Duality in Bandwidth Problems[1]

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Abstract

The bandwidth is an important invariant in graph theory. However, the problem to determine the bandwidth of a general graph is NP-complete. To get sharp bounds, we propose to pay attention to various duality properties or minimax relations related to the bandwidth problems. This paper presents a study of this point of view.

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1 Introduction

The bandwidth problem for graphs and matrices originates from sparse matrix computation, circuit layout of VLSI designs and other areas. Mathematically, we can describe it as follows.[2] Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. A labeling of G is a bijection $f : V \to \{1, 2, \cdots, |V|\}$. The bandwidth of a labeling f for G is defined by

$$B(G, f) = \max\{|f(u) - f(v)| : uv \in E\};$$

and the bandwidth of G is defined by

$$B(G, f) = \min\{B(G) : f \text{ is a labeling of } G\}.$$

A labeling that attains this minimum value is called an optimal labeling.

To determine $B(G)$ is a minimization problem which is NP-complete.[8] So a direction of research is to find sharp bounds for $B(G)$. It is well-known in combinatorial optimization that minimax relations can usually provide good lower or upper bounds. For bandwidth problem, as a minimization problem, we could establish relations with some maximization problems. In this paper, we study this type of duality in several aspects. In Section 2, we illustrate how a duality relation about degrees implies a series of bounds. In Section 3 we discuss the so-called density lower bounds. In Section 4, we describe a method based on the boundary lower bounds.

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2 Duality on Degrees

The degree sequence of G is denoted by $d(G) = (d_1, d_2, \cdots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose that G and G' have the same order n. G is said to be degree-majorized by G', denoted by $d(G) \leq d(G')$ for $i = 1, 2, \cdots, n$ (see for example[1]).

Lemma 2.1 *For any graph G with n vertices.*

$$B(G) \geq \min\{k : d(G) \leq d(P_n^k)\}.$$

where P_n^k stands for the k-th power of a path on n vertices.

Proof Suppose that $B(G) = k$. Then, by definition, G can be embedded in P_n^k., i.e. $G \subseteq P_n^k$, which implies $d(G) \leq d(P_n^k)$ Hence the lower bound follows. ∎

Next, we have the following minimax duality.

Lemma 2.2 *If $d(G) = (d_1, d_2, \cdots, d_n)$, then*

$$\min\{k : d(G) \leq d(P_n^k)\} = \max_{1 \leq i \leq n} \max\left\{d_i - \left\lfloor \frac{i-1}{2} \right\rfloor, \left\lceil \frac{d_i}{2} \right\rceil\right\}.$$

Proof It is clear that

$$d(P_n^k) = \begin{cases} (k, k, k+1, k+1, \cdots, 2k, 2k) & \text{if} \quad 1 \leq k \leq \dfrac{2}{n}, \\ (k, k, k+1, k+1, \cdots, n-1, \cdots, n-1) & \text{if} \quad \dfrac{2}{n} \leq k < n. \end{cases}$$

If we let $d(P_n^k) = (d_1^*, d_2^*, \cdots, d_n^*)$, then

$$d_i^* = k + \min\left\{\left\lfloor \frac{i-1}{2} \right\rfloor, k, n-k-1\right\}.$$

If $d(G) \leq d(P_n^k)$, then

$$d_i \leq k + \left\lfloor \frac{i-1}{2} \right\rfloor, \quad d_i \leq 2k, \quad d_i \leq n-1 \quad (1 \leq i \leq n),$$

and therefore

$$k \geq \max\left\{d_i - \left\lfloor \frac{i-1}{2} \right\rfloor, \left\lceil \frac{d_i}{2} \right\rceil\right\}.$$

It follows that

$$\min\{k : d(G) \leq d(P_n^k)\} \geq \max_{1 \leq i \leq n} \max\left\{d_i - \left\lfloor \frac{i-1}{2} \right\rfloor, \left\lceil \frac{d_i}{2} \right\rceil\right\}.$$

Conversely, if $k' = \max_{1\le i\le n} \max\left\{d_i - \left\lfloor\frac{i-1}{2}\right\rfloor, \left\lceil\frac{d_i}{2}\right\rceil\right\}$, then we can retrack the above argument to show that $d(G) \le d(P_n^k)$. ∎

By these two lemmas, we have the following minimax-type lower bound due to Chvátal.[4]

Theorem 2.3 *For any graph G with degree sequence $d_1, d_2, \cdots, d_n$ satisfying $d_1 \le d_2 \le \cdots \le d_n$,*

$$B(G) \ge \max_{1\le i\le n} \max\left\{d_i - \left\lfloor\frac{i-1}{2}\right\rfloor, \left\lceil\frac{d_i}{2}\right\rceil\right\}.$$

The survey[2] said "it is not so clear which graphs realize the bound". By the foregoing lemmas, it is clear that the bound is attained when $G = K_{1,n}$, P_n^k and K_n. Furthermore, we have an improvement of this lower bound when a part of vertices have been labeled. For example, we need this kind of lower bounds in the branch-and-bound algorithms.

