On the spectral radius of unicyclic graphs with fixed girth*

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Abstract

The spectral radius of a graph is the largest eigenvalue of its adjacency matrix. Let $\mathscr{U}_n^g$ be the set of unicyclic graphs of order n with girth g. For all integers n and g with $5 \leq g \leq n-6$, we determine the first $\lfloor \frac{g}{2} \rfloor + 3$ spectral radii of unicyclic graphs in the set $\mathscr{U}_n^g$.
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1 Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Its adjacency matrix is defined to be the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i is adjacent to v_j; and $a_{ij} = 0$, otherwise. The characteristic polynomial $det(xI - A(G))$ of $A(G)$ is called the characteristic polynomial of G, and is denoted by $\Phi(G, x)$ (or $\Phi(G)$ for short).

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The eigenvalues of G are the eigenvalues of $A(G)$; they are real numbers (since $A(G)$ is symmetric). As usual, $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$ are the eigenvalues of G in non-increasing order. The largest eigenvalue of G, *i.e.* $\rho_1(G)$, is also called the spectral radius of G, denoted by $\rho(G)$. When G is connected, $A(G)$ is irreducible and by the Perron-Frobenius Theorem, the spectral radius is simple and has a unique positive unit eigenvector. We will refer to such an eigenvector as the Perron vector of G. For $v \in V(G)$, let $N_G(v)$ (or $N(v)$ for short) be the set of vertices which are adjacent to v in G and $d_G(v) = |N(v)|$ (or $d(v)$ for short) be the degree of v. A pendent vertex is a vertex of degree 1. Let $d(u,v)$ and $\Delta(G)$ denote the distance between vertices u and v (in G) and the maximum degree of G. We use $G-x$ or $G-xy$ to denote the graph that arises from G by deleting the vertex $x \in V(G)$ and the edges incident with x or the edge $xy \in E(G)$. Similarly, $G+xy$ is a graph that arises from G by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$. Let S_n, C_n and P_n be the star, the cycle and the path of order n, respectively. Readers are referred to [2] for undefined terms.

The investigation on the spectral radius of graphs is an important topic in the theory of graph spectra. Recently, the problem concerning graphs with maximum and minimum spectral radius of a given class of graphs has been studied extensively (see, e.g., [1,4,7,8,10–15,17]).

Let $\mathscr{T}_n^d$ be the set of trees of order n with diameter d $(2 \leq d \leq n-1)$. Obviously, if $T \in \mathscr{T}_n^2$, then T is a star S_n, and if $T \in \mathscr{T}_n^{n-1}$, then T is a path P_n. Very recently, Guo *et al.* [10] and Simić *et al.* [13] characterized the first $\lfloor \frac{d}{2} \rfloor + 1$ spectral radii of trees in the set $\mathscr{T}_n^d$ $(3 \leq d \leq n-4)$, respectively.

In light of the information available for spectral radius of trees, it is natural to consider other classes of graphs, and the unicyclic graphs are a reasonable starting point for such an investigation. Indeed, The spectral radius of unicyclic graphs has been studied by many authors (see Belardo *et al.* [16], Chang *et al.* [4], Simić [14]).

Let $\mathscr{U}_n$ be the set of unicyclic graphs of order n, and $\mathscr{U}_n^g$ be the set of unicyclic graphs of order n with girth g $(3 \leq g \leq n)$. Obviously, if $U \in \mathscr{U}_n^n$, then U is a cycle C_n; if $U \in \mathscr{U}_n^{n-1}$, then $U \cong U^*_{n,n-1}$(shown in Figure 3). In this paper, the first $\lfloor \frac{g}{2} \rfloor + 3$ spectral radii of unicyclic graphs in the set $\mathscr{U}_n^g$ $(5 \leq g \leq n-6)$ are determined, where $\lfloor x \rfloor$ denotes the largest integer no more than x.

2 Preliminaries

In this section, we present some known results which will be used in this paper.

Lemma 2.1 ([5]) *Let G be a graph. Then the eigenvalues of G and $G-v$ interlace, that is*
$\rho_1(G) \geq \rho_1(G-v) \geq \rho_2(G) \geq \rho_2(G-v) \geq \cdots \geq \rho_{n-1}(G-v) \geq \rho_n(G)$.

Lemma 2.2 ([5]) *Let $e = uv$ be an edge of G, and $\mathscr{C}(e)$ be the set of all cycles containing e. Then the characteristic polynomial of G satisfies*
$\Phi(G) = \Phi(G-e) - \Phi(G-u-v) - 2\sum_{C\in\mathscr{C}(e)} \Phi\left(G \setminus V(C)\right)$,
where the summation extends over all $C \in \mathscr{C}(e)$.

A special case of Lemma 2.2 is when $e = uv$ is a cut edge.

Corollary 2.3 *Let $e = uv$ be a cut edge of G. Then*
$\Phi(G) = \Phi(G-e) - \Phi(G-u-v)$.

Note that the spectral radius $\rho(G)$ is just the largest root of $\Phi(G,x) = 0$. Hence, $\Phi(G,x) > 0$ for all $x > \rho(G)$. Accordingly, the following lemma is often used to compare the spectral radii of graphs.

Lemma 2.4 ([6,17]) *Let G_1 and G_2 be two graphs.*
(i) If G_2 is a proper spanning subgraph of G_1, then $\rho(G_1) > \rho(G_2)$ and $\Phi(G_2) > \Phi(G_1)$ for $x \geq \rho(G_1)$.
(ii) If $\Phi(G_2) > \Phi(G_1)$ for $x \geq \rho(G_2)$, then $\rho(G_1) > \rho(G_2)$.

Recall that $\rho(S_n) = \sqrt{n-1}$. Then Lemma 2.4 implies that $\rho(G) \geq \rho(S_{\Delta+1}) = \sqrt{\Delta}$ for any G of order n with maximum degree Δ, since $S_{\Delta+1} \cup (n-\Delta-1)S_1$ is a spanning subgraph of G.

