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SOME RESULTS ON k-EDGE-MAGIC BROKEN WHEEL GRAPHS

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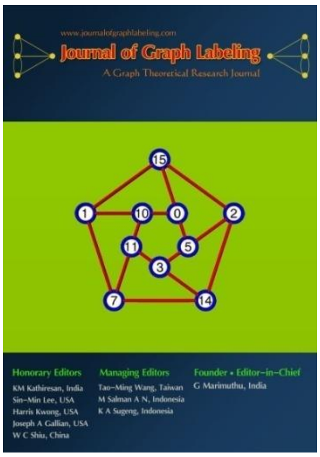

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Abstract

Let G be a (p,q)-graph in which the edges are labeled $k, k+1, \dots, k+q-1$, where $k \in \mathbb{Z}$. The vertex sum for a vertex v is the sum of the labels of the incident edges at v. If the vertex sums are constant modulo p, then G is said to be k-edge-magic. In this paper, we give necessary conditions for a family of regular broken wheel graphs to admit k-edge-magic labelings. Consequently, we show that some of these conditions are also sufficient.

1 Introduction

All undefined symbols and concepts may be looked up from [1]. Let $G = (V, E)$ be a (p,q)-graph, i.e., $|V| = p$ and $|E| = q$. Let $f : E \to \{k, k+1, \dots, k+q-1\}$ be a bijection for some $k \in \mathbb{Z}$. The *induced mapping* $f^+ : V \to \mathbb{Z}_r$ of f is defined by $f^+(u) = \sum\limits_{uv \in E} f(uv)$ for $u \in V$, the sum is taken in $\mathbb{Z}_r$ for some $r \geq 0$. Note that we fix $\mathbb{Z}_r = \{0, 1, \dots, r-1\}$ for $r \geq 1$ and denote $\mathbb{Z}$ by $\mathbb{Z}_0$. If f^+ is a constant mapping, then G is called *k-edge-magic over $\mathbb{Z}_r$*. If $k = 1$, then G is simply called *edge-magic over $\mathbb{Z}_r$*, f an *edge-magic labeling of G over $\mathbb{Z}_r$* and the value of f^+ an *edge-magic value of G over $\mathbb{Z}_r$*. This concept was first introduced by Shiu and Lee [18] in 2002. Moreover, G being edge-magic over $\mathbb{Z}_p$ or $\mathbb{Z}$ is called *edge-magic* or *supermagic*, the labeling f is called an *edge-magic labeling* or *supermagic labeling*, respectively. These concepts were introduced by Lee, Seah and Tan [9] in 1992 and Stewart [22] in 1966, respectively. Note that edge-magic value is not unique in general.

The necessary condition (see [16]) for (p,q)-graph being k-edge-magic is

$$q(q + 2k - 1) \equiv 0 \pmod{p}. \tag{1.1}$$

Some edge-magic or supermagic graphs were found [5, 9–11, 14–23]. More about supermagic graphs can be found in [2, 4, 6, 7]. For regular graph, there is no different between k-edge-magic and edge-magic (see [16, 18, 19]).

Let $[\boldsymbol{a}, \boldsymbol{b}]$ denote the set of integers from a to b. Let S and T be multisets of integers. $S \equiv T \pmod{r}$ means that two sets are equal after their elements are taken modulo r, where $r \geq 2$. From now on, the term "set" means multiset. Set operations are viewed as multiset operations.

A *wheel graph* (or *wheel*, for short) $W_p = C_{p-1} \vee K_1$ is the join graph of the cycle C_{p-1} and the complete graph K_1, where $p \geq 4$. So W_p is a $(p, 2p-2)$-graph and hence the necessary condition for W_p-graph being k-edge-magic is

$$4k \equiv 6 \pmod{p}. \tag{1.2}$$

Fukuchi [3] showed that wheel graph with p vertices is edge-magic if integer $p \geq 5$, and $p \not\equiv 0 \pmod 4$. However, the term 'edge-magic' in [3] is different from edge-magic in this paper. Lee, Wong and Lo [12] studied on the $Q(a)$-balance edge-magic graphs and provided some results, such as W_5, W_7, W_8 and W_9 are strong balance edge-magic. They also showed that all wheels are not edge-magic since (1.1) is not satisfied. By using (1.2) it is easy to obtain:

Proposition 1.1 ([13]). *If the wheel graph W_p is k-edge-magic, then we have the following three cases:*

1. $p = 4h + 1$ *and* $k \equiv 2h + 2 \pmod{p}$*;*
2. $p = 4h + 3$ *and* $k \equiv 2h + 3 \pmod{p}$*;*
3. $p = 4h + 2$ *and* $k \equiv h + 2$ *or* $3h + 3 \pmod{p}$*.*

Lee, Su and Wang [13] showed the converse for cases 1 and 2. They also showed that W_6 is k-edge-magic for all $k \equiv 0, 3 \pmod 6$, as an example. They conjectured that the converse of case 3 holds. It is still open.

