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FREQUENCY OF FIBONACCI NUMBERS MODULO 3^k THAT ARE CONGRUENT TO 8 (mod 27)

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Abstract: In this paper, the frequency in $\mathbb{Z}_{\pi(3^k)}$ of some Fibonacci numbers that are congruent to 8 (mod 27) will be discussed, where $\pi(3^k)$ is the period of the Fibonacci sequence modulo 3^k. The frequency of 8 modulo 3^k is determined.

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1. Introduction

Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, denote the sequence of classical Fibonacci numbers. We recall some useful formulas for F_n:

$$F_{-n} = (-1)^{n+1} F_n; \tag{1.1}$$

$$F_{n+m} = F_{m-1}F_n + F_m F_{n+1}; \tag{1.2}$$

$$F_{qn+r} = \sum_{i=0}^{q} \binom{q}{i} (F_n)^i (F_{n-1})^{q-i} F_{r+i}, \text{ for } q \geq 0. \tag{1.3}$$

In this paper, we shall be interested in the cases $q = 3, r = -1$ and $q = 3, r = 0$:

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$$F_{3n-1} = (F_{n-1})^3 + 3(F_n)^2 F_{n-1} + (F_n)^3; \tag{1.4}$$
$$F_{3n} = F_n[3(F_{n-1})^2 + 3F_n F_{n-1} + 2(F_n)^2]. \tag{1.5}$$

It is well known that for $m \geq 2$, the sequence $\{F_n \pmod{m}\}_{n \in \mathbb{Z}}$ is a periodic sequence in $\mathbb{Z}_m$ [2]. Let $\pi(m)$ be the period of this sequence. It was shown in [2] that, for $m = 3^k$ we have the following theorem.

Theorem 1.1. $\pi(3^k) = 8 \times 3^{k-1}$.

For $b \in \mathbb{Z}_{3^k}$, let $\nu(3^k, b)$ denote the number of frequency of b as a residue in one period of the sequence $\{F_n \bmod 3^k\}$. In [1], it was proved the following result.

Theorem 1.2. *For $k \geq 3$, we have*

$$\begin{cases} \nu(3^k, b) = 2 & \text{if } b \not\equiv 1, 8, 19, 26 \pmod{27}; \\ \nu(3^k, b) = 8 & \text{if } b \equiv 1 \text{ or } 26 \pmod{27}. \end{cases}$$

It remains to consider $\nu(3^k, b)$ with $b \equiv 8$ or $19 \pmod{27}$. Since $\nu(3^k, b) = \nu(3^k, 3^k - b)$ (see [1]), we only consider the case $b \equiv 8 \pmod{27}$ in this paper.

We shall prove that $\nu(3^k, 8) = 20$ for $k \geq 5$ and derive formulas for certain b such that $b \equiv 8 \pmod{27}$. Also we give a proof that the set of certain generalized Fibonacci numbers modulo 3^k is $\mathbb{Z}_{3^k}$.

2. $F_{6+24\ell}$ mod 3^k

In [1] we have a complete answer for $\nu(3^k, b)$, where $b \equiv F_n \pmod{27}$ and n satisfies $F_{n+3^{-1}\pi} \not\equiv F_n \pmod{3^k}$ or $F_{n+3^{-1}\pi} \equiv F_n \pmod{3^k}$ but $F_{n+3^{-2}\pi} \not\equiv F_n \pmod{3^k}$. In this paper we shall be interested in $\nu(3^k, b)$, where $b \equiv F_n \pmod{3^k}$ and n satisfies $F_{n+3^{-r}\pi} \equiv F_n \pmod{3^k}$ and $r \geq 2$.

Recall the following propositions in [1].

Proposition 2.1. *Let $k \geq 2$. Then $F_{n+3^{-1}\pi(3^k)} \equiv F_n \pmod{3^k}$ if and only if $n \equiv 2,\ 6 \pmod{8}$.*

Proposition 2.2. *Let $k \geq 4$. Then $F_{n+3^{-2}\pi(3^k)} \equiv F_n \pmod{3^k}$ if and only if $n \equiv 6,\ 18 \pmod{24}$.*

Thus to find a necessary and sufficient condition for n to satisfy $F_{n+3^{-r}\pi(3^k)} \equiv F_n \pmod{3^k}$ with $k \geq 2r \geq 6$ we need to consider $n \equiv 6$ or $18 \pmod{24}$. Since if $n \equiv 18 \pmod{24}$, then $F_n \equiv 19 \pmod{27}$, so we only consider $n \equiv 6 \pmod{24}$. We shall use $n(\ell) = 6 + 24\ell$ to denote this n for non-negative integer ℓ.

Using (1.3) one can establish the following result.

Proposition 2.3. *Let $F_{24} = 9a$ and $F_{23} = 1 + 9b$. Then for positive integer m, we have*

$$F_{24m} = \sum_{j=1}^{m} A_j \binom{m}{j} 3^{2j}, \quad F_{24m-1} = 1 + \sum_{j=1}^{m} B_j \binom{m}{j} 3^{2j},$$

where $A_j = \sum_{i=1}^{j} \binom{j}{i} F_i a^i b^{j-i}$, $B_j = \sum_{i=0}^{j} F_{i-1} \binom{j}{i} a^i b^{j-i}$, $a = 5152$ and $b = 3184$.

By using (1.2) and Proposition 2.3, we have the following result.

Proposition 2.4. *Let $\ell \geq 1$, $a = 5152$ and $b = 3184$. Then*

$$F_{n(\ell)} = F_{6+24\ell} = 8 + \sum_{j=1}^{\ell} \left(\sum_{i=0}^{j} \binom{j}{i} F_{6+i}\, a^i b^{j-i} \right) \binom{\ell}{j} 3^{2j}.$$

After taking modulo 3^k we have the following result.

Corollary 2.5. *Let $\ell \geq 1$. Then*

$$F_{n(\ell)} \equiv 8 + \sum_{j=1}^{\min\{\ell, \lfloor \frac{k-1}{2} \rfloor\}} \left(\sum_{i=0}^{j} \binom{j}{i} F_{6+i}\, a^i b^{j-i} \right) \binom{\ell}{j} 3^{2j} \pmod{3^k}.$$

For $k = 6$ we get $F_{6+24} \equiv 3^5 + 8 \pmod{3^6}$. Combining with Corollary 2.5 we get the following result.

Corollary 2.6. $F_{6+24\ell} \equiv 3^5\ell + 8\ell(\ell-1)3^4 + 8 \equiv 3^5\ell - 3^4\ell(\ell-1) + 8 \pmod{3^6}$, $\ell \geq 0$.

Remark 2.1. One can easily check that there are four different types of $F_{6+24\ell} \pmod{3^6}$, namely 8, 251, 332 and 494.