Lemma 2.5 ([17]) *Let v be a vertex of a graph G and suppose that two new paths $P: vv_1v_2\cdots v_k$ and $Q: vu_1u_2\cdots u_m$ of length k, m $(k \geq m \geq 1)$ are attached to G at v, respectively, to form a new graph $G_{k,m}$, where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_m$ are distinct. Then for $x > \rho(G_{k,m})$, we have $\Phi(G_{k+1,m-1}) > \Phi(G_{k,m})$. In particular, $\rho(G_{k,m}) > \rho(G_{k+1,m-1})$.*

Lemma 2.6 ([10]) *Let $a = \frac{x+\sqrt{x^2-4}}{2}$ and $b = \frac{x-\sqrt{x^2-4}}{2}$. Then*
$\Phi(P_n) = \frac{1}{\sqrt{x^2-4}}(a^{n+1} - b^{n+1})$.

Lemma 2.7 ([6]) *Let u, v be two vertices of a connected graph G. Suppose that $v_1, v_2, \ldots, v_s \in N(v) \backslash N(u)$ $(1 \leq s \leq d(v))$ and $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ is the Perron vector of G, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\rho(G) < \rho(G^*)$.*

From Lemma 2.7, the following corollary is immediate.

Corollary 2.8 *Let $e = uv$ (does not belong to C_3) be an edge of a connected graph G with $d(u) \geq 2$ and $d(v) \geq 2$. Let G' be the graph obtained from $G - uv$ by identifying u with v to form a new vertex w together with attaching a new pendant vertex w' to w. Then $\rho(G') > \rho(G)$.*

To state the next result (due to Hoffman and Smith [9]), we need more definitions. An internal path in a graph, denoted by $v_1, v_2, \ldots, v_{r-1}, v_r$, is a path joining vertices v_1 and v_r which are both of degree greater than two (not necessarily distinct), while all other vertices (i.e. $v_2, \ldots, v_{r-1}$) are of degree equal to 2. We denote by C_n and W_n the cycle and the double-snake (the tree on n vertices having two vertices of degree three which are at distance $n-5$).

Lemma 2.9 ([9]) *Let G' be a graph obtained from a graph $G \neq C_n, W_n$ by inserting a vertex of degree 2 in an edge e. Then the following holds:*

(i) If e belongs to an internal path, then $\rho(G) > \rho(G')$;

(ii) If e does not belong to an internal path, then $\rho(G) < \rho(G')$.

If $G = C_n$ (W_n) and $G' = C_{n+1}$(resp. W_{n+1}), then $\rho(G') = \rho(G) = 2$.

Lemma 2.10 ([3, 14]) *For any $U \in \mathscr{U}_n$, we have $\rho(C_n) = 2 \leq \rho(U) \leq \rho(S_n^*)$, where S_n^* denotes the graph obtained from S_n by joining any two vertices of degree one in S_n. The lower bound is attained if and only if $U \cong C_n$; the upper bound is attained if and only if $U \cong S_n^*$.*

Note that the characteristic polynomial of S_n^* is $\Phi(S_n^*) = x^{n-4}(x+1)[x^3 - x^2 - (n-1)x + n - 3]$. Let $f(x) = x^3 - x^2 - (n-1)x + n - 3$ and $x_1 \geq x_2 \geq x_3$ be three roots of $f(x) = 0$. Then $\rho(S_n^*) = x_1$. Note that $f(-\sqrt{n-1}) = -2 < 0$, $f(0) = n - 3 > 0$ for $n > 3$, $f(1) = -2 < 0$ and $f(\sqrt{n}) = \sqrt{3} - 3 \geq 0$ for $n \geq 9$. Then we have $x_3 < 0$, $x_2 \in (0, 1)$ and $x_1 = \rho(S_n^*) \leq \sqrt{n}$ for $n \geq 9$.

Recall that $\mathscr{U}_n^g$ is the set of all unicyclic graphs of order n with girth g. Then for each $U \in \mathscr{U}_n^g$, U consists of the (unique) cycle (say C_g) of length g and a certain number of trees attached at vertices of C_g having (in total) $n-g$ edges. We assume that the vertices of C_g are $v_1, v_2, \ldots, v_g$ (order in a natural way around C_g, say in the clockwise direction). Then U can be written as $C(T_1, T_2, \ldots, T_g)$, which is obtained from a cycle C_g on vertices $v_1, v_2, \ldots, v_g$ by identifying v_i with the root of a tree T_i of order n_i for each $i = 1, 2, \ldots, g$, where $n_i \geq 1$ and $\sum_{i=1}^{g} n_i = n$. If T_i, for each i, is a path of order n_i, whose root is a vertex of minimum degree, then we write $U = P(n_1, n_2, \ldots, n_g)$; If T_i, for each i, is a star of order n_i, whose root is a vertex of maximum degree, then we write $U = S(n_1, n_2, \ldots, n_g)$.

Lemma 2.11 ([16]) *Let $U \in C(T_1, T_2, \ldots, T_g)$, where $|V(T_i)| = n_i$ for $i = 1, 2, \ldots, g$ and $\sum_{i=1}^{g} n_i = n$. Then*

$$\rho(P(n_1, n_2, \ldots, n_g)) \leq \rho(U) \leq \rho(S(n_1, n_2, \ldots, n_g)),$$

where the degree of the root in P_{n_i} (S_{n_i}) is 1 (resp. $n_i - 1$). Moreover, both extremal graphs are unique.

3 Main results

Firstly, we introduce some notation to be used in this section. Let $T_{n,d}(i)$(shown in Figure 1) be the tree on n vertices (with diameter d) obtained from a path $P_{d+1} : v_1 v_2 \cdots v_d v_{d+1}$ (of length d) by attaching $n-d-1$ new pendant edges $v_i v_{d+2}, v_i v_{d+3}, \ldots v_i v_n$ to the vertex v_i, where $2 \leq d \leq n-2$ and $2 \leq i \leq d$. . Clearly, $T_{n,d}(i) = T_{n,d}(d+2-i)$ for $i = 2, \ldots, d$.

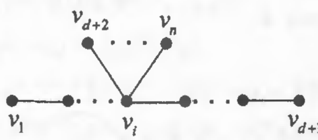

Figure 1: Trees $T_{n,d}(i)$, where $i = 2, 3, \ldots, d$.