It is easy to obtain the following proposition.

Proposition 1.2. *Suppose G is a graph of order p and $k \in \mathbb{Z}$. Then G is k-edge-magic if and only if G is $(k + pt)$-edge-magic for $t \in \mathbb{Z}$.*

So from now on we always assume that $1 \leq k \leq p$.

2 Broken wheel graphs

In this paper, we shall study the k-edge-magicness of some broken wheel graphs. Let us introduce some definitions and notation first.

Let $V(W_p) = \{c, u_1, \ldots, u_{p-1}\}$, where $u_1u_2 \cdots u_{p-1}u_1$ is a $(p-1)$-cycle and c is the *hub* (or the *center*) of the wheel, i.e., $\deg(c) = p - 1 = \Delta(W_p)$. The edge cu_i, $1 \leq i \leq p - 1$ is called a *spoke* of the wheel. Let $S = \{cu_i \mid 1 \leq i \leq p - 1\}$ be the set of all spokes. Let $\varnothing \neq A \subset S$. The graph $W_p(A) = W_p - (S \setminus A)$ is called a *broken wheel graph* (or *broken wheel*, for short). We shall keep these notation throughout this paper.

Let w be a factor of $p - 1$ with $w \geq 2$. Let $A_w = \{cu_{iw+1} \mid 0 \leq i \leq (p - 1)/w - 1\}$. The graph $RW_p(w) = W_p(A_w)$ is called a *regular broken wheel*. Note that $RW_p(w)$ is a

(p,q)-graph, where $q = (w+1)(p-1)/w$. Clearly $p \le q < 2p$. The following figures show W_9, $RW_9(2)$, $RW_9(4)$ and $RW_9(8)$.

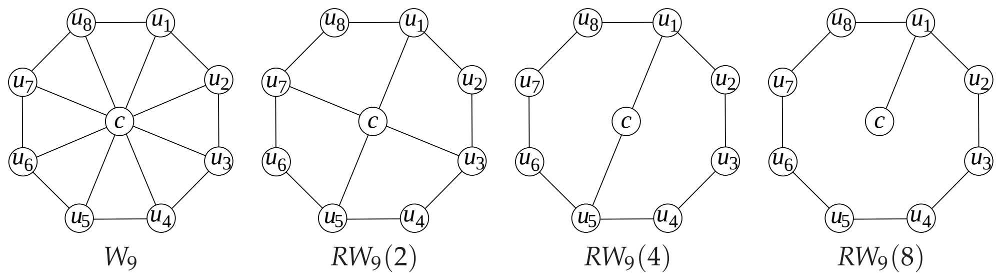


Figure 1

By pigeonhole principle, it is easy to show that

Lemma 2.1. *Let $p \ge 4$, $w|(p-1)$ with $w \ge 2$ and $q = (w+1)(p-1)/w$. Let $L = [\boldsymbol{k}, \boldsymbol{k+q-1}]$ for some k. Then every element in L is congruent modulo p to at most one other element in L. Moreover, $L \equiv \mathbb{Z}_p \cup R$, where $R \subset \mathbb{Z}_p$.*

Theorem 2.2. *If $RW_p(w)$ is k-edge-magic, then $w = 2$.*

Proof. Suppose $w \ge 5$ and f is a k-edge-magic labeling of $RW_p(w)$. Since $f^+(u_2) = f^+(u_3) = f^+(u_4) = f^+(u_5)$ in $\mathbb{Z}_p$, $f(u_1u_2) = f(u_3u_4) = f(u_5u_6)$ in $\mathbb{Z}_p$. But since the image of f is $L = [\boldsymbol{k}, \boldsymbol{k+q-1}]$, where $q = (w+1)(p-1)/w$, by Lemma 2.1 we get a contradiction. So $2 \le w \le 4$.

Suppose $w = 4$. Consider the subpath $u_{1+4i}u_{2+4i}u_{3+4i}u_{4+4i}u_{5+4i}$ of the cycle $C = u_1u_2\cdots u_{p-1}u_1$, $0 \le i \le (p-5)/4$ (here $u_p = u_1$). By a similar argument above we obtain that $f(u_{1+4i}u_{2+4i}) = f(u_{3+4i}u_{4+4i})$ and $f(u_{2+4i}u_{3+4i}) = f(u_{4+4i}u_{5+4i})$ in $\mathbb{Z}_p$. Since f is a bijection, by Lemma 2.1 $L \equiv Q \cup Q \pmod{p}$ for some $Q \subseteq \mathbb{Z}_p$. But it is impossible by Lemma 2.1 again.

