THE INTEGER-MAGIC SPECTRA OF BICYCLIC GRAPHS WITHOUT PENDANT

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ABSTRACT. Let A be a non-trivial, finitely-generated abelian group and $A^* = A - \{0\}$. A graph is A-magic if there exists an edge labeling using elements of A^* which induces a constant vertex labeling of the graph. For connected bicyclic graphs G (without pendant), we determine the set of numbers $k \geq 2$, where G has a $\mathbb{Z}_k$-magic labeling.

1. INTRODUCTION

Let $G = (V, E)$ be a connected simple graph. For any non-trivial, finitely generated abelian group A (written additively), let $A^* = A - \{0\}$. A mapping $f : E \to A^*$ is called a *labeling* of G. Any such labeling induces a map $f^+ : V \to A$, defined by $f^+(v) = \sum_{uv \in E} f(uv)$. If there exists a labeling f whose induced map on V is a constant map, we say that f is an *A-magic labeling* of G and that G is an *A-magic* graph. The corresponding constant is called an *A-magic value.* The *integer-magic spectrum* of a graph G is the set $\mathrm{IM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-magic and } k \geq 2\}$. Note that the integer-magic spectrum of a graph is not to be confused with the set of achievable magic values. $\mathbb{Z}$-magic (or $\mathbb{Z}_1$-magic) graphs were considered by Stanley [29, 30], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1–3] and others [7, 9, 15, 16, 26] have studied A-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [4,6,8,10–14,17–20,27,28].

2. BICYCLIC GRAPHS

A connected $(p, p+1)$-graph G is called a *bicyclic* graph. A characterization of bicyclic graphs without pendant is given in [25]. For the sake of completeness, we include the results and their proofs (Lemma 2.1, Theorems 2.2 and 2.3, and Corollary 2.4).

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Definition. A vertex of degree k is called a *k-vertex*. A vertex of degree greater than k is called a *k^+-vertex*.

Lemma 2.1. *Let G be a $(p, p+1)$-bicyclic graph without pendant. Then, the number of 2^+-vertices in G is at most two.*

Proof. It is known that $\sum_{v \in V(G)} d(v) = 2(p+1)$, where $d(v)$ denotes the degree of v. Let x be the number of 2^+-vertices in G. Then, $2(p-x)+3x \leq 2(p+1)$. Hence, $x \leq 2$. □

Definition. A *one-point union of two cycles* is a simple graph obtained from two cycles, say C_m and C_n where $m, n \geq 3$, by identifying one vertex from each cycle. Without loss of generality, we may assume the m-cycle to be $u_0 u_1 \cdots u_{m-1} u_0$ and the n-cycle to be $u_0 u_m u_{m+1} \cdots u_{n+m-2} u_0$. We denote this graph by $U(m, n)$.

Definition. A *theta graph* is the union of three internally disjoint (simple) paths that have the same two distinct end vertices. Without loss of generality, we may assume that $s, t, r \geq 1$ such that $P_s = u_0 u_1 \cdots u_s$, $P_t = u_0 u_{s+1} u_{s+2} \cdots u_{s+t-1} u_s$ and $P_r = u_0 u_{s+t} u_{s+t+1} \cdots u_{s+t+r-2} u_s$. Note that in this paper, P_i denotes a path of length i. We denote this graph by $\Theta(s, t, r)$.

Definition. A *long dumbbell graph* is a simple graph obtained from two cycles C_m and C_n, by joining a path of length l (again, denoted by P_l in this paper) for $m, n \geq 3$ and $l \geq 1$. Without loss of generality, we may assume

$$C_m = u_0 u_1 \cdots u_{m-1} u_0, \quad P_l = u_{m-1} u_m \cdots u_{m+l-1}$$
$$\text{and } C_n = u_{m+l-1} u_{m+l} \cdots u_{m+n+l-2} u_{m+l-1}.$$

We denote this graph by $D(m, n; l)$.

Theorem 2.2. *Let G be a $(p, p+1)$-bicyclic graph without pendant. Then, G contains only one 2^+-vertex if and only if G is a one-point union of two cycles.*

Proof. Suppose G contains only one 2^+-vertex. Let d be the degree of the 2^+-vertex. Since $2(p-1)+d = 2(p+1)$, $d = 4$. Since G contains one 4-vertex and $(p-1)$ 2-vertices, G is eulerian and contains two cycles. Hence, G is a one-point union of two cycles. The converse is clear. □

Theorem 2.3. *Let G be a $(p, p+1)$-bicyclic graph without pendant. Then, G contains two 2^+-vertices if and only if G is either a long dumbbell graph or a theta graph.*

Proof. Suppose G contains only two 2^+-vertices. Let d be the sum of the degrees of the 2^+-vertices. Since $2(p-2)+d = 2(p+1)$, $d = 6$. Since the degree of the 2^+-vertices is greater than 2, the two 2^+-vertices must be 3-vertices. Since G contains two 3-vertices and $(p-2)$ 2-vertices, G is edge-traceable. The two 3-vertices are connected to each other by joining either one path or three disjoint paths. If the 3-vertices are connected by one path, then two disjoint cycles are incident with these 3-vertices respectively. Hence, G is a long dumbbell graph. If the vertices are connected by three paths, then G is a theta graph. The converse is clear. $\square$

Corollary 2.4. *A bicyclic graph without pendant is either a one-point union of two cycles, a long dumbbell graph or a theta graph.*

3. Main results

We first recall the the following useful result, found in [16].