Let $U_{n,g}(i)$(shown in Figure 2) be the unicyclic graph on n vertices (with girth g) obtained from a cycle $C_g : v_1 v_2 \cdots v_g v_1$ (of length g) by attaching $n-g-1$ new pendant edges $v_1 v_{g+1}, v_1 v_{g+2}, \ldots, v_1 v_{n-1}$ to the vertex v_1 and a new pendant edge v_iv_n to the vertex v_i, respectively, where $3 \le g \le n-2$ and $2 \le i \le g$. Clearly, $U_{n,g}(i) = U_{n,d}(g+2-i)$ for $i = 2, \ldots, g$. Let $\mathscr{U}_{n,g} = \{U_{n,g}(i) : i = 2, 3, \ldots, \lfloor \frac{g}{2} \rfloor + 1\}$. Then $\mathscr{U}_{n,g} \subset \mathscr{U}_n^g$. Let $U'_{n,g}(i)$(shown in Figure 2) be the unicyclic graph on n vertices (with girth g) obtained from a cycle $C_g : v_1v_2 \cdots v_gv_1$ (of length g) by attaching $n-g-2$ new pendant edges $v_1v_{g+1}, v_1v_{g+2}, \ldots, v_1v_{n-2}$ to the vertex v_1 and two new pendant edges v_iv_{n-1}, v_iv_n to the vertex v_i, respectively, where $3 \le g \le n-3$ and $2 \le i \le g$. Clearly $U'_{n,g}(i) = U'_{n,d}(g+2-i)$ for $i = 2, \ldots, g$. Let $\mathscr{U}'_{n,g} = \{U'_{n,g}(i) : i = 2, 3, \ldots, \lfloor \frac{g}{2} \rfloor + 1\}$. Clearly, $\mathscr{U}'_{n,g} \subset \mathscr{U}_n^g$, and $\mathscr{U}'_{n,n-3} = \mathscr{U}_{n,n-3}$.

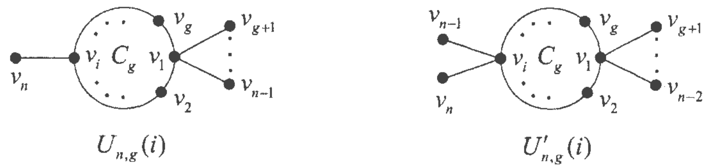

Figure 2: Unicyclic graphs $U_{n,g}(i)$ and $U'_{n,g}(i)$, where $i = 2, \ldots, \lfloor \frac{g}{2} \rfloor + 1$.

Let $U^*_{n,g}$ be the unicyclic graph on n vertices (with girth g) obtained from a cycle $C_g : v_1v_2 \cdots v_gv_1$ (of length g) by attaching $n-g$ new pendant edges $v_1v_{g+1}, v_1v_{g+2}, \ldots, v_1v_n$ to the vertex v_1. Let $U^+_{n,g}$ be the unicyclic graph on n vertices (with girth g) obtained from a cycle $C_g : v_1v_2 \cdots v_gv_1$ (of length g) by attaching a new path $P_3 : v_1v_{g+1}v_n$(of length 2) and $n-g-2$ new pendant edges $v_1v_{g+2}, v_1v_{g+3}, \ldots, v_1v_{n-1}$ to the vertex v_1, respectively. $U^*_{n,g}$ and $U^+_{n,g}$ are shown in Figure 3. Similarly, we can define the following four unicyclic graphs(of order n with girth g) $U^1_{n,g}, U^2_{n,g}, U^3_{n,g}$ and $U^4_{n,g}$, which are shown in Figure 4.

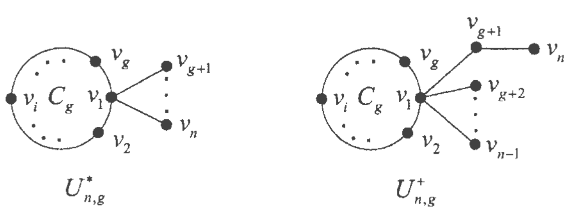

Figure 3: Unicyclic graphs $U^*_{n,g}$ and $U^+_{n,g}$.

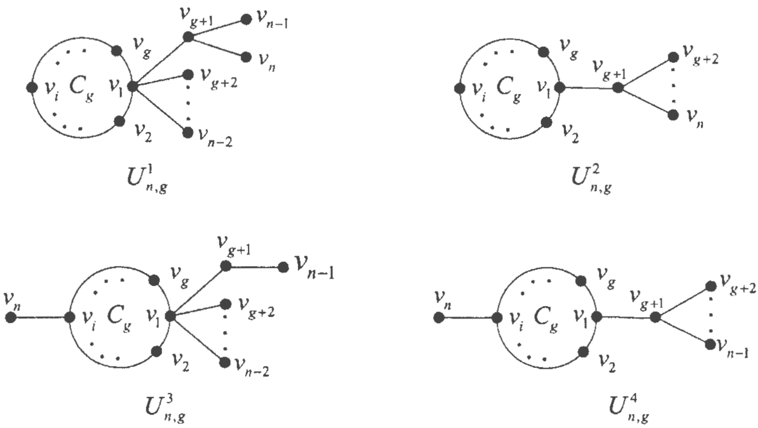

Figure 4: Unicyclic graphs $U^1_{n,g}$, $U^2_{n,g}$, $U^3_{n,g}$ and $U^4_{n,g}$.