Suppose $w = 3$. Consider the subpath $u_{1+3i}u_{2+3i}u_{3+3i}u_{4+3i}$ of the cycle C, $0 \le i \le (p-4)/3$. Similarly, we have $f(u_{1+3i}u_{2+3i}) = f(u_{3+3i}u_{4+3i})$ in $\mathbb{Z}_p$. Then

$$\{f(u_{1+3i}u_{2+3i}), f(u_{3+3i}u_{4+3i}) \mid 0 \le i \le (p-4)/3\} \equiv Q \cup Q \pmod{p},$$

where $Q \subseteq \mathbb{Z}_p$ with $|Q| = (p-1)/3$. By viewing Q as a subset of $\mathbb{Z}$, let r be the largest integer in Q. Then there is an edge e such that $f(e) = r + p$. Since $|Q| = (p-1)/3$, $r + p \ge (k + (p-1)/3 - 1) + p = k + q$ which is impossible. $\square$

By the above theorem, we only focus on $w = 2$. Thus if $RW_p(2)$ is k-edge-magic, then $p = 2n+1$ for some $n \ge 2$. In some articles, for example [8], $RW_{2n+1}(2)$ is also called a *gear graph* and denoted by G_n. For simplicity, we shall use this notation for the rest of this paper.

3 Property of edge-magic gear graphs

Suppose $f : E(G_n) \to L = [\boldsymbol{k}, \boldsymbol{k} + \boldsymbol{3n} - \boldsymbol{1}]$ is a k-edge-magic labeling of G_n. We let $L \equiv \mathbb{Z}_{2n+1} \cup R \pmod{2n+1}$ for some $R \subset \mathbb{Z}_{2n+1}$. Note that $|R| = n - 1$. For convenience, we let $u_0 = u_{2n}$ and $u_{2n+1} = u_1$. We shall keep these notation throughout this paper.

Theorem 3.1. *For $n \geq 2$, if f is a k-edge-magic labeling of G_n, then $f^+ = 0$.*

Proof. Suppose $f^+ = s$ for some $s \in \mathbb{Z}_{2n+1}$. Then we have

$$f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) \equiv s \pmod{2n+1},\ 1 \leq i \leq n; \tag{3.1}$$

$$f(u_{2i-2}u_{2i-1}) + f(u_{2i-1}u_{2i}) + f(cu_{2i-1}) \equiv s \pmod{2n+1},\ 1 \leq i \leq n; \tag{3.2}$$

$$\sum_{i=1}^{n} f(cu_{2i-1}) \equiv s \pmod{2n+1}. \tag{3.3}$$

Subtracting (3.2) by (3.1) we have

$$f(u_{2i-2}u_{2i-1}) + f(cu_{2i-1}) - f(u_{2i}u_{2i+1}) \equiv 0 \pmod{2n+1}.$$

Hence

$$\sum_{i=1}^{n} f(u_{2i-2}u_{2i-1}) + \sum_{i=1}^{n} f(cu_{2i-1}) - \sum_{i=1}^{n} f(u_{2i}u_{2i+1}) \equiv 0 \pmod{2n+1}.$$

So we have $\sum\limits_{i=1}^{n} f(cu_{2i-1}) \equiv 0 \pmod{2n+1}$.

Hence from (3.3) we have $s \equiv 0 \pmod{2n+1}$. □

Following we show the necessary condition for $G_{(p-1)/2} = RW_p(2)$ being k-edge-magic.

Lemma 3.2. *Suppose G_n is k-edge-magic for $n \geq 2$, where $1 \leq k \leq 2n+1$. We have*

1. *if $n = 6h$, then $k = 9h + 2$;*
2. *if $n = 6h + 1$, then $k = 3h + 2$, $7h + 3$ or $11h + 4$;*
3. *if $n = 6h + 2$, then $k = 9h + 5$;*
4. *if $n = 6h + 3$, then $k = 3h + 3$;*
5. *if $n = 6h + 4$, then $k = h + 2$, $5h + 5$ or $9h + 8$;*
6. *if $n = 6h + 5$, then $k = 3h + 4$.*

Proof. From (1.1) we have

$$\frac{3}{2}(p-1)\left(\frac{3}{2}(p-1)+2k-1\right) \equiv 0 \pmod{p}. \tag{3.4}$$

1. When $n = 6h$, from (3.4), we have

   $$(18h)(18h+2k-1) \equiv 0 \pmod{12h+1}.$$

   Equivalently, $$(6h-1)(6h+2k-2) \equiv 0 \pmod{12h+1}.$$

   This implies that $$(-3)(3h+k-1) \equiv 0 \pmod{12h+1}.$$

   Thus $$k \equiv 9h+2 \pmod{12h+1}.$$

2. When $n = 6h+1$, from (3.4), we have

   $$(18h+3)(18h+2k+2) \equiv 0 \pmod{12h+3}$$

   which implies that $$3k \equiv -3h+3 \pmod{12h+3}.$$

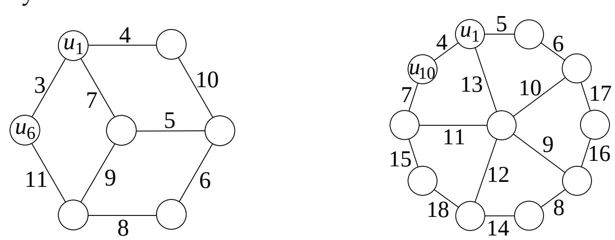
   Thus $$k \equiv -h+1 \equiv 3h+2 \pmod{4h+1}.$$

   So $k = 3h+2$, $7h+3$ or $11h+4$.

