# The integer-antimagic spectra of dumbbell graphs

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## 1. Introduction

Let $G$ be a connected simple graph. For any non-trivial abelian group $A$ (written additively), let $A^* = A \setminus \{0\}$, where 0 is the additive identity of $A$. Let a function $f : E(G) \to A^*$ be an edge labeling of $G$. Any such labeling induces a map $f^+ : V(G) \to A$, defined by $f^+(v) = \sum_{uv \in E(G)} f(uv)$. If there exists an edge labeling $f$ whose induced map $f^+$ on $V(G)$ is one-to-one, we say that $f$ is an *$A$-antimagic labeling* and that $G$ is an *$A$-antimagic* graph. The *integer-antimagic spectrum* of a graph $G$ is the set $\mathrm{IAM}(G) = \{k \mid G \text{ is } \mathbb{Z}_k\text{-antimagic and } k \geq 2\}$.

The concept of the $A$-antimagicness property for a graph $G$ (introduced in [1]) naturally arises as a variation of the $A$-magic labeling problem (where the induced vertex labeling is a constant map). $\mathbb{Z}$-magic (or $\mathbb{Z}_1$-magic) graphs were considered by Stanley [24, 25], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [2, 3, 4] and others [7, 9, 15, 16, 21] have studied $A$-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [5, 6, 8, 10, 11, 12, 13, 14, 17, 18, 19, 20, 22].

## 2. Some Known Results

The following two lemmas (found in [1]) will be used throughout this paper.

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**Lemma 2.1** ([1, Lemma 1]). *For $m \geq 1$, a graph of order $4m+2$ is not $\mathbb{Z}_{4m+2}$-antimagic.*

**Lemma 2.2** ([1, Theorem 4]). *For $m \geq 1$, $C_{4m+r}$ and $P_{4m+r}$ are $\mathbb{Z}_k$-antimagic, for all $k \geq 4m+r$ if $r = 0, 1, 3$; $C_{4m+2}$ and $P_{4m+2}$ are $\mathbb{Z}_k$-antimagic, for all $k \geq 4m+3$.*

Also, we will use the following $\mathbb{Z}_k$-antimagic labelings $g$ of cycles, found in [1].

**Remark 2.1.** Let $C_n = v_1v_2\cdots v_nv_1$ and $e_1 = v_1v_2$, $e_2 = v_2v_3, \ldots$, $e_n = v_nv_1$ be its edges. For integers $a \leq b$, $[a, b]$ denotes the set of integers from $a$ to $b$ inclusive.

Case 1. $n = 4m$:

$$g(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m; \\ 3 + 2(2m - \lceil \frac{i}{2} \rceil) & \text{if } 2m+1 \leq i \leq 4m. \end{cases}$$

The range of $g$ is $[1, 2m+1]$.

Case 2. $n = 4m+1$:

$$g(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m; \\ 3 + 2(2m - \lceil \frac{i}{2} \rceil) & \text{if } 2m+1 \leq i \leq 4m+1. \end{cases}$$

The range of $g$ is $[1, 2m+1]$.

Case 3. $n = 4m+2$:

$$g(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m+3; \\ 3 + 2(2m - \lceil \frac{i-2}{2} \rceil) & \text{if } 2m+4 \leq i \leq 4m+2. \end{cases}$$

The range of $g$ is $[1, 2m+3]$.

Case 4. $n = 4m-1$:

$$g(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2m+1; \\ 3 + 2(2m - \lceil \frac{i+1}{2} \rceil) & \text{if } 2m+2 \leq i \leq 4m-1. \end{cases}$$

The range of $g$ is $[1, 2m+1]$.

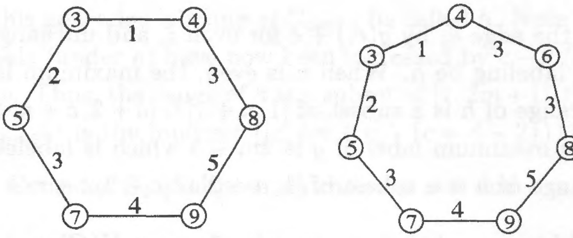
In this paper, we focus on the image of the induced mapping $g^+$. We denote the image of $g^+$ by $I_g(G)$, that is,

$$I_g(G) = \{g^+(v) \mid v \in V(G)\},$$

where $G$ is the graph under consideration. Note that if we label the edges of $C_3$ by 1,2,3, then $I_g(C_3) = [3, 5]$. Hence, $C_3$ is $\mathbb{Z}_k$-antimagic for $k \geq 4$.

Let $S$ and $T$ be multi-sets. If $S$ and $T$ are equal (as sets) modulo $k$, then this is denoted by $S \equiv T \pmod{k}$.

*Example.* From the labelings $g$ provided in [1], we have the following $\mathbb{Z}_k$-antimagic labelings of $C_6$ and $C_7$, for $k \geq 7$.




First, we view all labels on edges as integers. Then,

$$I_g(C_6) = [3, 9] \setminus \{6\} \text{ and } I_g(C_7) = [3, 9].$$

After taking modulo $k$ for $k \geq 10$, the numbers do not change. For $k = 9$, we have $I_g(C_6) \equiv [0, 8] \setminus \{1, 2, 6\} \pmod 9$ and $I_g(C_7) \equiv [0, 8] \setminus \{1, 2\} \pmod 9$. For $k = 8$, $I_g(C_6) \equiv [0, 7] \setminus \{2, 6\} \pmod 8$ and $I_g(C_7) \equiv [0, 7] \setminus \{2\} \pmod 8$. For $k = 7$, $I_g(C_6) \equiv [0, 5] \pmod 7$ and $I_g(C_7) \equiv [0, 6] \pmod 7$. So, $C_6$ and $C_7$ are $\mathbb{Z}_k$-antimagic for $k \geq 7$.

**Corollary 2.3.** *For $m \geq 1$ and labelings $g$ for cycles provided in Remark 2.1, we have $I_g(C_{4m-1}) = [3, 4m+1]$, $I_g(C_{4m}) = [3, 4m+2]$, $I_g(C_{4m+1}) = [2, 4m + 2]$ and $I_g(C_{4m+2}) = [3, 4m + 5] \setminus \{4m + 2\}$.*

**Proposition 2.4.** *All elements in $[a, b]$ are distinct after taking modulo $k$, for $k \geq b - a + 1$.*

## 3. Some Useful Lemmas

Throughout this paper, we use the labelings $g$ (defined in Remark 2.1) and view all values of $g$ as integers first. The notation in the preceding section will also be used. For $S \subset \mathbb{Z}$ and $a \in \mathbb{Z}$, we let the set $a + S = \{a + s \mid s \in S\}$.

**Lemma 3.1.** *For $n \geq 3$, suppose $g : E(C_n) \to \mathbb{Z}$ be a labeling and $c \in \mathbb{Z}$. Then, there is a labeling $h$ such that $I_h(C_n) = 2c + I_g(C_n)$. Note that the range of $h$ is the set $[c + 1, c + \lfloor n/2 \rfloor + 1]$ if $n \equiv 0, 1 \pmod 4$, and $[c + 1, c + \lfloor n/2 \rfloor + 2]$ otherwise.*

*Proof.* The required labeling is $h = g + c$. $\square$

**Lemma 3.2.** *Suppose that $n \geq 2$ and let $g : E(C_{2n}) \to \mathbb{Z}$, $c \in \mathbb{Z}$. Then, there is a labeling $h$ such that $I_h(C_{2n}) = c + I_g(C_{2n})$. Note that the range of $h$ is a subset of $[1, n+2] \cup [c+2, c+n+1]$.*

*Proof.* Relabel the edge $e_i$ by $g(e_i) + c$ for even $i$, and unchanged for odd $i$. Let this new labeling be $h$. When $n$ is even, the maximum label of $g$ is $n+1$. So, the range of $h$ is a subset of $[1, n+1] \cup [c+2, c+n+1]$. When $n = 2m+1$, the maximum label of $g$ is $2m+3$ which is labeled at $e_{2m+3}$ only. So, the range of $h$ is a subset of $[1, n+2] \cup [c+2, c+n+1]$. □

Let $G$ and $H$ be connected simple graphs. Let $u \in V(G)$ and $v \in V(H)$. The graph $G^{uv}H$ is obtained from $G$ and $H$ by add a new edge (bridge) $uv$.