Lemma 3.1 *For $n \geq g+4$, we have $\rho(U^1_{n,g}) > \rho(U^2_{n,g})$.*

Proof. Take $e = v_1v_{g+1}$ in Corollary 2.3, we have

$$\Phi(U^1_{n,g}) = \Phi(P_3)\Phi(U^*_{n-3,g}) - x^{n-g-1}\Phi(P_{g-1}) \text{ and}$$
$$\Phi(U^2_{n,g}) = \Phi(S_{n-g})\Phi(C_g) - x^{n-g-1}\Phi(P_{g-1}).$$

Note that $\Phi(C_g) = x\Phi(P_{g-1}) - 2\Phi(P_{g-2}) - 2$ and $P_{g-2} \cup K_1$ is a proper spanning subgraph of P_{g-1}. Therefore, by Lemmas 2.2 and 2.4(i), for $x \geq \rho(U^2_{n,g}) > \rho(P_{g-2})$, we have

$$\begin{aligned}
&\Phi(U^2_{n,g}) - \Phi(U^1_{n,g})\\
=&\Phi(S_{n-g})\Phi(C_g) - \Phi(P_3)\Phi(U^*_{n-3,g})\\
=&x^{n-g-2}[x^2 - (n-g-1)]\Phi(C_g)\\
&- x(x^2-2)\left[x^{n-g-3}\Phi(C_g) - (n-g-3)x^{n-g-4}\Phi(P_{g-1})\right]\\
=&(n-g-3)x^{n-g-3}\left[(x^2-2)\Phi(P_{g-1}) - x\Phi(C_g)\right]\\
=&2(n-g-3)x^{n-g-3}\left[\underbrace{x\Phi(P_{g-2}) - \Phi(P_{g-1})}_{>0} + x\right] > 0.
\end{aligned}$$

Thus Lemma 2.4(ii) implies that $\rho(U^1_{n,g}) > \rho(U^2_{n,g})$. □

Theorem 3.2 *For any $C(T_1,\ldots,T_g) \in \mathscr{U}_n^g$ $(n \geq g+4)$ with $n_2 = \cdots = n_g = 1$, where $|V(T_i)| = n_i$ for $i = 1,\ldots,g$, if $C(T_1,\ldots,T_g) \notin \{U^*_{n,g}, U^+_{n,g}, U^1_{n,g}\}$, then we have $\rho(C(T_1,\ldots,T_g)) < \rho(U^1_{n,g})$. Moreover, $\rho(U^*_{n,g}) > \rho(U^+_{n,g}) > \rho(U^1_{n,g})$.*

Proof. By Lemma 2.7, we have $\rho(U^*_{n,g}) > \rho(U^+_{n,g}) > \rho(U^1_{n,g})$ since $n \geq g+4$. Let $v_n \in V(T_1)$ be a pendent vertex such that $d(v_1, v_n)$ is maximum. Then $d(v_1, v_n) \geq 2$ since $C(T_1,\ldots,T_g) \neq U^*_{n,g}$. We consider the following two cases.

Case 1. $d(v_1, v_n) = 2$.

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be the Perron vector of $C(T_1,\ldots,T_g)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$, and $N(v_1) = \{v_2, v_g, v_{g+1}, \ldots, v_{g+t}\}$, where $t = d(v_1) - 2$. Then $t \geq 1$.

If $t = 1$, then $C(T_1,\ldots,T_g) \cong U^2_{n,g}$ since $d(v_1, v_n) = 2$. Then the result follows from Lemma 3.1;

If $t \geq 2$, then let $x_{g+i} = \max\{x_{g+1},\ldots,x_{g+t}\}$ $(i \in \{1,\ldots,t\})$. By Lemma 2.7, we can construct a new graph U^* such that $\rho(C(T_1,\ldots,T_g)) < \rho(U^*)$, where U^* obtained from $C(T_1,\ldots,T_g)$ by deleting all pendent edges, and then adding new edges between v_{g+i} and all isolated vertices. Then $d_{U^*}(v_{g+i}) \geq 2$ since $C(T_1,\ldots,T_g) \neq U^+_{n,g}$. If $U^* \cong U^1_{n,g}$, then the result follows. If $U^* \neq U^1_{n,g}$, then $d_{U^*}(v_{g+i}) \geq 3$. Moreover, Lemma 2.7 implies that

$$\rho(U^*) < \begin{cases} \rho(U^1_{n,g}) & \text{if } x_1 \geq x_{g+i}, \\ \rho(U^2_{n,g}) & \text{if } x_1 \leq x_{g+i}. \end{cases}$$

Then by Lemma 3.1, we have $\rho(U^*) < \rho(U^1_{n,g})$, the result follows.

Case 2. $d(v_1, v_n) \geq 3$.

By Corollary 2.8, we can construct a new graph U' such that the maximum distance between v_1 and a pendent vertex in U' is 2 and $\rho(C(T_1,\ldots,T_g)) < \rho(U')$. Then by a same argument as Case 1, we have $\rho(U') < \rho(U^1_{n,g})$, the result follows.

The proof is completed. □

Lemma 3.3 *For $n \geq g+5$, we have $\rho(U^3_{n,g}) > \rho(U^4_{n,g})$.*

Proof. Recall that $a = \frac{x+\sqrt{x^2-4}}{2}$, $b = \frac{x-\sqrt{x^2-4}}{2}$ and $i \leq \lfloor \frac{g}{2} \rfloor + 1$. Then $ab = 1$ and $g - i + 1 \geq i$. And for $x \geq \frac{3\sqrt{2}}{2}$, we have

$2x^2\sqrt{x^2-4}-(x^2-1)a > \frac{(x^2-1)}{2}\left[3\sqrt{x^2-4}-x\right] > 0.$

Then take $e = v_1v_{g+1}$ in Corollary 2.3, and by Lemmas 2.2 and 2.6, for $x > \rho(U^3_{n,g}) > \rho(S_6) = \sqrt{5} > \frac{3\sqrt{2}}{2}$ since $n \geq g+5$, we have

$$
\begin{aligned}
&\Phi(U^4_{n,g}) - \Phi(U^3_{n,g})\\
=&\Phi(S_{n-g-1})\Phi(U^*_{g+1,g}) - \Phi(P_2)\Phi(U_{n-2,g}(i))\\
=&(n-g-3)x^{n-g-4}\left[2x^2\Phi(P_{g-2}) + 2x^2 - (x^2-1)\Phi(P_{i-2})\Phi(P_{g-i})\right]\\
=&\frac{(n-g-3)x^{n-g-4}}{x^2-4}[2x^2\sqrt{x^2-4}(a^{g-1}-b^{g-1})\\
&-(x^2-1)(a^{i-1}-b^{i-1})(a^{g-i+1}-b^{g-i+1}) + \underbrace{2x^2(x^2-4)}_{>0}]\\
>&\frac{(n-g-3)x^{n-g-4}}{x^2-4}[2x^2\sqrt{x^2-4}(a^{g-1}-b^{g-1})\\
&-(x^2-1)(a^g - a^{g-i+1}b^{i-1} + \underbrace{b^g - a^{i-1}b^{g-i+1}}_{<0})]\\
>&\frac{(n-g-3)x^{n-g-4}}{x^2-4}\{[2x^2\sqrt{x^2-4}-(x^2-1)a]a^{g-1}\\
&-2x^2\sqrt{x^2-4}b^{g-1} + (x^2-1)\underbrace{a^{g-i+1}b^{i-1}}_{>ab^{g-1}}\}\\
>&\frac{(n-g-3)x^{n-g-4}}{x^2-4}\left[2x^2\sqrt{x^2-4}-(x^2-1)a\right](a^{g-1}-b^{g-1}) > 0.
\end{aligned}
$$

Thus Lemma 2.4(ii) implies that $\rho(U^3_{n,g}) > \rho(U^4_{n,g})$. □

By Lemmas 2.7 and 3.3, and a similar argument as Theorem 3.2, we have the following theorem.

