) \text{ by } 6.$$

Note that after performing CCP, no two consecutive edges are recolored. Let the new coloring be denoted by $c$. Then $c$ is still a proper coloring of $T$. We can see that the edges of triangles in $T$ are not colored by *6* except $e_{n-1}$ may be. Without loss of generality, we may assume $c(e_1) = 1$, $c(e_2) = 2$. Also we may assume $\triangle_1$, $\triangle_i$ and $\triangle_j$ are three consecutive triangles along the adjoint cycle $C_n$, $3 \leq i$, $i+2 \leq j \leq n-2$.

Case 1: Suppose $n \equiv 0 \pmod 3$. We color the edges of $C_n$ starting at $\{v_1, v_2\}$ clockwisely by the colors *7, 8, 9* cyclically.

Case 2: Suppose $n \equiv 1 \pmod 3$. We have the following cases:

Case 2-1: Suppose $c(e_3) \neq 6$ and $c(e_n) \neq 6$. Then define $c(\{v_1, v_2\}) = 6$ and color the remaining edges of $C_n$ starting at $\{v_2, v_3\}$ clockwisely by the colors *7, 8, 9* cyclically.

Case 2-2: Suppose $c(e_3) \neq 6$ and $c(e_n) = 6$. It means that $c(e_3) \neq 2$. We redefine $c(e_2) = 6$ and define $c(\{v_1, v_2\}) = 2$. And then color the remaining edges of $C_n$ starting at $\{v_2, v_3\}$ clockwisely by the colors *7, 8, 9* cyclically.

Case 2-3: Suppose $c(e_3) = 6$ and $c(e_n) \neq 6$. We redefine $c(e_1) = 6$ and define $c(\{v_1, v_2\}) = 1$. The rest is same as Case 2-2.

Case 2-4: Suppose $c(e_3) = 6$ and $c(e_n) = 6$. Consider $\triangle_i$. If $c(e_{i-1}) \neq 6$, then the case can be referred to Case 2-1 or 2-3. If $c(e_{i-1}) = 6$, then change back the original color assigned to $e_k$ for $3 \leq k \leq i-1$ first. And then exchange the colors of $e_1$ and $e_2$. Perform CCP for edges from $e_3$ to $e_i$. Then the case can be referred to Case 2-2.

Case 3: Suppose $n \equiv 2 \pmod 3$. From Case 2 we can see that the recoloring procedure only influences the edge $\{v_1, v_2\}$ and edges $e_k$ for $1 \leq k \leq i$. Thus we apply the recoloring procedure described in Case 2 from $\Delta_1$ to $\Delta_i$ and from $\Delta_j$ to $\Delta_i$ (anti-clockwisely). The new coloring is still denoted by $c$.

After that we have colored the $\{v_1, v_2\}$, $\{v_j, v_{j+1}\}$ and edges $e_k$ for $1 \leq k \leq j+1$ by colors in $[\![\mathbf{6}]\!]$. But the new colors assigned to $e_i$ and $e_{i+1}$ may be the same and equal to *6*. If it happens, then it means that the previous colors assigned to $e_{i-1}$ and $e_i$ are the same, also the previous colors assigned to $e_{i+1}$ and $e_{i+2}$ are the same. Since $e_i$ and $e_{i+1}$ are adjacent, the previous colors assigned to $e_i$ and $e_{i+1}$ are different. Recolor $e_i$ by $c(e_{i+2})$ and $e_{i+1}$ by $c(e_{i-1})$.

Up to now, there are $n-2$ edges in $C_n$ that have not been colored. Color those edges starting at any edge clockwisely by the colors *7, 8, 9* cyclically.

So we get a strong 9-edge-coloring for $G$. The proof is complete. □

**Remark:** From Theorem 3.4 we get that every cubic Halin graph is edge-decomposable into at most 9 induced matchings.

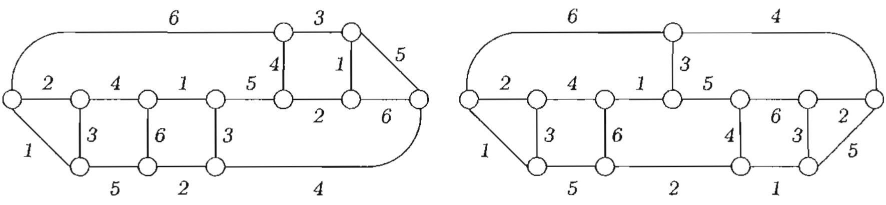


Figure 3.16. Strong 6-edge-colorings of the other two graphs in $\mathcal{G}_5$

There are only two (non-isomorphic) graphs contained in $\mathcal{G}_4$. We showed in Figure 1.1 that $s\chi'(G) = 7$, where $G$ is described in Figure 1.1(b). Also there are only three graphs contained in $\mathcal{G}_5$. We show in Figures 3.14 and 3.16 that $s\chi'(G) = 6$ for $G \in \mathcal{G}_5$. It can be checked that $\mathcal{G}_6$ has 6 members. The strong chromatic indices of 5 graphs in $\mathcal{G}_6$ are 7 and the strong chromatic index of the remaining one is 6. We wonder whether the graphs in $\mathcal{G}_h$ are strong 7-edge-colorable for $h \geq 5$. So we conclude by presenting the following conjectures:

**Conjecture 2:** *For $h \geq 5$, $s\chi'(G) \leq 7$ for any $G \in \mathcal{G}_h$.*

**Conjecture 3:** *For $h \geq 5$ and $h$ odd, $s\chi'(G) = 6$ for any $G \in \mathcal{G}_h$.*

**Conjecture 4:** *Suppose $G = T \cup C$ is a Halin graph. Then $s\chi'(G) \leq s\chi'(T) + 4$.*

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