# On Generalized Ramsey Numbers*

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**Abstract**

Let $f_1$ and $f_2$ be graph parameters. The Ramsey number $r(f_1 \geq m; f_2 \geq n)$ is defined as the minimum integer $N$ such that any graph $G$ on $N$ vertices, either $f_1(G) \geq m$ or $f_2(\overline{G}) \geq n$. A general existence condition is given and a general upper bound is shown in this paper. In addition, suppose the number of triangles in $G$ is denoted by $t(G)$. We verify that $(1 - o(1))(24n)^{1/3} \leq r(t \geq n; t \geq n) \leq (1 + o(1))(48n)^{1/3}$ as $n \to \infty$.

**Key words and phrases:** Ramsey number, mixed Ramsey number.

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## 1 Introduction

In this paper, all graphs are simple. Let $F$ and $H$ be graphs. The Ramsey number $r(F, H)$ is the smallest integer $N$ such that for any graph $G$ of order $N$, either $G$ contains $F$ as a subgraph or the complement $\overline{G}$ of $G$ contains $H$ as a subgraph. The *classical Ramsey number* $r(K_m, K_n)$, which is often denoted by $r(m, n)$, is difficult to compute or to estimate its bound. Many researchers made some generalization on the classical Ramsey numbers. For a graph parameter $f$ and a number $m$ and a graph $H$, Benedict, Chartrand and Lick [2] defined the *mixed Ramsey number* $\nu(f; m; H)$ to be the least positive integer $N$ such that for any graph $G$ of order $N$, either $f(G) \geq m$ or $\overline{G}$ contains $H$ as a subgraph. Thus $r(m, n) = \nu(\omega; m; K_n)$, where $\omega(G)$ is the clique number of $G$. Several relations between Ramsey number and mixed Ramsey number have been obtained by N. Achuthan, N. R. Achuthan and L. Caccetta [1]. The mixed Ramsey numbers involving some specified parameters have been much studied. Examples include chromatic number by Benedict,

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Chartrand and Lick [2]; edge chromatic number by Lesniak, Polimeni and VanderJagt [10]; total chromatic number by Fink [6] and by Cleves and Jacobson [4]; and vertex arboricity by Lesniak-Foster, by N. Achuthan, N. R. Achuthan and L. Caccetta [1] and by Xu and Zhang [11]. In this paper, we shall introduce a more generalized concept on Ramsey numbers.

## 2 Ramsey number on two parameters

We shall use the following notations. Let $f_1$ and $f_2$ be graph parameters of positive integers. Let $m$ and $n$ be positive integers. Define the *Ramsey number* $r(f_1 \geq m; f_2 \geq n)$ as the least positive integer $N$ such that for any graph $G$ of order $N$ either $f_1(G) \geq m$ or $f_2(\overline{G}) \geq n$. We call $r(f_1 \geq m; f_2 \geq n)$ the *Ramsey number on* $f_1$ *and* $f_2$, or the *mixed Ramsey number* if $f_1$ and $f_2$ are different. For simplicity, if $f_H(G)$ is the number of subgraphs $H$ in $G$, we write $r(H; \cdots)$ for $r(f_H \geq 1; \cdots)$. Thus $r(f_F \geq 1; f_H \geq 1)$ can be written as $r(F; H)$. Undefined symbols and concepts may be found in [3].

If we consider multicoloring the edge set of $K_N$ instead of $G$ and its complement, we can adopt the notation for more than two parameters.

Since the adjacency matrices of graphs form a subspace in a linear space (over $\mathbb{Z}_2$, the field consisting of two elements), the above Ramsey number on parameters can be generalized to linear spaces, or even more general set with some algebraic or analytic structures.

The most natural question is that for which pair $f_1$ and $f_2$, $r(f_1 \geq m; f_2 \geq n)$ exists. This is not always valid. For example, let $\chi(G)$ be the chromatic number of $G$ and let $\alpha(G)$ be the independence number of $G$, then $r(\chi \geq 2; \alpha \geq 2)$ does not exist since the null graph $N_n$ on $n$ vertices, $\chi(N_n) = 1$ and $\alpha(\overline{N}_n) = 1$. However, for most pairs of parameters, the existence of the Ramsey number on these parameters can easily be verified.

**Theorem 2.1** *Let $f_1$ and $f_2$ be graph parameters. Then for any positive integers $m$ and $n$, $r(f_1 \geq m; f_2 \geq n)$ exists if and only if*

$$\lim_{k\to\infty} \left\{ \min_{|V(G)|=k} \{f_1(G) + f_2(\overline{G})\} \right\} = \infty. \tag{1}$$

**Proof:** Suppose for any positive integers $m$ and $n$, $r(f_1 \geq m; f_2 \geq n)$ exists. Then for any integer $M > 0$, $K(M) = r(f_1 \geq M; f_2 \geq M)$ exists. For any graph $G$ with order $k \geq K(M)$, by the definition of $r(f_1 \geq M; f_2 \geq M)$, either $f_1(G) \geq M$ or $f_2(\overline{G}) \geq M$ thus $f_1(G) + f_2(\overline{G}) \geq M + 1$

since $f_i$ takes positive integer value. Therefore

$$\min_{|V(G)|=k}\{f_1(G)+f_2(\overline{G})\} \geq M+1$$

for $k \geq K(M)$ so (1) holds.