Theorem 3.4 *For any $C(T_1,\ldots,T_g) \in \mathscr{U}^g_n$ $(n \geq g+5)$ with $n_i = 2(i \in \{2,\ldots,\lfloor\frac{g}{2}\rfloor+1\})$ and $n_2 = \cdots = n_{i-1} = n_{i+1} = \cdots = n_g = 1$, where $|V(T_j)| = n_j$ for $j = 1,\ldots,g$, if $C(T_1,\ldots,T_g) \notin \{U_{n,g}(i), U^3_{n,g}\}$, then we have $\rho(U) < \rho(U^3_{n,g})$. Moreover, $\rho(U_{n,g}(i)) > \rho(U^3_{n,g})$.*

Lemma 3.5 *For any $2 \leq i < j \leq \lfloor\frac{g}{2}\rfloor+1$, we have $\rho(U_{n,g}(i)) > \rho(U_{n,g}(j))$. Moreover, $\rho(U^*_{n,g}) > \rho(U_{n,g}(2)) > \rho(U_{n,g}(3)) > \cdots > \rho\left(U_{n,g}\left(\lfloor\frac{g}{2}\rfloor+1\right)\right)$.*

Proof. In order to obtain the desired result, we only need to prove the case $j = i+1$. From Corollary 2.3, we have

$$\Phi(U_{n,g}(i)) = x\Phi(U_{n,g}(i) - v_iv_n) - \Phi(U_{n,g}(i)\setminus\{v_i,v_n\}) \quad \text{and}$$
$$\Phi(U_{n,g}(i+1)) = x\Phi(U_{n,g}(i+1) - v_{i+1}v_n) - \Phi(U_{n,g}(i+1)\setminus\{v_{i+1},v_n\}).$$

Note that $U_{n,g}(i)\setminus\{v_i, v_n\} \cong T_{n-2,g-2}(i-1)$ and $U_{n,g}(i+1)\setminus\{v_{i+1}, v_n\} \cong T_{n-2,g-2}(i)$. Then take $v = v_1$ in Lemma 2.5, we have $\Phi(U_{n,g}(i)\setminus\{v_i, v_n\}) > \Phi(U_{n,g}(i+1)\setminus\{v_{i+1}, v_n\})$, for $x > \rho(U_{n,g}(i+1)\setminus\{v_{i+1}, v_n\})$. Moreover, $U_{n,g}(i) - v_i v_n = U_{n,g}(i+1) - v_{i+1}v_n$. Then for $x > \rho(U_{n,g}(i+1)) > \rho(U_{n,g}(i+1)\setminus\{v_{i+1}, v_n\})$, we have $\Phi(U_{n,g}(i+1)) - \Phi(U_{n,g}(i)) = \Phi(U_{n,g}(i)\setminus\{v_i, v_n\}) - \Phi(U_{n,g}(i+1)\setminus\{v_{i+1}, v_n\}) > 0$. Thus, Lemma 2.4(ii) implies that $\rho(U_{n,g}(i)) > \rho(U_{n,g}(i+1))$ for $i = 2, \ldots, \lfloor \frac{g}{2} \rfloor$. Moreover, Lemma 2.7 implies that $\rho(U^*_{n,g}) > \rho(U_{n,g}(2))$. Then the result follows. □

Note that

$$\begin{aligned}\Phi(U'_{n,g}(i)) =& x^2\Phi(U'_{n,g}(i) - v_i v_{n-1} - v_i v_n)\\ &- 2x\Phi(U'_{n,g}(i)\setminus\{v_i, v_{n-1}, v_n\}) \quad \text{and}\\ \Phi(U'_{n,g}(i+1)) =& x^2\Phi(U'_{n,g}(i+1) - v_{i+1}v_{n-1} - v_{i+1}v_n)\\ &- 2x\Phi(U'_{n,g}(i+1)\setminus\{v_{i+1}, v_{n-1}, v_n\});\end{aligned}$$

$U'_{n,g}(i) - v_i v_{n-1} - v_i v_n = U'_{n,g}(i+1) - v_{i+1}v_{n-1} - v_{i+1}v_n$ and $U'_{n,g}(i)\setminus\{v_i, v_{n-1}, v_n\} \cong T_{n-3,g-2}(i-1)$ and $U'_{n,g}(i+1)\setminus\{v_{i+1}, v_{n-1}, v_n\} \cong T_{n-3,g-2}(i)$. Then similar reasoning implies that the following result holds.