Corollary 2.7. *There are at least 3^{k-2} of ℓ's with $0 < n(\ell) < \pi(3^k)$ such that $F_{n(\ell)} \equiv 8 \pmod{27}$. In fact $F_{n(\ell)} \equiv 8 \pmod{81}$.*

We recall the following lemma, see [1].

Lemma 2.8. *Let $k \geq 1$. Suppose $1 \leq n \leq \pi(3^k)$ with $n \not\equiv 2, 6 \pmod 8$. If $F_n \equiv b \pmod{3^k}$, then there exists n' in $\{n, n+\pi(3^k), n+2\pi(3^k)\}$ such that $F_{n'} \equiv b \pmod{3^{k+1}}$. Moreover, the sets $\{F_n,\ F_{n+\pi(3^k)},\ F_{n+2\pi(3^k)}\}$ and $\{b,\ b+3^k,\ b+2\times 3^k\}$ are equal in $\mathbb{Z}_{3^{k+1}}$.*

Theorem 2.9. *If $k \geq 3$ and $0 \leq b \leq 3^k - 1$ is such that $b \equiv 8 \pmod{27}$, then there are at least two distinct odd numbers n, n' with $1 \leq n, n' \leq \pi(3^k)$ with $F_n \equiv F_{n'} \equiv b \pmod{3^k}$.*

Proof. Write b uniquely as $8 + 3^3 a_3 + 3^4 a_4 + \cdots + 3^{k-1} a_{k-1}$, where $0 \le a_i \le 2$.

Since $F_{11} \equiv 8 \pmod{3^3}$, by Lemma 2.8, $\{8, 8+3^3, 8+2\times 3^3\}$ and $\{F_{11}, F_{11+\pi(3^3)}, F_{11+2\pi(3^3)}\}$ are equal in $\mathbb{Z}_{3^4}$. Thus we can choose $n_1 \in \{11, 11+\pi(3^3), 11+2\pi(3^3)\}$ which is clearly odd such that $1 \le n_1 \le \pi(3^4)$ and $b_1 = 8 + 3^3 a_3 \equiv F_{n_1} \pmod{3^4}$. Then apply Lemma 2.8 to $\{b_1, b_1 + 3^4, b_1 + 2\times 3^4\}$ to find n_2 which is odd and $1 \le n_2 \le \pi(3^5)$ so that $b_2 = b_1 + 3^4 a_4 \equiv F_{n_2} \pmod{3^5}$. Continuing in this fashion, we can find odd n such that $1 \le n \le \pi(3^k)$ and $F_n \equiv b \pmod{3^k}$. Take $n' = \pi(3^k) - n$. Clearly n' is odd and $1 \le n' \le \pi(3^k)$. Hence $F_{n'} = F_{\pi(3^k)-n} \equiv F_{-n} \equiv (-1)^{n+1} F_n \equiv F_n \pmod{3^k}$. □

Remark 2.2. We shall see later that there are exactly two such odd integers n and n'.

For each b, $1 \le b \le 27$, define $\omega(3^k, b) = |\{n \mid F_n \equiv b \pmod{27},\ 1 \le n \le \pi(3^k)\}|$. We have in [1] the following result.

Lemma 2.10. *For $k \ge 3$, $\omega(3^k, 8) = 5 \times 3^{k-3}$.*

Theorem 2.11. *For $k \ge 3$, there are exactly 3^{k-2} even n's in $[0, \pi(3^k))$ such that $F_n \equiv 8 \pmod{27}$. Indeed $F_n \equiv 8 \pmod{81}$. Moreover, such even number n must be of the form $6 + 24\ell$, $0 \le \ell \le 3^{k-2} - 1$*

Proof. Since there are exactly two odd n's, namely, 11 and 61 in $\mathbb{Z}_{\pi(3^3)}$ such that $F_n \equiv 8 \pmod{27}$. Thus there are at least $2 \times 3^{k-3}$ odd n's in $\mathbb{Z}_{\pi(3^k)}$ such that $F_n \equiv 8 \pmod{27}$. Also by Corollary 2.7, there are at least 3^{k-2} even n's of the form $6 + 24\ell$ such that $F_n \equiv 8 \pmod{27}$. Hence by the definition of $\omega(3^k, 8)$ and Lemma 2.10 we have $5 \times 3^{k-3} = \omega(3^k, 8) \ge 2 \times 3^{k-3} + 3^{k-2} = 5 \times 3^{k-3}$. Hence we have exactly $2 \times 3^{k-2}$ odd n's and exactly 3^{k-2} even n's of the form $6 + 24\ell$ in $\mathbb{Z}_{\pi(3^k)}$ with

$$F_n \equiv 8 \pmod{27}.$$ □

Those $2 \times 3^{k-2}$ odd n's with $F_n \equiv 8 \pmod{27}$ in the proof above are those described in the proof of Theorem 2.9. Since those odd n's come from $n = 11$ or 61 in $\mathbb{Z}_{\pi(3^3)}$, we have the corollary.

Corollary 2.12. *For $k \ge 3$, $b \equiv 8 \pmod{27}$ there are exactly two distinct odd numbers n, n' such that $1 \le n, n' \le \pi(3^k)$ with the property that $F_n \equiv F_{n'} \equiv b \pmod{3^k}$.*

3. Definition of a_r and b_r

Since $3^{-r}\pi(3^k) = \pi(3^{k-r})$ if $k > r$, we have therefore

$$F_{3^{-r}\pi(3^k)} \equiv 0 \pmod{3^{k-r}}, \text{ and } F_{3^{-r}\pi(3^k)-1} \equiv 1 \pmod{3^{k-r}}. \tag{3.1}$$

Using mathematical induction we have the following result.

Theorem 3.1. *For $k \geq 2r$, $F_{3^{-r}\pi(3^k)-1} \equiv 3^{k-r}b_r + 1 \pmod{3^k}$ and $F_{3^{-r}\pi(3^k)} \equiv 3^{k-r}a_r \pmod{3^k}$, where $a_r, b_r \in \mathbb{Z}_{3^r}$ depend only on r.*

We shall fix a_r and b_r as integers in the range $[0, 3^r - 1]$. It was shown in [1] that $a_1 = 1$ and $b_1 = 1$; $a_2 = 4$ and $b_2 = 7$.

