Ring-magic Labeling of Graphs

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Outline

- ⋆ Background
- ⋆ Ring-magic
- ⋆ General Results
- ⋆ group-magic but not ring-magic
- ⋆ $\mathbb{Z}_3$-ring-magic trees
- ⋆ V_4-ring-magic trees
- ⋆ Further studies

Group-magic Labeling of Graphs

Let $G = (V, E)$ be a connected, simple graph. Let A be a finite abelian group.

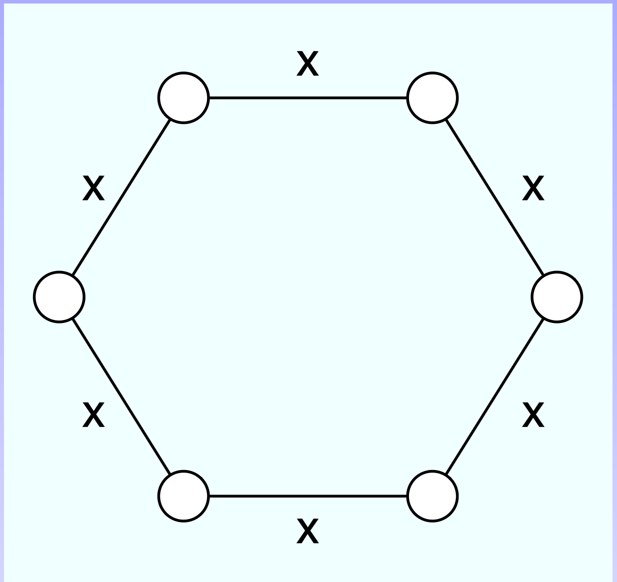
Definition 1 Suppose that there exists a labeling $f : E \to A \setminus \{0\}$ such that the induced vertex labeling $f^+ : V \to A$, defined by

$$f^+(v) = \sum_{uv \in E} f(uv),$$

is a constant map. Then, f is an ***A-magic labeling*** of G and G is ***A-magic***.

Some Examples

Ex 1. C_6 is A-magic, for all abelian groups A.



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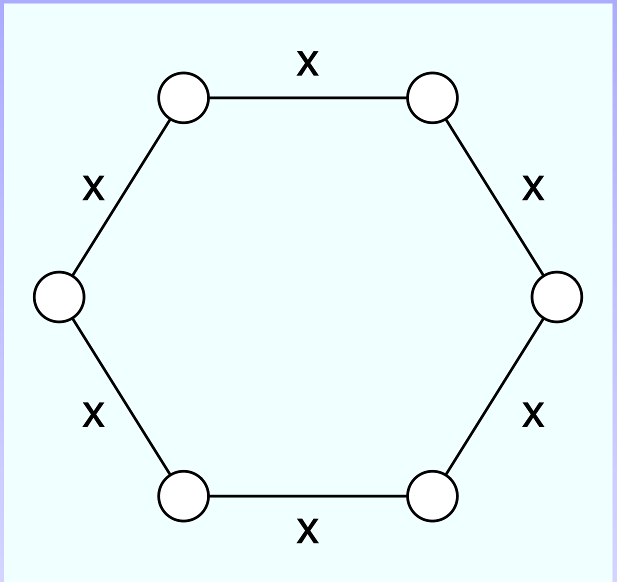
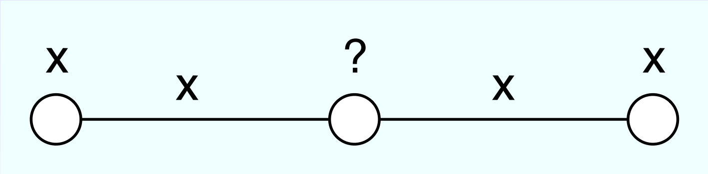
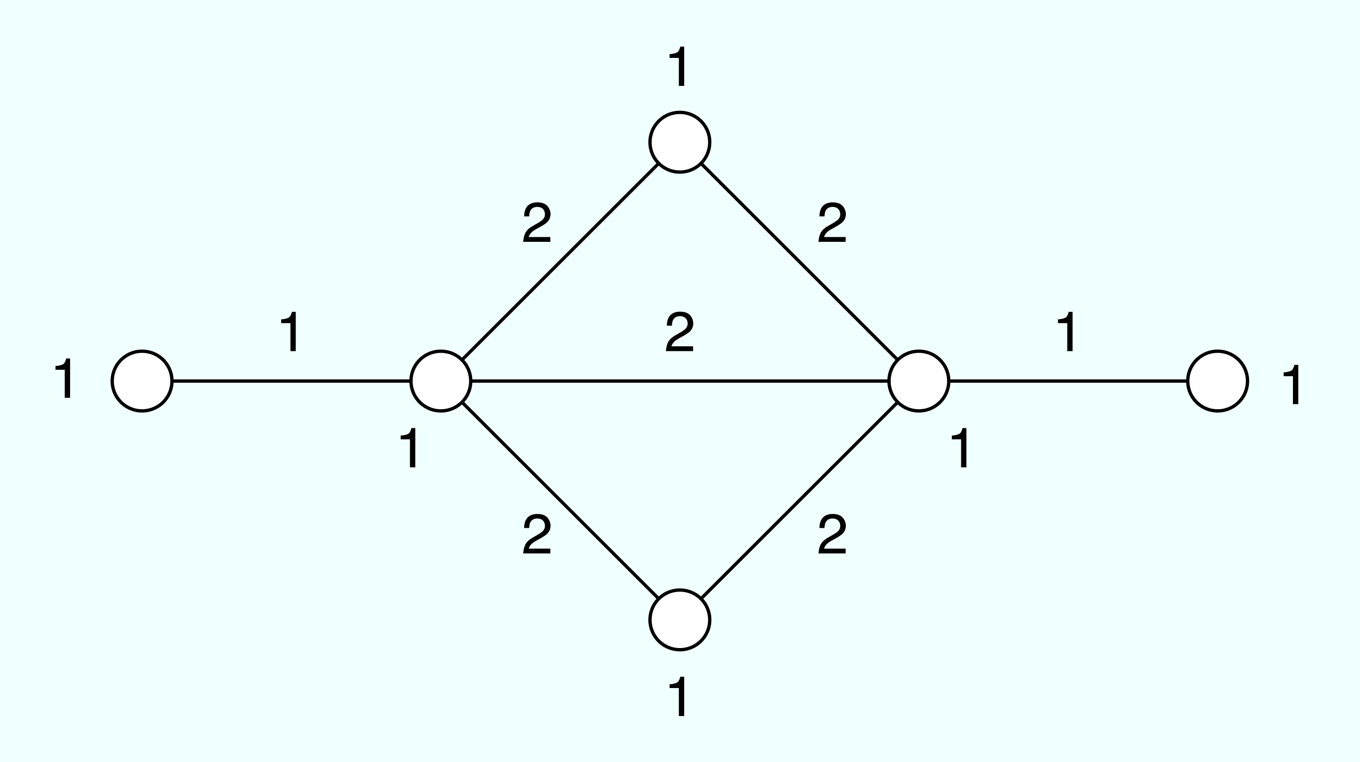