3. The proofs of the remaining cases are similar. We leave to readers.

□

Example 3.1. Following are two k-edge-magic labelings of the graphs G_3 and G_5 for a suitable k, respectively.



3-edge-magic labeling for G_3. 4-edge-magic labeling for G_5.

Figure 2

For convenience, we will represent the labeling f of the gear graph G_n by the following list:

$[f(u_{2n}u_1)]$	$f(u_1u_2)$	$f(u_2u_3)$	$f(u_3u_4)$	$\cdots$	$f(u_{2n-1}u_{2n})$	$f(u_{2n}u_1)$
$\cdot$	$f(cu_1)$	$\cdot$	$f(cu_3)$	$\cdots$	$f(cu_{2n-1})$	$\cdot$

So for the 4-edge-magic labeling of G_5 above, we represent it as

[4]	5	6	17	16	8	14	18	15	7	4
·	13	·	10	·	9	·	12	·	11	·

After taking modulo 11, the set of edge labels $L = [\mathbf{4}, \mathbf{18}] \equiv \mathbb{Z}_{11} \cup [\mathbf{4}, \mathbf{7}] \pmod{11}$. So we will represent the list above as

[4]	5	6	6	5	8	3	7	4	7	4
·	2	·	10	·	9	·	1	·	0	·

Lemma 3.3. *Suppose $f : E(G_n) \to \mathbb{Z}_{2n+1} \cup R$ is a k-edge-magic labeling. Suppose $0 \notin R$. Then $f(cu_{2i+1}) = 0$ for some i.*

Proof. Suppose an edge e is incident with a vertex v of degree 2 and $f(e) = 0$. Since $f^+ = 0$, another edge incident with v must be labeled by 0. Thus, $f(cu_{2i+1}) = 0$ for some i. □

Lemma 3.4. *Suppose $f : E(G_n) \to \mathbb{Z}_{2n+1} \cup R$ is a k-edge-magic labeling. Suppose $0 \in R$. Then either $f(cu_{2i+1}) = 0 = f(cu_{2j+1})$ for some distinct i, j or $f(u_{2i-1}u_{2i}) = f(u_{2i}u_{2i+1}) = 0$ for some i.*

Proof. By the proof of Lemma 3.3, if an edge e in the outer cycle of G_n is labeled by 0, then the edge with a common vertex of degree 2 with e is also labeled by 0. This completes the proof since there are exactly two zeros in the list of edge labels. □

For some cases, the necessary condition showed in Lemma 3.2 are not sufficient. Let us show you some examples. All arithmetics about the labels are taken in $\mathbb{Z}_{2n+1}$.

Example 3.2. Let f be a k-edge-magic labeling of G_2. According to the notation in Lemma 3.2, $h = 0$ and $k = 5$. Then $L \equiv \mathbb{Z}_5 \cup \{0\} \pmod 5$. Up to isomorphic and by Lemma 3.4 we may assume that $f(cu_1) = f(cu_3) = 0$. It is easy to see that f does not exist.

Example 3.3. Let f be a k-edge-magic labeling of G_4. According to the notation in Lemma 3.2, $h = 0$ and $k = 2$, 5 or 8.

Suppose $k = 2$. Then $L \equiv \mathbb{Z}_9 \cup \{2, 3, 4\}$. By Lemma 3.3, without loss of generality we may assume $f(cu_1) = 0$. Suppose $f(u_1u_2) = a \not\equiv 0$. Then $f(u_2u_3) \equiv f(u_8u_1) \equiv -a$ and $f(u_7u_8) \equiv a$. But there is no such a in L.

By a similar proof as above, we can show that there is no 5-edge-magic labeling of G_4.

Suppose $k = 8$. Then $L \equiv \mathbb{Z}_9 \cup \{8, 0, 1\} = \mathbb{Z}_9 \cup R$. By Lemma 3.4, without loss of generality, either $f(cu_1) = 0 = f(cu_{2i-1})$ with $i = 2, 3$ or $f(u_1u_2) = f(u_2u_3) = 0$. If $f(cu_1) = 0 = f(cu_3)$, then $f(u_1u_2) = f(u_3u_4) = f(u_7u_8)$ which is impossible. If $f(cu_1) = 0 = f(cu_5)$, then $f(u_1u_2) = f(u_3u_4)$, $f(u_8u_1) = f(u_2u_3)$, $f(u_3u_4) = f(u_5u_6)$ and $f(u_4u_5) = f(u_6u_7)$. It is impossible since $|R| = 3$. Suppose $f(u_1u_2) = f(u_2u_3) = 0$. Let $f(u_3u_4) = a$, and $f(u_7u_8) = b$. Then $f(u_4u_5) = -a$, $f(u_8u_1) = -b$, $f(cu_1) = b$ and $f(cu_3) = -a$. So $\{-a, b\} \equiv \{1, 8\}$. That means $a - b = 0$ or equivalent to $a = b$. It is impossible by Lemma 2.1. So there is no 8-edge-magic labeling of G_4.