Lemma 3.6 *For any $2 \le i < j \le \lfloor \frac{g}{2} \rfloor + 1$, we have $\rho(U'_{n,g}(i)) > \rho(U'_{n,g}(j))$. Moreover, $\rho(U^*_{n,g}) > \rho\left(U'_{n,g}(2)\right) > \rho\left(U'_{n,g}(3)\right) > \cdots > \rho\left(U'_{n,g}\left(\lfloor \frac{g}{2} \rfloor + 1\right)\right)$.*

Lemma 3.7 *For $n \ge g+6$ and $g \ge 5$, we have $\rho(U_{n,g}(2)) > \rho(U^+_{n,g}) > \rho(U_{n,g}(3))$.*

Proof. Note that $U_{n,g}(2)\setminus\{v_2, v_n\} \cong T_{n-2,g-1}(2)$ and $U_{n,g}(3)\setminus\{v_3, v_n\} \cong T_{n-2,g-2}(2)$. Then from Corollary 2.3, we have

$$\begin{aligned}\Phi(U_{n,g}(2)) &= x\Phi(U^*_{n-1,g}) - \Phi(T_{n-2,g-1}(2)),\\ \Phi(U^+_{n,g}) &= x\Phi(U^*_{n-1,g}) - \Phi(U^*_{n-2,g}) \quad \text{and}\\ \Phi(U_{n,g}(3)) &= x\Phi(U^*_{n-1,g}) - \Phi(T_{n-2,g-2}(2)).\end{aligned}$$

Since $T_{n-2,g-1}(2)$ is a proper spanning subgraph of $U^*_{n-2,g}$, Lemma 2.4(i) implies that $\Phi(T_{n-2,g-1}(2)) - \Phi(U^*_{n-2,g}) > 0$ for $x \ge \rho(U^*_{n-2,g})$. Thus for $x > \rho(U^+_{n,g}) > \rho(U^*_{n-2,g})$, we have $\Phi(U^+_{n,g}) - \Phi(U_{n,g}(2)) = \Phi(T_{n-2,g-1}(2)) - \Phi(U^*_{n-2,g}) > 0$. Then Lemma 2.4(ii) implies that $\rho(U^+_{n,g}) < \rho(U_{n,g}(2))$.

Recall that $\Phi(C_n) = \Phi(P_n) - \Phi(P_{n-2}) - 2$ and $\Phi(P_n) = x\Phi(P_{n-1}) - \Phi(P_{n-2})$. By Lemma 2.2, we have

$$\begin{aligned}
&\Phi(U_{n,g}(3)) - \Phi(U_{n,g}^{+}) \\
=&\Phi(U_{n-2,g}^{*}) - \Phi(T_{n-2,g-2}(2)) \\
=& \left[x^{n-g-2}\Phi(C_g) - (n-g-2)x^{n-g-3}\Phi(P_{g-1})\right] \\
&- \left[x^{n-g-1}\Phi(P_{g-1}) - (n-g-1)x^{n-g-1}\Phi(P_{g-3})\right] \\
=&x^{n-g-3}\left\{(n-g)x\Phi(P_{g-4}) + [(n-g-2) - x^2]\Phi(P_{g-3}) - 2x\right\} \qquad (3.1)
\end{aligned}$$

Note that $\Delta(U_{n,g}^{+}) = \Delta(U_{n,g}(3)) = n-g+1$. Then $\rho(U_{n,g}^{+}), \rho(U_{n,g}(3)) \geq \sqrt{n-g+1}$. Moreover, Lemmas 2.7 and 2.9 imply that $\rho(U_{n,g}^{+}), \rho(U_{n,g}(3)) < \rho(U_{n,g}^{*}) \leq \rho(S_{n-g+3}^{*}) \leq \sqrt{n-g+3}$. That is $\sqrt{7} \leq \sqrt{n-g+1} \leq \rho(U_{n,g}^{+})$, $\rho(U_{n,g}(3)) \leq \sqrt{n-g+3}$ since $n-g \geq 6$. Recall that $a = \frac{x+\sqrt{x^2-4}}{2}$ and $b = \frac{x-\sqrt{x^2-4}}{2}$. Then $ab = 1$, $a^{n+1} - b^{n+1} \geq a^n - b^n$ and $x(a^2-b^2) - 2x\sqrt{x^2-4} = x\sqrt{x^2-4}(x-2) \geq 0$ for $x \geq 2$. Then by (3.1) and Lemma 2.6, for $\sqrt{7} \leq \sqrt{n-g+1} \leq x \leq \sqrt{n-g+3}$, we have

$$\begin{aligned}
&\Phi(U_{n,g}(3)) - \Phi(U_{n,g}^{+}) \\
=&\frac{x^{n-g-3}}{\sqrt{x^2-4}}\{(n-g)x(a^{g-3} - b^{g-3}) + [(n-g-2) - x^2](a^{g-2} - b^{g-2}) \\
&- 2x\sqrt{x^2-4}\} \\
=&\frac{x^{n-g-3}}{\sqrt{x^2-4}}\{(n-g-1)x(a^{g-3} - b^{g-3}) + [(n-g-2) - x^2](a^{g-2} - b^{g-2}) \\
&+ x\underbrace{(a^{g-3} - b^{g-3})}_{\geq a^2-b^2} - 2x\sqrt{x^2-4}\} \\
\geq&\frac{x^{n-g-3}}{\sqrt{x^2-4}}\{(n-g-1)x(a^{g-3} - b^{g-3}) + [(n-g-2) - x^2](a^{g-2} - b^{g-2}) \\
&+ \underbrace{x(a^2-b^2) - 2x\sqrt{x^2-4}}_{\geq 0}\} \\
\geq&\frac{x^{n-g-3}}{\sqrt{x^2-4}}\{\underbrace{(n-g-1)}_{\geq 5}x(a^{g-3} - b^{g-3}) - \underbrace{[x^2 - (n-g-2)]}_{\leq 5}(a^{g-2} - b^{g-2})\} \\
\geq&\frac{x^{n-g-3}}{\sqrt{x^2-4}}\left\{5x(a^{g-3} - b^{g-3}) - 5(a^{g-2} - b^{g-2})\right\} \\
=&\frac{5x^{n-g-3}}{\sqrt{x^2-4}}(a^{g-4} - b^{g-4}) > 0. \qquad (3.2)
\end{aligned}$$

Moreover, note that $U_{n,g}(3)-v_1 = T_{g,g-2}(2)\cup(n-g-1)S_1$ and $U^+_{n,g}-v_1 = P_{g-1}\cup P_2\cup(n-g-2)S_1$. Then Lemma 2.1 implies that $\rho_2(U_{n,g}(3)) \le \rho(U_{n,g}(3)-v_1) = \rho(T_{g,g-2}(2)) < 2$ and $\rho_2(U^+_{n,g}) \le \rho(U^+_{n,g}-v_1) = \rho(P_{g-1}) < 2$. Recall that $\rho(U_{n,g}(3)), \rho(U^+_{n,g}) \ge \sqrt{7} > 2$. Then (3.2) implies that $\rho(U^+_{n,g}) > \rho(U_{n,g}(3))$. $\square$

