Edge-magic Labeling Matrices of the Composition of Paths with Null Graphs

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Abstract

Given two graphs G and H, the composition of G with H is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. In this paper, we shall construct some matrices with some special row sums, column sums or diagonal sums. By using these matrices we obtain an edge-magic labeling of the composition of P_n with N_n. Also we obtain an edge-magic labeling of the composition of P_m with N_{mk} for odd mk with $m \geq 3$ and $k \geq 1$.

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1. Introduction

Let $G = (V, E)$ be a (p, q)-graph, i.e., $|V| = p$ and $|E| = q$. If there exists a bijection

$$f : E \to \{k,\ k+1,\ \cdots,\ k+q-1\}$$

for some $k \in \mathbb{Z}$ such that the *induced mapping* $f^+(u) = \sum\limits_{v \in N(u)} f(uv)$ is a constant mapping from V to $\mathbb{Z}_p$, then G is called *k-edge-magic* and f is called a *k-edge-magic labeling* of G, where $u \in V$ and $N(u)$ is the neighborhood of u. If $k = 1$, then G is simply called *edge-magic* and f an *edge-magic labeling* of G. This concept was introduced by Lee, Seah and Tan [9] in 1992. If the induced mapping is a constant mapping from V to $\mathbb{Z}$ then G is called *supermagic*. Many researchers consider magic graphs in different ways. The interested reader is referred to [12, 15, 17] and more references are listed in [14, 16, 19]. The reader should be aware that alternative uses of the term edge-magic appear in same literature, for example [3, 4, 11].

A dual concept called *edge-graceful* was introduced by S.H. Lo [10] in 1985. Let f be the bijection defined in the previous paragraph. If the induced mapping f^+ is a bijection from V to $\mathbb{Z}_p$ then the graph G is called *k-edge-graceful.* If $k = 1$ then G is simply called *edge-graceful* and f is called an *edge-graceful labeling.* Many researchers investigated on certain families graphs. The interested reader is referred to [8]. Also some conjectures were held out in [8].

Given two graphs G and H. The *composition of G with H*, denoted as $G \circ H$, is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

In Section 4 we shall use some special matrices with some particular row sums and column sums to construct an edge-magic labeling for $P_n \circ N_n$, where P_n is the path with n vertices, N_n is the null graph with n vertices and n is odd. In Section 5 we shall use a special matrix with the same diagonal sum to construct an edge-magic labeling for $P_n \circ N_n$ with even positive integer n.

2. Some Concepts and Notations

We shall use $S \times n$ to denote the multiset which is an n-copies of a set S. Note that S may be a multiset itself. We let $[\boldsymbol{k}] = \{1, 2, \ldots, k\}$ for a positive integer k. Throughout, the unadorned term set will mean a multiset. The set operations are considered as multiset operations.

Let $G = (V, E)$ be a simple graph and S be a set. Suppose $f : E \to S$ is a mapping. A *labeling matrix* Ω of f is a matrix whose rows and columns are named by the vertices of G and the (u, v)-entry is $f(uv)$ if $uv \in E$, and is 0 otherwise. If f is an edge-magic (respectively, edge-graceful) labeling then Ω is called an *edge-magic* (respectively, *edge-graceful*) *labeling matrix.*

Thus, a simple (p, q)-graph $G = (V, E)$ is k-edge-magic (respectively, k-edge-graceful) if and only if there exists a bijection $f : E \to \{k, k+1, \cdots, k+q-1\}$ such that the row sums and the column sums of the labeling matrix of f are the same (respectively, distinct) modulo p.

Let $G = P_m \circ N_n$. Let $V = V(G) = [\boldsymbol{m}] \times [\boldsymbol{n}]$. If $f : E(G) \to S$ is a mapping, then under the lexicographic order the labeling matrix of f is formed by

$$\begin{pmatrix} O & A_1 & O & \cdots & O & O \\ A_1^T & O & A_2 & \ddots & \ddots & O \\ O & A_2^T & O & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ O & \ddots & \ddots & \ddots & O & A_{m-1} \\ O & O & \cdots & \cdots & A_{m-1}^T & O \end{pmatrix}, \tag{2.1}$$

where O is the $n \times n$ zero matrix and A_i is an $n \times n$ matrix.

3. Necessary Condition of Edge-magicness and Edge-gracefulness of $P_m \circ N_n$

Let $G = (V, E)$ be a (p, q)-graph. Using a simple counting argument, it is easy to show that if G is k-edge-magic then p divides $q(q + 2k - 1)$, and if G is k-edge-graceful then p divides $q(q + 2k - 1) - \frac{1}{2}p(p - 1)$. Lee, Lee and Murthy [5] showed that a graph with $p \equiv 2 \pmod 4$ is not edge-graceful.