Theorem 2.4 *Suppose that $v_1, v_2, \cdots, v_j$ have been labeled by $i, 2, \cdots, j$ respectively and that the remaining vertices have degree sequence $d_{j+1} \le d_{j+2} \le \cdots \le d_n$. Then*

$$B(G) \ge \begin{cases} \max\{b_1, b_2, b_3\}, & \text{if} \quad j < \frac{n}{2} \\ \max\{b_1, b_2\}, & \text{if} \quad j \ge \frac{n}{2} \end{cases}$$

where

$$\begin{aligned} b_1 &= \max_{1\le i\le j} \max\left\{d(v_i) - i + 1, \left\lceil\frac{d(v_i)}{2}\right\rceil\right\} \\ b_2 &= \max_{1\le i\le j} \max\left\{d_{j+i} - i + 1, \left\lceil\frac{d_{j+i}}{2}\right\rceil\right\} \\ b_2' &= \max_{1\le i\le n-j} \max\left\{d_{j+i} - i + 1, \left\lceil\frac{d_{j+i}}{2}\right\rceil\right\} \\ b_3 &= \max_{2j+1\le i\le n} \max\left\{d_i - \left\lfloor\frac{i-1}{2}\right\rfloor, \left\lceil\frac{d_i}{2}\right\rceil\right\} \end{aligned}$$

Proof Suppose that $B(G) = k$. Then G can be embedded in P_n^k such that $v_1, v_2, \cdots, v_j$ are put in its first j vertices. Hence

$$d(v_i) \le \min\{k + i - 1, 2k, n - 1\} \quad (1 \le i \le j).$$

Thus $B(G) = k \ge b_1$. When $j < \frac{n}{2}$, the degree sequence for the remaining vertices of P_n^k will be

$$k, k+1, \cdots, k+j-1, k+j, k+j, k+j+1, k+j+1, \cdots, n-1, n-1$$

(assume that $k \geq \frac{n}{2}$, say). For the first j terms of the sequence, it follows that $B(G) \geq b_2$ (similar to the case of b_1). For the last $n-2j$ terms of the sequence, we can obtain that $B(G) \geq b_3$ (similar to the case of Theorem 2.3). Therefore $B(G) \geq \max\{b_1, b_2, b_3\}$. When $j \geq \frac{n}{2}$, the remaining degree sequence of P_n^k will be

$$-k, k+1, k+2, \cdots, n-1.$$

So, we can obtain that $B(G) \geq b_2$. Thus $B(G) \geq \max\{b_1, b_2\}$. ▮

The efficiency of these minimax-type lower bounds can be illustrated by the fact that they can imply most lower bounds in terms of other graph invariants listed in the survey.[2] We mention some typical examples as follows.

Corollary 2.5 (1) *If* $n = |V(G)|, m = |E(G)|$, *then*

$$B(G) \geq n - \frac{1 + \left[(2n_1)^2 - 8m\right]^{\frac{1}{2}}}{2}.$$

(2) $B(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

(3) $B(G) \geq \delta(G)$.

(4) $B(G) \geq \kappa(G)$, *where* $\kappa(G)$ *is the connectivity of* G.

(5) $B(G) \geq \chi(G) - 1$, *where* $\chi(G)$ *is the chromatic number of* G.

(6) $B(G) \geq \left\lceil \frac{|V(G)|}{\beta_0(G)} \right\rceil - 1$, *where* $\beta_0(G)$ *is the vertex independence number of* G.