Proposition 3.2. *For $r \geq 2$, we have*

$$F_{\pi(3^r)} = \sum_{j=1}^{3^r-2} A_j \binom{3^r-2}{j} 3^{2j}, \quad F_{\pi(3^r)-1} = 1 + \sum_{j=1}^{3^r-2} B_j \binom{3^r-2}{j} 3^{2j};$$

and

$$F_{\pi(3^r)} \equiv \sum_{j=1}^{r-1} A_j \binom{3^r-2}{j} 3^{2j} \pmod{3^{2r}},$$

$$F_{\pi(3^r)-1} \equiv 1 + \sum_{j=1}^{r-1} B_j \binom{3^r-2}{j} 3^{2j} \pmod{3^{2r}}.$$

Proof. This follows from Proposition 2.3 and the fact that $\pi(3^r) = 24 \times 3^{r-2}$. □

Lemma 3.3. *Let $r \geq 2$. For $j \geq 1$, $\binom{3^r-2}{j}3^{2j}$ is divisible by 3^r.*

Proof. Note that $\binom{3^r-2}{j} = \frac{3^r-2}{j}\binom{3^r-2-1}{j-1}$. If 3 and j are relatively prime, then $\binom{3^r-2}{j}$ has 3^{r-2} as a factor. Hence $\binom{3^r-2}{j}3^{2j}$ is divisible by 3^r.

Next, if $j = 3^m c$ for some positive integers c and m with g.c.d.$(3, c) = 1$, then one can show that $2c \cdot 3^m - 2 - m \geq 0$. Thus $\binom{3^r-2}{j}3^{2j} = \frac{3^{r-2+2j}}{j}\binom{3^r-2-1}{j-1}$ has 3^r as a factor. □

To compute a_r and b_r for $r \geq 3$ we can simply take $k = 2r$ and then calculate

$$a_r \equiv 3^{-r}F_{3^{-r}\pi(3^{2r})} \pmod{3^r} \quad \text{and} \quad b_r \equiv 3^{-r}\left(F_{3^{-r}\pi(3^{2r})-1} - 1\right) \pmod{3^r}.$$

Proposition 3.4. *For $r \geq 3$, we have*

$$a_r \equiv \sum_{j=1}^{r-1} \frac{A_j}{j} \binom{3^{r-2}-1}{j-1} 3^{2j-2} \pmod{3^r},$$

$$b_r \equiv \sum_{j=1}^{r-1} \frac{B_j}{j} \binom{3^{r-2}-1}{j-1} 3^{2j-2} \pmod{3^r},$$

where A_j and B_j are as defined in Proposition 2.3.

Proof. These follow from Lemma 3.3 and the fact that if $r \geq 3$, then $3^{r-2} \geq r$. □

Remark 3.1. We have $a_r \equiv A_1 \pmod{3^2}$. We can directly compute $A_1 \equiv 4 \pmod 9$. So $a_r \equiv 4 \pmod 9$ for $r \geq 2$. Similarly, one can show that $b_r \equiv 7 \pmod 9$ for $r \geq 2$.

More generally for $s, \lambda \in \mathbb{N}$, since $F_{\lambda\pi(3^s)} \equiv 0 \pmod{3^s}$ and $F_{\lambda\pi(3^s)-1} \equiv 1 \pmod{3^s}$, we let

$$F_{\lambda\pi(3^s)} = 3^s a_{s,\lambda} \text{ and } F_{\lambda\pi(3^s)-1} = 1 + 3^s b_{s,\lambda}.$$

Applying equations (1.4) and (1.5), we have

$$\begin{aligned} a_{r,\lambda} &= a_{r-1,\lambda} + 3^{r-1}(2a_{r-1,\lambda}b_{r-1,\lambda} + a_{r-1,\lambda}^2) \\ &\quad + 2 \times 3^{2r-3}a_{r-1,\lambda}^3 + 3^{2r-2}(a_{r-1,\lambda}b_{r-1,\lambda}^2 + a_{r-1,\lambda}^2 b_{r-1,\lambda}); \qquad (3.2) \\ b_{r,\lambda} &= b_{r-1,\lambda} + 3^{r-1}(a_{r-1,\lambda}^2 + b_{r-1,\lambda}^2) \\ &\quad + 3^{2r-3}(a_{r-1,\lambda}^3 + b_{r-1,\lambda}^3) + 3^{2r-2}a_{r-1,\lambda}^2 b_{r-1,\lambda}. \qquad (3.3) \end{aligned}$$

For $r \geq 3$, we have

$$a_{r,\lambda} \equiv a_{r-1,\lambda} + 3^{r-1}(2a_{r-1,\lambda}b_{r-1,\lambda} + a_{r-1,\lambda}^2) \pmod{3^r}; \qquad (3.4)$$

$$b_{r,\lambda} \equiv b_{r-1,\lambda} + 3^{r-1}(a_{r-1,\lambda}^2 + b_{r-1,\lambda}^2) \pmod{3^r}. \qquad (3.5)$$

Lemma 3.5. *For $r \geq 1$ and $\lambda \in \mathbb{N}$, we have $a_{r,\lambda} \equiv \lambda a_r \pmod{3^r}$ and $b_{r,\lambda} \equiv \lambda b_r \pmod{3^r}$.*

Proof. Suppose that $k \geq 2r$. For convenience let $s = k - r$. From Theorem 3.1, we have $F_{\pi(3^s)} \equiv 3^s a_r \pmod{3^k}$ and $F_{\pi(3^s)-1} \equiv 1 + 3^s b_r \pmod{3^k}$. From (1.3) we have

$$F_{\lambda\pi(3^s)} = \sum_{i=0}^{\lambda} \binom{\lambda}{i} (F_{\pi(3^s)})^i (F_{\pi(3^s)-1})^{\lambda-i} F_i$$

$$= \sum_{i=1}^{\lambda} \binom{\lambda}{i} (F_{\pi(3^s)})^i (F_{\pi(3^s)-1})^{\lambda-i} F_i.$$

In particular when $k = 2r$, we have

$$3^r a_{r,\lambda} = F_{\lambda\pi(3^r)} \equiv \sum_{i=1}^{\lambda} \binom{\lambda}{i} (3^r a_r)^i (1 + 3^r b_r)^{\lambda-i} F_i \pmod{3^{2r}}.$$

Hence we have $a_{r,\lambda} \equiv \lambda a_r (1 + 3^r b_r)^{\lambda-1} F_1 \equiv \lambda a_r \pmod{3^r}$.