Theorem 3.5. *For $h \geq 0$, G_{6h+3} is $(3h+3)$-edge-magic.*

Proof. For $h = 0$, it is shown in Example 3.1. So we assume that $h \geq 1$. Again the arithmetics about the labels are taken in $\mathbb{Z}_{12h+7}$. So we omit to write $\pmod{12h+7}$.

Case 1. Suppose $h = 2m$ for $m \geq 1$. Then $L = [\mathbf{6m+3, 42m+11}]$. An edge-labeling f is defined by

$$f(u_{4t+1}u_{4t+2}) = \begin{cases} 9m+3+t, \text{ if } 0 \leq t \leq 3m; \\ 39m+11-t, \text{ if } 3m+1 \leq t \leq 6m+1, \end{cases} \tag{3.5}$$

$$f(u_{4t+2}u_{4t+3}) = \begin{cases} 15m+4-t, \text{ if } 0 \leq t \leq 3m; \\ 33m+10+t, \text{ if } 3m+1 \leq t \leq 6m+1, \end{cases} \tag{3.6}$$

$$f(u_{4t+3}u_{4t+4}) = \begin{cases} 9m+2-t, \text{ if } 0 \leq t \leq 3m-1; \\ 27m+9+t, \text{ if } 3m \leq t \leq 6m, \end{cases} \tag{3.7}$$

$$f(u_{4t+4}u_{4t+5}) = \begin{cases} 15m+5+t, \text{ if } 0 \leq t \leq 3m; \\ 45m+12-t, \text{ if } 3m+1 \leq t \leq 6m, \end{cases} \tag{3.8}$$

$$f(cu_j) = \begin{cases} 24m+7-2i, \text{ if } j = 4i+1, 1 \leq i \leq 3m; \\ 12m+4+2i, \text{ if } j = 4i+1, 3m+1 \leq i \leq 6m+1; \\ 24m+8+2i, \text{ if } j = 4i+3, 0 \leq i \leq 3m; \\ 36m+9-2i, \text{ if } j = 4i+3, 3m+1 \leq i \leq 6m+1. \end{cases} \tag{3.9}$$

We can check that the image of f is
$[\mathbf{9m+3, 12m+3}] \cup [\mathbf{33m+10, 36m+10}]$ from (3.5);
$[\mathbf{12m+4, 15m+4}] \cup [\mathbf{36m+11, 39m+11}]$ from (3.6);
$[\mathbf{6m+3, 9m+2}] \cup [\mathbf{30m+9, 33m+9}]$ from (3.7);
$[\mathbf{15m+5, 18m+5}] \cup [\mathbf{39m+12, 42m+11}]$ from (3.8); and
$[\mathbf{18m+6, 30m+8}]$ from (3.9).
So f is a bijection.

Now we are going to check that $f^+ = 0$.
$f^+(u_{4t+4}) = f(u_{4t+3}u_{4t+4}) + f(u_{4t+4}u_{4t+5})$

$$= \left\{ \begin{array}{ll} (9m+2-t) + (15m+5+t) = 24m+7, & \text{if } 0 \leq t \leq 3m-1 \\ (27m+9+3m) + (15m+5+3m) = 48m+14, & \text{if } t = 3m \\ (27m+9+t) + (45m+12-t) = 72m+21, & \text{if } 3m+1 \leq t \leq 6m \end{array} \right\} \equiv 0.$$

Similarly, we can verify that $f^+(u_{4t+2}) = 0$ for $0 \leq t \leq 6m+1$.

$$\begin{aligned} f^+(u_1) = f^+(u_{24m+7}) &= f(u_{24m+6}u_1) + f(cu_{24m+7}) + f(u_1u_2) \\ &= (33m+10+6m+1) + (36m+9-12m-2) + (9m+3+0) \\ &= 72m+21 \equiv 0. \end{aligned}$$

$$
\begin{aligned}
f^+(u_{4t+3}) &= f(u_{4t+2}u_{4t+3}) + f(cu_{4t+3}) + f(u_{4t+3}u_{4t+4})\\
&= \begin{cases} (15m+4-t)+(24m+8+2t)+(9m+2-t), & \text{if } 0 \le t \le 3m-1\\ (15m+4-3m)+(24m+8+6m)+(27m+9+3m), & \text{if } t = 3m\\ (33m+10+t)+(36m+9-2t)+(27m+9+t), & \text{if } 3m+1 \le t \le 6m \end{cases}\\
&= \begin{Bmatrix} 48m+14\\ 72m+21\\ 96m+28 \end{Bmatrix} \equiv 0.
\end{aligned}
$$

Similarly, we can verify that $f^+(u_{4t+1}) = 0$ for $0 \le t \le 6m+1$. Finally, since the order of any gear graph is odd and from (1.1), the sum of labels is 0. Thus $f^+(c) = 0$.

