On The Equitable Chromatic Number of Complete n-Partite Graphs[1]

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Abstract

In this note, we derive an explicit formula for the equitable chromatic number of a complete n-partite graph $K_{p_1,p_2,\cdots,p_n}$. Namely, if M is the largest integer such that

$$p_i \pmod M < \left\lceil \frac{p_i}{M} \right\rceil, \ \ (i = 1, 2, \cdots, n)$$

then

$$\chi_e(K_{p_1,p_2,\cdots,p_n}) = \sum_{i=1}^{n} \left\lceil \frac{p_i}{M+1} \right\rceil,$$

where $\chi_e(G)$ is the equitable chromatic number of graph G.

Key words and phrases: Complete n-partite graphs, Equitable chromatic number.

AMS 1991 Subject Classifications: 05C15, 68Q20.

1. Introduction

In this note, all graphs are simple and undirected. A graph $G = (V, E)$ is said to be *equitably k-colorable*, if V may be partitioned into independent sets $V_1, \cdots, V_k$ such that for any $i \neq j$, $||V_i| - |V_j|| \leq 1$. The *equitable chromatic number* of G, denoted as $\chi_e(G)$, is defined as the smallest k such that G is equitably k-colorable.

In 1973, W. Meyer [5] proposed the conjecture:

$$\chi_e(G) \leq \Delta(G)$$

for simple graphs G which are neither complete graphs K_p nor odd cycles C_{2n+1}, where $\Delta(G)$ denotes the maximal degree of G. In 1970, Hajnal and Szemeredi [3] proved that if $k > \Delta(G)$, then G is equitably k-colorable. In 1983, B. Bollobás and R.K. Guy verified Meyer's conjecture for trees [1]. Recently, K. W. Lih *et al* [2, 4] proved the validity of Meyer's conjecture for the cases when $\Delta(G) \leq 3$, $\Delta(G) \geq |V|/2$, and G is a bipartite graph. Yap and Zhang [6–8] proved

[1]This work was partially supported by RGC, Hong Kong; and FRG, Hong Kong Baptist University.

Meyer's conjecture for outerplanar graphs, for planar graphs G with $\Delta(G) \geq 13$, and for graphs $G(V, E)$ with $\Delta(G) \geq \frac{|V|}{3} + 1$.

In this note, we derive an explicit formula for the equitable chromatic number of complete n-partite graphs ($n \geq 2$).

2. Main Results

We shall first consider a related combinatoric problem. Let n natural numbers $p_1, p_2, \cdots, p_n$ be given. For each $i,\ 1 \leq i \leq n$, we decompose p_i into λ_i non-negative integers $p_{ij},\ j = 1, \cdots, \lambda_i$, such that

$$p_i = \sum_{j=1}^{\lambda_i} p_{ij}$$

and $|p_{ij} - p_{kl}| \leq 1$ for $i,\ k = 1, \cdots, n,\ j = 1, \cdots, \lambda_i$, and $l = 1, \cdots, \lambda_k$. In this way, the natural numbers $p_1, \cdots, p_n$ are said to be λ-*equitably partitioned*, where

$$\lambda = \sum_{i=1}^{n} \lambda_i.$$

The minimum value of λ for which $p_1, \cdots, p_n$ may be λ-equitably partitioned is called the *equitable partition number* of $p_1, \cdots, p_n$ and is denoted by $e(p_1, \cdots, p_n)$.

Lemma 1 *The equitable partition number of* $p_1, \cdots, p_n$ *is* $\sum_{i=1}^{n} \left\lceil \frac{p_i}{M+1} \right\rceil$, *where* M *is the largest integer such that*

$$p_i \pmod M < \left\lceil \frac{p_i}{M} \right\rceil$$

and $0 \leq p_i \pmod M < M$, *for each* $i = 1, \cdots, n$.

Proof. To achieve an equitable partition of the natural numbers $p_1, p_2, \cdots, p_n$, each part p_{ij} must be of size M or $M+1$, for some integer M. Suppose the number p_i is decomposed into x_i numbers M and y_i numbers $M+1$. For each p_i, we write

$$\begin{aligned} p_i &= Mx_i + (M+1)y_i, \\ &= M(x_i + y_i) + y_i, \quad (1) \\ &= (M+1)(x_i + y_i) - x_i. \quad (2) \end{aligned}$$

where $x_i,\ y_i \geq 0$.