Similarly from $1 + 3^r b_{r,\lambda} = F_{\lambda\pi(3^r)-1} \equiv (1 + 3^r b_r)^{\lambda} F_{-1} \pmod{3^{2r}}$, we get $b_{r,\lambda} \equiv \lambda b_r \pmod{3^r}$. □

Lemma 3.6. *For $r \geq 3$ and $\lambda \in \mathbb{N}$, we have $a_{r-1,\lambda} \equiv \lambda a_r \pmod{3^r}$ and $b_{r-1,\lambda} \equiv \lambda b_r + \lambda^2 3^{r-1} \pmod{3^r}$.*

Proof. From Remark 3.1 we have $a_{r-1} \equiv 1 \pmod 3$ and $b_{r-1} \equiv 1 \pmod 3$. By Lemma 3.5 we have $a_{r-1,\lambda} \equiv \lambda a_{r-1} \pmod{3^{r-1}}$ and $b_{r-1,\lambda} \equiv \lambda b_{r-1} \pmod{3^{r-1}}$. By (3.4) and (3.5) we have $a_{r,\lambda} \equiv a_{r-1,\lambda} \pmod{3^r}$ and $b_{r,\lambda} \equiv b_{r-1,\lambda} + 2\lambda^2 \times 3^{r-1} \pmod{3^r}$. □

Remark 3.2. By putting $\lambda = 1$ into Lemma 3.5 and Lemma 3.6 we get $a_r \equiv a_{r-1} \pmod{3^{r-1}}$ and $b_r \equiv b_{r-1} \pmod{3^{r-1}}$ for $r \geq 3$. One may check that both congruences hold for $r = 2$. Hence $a_r \equiv a_s \pmod{3^s}$ and $b_r \equiv b_s \pmod{3^s}$ for $r \geq s \geq 1$. Also by equations (3.2) and (3.3) we get $a_{r,\lambda} \equiv a_{s,\lambda} \pmod{3^s}$ and $b_{r,\lambda} \equiv b_{s,\lambda} \pmod{3^s}$ for $r \geq s \geq 1$ and $\lambda \in \mathbb{N}$.

4. Necessary and Sufficient Condition

Lemma 4.1. *For $k \geq 2r$ and $\lambda \geq 0$, we have $F_{n+3^{-r}\lambda\pi} \equiv F_n + 3^{k-r}\lambda(b_r F_n + a_r F_{n+1}) \pmod{3^k}$. Here $\pi = \pi(3^k)$ and a_r and b_r are defined in Theorem 3.1.*

Proof. From (1.2) we have

$$F_{n+\lambda\pi(3^s)} = F_n F_{\lambda\pi(3^s)-1} + F_{n+1} F_{\lambda\pi(3^s)} = F_n + 3^s(b_{s,\lambda} F_n + a_{s,\lambda} F_{n+1}). \quad (4.1)$$

By putting $s = k - r$, we have

$$F_{n+3^{-r}\lambda\pi} = F_{n+\lambda\pi(3^{k-r})} = F_n + 3^{k-r}(b_{k-r,\lambda} F_n + a_{k-r,\lambda} F_{n+1}).$$

Since $k-r \geq r$, from Remark 3.2 we have $b_{k-r,\lambda}F_n + a_{k-r,\lambda}F_{n+1} \equiv b_{r,\lambda}F_n + a_{r,\lambda}F_{n+1} \pmod{3^r}$. By Lemma 3.5 we have $F_{n+3^{-r}\lambda\pi} \equiv F_n + 3^{k-r}\lambda(b_rF_n + a_rF_{n+1}) \pmod{3^k}$. □

Corollary 4.2. *For $k \geq 2r$, $F_{n+3^{-r}\pi(3^k)} \equiv F_n \pmod{3^k}$ if and only if*

$$b_rF_n + a_rF_{n+1} \equiv 0 \pmod{3^r}. \qquad (4.2)_r$$

Corollary 4.3. *For $k \geq 2r$, $F_{n+3^{-r}\pi(3^k)} \equiv F_n \pmod{3^k}$ if and only if $F_{n+3^{-r}\ell\pi(3^k)} \equiv F_n \pmod{3^k}$ for any integer ℓ.*

Proof. For $\ell \geq 0$, it is obvious. For $\ell < 0$, the corollary follows from $F_{n-3^{-r}\pi(3^k)} \equiv F_{n+3^{-r}(3^r-1)\pi(3^k)} \pmod{3^k}$. □

Proposition 4.4. *For $k \geq 2r \geq 6$, suppose $n = n(\ell)$ satisfies $(4.2)_r$. If $F_n \equiv F_{n+\lambda\pi(3^s)} \pmod{3^k}$ for $s \leq r-1$, then $\lambda \equiv 0 \pmod 3$. Moreover, when $s = r-1$ and $k = 2r$, the converse holds.*

Proof. From (4.1) and Lemma 3.6 we have

$$\begin{aligned} F_{n+\lambda\pi(3^s)} - F_n &= 3^s(b_{s,\lambda}F_n + a_{s,\lambda}F_{n+1}) \\ &\equiv 3^s[\lambda(b_{s+1}F_n + a_{s+1}F_{n+1}) + \lambda^2 3^s F_n] \pmod{3^{2s+1}}. \end{aligned}$$

If $F_n \equiv F_{n+\lambda\pi(3^s)} \pmod{3^k}$, then $F_n \equiv F_{n+\lambda\pi(3^s)} \pmod{3^{2s+1}}$. Since n satisfies $(4.2)_r$, by Remark 3.2 n satisfies $(4.2)_{s+1}$ as well. So we have $\lambda^2F_n \equiv 0 \pmod 3$. Since $n = n(\ell)$ is not divisible by 4, $F_n \not\equiv 0 \pmod 3$. We have $\lambda^2 \equiv 0 \pmod 3$. Hence $\lambda \equiv 0 \pmod 3$.

When $s = r-1$, the converse is clear. □

By the same argument used in the above proof we have the following proposition.

Proposition 4.5. *For $k \geq 2r+1 \geq 7$, suppose $n = n(\ell)$ satisfies $(4.2)_{r+1}$. If $F_n \equiv F_{n+\lambda\pi(3^s)} \pmod{3^k}$ for $s \leq r$, then $\lambda \equiv 0 \pmod 3$. Moreover, when $s = r$ and $k = 2r+1$, the converse holds.*

5. Existence of Solutions of $(4.2)_r$

In this section, we shall prove that $(4.2)_r$ has solutions. For a, b positive and $3/\text{g.c.d.}(a,b)$, we define the so-called *generalized Fibonacci sequence* $G_n = bF_n + aF_{n+1}$, $n = 0, 1, 2, \ldots$. It satisfies $G_{n+2} = G_{n+1} + G_n$ with initial values $G_0 = a$ and $G_1 = a + b$. Then it is easy to establish the following properties.

Proposition 5.1.