For $k = 1$, these two necessary conditions become

$$q(q+1) \equiv 0 \pmod{p}. \tag{NCEM}$$

$$p \not\equiv 2 \,(\bmod\ 4) \text{ and } q(q+1) \equiv \begin{cases} 0 \pmod{p} & \text{if } p \text{ is odd ,} \\ \frac{1}{2}p \,(\bmod\ p) & \text{if } p \equiv 0 \,(\bmod\ 4). \end{cases} \tag{NCEG}$$

Now we consider the graph $P_m \circ N_n$, $m \geq 2$ and $n \geq 1$. It is an $(mn, n^2(m-1))$-graph. For both edge-graceful and edge-magic cases, there are very few known results. When $n = 1$, $P_m \circ N_1 \cong P_m$. Lo [10] showed that P_m is edge-graceful if and only if m is odd. Lee, Seah and Tan [9] showed that all tree except P_2 are not edge-magic. When $n = 2$, for the edge-graceful case m is 4 or 12; for the edge-magic case m is 2, 3 or 6. Shiu [13] showed that $P_4 \circ N_2$ and $P_{12} \circ N_2$ are edge-graceful. We shall show that $P_3 \circ N_2$ and $P_6 \circ N_2$ are edge-magic (please see Examples 5.3 and 5.4). When $m = 2$ and $n \geq 2$, $P_2 \circ N_n \cong K_{n,n}$. It is easy to see that $K_{n,n}$ does not satisfy NCEG. So it is not edge-graceful. It is also easy to see that $K_{2,2}$ is not edge-magic and $K_{n,n}$ is supermagic if $n \geq 3$ (see for example [12, 17]). In general, Lee and Seah [6] showed that the regular complete k-partite graph, $K_{\underbrace{n,\ldots,n}_{k \text{ times}}}$, is edge-graceful if and only if n is odd, and k is either odd or a multiple of 4. So we may assume $m \geq 3$ and $n \geq 3$. In [7], there is an edge-graceful labeling of $P_3 \circ N_3$. Another edge-graceful labeling of $P_3 \circ N_3$ and an edge-graceful labeling of $P_3 \circ N_5$ were given in [13]. Shiu *et al.* [18] showed that $P_3 \circ N_n$ with odd $n \geq 7$ is edge-graceful.

Edge-gracefulness of $P_m \circ N_n$ is not the main concern in this paper. So we change our consideration to the edge-magicness of $P_m \circ N_n$. In this case, the condition NCEM is equivalent to

$$n(n^2 - 1) \equiv 0 \pmod{m} \tag{3.2}$$

Thus, if $m = n$ then (3.2) always holds. In the following we will concentrate on this particular case. The general case seems to be elusive.

4. Edge-magic Labeling of $P_n \circ N_n$ with odd n

Now we assume $m = n$ and n is odd. From now on unless stated otherwise, all arithmetic are taken in $\mathbb{Z}_{n^2}$.

Lemma 4.1: *For a given j, there is a unique bijection σ on $[n]$ such that $jn + \sigma(j)n = \frac{1}{2}n(n+1)$.*

Proof: It is easy to see that σ must be defined as $\sigma(j) \equiv \frac{1}{2}(n+1) - j \pmod{n}$ for all $j \in [n]$. Clearly it is a bijection. □

Lemma 4.2: *For a given i, there is a unique bijection τ on $[n]$ such that $\frac{1}{2}n(n+1) - in + \tau(i)n = in$.*

Proof: Clearly, $\tau(i) \equiv 2i - \frac{1}{2}(n+1) \pmod{n}$. Since n is odd, $\tau(i)$ is unique. The mapping τ is clearly a bijection. □

For any $n \times n$ matrix M, we let $\sigma(M)$ be the matrix such that the j-th column of $\sigma(M)$ is the $\sigma(j)$-th column of M and let $\tau(M)$ be the matrix obtained from M by replacing the (i, n)-th entry by the $(\tau(i), n)$-th entry of M. We shall denote $\sigma(\tau(M))$ by $\sigma\tau(M)$. We denote $r_i(M)$ and $c_j(M)$ to be the i-th row sum and the j-th column sum of M, $1 \le i, j \le n$, respectively.

To find an edge-magic labeling of $P_n \circ N_n$ is equivalent to fill all the elements of $[\boldsymbol{n^2(n-1)}] = [\boldsymbol{n^2}]\times(n-1)$ into the matrices A_j's in (2.1) such that all row sums are equal. Note that, for two sets S and T, $S = T$ means that S and T are equal under taking modulo n^2.