Theorem A. *Let G be a $\mathbb{Z}_k$-magic graph, with $k|n$. Then, G is a $\mathbb{Z}_n$-magic graph.*

Using Theorem A, we obtain the following two theorems which describe the integer-magic spectra of $U(m,n)$.

Theorem 3.1. *Let $m, n \geq 3$. If m and n have the same parity, then* $\text{IM}[U(m,n)] = \{2,3,4,\dots\}$.

Proof. Since m and n have the same parity, this implies that $U(m,n)$ is an eulerian graph with an even number of edges. Traveling along an eulerian circuit of $U(m,n)$, we label its edges with $a, -a, a, -a, \dots, a, -a$, where $a \neq 0$. This yields a $\mathbb{Z}_k$-magic labeling, for all $k \in \{2,3,4,\dots\}$. $\square$

Note that $U(m,n)$ is also $\mathbb{Z}$-magic, when m and n have the same parity.

Theorem 3.2. *Let $m, n \geq 3$. If m and n have different parity, then* $\text{IM}[U(m,n)] = 2\mathbb{N}$.

Proof. Without loss of generality, suppose that m is even and n is odd. In order for $U(m,n)$ to be $\mathbb{Z}_k$-magic, we must consecutively label the edges of C_m with non-zero $x, y, x, y, \dots, x, y$. The edges of C_n must then be consecutively labeled with non-zero $z, x+y-z, z, x+y-z, \dots, z, x+y-z, z$. See Figure 1. Then, $U(m,n)$ has a $\mathbb{Z}_k$-magic labeling $\Longleftrightarrow x+y \equiv x+y+2z \pmod{k}$. In order for $0 \equiv 2z \pmod{k}$, k must be even. Clearly, $U(m,n)$ is $\mathbb{Z}_2$-magic, since every vertex is of even degree. Using Theorem A, this implies that $U(m,n)$ is $\mathbb{Z}_4$-magic, $\mathbb{Z}_6$-magic, etc. $\square$

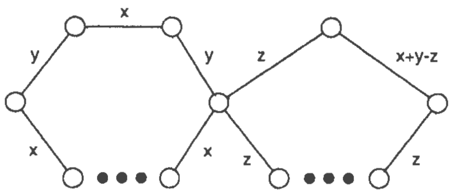

FIGURE 1

Next, we establish the integer-magic spectra of long dumbbell graphs.

Theorem 3.3. *Let* $m, n \geq 3$*, and* $l \geq 1$*. If* $m \cdot n$ *is even, then* $\mathrm{IM}[D(m,n;l)] = \varnothing$.

Proof. Since $m \cdot n$ is even, (WLOG) we can assume that m is even. If $D(m,n;l)$ has a $\mathbb{Z}_k$-magic labeling, then the edges of subgraph C_m must be consecutively labeled with non-zero $x, y, x, y, \ldots, x, y$ (with its vertices having magic value $x+y$). However, this would force the edge (from the path of length l) connecting C_m to be labeled 0. Hence, $D(m,n;l)$ is not $\mathbb{Z}_k$-magic, for all k. □

Theorem 3.4. *Let* $m, n \geq 3$*, and* $l \geq 1$*. If* $m \cdot n$ *is odd, then* $\mathrm{IM}[D(m,n;l)] = \{3, 4, 5, \ldots\}$.

Proof. Since $m \cdot n$ is odd, m and n must both be odd. We first note that $D(m,n;l)$ is not $\mathbb{Z}_2$-magic, since it contains vertices of different parity. Figure 2 provides a $\mathbb{Z}_k$-magic labeling of $D(m,n;l)$, for $k = 4, 5, 6, \ldots$. Figures 3 and 4 provide $\mathbb{Z}_3$-magic labelings of $D(m,n;l)$, for l odd and even, respectively.