1. *$\{G_n \pmod{3^k}\}$ is periodic with period $\pi(3^k)$.*
2. *$G_{n+3^{-1}\pi(3^k)\lambda} \equiv 3^{k-1}\lambda G_{n+2} + G_n \pmod{3^k}$, for $\lambda \geq 1$.*
3. *There exist three distinct $n_1, n_2, n_3 \in \mathbb{Z}_8$ such that $G_{n_1} \equiv 0 \pmod 3$, $G_{n_2} \equiv 1 \pmod 3$, $G_{n_3} \equiv 2 \pmod 3$ and $G_{n_i+2} \not\equiv 0 \pmod 3$ for $i = 1, 2, 3$.*

Proof. (1) and (2) follow from the definition of G_n and Lemma 4.1. For (3), observe that for all ordered pairs $(a, b) \in \mathbb{Z}_3^2 \setminus \{(0,0)\}$ the sequence $\{G_n \pmod 3\}_{n\in\mathbb{Z}} = \{0, 1, 1, 2, 0, 2, 2, 1, \dots\}$. Hence we can choose n_i, $i = 1, 2, 3$, satisfying the requirements. □

Theorem 5.2. *For $k \geq 1$, the set of all the generalized Fibonacci numbers modulo 3^k is $\mathbb{Z}_{3^k}$. Here we assume that $3 \not| \; g.c.d.(a, b)$.*

Proof. By Proposition 5.1, there are three distinct $n_i \in \mathbb{Z}_8$ so that $G_{n_i+2} \not\equiv 0 \pmod 3$, $i = 1, 2, 3$. Let $A = \{G_s \pmod{3^k} \mid s \equiv n_i \pmod 8,\ 1 \leq i \leq 3,\ 0 \leq s < \pi(3^k)\}$. We want to show that $A = \mathbb{Z}_{3^k}$. Since A has at most $3 \times \frac{\pi(3^k)}{8} = 3^k$ elements, we only need to show that A has 3^k distinct elements.

It suffices to show that $G_s \not\equiv G_t \pmod{3^k}$ for all $s \not\equiv t \pmod{\pi(3^k)}$ and $s \pmod 8, t \pmod 8 \in \{n_1, n_2, n_3\}$. We shall prove this by induction on k.

The assertion clearly holds for $k = 1$. Assume the assertion holds for $k - 1$ with $k > 1$. Let s, t be taken from $\{n_1, n_2, n_3\} \pmod 8$ with $s \not\equiv t \pmod{\pi(3^k)}$. If $s \not\equiv t \pmod{\pi(3^{k-1})}$, then by the induction hypothesis, $G_s \not\equiv G_t \pmod{3^{k-1}}$, hence $G_s \not\equiv G_t \pmod{3^k}$.

Thus we assume that $s \equiv t \pmod{\pi(3^{k-1})}$ but $s \not\equiv t \pmod{\pi(3^k)}$. Hence $s = t + \lambda\pi(3^{k-1})$ with $\lambda = 1$ or 2. Applying Lemma 4.1 and the definition of G_n we have

$$G_{t+\lambda\pi(3^{k-1})} = G_{t+\frac{\lambda\pi(3^k)}{3}} \equiv 3^{k-1}\lambda G_{t+2} + G_t \pmod{3^k}.$$

By the choice of t we have that $G_{t+2} \not\equiv 0 \pmod 3$, hence $G_s = G_{t+\lambda\pi(3^{k-1})} \not\equiv G_t \pmod{3^k}$. This completes the induction. □

Corollary 5.3. *For $k \geq 1$, the set of all Fibonacci numbers modulo 3^k is $\mathbb{Z}_{3^k}$.*

Corollary 5.4. *The equation $(4.2)_r$, i.e., $b_r F_n + a_r F_{n+1} \equiv 0 \pmod{3^r}$, has solutions $n \in \mathbb{Z}_{\pi(3^r)}$ for $r \geq 1$.*

Later we shall prove that there are exactly two $n \in \mathbb{Z}_{3^r}$ satisfying the equation $(4.2)_r$.

Consider a generalized Fibonacci sequence $G_n = bF_n + aF_{n+1}$ for $3 \nmid \text{g.c.d.}(a,b)$. By Proposition 5.1 (2) we get $G_{n+\frac{\pi}{3}} \equiv G_n + 3^{k-1}(ax_n + by_n) \pmod{3^k}$, where $\pi = \pi(3^k)$, $n \in \mathbb{Z}_8$, $0 \le x_n, y_n \le 2$ with $(x_n, y_n) \ne (0,0)$. Moreover, x_n and y_n depend only on n mod 8. It is easy to check that

$$\{ax_n + by_n\}_{n=0}^{7} \equiv \{2a+b, 2b, 2a, 2a+2b, a+2b, b, a, a+b\} \pmod{3}.$$

It is just a shift of the sequence $\{F_n\}_{n\ge 0} \equiv \{0,1,1,2,0,2,2,1,\dots\} \pmod{3}$. Thus, for $n \in \mathbb{Z}_8$, there are two n's such that $G_{n+\frac{\pi}{3}} \equiv G_n \pmod{3^k}$; three n's such that $G_{n+\frac{\pi}{3}} \equiv G_n + 3^{k-1} \pmod{3^k}$; and three n's that satisfy $G_{n+\frac{\pi}{3}} \equiv G_n + 2\times 3^{k-1} \pmod{3^k}$.

Theorem 5.5. *Let $G_n = 7F_n + 4F_{n+1}$ and $b \not\equiv 4, 5 \pmod{9}$. For $k \ge 2$, there are only two n's in $\mathbb{Z}_{\pi(3^k)}$ such that $G_n \equiv b \pmod{3^k}$.*

Proof. Since $\{G_n\}_{n\ge 0} \equiv \{4,2,6,8,5,4,0,4,4,8,3,2,5,7,3,1,4,5,0,5,5,1,6,7;4,2,\dots\} \pmod{9}$, the theorem is true for $k = 2$.

By the above discussion, $G_{n+\frac{\pi}{3}} \equiv G_n \pmod{3^k}$ only if $n \equiv 0$ or $4 \pmod{8}$. For these cases, $G_n \equiv 4$ or $5 \pmod{9}$.

Using the same approach that leads to the proofs of Theorems 4.6 and 4.7 in [1], replacing the Fibonacci sequence $\{F_n\}$ by $\{G_n\}$ we get the assertion.□

Corollary 5.6. *For $r \ge 1$, let $G_n(r) = b_rF_n + a_rF_{n+1}$, where a_r and b_r are defined in Theorem 3.1. Then $G_n(r) \equiv 0 \pmod{3^k}$ has exactly two solutions $n \in \mathbb{Z}_{\pi(3^k)}$ for $k \ge 1$.*

Proof. For $r = 1$, $G_n(1) = b_1F_n + a_1F_{n+1} = F_{n+2}$. From Theorem 1.2, $\nu(3^k, b) = 2$ for $b \equiv 0 \pmod{3}$, for $k \ge 1$. Hence the corollary is true in this case.

For $r \ge 2$, from Remark 3.1 we have $a_r \equiv 4$ and $b_r \equiv 7 \pmod{9}$. If $k = 1$, then $G_n(r) \equiv F_{n+2} \pmod{3}$ and corollary holds. For $k \ge 2$, the corollary follows from Theorem 5.5. □

Remark 5.1. One can easily see that $G_6 \equiv G_{18} \equiv 0 \pmod{9}$. Thus, $G_n \equiv 0 \pmod{3^k}$ only if $n = 6 + 24\ell$ or $18 + 24\ell$ for some ℓ. Thus there is a unique n in $\mathbb{Z}_{3^r}$ of the form $6 + 24\ell$ satisfying equation $(4.2)_r$. In the following we shall find the unique value of $\ell_r \in [0, 3^{r-2})$ such that $n(\ell_r)$ satisfies the equation $(4.2)_r$.