Ex 2. P_3 is not A-magic, for all abelian groups A.



Some Examples (continued)

Ex 3. The following graph is $\mathbb{Z}_3$-magic, but not $\mathbb{Z}_2$-magic.



A graph is $\mathbb{Z}_2$-magic if and only if the degree of each vertex is of the same parity.

$\mathbb{Z}_n$-magic Labelings of Graphs

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- ⋆ Sun graphs (W.C. Shiu, R.M. Low, 2007).

Ring-magic Labelings of Graphs

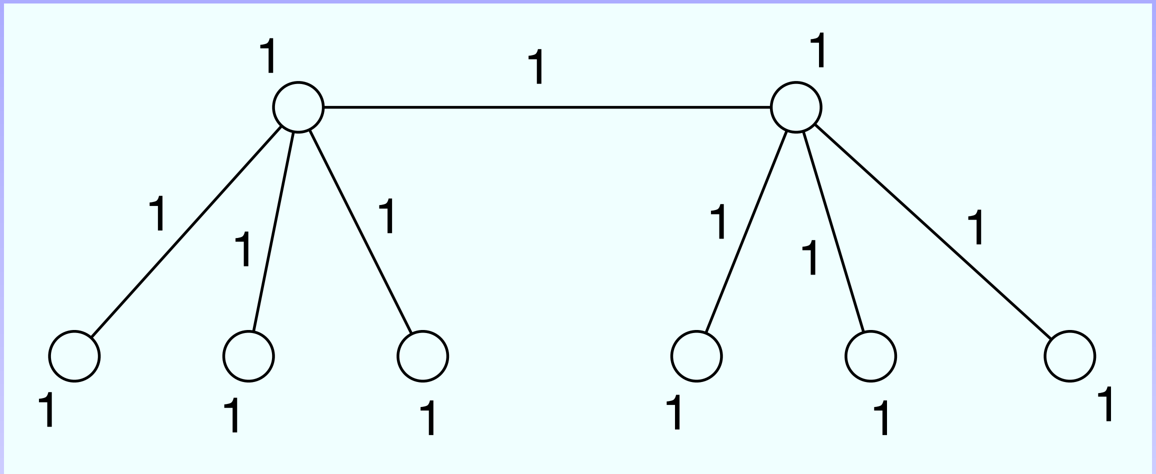
Let $G = (V, E)$ be a connected, simple graph. Let R be a commutative ring with unity 1.