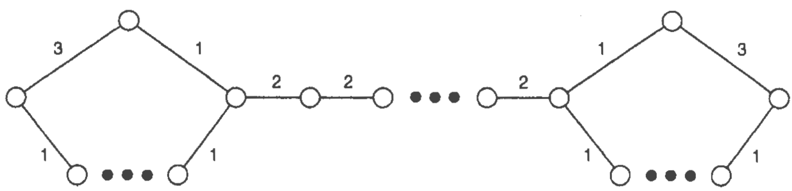

FIGURE 2. $\mathbb{Z}_k$-magic labeling of $D(m,n;l)$, for $k = 4, 5, 6, \ldots$

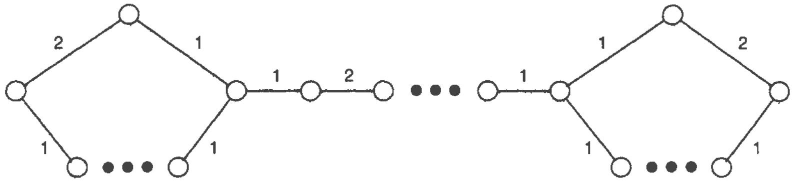

FIGURE 3. $\mathbb{Z}_3$-magic labeling of $D(m,n;l)$, for odd l

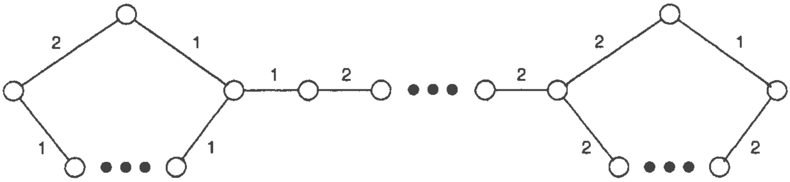

FIGURE 4. $\mathbb{Z}_3$-magic labeling of $D(m,n;l)$, for even l

$\square$

Lastly, we focus on the integer-magic spectra of theta graphs.

Theorem 3.5. *$\Theta(s,t,r)$ is $\mathbb{Z}_2$-magic if and only if $s=t=r=1$.*

Proof. A graph G is $\mathbb{Z}_2$-magic if and only if the degrees of the vertices are of the same parity. $\square$

Theorem 3.6. *Let $s\cdot t\cdot r>1$. If exactly one of s,t and r is odd, then $\mathrm{IM}[\Theta(s,t,r)]=\{2k\mid k\geq 2\}$. If exactly one of s,t and r is even, then $\mathrm{IM}[\Theta(s,t,r)]=\{4,5,6,\dots\}$. Otherwise, $\mathrm{IM}[\Theta(s,t,r)]=\{3,4,5,\dots\}$.*

Proof. Let $G=\Theta(s,t,r)$, where $s,t,r\geq 1$. Without loss of generality, let the three internally disjoint (simple) paths (having the same two distinct end vertices) be $P_s=u_0u_1\cdots u_s$, $P_t=u_0u_{s+1}u_{s+2}\cdots u_{s+t-1}u_s$ and $P_r=u_0u_{s+t}u_{s+t+1}\cdots u_{s+t+r-2}u_s$. Suppose that G is $\mathbb{Z}_k$-magic (with magic labeling f) and has $\mathbb{Z}_k$-magic value m. Furthermore, let $x_1,x_2,\dots,x_s$ be the edge labeling of P_s (under f, read from left to right). Similarly, let $y_1,y_2,\dots,y_t$ and $z_1,z_2,\dots,z_r$ be the edge labelings of P_t and P_r, respectively (under f, read from left to right).

CASE 1. (s is odd and t,r are even): In this case, we must have that $x_1+y_1+z_1=m$ and $x_1+(m-y_1)+(m-z_1)=m$. This implies that

$2x_1 = 0$. Thus, $\mathrm{IM}(G) \subseteq \{4, 6, 8, \ldots\}$. Figure 5 gives a $\mathbb{Z}_{2n}$-magic labeling of G, for $n \geq 2$.

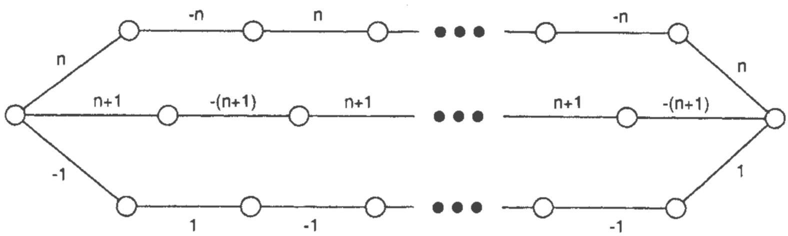

FIGURE 5. $\mathbb{Z}_{2n}$-magic labeling of $\Theta(s, t, r)$, $n \geq 2$, for s odd and t, r even.

