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BOUNDS ON $L(2,1)$-CHOICE NUMBER OF CARTESIAN PRODUCTS OF PATHS AND SPIDERS*

***Haiying Zhou*[1,†], *Wai Chee Shiu*[2,‡] *and Peter, Che Bor Lam*[3,§]**
[1]School of Applied Mathematics, Beijing Normal University,
Zhuhai, China
[2]Department of Mathematics, Hong Kong Baptist University,
Kowloon Tong, Hong Kong, China
[3]Division of Science and Technology,
BNU-HKBU United International College, Zhuhai, China

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Abstract

For a given graph $G=(V,E)$, let $\mathscr{L}(G)=\{L(v):v\in V\}$ be a prescribed list assignment. G is $\mathscr{L}$-$L(2,1)$-colorable if there exists a vertex labeling f of G such that $f(v)\in L(v)$ for all $v\in V$; $|f(u)-f(v)|\geq 2$ if $d_G(u,v)=1$; and $|f(u)-f(v)|\geq 1$ if $d_G(u,v)=2$. If G is $\mathscr{L}$-$L(2,1)$-colorable for every list assignment $\mathscr{L}$ with $|L(v)|\geq k$ for all $v\in V$, then G is said to be k-$L(2,1)$-choosable. The $L(2,1)$-choice number $\lambda_\ell(G)$ of a graph G is the smallest number k such that G is k-$L(2,1)$-choosable. In this paper, we provide a lower bound on λ_ℓ for the Cartesian products of a path and a spider. Also we provide an upper bound on λ_ℓ for the Cartesian products of K_2 and a spider. The $L(2,1)$-choice number of spider is determined.

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1. Introduction

Let $G=(V,E)$ be a graph of order n. Sometimes, we use $V(G)$ and $E(G)$ to denote V and E, respectively.

As a variation of Hale's channel assignment problem [8], the $L(2,1)$-labeling of a simple graph with a condition at distance two was first proposed and studied by Griggs and Yeh [1]. An $L(2,1)$-labeling of a graph G is a function f from the vertex set of G to the set of nonnegative integers such that $|f(u)-f(v)|\geq 2$ if $d_G(u,v)=1$; and $|f(u)-f(v)|\geq 1$

*This work is partially supported by the Faculty Research Grant, Hong Kong Baptist University.
†E-mail address: zhouhaiying@bnuz.edu.cn
‡E-mail address: wcshiu@hkbu.edu.hk (Corresponding author)
§E-mail address: cblam@uic.edu.hk

if $d_G(u,v) = 2$. The *span* of f is the difference between the maximum and the minimum numbers assigned by f. The $L(2,1)$-*labeling number* $\lambda(G)$ of G is the minimum span over all $L(2,1)$-labelings of G. $L(2,1)$-labelings have been studied extensively during the past years. See surveys [3, 4] for details. Vizing [9] and Erdős et al. [10] generalized the graph coloring problem and introduced the list coloring problem independently more than three decades ago. We shall consider a new variation of the $L(2,1)$-labeling problem, the *list*-$L(2,1)$-*labeling* problem, hopefully, that will help us to solve some $L(2,1)$-labeling problems.

Let $\mathbb{N}$ be the set of all non-negative integers. A list coloring of a graph G is an assignment of labels (colors) to the vertices such that each vertex v receives a label from a prescribed list $L(v) \subseteq \mathbb{N}$ and adjacent vertices receive distinct labels. $\mathscr{L}(G) = \{L(v) : v \in V(G)\}$ is called a *list assignment* of G. G is called k-choosable if G admits a list coloring for all list assignments $\mathscr{L}$ with at least k labels in each list. For list coloring of plane graphs, some results have obtained. All 2-choosable graphs have been characterized by Erdős et al. [10]. Thomassen [11] proved that every plane graph is 5-choosable, whereas Voigt [17] presented examples of plane graphs which are not 4-choosable. Please refer to [12–16] for other results on the choosability of planar graph.