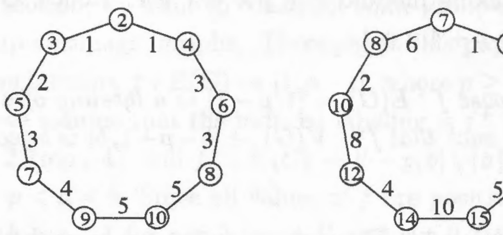
**Theorem 3.3.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $1 \leq b-p < a < b$. Then, $b-a$ is odd.*

*Proof.* Assume that $b-a$ is even. We choose an integer $m$ such that $b-p \leq 2m$. Let $g$ be the labeling of $C_{4m}$ provided in Remark 2.1. Recall that $g : E(C_{4m}) \to [1, 2m+1]$ and $I_g(C_{4m}) = [3, 4m+2]$. By putting $c = b-3$ into Lemma 3.2, we have a labeling $h$ of $C_{4m}$ such that $I_h(C_{4m}) = [b, 4m+b-1]$. Choose the vertex $u$ from $G$ with $f^+(u) = (b+a)/2$ [which exists, since $b-a$ is even] and the vertex $v$ from $C_{4m}$ with $h^+(v) = b$. Join $u$ and $v$ by a bridge $uv$ with label $(a-b)/2$. Let the final labeling be $\phi$. Note that $b \geq p+1 \geq 7$ [since $p \geq 6$ and thus, $b-p \geq 1 \geq 7-p$] and hence, $c \geq 4$. The set of edge-labels is a subset of $[1, p-1] \cup \{\frac{a-b}{2}\} \cup [1, 2m+2] \cup [b-1, b-2+2m]$. Note that $b-p \leq 2m < 2m+1$ and thus, $b-2+2m \leq 4m+p-1$. Hence, all of the edge-labels of $\phi$ are non-zero $\pmod k$, for all $k \geq p+4m$. Also, $I_\phi(G^{uv}C_{4m}) = [b-p, b] \setminus \{b\} \cup [b, 4m+b-1] = [b-p, 4m+b-1]$. Hence, $G^{uv}C_{4m}$ is $\mathbb{Z}_{4m+p}$-antimagic, which contradicts Lemma 2.1. □

**Lemma 3.4.** *For $d \in [2, 4m+2]$ and any integer $c$, there is a labeling $h$ such that $I_h(C_{4m+1})$ is the multiset $([c, 4m+c] \setminus \{c+d-2\}) \cup \{d\}$. Note that the range of $h$ is a subset of $[1, 2m+1] \cup [c-1, c-1+2m]$.*

*Proof.* Let $g$ be the labeling of $C_{4m+1}$ defined in Remark 2.1 and let $v = g^{-1}(d)$. We modify the edge labeling under $g$ in the following way: Note that $C_{4m+1} - v \cong P_{4m}$ and let $e_1, e_2, e_3, \ldots, e_{4m-1}$ be the consecutive adjacent edges of $C_{4m+1}-v$. Relabel edges $e_1, e_3, e_5, \ldots, e_{4m-1}$ by adding $c-2$ to the original labels under $g$. Keep the original edge labels (under $g$) for $e_2, e_4, \ldots, e_{4m-2}$, as well as for the two edges adjacent to $v$ in $C_{4m+1}$. Now, let this new edge labeling of $C_{4m+1}$ be called $h$. Note that the induced vertex labels (under $g$) have now been increased by $c-2$ (under $h$), except at vertex $v$. Thus, the range of $h$ is a subset of $[1, 2m+1] \cup [c-1, c-1+2m]$ and $I_h(C_{4m+1})$ is the multiset $([c, 4m+c] \setminus \{c+d-2\}) \cup \{d\}$. □

*Example.* Consider $C_9$. Suppose we choose $d = 8$ and $c = 7$. We have




Labeling $g$, $I_g(C_9) = [2, 10]$. Labeling $h$, $I_h(C_9) = ([7, 15] \setminus \{13\}) \cup \{8\}$.

**Lemma 3.5.** *For $d \in [3, 4m+1]$ and any integer $c$, there is a labeling $h$ such that $I_h(C_{4m-1})$ is the multiset $([c, 4m+c-2] \setminus \{c+d-3\}) \cup \{d\}$. Note that the range of $h$ is a subset of $[1, 2m+1] \cup [c-2, c-2+2m]$.*

*Proof.* Let $g$ be the labeling of $C_{4m-1}$ defined in Remark 2.1 and let $v = g^{-1}(d)$. We modify the edge labeling under $g$ in the following way: Note that $C_{4m-1} - v \cong P_{4m-2}$ and let $e_1, e_2, e_3, \ldots, e_{4m-3}$ be the consecutive adjacent edges of $C_{4m-1}-v$. Relabel edges $e_1, e_3, e_5, \ldots, e_{4m-3}$ by adding $c-3$ to the original labels under $g$. Keep the original edge labels (under $g$) for $e_2, e_4, \ldots, e_{4m-4}$, as well as for the two edges adjacent to $v$ in $C_{4m-1}$. Now, let this new edge labeling of $C_{4m-1}$ be called $h$. Note that the induced vertex labels (under $g$) have now been increased by $c-3$ (under $h$), except at vertex $v$. Thus, the range of $h$ is a subset of $[1, 2m+1] \cup [c-2, c-2+2m]$ and $I_h(C_{4m-1})$ is the multiset $([c, 4m+c-2] \setminus \{c+d-3\}) \cup \{d\}$. □

**Lemma 3.6.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \not\equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $b - p \leq 4m + 1 + p$, and $b$ is odd. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m+1})$ such that $G^{uv}C_{4m+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m + 1$.*

*Proof.* By putting $c = (b-3)/2$ into Lemma 3.1, we have a labeling $h$ such that $I_h(C_{4m+1}) = [b-1, 4m+b-1]$. Note that the range of $h$ is the set $[\frac{b-3}{2}+1, \frac{b-3}{2}+\lfloor\frac{4m+1}{2}\rfloor+1] = [\frac{b-1}{2}, \frac{b-1}{2}+\lfloor\frac{4m+1}{2}\rfloor]$. Choose $u \in V(G)$ with $f^+(u) = b$ and $v \in V(C_{4m+1})$ with $h^+(v) = b-1$. Join $u$ and $v$ by a bridge $uv$ with label $-p$. Let the final labeling be $\phi$. Then, $I_\phi(G^{uv}C_{4m+1}) = [b-p+1, b-1] \cup \{b-p\} \cup \{b-p-1\} \cup [b-2, 4m+b-1] = [b-p-1, 4m+b-1]$. The set of edge-labels of $\phi$ is the set $[1, p-1] \cup \{-p\} \cup [\frac{b-1}{2}, \frac{b-1}{2}+\lfloor\frac{4m+1}{2}\rfloor]$. Note that $b-p \leq 4m+1+p$ which implies $\frac{b-1}{2} \leq 2m+p$. Thus, $\frac{b-1}{2}+\lfloor\frac{4m+1}{2}\rfloor] < \frac{b-1}{2}+2m+1 \leq (2m+p)+(2m+1) = 4m+p+1$. Thus, all of the edge-labels are non-zero, after taking modulo $k \geq p+4m+1$. Hence, $G^{uv}C_{4m+1}$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m+1$. □

**Theorem 3.7.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 1 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective. Then, $b$ must be even.*

*Proof.* Assume that $b$ is odd. There exists an integer $m$ such that $b-p \leq 4m+1+p$. By Lemma 3.6, there exist vertices $u \in V(G)$ and $v \in V(C_{4m+1})$ such that $G^{uv}C_{4m+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p+4m+1$. In particular, $G^{uv}C_{4m+1}$ is $\mathbb{Z}_{p+4m+1}$-antimagic. But the order of $G^{uv}C_{4m+1}$ is $p+4m+1 \equiv 2 \pmod 4$. This contradicts Lemma 2.1. □

**Lemma 3.8.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \not\equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $b-p \leq 4m-2+p$. Let $b$ be even. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m-1})$ such that $G^{uv}C_{4m-1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p+4m-1$.*

*Proof.* By putting $c = b/2-2$ into Lemma 3.1, we have a labeling $h$ such that $I_h(C_{4m-1}) = [b-1, 4m+b-3]$. Note that the range of $h$ is the set $[\frac{b}{2}-2+1, \frac{b}{2}-2+\lfloor\frac{4m-1}{2}\rfloor+2] = [\frac{b}{2}-1, \frac{b}{2}+\lfloor\frac{4m-1}{2}\rfloor]$. Choose $u \in V(G)$ with $f^+(u) = b$ and $v \in V(C_{4m-1})$ with $h^+(v) = b-1$. Join $u$ and $v$ by a bridge $uv$ with label $-p$. Let the final labeling be $\phi$. Then, $I_\phi(G^{uv}C_{4m-1}) = [b-p+1, b-1] \cup \{b-p\} \cup [b, 4m+b-3] \cup \{b-p-1\} = [b-p-1, 4m+b-3]$. The set of edge-labels of $\phi$ is the set $[1, p-1] \cup \{-p\} \cup [\frac{b}{2}-1, \frac{b}{2}+\lfloor\frac{4m-1}{2}\rfloor]$. Note that $b-p \leq 4m-2+p$ which implies that $\frac{b}{2} \leq 2m+p-1$. Thus, $\frac{b}{2}+\lfloor\frac{4m-1}{2}\rfloor < \frac{b}{2}+2m-1 \leq (2m+p-1)+(2m-1) = 4m+p-2 < 4m+p-1$. Hence, $G^{uv}C_{4m-1}$ is $\mathbb{Z}_k$-antimagic for $k \geq p+4m-1$. □

