Edge-gracefulness of the Composition of Paths with Null Graphs*

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Abstract

Let $G = (V, E)$ be a (p, q)-graph. Let $f : E \to \{1, 2, \ldots, q\}$ be a bijection. The *induced mapping* $f^+ : V \to \mathbb{Z}_p$ of f is defined by $f^+(u) \equiv \sum_{uv \in E} f(uv) \pmod{p}$ for $u \in V$. If f^+ is a bijection, then G is called *edge-graceful*. In this paper, we investigate the edge-gracefulness of the composition of paths with null graphs $P_m \circ N_n$, where there are mn vertices and $(m-1)n^2$ edges. We show that $P_3 \circ N_n$ is edge-graceful if n is odd.

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1 Introduction, Notations and Basic Concepts

In this paper, the term "graph" means finite multigraph (not necessary connected) having no loop and no isolated vertex. The term "set" means multiset. Set operations are viewed as multiset operations. For any positive integer r, we denote by $[\boldsymbol{r}]$ the set $\{1, 2, \cdots, r\}$ by $[\boldsymbol{r}]$. All undefined symbols and concepts may be looked up from [1]. A graph $G = (V, E)$ is called a (p, q)-graph if $|V| = p$ and $|E| = q$.

Let $G = (V, E)$ be a (p, q)-graph. Let $f : E \to \{d, d + 1, \ldots, d + q - 1\}$ be a bijection for some $d \in \mathbb{Z}$. The *induced mapping* $f^+ : V \to \mathbb{Z}_p$ of f is defined by $f^+(u) \equiv \sum_{uv \in E} f(uv) \pmod{p}$ for $u \in V$. If f^+ is a bijection, then G is called *d-edge-graceful*. If $d = 1$, then G is simply called *edge-graceful*, and f an *edge-graceful labeling of* G. A necessary condition for a (p, q)-graph to be edge-graceful is:

$$q(q + 1) \equiv \tfrac{1}{2}p(p - 1) \pmod{p}. \tag{1.1}$$

Edge-graceful was introduced by Lo [6] in 1985. Many researchers investigated on certain families of graphs [4]. It is known that a graph with $p \equiv 2 \pmod 4$ is not edge-graceful [2].

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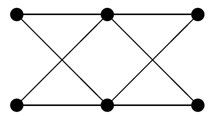
Let $G=(V,E)$ be a simple graph and S be a set. A *labeling matrix* Ω of a mapping $f : E \to S$ is a matrix whose rows and columns are named by the vertices of G and the (u,v)-entry is $f(uv)$ if $uv \in E$ and is 0 otherwise. If f is an edge-graceful labeling, then Ω is called an *edge-graceful labeling matrix*. This representation of a labeling was first introduced in [7]. Therefore, f is an edge-graceful labeling of G if and only if row sums and column sums, modulo p, of the labeling matrix of f are all distinct.

2 A Necessary Condition for $P_m \circ N_n$ to be Edge-graceful

Given two graphs G and H. The *composition* of G and H, denoted by $G \circ H$, is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent with (u_2, v_2) if and only if $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The following figure is the graph $P_3 \circ N_2$.



Let $G = P_m \circ N_n$ and let $V = V(G) = [\boldsymbol{m}] \times [\boldsymbol{n}]$. We use the lexicographic order on V. If $f : E(G) \to S$ is a mapping, then the labeling matrix of f is formed by

$$\Omega = \begin{pmatrix} O & A_1 & O & \cdots & O & O \\ A_1^T & O & A_2 & \ddots & \ddots & O \\ O & A_2^T & O & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & O \\ O & \ddots & \ddots & \ddots & O & A_{m-1} \\ O & O & \cdots & O & A_{m-1}^T & O \end{pmatrix}, \tag{2.1}$$

where O is the $n \times n$ zero matrix and A_i is an $n \times n$ matrix. Note that G is an $(mn, (m-1)n^2)$-graph. From (1.1), we have

$$(m-1)n^2(mn^2 - n^2 + 1) \equiv \tfrac{1}{2}mn(mn-1) \pmod{mn} \tag{2.2}$$

It is easy to see that if $mn \equiv 0 \pmod 4$ and (2.2) holds, then m is even. Therefore, a necessary condition for $P_m \circ N_n$ to be edge-graceful is:

$$mn \not\equiv 2 \pmod 4 \quad \text{and} \quad n(n^2-1) \equiv \begin{cases} 0 & \pmod m \quad \text{if } mn \text{ is odd,} \\ \frac{m}{2} & \pmod m \quad \text{if } mn \equiv 0 \pmod 4. \end{cases} \tag{2.3}$$

It is easy to see that for a fixed $n > 1$, there are finitely many m satisfying (2.3). The following conjecture was posed in [3] and modified in [8].

Conjecture 2.1: *$P_m \circ N_n$ is edge-graceful if m and n satisfy* (2.3).

Lo [6] showed that $P_m \cong P_m \circ N_1$ is edge-graceful if and only if m is odd. Shiu [8] showed that $P_m \circ N_2$ is edge-graceful if and only if $m = 4$ or $m = 12$ by constructing some edge-graceful labeling matrices. Lee, Lee and Murthy [2] and Shiu [8] also showed that $P_3 \circ N_3$ and $P_3 \circ N_5$ are edge-graceful, respectively. Here we list one edge-graceful labeling matrix for each of these graphs.