Case 2. Suppose $h = 2m+1$ for $m \ge 0$. We only define the labeling f for this case. To show f^+ being an edge-magic labeling is similar to Case 1. It is left to readers.

$$
f(u_{4t+1}u_{4t+2}) = \begin{cases} 9m+7-t, & \text{if } 0 \le t \le 3m+1;\\ 27m+22+t, & \text{if } 3m+2 \le t \le 6m+4, \end{cases}
$$

$$
f(u_{4t+2}u_{4t+3}) = \begin{cases} 15m+12+t, & \text{if } 0 \le t \le 3m+2;\\ 45m+35-t, & \text{if } 3m+3 \le t \le 6m+4, \end{cases}
$$

$$
f(u_{4t+3}u_{4t+4}) = \begin{cases} 9m+8+t, & \text{if } 0 \le t \le 3m+1;\\ 39m+30-t, & \text{if } 3m+2 \le t \le 6m+3, \end{cases}
$$

$$
f(u_{4t+4}u_{4t+5}) = \begin{cases} 15m+11-t, & \text{if } 0 \le t \le 3m+1;\\ 33m+27+t, & \text{if } 3m+2 \le t \le 6m+3, \end{cases}
$$

$$
f(cu_j) = \begin{cases} 24m+19+2i, & \text{if } j = 4i+1, 0 \le i \le 3m+2;\\ 36m+28-2i, & \text{if } j = 4i+1, 3m+3 \le i \le 6m+4;\\ 24m+18-2i, & \text{if } j = 4i+3, 0 \le i \le 3m+1;\\ 12m+11+2i, & \text{if } j = 4i+3, 3m+2 \le i \le 6m+3. \end{cases}
$$

This completes the proof. □

Example 3.4. According to the proof above, we have a 9-edge-magic labeling for G_{15}

[50]	12	19	11	20	13	18	10	21	14	17	9	22	15	16	39
·	31	·	32	·	29	·	34	·	27	·	36	·	25	·	38

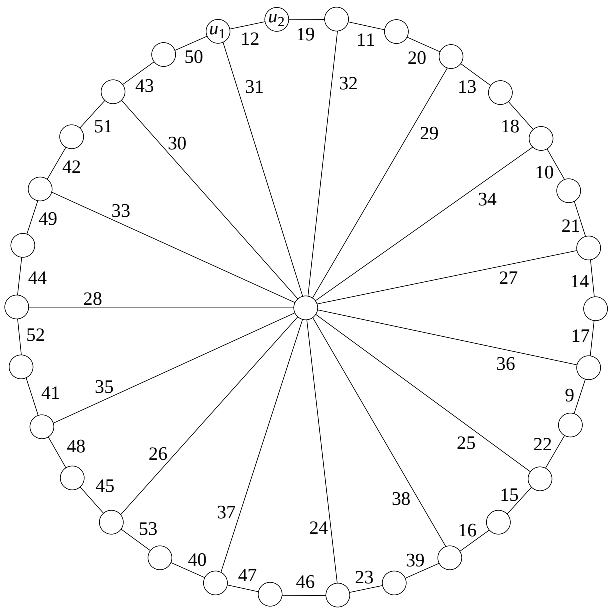
--→

--→

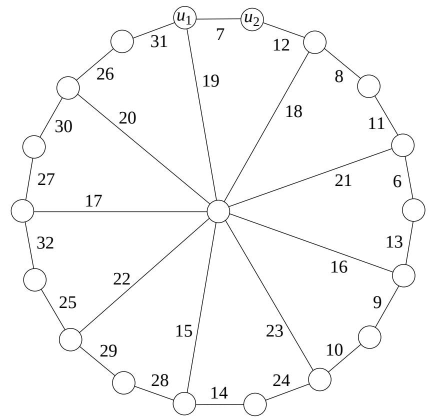
23	46	47	40	53	45	48	41	52	44	49	42	51	43	50
·	24	·	37	·	26	·	35	·	28	·	33	·	30	·

and a 6-edge-magic labeling of G_9

[31]	7	12	8	11	6	13	9	10	24	14	28	29	25	32	27	30	26	31
·	19	·	18	·	21	·	16	·	23	·	15	·	22	·	17	·	20	·



9-edge-magic labeling for G_{15}.



6-edge-magic labeling for G_9.

Figure 3

The proofs of the following theorems are similar to that of Theorem 3.5, so we only provide corresponding labelings and omit the proofs.