Definition 2 Suppose that there exists a labeling $f : E \to R \setminus \{0\}$ such that the induced vertex labelings $f^+ : V \to R$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$, and $f^\times : V \to R$, defined by $f^\times(v) = \prod_{uv \in E} f(uv)$, are constant maps. Then, f is an ***R-ring-magic labeling*** of G and G is ***R-ring-magic***.

In this case, the values of f^+ and $f^\times$ are called the ***additive*** and ***multiplicative R-magic values*** of f, respectively.

Examples of $\mathbb{Z}_3$-ring-Magic Graphs

Ex 1. A $\mathbb{Z}_3$-ring-magic tree.



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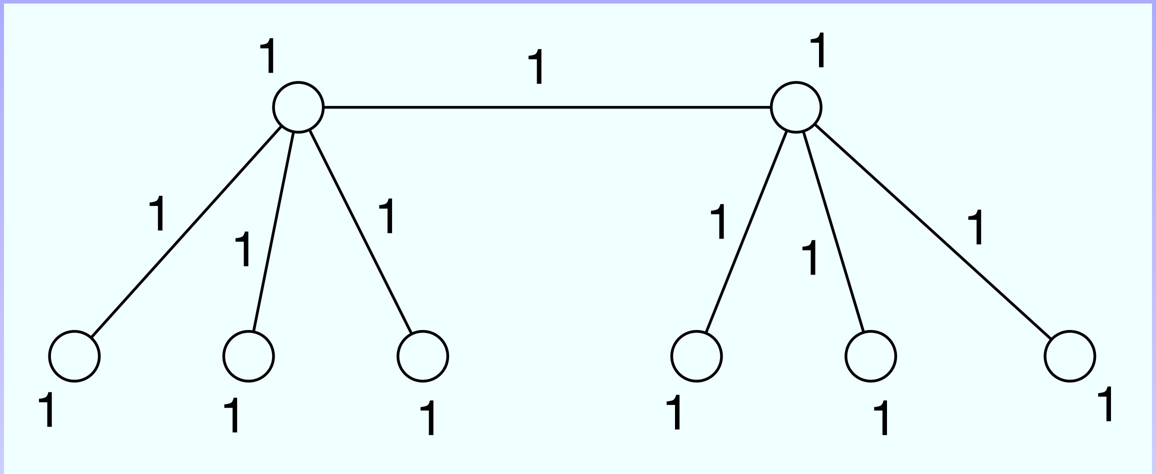

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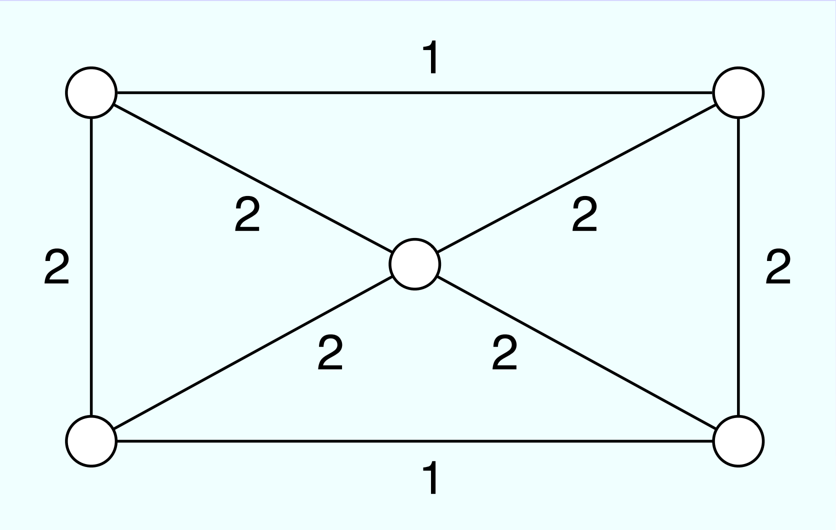

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- ⋆ Let A be the abelian group associated with ring R. If G is not A-magic, then G is not R-ring magic.
- ⋆ G is $\mathbb{Z}_2$-ring magic $\Longleftrightarrow$ the degree of each vertex of G is of the same parity.

A Few General Results

Theorem 1 *Let R be a ring and $G = (V, E)$ be an R-ring magic graph of order p. Let h and k be the additive and multiplicative R-magic values respectively, of an R-ring magic labeling f. Then,*

$$hp = 2a \text{ and } k^p = b^2$$

for some $a, b \in R$.