Lemma 3.8 *For $n \ge g+5$, we have $\rho\left(U_{n,g}(\lfloor \frac{g}{2}\rfloor+1)\right) > \rho(U'_{n,g}(2))$.*

Proof. From Corollary 2.3, we have

$$\Phi\left(U_{n,g}(\lfloor\frac{g}{2}\rfloor+1)\right) = x^2\Phi(U^*_{n-2,g}) - x^{n-g-1}\Phi(P_{g-1}) - \Phi\left(T_{n-2,g-2}(\lfloor\frac{g}{2}\rfloor)\right),$$

$$\Phi(U'_{n,g}(2)) = x^2\Phi(U^*_{n-2,g}) - 2x\Phi(T_{n-3,g-1}(2)).$$

Recall that $a = \frac{x+\sqrt{x^2-4}}{2}$. Then $4\left(2\sqrt{x^2-4}-a\right) - 2\sqrt{x^2-4} = 4\sqrt{x^2-4} - 2x > 0$ for $x > \frac{4\sqrt{3}}{3}$. Note that $\rho(U_{n,g}(\lfloor\frac{g}{2}\rfloor+1)) \ge \rho(S_7) = \sqrt{6} > \frac{4\sqrt{3}}{3}$ since $n \ge g+5$. Thus by Corollary 2.3 and Lemma 2.6, for $x \ge \rho(U_{n,g}(\lfloor\frac{g}{2}\rfloor+1)) \ge \sqrt{6}$, we have

$$\begin{aligned}
&\Phi(U'_{n,g}(2)) - \Phi(U_{n,g}(\lfloor\frac{g}{2}\rfloor+1))\\
=&x^{n-g-1}\Phi(P_{g-1}) + \Phi(T_{n-2,g-2}(\lfloor\frac{g}{2}\rfloor)) - 2x\Phi(T_{n-3,g-1}(2))\\
=&x^{n-g-1}\Phi(P_{g-1}) + [x^{n-g-1}\Phi(P_{g-1}) - (n-g-1)x^{n-g-2}\Phi(P_{\lceil\frac{g}{2}\rceil-1})\Phi(P_{\lfloor\frac{g}{2}\rfloor-1})]\\
&-2x[x^{n-g-2}\Phi(P_{g-1}) - (n-g-2)x^{n-g-3}\Phi(P_{g-2})]\\
=&2(n-g-2)x^{n-g-2}\Phi(P_{g-2}) - (n-g-1)x^{n-g-2}\Phi(P_{\lceil\frac{g}{2}\rceil-1})\Phi(P_{\lfloor\frac{g}{2}\rfloor-1})\\
=&\frac{x^{n-g-2}}{x^2-4}[2(n-g-2)\sqrt{x^2-4}(a^{g-1}-b^{g-1})\\
&-(n-g-1)(a^{\lceil\frac{g}{2}\rceil}-b^{\lceil\frac{g}{2}\rceil})(a^{\lfloor\frac{g}{2}\rfloor}-b^{\lfloor\frac{g}{2}\rfloor})]\\
=&\frac{x^{n-g-2}}{x^2-4}\{(n-g-1)[2\sqrt{x^2-4}(a^{g-1}-b^{g-1})\\
&-(a^g - a^{\lceil\frac{g}{2}\rceil}b^{\lfloor\frac{g}{2}\rfloor} + \underbrace{b^g - a^{\lfloor\frac{g}{2}\rfloor}b^{\lceil\frac{g}{2}\rceil}}_{<0})] - 2\sqrt{x^2-4}(a^{g-1}-b^{g-1})\}\\
>&\frac{x^{n-g-2}}{x^2-4}\{(n-g-1)[a^{g-1}(2\sqrt{x^2-4}-a) - 2\sqrt{x^2-4}b^{g-1} + \underbrace{a^{\lceil\frac{g}{2}\rceil}b^{\lfloor\frac{g}{2}\rfloor}}_{>ab^{g-1}}]\\
&-2\sqrt{x^2-4}(a^{g-1}-b^{g-1})\}\\
>&\frac{x^{n-g-2}}{x^2-4}[(n-g-1)(2\sqrt{x^2-4}-a)(a^{g-1}-b^{g-1})\\
&-2\sqrt{x^2-4}(a^{g-1}-b^{g-1})]
\end{aligned}$$

$$=\frac{x^{n-g-2}}{x^2-4}(a^{g-1}-b^{g-1})[\underbrace{(n-g-1)}_{\geq 4}(2\sqrt{x^2-4}-a)-2\sqrt{x^2-4}]$$
$$\geq\frac{x^{n-g-2}}{x^2-4}\left(a^{g-1}-b^{g-1}\right)\left[4\left(2\sqrt{x^2-4}-a\right)-2\sqrt{x^2-4}\right]$$
$$=\frac{x^{n-g-2}}{x^2-4}\left(a^{g-1}-b^{g-1}\right)\left(4\sqrt{x^2-4}-2x\right)>0.$$

Then Lemma 2.4(ii) implies that $\rho(U_{n,g}(\lfloor\frac{g}{2}\rfloor+1))>\rho(U'_{n,g}(2))$. □

Lemma 3.9 *For $n\geq g+4$, we have $\rho(U'_{n,g}(2))>\rho(U^1_{n,g})$.*

Proof. From Corollary 2.3, we have

$$\Phi(U^1_{n,g})=x^2\Phi(U^*_{n-2,g})-2x\Phi(U^*_{n-3,g})\ \text{and}$$
$$\Phi(U'_{n,g}(2))=x^2\Phi(U^*_{n-2,g})-2x\Phi(T_{n-3,g-1}(2)).$$

Since $T_{n-3,g-1}(2)$ is a proper spanning subgraph of $U^*_{n-3,g}$, Lemma 2.4(i) implies that $\Phi(T_{n-3,g-1}(2))-\Phi(U^*_{n-3,g})>0$ for $x\geq\rho(U^*_{n-3,g})$. Thus for $x>\rho(U^1_{n,g})>\rho(U^*_{n-3,g})$, we have

$$\Phi(U^1_{n,g})-\Phi(U'_{n,g}(2))=2x\left[\Phi(T_{n-3,g-1}(2))-\Phi(U^*_{n-3,g})\right]>0.$$