Theorem 4.3: *$P_n \circ N_n$ is edge-magic if n is odd and $n \ge 3$.*

Proof: Arrange the elements of $[\boldsymbol{n^2}]$ into an $n \times n$ matrix A in the natural order, i.e.,

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & n^2 \end{pmatrix}. \tag{4.1}$$

Then

$$\begin{aligned} r_i(A) &= \tfrac{1}{2}n(n+1) + (i-1)n^2 = \tfrac{1}{2}n(n+1), \\ r_i(\sigma(A)) &= \tfrac{1}{2}n(n+1) = r_i(A), \\ r_i(\tau(A)) &= \tfrac{1}{2}n(n+1) - in + \tau(i)n = in \text{ (by Lemma 4.2)}, \\ r_i(\sigma\tau(A)) &= r_i(\tau(A)) = in. \end{aligned}$$

$$\begin{aligned} c_j(A) &= \tfrac{1}{2}n^2(n-1) + jn = jn, \\ c_j(\sigma(A)) &= \sigma(j)n, \\ c_j(\tau(A)) &= jn = c_j(A), \\ c_j(\sigma\tau(A)) &= c_j(\sigma(A)) = \sigma(j)n. \end{aligned}$$

Now let $A_1 = A$, $A_{n-1} = \sigma(A)^T$ and $A_i = \sigma\tau(A)^T$ for $2 \le i \le n-2$. Substitute these A_j's into (2.1) to obtain an $n^2 \times n^2$ (symmetric) matrix Ω, say. Clearly, all elements of $[\boldsymbol{n^2}]\times(n-1)$ are occupied. Then when $n \ge 5$ we have,

$$
\begin{aligned}
&\text{for } 1 \le k \le n, && r_k(\Omega) = r_k(A);\\
&\text{for } n+1 \le k \le 2n, && r_k(\Omega) = r_k(A^T) + r_k(\sigma\tau(A)^T)\\
& && \quad = c_k(A) + c_k(\sigma\tau(A))\\
&\quad\text{(by Lemma 4.1)} && \quad = kn + \sigma(k)n = \tfrac{1}{2}n(n+1)\\
& && \quad = r_k(A);\\
&\text{for } 2n+1 \le k \le n(n-2), && r_k(\Omega) = r_k(\sigma\tau(A)) + r_k(\sigma\tau(A)^T)\\
& && \quad = kn + c_k(\sigma\tau(A))\\
& && \quad = kn + \sigma(k)n = r_k(A);\\
&\text{for } n(n-2)+1 \le k \le n(n-1), && r_k(\Omega) = r_k(\sigma\tau(A)) + r_k(\sigma(A)^T)\\
& && \quad = kn + c_k(\sigma(A))\\
& && \quad = kn + \sigma(k)n = r_k(A);\\
&\text{for } n(n-1)+1 \le k \le n^2, && r_k(\Omega) = r_k(\sigma(A)) = r_k(A).
\end{aligned}
$$

When $n = 3$ the proof is similar. Hence, Ω is an edge-magic labeling matrix of $P_n \circ N_n$. □

Example 4.1: For $n = 3$, we have

$$
A_1 = A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \qquad A_2 = \sigma(A)^T = \begin{pmatrix} 1 & 4 & 7 \\ 3 & 6 & 9 \\ 2 & 5 & 8 \end{pmatrix}.
$$

Example 4.2: For $n = 5$, we have

$$
A = \begin{array}{c}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{pmatrix} \begin{array}{l} 15 \\ 15 \\ 15 \\ 15 \\ 15 \end{array} \Big\downarrow \text{row sum} \\ \begin{array}{ccccc} 5 & 10 & 15 & 20 & 25 \end{array} \leftarrow \text{column sum} \end{array}
$$

$$
\sigma(A) = \begin{array}{c}\begin{pmatrix} 2 & 1 & 5 & 4 & 3 \\ 7 & 6 & 10 & 9 & 8 \\ 12 & 11 & 15 & 14 & 13 \\ 17 & 16 & 20 & 19 & 18 \\ 22 & 21 & 25 & 24 & 23 \end{pmatrix} \begin{array}{l} 15 \\ 15 \\ 15 \\ 15 \\ 15 \end{array} \Big\downarrow \text{row sum} \\ \begin{array}{ccccc} 10 & 5 & 25 & 20 & 15 \end{array} \leftarrow \text{column sum} \end{array}
$$

$$
\tau(A) = \begin{array}{c}\begin{pmatrix} 1 & 2 & 3 & 4 & 20 \\ 6 & 7 & 8 & 9 & 5 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 25 \\ 21 & 22 & 23 & 24 & 10 \end{pmatrix} \begin{array}{l} 5 \\ 10 \\ 15 \\ 20 \\ 25 \end{array} \Big\downarrow \text{row sum} \\ \begin{array}{ccccc} 5 & 10 & 15 & 20 & 25 \end{array} \leftarrow \text{column sum} \end{array}
$$

$$\sigma\tau(A) = \begin{array}{c} \begin{pmatrix} 2 & 1 & 20 & 4 & 3 \\ 7 & 6 & 5 & 9 & 8 \\ 12 & 11 & 15 & 14 & 13 \\ 17 & 16 & 25 & 19 & 18 \\ 22 & 21 & 10 & 24 & 23 \end{pmatrix} \begin{array}{l} 5 \\ 10 \\ 15 \ \downarrow \text{row sum} \\ 20 \\ 25 \end{array} \\ \begin{array}{ccccc} 10 & 5 & 25 & 20 & 15 \end{array} \leftarrow \text{column sum} \end{array}$$