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Proof:

$$hp = \sum_{v \in V} f^+(v) = \sum_{v \in V} \sum_{uv \in E} f(uv) = 2 \sum_{e \in E} f(e)$$

$$k^p = \prod_{v \in V} f^\times(v) = \prod_{v \in V} \prod_{uv \in E} f(uv) = \left(\prod_{e \in E} f(e) \right)^2$$

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Theorem 2 *Let R_1 be a ring, which contains a subring isomorphic to ring R_2. If graph G is R_2-ring magic, then G is R_1-ring magic.*

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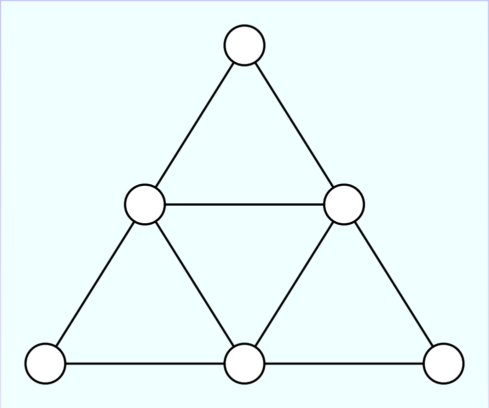
The converse of Theorem 2 does not always hold.



This graph is $\mathbb{Z}_2$-ring-magic. Hence it is $\mathbb{Z}_6$-ring-magic. However, it is straight-forward to show (using an exhaustive case analysis) that G is not $\mathbb{Z}_3$-group-magic and hence, cannot be $\mathbb{Z}_3$-ring-magic.

A Few General Results (continued)

For a commutative ring R, let $U(R)$ denote the group of units.

Theorem 3 *Suppose that f is an R-ring magic labeling of a graph G and $u \in U(R)$, where R is an integral domain. Then, uf is an R-ring magic labeling of $G \iff o(u)|[d(v_i) - d(v_j)]$, for all v_i, $v_j \in V(G)$, where $o(u)$ is the order of u in $U(R)$.*

Proof of Theorem 3

Proof: $(uf)^{\times}(v) = u^{d(v)}k$, where k is the multiplicative magic value.

$$
\begin{aligned}
u^{d(v_i)}k = u^{d(v_j)}k &\Longleftrightarrow (u^{d(v_i)} - u^{d(v_j)})k = 0 \\
&\Longleftrightarrow u^{d(v_i)} - u^{d(v_j)} = 0 \\
&\Longleftrightarrow u^{d(v_i)-d(v_j)} = 1 \\
&\Longleftrightarrow o(u) | [d(v_i) - d(v_j)].
\end{aligned}
$$

Example

Ex. Consider the ring $\mathbb{Z}_5$. Then, $U(\mathbb{Z}_5) = \{1, 2, 3, 4\}$ is isomorphic (as a group) to $\mathbb{Z}_4$. Element in $U(\mathbb{Z}_5)$ is of order 1 or 2.

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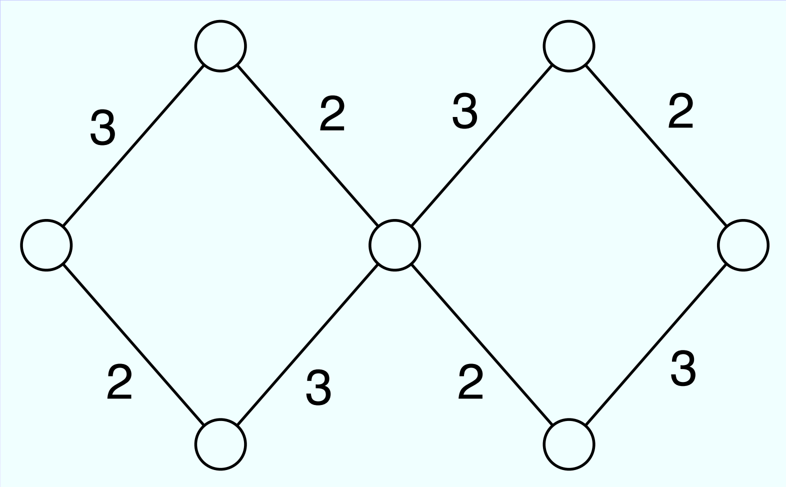
In particular, $u = 4$ which has order 2.