6. Formula of ℓ_r

Let $k \geq 4$. For each $0 \leq \ell < 3^{k-4}$ and $0 \leq i \leq 8$, we note that $n(\ell + 3^{k-4}i) = 6+24(\ell+3^{k-4}i) = n(\ell)+3^{-2}\pi(3^k)i$. So by Proposition 2.2 $F_{n(\ell+3^{k-4}i)} \equiv F_{n(\ell)} \pmod{3^k}$.

Now we put all those ℓ such that $0 \leq \ell < 3^{k-2}$ in the array:

$$\begin{array}{cccccc}
0 & 1 & 2 & \cdots & 3^{k-4}-1 \\
3^{k-4} & 3^{k-4}+1 & \cdots & \cdots & 2\times 3^{k-4}-1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
8\times 3^{k-4} & 8\times 3^{k-4}+1 & \cdots & \cdots & 9\times 3^{k-4}-1
\end{array}$$

Each of the nine elements in each column when taking $n(\ell)$ will all yield the same F-value $\pmod{3^k}$. For abbreviation, we shall say ℓ yields $F_{n(\ell)}$ value. Therefore, we only need to consider those ℓ such that $0 \leq \ell \leq 3^{k-4}-1$.

Proposition 6.1. *Suppose $k \geq 4$.*

1. Suppose the integer $\ell \in [0, \frac{1}{2}(3^{k-4}-1)]$. Then there exists an integer $\overline{\ell} \in [0, \frac{1}{2}(3^{k-4}-1)]$ with $\ell+\overline{\ell} = \frac{1}{2}(3^{k-4}-1)$ such that $F_{n(\ell)} \equiv F_{n(\overline{\ell})} \pmod{3^k}$. Moreover, $\ell \neq \overline{\ell}$ unless k is even and $\ell = \frac{1}{4}(3^{k-4}-1)$.

2. If the integer $\ell \in (\frac{1}{2}(3^{k-4}-1), 3^{k-4})$, then there exists an integer $\overline{\ell} \in (\frac{1}{2}(3^{k-4}-1), 3^{k-4})$ with $\ell+\overline{\ell} = \frac{1}{2}(3^{k-3}-1)$ such that $F_{n(\ell)} \equiv F_{n(\overline{\ell})} \pmod{3^k}$. Moreover, $\ell \neq \overline{\ell}$ unless k is odd and $\ell = \frac{1}{4}(3^{k-3}-1)$.

Proof. Let $\pi = \pi(3^k)$. By (1.2) we note that

$$\begin{aligned} F_{\frac{\pi}{2}-m} = F_{\frac{\pi}{2}-1}F_{-m} + F_{\frac{\pi}{2}}F_{-m+1} \equiv -F_{-m} \equiv (-1)^{m+2}F_m \\ \equiv (-1)^m F_m \pmod{3^k}. \end{aligned}$$

Thus m is even if and only if $F_{\frac{\pi}{2}-m} \equiv F_m \pmod{3^k}$.

1. Suppose $\ell \in [0, \frac{1}{2}(3^{k-4}-1)]$. Put $\overline{\ell} = \frac{1}{2}(3^{k-4}-1)-\ell$. Then it is straightforward to verify that $n(\overline{\ell}) + 4 \times \frac{\pi}{3^2} = \frac{\pi}{2} - n(\ell)$. Since $n(\ell)$ is even, $F_{n(\overline{\ell})} \equiv F_{n(\ell)} \pmod{3^k}$.

2. Suppose $\ell \in (\frac{1}{2}(3^{k-4}-1), 3^{k-4})$. Put $\overline{\ell} = \frac{1}{2}(3^{k-3}-1)-\ell$. Then we have $n(\overline{\ell}) + \frac{\pi}{3} = \frac{\pi}{2} - n(\ell)$. Hence $F_{n(\overline{\ell})} \equiv F_{n(\ell)} \pmod{3^k}$. □

The integer $\overline{\ell}$ defined in Proposition 6.1 is called the *conjugate* of ℓ.

Proposition 6.1 implies that integers $\ell \in [0, 3^{k-4})$ are in pairs in the above sense except for the special one $\frac{1}{4}(3^{k-4}-1)$ if k is even or $\frac{1}{4}(3^{k-3}-1)$ if k is odd.

Note that if $n(\ell)$ is a solution of $(4.2)_r$, then since $n(\ell + 3^{k-r-2}) = n(\ell) + \frac{\pi(3^k)}{3^r}$ by Corollary 4.2, $n(\ell + 3^{k-r-2})$ is also a solution of $(4.2)_r$.

Suppose $k = 2r$ and $r \geq 3$. Let $l_i = \ell_r + 3^{r-2}i$, $0 \leq i < 3^{r-2}$, where $\ell_r \in [0, 3^{r-2})$ is such that $n(\ell_r) = 6 + 24\ell_r$ is the unique solution of $(4.2)_r$ in $\mathbb{Z}_{\pi(3^r)}$. Then by Corollary 4.3 $\{l_0, l_1, \ldots, l_{3^{r-2}-1}\}$ is a sequence in $[0, 3^{2r-4})$ such that $F_{n(l_i)}$ all have the same value modulo 3^{2r}. By the uniqueness of the solution (in $\mathbb{Z}_{\pi(3^r)}$) of $(4.2)_r$, these $n(l_i)$'s constitute all the solutions of $(4.2)_r$ in $[0, 3^{2r-4})$.

Let ℓ be one of the l_i's and $\overline{\ell}$ be its conjugate. Suppose $\ell \in [0, \frac{1}{2}(3^{k-4} - 1)]$. By the proof of Proposition 6.1, Proposition 2.1, Proposition 2.2 and Corollary 4.3 we have

$$\begin{aligned} F_{n(\overline{\ell})+\frac{\pi}{3^r}} &= F_{-n(\ell)+\frac{\pi}{2}-4\times\frac{\pi}{3^2}+\frac{\pi}{3^r}} \equiv F_{\frac{\pi}{2}-(n(\ell)-\frac{\pi}{3^r})} \equiv F_{n(\ell)-\frac{\pi}{3^r}} \\ &\equiv F_{n(\ell)} \equiv F_{n(\overline{\ell})} \pmod{3^k}. \end{aligned}$$

We have the same result if $\ell \in (\frac{1}{2}(3^{k-4} - 1), 3^{k-4})$. Hence $n(\overline{\ell})$ is also a solution of equation $(4.2)_r$ in $[0, 3^{2r-4})$.