For the graph $P_4 \circ N_2$, according the notations described in (2.1)

$$A_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 5 & 8 \\ 6 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 2 & 7 \\ 4 & 3 \end{pmatrix}.$$

For the graph $P_{12} \circ N_2$,

$$A_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 11 & 12 \\ 9 & 10 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 13 & 14 \\ 15 & 16 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 19 & 20 \\ 17 & 18 \end{pmatrix}, \qquad A_6 = \begin{pmatrix} 19 & 18 \\ 23 & 24 \end{pmatrix}, \qquad A_7 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}, \qquad A_8 = \begin{pmatrix} 7 & 8 \\ 6 & 5 \end{pmatrix},$$

$$A_9 = \begin{pmatrix} 9 & 10 \\ 11 & 12 \end{pmatrix}, \qquad A_{10} = \begin{pmatrix} 14 & 16 \\ 15 & 17 \end{pmatrix}, \qquad A_{11} = \begin{pmatrix} 13 & 20 \\ 21 & 22 \end{pmatrix}.$$

For the graph $P_3 \circ N_3$,

$$A_1 = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 2 & 5 \\ 3 & 1 & 6 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 4 & 8 & 7 \\ 9 & 8 & 9 \\ 7 & 5 & 6 \end{pmatrix}.$$

For the graph $P_3 \circ N_5$,

$$A_1 = \begin{pmatrix} 1 & 20 & 21 & 15 & 16 \\ 2 & 19 & 22 & 14 & 17 \\ 3 & 18 & 23 & 13 & 18 \\ 4 & 17 & 24 & 12 & 19 \\ 5 & 16 & 25 & 11 & 20 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 6 & 7 & 8 & 1 & 10 \\ 15 & 14 & 13 & 5 & 11 \\ 11 & 12 & 13 & 14 & 15 \\ 5 & 4 & 3 & 2 & 9 \\ 1 & 2 & 3 & 4 & 12 \end{pmatrix}.$$

In this paper, we work on $P_3 \circ N_n$. Since $m = 3$, n is odd. For n is odd, $n(n^2 - 1) \equiv 0 \pmod 3$ is always true. Thus $P_3 \circ N_n$ satisfies (2.3) for any odd n. In the next section, we shall show that $P_3 \circ N_n$ is edge-graceful if n is odd and $n \geq 7$.

3 Main Theorems

Let $G = P_3 \circ N_n$, where n is odd and $n \geq 7$. In this case (2.1) becomes

$$\Omega = \begin{pmatrix} O & A^T & O \\ A & O & B \\ O & B^T & O \end{pmatrix}. \tag{3.1}$$

We have to fill A and B with elements of $[\boldsymbol{2n^2}]$ so that

$$R_{A+B} \cup C_A \cup C_B = \mathbb{Z}_{3n}, \tag{3.2}$$

where R_{A+B} is the set of row sums of $A+B$, C_A and C_B are the set of column sums of A and B respectively.

First, we arrange the elements of $[\boldsymbol{2n^2}]$ as the following $n \times (2n)$ matrix

$$\left(\begin{array}{cccc|cccc}
1 & 2 & \cdots & n & n+1 & \cdots & 2n-1 & 2n \\
4n & 4n-1 & \cdots & 3n+1 & 3n & \cdots & 2n+2 & 2n+1 \\
4n+1 & 4n+2 & \cdots & 5n & 5n+1 & \cdots & 6n-1 & 6n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2n^2-2n+1 & 2n^2-2n+2 & \cdots & 2n^2-n & 2n^2-n+1 & \cdots & 2n^2-1 & 2n^2
\end{array}\right).$$

After taking the entries modulo $3n$, the above matrix becomes

$$\Psi = \left(\begin{array}{cccc|cccc}
1 & 2 & \cdots & n & n+1 & \cdots & 2n-1 & 2n \\
n & n-1 & \cdots & 1 & 3n & \cdots & 2n+2 & 2n+1 \\
n+1 & n+2 & \cdots & 2n & 2n+1 & \cdots & 3n-1 & 3n \\
2n & 2n-1 & \cdots & n+1 & n & \cdots & 2 & 1 \\
2n+1 & 2n+2 & \cdots & 3n & 1 & \cdots & n-1 & n \\
3n & 3n-1 & \cdots & 2n+1 & 2n & \cdots & n+2 & n+1 \\
1 & 2 & \cdots & n & n+1 & \cdots & 2n-1 & 2n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right). \tag{3.3}$$

From now on, we shall use "=" to denote "$\equiv \pmod{3n}$", i.e., unless other state, the arithmetic is taken in $\mathbb{Z}_{3n}$.