**Theorem 3.9.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 3 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective. Then, $b$ must be odd.*

*Proof.* Assume that $b$ is even. There exists an integer $m$ such that $b - 2p \leq 4m - 2$. By Lemma 3.8, there exists a graph of order $p + 4m - 1 \equiv 2 \pmod 4$, which is $\mathbb{Z}_{p+4m-1}$-antimagic. This contradicts Lemma 2.1. $\square$

## 4. $\mathbb{Z}_k$-ANTIMAGICNESS OF $G^{uv}C_s$

In this section, we want to construct some group-antimagic graphs from other group-antimagic graphs. Throughout this paper, we assume that $G$ has an edge labeling $f : E(G) \to [1, p-1]$, where $p \geq 3$ is the order of $G$. In addition, we assume that the induced labeling is $f^+ : V(G) \to [b-p+1, b]$ when $p \not\equiv 2 \pmod 4$, and $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ when $p \equiv 2 \pmod 4$, where $b - p < a < b$. Since all values of $f$ are positive and $G$ is connected, $b \geq p$ and $b \geq p+1$ for $p \not\equiv 2 \pmod 4$ and $p \equiv 2 \pmod 4$, respectively.

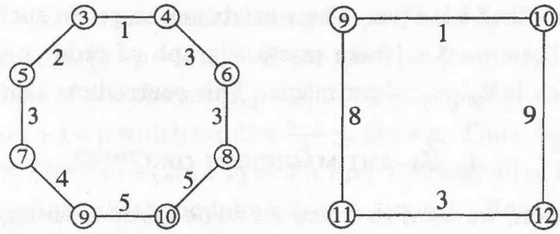
We use the following construction in this paper: First, we relabel some edges of $C_s$ obtained from $g$ (defined in Remark 2.1) to get a new labeling $h$. Then, we choose suitable vertices $u \in V(G)$ and $v \in V(C_s)$ to construct the graph $G^{uv}C_s$. Lastly, we label this bridge $uv$ to construct a $\mathbb{Z}_k$-antimagic labeling $\phi$ of $G^{uv}C_s$.

### 4.1. $\mathbb{Z}_k$-antimagic Labelings of $G^{uv}C_{4m}$.

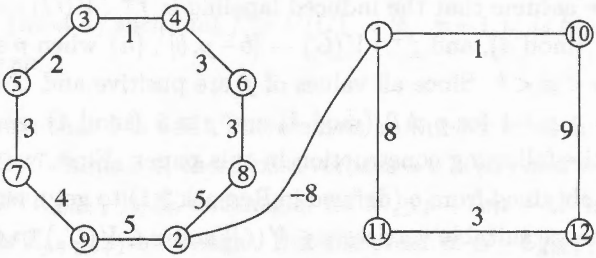
**Theorem 4.1.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \not\equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $b - p \leq 2m + 2$. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m})$ such that $G^{uv}C_{4m}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m$.*

*Proof.* By putting $c = b - 4$ into Lemma 3.2, we have a labeling $h$ such that $I_h(C_{4m}) = [b-1, 4m+b-2]$. Note that the range of $h$ is a subset of $[1, 2m+2] \cup [b-2, b+2m-3]$. Choose $u \in V(G)$ with $f^+(u) = b$ and $v \in V(C_{4m})$ with $h^+(v) = b - 1$. Join $u$ and $v$ with a bridge $uv$ and label it $-p$. Then, $I_\phi(G^{uv}C_{4m}) = [b-p+1, b-1] \cup \{b-p\} \cup \{b-p-1\} \cup [b, 4m+b-2] = [b-p-1, 4m+b-2]$. The set of edge-labels of $\phi$ is a subset of $[1, p-1] \cup \{-p\} \cup [1, 2m+2] \cup [b-2, b+2m-3]$. Note that $b - p \leq 2m + 2 < 2m + 3$ which implies that $b + 2m - 3 < 4m + p$. Hence, $G^{uv}C_{4m}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 4m + p$. $\square$

*Example.* Here are $\mathbb{Z}_k$-antimagic labelings for the dumbbell graph $D(8,4) = C_8^{uv}C_4$, for $k \geq 12$. In this case, $p = 8$, $b = 10$ and $m = 1$. We choose $h$ (defined in Lemma 3.2) with $c = 6$.




Labeling for $C_8$ under $g$. Labeling for $C_4$ under $h$.

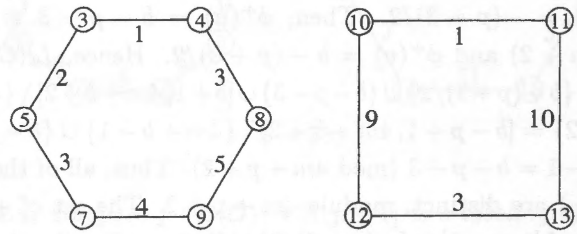



$\mathbb{Z}_k$-antimagic labeling for $D(8,4)$ under $\phi$.

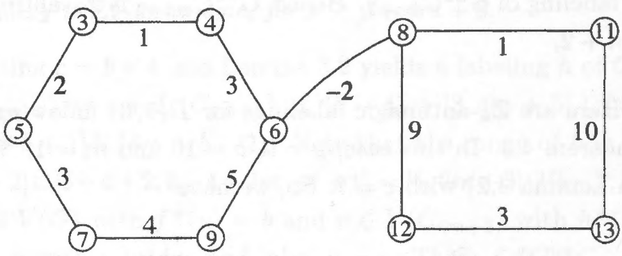
**Theorem 4.2.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $b - p \leq 2m + 1$ Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m})$ such that $G^{uv}C_{4m}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m + 1$.*

*Proof.* By Theorem 3.3, $b - a$ is odd. Putting $c = b - 2$ into Lemma 3.2 yields a labeling $h$ of $C_{4m}$ such that $I_h(C_{4m}) = [b+1, 4m+b]$. Note that the range of $h$ is a subset of $[1, 2m+2] \cup [b, b+2m-1]$. Choose $u \in V(G)$ with $f^+(u) = (b+a+1)/2$ and $v \in V(C_{4m})$ with $h^+(v) = b+1$. Join $u$ and $v$ with a bridge and label it $(a-b-1)/2$. Then, $I_\phi(G^{uv}C_{4m}) = [b-p, b] \setminus \{(a+b+1)/2\} \cup [b+2, 4m+b] \cup \{(a+b+1)/2\} = [b-p, 4m+b] \setminus \{b+1\}$, since we have $(a+b+1)/2 \neq b+1$. The set of edge-labels of $\phi$ is a subset of $[1, p-1] \cup \{(a-b-1)/2\} \cup [1, 2m+2] \cup [b, b+2m-1]$. Note that $b - p \leq 2m+1 < 2m+2$ which implies that $b + 2m - 1 < p + 4m + 1$. Hence, $G^{uv}C_{4m}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m + 1$. $\square$

*Example.* Here are $\mathbb{Z}_k$-antimagic labelings for $D(6,4)$ following the procedure of Theorem 4.2, for $k \geq 11$. In this case, $p = 6$, $b = 9$, $a = 6$ and $m = 1$. We choose $h$ (defined in Lemma 3.2) with $c = 7$.




Labeling for $C_6$ under $g$. Labeling for $C_4$ under $h$.




$\mathbb{Z}_k$-antimagic labeling for $D(6,4)$ under $\phi$.