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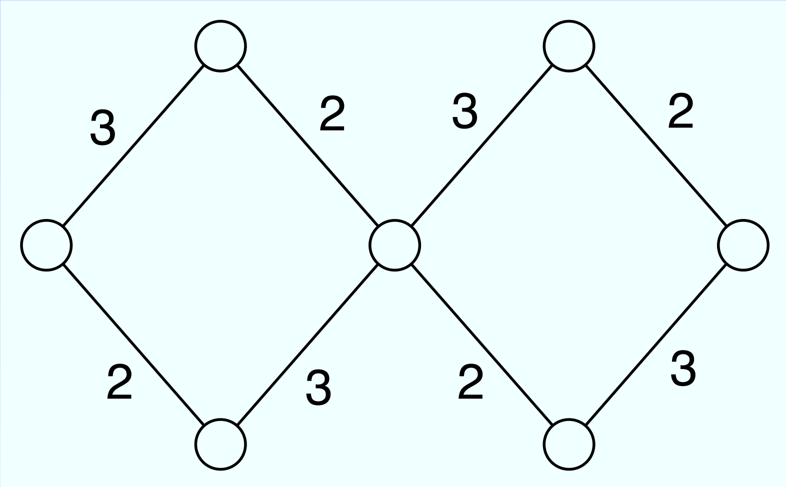

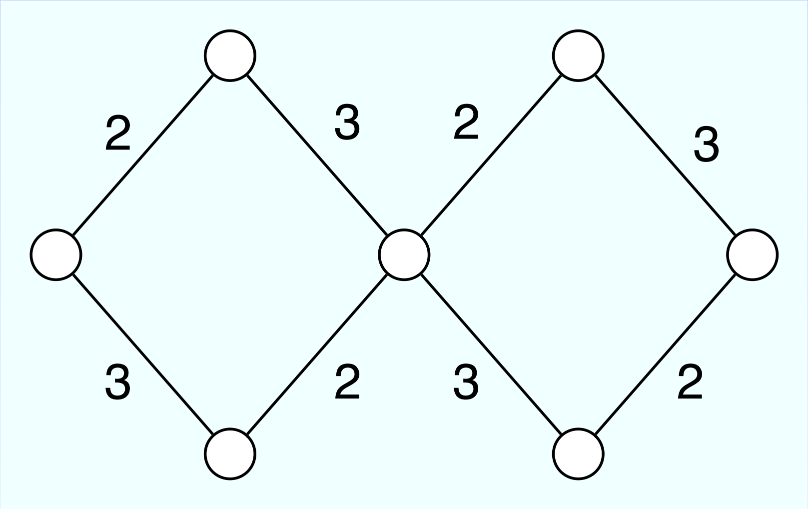

$\mathbb{Z}_n$-ring Magic Graphs

Corollary 4 *Let G be a $\mathbb{Z}_n$-ring magic graph of odd order. Then, for each odd prime factor r of n, the multiplicative magic value k must be a square in $\mathbb{Z}_r$.*

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Corollary 4 *Let G be a $\mathbb{Z}_n$-ring magic graph of odd order. Then, for each odd prime factor r of n, the multiplicative magic value k must be a square in $\mathbb{Z}_r$.*

Proof: By Theorem 1, there exists $b \in \mathbb{Z}_n$ such that $\left(\frac{k}{r}\right) = \left(\frac{k}{r}\right)^p = \left(\frac{k^p}{r}\right) = \left(\frac{b^2}{r}\right) = \left(\frac{b}{r}\right)^2$. Here $\left(\frac{a}{r}\right)$ is the Legendre symbol.

A Natural Question

At this point, it is very natural for us to ask the following question. For a given n, are there graphs which are $\mathbb{Z}_n$-group-magic but not $\mathbb{Z}_n$-ring-magic?

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The answer to this question is yes for n being odd prime. Let $C_m(y)$ denote the m-cycle with y pendants attached to each vertex.

Theorem 5 *Let n be an odd prime. Then, there exists an integer y such that $C_4(y)$ is $\mathbb{Z}_n$-group magic but which is not $\mathbb{Z}_n$-ring magic.*

An Example

Before to prove Theorem 5, we need a technical lemma.

Lemma 6 *Let n be an odd prime. Then, there exists $y \geq 1$ such that $y \equiv -1 \pmod{n-1}$ and $\left(\frac{y^2-2y-3}{n}\right) = -1$.*

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We also need the following well-known result:

Theorem A *Let n be an odd prime. Then, there are exactly $(n-1)/2$ quadratic residues of n and $(n-1)/2$ quadratic nonresidues of n among the integers $1, 2, \ldots, n-1$.*

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Proof: $y \equiv -1 \pmod{n-1} \Leftrightarrow y = -1 + k(n-1)$.

$$\left(\frac{y^2-2y-3}{n}\right) = \left(\frac{(-1-k)^2-2(-1-k)-3}{n}\right) = \left(\frac{k(k+4)}{n}\right).$$

Theorem A implies the existence of an integer w $(1 \leq w \leq n-1)$ such that $\left(\frac{w}{n}\right) = 1$ and $\left(\frac{w+1}{n}\right) = -1$.