So, we have $\{l_i \mid 0 \leq i \leq 3^{r-2} - 1\} = \{\overline{l_i} \mid 0 \leq i \leq 3^{r-2} - 1\}$. Thus $\min\{\overline{l_i} \mid 0 \leq i \leq 3^{r-2} - 1\} = \ell_r$. Let $m = \left\lfloor \frac{1}{3^{r-2}}(\frac{1}{2}(3^{2r-4} - 1) - \ell_r)\right\rfloor$. By Proposition 6.1, $\overline{l_m} = \min\{\overline{l_i} \mid 0 \leq i \leq 3^{r-2} - 1\}$. Since $0 \leq \ell_r \leq 3^{r-2} - 1$, $\frac{1}{2}(3^{r-2} - 1) - 1 \leq m \leq \frac{1}{2}(3^{r-2} - 1)$.

Suppose $m = \frac{1}{2}(3^{r-2} - 1)$. Since $\ell_r = \overline{l_m} = \frac{1}{2}(3^{2r-4} - 1) - \ell_r - 3^{r-2}m$, $\ell_r = \frac{1}{4}(3^{r-2} - 1)$. Moreover, since ℓ_r is an integer, r must be even. Suppose $m = \frac{1}{2}(3^{r-2} - 1) - 1$. Similarly, we have $\ell_r = \frac{1}{4}(3^{r-1} - 1)$ and hence r is odd. Combining the cases of $r = 1$ and $r = 2$, we have the following result.

Theorem 6.2. *For $r \geq 1$, let $n(\ell_r)$ be the unique solution of equation $(4.2)_r$ in $\mathbb{Z}_{\pi(3^r)}$. Then*

$$\ell_r = \begin{cases} \frac{1}{4}(3^{r-1} - 1) & \text{if } r \text{ is odd;} \\ \frac{1}{4}(3^{r-2} - 1) & \text{if } r \text{ is even.} \end{cases}$$

Thus, $\ell_r = \ell_{r+1}$ for odd r, while $\ell_{r+1} = \ell_r + 2 \times 3^{r-2}$ for even r.

7. Frequency of $F_{n(\ell_r)}$ in $\mathbb{Z}_{3^k}$ for $k = 2r$ or $2r + 1$

Let $L_k(b)$ be the set of ℓ in $[0, 3^{k-4})$ that yields F-value b modulo 3^k. From Corollary 2.12, we have $\nu(3^k, b) = 9|L_k(b)| + 2$.

Theorem 7.1. *For $r \geq 3$, $L_{2r}(b) = \{\ell_r + 3^{r-2}i \mid 0 \leq i \leq 3^{r-2} - 1\}$, where $b \equiv F_{n(\ell_r)} \pmod{3^{2r}}$ and ℓ_r is defined in Section 4. Hence $\nu(3^{2r}, b) = 3^r + 2$.*

Proof. It is straightforward to verify that $\{\ell_r + 3^{r-2}i \mid 0 \leq i \leq 3^{r-2} - 1\} \subseteq L_{2r}(b)$. For $r = 3$, $\ell_3 = 2$ and $n(2) = 54$. It is clear that $L_6(F_{54}) = \{2, 5, 8\}$. Assume that $L_{2r}(b) = \{\ell_r + 3^{r-2}i \mid 0 \leq i \leq 3^{r-2} - 1\}$ for $r \geq 3$, where $b \equiv F_{n(\ell_r)} \pmod{3^{2r}}$.

Now we let $b \equiv F_{n(\ell_{r+1})} \pmod{3^{2r+2}}$. If r is odd, then $\ell_r = \ell_{r+1}$ and hence $F_{n(\ell_{r+1})} = F_{n(\ell_r)}$. If r is even, then $\ell_{r+1} = \ell_r + 2 \times 3^{r-2}$. Hence $n(\ell_{r+1}) = n(\ell_r) + 2\pi(3^r)$. By (4.1) and Lemma 3.5 we have

$$\begin{aligned} F_{n(\ell_{r+1})} &= F_{n(\ell_r)} + 3^r(b_{r,2}F_{n(\ell_r)} + a_{r,2}F_{n(\ell_r)+1}) \\ &\equiv F_{n(\ell_r)} + 2 \times 3^r(b_r F_{n(\ell_r)} + a_r F_{n(\ell_r)+1}) \equiv F_{n(\ell_r)} \pmod{3^{2r}}. \end{aligned}$$

So $b \equiv F_{n(\ell_{r+1})} \equiv F_{n(\ell_r)} \pmod{3^{2r}}$ for both cases.

Let $c \in L_{2r+2}(b)$. We have $F_{n(c)} \equiv b \pmod{3^{2r}}$. Thus $c \equiv c' \pmod{3^{2r-4}}$ for some $c' \in L_{2r}(b)$. So $c = c' + 3^{2r-4}j$ for some j with $0 \leq j \leq 8$. Also $c' = \ell_r + 3^{r-2}i$ for some i with $0 \leq i \leq 3^{r-2} - 1$. Hence $c = \ell_r + 3^{r-2}i + 3^{2r-4}j = \ell_{r+1} + 3^{r-2}(i + 3^{r-2}j)$. Since $n(c) = n(\ell_{r+1}) + (i + 3^{r-2}j)\pi(3^r)$ and $F_{n(c)} \equiv b \pmod{3^{2r+2}}$, by Proposition 4.4 $\lambda = i + 3^{r-2}j \equiv 0 \pmod 3$. That is, $i \equiv 0 \pmod 3$. Hence $c \in \{\ell_{r+1} + 3^{r-1}i' \mid 0 \leq i' \leq 3^{r-1} - 1\}$. This completes the induction. □

For $k = 2r+1$, we can verify that the set $\{\ell \in [0, 3^{k-4}) \mid n(\ell)$ is a solution of $(4.2)_r\}$ is the union of the following sets:

$$\begin{aligned} A_0 &= \{\ell_r + 3^{r-1}i \mid 0 \leq i \leq 3^{r-2} - 1\}; \\ A_1 &= \{\ell_r + 3^{r-2} + 3^{r-1}i \mid 0 \leq i \leq 3^{r-2} - 1\}; \\ A_2 &= \{\ell_r + 2 \times 3^{r-2} + 3^{r-1}i \mid 0 \leq i \leq 3^{r-2} - 1\}. \end{aligned}$$

All members in each sets yield the same F-value modulo 3^{2r+1}.

Case 1. Suppose r is odd. Then $\ell_r = \ell_{r+1} = \frac{1}{4}(3^{r-1} - 1) \in A_0$. Let $c = \ell_{r+1} + 3^{r-2} \in A_1$. Then it is easy to see that the conjugate of c is $\bar{c} = \ell_r + 2 \times 3^{r-2} + 3^{r-1}i$, where $i = \frac{1}{2}(3^{r-2} - 3)$. Thus, $A_1 \cup A_2 \subseteq L_{2r+1}(b)$, where $b \equiv F_{n(c)} \pmod{3^{2r+1}}$.