## 4.2. $\mathbb{Z}_k$-antimagic Labelings of $G^{uv}C_{4m+2}$.

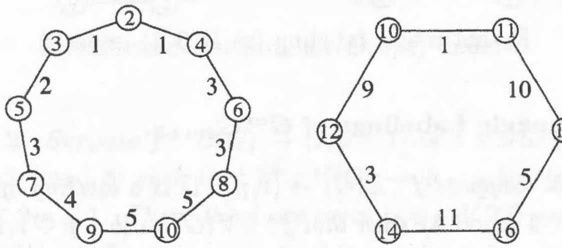
**Theorem 4.3.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 1$ or $3 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $b - p \leq 2m+2$. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m+2})$ such that $G^{uv}C_{4m+2}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m + 2$.*

*Proof.* Putting $c = b - 3$ into Lemma 3.2 yields a labeling $h$ of $C_{4m+2}$ such that $I_h(C_{4m+2}) = c + I_g(C_{4m+2}) = (b-3) + [3, 4m+5] \setminus \{4m+2\} = [b, 4m+b+2] \setminus \{4m+b-1\}$. Note that the range of $h$ is a subset of $[1, 2m+1+2] \cup [b-3+2, b-3+2m+1+1] = [1, 2m+3] \cup [b-1, b-1+2m]$. Choose $u \in V(G)$ with $f^+(u) = b-4$ (this label exists, since $p \geq 5$) and $v \in V(C_{4m+2})$ with $h^+(v) = b$. Join $u$ and $v$ with a bridge and label it $4m+3$. Then, $I_\phi(G^{uv}C_{4m+2}) = [b-p+1, b] \setminus \{b-4\} \cup \{b-4+4m+3\} \cup [b+1, 4m+b+2] \setminus \{4m+b-1\} \cup \{4m+b+3\} = [b-p+1, 4m+b+3] \setminus \{b-4\}$. The set of edge-labels of $\phi$ is a subset of $[1, p-1] \cup \{4m+3\} \cup [1, 2m+3] \cup [b-1, b-1+2m]$.

Note that $b-p \le 2m+2 < 2m+3$, which implies that $b-1+2m < p+4m+2$. Hence, $G^{uv}C_{4m+2}$ is $\mathbb{Z}_k$-antimagic for $k \ge p+4m+3$.

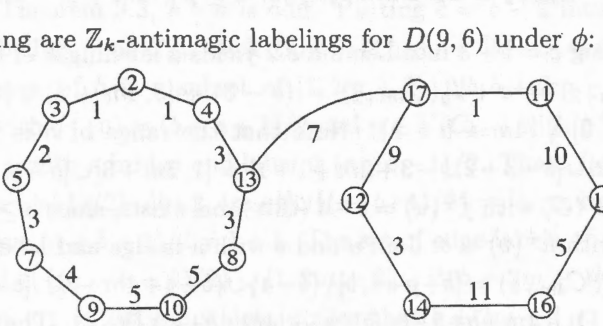
Now, let $k = p+4m+2$. Choose $u \in V(G)$ with $f^+(u) = b-(p+3)/2$ (note that $b-(p+3)/2 \in [b-p+1, b]$) and $v \in V(C_{4m+2})$ with $h^+(v) = b$. Define $\phi(uv) = -(p+3)/2$. Then, $\phi^+(u) = b-p-3 \equiv 4m+b-1 \pmod{p+4m+2}$ and $\phi^+(v) = b-(p+3)/2$. Hence, $I_\phi(G^{uv}C_{4m+2}) = [b-p+1, b] \setminus \{b-(p+3)/2\} \cup \{b-p-3\} \cup [b+1, 4m+b+2] \setminus \{4m+b-1\} \cup \{b-(p+3)/2\} = [b-p+1, 4m+b+2] \setminus \{4m+b-1\} \cup \{b-p-3\}$. Note that $4m+b-1 = b-p-3 \pmod{4m+p+2}$. Thus, all of the elements of $I_\phi(G^{uv}C_{4m+2})$ are distinct, modulo $4m+p+2$. The set of edge-labels of $\phi$ is a subset of $[1, p-1] \cup \{-(p+3)/2\} \cup [1, 2m+3] \cup [b-1, b-1+2m]$. Note that $b-1+2m < p+4m+2$ (as before). Thus, $\phi$ gives a $\mathbb{Z}_{4m+p+2}$-antimagic labeling of $G^{uv}C_{4m+2}$. Hence, $G^{uv}C_{4m+2}$ is $\mathbb{Z}_k$-antimagic, for all $k \ge p+4m+2$. $\square$

*Example.* Here are $\mathbb{Z}_k$-antimagic labelings for $D(9,6)$ following the procedure of Theorem 4.3. In this case, $p=9$, $b=10$ and $m=1$. We choose $h$ (defined in Lemma 3.2) with $c=7$. So, we have

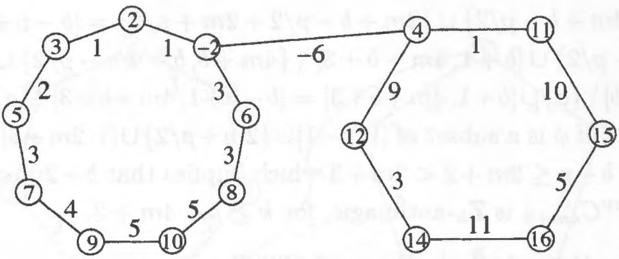


Labeling for $C_9$ under $g$. Labeling for $C_6$ under $h$.

The following are $\mathbb{Z}_k$-antimagic labelings for $D(9,6)$ under $\phi$:



$k \ge 16$.



$k = 15.$

**Theorem 4.4.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 0 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $b - p \leq 2m+4$. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m+2})$ such that $G^{uv}C_{4m+2}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m + 3$.*

*Proof.* Putting $c = b - 4$ into Lemma 3.2 yields a labeling $h$ of $C_{4m+2}$ such that $I_h(C_{4m+2}) = c + I_g(C_{4m+2}) = (b-4) + [3, 4m+5] \setminus \{4m+2\} = [b-1, 4m+b+1] \setminus \{4m+b-2\}$. Note that the range of $h$ is a subset of $[1, 2m+1+2] \cup [b-4+2, b-4+2m+1+1] = [1, 2m+3] \cup [b-2, b-2+2m]$. Choose $u \in V(G)$ with $f^+(u) = b$ and $v \in V(C_{4m+2})$ with $h^+(v) = b-1$. Join $u$ and $v$ with a bridge and label it $-p$. Then, $I_\phi(G^{uv}C_{4m+2}) = [b-p+1, b-1] \cup \{b-p\} \cup [b, 4m+b+1] \setminus \{4m+b-2\} \cup \{b-p-1\} = [b-p-1, 4m+b+1] \setminus \{4m+b-2\}$. The set of edge-labels of $\phi$ is a subset of $[1, p-1] \cup \{-p\} \cup [1, 2m+3] \cup [b-2, b-2+2m]$. Note that $b - p \leq 2m+4 < 2m+5$ which implies that $b - 2 + 2m < 4m + p + 3$. Hence, $G^{uv}C_{4m+2}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m + 3$. □

**Theorem 4.5.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $b - p \leq 2m+2$. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m+2})$ such that $G^{uv}C_{4m+2}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m + 3$.*

*Proof.* Putting $c = b - 2$ into Lemma 3.2 yields a labeling $h$ of $C_{4m+2}$ such that $I_h(C_{4m+2}) = c + I_g(C_{4m+2}) = (b-2) + [3, 4m+5] \setminus \{4m+2\} = [b+1, 4m+b+3] \setminus \{4m+b\}$. Note that the range of $h$ is a subset of $[1, 2m+1+2] \cup [b-2+2, b-2+2m+1+1] = [1, 2m+3] \cup [b, b+2m]$. Choose $u \in V(G)$ with $f^+(u) = b-p$ and $v \in V(C_{4m+2})$ with $h^+(v) = 2m + b - p/2$. Join $u$ and $v$ with a bridge and label it $2m + p/2$. Then, $I_\phi(G^{uv}C_{4m+2}) = [b-p+1, b] \setminus \{a\} \cup \{b-p+2m+p/2\} \cup [b+1, 4m+b+3] \setminus$

$\{4m+b, 2m+b-p/2\} \cup \{2m+b-p/2+2m+p/2\} = [b-p+1, b] \setminus \{a\} \cup \{b+2m-p/2\} \cup [b+1, 4m+b+3] \setminus \{4m+b, b+2m-p/2\} \cup \{4m+b\} = [b-p+1, b] \setminus \{a\} \cup [b+1, 4m+b+3] = [b-p+1, 4m+b+3] \setminus \{a\}$. The set of edge-labels of $\phi$ is a subset of $[1, p-1] \cup \{2m+p/2\} \cup [1, 2m+3] \cup [b, b+2m]$. Note that $b-p \leq 2m+2 < 2m+3$ which implies that $b+2m < p+4m+3$. Hence, $G^{uv}C_{4m+2}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p+4m+3$. □

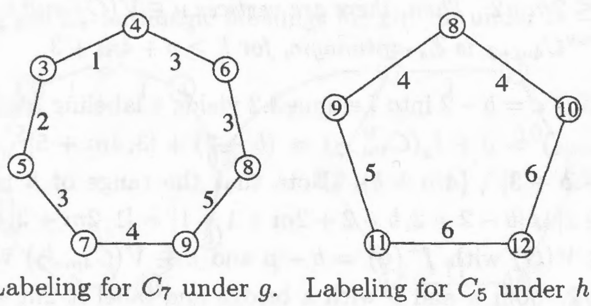
### 4.3. $\mathbb{Z}_k$-antimagic Labelings of $G^{uv}C_{4m+1}$.