Thus, $\left(\frac{4w}{n}\right)\left(\frac{4(w+1)}{n}\right) = -1$. Hence, $\left(\frac{k(k+4)}{n}\right) = -1$, where $k = 4w$.

Proof of Theorem 5

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$C_4(y)$ is $\mathbb{Z}_n$-group-magic, as all of the pendants can be labeled with 1 and the edges in the cycle labeled a, $1 - a - y$, a, and $1 - a - y$ respectively.

Proof of Theorem 5

We now claim that $C_4(y)$ is not $\mathbb{Z}_n$-ring-magic. Assume that $C_4(y)$ has a $\mathbb{Z}_n$-ring-magic labeling.

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Degree of each vertex of $C_4(y)$ is either 1 or $y + 2$. Because of Theorem 3 and the first condition in Lemma 6, we can assume without loss of generality that the pendants are labeled 1.

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Degree of each vertex of $C_4(y)$ is either 1 or $y+2$.
Because of Theorem 3 and the first condition in Lemma 6, we can assume without loss of generality that the pendants are labeled 1.
Let a and b be the labels of the two non-pendant edges incident to a vertex of degree $y+2$ in $C_4(y)$. Then, the following two relationships must hold:

- $a+b+y \equiv 1 \pmod{n}$.
- $ab \equiv 1 \pmod{n}$.

Proof of Theorem 5

Equivalent to saying that $x(1-y-x) \equiv 1 \pmod{n}$ has a solution, which implies that $x^2 + (y-1)x + 1 \equiv 0 \pmod{n}$ has a solution.

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However, this equation has discriminant $D=(y-1)^2-4(1)=y^2-2y-3$. Since y was chosen as to satisfy the conditions of Lemma 6, D is not a square in $\mathbb{Z}_n$.

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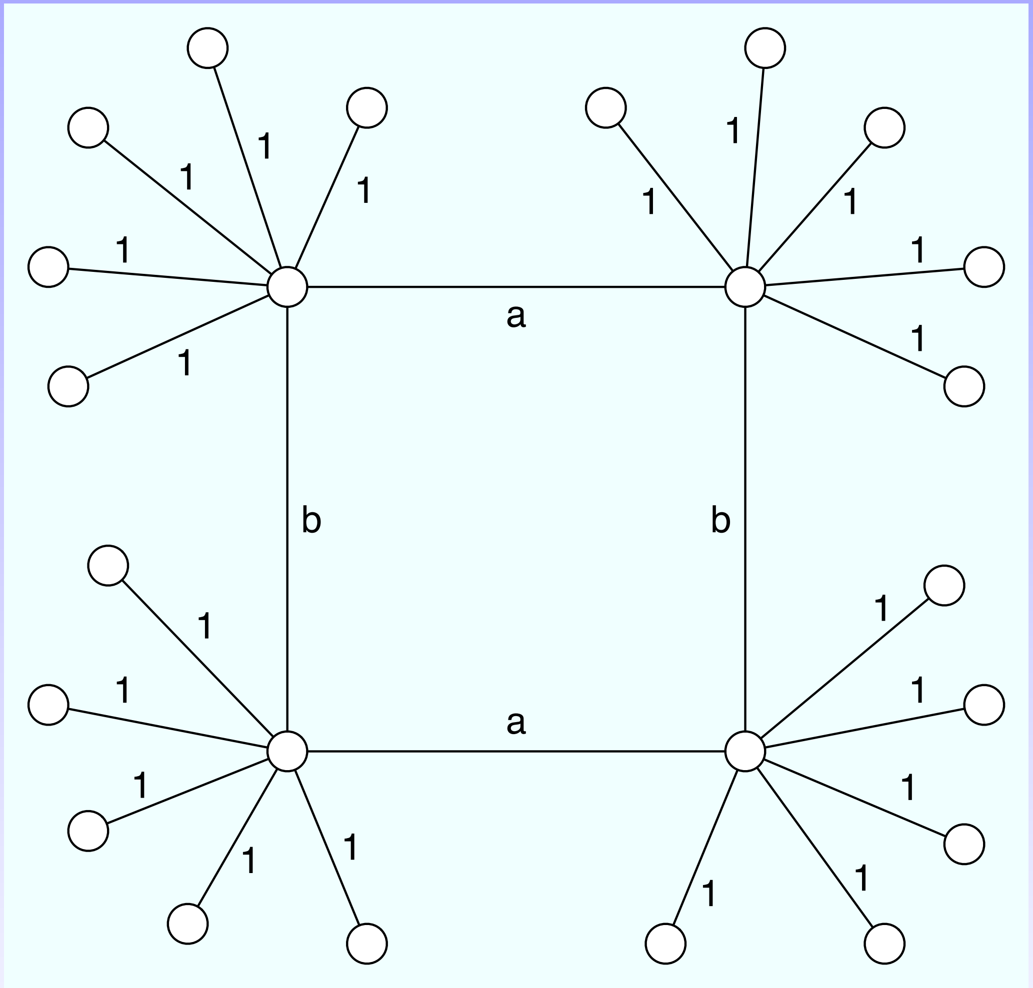
Thus, $x^2 + (y-1)x + 1 \equiv 0$ has no solutions in $\mathbb{Z}_n$. A contradiction.