The rows of Ψ appear periodically with a period of 6. The left and the right $n \times n$ submatrices of Ψ are the matrices A and B in (3.1), respectively. We can compute the i-th row sum $r_i(\Psi)$ and the j-th column sum $c_j(\Psi)$ of Ψ as:

$$r_i(\Psi) = \begin{cases} -n^2+n, & \text{if } i \equiv 1 \pmod 3 \\ n, & \text{if } i \equiv 2 \pmod 3 \\ n^2+n, & \text{if } i \equiv 0 \pmod 3 \end{cases}, \tag{3.4}$$

for $1 \le i \le n$ and

$$c_j(\Psi) = \tfrac{1}{2}(n-1)(2n^2+2n+1)+j,\ 1 \le j \le 2n. \tag{3.5}$$

Moreover,

$$\tfrac{1}{2}(n-1)(2n^2+2n+1) = \begin{cases} 3k, & \text{if } n = 6k+1 \\ 15k+7, & \text{if } n = 6k+3 \\ 3k+2, & \text{if } n = 6k+5 \end{cases}. \tag{3.6}$$

and

$$n^2 = \begin{cases} n, & \text{if } n = 6k+1 \\ 0, & \text{if } n = 6k+3 \\ 2n, & \text{if } n = 6k+5 \end{cases}, \tag{3.7}$$

where $k \geq 1$.

Theorem 3.1: *$P_3 \circ N_n$ is edge-graceful if n is odd.*

By swapping pairs of entries within same columns of A and B, the column sums of matrices A and B will not change. We shall show that we can choose suitable pairs of elements to swap so that the resulting matrix Ψ satisfies the requirement (3.2).

For convenience, we let $A_1, \ldots, A_6$ be the first 6 rows of A respectively, and let $B_1, \ldots, B_6$ be the first 6 rows of B respectively. Since the i-th row of Ψ is equal to the $(i+6)$-th row of Ψ, Ψ contains a certain copies of A_1 and B_1; a certain copies of A_2 and B_2; and so on. For a row vector α, we denote by $r(\alpha)$ the sum of entries in α. Sometimes, we shall view α as a set.

Theorem 3.1A: *$P_3 \circ N_{6k+1}$ is edge-graceful for $k \geq 1$.*

Proof: Keep the notation defined above in this section. The sets of rows of A and rows of B consist of $k+1$ copies of A_1 and k copies of $A_2, \ldots, A_6$ each and $k+1$ copies of B_1 and k copies of $B_2, \ldots, B_6$ each, respectively. From (3.5) and (3.6),

$$C_A \cup C_B = \{3k+1, 3k+2, \ldots, 15k+1, 15k+2\}.$$

Thus we have to swap the entries of the same column of A and B such that the set of row sums of the resulting matrix Ψ is $\{1, 2, \ldots, 3k\} \cup \{15k+3, 15k+4, \ldots, 18k+3\}$. From (3.4) and (3.7),

$$r_i(\Psi) = \begin{cases} 0, & \text{if } i \equiv 1 \pmod 3 \\ n, & \text{if } i \equiv 2 \pmod 3 \\ 2n, & \text{if } i \equiv 0 \pmod 3 \end{cases}, \quad 1 \leq i \leq n.$$

First we consider the following differences.

$$A_{3,2} = A_3 - A_2 = (1, 3, 5, \ldots, n-2, n, n+2, \ldots, 2n-3, 2n-1);$$

$$B_{6,5} = B_6 - B_5 = (2n-1, 2n-3, 2n-5, \ldots, n+2, n, n-2, \ldots, 5, 3, 1).$$

Since $n = 6k+1$ is odd, $n \in A_{3,2}$. Suppose n is the j-th entry of $A_{3,2}$. Then we swap the j-th entries of A_2 and A_3. Let the resulting rows be A_2^* and A_3^* respectively. Also, $n \in B_{6,5}$. Suppose

n is the l-th entry of $B_{6,5}$. Then we swap the l-th entries of B_5 and B_6. Let the resulting rows be B_5^* and B_6^* respectively. After swapping some entries of k copies of A_2 and A_3 and k copies of B_5 and B_6, the row sums of matrix Ψ become:

$$r_i(\Psi) = \begin{cases} 0, & \text{if } i \equiv 1 \pmod 3 \\ 2n, & \text{if } i \equiv 2 \pmod 3 \\ n, & \text{if } i \equiv 0 \pmod 3 \end{cases}, \; 1 \leq i \leq n.$$

Now we consider the following three differences.

$$\begin{aligned} B_{1,4} &= B_1 - B_4 &&= (1, 3, 5, \ldots, 2n-5, 2n-3, 2n-1) \\ & &&= (1, 3, 5, \ldots, 12k-3, 12k-1, 12k+1); \\ B_{3,2} &= B_3 - B_2 &&= (2n+1, 2n+3, 2n+5, \ldots, 3n, 2, 4, \ldots, n-5, n-3, n-1) \\ & &&= (12k+3, 12k+5, 12k+7, \ldots, 18k+3, 2, 4, \ldots, 6k-4, 6k-2, 6k); \\ B_{6,5}^* &= B_6^* - B_5^* &&= (2n-1, 2n-3, 2n-5, \ldots, n+2, -n, n-2, \cdots, 5, 3, 1) \\ & &&= (12k+1, 12k-1, 12k-3, \ldots, 6k+3, 12k+2, 6k-1, \ldots, 5, 3, 1). \end{aligned}$$