Conversely, suppose (1) holds. For any positive integers $m$ and $n$, there exists $K = K(m,n)$ such that if $k \geq K$,

$$\min_{|V(G)|=k}\{f_1(G)+f_2(\overline{G})\} \geq m+n.$$

So for any graph with order $k \geq K$, $f_1(G)+f_2(\overline{G}) \geq m+n$, hence either $f_1(G) \geq m$ or $f_2(\overline{G}) \geq n$. Minimizing such $K$ shows the existence of $r(f_1 \geq m; f_2 \geq n)$. ∎

The graph parameter $f$ is said to be *increasing* if when $G_1$ is a subgraph of $G_2$, then $f(G_1) \leq f(G_2)$. Using the elementary technique [5] in Ramsey theory, we have

**Theorem 2.2** *Let $f_1$ and $f_2$ be increasing graph parameters satisfying $f_i(G \vee v) = f_i(G)+1$ for $v \notin V(G)$ and $f_i(K_1) = 1$, where $G \vee v$ is the graph obtained by completely joining vertices of $G$ with vertex $v$. Then $r(f_1 \geq m; f_2 \geq n)$ exists for any integers $m,n \geq 1$. Moreover,*

$$r(f_1 \geq m; f_2 \geq n) \leq r(f_1 \geq m-1; f_2 \geq n) + r(f_1 \geq m; f_2 \geq n-1) \tag{2}$$

*for any integers $m,n \geq 2$. Consequently for $m,n \geq 1$,*

$$r(f_1 \geq m; f_2 \geq n) \leq \binom{m+n-2}{m-1} \tag{3}$$

*and hence*

$$r(f_1 \geq m; f_2 \geq n) \leq \min\left\{\binom{m+n-2}{m-1}, \binom{m+n-2}{n-1}\right\}. \tag{4}$$

**Proof:** We shall prove the existence of $r(f_1 \geq m; f_2 \geq n)$ and the inequality (3) by induction on $m+n$ with $m,n \geq 1$.

Since $f_1(K_1) = f_2(K_1) = 1$, we have $r(f_1 \geq 1; f_2 \geq n) = r(f_1 \geq m; f_2 \geq 1) = 1$ for $m,n \geq 1$. Thus $r(f_1 \geq m; f_2 \geq n)$ exists and (3) holds in this case. This result includes the case when $2 \leq m+n \leq 3$.

For $k \geq 3$ and $m,n \geq 1$, we assume $r(f_1 \geq m; f_2 \geq n)$ exists and (3) holds when $m+n = k$.

Suppose $m+n = k+1$. For $m = 1$ or $n = 1$, then we are done. For $m,n \geq 2$, by induction assumption $r(f_1 \geq m-1; f_2 \geq n)$ and $r(f_1 \geq m; f_2 \geq n-1)$ exist. Let $N_1 = r(f_1 \geq m-1; f_2 \geq n)$, $N_2 = r(f_1 \geq m; f_2 \geq n-1)$, and $G$ be a graph of order $N_1+N_2$. For any vertex $v$ in $G$, either

$|N_G(v)| \geq N_1$ or $|N_{\overline{G}}(v)| \geq N_2$. By symmetry we assume the first case holds. Let $V' = N_G(v)$ and $H = G[V']$, the vertex-induced subgraph of $G$ by $V'$ (c.f. [3]). Then $H$ is of order at least $N_1$. By induction assumption, either $f_1(H) \geq m-1$ or $f_2(\overline{H}) \geq n$. If $f_1(H) \geq m-1$, then $f_1(H \vee v) \geq m$. Since $H \vee v$ is a subgraph of $G$ and by the monotonicity of $f_1$, $f_1(G) \geq f_1(H \vee v) \geq m$. Similarly, if $f_2(\overline{H}) \geq n$, then since $\overline{H} = \overline{G}[V']$, $f_2(\overline{G}) \geq f_2(\overline{H}) \geq n$.

Hence we have (2) and

$$\begin{aligned} r(f_1 \geq m; f_2 \geq n) &\leq r(f_1 \geq m-1; f_2 \geq n) + r(f_1 \geq m; f_2 \geq n-1) \\ &\leq \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} \\ &= \binom{m+n-2}{m-1}. \end{aligned}$$

■

By symmetry, we have the inequality (4).