Then

$$\begin{pmatrix} O & A & O & O & O \\ A^T & O & \sigma\tau(A)^T & O & O \\ O & \sigma\tau(A) & O & \sigma\tau(A)^T & O \\ O & O & \sigma\tau(A) & O & \sigma(A)^T \\ O & O & O & \sigma(A) & O \end{pmatrix}$$

is an edge-magic labeling matrix of $P_5 \circ N_5$. ∎

5. Edge-magic Labeling of $P_n \circ N_n$ with even n

In this section, we assume n is even. It will be separated into two cases: $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$. Let us give a lemma first. Although this lemma is not used to prove the main theorem, it provides us a direction.

Lemma 5.1: *Let Ω be an edge-magic labeling matrix of $P_n \circ N_n$ formed as* (2.1). *Let a be the row sum of Ω. Then*

$$r_j(A_1) = a, \qquad c_j(A_{n-1}) = a \qquad \textit{for } 1 \le j \le n; \tag{5.1}$$

$$c_j(A_{i-1}) + r_j(A_i) = a, \qquad \textit{for } 2 \le i \le n-2,\ 1 \le j \le n; \tag{5.2}$$

$$\sum_{k=1}^{n} r_k(A_i) \left(= \sum_{k=1}^{n} c_k(A_i) \right) = \begin{cases} na & \textit{if } i \textit{ is odd,} \\ 0 & \textit{if } i \textit{ is even.} \end{cases} \tag{5.3}$$

Moreover, a is odd (viewed as an integer).

Proof: It is easy to obtain (5.1) and (5.2). Note that $\sum\limits_{k=1}^{n} r_k(A_i) = \sum\limits_{k=1}^{n} c_k(A_i)$, which is the sum of all entries of A_i, for all i. From (5.1) we have $\sum\limits_{k=1}^{n} c_k(A_1) = na = \sum\limits_{k=1}^{n} r_k(A_{n-1})$. Combining with (5.2) and by a simple calculation we have (5.3).

Consider the sum

$$\sum_{k=1}^{n}\sum_{i=1}^{n-1} r_k(A_i) = \sum_{\substack{1\le i\le n-1 \\ i \text{ is odd}}} na = \tfrac{1}{2}n^2 a.$$

On the other hand, since all elements of $[n^2(n-1)]$ are filled into A_i's,

$$\sum_{k=1}^{n}\sum_{i=1}^{n-1} r_k(A_i) = \tfrac{1}{2}n^2(n-1)[n^2(n-1)+1] = -\tfrac{1}{2}n^2 = \tfrac{1}{2}n^2.$$

Thus $\frac{1}{2}n \equiv \frac{1}{2}na \pmod{n}$. Hence we have $n \equiv na \pmod{2n}$ and then $a \equiv 1 \pmod{2}$. □

Before proving the existence of edge-magic labeling matrix of $P_n \circ N_n$, we introduce some notations and terminologies.

Let R be a ring (or an additive group). Let $\alpha = (a_1, a_2, \ldots, a_p)^T$ be a column vector over R. We write $\overline{\alpha} = (a_p, \ldots, a_2, a_1)^T$. Let $M = (\alpha_1\ \alpha_2\ \cdots\ \alpha_{2s})$ be a $p\times 2s$ matrix. We write $\overline{M} = (\alpha_1\ \cdots\ \alpha_s\ \overline{\alpha_{s+1}}\ \cdots\ \overline{\alpha_{2s}})$. Suppose $M = (m_{i,j})$ be a $p \times p$ matrix over R. For $1 \le j \le p$, the *j-th diagonal* of M and the *j-th anti-diagonal* of M are the ordered sequences

$$(m_{1,j},\ m_{2,j+1},\ \ldots,\ m_{p-1,j-2},\ m_{p,j-1})$$

$$\text{and}\quad (m_{1,j},\ m_{2,j-1},\ \ldots,\ m_{p-1,j+2},\ m_{p,j+1})$$

respectively, where the indices are taken modulo p. The j-th *diagonal* (*anti-diagonal*) *sum* of M is the sum of entries of the j-th diagonal (anti-diagonal) of M and is denoted by $d_j(M)$ ($\tilde{d}_j(M)$).

Lemma 5.2: *Let R be a ring (or group) and let $d \in R$. Suppose $\alpha_i = (a_i, a_i+d, \ldots, a_i+(2s-1)d)^T$ for some $a_i \in R$, $1 \le i \le 2s$. Let $M = (\alpha_1\ \alpha_2\ \cdots\ \alpha_{2s})$. Then*

$$r_j(\overline{M}) = s(2s-1)d + K \quad \textit{for all } 1 \le j \le 2s;$$

$$d_j(\overline{M}) = \begin{cases} (s^2-3s+2sj)d + K & \textit{if } 1 \le j \le s, \\ (5s^2+s-2sj)d + K & \textit{if } s+1 \le j \le 2s; \end{cases}$$

$$\tilde{d}_j(\overline{M}) = \begin{cases} (3s^2-s-2sj)d + K & \textit{if } 1 \le j \le s, \\ (-s^2-s+2sj)d + K & \textit{if } s+1 \le j \le 2s, \end{cases}$$

where $K = \sum_{i=1}^{2s} a_i$.