**Theorem 4.6.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 0$ or $3 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $b - 2p \leq 4m+1$. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m+1})$ such that $G^{uv}C_{4m+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p+4m+1$.*

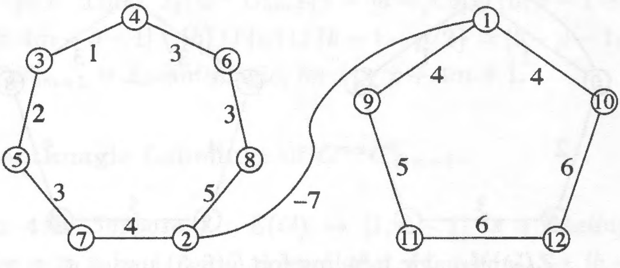
*Proof.* For odd $b$, the claim follows from Lemma 3.6.

For even $b$, the contrapositive of Theorem 3.9 implies that $p \equiv 0 \pmod 4$. Hence, $b - 2p \leq 4m$. Putting $c = b/2 - 1$ into Lemma 3.1 yields a labeling $h$ of $C_{4m+1}$ such that $I_h(C_{4m+1}) = b-2+[2, 4m+2] = [b, 4m+b]$. Note that the range of $h$ is $[\frac{b}{2}, \frac{b}{2}+\lfloor\frac{4m+1}{2}\rfloor]$. Choose $u \in V(G)$ with $f^+(u) = b-p/2$ and $v \in V(C_{4m+1})$ with $h^+(v) = b$. Let $\phi(uv) = -p/2$. Then, $I_\phi(G^{uv}C_{4m+1}) = [b-p+1, b] \setminus \{b-p/2\} \cup \{b-p\} \cup [b+1, 4m+b] \cup \{b-p/2\} = [b-p, 4m+b]$. The set of edge-labels of $\phi$ is $[1, p-1] \cup \{-p/2\} \cup [\frac{b}{2}, \frac{b}{2}+\lfloor\frac{4m+1}{2}\rfloor]$. Note that $b-2p \leq 4m+1 < 4m+2$. This implies that $b+4m < 2p+8m+2$ and hence, $\frac{b}{2}+2m < p+4m+1$. Thus, $\frac{b}{2}+\lfloor\frac{4m+1}{2}\rfloor < p+4m+1$. Therefore, $G^{uv}C_{4m+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p+4m+1$. □

*Example.* Here are $\mathbb{Z}_k$-antimagic labelings for $D(7, 5)$ following the procedure of Theorem 4.6, for $k \geq 12$. In this case, $p = 7$, $b = 9$ and $m = 1$. We choose $h$ (defined in Lemma 3.1) with $c = 3$. So, we have




Labeling for $C_7$ under $g$. Labeling for $C_5$ under $h$.

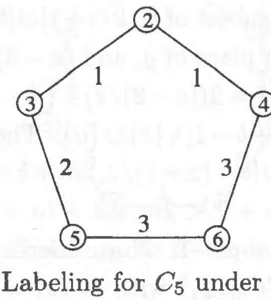


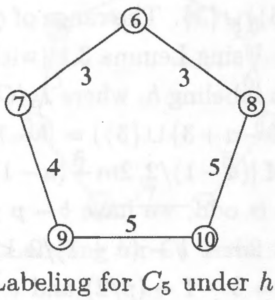

$\mathbb{Z}_k$-antimagic labeling for $D(7,5)$ under $\phi$.

**Theorem 4.7.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 1 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $b - 2p \leq 4m+2$. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m+1})$ such that $G^{uv}C_{4m+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m + 2$.*

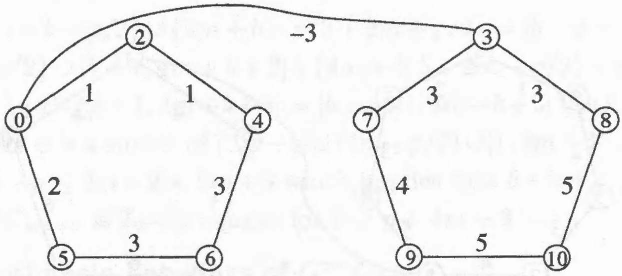
*Proof.* By Theorem 3.7, $b$ is even. Putting $c = b/2 - 1$ into Lemma 3.1 yields a labeling $h$ such that $I_h(C_{4m+1}) = (b-2) + [2, 4m+2] = [b, 4m+b]$. Note that the range of $h$ is $[\frac{b}{2}, \frac{b}{2} + \lfloor \frac{4m+1}{2} \rfloor]$. Choose $u \in V(G)$ with $f^+(u) = b - (p+1)/2$ and $v \in V(C_{4m+1})$ with $h^+(v) = b$. Let $\phi(uv) = -(p+1)/2$. Then, $I_\phi(G^{uv}C_{4m+1}) = [b-p+1, b] \setminus \{b-(p+1)/2\} \cup \{b-p-1\} \cup [b, 4m+b] \setminus \{b\} \cup \{b-(p+1)/2\} = [b-p-1, 4m+b] \setminus \{b-p\}$. The set of edge-labels of $\phi$ is $[1, p-1] \cup \{-(p+1)/2\} \cup [\frac{b}{2}, \frac{b}{2} + \lfloor \frac{4m+1}{2} \rfloor]$. Note that $b - 2p \leq 4m+2 < 4m+4$. This implies that $\frac{b}{2} + 2m < p + 4m + 2$. Thus, $\frac{b}{2} + \lfloor \frac{4m+1}{2} \rfloor < p + 4m + 2$. Therefore, $G^{uv}C_{4m+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m + 2$. □

*Example.* Here are $\mathbb{Z}_k$-antimagic labelings for $D(5,5)$ following the procedure of Theorem 4.7, for $k \geq 11$. In this case, $p = 5$, $b = 6$ and $m = 1$. We choose $h$ (defined in Lemma 3.1) with $c = 2$. So, we have




Labeling for $C_5$ under $g$. Labeling for $C_5$ under $h$.




$\mathbb{Z}_k$-antimagic labeling for $D(5,5)$ under $\phi$.

**Theorem 4.8.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \geq 6$ such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $p \equiv 2 \pmod 4$ and $b - p \leq 2m+2$. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m+1})$ such that $G^{uv}C_{4m+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq 4m+p+1$.*

*Proof.* Suppose that $a \geq 2$ is even. Putting $d = 2$ and $c = b - a$ into Lemma 3.4 yields a labeling $q$, where $I_q(C_{4m+1})$ is the multi-set $([b-a, 4m+b-a] \setminus \{b-a\}) \cup \{2\} = \{2\} \cup [b-a+1, 4m+b-a]$. The range of $q$ is a subset of $[1, 2m+1] \cup [b-a-1, b-a-1+2m]$. Using Lemma 3.1 (with $q$ in place of $g$, and $(a/2)-1$ in place of $c$) yields a labeling $h$, where $I_h(C_{4m+1}) = 2(a/2-1) + ([b-a+1, 4m+b-a] \cup \{2\}) = \{a\} \cup [b-1, 4m+b-2]$. The range of $h$ is a subset of $[a/2, 2m+a/2] \cup [b-a/2-2, b-a/2-2+2m]$. Note that $b-p \leq 2m+2 < 2m+a/2+3$, which implies that $b-a/2-2+2m < 4m+p+1$. Now, choose $u \in V(G)$ with $f^+(u) = b$ and $v \in V(C_{4m+1})$ with $h^+(v) = b-1$. Let $\phi(uv) = 4m$. Then, $I_\phi(G^{uv}C_{4m+1}) = [b-p, b-1] \setminus \{a\} \cup \{b+4m\} \cup [b, 4m+b-2] \cup \{a\} \cup \{b-1+4m\} = [b-p, 4m+b]$. Thus, $G^{uv}C_{4m+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p+4m+1$.