Proof (1) Let k be the lower bound of Theorem 2.3. Then, by Lemma 2.2, $d(P_n^k) \geq d(G)$. By $\sum_{i=1}^{n} d_i^* \geq \sum_{i=1}^{n} d_i$ and the degree sequence of P_n^k (see the proof of Lemma 2.2.), we have $k^2 - (2n-1)k + 2m \leq 0$. Thus

$$B(G) \geq k \geq \frac{(2n-1) - \sqrt{(2n-1)^2 - 8m}}{2}.$$

(2) By Theorem 2.3, $B(G) \geq \max_{1 \leq i \leq n} \left\lceil \frac{d_i}{2} \right\rceil = \left\lceil \frac{\Delta(G)}{2} \right\rceil$.
(3) For i = 1, $B(G) \geq d_1 = \delta(G)$.
(4) $B(G) \geq \delta(G) \geq \kappa(G)$.

(5) For the chromatic number of G, there must be a critical subgraph H such that $\delta(H) \geq \chi(G) - 1$. Hence $B(G) \geq B(H) \geq \delta(H) \geq \chi(G) - 1$.

(6) $B(G) \geq \chi(G) - 1 \geq \left\lceil \frac{|V(G)|}{\beta_0(G)} \right\rceil - 1$. ∎

Note that any graph attaining one of the six bounds in Corollary 2.5 also attains the one in the Theorem 2.3.

3 Duality on Diameters

Theorem 3.1 *For any vertices u, v in G, if there is a (u, v) path with length k, then*

$$B(G, f) \geq \frac{|f(u) - f(v)|}{k}.$$

Proof Let $P = v_0 v_1 v_2 \cdots v_k$ be a path between u and v in G (where $u = v_0, v = v_k$). Then

$$\begin{aligned} B(G, f) &\geq \max\Big\{|f(x) - f(y)| : xy \in E(P)\Big\} \\ &\geq \frac{1}{k}\sum_{i=0}^{k-1} |f(v_{i+1}) - f(v_i)| \geq \frac{|f(u) - f(v)|}{k}. \end{aligned}$$

∎

From this, we have a useful lower bound due to V. Chvátal[4] as follows:

Corollary 3.2 *For any connected graph G with diameter $D(G)$*

$$B(G) \geq \left\lceil \frac{|V(G)| - 1}{D(G)} \right\rceil.$$

Proof We may choose u, v satisfying $f(u) = 1$, $f(v) = |V(G)|$ for any labeling f. ∎

To strengthen this bound, we can establish a minimax form as follows.[3,7]

Corollary 3.3 *For any connected graph G*

$$B(G) \geq \max_k \max \left\{ \frac{|S| - 1}{D(S)} : S \subseteq V(G), D(S) = k \right\}$$

where the diameter, of subset S is defined by $D(S) = \max\{d_G(x, y) : x, y \in S\}$.

Proof We may choose u, v satisfying $f(u) = \min\{f(x) : x \in S\}$ and $f(v) = \max\{f(x) : x \in S\}$. ∎

This minimax lower bound is sharp for many classes of special graphs. Two examples are given below.

A caterpillar is a tree which yields a path (the spine) when all its pendant vertices are removed.

Proposition 3.4 [9] *For a caterpillar T with spine $u_1u_2\cdots u_m$, let T_{ij} be the subtree induced by $u_i, u_{i+1}, \cdots, u_j$ and their neighbors. Then*

$$B(T) = \max_{1\leq i\leq j\leq m} \left\lceil \frac{|V(T_{ij})| - 1}{j - i + 2} \right\rceil.$$

A tree of diameter 4 consists of a central star $S_0 \cong K_{1,r}$ and r stars $S_1, S_2, \cdots, S_r$, each of which has an end vertex of S_0 as its center. We denote this tree T by $(S_0, S_1, S_2, \cdots, S_r)$.

Proposition 3.5 [7] *For a tree of diameter 4, $T = (S_0, S_1, S_2, \cdots, S_r)$, let n_i and n_{oi} be the vertex numbers of S_i and $S_0 \cup S_i$ respectively (n is the vertex number of T). Then*

$$B(T) = \max\left\{ \max_{0\leq i\leq r} \left\lceil \frac{n_i - 1}{2} \right\rceil, \max_{1\leq i\leq r} \left\lceil \frac{n_{oi} - 1}{3} \right\rceil, \left\lceil \frac{n-1}{4} \right\rceil \right\}.$$

4 Duality on Boundaries

For a subset $S \subseteq V(G)$, the interior and outer boundaries of S are defined respectively as

$$\begin{aligned} \partial(S) &= \{u \in S : \exists v \in V\backslash S \text{ such that } uv \in E(G)\} \\ N(S) &= \{u \in V/S : \exists v \in S \text{ such that } u \in E(G)\}. \end{aligned}$$

The latter is also called the neighboring set of S. L.H. Harper[5] first established the following duality relation.