Proof: Let $b_i = a_i + (2s-1)d$. Then $\overline{\alpha_i} = (b_i, b_i-d, \ldots, b_i-(2s-1)d)$. Thus

$$\begin{aligned} r_j(\overline{M}) &= \sum_{i=1}^{s}[a_i+(j-1)d] + \sum_{i=s+1}^{2s}[b_j-(j-1)d] \\ &= \sum_{i=1}^{s} a_i + \sum_{i=s+1}^{n} b_i = s(2s-1)d + K. \end{aligned}$$

For $1 \le j \le s$,

$$\begin{aligned}
d_j(\overline{M}) &= \sum_{\substack{1 \le i \le 2s \\ k-i \equiv j-1 \pmod{2s}}} (\overline{M})_{i,k} \\
&= \sum_{i=1}^{s} a_i + \sum_{i=s+1}^{2s} b_i + d\left(\sum_{i=1}^{s-j+1} (i-1) - \sum_{i=s-j+2}^{2s-j+1} (i-1) + \sum_{i=2s-j+2}^{2s} (i-1) \right) \\
&= (-s^2 - 2s + 2sj)d + s(2s-1)d + K = (s^2 - 3s + 2sj)d + K.
\end{aligned}$$

For $s+1 \le j \le 2s$,

$$\begin{aligned}
d_j(\overline{M}) &= \sum_{\substack{1 \le i \le 2s \\ k-i \equiv j-1 \pmod{2s}}} (\overline{M})_{i,k} \\
&= \sum_{i=1}^{s} a_i + \sum_{i=s+1}^{2s} b_i + d\left(- \sum_{i=1}^{2s-j+1} (i-1) + \sum_{i=2s-j+2}^{3s-j+1} (i-1) - \sum_{i=3s-j+2}^{2s} (i-1) \right) \\
&= (5s^2 + s - 2sj)d + K.
\end{aligned}$$

By a similar calculation, we have $\tilde{d}_j(\overline{M})$. □

Corollary 5.3: *Keep the notations as in Lemma 5.2. If $R = \mathbb{Z}_{16m^2}$, $s = 2m$ and $d = 4m$ then the row sums, the diagonal sums and the anti-diagonal sums of $\overline{M}$ are equal to $8m^2 + \sum_{i=1}^{4m} a_i$.*

If we can fill all elements of $[\boldsymbol{n^2(n-1)}]$ into A_j's of (2.1) such that A_j's satisfy (5.1) and (5.2), then $P_n \circ N_n$ is edge-magic. For concentrating to find such matrices, we add an extra condition

$$c_j(A_i) = r_j(A_i) \qquad \text{for } 1 \le i \le n-1,\ 1 \le j \le n. \tag{5.4}$$

Combine (5.1), (5.2) and (5.4) we have

$$r_j(A_i) = c_j(A_i) = \begin{cases} a & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases} \quad \text{for all } 1 \le j \le n.$$

Therefore, the problem becomes to find $\frac{n}{2}$ matrices whose row sums and column sums are equal to an odd integer and $\frac{n}{2} - 1$ matrices whose row sums and column sums are equal to 0.

Theorem 5.4: *$P_{4m} \circ N_{4m}$ is edge-magic for $m \ge 1$.*

Proof: Let A be defined as (4.1) when $n = 4m$. Using the elements of $[16m^2]\times(4m-1)$ we have $4m-1$ copies of matrix A. Let α_i be the i-th column of A. From the proof of Theorem 4.3 we have $r_j(A) = 2m(4m+1)$ which is even.

Case 1: m is odd.

For $1 \le i \le 2m$, let B_i be the matrix obtained from A by replacing the $(2i-1)$-th column by α_{2i}, i.e., the $(2i-1)$-th and $(2i)$-th columns of B_i are α_{2i}. Then $r_j(B_i) = 2m(4m+1)+1$. By Corollary 5.3 the row sums, the diagonal sums and the anti-diagonal sums of $\overline{B_i}$ are equal to $2m(4m+1)+1+8m^2 = 2m+1$. Let A_{2i-1} be the matrix whose j-th column is the j-th diagonal of $\overline{B_i}$. Note that one can use the anti-diagonals.

Let us define a notation $S(a) = \begin{pmatrix} a & -a \\ -a & a \end{pmatrix}$, which will be used frequently.