Let $\mathscr{L}(G) = \{L(v) : v \in V(G)\}$ be a list assignment of a graph $G = (V, E)$. G is $\mathscr{L}$-$L(2,1)$-*colorable* if there exists a vertex labeling f of G such that $f(v) \in L(v)$ for all $v \in V$; $|f(u) - f(v)| \geq 2$ if $d_G(u,v) = 1$; and $|f(u) - f(v)| \geq 1$ if $d_G(u,v) = 2$. Such labeling f is called a $\mathscr{L}$-$L(2,1)$-*labeling* of G. If G is $\mathscr{L}$-$L(2,1)$-colorable for every list assignment $\mathscr{L}$ with $|L(v)| \geq k$ for all $v \in V$, then G is said to be k-$L(2,1)$-*choosable.* The $L(2,1)$-*choice number* $\lambda_\ell(G)$ of a graph G is the smallest number k such that G is k-$L(2,1)$-choosable. Obviously, $\lambda_\ell(G) \geq \lambda(G) + 1$. In this paper, we consider the λ_ℓ for the Cartesian products of a path and a spider.

The Cartesian product $G \times H$ of two graphs G and H is defined as the graph with vertex set $V(G \times H) = V(G) \times V(H)$, where two vertices (a, x) and (b, y) are adjacent if and only if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$. The $L(2,1)$-labeling of Cartesian products of some graphs have been studied recently [5, 6].

Let $n, p, m_1, m_2, \ldots, m_p$ be integers, where $n \geq 1, p \geq 3$ and $m_i \geq 2$ for $1 \leq i \leq p$. A graph $S(m_1, m_2, \ldots, m_p)$ is called a *spider* (also called *superstar*) *with* p *legs* if it is obtained from the union of p paths, $P_{m_k}, 1 \leq k \leq p$, with one of the end vertices of each path identified. The identified vertex is called the *center* of the spider graph. The path P_{m_k} is called the k-*th leg* of the spider, $1 \leq k \leq p$. Up to isomorphic, we always assume $m_1 \geq \cdots \geq m_p \geq 2$. We consider the Cartesian product of a path and a spider. We call $P_n \times S(m_1, m_2, \ldots, m_p)$ a *book* and denote it by $B_n(m_1, m_2, \ldots, m_p)$. In particular, $B_1(m_1, \ldots, m_p) = K_{1,p}$ if $m_1 = 2$.

For a fix k, the Cartesian product of P_n and the k-th leg P_{m_k} is a rectangular grid of $n \times m_k$ vertices organized into n rows and m_k columns where the m_k (resp., n) vertices in each row (resp., column) induce a subgraph isomorphic to P_{m_k} (resp., P_n). We call $P_n \times P_{m_k}$ the k-*th page* of the book $B_n(m_1, m_2, \ldots, m_p)$ and each vertex on the k-th page will be represented by an ordered triple (i, k, j) with $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m_k - 1$. Note that all the first column of these p pages must be the same as they represent the *spine* of the book. So, we denote all vertices on the spine $(i, k, 0)$ as $(i, 0, 0)$. Fig. 1 shows the graph of $B_3(4, 3, 2)$.

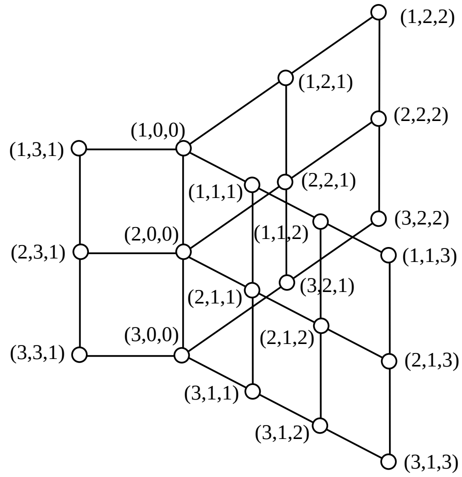

Figure 1. The graph of $B_3(4,3,2)$.

Suppose $\mathscr{L} = \{L(v) \;:\; v \in V(G)\}$ is a list assignment for a graph G. Suppose a vertex v has been labeled by $\ell \in L(v)$. Then ℓ cannot be used to label the vertex u with $d(u,v) \le 2$ and $\ell \pm 1$ cannot be used to label the vertex w with $d(w,v) = 1$. In this case, we remove those corresponding labels from the list of the vertices of distance at most 2 from v. That is, $L'(u) = L(u) \setminus \{\ell\}$ if $d(u,v) = 2$ and $L'(w) = L(w) \setminus \{\ell, \ell+1, \ell-1\}$ if $d(w,v) = 1$. The resulting list assignment $\mathscr{L}' = \{L'(x) \;:\; x \ne v\}$ is called the *residual list assignment* (*RLA* for abbreviation) for the graph $G - v$. This concept can be extended to more vertices have been labeled.