If x is an even number and $3k+1 \leq x \leq 5k$, then x is the j-th entry of $B_{3,2}$ for some j. We swap the j-th entries of B_2 and B_3, and let the resulting rows be B_2^x and B_3^x respectively. Hence,

$$15k+3 \leq r(A_2^* + B_2^x) = 12k+2+x \leq 17k+2$$

and

$$k+1 \leq r(A_3^* + B_3^x) = 6k+1-x \leq 3k.$$

Similarly, if x is an odd number and $3k+1 \leq x \leq 5k$, then x is the j-th entry of $B_{6,5}^*$ for some j. We swap the j-th entries of B_5^* and B_6^*, and let the resulting rows be B_5^x and B_6^x respectively. Then we have

$$15k+3 \leq r(A_5 + B_5^x) = 12k+2+x \leq 17k+2$$

and

$$k+1 \leq r(A_6 + B_6^x) = 6k+1-x \leq 3k.$$

After swapping k copies of B_2 and B_3; and the k copies of B_5^* and B_6^* for different x chosen suitably, we obtain $k+1, k+2, \cdots, 3k$, and $15k+3, 15k+4, \ldots, 17k+2$ as row sums of the resulting matrix Ψ.

Now, we handle the swapping of some entries of B_1 and B_4. We want to obtain row sums of the resulting Ψ consisting of $1, 2, \ldots, k$ and $17k+3, 17k+4, \ldots, 18k+2$. It is clear that we cannot reach our goal if only swap a pair of entries of B_1 and B_4.

Let $y \in \mathbb{Z}$ and $1 \le y \le k$. If we can find some pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_s, b_s)$ in $B_1 \times B_4$ such that $\sum_{i=1}^{s}(a_i - b_i) = y$ where a_i, b_i are in the same column of B. Then we swap those entries and let the resulting rows be B_1^y and B_4^y respectively. Hence,

$$1 \le r(A_4 + B_4^y) = y \le k \quad \text{and} \quad 17k + 3 \le r(A_1 + B_1^y) = 18k + 3 - y \le 18k + 2.$$

$\sum_{i=1}^{s}(a_i - b_i)$ is the sum of some entries of $B_{1,4}$. It is easy to see that for $1 \le y \le k$ and y is odd, $y \in B_{1,4}$ (i.e., $s = 1$). For $4 \le y \le k$ and y is even, we choose 1 and $y - 1$ from $B_{1,4}$ (i.e., $s = 2$). For $y = 2$, we choose 1, $n + 2$ and $2n - 1$ from $B_{1,4}$ (i.e., $s = 3$). Thus it is possible to find such pairs. After swapping the k copies of B_1 and B_4 for different y suitably, we obtain $1, 2, \ldots, k$ and $17k + 3, 17k + 4, \ldots, 18k + 2$ as row sums of the resulting matrix Ψ.

Up to now, only $18k+3$ is missing as a row sum. But the sum of the last row of Ψ is $0 = 18k+3$. Therefore we keep the last row of Ψ. Hence we obtain the required arrangement. ■

We use the following example to demonstrate the above proof.

Example 3.1: Consider $P_3 \circ N_7$. Then

$$\Psi = (A|B) = \begin{array}{cc} \left(\begin{array}{ccccccc|ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 21 & 20 & 19 & 18 & 17 & 16 & 15 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 15 & 16 & 17 & 18 & 19 & 20 & 21 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{array}\right) & \begin{array}{c} \mathit{0} \\ \mathit{7} \\ \mathit{14} \\ \mathit{0} \\ \mathit{7} \\ \mathit{14} \\ \mathit{0} \end{array} \\ \begin{array}{ccccccc|ccccccc} \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} & \mathit{10} & \mathit{11} & \mathit{12} & \mathit{13} & \mathit{14} & \mathit{15} & \mathit{16} & \mathit{17} \end{array} & \end{array}$$

The last row and the rightmost column in italics are the column sums and the row sums of Ψ respectively. First, we consider

$$A_{3,2} = (1, 3, 5, \underline{7}, 9, 11, 13), \qquad B_{6,5} = (13, 11, 9, \underline{7}, 5, 3, 1).$$

Since $n = 7$ is the 4th entry of $A_{3,2}$, we swap the 4th entry of A_2 and A_3. Also, because n is the 4th entry of $B_{6,5}$, we swap the 4th entry of B_5 and B_6. Ψ becomes

$$\begin{array}{cc} \left(\begin{array}{ccccccc|ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 7 & 6 & 5 & \mathbf{11} & 3 & 2 & 1 & 21 & 20 & 19 & 18 & 17 & 16 & 15 \\ 8 & 9 & 10 & \mathbf{4} & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 15 & 16 & 17 & 18 & 19 & 20 & 21 & 1 & 2 & 3 & \mathbf{11} & 5 & 6 & 7 \\ 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & \mathbf{4} & 10 & 9 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{array}\right) & \begin{array}{c} \mathit{0} \\ \mathit{14} \\ \mathit{7} \\ \mathit{0} \\ \mathit{14} \\ \mathit{7} \\ \mathit{0} \end{array} \end{array}$$

