Strong vertex-graceful labelings for some double cycles*

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Abstract

A graph G with p vertices and q edges is said to be vertex-graceful if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p\}$ such that the induced labeling $f^+ : E(G) \to \mathbb{Z}_q$ defined by $f^+(uv) \equiv f(u) + f(v) \pmod{q}$, for each edge uv, is a bijection. Lee, Pan and Tsai showed some double cycles to be vertex-graceful with small order in 2005. In this paper, we will extend their result. In particular, a necessary condition for the vertex-gracefulness of double cycles is provided.

Keywords: Strong vertex-graceful, strongly indexable, vertex-graceful, total edge-magic, super edge-magic, double cycle.
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1 Introduction

All graphs in this paper are finite, connected and simple. A graph G with p vertices and q edges is said to be *vertex-graceful* if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p\}$ such that the induced labeling $f^+ : E(G) \to \mathbb{Z}_q$ defined by $f^+(uv) \equiv f(u) + f(v) \pmod{q}$, for each edge uv, is a bijection. In this case, f is called a *vertex-graceful labeling* of G. This concept was first introduced by Lee, Pan and Tsai [8] in 2005. Another induced labeling $f^* : E(G) \to \mathbb{Z}$, defined by $f^*(uv) = f(u) + f(v)$. If $f^*(E(G))$ consists of consecutive integers, then f is called a *strong vertex-graceful labeling.* A graph G is said to be *strong vertex-graceful* if it admits a strong vertex-graceful labeling. This concept was first introduced by Acharya and Hegde [1] in 1991. They called a strong vertex-graceful graph

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as *strongly s-indexable graph*, where s is the minimum value of the mapping f^*. For further results on strongly indexable graphs, one may refer to [6,9]. It is clear that a strong vertex-graceful graph is vertex-graceful.

A graph G with p vertices and q edges is said to be *total edge-magic* if there a bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $f(u) + f(uv) + f(v)$ is a constant, for each edge uv. The study of total edge-magic graphs is initially introduced by Kotzig and Rosa [10, 11]. They called the total edge magic graph as magic graph. In 1998, Enomoto *et al.* [3] called a total edge-magic graph as *super edge-magic* if $f(V(G)) = \{1, 2, \ldots, p\}$. More results on super edge-magic graphs, one may see [2–5, 7].

A set consists of consecutive integers is called *consecutive.* Chen [2] claimed that a graph is super edge-magic if and only if there exists a vertex labeling such that two sets $f(V(G))$ and $\{f(u) + f(v) \mid uv \in E(G)\}$ are both consecutive. Independently, Figueroa-Centeno *et al.* [4] showed that

A graph G with p vertices and q edges is super edge-magic if only if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p\}$ such that $S = \{f(u)+f(v) \mid uv \in E(G)\}$ is consecutive. In such a case, f extends to a super edge-magic labeling of G by defined $f(uv) = p+q+s-f(u)-f(v)$ for each $uv \in E(G)$, where $s = \min(S)$.

So strong vertex-graceful (strongly s-indexable) and super edge-magic are equivalent.

2 Necessary condition for vertex-graceful double cycles

In this section, we will provide a necessary condition on vertex-gracefulness of double cycles.

Let f be any vertex labeling of a graph G which contains p vertices and q edges. Then

$$\sum_{e \in E(G)} f^*(e) = \sum_{uv \in E(G)} (f(u) + f(v)) = \sum_{x \in V(G)} \deg(x) f(x). \tag{2.1}$$

If f is a vertex-graceful labeling of G, then by (2.1) we have

$$\sum_{x\in V(G)} \deg(x)f(x) \equiv \sum_{e\in E(G)} f^+(e) \equiv \sum_{i=1}^{q} i \equiv \frac{q(q+1)}{2}$$
$$\equiv \begin{cases} 0 & \text{if } q \text{ is odd} \\ \frac{q}{2} & \text{if } q \text{ is even} \end{cases} \pmod{q}. \tag{2.2}$$

A *double cycle* (or *one-point union of two cycles*) is a simple graph obtained from two cycles, say C_m and C_n where $m, n \geq 3$, by identifying one vertex from each cycle. Without loss of generality, we may assume that the m-cycle is $u_0u_1\cdots u_{m-1}u_0$ and the n-cycle is $v_0v_1\cdots v_{n-1}v_0$, where $u_0 = v_0$. We denote this graph by $C(m,n)$. The unique vertex of degree 4 in the graph $C(m,n)$ is called the *coalesced vertex.*

Lee *et al.* [8] claimed without proof that $C(3,n)$ for $n = 4,6,7,8$; $C(4,m)$ for $m = 5,6,7,9$; and $C(5,k)$ for $k = 5,6,8,9$ are not vertex-graceful. They also showed that $C(3,5)$, $C(3,9)$, $C(4,4)$, $C(4,8)$, $C(5,7)$ are strong vertex-graceful. Now let us make a supplement on that result.