Theorem 3.6. *For $h \geq 0$, G_{6h+5} is $(3h+4)$-edge-magic.*

Proof. Suppose $h = 2m$ for $m \geq 0$. Define f by

$$f(u_{4t+1}u_{4t+2}) = \begin{cases} 9m+4-t, \text{ if } 0 \leq t \leq 3m; \\ 27m+13+t, \text{ if } 3m+1 \leq t \leq 6m+2, \end{cases}$$

$$f(u_{4t+2}u_{4t+3}) = \begin{cases} 15m+7+t, \text{ if } 0 \leq t \leq 3m+1; \\ 45m+20-t, \text{ if } 3m+2 \leq t \leq 6m+2, \end{cases}$$

$$f(u_{4t+3}u_{4t+4}) = \begin{cases} 9m+5+t, \text{ if } 0 \leq t \leq 3m; \\ 39m+17-t, \text{ if } 3m+1 \leq t \leq 6m+1, \end{cases}$$

$$f(u_{4t+4}u_{4t+5}) = \begin{cases} 15m+6-t, \text{ if } 0 \leq t \leq 3m; \\ 33m+16+t, \text{ if } 3m+1 \leq t \leq 6m+1, \end{cases}$$

$$f(cu_j) = \begin{cases} 24m+11+2i, \text{ if } j = 4i+1, 0 \leq i \leq 3m+1; \\ 36m+16-2i, \text{ if } j = 4i+1, 3m+2 \leq i \leq 6m+2; \\ 24m+10-2i, \text{ if } j = 4i+3, 0 \leq i \leq 3m; \\ 12m+7+2i, \text{ if } j = 4i+3, 3m+1 \leq i \leq 6m+1. \end{cases}$$

Suppose $h = 2m + 1$ for $m \geq 0$. Define f by

$$f(u_{4t+1}u_{4t+2}) = \begin{cases} 9m + 9 + t, & \text{if } 0 \leq t \leq 3m + 2; \\ 39m + 37 - t, & \text{if } 3m + 3 \leq t \leq 6m + 5, \end{cases}$$

$$f(u_{4t+2}u_{4t+3}) = \begin{cases} 15m + 14 - t, & \text{if } 0 \leq t \leq 3m + 2; \\ 33m + 32 + t, & \text{if } 3m + 3 \leq t \leq 6m + 5, \end{cases}$$

$$f(u_{4t+3}u_{4t+4}) = \begin{cases} 9m + 8 - t, & \text{if } 0 \leq t \leq 3m + 1; \\ 27m + 27 + t, & \text{if } 3m + 2 \leq t \leq 6m + 4, \end{cases}$$

$$f(u_{4t+4}u_{4t+5}) = \begin{cases} 15m + 15 + t, & \text{if } 0 \leq t \leq 3m + 2; \\ 45m + 42 - t, & \text{if } 3m + 3 \leq t \leq 6m + 4, \end{cases}$$

$$f(cu_j) = \begin{cases} 24m + 23 - 2i, & \text{if } j = 4i + 1, 0 \leq i \leq 3m + 2; \\ 12m + 12 + 2i, & \text{if } j = 4i + 1, 3m + 3 \leq i \leq 6m + 5; \\ 24m + 24 + 2i, & \text{if } j = 4i + 3, 0 \leq i \leq 3m + 2; \\ 36m + 33 - 2i, & \text{if } j = 4i + 3, 3m + 3 \leq i \leq 6m + 4. \end{cases}$$

□

Example 3.5. According to the construction above, we have a 4-edge-magic labeling for G_5

[18]	4	7	5	6	14	8	16	17	15	18
·	11	·	10	·	13	·	9	·	12	·

and a 7-edge-magic labeling for G_{11}

[37]	9	14	8	15	10	13	7	16	11	12	29	17	34	35	30	39	33	36	31	38	32	37
·	23	·	24	·	21	·	26	·	19	·	28	·	18	·	27	·	20	·	25	·	22	·

Theorem 3.7. *For $h \geq 1$, G_{6h+1} is $(3h + 2)$-edge-magic.*

Proof. Suppose $h = 2m$ for $m \geq 1$. Define f by

$$f(u_{4t+1}u_{4t+2}) = \begin{cases} 9m + 1 - t, & \text{if } 0 \leq t \leq 3m - 1; \\ 27m + 4 + t, & \text{if } 3m \leq t \leq 6m, \end{cases}$$

$$f(u_{4t+2}u_{4t+3}) = \begin{cases} 15m + 2 + t, & \text{if } 0 \leq t \leq 3m; \\ 45m + 5 - t, & \text{if } 3m + 1 \leq t \leq 6m, \end{cases}$$

$$f(u_{4t+3}u_{4t+4}) = \begin{cases} 9m + 2 + t, & \text{if } 0 \leq t \leq 3m - 1; \\ 39m + 4 - t, & \text{if } 3m \leq t \leq 6m - 1, \end{cases}$$

$$f(u_{4t+4}u_{4t+5}) = \begin{cases} 15m + 1 - t, & \text{if } 0 \leq t \leq 3m - 1; \\ 33m + 5 + t, & \text{if } 3m \leq t \leq 6m - 1, \end{cases}$$

$$f(cu_j)=\begin{cases}24m+3+2i, \text{ if } j=4i+1, 0\le i\le 3m;\\ 36m+4-2i, \text{ if } j=4i+1, 3m+1\le i\le 6m;\\ 24m+2-2i, \text{ if } j=4i+3, 0\le i\le 3m-1;\\ 12m+3+2i, \text{ if } j=4i+3, 3m\le i\le 6m-1.\end{cases}$$