The boldface numbers have been swapped. Since

$$B_{1,4} = (\underline{1}, 3, 5, 7, 9, 11, 13), \quad B_{3,2} = (15, 17, 19, 21, 2, \underline{4}, 6), \quad B^*_{6,5} = (13, 11, 9, 14, \underline{5}, 3, 1).$$

We have the last version of the matrix Ψ

$$\begin{array}{cc}
\left(\begin{array}{ccccccc|ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \boldsymbol{7} & 9 & 10 & 11 & 12 & 13 & 14 \\
7 & 6 & 5 & \mathbf{11} & 3 & 2 & 1 & 21 & 20 & 19 & 18 & 17 & \boldsymbol{20} & 15 \\
8 & 9 & 10 & \mathbf{4} & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & \boldsymbol{16} & 21 \\
14 & 13 & 12 & 11 & 10 & 9 & 8 & \boldsymbol{8} & 6 & 5 & 4 & 3 & 2 & 1 \\
15 & 16 & 17 & 18 & 19 & 20 & 21 & 1 & 2 & 3 & \mathbf{11} & \boldsymbol{10} & 6 & 7 \\
21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & \mathbf{4} & \mathbf{5} & 9 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14
\end{array}\right) & \begin{array}{c} \mathit{20} \\ \mathit{18} \\ \mathit{3} \\ \mathit{1} \\ \mathit{19} \\ \mathit{2} \\ \mathit{21} \end{array} \\
\begin{array}{ccccccc|ccccccc}
\mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} & \mathit{10} & \mathit{11} & \mathit{12} & \mathit{13} & \mathit{14} & \mathit{15} & \mathit{16} & \mathit{17}
\end{array} &
\end{array}$$

The italic boldface numbers have been swapped. ■

Theorem 3.1B: $P_3 \circ N_{6k+3}$ *is edge-graceful for* $k \geq 1$.

Proof: We shall keep the notation defined in the beginning of Theorem 3.1A, including the matrix A and the column sums of B. Rename the last 3 rows of B by L_1, L_2 and L_3 respectively. Then the set of rows of B consists of k copies of $B_1, \ldots, B_6$ each and L_1, L_2, L_3. From (3.5) and (3.6), $C_A \cup C_B = \{15k+8, \ldots, 18k+9, 1, \ldots, 9k+4\}$. Thus we have to swap the entries of B such that the set of row sums of the resulting matrix Ψ is $\{9k+5, 9k+6, \ldots, 15k+7\}$. From (3.4) and (3.7), $r_i(\Psi) = n$ for all $1 \leq i \leq n$.

First we swap the first entries of all B_1 with B_3; and the first entries of all B_4 with B_6. We denote the resulting rows by B^*_1, B^*_3, B^*_4 and B^*_6 respectively. Now,

$$r(A_1 + B^*_1) = 2n = r(A_4 + B^*_4),\ r(A_2 + B_2) = n = r(A_5 + B_5),\ r(A_3 + B^*_3) = 0 = r(A_6 + B^*_6).$$

We shall swap entries of the same column between B^*_1 and B^*_4; B_2 and B^*_3; B_5 and B^*_6. First we look for the following three differences

$$\begin{aligned}
B_{1,4} &= B_1^* - B_4^* &&= (1,3,5,\ldots,2n-5,2n-3,2n-1)\\
& &&= (1,3,5,\ldots,12k+1,12k+3,12k+5);\\
B_{3,2} &= B_3^* - B_2 &&= (-2n+1,-n+3,-n+5,\ldots,n-5,n-3,n-1)\\
& &&= (n+1,2n+3,2n+5,\ldots,3n-2,3n,2,4,\ldots,n-3,n-1)\\
& &&= (6k+4,12k+9,12k+11,\ldots,18k+7,18k+9,2,4,\ldots,6k,6k+2);\\
B_{6,5} &= B_6^* - B_5 &&= (n-1,2n-3,2n-5,\ldots,5,3,1)\\
& &&= (6k+2,12k+3,12k+1,\ldots,5,3,1).
\end{aligned}$$

If x is even and $3k+2 \le x \le 5k+1$ then x is the j-th entry of $B_{3,2}$ for some j. Swapping the j-th entries of B_2 and B_3^* and letting the resulting rows be B_2^x and B_3^x respectively, we have

$$9k+5 \le r(A_2+B_2^x) = 6k+3+x \le 11k+4$$

and

$$13k+8 \le r(A_3+B_3^x) = 18k+9-x \le 15k+7.$$

Similarly, if x is odd and $3k+2 \le x \le 5k+1$, then x is the j-th entry of $B_{6,5}$ for some j. Swapping the j-th entries of B_5 and B_6^* and letting the resulting rows be B_5^x and B_6^x respectively, we have

$$9k+5 \le r(A_5+B_5^x) \le 11k+4 \quad \text{and} \quad 13k+8 \le r(A_6+B_6^x) \le 15k+7.$$

After swapping the k copies of B_2 and B_3^*; and the k copies of B_5 and B_6^* for different x suitably, we obtain $9k+5, 9k+6, \ldots, 11k+4$ and $13k+8, 13k+9, \ldots, 15k+7$ as row sums of the resulting matrix Ψ.