Actual Example

Ex. Let $n = 7$.



The system $a + b = 3 \pmod 7$, $ab = 1 \pmod 7$ has no solutions.

R-ring-Magic Property for Trees

Let T be a tree. If f is an R-ring-magic labeling of T, then the additive and multiplicative magic values of f are the same. We call this value the ***R-ring-magic value*** of T.

Remark on Group-magic Labeling of Graphs

M. Doob studied group-magic graphs since 1974. The codomain of the edge labeling is the whole abelian group. That is, 0 was allowed to label on edges.

Under this concept of group-magic, R.P. Stanley considered (1973-1976) $\mathbb{Z}$-magic graphs. He pointed out that the theory of magic labelings could be studies in the general context of linear homogeneous diophantine equations.

R-ring-Magic Property for Trees

Definition 3 Let R be a commutative ring with unity. A graph $G = (V, E)$ is called ***R'-ring-magic*** if there exists a labeling $f : E \to R$ such that the induced vertex labelings $f^+ : V \to R$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$, and $f^\times : V \to R$, defined by $f^\times(v) = \prod_{uv \in E} f(uv)$, are constant maps.

R-ring-Magic Property for Trees

Theorem B (Shiu, Lam and Lee 2002) *Let T be a tree and A be a ring. Suppose that f is an A'-group-magic labeling of T. If there is an edge e which is incident to a leaf of T and $f(e) = 0$, then $f = 0$.*

Lemma 7 *Let T be a tree and A be a ring. Then, T has at most one A-ring-magic labeling with A-ring-magic value k.*

Lemma 8 *Let T be a tree. Suppose that f is a $\mathbb{Z}_n$-ring-magic labeling of T, with $\mathbb{Z}_n$-ring-magic value 1. Then, $f = 1$ (i.e., all the values of f are 1).*

$\mathbb{Z}_3$-ring Magic Property for Trees

Theorem 9 *Let T be a tree. Then, T is $\mathbb{Z}_3$-ring magic with ring-magic value $1 \iff d(v) \equiv 1 \pmod 3$, for all $v \in V(T)$. Moreover, for this case, suppose T is of order p. Then, $p \equiv 2 \pmod 3$.*

Proof: The first part is easy.

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Proof:
Let f be the $\mathbb{Z}_3$-ring-magic labeling with $\mathbb{Z}_3$-ring-magic value 1. By the proof of Theorem 1, we have

$$p = \sum_{v \in V} f^+(v) = 2(p-1) \pmod 3.$$

Hence, $p \equiv 2 \pmod 3$.

$\mathbb{Z}_3$-ring Magic Property for Trees

Corollary 10 *A tree T of odd order p is $\mathbb{Z}_3$-ring magic $\iff p \equiv 5 \pmod 6$ and $d(v) \equiv 1 \pmod 3$, for all $v \in V(T)$.*

$\mathbb{Z}_3$-ring Magic Property for Trees

Theorem 11 *Suppose a tree T has a $\mathbb{Z}_3$-ring magic labeling f with magic value 2. Let v be a vertex of T which is adjacent to pendants. Then, T is of even order and $d(v) \equiv 1$ or $0 \pmod 6$.*

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Proof: Order of T must be even.
Let a and b be the number of 1 and 2 labeled to the edges incident with v.

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Proof: Let a and b be the number of 1 and 2 labeled to the edges incident with v.
Clearly $a + b = d(v) = d$. Since the ring-magic value is 2, we have $2^b \equiv 2 \pmod 3$ and $a + 2b \equiv 2 \pmod 3$. Hence b is odd. Since $f(uv) = 2$ for all pendants u adjacent with v, this implies that $b = d$ or $b = d - 1$.