It remains $2m-2$ copies of A and 2 copies of α_{2i-1}, $1 \le i \le 2m$ that we have to handle. For the k-th coordinate a of α_j there is a unique number $b \neq a$ which is the k-th coordinate of $\overline{\alpha_{4m-j}}$ (where $\alpha_{4m} = \alpha_0$) such that $a+b = 16m^2 = 0$ except $a = 8m^2$ and $a = 16m^2$, i.e., $b = -a$. Thus for each 2 copies of α_i and α_{4m-j} (if $j = 2m$ or $4m$ we only need 2 copies of α_{2m} or 1 copy of α_{4m}, respectively), we can group a certain number of matrices $S(a)$. Now it remains $2m-2$ copies of $8m^2$ and $16m^2 = 0$ that we have to handle. Since $2m-2$ is a 4 multiple, we obtain $\frac{1}{2}(m-1)$ copies of matrices $S(8m^2)$ and $S(0)$. Combining any $4m^2$ of these 2×2 matrices as blocks we obtain a $16m^2 \times 16m^2$ matrix. Thus we obtain $2m-1$ matrices, say $A_2, A_4, \dots, A_{4m-2}$, whose row sums and column sums are equal to 0.

Substituting those A_j's into (2.1) we obtain an edge-magic labeling matrix of $P_{4m} \circ N_{4m}$ is edge-magic.

Case 2: m is even.

Similar to Case 1, but now we replace the $(2i)$-th column of A by α_{2i-1} to obtain B_i, $1 \le i \le 2m$. Then $r_j(B_i) = 2m(4m+1)-1$. By the same process as Case 1 we obtain A_{2i-1}.

It remains $2m-2$ copies of A and 2 copies of α_{2i}, $1 \le i \le 2m$. Similar to Case 1 we can group all the remaining numbers as matrices $S(a)$ except the numbers $8m^2$ and 0. Now we have $2m$ copies of $8m^2$ and 0. Since m is even, similar to Case 1, we obtain $\frac{1}{2}m$ copies of matrices $S(8m^2)$ and $S(0)$, and also obtain $2m-1$ matrices, say $A_2, A_4, \dots, A_{4m-2}$, whose row sums and column sums are equal to 0. Hence we obtain an edge-magic labeling matrix of $P_{4m} \circ N_{4m}$. □

Theorem 5.5: *$P_{4m+2} \circ N_{4m+2}$ is edge-magic for $m \ge 1$.*

Proof: Each $[(4m+2)^2]$ can be formed as a magic square M of order

$4m+2$ (since $4m+2 \geq 6$, the magic square exists*). Note that the magic number is $(2m+1)[(4m+2)^2+1]$ which is odd. Use $2m+1$ copies of $[(\mathbf{4m+2})^{\mathbf{2}}]$ to obtain $2m+1$ magic squares. It remains $2m$ copies of A which is defined as (4.1) when $n = 4m+2$ that we have to handle.

If m is even then similar to the proof of Theorem 5.4 we prove the theorem. If m is odd then after grouping elements as matrices $S(a)$ it remains two $h = \frac{1}{2}(4m+2)^2$'s and two 0's. There are three matrices $S(a)$, $S(a')$ and $S(b)$ such that $a + a' + h = 0$. Then we let

$$D = \left(\begin{array}{cc|cc} h & 0 & a & a' \\ 0 & h & -a & -a' \\ \hline a & -a & b & -b \\ a' & -a' & -b & b \end{array}\right).$$

Combine D with an appropriate number of matrices $S(c)$ to form a $(4m+2) \times (4m+2)$ matrix such that whose row sums and column sums are 0. The rest is same as the proof of Theorem 5.4. □

We use the following two examples to demonstrate the proofs of Theorems 5.4 and 5.5.

Example 5.1: Consider $P_4 \circ N_4$. First we have

$$B_1 = \begin{pmatrix} 2 & 2 & 3 & 4 \\ 6 & 6 & 7 & 8 \\ 10 & 10 & 11 & 12 \\ 14 & 14 & 15 & 16 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 2 & 4 & 4 \\ 5 & 6 & 8 & 8 \\ 9 & 10 & 12 & 12 \\ 13 & 14 & 16 & 16 \end{pmatrix}.$$

$$\overline{B_1} = \begin{pmatrix} 2 & 2 & 15 & 16 \\ 6 & 6 & 11 & 12 \\ 10 & 10 & 7 & 8 \\ 14 & 14 & 3 & 4 \end{pmatrix}, \quad \overline{B_2} = \begin{pmatrix} 1 & 2 & 16 & 16 \\ 5 & 6 & 12 & 12 \\ 9 & 10 & 8 & 8 \\ 13 & 14 & 4 & 4 \end{pmatrix}.$$

Then use the diagonals of $\overline{B_i}$ we have

$$A_1 = \begin{pmatrix} 2 & 2 & 15 & 16 \\ 6 & 11 & 12 & 6 \\ 7 & 8 & 10 & 10 \\ 4 & 14 & 14 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 2 & 16 & 16 \\ 6 & 12 & 12 & 5 \\ 8 & 8 & 9 & 10 \\ 4 & 13 & 14 & 4 \end{pmatrix}.$$