Thus $\lambda_i = x_i + y_i$. Now if $x_i \geq M+1$, then we may let $x_i = a(M+1) + x_i', (a > 0)$ and rewrite

$$
\begin{aligned}
p_i &= Mx_i + (M+1)y_i \\
&= Mx_i' + (M+1)(y_i + Ma)
\end{aligned}
$$

Hence the size of the partitioning can be decreased to $\lambda_i = x_i' + y_i + Ma = x_i + y_i - a$. Thus for the minimum partition, we may assume that $x_i \leq M$. Equation (2) then yields

$$
\begin{aligned}
\left\lceil \frac{p_i}{M+1} \right\rceil &= x_i + y_i - \left\lfloor \frac{x_i}{M+1} \right\rfloor \\
&= x_i + y_i.
\end{aligned}
$$

From equation (1), we have

$$
p_i \pmod M = y_i \pmod M \quad < \quad y_i + x_i + \left\lceil \frac{y_i}{M} \right\rceil = \left\lceil \frac{p_i}{M} \right\rceil. \tag{3}
$$

Clearly, $e(p_1, p_2, \cdots, p_n)$ is minimized when M is maximized. For given $K_{p_1,\cdots,p_n}$, $(p_i \neq 0)$, we select the largest integer M such that (3) holds for $i = 1,\ 2,\ \cdots,\ n$. Then

$$
\begin{aligned}
e(p_1, p_2, \cdots, p_n) &= \min \sum_{i=1}^{n} \lambda_i \quad (\text{w.r.t.}\, p_i) \\
&= \sum_{i=1}^{n} (x_i + y_i) \\
&= \sum_{i=1}^{n} \left\lceil \frac{p_i}{M+1} \right\rceil.
\end{aligned}
$$

∎

Let us illustrate the proof with an example: $e(3, 5, 9)$. Clearly, the minimum set size $M \leq 3$. We start by testing the case $M = 3$. But,

$$
p_2 \pmod M = 5 \pmod 3 = 2 \not< \left\lceil \frac{p_2}{M} \right\rceil = \left\lceil \frac{5}{3} \right\rceil = 2.
$$

Next we try $M = 2$, and it is easy to check that (3) holds for $p_1 = 3,\ p_2 = 5$ and $p_3 = 9$.

Hence the equitable partition number $e(3, 5, 9)$ is $\left\lceil \frac{3}{3} \right\rceil + \left\lceil \frac{5}{3} \right\rceil + \left\lceil \frac{9}{3} \right\rceil = 6$. Specifically, the 6 partitions are (3), (2+3), and (3+3+3).

Corollary 2 *Suppose M is the largest integer such that*

$$p_i (\text{mod } M) < \left\lceil \frac{p_i}{M} \right\rceil \quad for \quad i = 1, \cdots, n.$$

Then

$$\chi_e(K_{p_1, p_2, \cdots, p_n}) = \sum_{i=1}^{n} \left\lceil \frac{p_i}{M+1} \right\rceil.$$

3. Remarks

Let $G = K_{p_1, p_2, \cdots, p_n}$ be a complete n-partite graph, where $p_1 \leq p_2 \leq \cdots \leq p_n$. The order of G is then $N = p_1 p_2 \cdots p_n$. The proof of Lemma 1 provides an efficient algorithm for the explicit calculation of the equitable partition number of $p_1, p_2, \cdots, p_n$ and hence the equitable chromatic numbers of G. Suppose $M > p_1/2$ and $M + 1 \neq p_1$. Then (3) does not hold for $i = 1$. Hence we need only to consider the case $M \leq p_1/2$ or $M + 1 = p_1$. Consequently, the total number of steps required to determine the largest M to satisfy (3) is approximately $p_1 n/2$.

4. Acknowledgement

The authors wish to thank the referees for their help and suggested amendments to this note.

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