Suppose that $a \geq 3$ is odd. Putting $d = 3$ and $c = b-a+2$ into Lemma 3.4 yields a labeling $q$, where $I_q(C_{4m+1})$ is the multi-set $([b-a+2, 4m+b-a+2] \setminus \{b-a+3\}) \cup \{3\}$. The range of $q$ is a subset of $[1, 2m+1] \cup [b-a+1, b-a+1+2m]$. Using Lemma 3.1 (with $q$ in place of $g$, and $(a-3)/2$ in place of $c$) yields a labeling $h$, where $I_h(C_{4m+1}) = 2((a-3)/2) + ([b-a+2, 4m+b-a+2] \setminus \{b-a+3\} \cup \{3\}) = [b-1, 4m+b-1] \setminus \{b\} \cup \{a\}$. The range of $h$ is a subset of $[(a-1)/2, 2m+(a-1)/2] \cup [b-(a+1)/2, 2m+b-(a+1)/2]$. Since $a \geq 3$ is odd, we have $b-p \leq 2m+2 < 2m+1+(a+1)/2$. This implies that $2m+b-(a+1)/2 < 4m+p+1$. Now, choose $u \in V(G)$ with $f^+(u) = b-1-(p/2)$ and $v \in V(C_{4m+1})$ with $h^+(v) = b-1$. Let

$\phi(uv) = -p/2$. Then, $I_\phi(G^{uv}C_{4m+1}) = [b-p, b] \setminus \{a, b-1-p/2\} \cup \{b-p-1\} \cup [b, 4m+b-1] \setminus \{b\} \cup \{a\} \cup \{b-1-p/2\} = [b-p-1, 4m+b-1]$. Thus, $G^{uv}C_{4m+1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m + 1$. $\square$

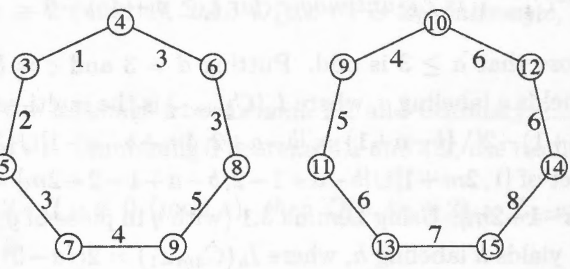
### 4.4. $\mathbb{Z}_k$-antimagic Labelings of $G^{uv}C_{4m-1}$.

**Theorem 4.9.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 0$ or $1 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $b - 2p \leq 4m - 2$. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m-1})$ such that $G^{uv}C_{4m-1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m - 1$.*

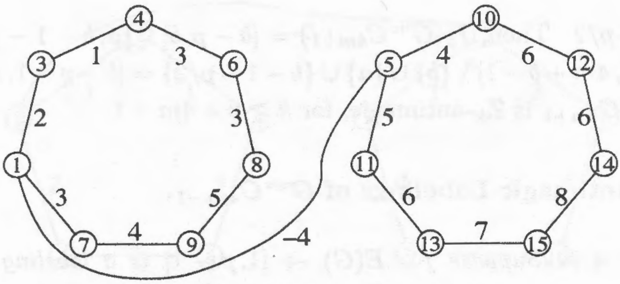
*Proof.* If $b$ is even, then the claim follows from Lemma 3.8.

Now, suppose that $b$ is odd. Then, $b - 2p \leq 4m - 3$. By Theorem 3.7, $p \not\equiv 1 \pmod 4$. Hence, $p \equiv 0 \pmod 4$. Putting $c = (b-3)/2$ into Lemma 3.1 yields a labeling $h$, where $I_h(C_{4m-1}) = (b-3) + [3, 4m+1] = [b, 4m+b-2]$. Note that the range of $h$ is the set $[1 + (b-3)/2, 2 + (b-3)/2 + \lfloor \frac{4m-1}{2} \rfloor] = [(b-1)/2, (b+1)/2 + 2m - 1]$. Now, choose $u \in V(G)$ with $f^+(u) = b - p/2$ and $v \in V(C_{4m-1})$ with $h^+(v) = b$. Let $\phi(uv) = -p/2$. Then, $I_\phi(G^{uv}C_{4m-1}) = [b-p, 4m+b-2]$. The set of edge-labels of $\phi$ is $[1, p-1] \cup \{-p/2\} \cup [(b-1)/2, (b+1)/2 + 2m - 1]$. Note that $b - 2p \leq 4m - 2 < 4m - 1$. This implies that $(b+1)/2 + 2m - 1 < 4m + p - 1$. Thus, $G^{uv}C_{4m-1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m - 1$ $\square$

*Example.* Here are $\mathbb{Z}_k$-antimagic labelings for $D(7,7)$ following the procedure of Theorem 4.9, for $k \geq 15$. In this case, $p = 7$, $b = 9$ and $m = 2$. We choose $h$ (defined in Lemma 3.1) with $c = 3$. So, we have




Labeling for $C_7$ under $g$. Labeling for $C_7$ under $h$.




$\mathbb{Z}_k$-antimagic labeling for $D(7,7)$ under $\phi$.

**Theorem 4.10.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 3 \pmod 4$ such that $f^+ : V(G) \to [b-p+1, b]$ is bijective, where $b - 2p \leq 4m - 1$. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m-1})$ such that $G^{uv}C_{4m-1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m$.*

*Proof.* By Theorem 3.9, $b$ is odd. As in the proof of Theorem 4.9, there is a labeling $h$, where $I_h(C_{4m-1}) = [b, 4m + b - 2]$. Note that the range of $h$ is $[(b-1)/2, (b+1)/2 + 2m - 1]$. Now, choose $u \in V(G)$ with $f^+(u) = b - (p+1)/2$ and $v \in V(C_{4m-1})$ with $h^+(v) = b$. Let $\phi(uv) = -(p+1)/2$. Then, $I_\phi(G^{uv}C_{4m-1}) = [b-p+1, b] \setminus \{b-(p+1)/2\} \cup \{b-p-1\} \cup [b+1, 4m+b-2] \cup \{b-(p+1)/2\} = [b-p-1, 4m+b-2] \setminus \{b-p\}$. The set of edge-labels of $\phi$ is $[1, p-1] \cup \{-(p+1)/2\} \cup [(b-1)/2, (b+1)/2 + 2m - 1]$. Note that $b - 2p \leq 4m - 1 < 4m + 1$. This implies that $(b+1)/2 + 2m - 1 < 4m + p$. Hence, $G^{uv}C_{4m-1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m$. □

**Theorem 4.11.** *Suppose $f : E(G) \to [1, p-1]$ is a labeling of a graph $G$ of order $p \equiv 2 \pmod 4$ such that $f^+ : V(G) \to [b-p, b] \setminus \{a\}$ is bijective, where $b - p \leq 2m$. Then, there are vertices $u \in V(G)$ and $v \in V(C_{4m-1})$ such that $G^{uv}C_{4m-1}$ is $\mathbb{Z}_k$-antimagic, for $k \geq p + 4m - 1$.*

*Proof.* Suppose that $a \geq 3$ is odd. Putting $d = 3$ and $c = b - a + 1$ into Lemma 3.5 yields a labeling $q$, where $I_q(C_{4m-1})$ is the multi-set $\{3\} \cup [b-a+1, 4m+(b-a+1)-2] \setminus \{b-a+1\} = [b-a+2, 4m+b-a-1] \cup \{3\}$. The range of $q$ is a subset of $[1, 2m+1] \cup [b-a+1-2, b-a+1-2+2m] = [1, 2m+1] \cup [b-a-1, b-a-1+2m]$. Using Lemma 3.1 (with $q$ in place of $g$, and $(a-3)/2$ in place of $c$) yields a labeling $h$, where $I_h(C_{4m-1}) = 2((a-3)/2) + ([b-a+2, 4m+b-a-1] \cup \{3\}) = [b-1, 4m+b-4] \cup \{a\}$. The range of $h$ is a subset of $[(a-3)/2+1, (a-3)/2+2m+1] \cup [(a-3)/2+b-a-1, (a-3)/2+b-a-1+2m] =$

$[(a-1)/2, (a-1)/2+2m] \cup [b-(a+5)/2, 2m+b-(a+5)/2]$. Note that $b-p \le 2m < (a+5)/2-1+2m$ (since $a \ge 3$ is odd), which implies that $2m+b-(a+5)/2 < 4m+p-1$. Now, choose $u \in V(G)$ with $f^+(u) = b$ and $v \in V(C_{4m-1})$ with $h^+(v) = b-1$. Let $\phi(uv) = 4m-2$. Then, $I_\phi(G^{uv}C_{4m-1}) = [b-p, b-1] \cup \{b+4m-2\} \cup [b, 4m+b-4] \cup \{b+4m-3\} = [b-p, 4m+b-2]$. Thus, $G^{uv}C_{4m-1}$ is $\mathbb{Z}_k$-antimagic, for all $k \ge p+4m-1$.