$\mathbb{Z}_3$-ring Magic Property for Trees

Theorem 11 *Suppose a tree* T *has a* $\mathbb{Z}_3$*-ring magic labeling* f *with magic value* 2*. Let* v *be a vertex of* T *which is adjacent to pendants. Then,* T *is of even order and* $d(v) \equiv 1$ *or* $0 \pmod 6$.

Proof: Let a and b be the number of 1 and 2 labeled to the edges incident with v.
$a + 2b \equiv 2 \pmod 3$, $b = d$ or $b = d - 1$.
(Case 1). If $b = d$, then $a = 0$ and $b \equiv 1 \pmod 3$.
Since b is odd, $d = b \equiv 1 \pmod 6$.
(Case 2). If $b = d - 1$, then $a = 1$ and $b \equiv 2 \pmod 3$. Since b is odd, $d = b + 1 \equiv 0 \pmod 6$.

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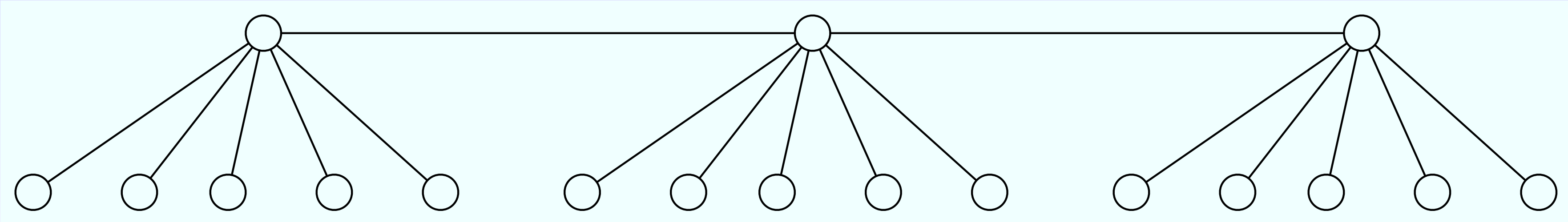
However, the following caterpillar G provides a counter-example which illustrates that the converse does not always hold.

G is of order 18, $d(v_i) \equiv 1$ or $0 \pmod 6$, but G is not $\mathbb{Z}_3$-group-magic. Hence, it is not $\mathbb{Z}_3$-ring-magic.



This caterpillar graph G is not $\mathbb{Z}_3$-ring-magic.

V_4-ring Magic Property for Trees

Let V_4 be the ring $Z_2 \times \mathbb{Z}_2$.

Lemma 12 **(Lee, Saba, Salehi and Sun, 2002).** *A tree* T *is* V_4*-group magic* $\Longleftrightarrow$ T *has no vertex of even degree.*

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Theorem 14 *Let T be a tree. Then, T is V_4-ring magic $\Longleftrightarrow$ $d(v) \equiv 1 \pmod 2$, for all $v \in V(T)$. Moreover, for this case, suppose T is of order p. Then, $p \equiv 0 \pmod 2$.*

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Proof:

$\Rightarrow$: By Lemma 12

$\Leftarrow$: Suppose that $d(v) \equiv 1 \pmod 2$, for all $v \in V(T)$. Then, the constant map f which labels every edge of T with x, where $x \in V_4 - \{0\}$, is a V_4-ring-magic labeling of T. Here, $f^+ = f^\times = x$.

Questions on Group-magic

- ⋆ Let $k \in \mathbb{Z}$ and $k \geq 2$. Are there graphs G whose integer-magic spectrum ($\{k \in \mathbb{N} \mid G$ is $\mathbb{Z}_k$-magic$\}$) is $k\mathbb{N}$?
- ⋆ Let $k_1, k_2, \ldots, k_n \in \mathbb{Z}$ and $k_i \geq 2$. Are there graphs G whose integer-magic spectrum is $k_1\mathbb{N} \cup k_2\mathbb{N} \cup \cdots \cup k_n\mathbb{N}$?
- ⋆ Are there classes of $\mathbb{Z}_k$-magic graphs having only magic-value 0?
- ⋆ Is it possible to construct $\mathbb{Z}_k$-magic graphs which have certain specified magic-values?
- ⋆ Let $k \in \mathbb{Z}$ and $k \geq 2$. What is the "smallest" graph which has integer-magic spectrum $\{2, 3, \ldots, \} - \{k\mathbb{N}\}$?
- ⋆ Find a V_4-magic graph which is not Z_4-magic.

Questions on Ring-magic

- ⋆ We can define integer-ring-magic spectrum and ask the same questions as group-magic.
- ⋆ Find the integer-ring-magic spectrum of some classes of graphs.

$\star$ **Thank you** $\star$