CASE 2. (s, t are odd and r is even): In this case, we must have that $x_1+y_1+z_1 = m$ and $x_1+y_1+(m-z_1) = m$. This implies that $x_1+y_1+z_1 = m$ and $x_1+y_1-z_1 = 0$. Figure 6 gives a $\mathbb{Z}_{2n}$-magic labeling of G, for $n \geq 2$. Figure 7 gives a $\mathbb{Z}_{2n+1}$-magic labeling of G, for $n \geq 3$. Figure 8 gives a $\mathbb{Z}_5$-magic labeling of G. It is straight-forward to show (using indirect proof) that G is not $\mathbb{Z}_3$-magic.

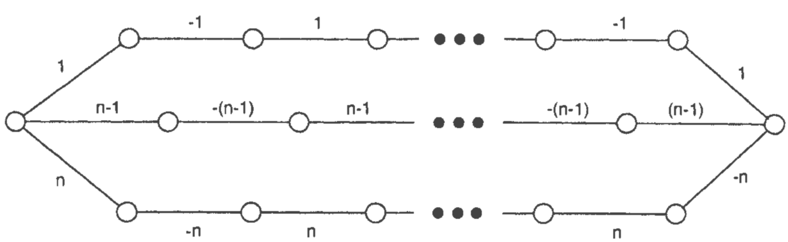

FIGURE 6. $\mathbb{Z}_{2n}$-magic labeling of $\Theta(s, t, r)$, $n \geq 2$, for s, t odd and r even.

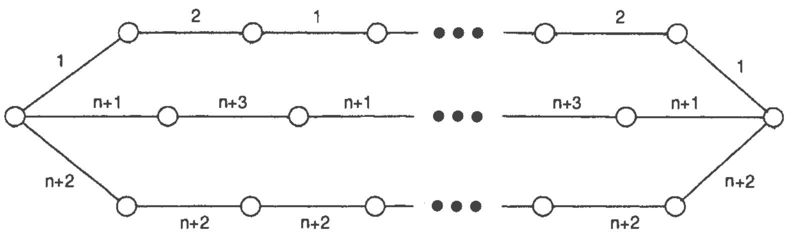

FIGURE 7. $\mathbb{Z}_{2n+1}$-magic labeling of $\Theta(s,t,r)$, $n \geq 3$, for s,t odd and r even.

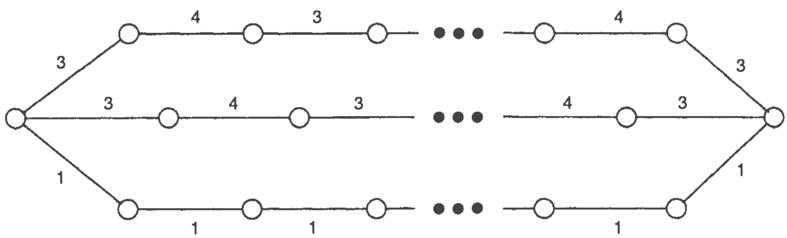

FIGURE 8. $\mathbb{Z}_5$-magic labeling of $\Theta(s,t,r)$, for s,t odd and r even.

CASE 3. (s,t, and r are odd): Figure 9 gives a $\mathbb{Z}_k$-magic labeling of G, for $k \geq 3$.

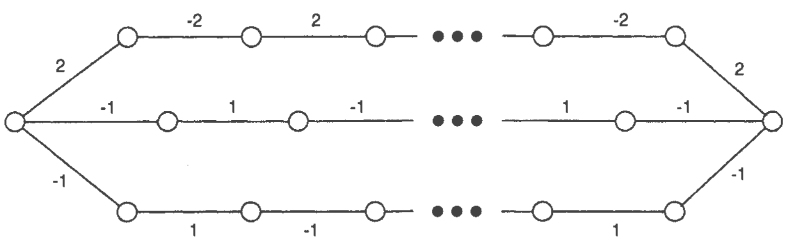

FIGURE 9. $\mathbb{Z}_k$-magic labeling of $\Theta(s,t,r)$, $k \geq 3$, for s,t, and r odd.

CASE 4. (s, t, and r are even): Figure 10 gives a $\mathbb{Z}_k$-magic labeling of G, for $k \geq 3$.

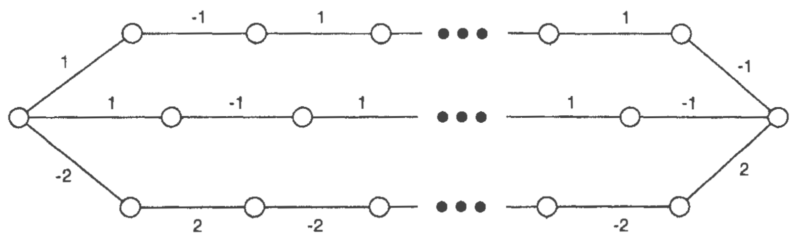

FIGURE 10. $\mathbb{Z}_k$-magic labeling of $\Theta(s, t, r)$, $k \geq 3$, for s, t, and r even.

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