If use the anti-diagonals of $\overline{B_i}$ we have

$$A_1 = \begin{pmatrix} 2 & 2 & 15 & 16 \\ 12 & 6 & 6 & 11 \\ 7 & 8 & 10 & 10 \\ 14 & 3 & 4 & 14 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 2 & 16 & 16 \\ 12 & 5 & 6 & 12 \\ 8 & 8 & 9 & 10 \\ 14 & 4 & 4 & 13 \end{pmatrix}.$$

*The existence of magic squares can be found in [1, 2]

It remains 2 copies of $\alpha_1 = (1,\ 5,\ 9,\ 13)^T$ and $\alpha_3 = (3,\ 7,\ 11,\ 15)^T$. Let

$$A_2 = \left(\begin{array}{cc|cc} 1 & 15 & 3 & 13 \\ 15 & 1 & 13 & 3 \\ \hline 5 & 11 & 7 & 9 \\ 11 & 5 & 9 & 7 \end{array}\right)$$

Substituting A_i's into (2.1) we have an edge-magic labeling of $P_4 \circ N_4$.

Alternatively, following is an ad hoc edge-magic labeling matrix.

$$A_1 = \begin{array}{l} \begin{pmatrix} 2 & 2 & 3 & 4 \\ 6 & 6 & 7 & 8 \\ 10 & 10 & 11 & 12 \\ 14 & 14 & 15 & 16 \end{pmatrix} \begin{matrix} 11 \\ 11 \\ 11 \\ 11 \end{matrix} \\ \ \ \begin{matrix} 0 & \ 0 & \ 4 & \ 8 \end{matrix} \end{array}, \quad A_2 = \begin{array}{l} \begin{pmatrix} 1 & 4 & 5 & 1 \\ 5 & 12 & 8 & 2 \\ 10 & 14 & 6 & 9 \\ 13 & 9 & 16 & 13 \end{pmatrix} \begin{matrix} 11 \\ 11 \\ 7 \\ 3 \end{matrix} \\ \ \ \begin{matrix} 13 & 7 & 3 & 9 \end{matrix} \end{array},$$

$$A_3 = \begin{array}{l} \begin{pmatrix} 3 & 5 & 9 & 13 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \\ 1 & 7 & 11 & 15 \end{pmatrix} \begin{matrix} 14 \\ 4 \\ 8 \\ 2 \end{matrix} \\ \ \ \begin{matrix} 11 & 11 & 11 & 11 \end{matrix} \end{array},$$

where the columns besides the matrices are the row sums and the rows under the matrices are the column sums. ∎

Example 5.2: Consider $P_6 \circ N_6$. Choose a magic square

$$M = \begin{pmatrix} 1 & 35 & 34 & 3 & 32 & 6 \\ 30 & 8 & 28 & 27 & 11 & 7 \\ 24 & 23 & 15 & 16 & 14 & 19 \\ 13 & 17 & 21 & 22 & 20 & 18 \\ 12 & 26 & 9 & 10 & 29 & 25 \\ 31 & 2 & 4 & 33 & 5 & 36 \end{pmatrix} = A_1 = A_3 = A_5.$$

$$A_2 = \left(\begin{array}{cc|cc|cc} 18 & 36 & 1 & 17 & 3 & 33 \\ 36 & 18 & 35 & 19 & 33 & 3 \\ \hline 1 & 35 & 2 & 34 & 4 & 32 \\ 17 & 19 & 34 & 2 & 32 & 4 \\ \hline 5 & 31 & 6 & 30 & 7 & 29 \\ 31 & 5 & 30 & 6 & 29 & 7 \end{array}\right), \quad A_4 = \left(\begin{array}{cc|cc|cc} 8 & 28 & 9 & 27 & 10 & 26 \\ 28 & 8 & 27 & 9 & 26 & 10 \\ \hline 11 & 25 & 12 & 24 & 13 & 23 \\ 25 & 11 & 24 & 12 & 23 & 13 \\ \hline 4 & 22 & 15 & 21 & 16 & 20 \\ 22 & 4 & 21 & 15 & 20 & 16 \end{array}\right).$$

Substituting A_i's into (2.1) we obtain an edge-magic labeling matrix of $P_6 \circ N_6$. ∎

6. Edge-magic Labeling of $P_m \circ N_{mk}$ with odd mk

In this section, we apply the method described in Section 4 to the case that $n = mk$ for some positive integer k. So now, unless stated otherwise all arithmetic are taken in $\mathbb{Z}_{m^2k}$.

We let $n = mk$ with $m \geq 3$ for some positive integer k. We have to fill all the elements of $[\boldsymbol{n^2(m-1)}] = [\boldsymbol{m^2k^2(m-1)}] = [\boldsymbol{m^2k}] \times k(m-1)$ into the matrices A_j's in (2.1) such that all row sums are equal.