Then Lemma 2.4(ii) implies that $\rho(U'_{n,g}(2))>\rho(U^1_{n,g})$. □

Theorem 3.10 *For any unicyclic graph $U\in\mathscr{U}^g_n\backslash\{U^*_{n,g},\mathscr{U}_{n,g},U^+_{n,g},U'_{n,g}(2)\}$ with $n\geq g+5$ and $g\geq 5$, we have $\rho(U)<\rho(U'_{n,g}(2))$.*

Proof. Since for each $U\in\mathscr{U}^g_n$, as mentioned before, U can be re-written as the form $C(T_1,T_2,\ldots,T_g)$. Let $|V(T_i)|=n_i$ for $i=1,2,\ldots,g$. Then $n_i\geq 1$ and $\sum_{i=1}^{g} n_i=n$. We consider the following three cases.

Case 1. At least three of $n_1,n_2,\ldots,n_g$ are greater than or equal to 2.

Lemma 2.11 implies that $\rho(U)\leq\rho(S(n_1,n_2,\ldots,n_g))$. Moreover, by Lemma 2.7, we can construct a graph $U^*\in\mathscr{U}'_{n,g}$ such that $\rho(S(n_1,n_2,\ldots,n_g))<\rho(U^*)$ since at least three of $n_1,n_2,\ldots,n_g$ are greater than or equal to 2. Thus, the result follows from Lemma 3.6.

Case 2. Exactly two of $n_1,n_2,\ldots,n_g$ are greater than or equal to 2.

Without loss of generality, we assume that $n_j\geq n_i\geq 2$, where $i,j\in\{1,2,\ldots,g\}$ and $i\neq j$.

If $n_i\geq 3$, then Lemma 2.11 implies that
$\rho(U)\leq\rho(S(1,\ldots,1,n_i,1,\ldots,1,n_j,1,\ldots,1))$.

Let $x = (x_1, x_2, \ldots, x_n)$ be the Perron vector of $S(1, \ldots, 1, n_i, 1, \ldots, 1, n_j, 1, \ldots, 1)$, where x_i corresponds to the vertex v_i $(1 \le i \le n)$. Then Lemma 2.7 implies that

$$\rho(S(1, \ldots, 1, n_i, 1, \ldots, 1, n_j, 1, \ldots, 1)) < \begin{cases} \rho(S(1, \ldots, 1, 3, 1, \ldots, 1, n_j + n_i - 3, 1, \ldots, 1)) & \text{if } x_i \le x_j, \\ \rho(S(1, \ldots, 1, n_i + n_j - 3, 1, \ldots, 1, 3, 1, \ldots, 1)) & \text{if } x_i \ge x_j. \end{cases}$$

Since $S(1, \ldots, 1, 3, 1, \ldots, 1, n_j + n_i - 3, 1, \ldots, 1) \cong S(1, \ldots, 1, n_i + n_j - 3, 1, \ldots, 1, 3, 1, \ldots, 1) \in \mathscr{U}'_{n,g}$, the result follows from Lemma 3.6.

If $n_i = 2$, then Theorem 3.4 implies that $\rho(U) \le \rho(U^3_{n,g})$ since $U \notin \mathscr{U}_{n,g}$. Moreover, Lemma 2.7 implies that $\rho(U^3_{n,g}) < \max\{\rho(U'_{n,g}(i)), \rho(U^1_{n,g})\}$. By Lemmas 3.6 and 3.9, we have $\max\{\rho(U'_{n,g}(i)), \rho(U^1_{n,g})\} \le \rho(U'_{n,g}(2))$. Thus the result follows.

Case 3. Only one of $n_1, n_2, \ldots, n_g$ is greater than or equal to 2.

By Theorem 3.2, we have $\rho(U) \le \rho(U^1_{n,g})$ since $U \notin \{U^*_{n,g}, U^+_{n,g}\}$. Then the result follows from Lemma 3.9.

The proof is completed. □

Combing Theorem 3.10 with Lemmas 3.5, 3.7 and 3.8, the main result of this paper is immediate.

Theorem 3.11 *The first $\lfloor \frac{g}{2} \rfloor + 3$ spectral radii of unicyclic graphs in the set $\mathscr{U}^g_n$ with $n - g \ge 6$ and $g \ge 5$ are as follows:*

$\rho(U^*_{n,g}) > \rho(U_{n,g}(2)) > \rho(U^+_{n,g}) > \rho(U_{n,g}(3)) > \rho(U_{n,g}(4)) > \cdots > \rho\left(U_{n,g}\left(\lfloor \frac{g}{2} \rfloor + 1\right)\right) > \rho(U'_{n,g}(2))$.

4 Concluding remarks

In this paper, we determine the first $\lfloor \frac{g}{2} \rfloor + 3$ spectral radii of unicyclic graphs in the set $\mathscr{U}^g_n$ with $n \ge g + 6$ and $g \ge 5$. But for case $n - g = 2$, giving a total ordering (i.e, the first $\lfloor \frac{g}{2} \rfloor + 2$ spectral radii) on the set $\mathscr{U}^{n-2}_n$ is difficult, since computer experiments show that $\rho(U_{n,n-2}(\lfloor \frac{n-2}{2} \rfloor + 1)) > \rho(U^+_{n,n-2})$ when $n \le 13$; $\rho(U_{n,n-2}(\lfloor \frac{n-2}{2} \rfloor + 1)) \approx \rho(U^+_{n,n-2})$ when $n = 14$; $\rho(U_{n,n-2}(5)) > \rho(U^+_{n,n-2}) > \rho(U_{n,n-2}(6))$ when $15 \le n < 40$. For Cases $n - g = 3, 4, 5$, a lot of computational results show that $\rho(U_{n,g}(2)) > \rho(U^+_{n,g}) > \rho(U_{n,g}(3))$. But providing a mathematical proof seems to be difficult. We close the paper with the following conjecture.

Conjecture 4.1 *For $n-g=3,4,5$ and $g\geq 5$, we have $\rho(U_{n,g}(2)) > \rho(U^{+}_{n,g}) > \rho(U_{n,g}(3))$.*

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