Theorem 2.1. *A double cycle $C(m,n)$ is vertex-graceful only if $m+n \equiv 0 \pmod 4$.*

Proof: Note that $C(m,n)$ contains $m+n-1$ vertices and $m+n$ edges. Let c be the coalesced vertex. Then

$$\sum_{x\in V(G)} \deg(x)f(x) = 2f(c) + 2\sum_{x\in V(G)} f(x) = 2f(c) + (m+n-1)(m+n). \tag{2.3}$$

By (2.2) we have

$$2f(c) \equiv \begin{cases} 0 & \text{if } m+n \text{ is odd} \\ \frac{m+n}{2} & \text{if } m+n \text{ is even.} \end{cases} \pmod{m+n}$$

Suppose $m+n$ is odd. Then $f(c) \equiv 0 \pmod{m+n}$. But it is impossible since $f(c) \in \{1,2,\ldots,m+n-1\}$.

Suppose $m+n$ is even. Then $4f(c) \equiv m+n \pmod{2(m+n)}$. This implies $m+n \equiv 0 \pmod 4$. □

Note that, from the proof above we can see that $f(c) = \frac{m+n}{4}$ or $\frac{3(m+n)}{4}$ for a vertex-graceful labeling of $C(m,n)$. But these cases are equivalent.

It is because that if f is a vertex-graceful labeling, then $(m+n)-f$ is also a vertex-graceful labeling. This result also holds for any strong vertex-graceful labeling of $C(m,n)$.

3 Some strong vertex-graceful double cycles

In this section we will show some special double cycles to be strong vertex-graceful. For convenience we use $[\boldsymbol{a},\boldsymbol{b}]$ to denote the set $\{x \in \mathbb{Z} \mid a \leq x \leq b\}$ for integers $a < b$.

Suppose that f is a strong vertex-graceful labeling of $C(m,n) = (V,E)$ with $f(c) = \frac{m+n}{4}$ (of course, $m+n \equiv 0 \pmod 4$), where c is the coalesced vertex. From Equations (2.1) and (2.3) we have

$$2\left(\frac{m+n}{4}\right) + (m+n-1)(m+n) = \sum_{i=s}^{s+m+n-1} i, \text{ for some } s.$$

It is easy to solve that $s = \frac{m+n}{2}$. That is $f^*(E) = \left[\frac{\boldsymbol{m+n}}{\boldsymbol{2}}, \frac{\boldsymbol{3(m+n)}}{\boldsymbol{2}} - \boldsymbol{1}\right]$.

Theorem 3.1. *For $k \geq 2$, $C(3,4k-3)$ is strong vertex-graceful.*

Proof: Let the two cycles in $C(3,4k-3) = (V,E)$ be $u_0u_1u_2u_0$ and $v_0v_1\cdots v_{4k-4}v_0$, where $u_0 = v_0$. Define $f : V \to \{1,2,\ldots,4k-1\}$ by $f(v_{2i}) = k+i$ for $0 \leq i \leq 2k-2$; $f(v_{2j+1}) = 3k+1+j$ for $0 \leq j \leq k-2$; $f(v_{2j+1}) = j+2-k$ for $k-1 \leq j \leq 2k-3$; $f(u_1) = 3k-1$ and $f(u_2) = 3k$.

It is easy to see that

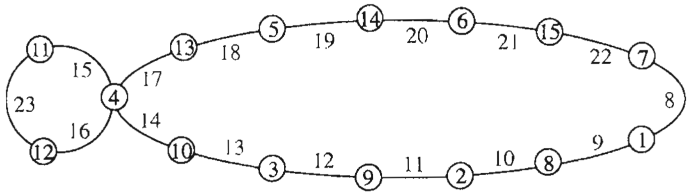
$\{f(v_{2i}) \mid 0 \leq i \leq 2k-2\} = [\boldsymbol{k}, \boldsymbol{3k-2}]$;

$\{f(v_{2j+1}) \mid 0 \leq j \leq k-2\} = [\boldsymbol{3k+1}, \boldsymbol{4k-1}]$;

$\{f(v_{2j+1}) \mid k-1 \leq j \leq 2k-3\} = [\boldsymbol{1}, \boldsymbol{k-1}]$. Thus, f is a bijection.