Note that, for any graph $F$, $u \in V(F)$ and $v \notin V(F)$, there is a subgraph of the graph $(F-u) \vee v$ isomorphic to $F$. Thus by a similar argument of the above proof, we have the following result.

**Theorem 2.3** *For a fixed graph $F$, define $f_F(G)$ to be the number of subgraphs isomorphic to $F$ with different vertex sets in graph $G$. Then for graphs $F$ and $H$ of order at least 2 and any positive integers $m$ and $n$,*

$$r(f_F \geq m; f_H \geq n) \leq r(f_{F-u} \geq m; f_H \geq n) + r(f_F \geq m; f_{H-v} \geq n)$$

*for any vertices $u \in V(F)$ and $v \in V(H)$.*

By setting $m = n = 1$ we have the following corollary which is one of main results of Huang and Zhang in [8].

**Corollary 2.1** *Let $G$ and $H$ be graphs of order at least two. Then*

$$r(G, H) \leq r(G-u, H) + r(G, H-v)$$

*for any vertices $u \in V(G)$ and $v \in V(H)$.*

The upper bound $\binom{m+n-2}{m-1}$ in Theorem 2.2 may be far from the truth. As an example, we shall prove that

$$r(\chi \geq m; \chi \geq n) = (m-1)(n-1) + 1$$

for any positive integers $m$ and $n$.

In fact, it is obviously true if $m$ or $n$ is one. Assume that $m, n \geq 2$. Let $G$ be the graph consisting of disjoint $n-1$ copies of $K_{m-1}$, then the facts that $\chi(G) = m-1$ and $\chi(\overline{G}) = n-1$ verify

$$r(\chi \geq m; \chi \geq n) \geq (m-1)(n-1)+1.$$

On the other hand, if $G$ is a graph of order $N = (m-1)(n-1)+1$ with $\chi(G) \leq m-1$, then

$$\alpha(G) \geq \left\lceil \frac{N}{\chi(G)} \right\rceil = \left\lceil \frac{(m-1)(n-1)+1}{\chi(G)} \right\rceil \geq n.$$

Therefore $\chi(\overline{G}) \geq \omega(\overline{G}) = \alpha(G) \geq n$. This proves that

$$r(\chi \geq m; \chi \geq n) \leq (m-1)(n-1)+1.$$

Note that $\chi(G) \geq \omega(G)$ and the inequality is strict unless $G$ is perfect (i.e., for any induced subgraph $H$ of $G$, $\omega(H) = \chi(H)$) . Thus $r(\chi \geq m; \chi \geq n) \leq r(m,n)$. Since $r(m,n)$ is much larger than $(m-1)(n-1)+1$ for $n \geq m \geq 3$, this indicates that extremal graphs for $r(m,n)$ is far from being perfect graphs.

As a Ramsey number on specific parameters, we obtain the following theorem.

**Theorem 2.4** *Let $t(G)$ be the number of triangles in $G$. Then*

$$(1-o(1))(24n)^{1/3} \leq r(t \geq n; t \geq n) \leq (1+o(1))(48n)^{1/3},$$

*as $n \to \infty$.*

**Proof:** Let $N = r(t \geq n; t \geq n) - 1$. By definition there is a graph $G$ of order $N$ such that $t(G) \leq n-1$ and $t(\overline{G}) \leq n-1$. Let $v_1, v_2, \ldots, v_N$ be vertices of $G$. We color all edges of $G$ by red and all edges of $\overline{G}$ by blue. Employing a well known technique of Goodman [7], the number of non-monochromatic triangles containing vertex $v_i$ which is incident to edges in different colors is $d_i(N-d_i-1)$, where $d_i = \deg_G(v_i)$. Each triangle of $G$ is counted exactly twice in the summation $\sum_{i=1}^{N} d_i(N-d_i-1)$. Thus the total number of monochromatic triangles is given by $\binom{N}{3} - \frac{1}{2}\sum_{i=1}^{N} d_i(N-d_i-1)$. Thus

$$2(n-1) \geq \binom{N}{3} - \frac{1}{2}\sum_{i=1}^{N} d_i(N-d_i-1).$$

Minimizing the right side, we obtain

$$2(n-1) \geq \binom{N}{3} - \frac{N}{8}(N-1)^2.$$

Simplifying, we have

$$48n \geq N^3 - 6N^2 + 5N + 48 > (N-6)^3,$$

which yields $r(t \geq n; t \geq n) = N + 1 \leq (48n)^{1/3} + 7 \leq (1 + o(1))(48n)^{1/3}$. On the other hand, for any $\epsilon > 0$, set $N = \lceil (1-\epsilon)(24n)^{1/3} \rceil$. Let $G$ be the complete bipartite graph on $N$ vertices and the cardinalities of two classes differ by at most one. Then $t(G) = 0$ and $t(\bar{G}) < n$, which shows that $r(t \geq n; t \geq n) \geq (1 - o(1))(24n)^{1/3}$. ■

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