Theorem 4.1 *For any connected graph G,*

$$B(G) \geq \max_{1\leq k\leq n} \min\{|\partial(S)| : S \subseteq V(G), |S| = k\}.$$

By symmetry, an equivalent form is

$$B(G) \geq \max_{1\leq k\leq n} \min\{|N(S)| : S \subseteq V(G), |S| = k\}.$$

This lower bound is of significance in solving the bandwidth problem for some special graphs, e.g. n-cubes, $P_m \times P_n$ and others (see[2,5,6]). Nevertheless, we can make it sharper in some cases.

For a labeling f, let $u_i = f^{-1}(i)$, $1 \leq i \leq n$, and $S_k(f) = \{u_1, u_2, \cdots, u_k\} = f^{-1}(\{1, 2, \cdots, k\})$. We have the following.

Theorem 4.2 *For any connected graph G,*

$$B(G) \geq \min_f \max_{1 \leq k \leq n} |N(S_k(f))|,$$

where min *is taken over all labelings.*

The proof is similar to that of Theorem 4.1 (see[2], p. 225) However, this lower bound is greater than or equal to that of Theorem 4.1. In fact, for any labeling f and any integer K,

$$|N(S_k(f))| \geq \min\{|N(S)| : S \subseteq V(G), |S| = k\}.$$

Hence

$$\min_f \max_{1 \leq k \leq n} |N(S_k(f))| \geq \max_{1 \leq k \leq n} \min\{|N(S)| : S \subseteq V(G), |S| = k\}.$$

This is a quite good minimax duality relation since the equality holds for many families of graphs. We do not know if equality always holds.

From the above discussion, we may introduce two new invariants:

$$\begin{aligned} H(G) &= \max_{1 \leq k \leq n} \min\{|N(S)| : S \subseteq V(G), |S| = k\} \\ H'(G) &= \min_f \max_{1 \leq k \leq n} |N(S_k(f))|. \end{aligned}$$

Recently, J. Yuan[10] proved that $H'(G)$ is in fact the pathwidth of G. Furthermore, we may combine the interior and outer boundaries to get

$$H^*(G) = \min_f \max_{1 \leq k \leq n} \max\{|N(S_k(f))|, |\partial(S_k(f))|\}.$$

Then we have

Theorem 4.3 *For any connected graph G,*

$$B(G) \geq H^*(G) \geq H'(G) \geq H(G).$$

The lower bound $H^*(G)$ is sometimes greater than $H(G)$. For example, $G = K_{1,3}$, $H(G) = 1$, $B(G) = H^*(G) = 2$. As an application, consider the bandwidth of "fan meshes" $F_{m,n}$ in Figure 1. We originally studied this problem in[11]. Now, here is a revised version.

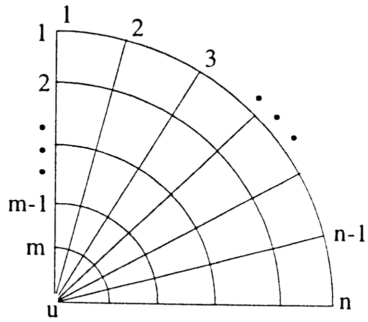

Figure 1

Proposition 4.4 *Let $F_{m,n}$ be a mesh $P_m \times P_n$ with an extra vertex u joined to n vertices of P_n at one side. Then*

$$B(F_{m,n}) = \min\left\{n, \left\lceil \frac{n}{2} \right\rceil + m - 1\right\}.$$

Proof Let $R_1, R_2, \cdots, R_m$ be the vertex sets of m copies of P_n; $T_1, T_2, \cdots, T_n$ be those of n copies of P_m in $F_{m,n}$ respectively. For any subset $S \subseteq V$, denote

$$\begin{aligned} \varphi(S) &= |\{R_i | R_i \cap S \neq \emptyset\}| \\ \psi(S) &= |\{T_j | T_j \cap S \neq \emptyset\}|. \end{aligned}$$

For a given labeling f, suppose $k = f(u)$.

Case 1 If $\varphi(S_k(f)) < m$ then every T_j has at least one vertex in $N(S_k(f))$. Thus $B(G, f) \geq |N(S_k(f))| \geq n$.