Suppose that $a \ge 2$ is even. Putting $d = 4$ and $c = b-a+3$ into Lemma 3.5 yields a labeling $q$, where $I_q(C_{4m-1})$ is the multi-set $\{4\} \cup [b-a+3, 4m+(b-a+3)-2] \setminus \{b-a+4\} = [b-a+3, 4m+b-a+1] \setminus \{b-a+4\} \cup \{4\}$. The range of $q$ is a subset of $[1, 2m+1] \cup [b-a+3-2, b-a+3-2+2m] = [1, 2m+1] \cup [b-a+1, b-a+1+2m]$. Using Lemma 3.1 (with $q$ in place of $g$, and $a/2-2$ in place of $c$) yields a labeling $h$, where $I_h(C_{4m-1}) = 2(a/2-2) + ([b-a+3, 4m+b-a+1] \setminus \{b-a+4\} \cup \{4\}) = [b-1, 4m+b-3] \setminus \{b\} \cup \{a\}$. The range of $h$ is a subset of $[a/2-2+1, a/2-2+2m+1] \cup [a/2-2+b-a+1, a/2-2+b-a+1+2m] = [(a-2)/2, (a-2)/2+2m] \cup [b-(a+2)/2, 2m+b-(a+2)/2]$. Note that $b-p \le 2m < (a+2)/2-1+2m$ (since $a \ge 2$ is even), which implies that $2m+b-(a+2)/2 < 4m+p-1$. Now, choose $u \in V(G)$ with $f^+(u) = b-1-p/2$ and $v \in V(C_{4m-1})$ with $h^+(v) = b-1$. Let $\phi(uv) = -p/2$. Then, $I_\phi(G^{uv}C_{4m-1}) = [b-p, b] \setminus \{a, b-1-p/2\} \cup \{b-p-1\} \cup [b, 4m+b-3] \setminus \{b\} \cup \{a\} \cup \{b-1-p/2\} = [b-p-1, 4m+b-3]$. Thus, $G^{uv}C_{4m-1}$ is $\mathbb{Z}_k$-antimagic, for all $k \ge p+4m-1$. □

## 5. Application to Dumbbell Graphs

The *dumbbell* graph $D(p, s)$ is obtained by joining two cycles $C_p$ and $C_s$ by a bridge, where $p, s \ge 3$.

**Theorem 5.1.** *If $p \not\equiv 2 \pmod 4$, then $D(p, 4m)$ is $\mathbb{Z}_k$-antimagic, for $k \ge p+4m$. If $p \equiv 2 \pmod 4$, then $D(p, 4m)$ is $\mathbb{Z}_k$-antimagic, for $k \ge p+4m+1$.*

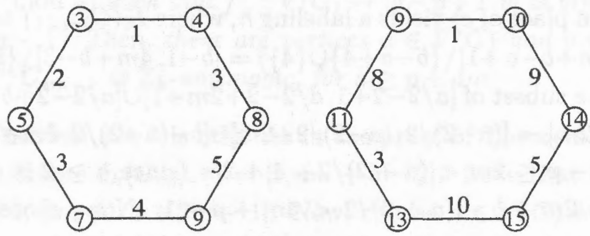
*Proof.* Using the labelings from Lemma 2.2 and Corollary 2.3, we see that $b-p \le 3 \le 2m+1$. Combining Theorems 4.1 and 4.2, the result follows. □

**Theorem 5.2.** *If $p \not\equiv 0 \pmod 4$, then $D(p, 4n+2)$ is $\mathbb{Z}_k$-antimagic, for $k \ge p+4n+2$.*

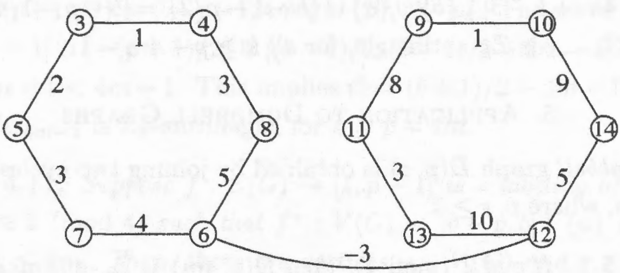
*Proof.* From Corollary 2.3, we see that $b-p \le 3 \le 2n+2$. By Theorem 4.3, the result follows for $r \equiv 1$ or $-1 \pmod 4$.

Suppose $p = 4m+2$ for some $m \geq 1$. In this case, $I_f(C_{4m+2}) = [3, 4m+5] \setminus \{4m+2\}$, where $f$ is the labeling defined in Remark 2.1 for $C_{4m+2}$. By putting $c = 4m+2$ into Lemma 3.2, we get a labeling $h$ of $C_{4n+2}$ such that $I_h(C_{4n+2}) = [4m+5, 4n+4m+7] \setminus \{4n+4m+4\}$. Choose $u \in V(C_{4m+2})$ with $f^+(u) = 4m+5$ and $v \in V(C_{4n+2})$ with $h^+(v) = 4n+4m+7$. Let $\phi(uv) = -3$. Then, $I_\phi(C^{uv}_{4m+2}C_{4n+2}) = [3, 4n+4m+6]$. Hence, $D(4m+2, 4n+2)$ is $\mathbb{Z}_k$-antimagic, for $k \geq 4n+4m+4$. $\square$

*Example.* Here are $\mathbb{Z}_k$-antimagic labelings for $D(6,6)$ following the procedure of Theorem 5.2, for $k \geq 12$. In this case, $p = 6$, $b = 9$ and $n = 1$. We choose $h$ (defined in Lemma 3.2) with $c = 6$. So, we have




Labeling for $C_6$ under $g$. Labeling for $C_6$ under $h$.




$\mathbb{Z}_k$-antimagic labeling for $D(6,6)$ under $\phi$.

**Theorem 5.3.** *$D(4m-1, 4n+1)$ is $\mathbb{Z}_k$-antimagic, for $k \geq 4m+4n$ and $D(4m+1, 4n+1)$ is $\mathbb{Z}_k$-antimagic, for $k \geq 4m+4n+3$.*

**Proof.** From Corollary 2.3 we see $b - 2p < 0$. Combining Theorems 4.6 and 4.7, the result follows. $\square$

**Theorem 5.4.** *For $m, n \geq 1$, $D(4m-1, 4n-1)$ is $\mathbb{Z}_k$-antimagic, for $k \geq 4m+4n-1$.*

**Proof.** From Corollary 2.3, we see $b - 2p < 0$. The result follows from Theorem 4.10. $\square$

Combining Theorems 5.1, 5.2, 5.3 and 5.4, we have

**Theorem 5.5.** *For* $r, s \geq 3$,

$$\mathrm{IAM}(D(r,s)) = \begin{cases} [r+s, \infty) & \text{if } r+s \not\equiv 2 \pmod 4; \\ [r+s+1, \infty) & \text{if } r+s \equiv 2 \pmod 4. \end{cases}$$

## 6. Application to Heavy Dumbbell Graphs and Semi-heavy Dumbbell Graphs

The *heavy dumbbell* graph $HD(p,s)$ is obtained by joining two complete graphs $K_p$ and $K_s$ by a bridge, where $p, s \geq 3$. The *semi-heavy dumbbell* graph $SD(p,s)$ is obtained by joining a cycle $C_p$ and a complete graph $K_s$ by a bridge, where $p, s \geq 3$. Note that $HD(3,s) = SD(3,s)$.

Chan *et al.* [1] proved that any regular Hamiltonian graph of order $p$ is $\mathbb{Z}_k$-antimagic, for all $k \geq p$ or $p+1$, when $p \not\equiv 2 \pmod 4$ or $p \equiv 2 \pmod 4$, respectively. Here, we provide another labeling of $K_p$ so that the image of the induced labeling is the same as that of $C_p$ (given by Lemma 2.2). The results in the preceding sections of this paper are then used to determine the integer-antimagic spectra of heavy dumbbell and semi-heavy dumbbell graphs.

Let the vertex set of $K_p$ be $\{u_1, \ldots, u_p\}$. Let $z$ be an integer with $1 \leq z \leq \lfloor p/2 \rfloor$. We construct a spanning subgraph $K_p(z)$ of $K_p$ in which two vertices $u_i$ and $u_j$ are adjacent if $j \equiv i + z \pmod p$. Then, $K_p(z)$ is a union of $\gcd(z,p)$ cycles (each of order $p/\gcd(z,p)$). Note that if $z = p/2$, then $K_p(z)$ is a perfect matching. Also, observe that $K_p = \bigcup_{z=1}^{\lfloor p/2 \rfloor} K_p(z)$.