Suppose $h=2m+1$ for $m\ge 0$. Define f by

$$f(u_{4t+1}u_{4t+2})=\begin{cases}9m+6+t, \text{ if } 0\le t\le 3m+1;\\ 39m+24-t, \text{ if } 3m+2\le t\le 6m+3,\end{cases}$$

$$f(u_{4t+2}u_{4t+3})=\begin{cases}15m+9-t, \text{ if } 0\le t\le 3m+1;\\ 33m+21+t, \text{ if } 3m+2\le t\le 6m+3,\end{cases}$$

$$f(u_{4t+3}u_{4t+4})=\begin{cases}9m+5-t, \text{ if } 0\le t\le 3m;\\ 27m+18+t, \text{ if } 3m+1\le t\le 6m+2,\end{cases}$$

$$f(u_{4t+4}u_{4t+5})=\begin{cases}15m+10+t, \text{ if } 0\le t\le 3m+1;\\ 45m+27-t, \text{ if } 3m+2\le t\le 6m+2,\end{cases}$$

$$f(cu_j)=\begin{cases}24m+15-2i, \text{ if } j=4i+1, 0\le i\le 3m+1;\\ 12m+8+2i, \text{ if } j=4i+1, 3m+2\le i\le 6m+3;\\ 24m+16+2i, \text{ if } j=4i+3, 0\le i\le 3m+1;\\ 36m+21-2i, \text{ if } j=4i+3, 3m+2\le i\le 6m+2.\end{cases}$$

□

Example 3.6. According to the construction above, we have an 8-edge-magic labeling for G_{13}

[44]	10	17	11	16	9	18	12	15	8	19	13	14	34
·	27	·	26	·	29	·	24	·	31	·	22	·	33

$\dashrightarrow$

20	40	41	35	46	39	42	36	45	38	43	37	44
·	21	·	32	·	23	·	30	·	25	·	28	·

and an 8-edge-magic labeling for G_7

[24]	6	9	5	10	7	8	19	11	22	23	20	25	21	24
·	15	·	16	·	13	·	18	·	12	·	8	·	14	·

We also find some ad hoc examples. We have six 11-edge-magic labelings for G_6. They are

[27]	12	14	20	19	11	15	24	28	16	23	25	27
·	13	·	18	·	22	·	26	·	21	·	17	·

[25]	14	12	17	22	11	15	24	28	21	18	27	25
·	13	·	23	·	19	·	26	·	16	·	20	·

[14]	25	27	17	22	20	19	15	11	28	24	12	14
·	26	·	21	·	23	·	18	·	13	·	16	·

[27]	12	14	15	24	28	11	20	19	17	22	25	27
·	13	·	23	·	26	·	21	·	16	·	18	·

[14]	13	26	11	15	18	21	17	22	23	16	12	14
·	25	·	28	·	19	·	27	·	20	·	24	·

[14]	13	26	11	15	17	22	16	23	18	21	12	14
·	25	·	28	·	20	·	27	·	24	·	19	·

We obtain two 14-edge-magic labelings for G_8. They are

[37]	14	20	35	16	18	33	27	24	15	19	23	28	36	32	31	37
·	17	·	30	·	34	·	25	·	29	·	26	·	21	·	22	·

[20]	17	34	18	16	32	36	23	28	30	21	22	29	15	19	14	20
·	31	·	33	·	37	·	26	·	27	·	25	·	24	·	35	·

We get one 23-edge-magic labeling for G_{14}:

[60]	27	31	34	24	26	32	35	23	25	33	41	46	30	28
·	29	·	51	·	37	·	49	·	39	·	42	·	40	·

$\dashrightarrow$

$\dashrightarrow$

59	57	43	44	54	62	64	52	55	61	63	53	56	60
58	··	45	·	47	·	48	·	38	·	50	·	36	·

From those examples, we have not discovered any regulation to obtain edge-magic labelings for other unsolved cases. We summarize those unsolved problems below:

Problem 3.1. Find a $(9h+2)$-edge-magic labeling for G_{6h}, $h \geq 1$.

Problem 3.2. Find a $(7h+3)$-edge-magic labeling and a $(11h+4)$-edge-magic labeling for G_{6h+1}, $h \geq 1$.

Problem 3.3. Find a $(9h+5)$-edge-magic labeling for G_{6h+2}, $h \geq 1$.

Problem 3.4. Find a $(h+2)$-edge-magic labeling,a $(5h+5)$-edge-magic labeling, and a $(9h+8)$-edge-magic labeling for G_{6h+4}, $h \geq 1$.

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