Case 2 If $\varphi(S_k(f)) = m$ and $\varphi(S_k(f)) = n$ then every T_j has at least one vertex in $\partial(S_{k-1}(f))$. Thus $B(G, f) \geq |\partial(S_{k-1}(f))| \geq n$

Case 3 Suppose that $\varphi(S_k(f)) = m$ and $\psi(S_k(f)) < n$. If $|R_1 \cap S_k(f)| \leq \lfloor \frac{n}{2} \rfloor$ then $R_1 \backslash S_k(f) \subseteq N(S_k(f))$, so $B(G, f) \geq |N(S_k(f))| \geq \lceil \frac{n}{2} \rceil + m - 1$. If $|R_1 \cap S_k(f)| \geq \lceil \frac{n}{2} \rceil$ then $B(G, f) \geq |\partial(S_{k-1}(f))| \geq \lceil \frac{n}{2} \rceil + m - 1$.

To sum up the above cases, we have

$$B(G) \geq \min\left\{n, \left\lceil \frac{n}{2} \right\rceil + m - 1\right\}.$$

On the other hand, we can construct a labeling f attaining the lower bound $\beta = \min\{n, \lceil \frac{n}{2} \rceil + m - 1\}$. For clarity, we may write down all labels in an $m \times n$ table with an extra row (row $m+1$) for $f(u)$. (see examples below). If $\beta = n$, we may simply label the table row by row. Assume in the sequel that $\beta = \lceil \frac{n}{2} \rceil + m - 1$. Denote $r = \lceil \frac{n}{2} \rceil - m$. We partition the table into levels:

$$\begin{aligned} L_1 &= \{(1,1), \cdots, (1, r+1)\} \\ L_2 &= \{(2,1), \cdots, (2, r+2), (1, r+2)\} \\ &\cdots \\ L_m &= \{(m,1), \cdots, (m, r+m), (m-1, r+m), \cdots, (1, r+m)\} \end{aligned}$$

where (i,j) stands for the element of row i and column j. Then, take the row $m+1$ as L_{M+1}. The remaining $\lfloor \frac{n}{2} \rfloor$ columns are partitioned symmetrically. After that, we put the labels $1, 2, \cdots, mn, mn+1$ into the table level by level (where $f(u) = m\lceil \frac{n}{2} \rceil + 1$). It is easy to check that $B(G,f) = \lceil \frac{n}{2} \rceil + m - 1$. This completes the proof. ∎

Example The optimal labelings for $F_{3,6}$ and $F_{3,7}$ are as follows.

1	4	9	13	17	19
2	3	8	12	16	18
5	6	7	11	14	15
10					

B$(F_{3,6})$ = 5 (r = 0)

1	2	6	12	16	23	22
3	4	5	11	15	19	21
7	8	9	10	14	17	18
13						

B$(F_{3,7})$ = 6 (r = 1)

5 Concluding Remarks

There may be two approaches to determine the bandwidth of a graph. (1) A static approach: first, give a sharp lower bound; then, construct a labeling attaining the bound. (2) A dynamic approach: during the labeling searching process, the lower bounds are improved successively until a bound is attained by a labeling. The former is to used to get explicit solutions for some special graphs; the latter is used to get numerical solutions (in branch-and-bound algorithms or dynamic programming algorithms, for examples). In either case, the greater a lower bound is found, the better. Therefore, it is natural to set up a lower bound through a maximization problem.

Furthermore, the bandwidth minimization problem can be stated as an integer programming problem. Let $x_{ij} = 1$ if $f(v_i) = j$ and 0 otherwise.

Then the label of v_i is $f(v_i) = \sum_{k=1}^{n} kx_{ik}$ $(1 \le i \le n)$, and we have the following IP problem:

$$\begin{aligned}
\text{Minimize} \quad & y \\
\text{subject to} \quad & \sum_{k=1}^{n} k(x_{ik} - x_{jk}) \le y \quad \text{for } v_iv_j \in E && (1)\\
& \sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1, \cdots, n && (2)\\
& \sum_{i=1}^{n} x_{ij} = 1 \text{ for } j = 1, \cdots, n && (3)\\
& x_{ij} \ge 0 \quad \text{integer} && (4)
\end{aligned}$$

We may apply the method of Lagrange relaxation with respect to (1) so as to include these constraints into the objective function as a term of "penalty". Then the re-formulation will be the well-known assignment problem (optimal bipartite matching problem). Thus we can obtain lower bounds by solving the dual of the assignment problem (a maximization problem). This is the so-called Lagrange relaxation and duality procedure, which would be considered as an efficient way in the algorithmic study of the bandwidth problem.

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