Case 2. Suppose r is even. Then $\ell_{r+1} = \frac{1}{4}(3^r - 1) = \ell_r + 2 \times 3^{r-2} \in A_2$. For $d = \ell_r$, the conjugate of d is $\bar{d} = \ell_r + 3^{r-2} + 3^{r-1}i \in A_1$, where $i = \frac{1}{2}(3^{r-2} - 1)$. In this case we have $A_0 \cup A_1 \subseteq L_{2r+1}(b)$, where $b \equiv F_{n(d)} \pmod{3^{2r+1}}$.

Theorem 7.2. *Same notation as above. For odd r,*

$$L_{2r+1}(b) = \begin{cases} A_1 \cup A_2 & \text{if } b \equiv F_{n(\ell_{r+1}+3^{r-2})} \pmod{3^{2r+1}}; \\ A_0 & \text{if } b \equiv F_{n(\ell_r)} = F_{n(\ell_{r+1})} \pmod{3^{2r+1}}. \end{cases}$$

For even r,

$$L_{2r+1}(b) = \begin{cases} A_0 \cup A_1 & \text{if } b \equiv F_{n(\ell_r)} \pmod{3^{2r+1}}; \\ A_2 & \text{if } b \equiv F_{n(\ell_{r+1})} \pmod{3^{2r+1}}. \end{cases}$$

Proof. For $b \equiv F_{n(\ell_{r+1}+3^{r-2}j)} \pmod{3^{2r+1}}$, where $0 \le j \le 2$, we have that $b \equiv F_{n(\ell_r)} \pmod{3^{2r}}$. Hence each $c \in L_{2r+1}(b)$ comes from $A_0 \cup A_1 \cup A_2$. Thus for odd r, we only need to show that $F_{n(\ell_{r+1})} \not\equiv F_{n(\ell_{r+1}+3^{r-2})} \pmod{3^{2r+1}}$; and for even r, we only need to show that $F_{n(\ell_{r+1})} \not\equiv F_{n(\ell_{r+1}-2\times 3^{r-2})} \pmod{3^{2r+1}}$.

Case 1. When r is odd, since $F_{n(\ell_{r+1}+3^{r-2})} = F_{n(\ell_{r+1})+\pi(3^r)}$, the assertion follows from Proposition 4.5.

Case 2. When r is even, since

$$\begin{aligned} F_{n(\ell_{r+1}-2\times 3^{r-2})} &\equiv F_{n(\ell_{r+1})-2\pi(3^r)+\pi(3^{2r+1})} \\ &= F_{n(\ell_{r+1})+(3^{r-1}-2)\pi(3^r)} \pmod{3^{2r+1}}, \end{aligned}$$

by Proposition 4.5, we have the assertion. □

Corollary 7.3. *Suppose $r \ge 3$. We have:*

$$\nu(3^{2r+1}, F_{n(\ell_{r+1})}) = 3^r + 2,$$

$$\nu(3^{2r+1}, F_{n(\ell_{r+1}+3^{r-2})}) = 2 \times 3^r + 2$$

for odd r and $\nu(3^{2r+1}, F_{n(\ell_r)}) = 2 \times 3^r + 2$ for even r.

8. Frequency of 8 in $\mathbb{Z}_{3^k}$

Suppose that $k \ge 6$ and $b \equiv 8 \pmod{27}$. If $b \not\equiv 8$, 251, 332 and 494 modulo 3^6, then by Remark 2.1 we have that $\nu(3^k, b) = 2$.

In general we do not have formula for $\nu(3^k, b)$, where $b \equiv 8 \pmod{27}$ and k arbitrary. In [1], we observed that $\nu(3^8, 332) = \nu(3^9, 332) = 83$ but $\nu(3^{10}, 332) = 2$. As one can verify from Corollary 2.5 that $F_{n(\ell)} \equiv 332$

(mod 3^{10}) has no solution in ℓ. This means that $b = 332$ is no longer generated from $n(\ell)$ for $k \geq 10$. Thus $\nu(3^k, 332) = 2$ if $k \geq 10$.

If $b = 8 = F_{n(0)}$ which appears in $\mathbb{Z}_{3^k}$ for every $k \geq 5$, then we have the following theorem.

Theorem 8.1. *For $k \geq 5$, $\nu(3^k, 8) = 20$.*

Proof. We first note that for $k = 5$, $\nu(3^5, 8) = 20$ (please see [1]). Since $F_{n(0)} = 8$, $\nu(3^k, 8) \geq 20$ for $k \geq 6$.

We shall prove the theorem by contradiction. Suppose k is the first positive integer such that $\nu(3^k, 8) > 20$. Thus $\nu(3^{k-1}, 8) = 20$. In $[0, 3^{k-5})$, 0 and $\frac{1}{2}(3^{k-5} - 1)$ are the only integers that yield F-value 8 (mod 3^{k-1}). Suppose $F_{n(\ell)} \equiv 8 \pmod{3^k}$, for $\ell \in [0, 3^{k-4})$. Then $F_{n(\ell)} \equiv 8 \pmod{3^{k-1}}$. Since $\ell \in [0, 3 \times 3^{k-5})$ and $F_{n(\ell)} \equiv 8 \pmod{3^{k-1}}$, $\ell = 0$, $\frac{1}{2}(3^{k-5} - 1)$, 3^{k-5}, $3^{k-5} + \frac{1}{2}(3^{k-5} - 1) = \frac{1}{2}(3^{k-4} - 1)$, $2 \times 3^{k-5}$, or $2 \times 3^{k-5} + \frac{1}{2}(3^{k-5} - 1)$. Thus, if there are integers in $[0, 3^{k-4})$ that yield F-value 8 (mod 3^k) except for these $\ell = 0$ and $\ell = \frac{1}{2}(3^{k-4} - 1)$, then they must be from the other four. Since they are in pairs, they must come from 3^{k-5} or $2 \times 3^{k-5}$.

Now $n(3^{k-5}) = 6 + 24 \times 3^{k-5} = 6 + \frac{\pi(3^k)}{27}$ and $n(2 \times 3^{k-5}) = 6 + \frac{2\pi(3^k)}{27}$. One can use Lemma 4.1 with $a_3 = 22$ and $b_3 = 16$ to obtain that

$$F_{n(3^{k-5})} \equiv 8 + 3^{k-1} \pmod{3^k} \quad \text{and} \quad F_{n(2\times 3^{k-5})} \equiv 8 + 2 \times 3^{k-1} \pmod{3^k},$$

which both cannot be congruent to 8 (mod 3^k). This is clearly a contradiction. $\square$

References

[1] W.C. Shiu, C.I. Chu, Distribution of Fibonacci numbers mod 3^k, *The Fibonacci Quarterly*, **43** (2005), 22-28.

[2] D.D. Wall, Fibonacci series modulo m, *Amer. Math. Monthly*, **67** (1960), 525-532.