**Antimagic labeling for $K_{4m}$:**

$K_{4m} = \bigcup_{z=1}^{2m} K_{4m}(z)$. We label $K_{4m}(1)$, using $g+1$. All edges of $K_{4m}(z)$ are labeled by $-1$ for even $z$, except $z = 2m$. All edges of $K_{4m}(z)$ are labeled by 1 for odd $z$, except $z = 1$. All edges of $K_{4m}(2m)$ are labeled by $-2$. Let this labeling be $f$. Then, $I_f(K_{4m}) = I_g(C_{4m})$.

**Antimagic labeling for $K_{4m+2}$:**

$K_{4m+2} = \bigcup_{z=1}^{2m+1} K_{4m+2}(z)$. We label $K_{4m+2}(1)$, using $g$. All edges of $K_{4m+2}(z)$ are labeled by 1 for even $z$. All edges of $K_{4m+2}(z)$ are labeled by $-1$ for odd $z$, except $z = 1$ and $2m+1$. All edges of $K_{4m+2}(2m+1)$ are labeled by $-2$. Let this labeling be $f$. Then, $I_f(K_{4m+2}) = I_g(C_{4m+2})$.

**Antimagic labeling for $K_{4m-1}$:**

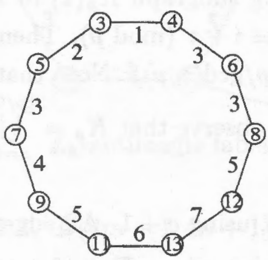
$K_{4m-1} = \bigcup_{z=1}^{2m-1} K_{4m-1}(z)$ for $m \geq 2$. We label $K_{4m-1}(1)$, using $g$. All edges of $K_{4m-1}(z)$ are labeled by 1 for even $z$. All edges of $K_{4m-2}(z)$ are labeled by $-1$ for odd $z$, except $z = 1$. Let this labeling be $f$. Then, $I_f(K_{4m-1}) = I_g(C_{4m-1})$.

**Antimagic labeling for $K_{4m+1}$:**

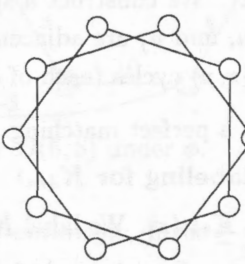
$K_{4m+1} = \bigcup_{z=1}^{2m} K_{4m+1}(z)$. We label $K_{4m+1}(1)$, using $g+1$. All edges of $K_{4m+1}(z)$ are labeled by $-1$ for even $z$. All edges of $K_{4m-2}(z)$ are labeled by 1 for odd $z$, except $z = 1$. Let this labeling be $f$. Then, $I_f(K_{4m+1}) = I_g(C_{4m+1})$.

**Observe that in all of these cases, the domain of $f$ is a subset of $[-2, p-1] \setminus \{0\}$, where $p$ is the order of the graph under consideration.**

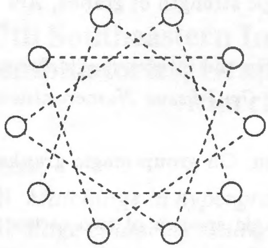
*Example.* $K_{10}$ can be decomposed into $K_{10}(1) = C_{10}$, $K_{10}(2)$, $K_{10}(3)$, $K_{10}(4)$ and $K_{10}(5)$. The labeling of these graphs are:



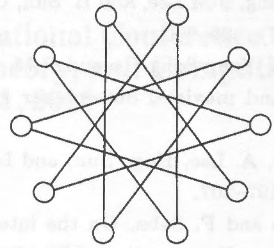

Labeled under $g$.



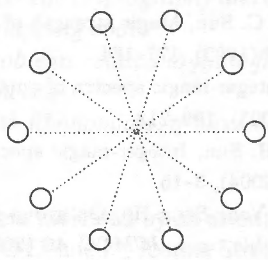
Labeled with 1.



Labeled with $-1$.



Labeled with $1$.



Labeled with $-2$.

Combining all of the subgraphs, we obtain a labeling for $K_{10}$. Clearly, the image of this labeling is $I_g(C_{10}) = [3, 13] \setminus \{10\}$.

If we change the domain of $f$ (described in the lemmas and theorems in Sections 3 and 4) to $[-2, p-1] \setminus \{0\}$, then those results continue to hold. By substituting $G$ by $K_r$ or $C_r$ and $H$ by $K_s$ and using these results and similar arguments as in Section 5, we see that

**Theorem 6.1.** *For $r, s \geq 3$,*

$$\mathrm{IAM}(HD(r,s)) = \mathrm{IAM}(SD(r,s)) = \begin{cases} [r+s, \infty) & \text{if } r+s \not\equiv 2 \pmod 4; \\ [r+s+1, \infty) & \text{if } r+s \equiv 2 \pmod 4. \end{cases}$$

## References


[1] W.H. Chan, R.M. Low and W.C. Shiu, Group-antimagic labelings of graphs, *Congressus Numerantium*, **217** (2013), 21–31.

[2] M. Doob, On the construction of magic graphs, *Proc. Fifth S.E. Conference on Combinatorics, Graph Theory and Computing* (1974), 361–374.

[3] M. Doob, Generalizations of magic graphs, *Journal of Combinatorial Theory, Series B*, **17** (1974), 205–217.

[4] M. Doob, Characterizations of regular magic graphs, *Journal of Combinatorial Theory, Series B*, **25** (1978), 94–104.

[5] M.C. Kong, S-M Lee, and H. Sun, On magic strength of graphs, *Ars Combinatoria*, **45** (1997), 193–200.

[6] S-M Lee, Yong-Song Ho and R.M. Low, On the integer-magic spectra of maximal planar and maximal outerplanar graphs, *Congressus Numerantium*, **168** (2004), 83–90.

[7] S-M Lee, A. Lee, Hugo Sun, and Ixin Wen, On group-magic graphs, JCMCC, **38** (2001), 197–207.

[8] S-M Lee and F. Saba, On the integer-magic spectra of two-vertex sum of paths, *Congressus Numerantium*, **170** (2004), 3–15.

[9] S-M Lee, F. Saba, E. Salehi, and H. Sun, On the $V_4$-group magic graphs, *Congressus Numerantium*, **156** (2002), 59–67.

[10] S-M Lee, F. Saba, and G.C. Sun, Magic strength of the k-th power of paths, *Congressus Numerantium*, **92** (1993), 177–184.

[11] S-M Lee and E. Salehi, Integer-magic spectra of amalgamations of stars and cycles, *Ars Combinatoria*, **67** (2003), 199–212.

[12] S-M Lee, E. Salehi and H. Sun, Integer-magic spectra of trees with diameters at most four, *JCMCC*, **50** (2004), 3–15.

[13] S-M Lee, L. Valdes, and Yong-Song Ho, On group-magic spectra of trees, double trees and abbreviated double trees, *JCMCC*, **46** (2003), 85–95.

[14] R.M. Low and S-M Lee, On the integer-magic spectra of tessellation graphs, *Australas. J. Combin.*, **34** (2006), 195–210.

[15] R.M. Low and S-M Lee, On the products of group-magic graphs, *Australas. J. Combin.*, **34** (2006), 41–48.

[16] R.M. Low and S-M Lee, On group-magic eulerian graphs, *JCMCC*, **50** (2004), 141–148.

[17] R.M. Low and L. Sue, Some new results on the integer-magic spectra of tessellation graphs, *Australas. J. Combin.*, **38** (2007), 255–266.

[18] E. Salehi, Zero-sum magic graphs and their null sets, *Ars Combinatoria*, **82** (2007), 41–53.

[19] E. Salehi, On zero-sum magic graphs and their null sets, *Bulletin of the Institute of Mathematics, Academia Sinica*, **3** (2008), 255–264.

[20] E. Salehi and P. Bennett, On integer-magic spectra of caterpillars, *JCMCC*, **61** (2007), 65–71.

[21] W.C. Shiu and R.M. Low, Group-magicness of complete $N$-partite graphs, *JCMCC*, **58** (2006), 129–134.

[22] W.C. Shiu and R.M. Low, Integer-magic spectra of sun graphs, *J. Comb. Optim.*, **14** (2007), 309–321.

[23] W.C. Shiu and R.M. Low, $Z_k$-magic labelings of fans and wheels with magic-value zero, *Australas. J. Combin.*, **45** (2009), 309–316.

[24] R.P. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.*, **40** (1973), 607–632.

[25] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings, *Duke Math. J.*, **40** (1976), 511–531.

[26] W.D. Wallis, *Magic Graphs*, Birkhauser Boston, (2001).