Now, we shall swap some entries of B_1^* and B_4^*. We want to obtain row sums of the resulting Ψ consisting of $11k+6, 11k+7, \ldots, 12k+5$ and $12k+7, 12k+8, \ldots, 13k+6$. Similar to the proof of Theorem 3.1A, for $y \in \mathbb{Z}$ and $1 \le y \le k$, we have to find some pairs $(a_1,b_1),(a_2,b_2),\ldots,(a_s,b_s)$ in $B_1^* \times B_4^*$ and a_i, b_i in the same column of B such that $\sum\limits_{i=1}^{s}(a_i,b_i) = y$. Then we swap those entries and let the resulting rows be B_1^y and B_4^y respectively. Hence,

$$11k+6 \le r(A_1+B_1^y) = 2n-y \le 12k+5$$

and

$$12k+7 \le r(A_4+B_4^y) = 2n+y \le 13k+6.$$

As in the proof of Theorem 3.1.A, for $1 \le y \le k$ and y is odd, $y \in B_{1,4}$ and $4 \le y \le k$ and y is even, we choose 1 and $y-1$ from $B_{1,4}$; for $y=2$ we choose 1, $n+2$ and $2n-1$ from $B_{1,4}$.

Up to now, $11k+5, 12k+6$ and $13k+7$ are missing as row sums of the resulting matrix Ψ. We have three unchanged rows L_1, L_2 and L_3. We swap the last two entries of L_1 and L_3, and denote the resulting rows by L_1^* and L_3^*. Then $r(A_1+L_1^*)=0$, $r(A_2+L_2)=n$ and $r(A_3+L_3^*)=2n=12k+6$.

Consider $L_2-L_1^*=(2n-1, 2n-3, 2n-5, \dots, 5, 2n+3, 2n+1)$. Let $z=11k+5$. If z is odd, then $z \in L_2-L_1^*$; if z is even, then choose 5 and $11k$ from $L_2-L_1^*$. After swapping the entries of L_1^* and L_2 suitably, we have two resulting rows L_1^z and L_2^z. Then

$$r(A_1+L_1^z)=0+z=11k+5 \quad \text{and} \quad r(A_2+L_2^z)=n-z=13k+7.$$

Hence we obtain the required arrangement. ■

We use the following example to demonstrate the above proof.

Example 3.2: Consider $P_3 \circ N_9$. Then $\Psi=(A|B)=$

$$\begin{array}{c c}
\left(\begin{array}{ccccccccc|ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 \\
10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\
18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 \\
10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27
\end{array}\right) &
\begin{array}{c} \mathit{9} \\ \mathit{9} \\ \mathit{9} \\ \mathit{9} \\ \mathit{9} \\ \mathit{9} \\ \mathit{9} \\ \mathit{9} \\ \mathit{9} \end{array} \\
\begin{array}{ccccccccc|ccccccccc}
\mathit{23} & \mathit{24} & \mathit{25} & \mathit{26} & \mathit{27} & \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} & \mathit{10} & \mathit{11} & \mathit{12} & \mathit{13}
\end{array} &
\end{array}$$

The last row and the rightmost column in italics are the column sums and the row sums of Ψ respectively. After swapping the first entries of B_1 and B_3; the first entries of B_4 and B_6; and the last two entries of L_1 and L_3, B becomes

$$B=\begin{array}{cc}
\left(\begin{array}{ccccccccc}
\mathbf{19} & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 \\
\mathbf{10} & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\
\mathbf{18} & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\mathbf{9} & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 \\
\hline
10 & 11 & 12 & 13 & 14 & 15 & 16 & \mathbf{26} & \mathbf{27} \\
27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 \\
19 & 20 & 21 & 22 & 23 & 24 & 25 & \mathbf{17} & \mathbf{18}
\end{array}\right) &
\begin{array}{c} \mathit{18} \\ \mathit{9} \\ \mathit{0} \\ \mathit{18} \\ \mathit{9} \\ \mathit{0} \\ \mathit{0} \\ \mathit{9} \\ \mathit{18} \end{array}
\end{array}$$

The boldface numbers have been swapped. Since

$$B_{1,4} = (\underline{1}, 3, 5, 7, 9, 11, 13, 15, 17), \qquad B_{3,2} = (10, 21, 23, 25, 0, 2, 4, \underline{6}, 8)$$

$$B_{6,5} = (8, 15, 13, 11, 9, 7, \underline{5}, 3, 1), \quad L_2 - L_1^* = (17, 15, 13, \underline{11}, 9, 7, \underline{5}, 21, 19), (z = 16 = 5 + 11).$$

We have the last version of the matrix B

$$B = \begin{array}{c} \left(\begin{array}{ccccccccc} \boldsymbol{18} & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 27 & 26 & 25 & 24 & 23 & 22 & 21 & \boldsymbol{26} & 19 \\ \mathbf{10} & 20 & 21 & 22 & 23 & 24 & 25 & \boldsymbol{20} & 27 \\ \boldsymbol{19} & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & \boldsymbol{12} & 8 & 9 \\ \mathbf{9} & 17 & 16 & 15 & 14 & 13 & \boldsymbol{7} & 11 & 10 \\ \hline 10 & 11 & 12 & \boldsymbol{24} & 14 & 15 & \boldsymbol{21} & \mathbf{26} & \mathbf{27} \\ 27 & 26 & 25 & \boldsymbol{13} & 23 & 22 & \boldsymbol{16} & 20 & 19 \\ 19 & 20 & 21 & 22 & 23 & 24 & 25 & \mathbf{17} & \mathbf{18} \end{array} \right) \end{array} \begin{array}{c} \mathit{17} \\ \mathit{15} \\ \mathit{21} \\ \mathit{19} \\ \mathit{14} \\ \mathit{22} \\ \mathit{16} \\ \mathit{20} \\ \mathit{18} \end{array}$$

The italic boldface numbers have been swapped. ■

Theorem 3.1C: $P_3 \circ N_{6k+5}$ *is edge-graceful for* $k \geq 1$.