Note that $[\boldsymbol{n^2}]=[\boldsymbol{m^2k}]\times k$. We let A be the same as (4.1). Then all the equalities described in the proof of Theorem 4.3 are still hold. So we use A, $\sigma\tau(A)^T$ ($m-3$ times) and $\sigma(A)^T$ for the block matrices above the diagonal, and their transposes for the blocks under the diagonal. Then we obtain an edge-magic labeling matrix for $P_m \circ N_{mk}$, where mk is odd. Hence we have

Theorem 6.1: *Suppose $m \geq 3$ and k are odd. Then $P_m \circ N_{mk}$ is edge-magic.*

Although the general case is elusive, we have some ad hoc edge-magic labelings (see the following two examples) and we believe the following conjecture is true.

Conjecture: For m, $n \geq 3$ if $n(n^2-1) \equiv 0 \pmod{m}$, then $P_m \circ N_n$ is edge-magic.

Example 6.1: For $n = 2$ and $m = 3$, substituting

$$A_1 = \begin{pmatrix} 1 & 8 \\ 4 & 5 \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 3 & 7 \\ 6 & 2 \end{pmatrix} \equiv \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix},$$

into (2.1) we obtain an edge-magic labeling matrix of $P_3 \circ N_2$. ∎

Example 6.2: For $n = 2$ and $m = 6$, substituting

$$A_1 = \begin{pmatrix} 2 & 11 \\ 4 & 9 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 5 & 2 \\ 4 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 10 & 6 \\ 7 & 3 \end{pmatrix},$$
$$A_4 = \begin{pmatrix} 12 & 8 \\ 1 & 3 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 5 & 7 \\ 8 & 6 \end{pmatrix},$$

into (2.1) we obtain an edge-magic labeling matrix of $P_6 \circ N_2$. ∎

References

[1] W.S. Andrews, *Magic Squares and Cubes*, Open Court Pub. Co., 1908; 2nd ed., Dover, 1960.

[2] W.H. Benson and O. Jacoby, *New Recreations with Magic Squares*, Dover, 1976.

[3] D. Craft, E.H. Tesar, On a question by Erdős about edge-magic graphs, *Discrete Mathematics*, **207**, 271-276, 1999.

[4] R.D. Godbold and R.J. Slater, All cycles are Edge-Magic, *Bull. ICA*, **22**, 93-97, 1998.

[5] Li-Ming Lee, Sin-Min Lee and Gita Murthy, On edge-graceful labelings of complete graphs — solutions of Lo's conjecture, *Congressus Numerantium*, **62**, 225-233, 1988.

[6] Sin-Min Lee and Eric Seah, Edge-graceful labeling of regular complete k-partite graphs, *Congressus Numerantium*, **75**, 41-50, 1990.

[7] Sin-Min Lee and Eric Seah, On Edge-Gracefulness of the Composition of Step Graphs with Null graphs, *Graph theory, Combinatorics, Algorithm, and Applications, Proc. 2nd Int. Conf., San Francisco, 1989*, 325-330, 1991.

[8] Sin-Min Lee, New Directions in the Theory of Edge-Graceful Graphs, *Proceedings of the 6th Caribbean Conference on Combinatorics and Computing*, 216-231, 1991.

[9] Sin-Min Lee, Eric Seah and Sie-Keng Tan, On Edge-magic Graphs, *Congressus Numerantium*, **86**, 179-191, 1992.

[10] S.P. Lo, On edge-graceful labelings of graphs, *Congressus Numerantium*, **50**, 231-241, 1985.

[11] G.Ringel and A.S. Llado, Another tree conjecture, *Bull. ICA*, **18**, 83-85, 1996.

[12] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge-magicness of the composition of a cycle with a null graph, *Congressus Numerantium*, **132**, 9-18, 1998.

[13] W.C. Shiu, Some Results on Edge-magic Labeling of the Composition of Paths with Null Graphs, *Technical Report of Department of Mathematics, Hong Kong Baptist University*, **206**, 1999.

[14] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge-magic Index Sets of (p,p)-graphs, *Congressus Numerantium*, **137**, 97-107, 1999.

[15] W.C. Shiu, P.C.B. Lam and H.L. Cheng, Supermagic Labeling of an s-duplicate of $K_{n,n}$, *Congressus Numerantium*, **146**, 119-124, 2000.

[16] W.C. Shiu and Sin-Min Lee, Some Edge-magic Cubic Graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **40**, 115-127, 2002.

[17] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, On a Construction of Supermagic of Graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **42**, 147-160, 2002.

[18] W.C. Shiu, P.C.B. Lam and H.L. Cheng, Edge-gracefulness of the Composition of Paths with Null Graphs, *Discrete Mathematics*, **253**, 63-76, 2002.

[19] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge-magic Indices of $(n, n-1)$-graphs, *Electronic Notes in Discrete Mathematics*, **11**, 443-458, 2002.