Now we have $f^*(v_{2i}v_{2i+1}) = (k+i)+(3k+1+i) = 4k+2i+1$ for $0 \leq i \leq k-2$; $f^*(v_{2i}v_{2i+1}) = 2i+2$ for $k-1 \leq i \leq 2k-3$; $f^*(v_{2i+1}v_{2i+2}) = (3k+1+i)+(k+i+1) = 4k+2i+2$ for $0 \leq i \leq k-2$; $f^*(v_{2i+1}v_{2i+2}) = 2i+3$ for $k-1 \leq i \leq 2k-3$; $f^*(v_{4k-4}v_0) = 4k-2$; $f^*(u_0u_1) = 4k-1$; $f^*(u_1u_2) = 6k-1$; and $f^*(u_0u_1) = 4k$. Thus, we have $f^*(E) = [\boldsymbol{2k}, \boldsymbol{6k-1}]$. Hence f is a strong vertex-graceful labeling of $C(3,4k-3)$. □

Example 3.1. Following is the strong vertex-graceful labeling for $C(3,13)$ constructed in the proof of Theorem 3.1.



Theorem 3.2. *The graph $C(2n+3, 2n+1)$ is strong vertex-graceful for $n \geq 1$.*

Proof: First we consider a cycle of length $4n+4$, namely

$$C_{4n+4} = x_1x_2\cdots x_{n+1}y_{n+1}\cdots y_2y_1z_1z_2\cdots z_{n+1}w_{n+1}\cdots w_2w_1x_1.$$

Now we want to define a labeling $f : V(C_{4n+4}) \to [\mathbf{1}, \mathbf{4n+3}]$ such that $f(x_1) = f(y_1) = n+1$. Namely,

$$\begin{aligned}
f(x_{2i+1}) &= n+1-i, & 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
f(x_{2i}) &= 3n+3-i, & 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor \\
f(y_{2i+1}) &= n+1+i, & 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
f(y_{2i}) &= 3n+2+i, & 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor \\
f(z_{2i+1}) &= 2n+2+i, & 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
f(z_{2i}) &= i, & 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor \\
f(w_{2i+1}) &= 4n+3-i, & 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
f(w_{2i}) &= 2n+2-i, & 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor
\end{aligned}$$

Clearly f is onto with $f(x_1) = f(y_1) = n+1$.

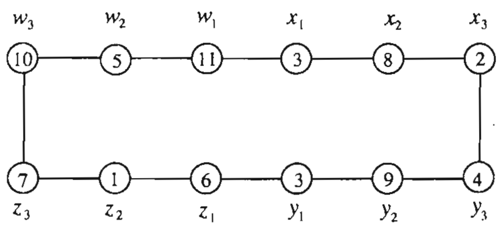
Now we merge x_1 with y_1 to get the graph $C(2n+3, 2n+1)$ and keep the labeling f. Then f is a bijection between $V(C(2n+3, 2n+1))$ and $[\mathbf{1}, \mathbf{4n+3}]$.

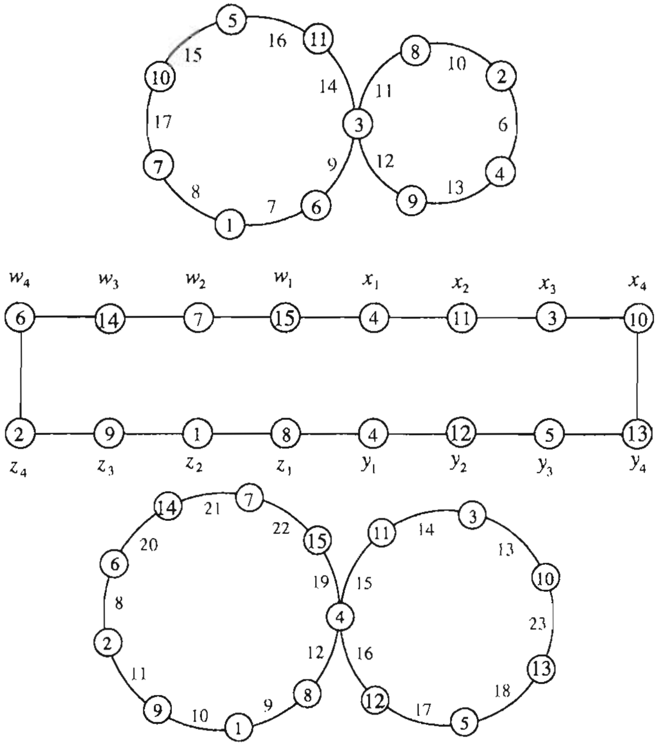
We separate the proof into two cases:

Case 1: $n = 2k+1$ for some $k \geq 0$. Then $f^*(x_{2i}x_{2i+1}) = 8k+8-2i$ for $1 \leq i \leq k$ (if $k = 0$, then there is no this case), $f^*(x_{2i-1}x_{2i}) = 8k+9-2i$ for $1 \leq i \leq k+1$. Clearly, the labels of these edges cover the set $[\mathbf{6k+7, 8k+7}]$. Similarly we have $f^*(y_{2i}y_{2i+1}) = 8k+7+2i$ for $1 \le i \le k$, $f^*(y_{2i-1}y_{2i}) = 8k+6+2i$ for $1 \le i \le k+1$. The labels of these edges cover the set $[\mathbf{8k+8, 10k+8}]$. $f^*(z_{2i}z_{2i+1}) = 4k+4+2i$ for $1 \le i \le k$, $f^*(z_{2i-1}z_{2i}) = 4k+3+2i$ for $1 \le i \le k+1$. These labels cover the set $[\mathbf{4k+5, 6k+5}]$. $f^*(w_{2i}w_{2i+1}) = 12k+11-2i$ for $1 \le i \le k$, $f^*(w_{2i-1}w_{2i}) = 12k+12-2i$ for $1 \le i \le k+1$. These labels cover the set $[\mathbf{10k+10, 12k+10}]$. Finally we have $f^*(x_{2k+2}y_{2k+2}) = 12k+11$, $f^*(w_{2k+2}z_{2k+2}) = 4k+4$, $f^*(x_1w_1) = 10k+9$ and $f^*(y_1z_1) = 6k+6$. Combining all labels of these $4n+4$ edges, we see that they cover the set $[\mathbf{4k+4, 12k+11}] = [\mathbf{2n+2, 6n+5}]$ once. Hence we obtain a strong vertex-graceful of $C(2n+3, 2n+1)$ for odd n.

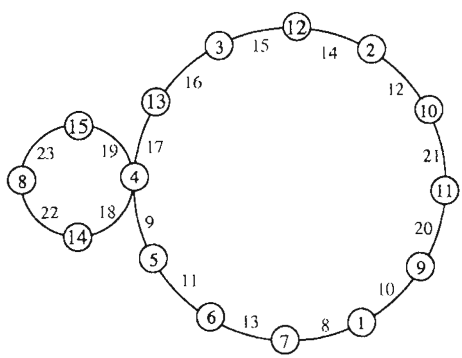
Case 2: $n = 2k$ for some $k \ge 1$. Then $f^*(x_{2i}x_{2i+1}) = 8k+4-2i$ for $1 \le i \le k$, $f^*(x_{2i-1}x_{2i}) = 8k+5-2i$ for $1 \le i \le k$. Clearly, the labels of these edges cover the set $[\mathbf{6k+4, 8k+3}]$. Similarly we have $f^*(y_{2i}y_{2i+1}) = 8k+3+2i$ for $1 \le i \le k$, $f^*(y_{2i-1}y_{2i}) = 8k+2+2i$ for $1 \le i \le k$. The labels of these edges cover the set $[\mathbf{8k+4, 10k+3}]$. $f^*(z_{2i}z_{2i+1}) = 4k+2+2i$ for $1 \le i \le k$, $f^*(z_{2i-1}z_{2i}) = 4k+1+2i$ for $1 \le i \le k$. These labels cover the set $[\mathbf{4k+3, 6k+2}]$. $f^*(w_{2i}w_{2i+1}) = 12k+5-2i$ for $1 \le i \le k$, $f^*(w_{2i-1}w_{2i}) = 12k+6-2i$ for $1 \le i \le k$. These labels cover the set $[\mathbf{10k+5, 12k+4}]$. Finally we have $f^*(x_{2k+1}y_{2k+1}) = 4k+2$, $f^*(w_{2k+1}z_{2k+1}) = 12k+5$, $f^*(x_1w_1) = 10k+4$ and $f^*(y_1z_1) = 6k+3$. Combining all labels of these $4n+4$ edges, we see that they cover the set $[\mathbf{4k+2, 12k+5}] = [\mathbf{2n+2, 6n+5}]$ once. Hence we obtain a strong vertex-graceful of $C(2n+3, 2n+1)$ for even n. □

Example 3.2. Following are the strong vertex-graceful labelings for $C(7,5)$ and $C(9,7)$ constructed in the proof of Theorem 3.2.





Example 3.3. Here is an ad hoc strong vertex-graceful labeling for $C(4, 12)$.



Finally, let us propose the following conjecture:

Conjecture 3.3. *If a double cycle is vertex-graceful, then it is also strong vertex-graceful.*

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