Proof: We shall keep the notation defined in the beginning of Theorem 3.1A, including the matrix A and the column sums of B. Let $L_1, \ldots, L_5$ be the last 5 rows of B respectively. Thus the set of rows of B consists of k copies of $B_1, \ldots, B_6$ each and $L_1, \ldots, L_5$. From (3.5) and (3.6), $C_A \cup C_B = \{3k+3, 3k+4, \ldots, 15k+11, 15k+12\}$. Hence, we have to swap the entries of B such that the set of row sums of the resulting matrix plus the matrix A is $\{1, 2, \ldots, 3k+2\} \cup \{15k+13, 15k+14, \ldots, 18k+15\}$. From (3.4) and (3.7),

$$r_i(\Psi) = \begin{cases} 2n, & \text{if } i \equiv 1 \pmod 3 \\ n, & \text{if } i \equiv 2 \pmod 3 \\ 0, & \text{if } i \equiv 0 \pmod 3 \end{cases}, 1 \leq i \leq n.$$

First, we consider the following three differences

$$\begin{aligned} B_{3,6} &= B_3 - B_6 = (1, 3, 5, \ldots, 2n-5, 2n-3, 2n-1) \\ &= (1, 3, 5, \ldots, 12k+5, 12k+7, 12k+9); \\ B_{2,1} &= B_2 - B_1 = (2n-1, 2n-3, 2n-5, \ldots, 5, 3, 1) \\ &= (12k+9, 12k+7, 12k+5, \ldots, 5, 3, 1); \\ B_{5,4} &= B_5 - B_4 = (2n+1, 2n+3, 2n+5, \ldots, 3n-2, 3n, 2, 4, \ldots, n-3, n-1) \\ &= (12k+11, 12k+13, \ldots, 18k+13, 18k+15, 2, 4, \ldots, 6k+2, 6k+4). \end{aligned}$$

For $y \in \mathbb{Z}$ and $1 \leq y \leq k$, the approach is same as the proof of Theorem 3.1A by considering $B_{3,6}$. After suitably swapping the k copies of B_3 and B_6 for different y, we obtain $1, 2, \ldots, k$ and $17k + 15, 17k + 16, \ldots, 18k + 14$ as row sums of the resulting matrix Ψ.

By considering $B_{2,1}$ and $B_{5,4}$ and using a similar approach of the proof of Theorem 3.1A, we obtain rows B_1^x and B_2^x for $3k + 3 \leq x \leq 5k + 2$. After suitably swapping the k copies of B_1 and B_2; and the k copies of B_4 and B_5 for different x, we obtain $k + 3, k + 4, \ldots, 3k + 2$, and $15k + 13, 15k + 14, \ldots, 17k + 12$ as row sums of the resulting matrix Ψ.

Up to now, $k + 1, k + 2, 17k + 13, 17k + 14$ and $18k + 15$ are missing as row sums. Consider
$L_{2,1} = L_2 - L_1 = (12k + 9, 12k + 7, 12k + 5, \ldots, 5, 3, 1)$
$L_{5,4} = L_5 - L_4 = (12k + 11, 12k + 13, 12k + 15, \ldots, 18k + 13, 18k + 15, 2, 4, \ldots, 6k + 2, 6k + 4)$.
Let $5k + 3 \leq z \leq 5k + 4$. If z is odd, then $z \in L_{2,1}$. Assume z is the l-th entry of $L_{2,1}$. We swap the l-th entry of L_1 and L_2, and let the resulting rows be L_1^z and L_2^z respectively. Hence,

$$17k + 13 \leq r(A_1 + L_1^z) = 12k + 10 + z \leq 17k + 14$$

and

$$k + 1 \leq r(A_2 + L_2^z) = 6k + 5 - z \leq k + 2.$$

Similarly, if z is even, then $z \in L_{5,4}$. We swap the suitable entries of L_4 and L_5. Let the resulting rows be L_4^z and L_5^z respectively. We have

$$17k + 13 \leq r(A_4 + L_4^z) = 12k + 10 + z \leq 17k + 14$$

and

$$k + 1 \leq r(A_5 + L_5^z) = 6k + 5 - z \leq k + 2.$$

The sum of the row L_3 is $0 = 18k + 15$. Hence, we obtain the required arrangement. ■

Example 3.3: Consider $P_3 \circ N_{11}$. Then $\Psi = (A|B)$, where matrix A is

$$A = \begin{array}{c} \left(\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \\ 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 \\ 33 & 32 & 31 & 30 & 29 & 28 & 27 & 26 & 25 & 24 & 23 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \\ 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 \end{array}\right) \\ \begin{array}{ccccccccccc} \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} & \mathit{10} & \mathit{11} & \mathit{12} & \mathit{13} & \mathit{14} & \mathit{15} & \mathit{16} \end{array} \end{array}$$

and matrix B is

$$B = \begin{array}{cc} \left(\begin{array}{ccccccccccc} 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 33 & 32 & 31 & 30 & 29 & 28 & 27 & 26 & 25 & 24 & 23 \\ 23 & 24 & 25 & 26 & 27 & 28 & 28 & 30 & 31 & 32 & 33 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \\ \hline 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 33 & 32 & 31 & 30 & 29 & 28 & 27 & 26 & 25 & 24 & 23 \\ 23 & 24 & 25 & 26 & 27 & 28 & 28 & 30 & 31 & 32 & 33 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array}\right) & \begin{array}{c} \mathit{22} \\ \mathit{11} \\ \mathit{0} \\ \mathit{22} \\ \mathit{11} \\ \mathit{0} \\ \mathit{22} \\ \mathit{11} \\ \mathit{0} \\ \mathit{22} \\ \mathit{11} \end{array} \\ \begin{array}{ccccccccccc} \mathit{17} & \mathit{18} & \mathit{19} & \mathit{20} & \mathit{21} & \mathit{22} & \mathit{23} & \mathit{24} & \mathit{25} & \mathit{26} & \mathit{27} \end{array} & \end{array}$$

The last rows of matrices A and B in italics represent the column sums of Ψ. The rightmost column of matrix B in italics represents the row sums of Ψ. We shall keep the matrix A and the column sums of B. First, we consider

$$B_{3,6} = (\underline{1}, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21),$$

$$B_{2,1} = (21, 19, 17, 15, 13, 11, 9, \underline{7}, 5, 3, 1),$$

$$B_{5,4} = (23, 25, 27, 29, 31, 33, 2, 4, \underline{6}, 8, 10)$$

Therefore, B becomes

$$B = \left(\begin{array}{ccccccccccc} 12 & 13 & 14 & 15 & 16 & 17 & 18 & \mathbf{26} & 20 & 21 & 22 \\ 33 & 32 & 31 & 30 & 29 & 28 & 27 & \mathbf{19} & 25 & 24 & 23 \\ \mathbf{22} & 24 & 25 & 26 & 27 & 28 & 28 & 30 & 31 & 32 & 33 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & \mathbf{9} & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \mathbf{3} & 10 & 11 \\ \mathbf{23} & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \\ \hline 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 33 & 32 & 31 & 30 & 29 & 28 & 27 & 26 & 25 & 24 & 23 \\ 23 & 24 & 25 & 26 & 27 & 28 & 28 & 30 & 31 & 32 & 33 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array}\right) \begin{array}{c} \mathit{29} \\ \mathit{4} \\ \mathit{32} \\ \mathit{28} \\ \mathit{5} \\ \mathit{1} \\ \mathit{22} \\ \mathit{11} \\ \mathit{0} \\ \mathit{22} \\ \mathit{11} \end{array}$$

Finally, we consider

$$L_{2,1} = (21, 19, 17, 15, 13, 11, \underline{9}, 7, 5, 3, 1), \qquad L_{5,4} = (23, 25, 27, 29, 31, 33, 2, 4, 6, \underline{8}, 10)$$

Hence, we have the last version of the matrix B, the italic boldface numbers have been swapped.

$$\begin{array}{cc} B = \left(\begin{array}{ccccccccccc} 12 & 13 & 14 & 15 & 16 & 17 & 18 & \mathbf{26} & 20 & 21 & 22 \\ 33 & 32 & 31 & 30 & 29 & 28 & 27 & \mathbf{19} & 25 & 24 & 23 \\ \mathbf{22} & 24 & 25 & 26 & 27 & 28 & 28 & 30 & 31 & 32 & 33 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & \mathbf{9} & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \mathbf{3} & 10 & 11 \\ \mathbf{23} & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 \\ \hline 12 & 13 & 14 & 15 & 16 & 17 & \boldsymbol{27} & 19 & 20 & 21 & 22 \\ 33 & 32 & 31 & 30 & 29 & 28 & \boldsymbol{18} & 26 & 25 & 24 & 23 \\ 23 & 24 & 25 & 26 & 27 & 28 & 28 & 30 & 31 & 32 & 33 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & \boldsymbol{10} & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \boldsymbol{2} & 11 \end{array}\right) & \begin{array}{c} \mathit{29} \\ \mathit{4} \\ \mathit{32} \\ \mathit{28} \\ \mathit{5} \\ \mathit{1} \\ \mathit{31} \\ \mathit{2} \\ \mathit{33} \\ \mathit{30} \\ \mathit{3} \end{array} \\ \begin{array}{ccccccccccc} \mathit{17} & \mathit{18} & \mathit{19} & \mathit{20} & \mathit{21} & \mathit{22} & \mathit{23} & \mathit{24} & \mathit{25} & \mathit{26} & \mathit{27} \end{array} & \end{array}$$

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