2. Main Results

Lemma 2.1 ([1]). *If a graph G contains three vertices with maximum degree $\Delta(G) \ge 2$ and one of them is adjacent to the other two vertices, then $\lambda(G) \ge \Delta(G) + 2$.*

Lemma 2.2. $\lambda_\ell(B_1(m_1, m_2, \ldots, m_p)) \ge p + 2$; $\lambda_\ell(B_2(m_1, m_2, \ldots, m_p)) \ge p + 3$; $\lambda_\ell(B_n(m_1, m_2, \ldots, m_p)) \ge p + 4$ *for* $3 \le n \le 4$; $\lambda_\ell(B_n(m_1, m_2, \ldots, m_p)) \ge p + 5$ *for* $n \ge 5$.

Proof. Since $B_1(m_1, m_2, \ldots, m_p)$ contains the star $K_{1,p}$, we have

$$\lambda(B_1(m_1, m_2, \ldots, m_p)) \ge \lambda(K_{1,p}) = p + 1.$$

So $\lambda_\ell(B_1(m_1, m_2, \ldots, m_p)) \ge \lambda(B_1(m_1, m_2, \ldots, m_p)) + 1 \ge p + 2$. Similarly, since $B_2(m_1, m_2, \ldots, m_p)$ contains the star $K_{1,p+1}$; $B_3(m_1, m_2, \ldots, m_p)$ and $B_4(m_1, m_2, \ldots, m_p)$ contain the star $K_{1,p+2}$, we have $\lambda(B_2(m_1, m_2, \ldots, m_p)) \ge \lambda(K_{1,p+1}) = p + 2$ and $\lambda(B_n(m_1, m_2, \ldots, m_p)) \ge \lambda(K_{1,p+2}) = p + 3$ for $3 \le n \le 4$. So $\lambda_\ell(B_2(m_1, m_2, \ldots, m_p)) \ge p + 3$; $\lambda_\ell(B_n(m_1, m_2, \ldots, m_p)) \ge p + 4$ for $3 \le n \le 4$.

Now we consider the case when $n \ge 5$.

Since $(2,0,0),(3,0,0),(4,0,0)$ are three vertices with maximum degree $p+2$; and $(3,0,0)$ is adjacent to $(2,0,0)$ and $(4,0,0)$. Then by Lemma 2.1, we get $\lambda(B_n(m_1,m_2,\ldots,m_p)) \geq p+4$. So $\lambda_\ell(B_n(m_1,m_2,\ldots,m_p)) \geq p+5$ for $n \geq 5$. □

We denote the path P_n by $v_1 \cdots v_n$. A list assignment $\mathscr{L}$ of P_n is of *order* $(a_1,\ldots,a_n)$ if $|L(v_i)| \geq a_i$, for all $i=1,\ldots,n$. P_n is said to be $(a_1,\ldots,a_n)$-$L(2,1)$-*choosable* if P_n is $\mathscr{L}$-$L(2,1)$-colorable for every list assignment $\mathscr{L}$ of order $(a_1,\ldots,a_n)$. For convenience, we use $(a,b,c^{[n]},d)$ to denote the sequence $(a,b,\overbrace{c,\ldots,c}^{n},d)$.

Lemma 2.3 ([19]). *P_2 is $(2,3)$-$L(2,1)$-choosable.*

Corollary 2.1. *Let $G = P_2 \times P_n$, where $n \geq 2$, $V(P_2) = \{u_1,u_2\}$ and $V(P_n) = \{v_1,\ldots,v_n\}$. Let $\mathscr{L}$ be a list assignment of G. Suppose $|L(u_i,v_1)| = 3$, $|L(u_i,v_2)| = 7$ and $|L(u_i,v_j)| = 8$ for $i=1,2$, $3 \leq j \leq n$ if any. Then G is $\mathscr{L}$-$L(2,1)$-colorable.* For short, we use $L(u_i,v_j)$ instead of $L((u_i,v_j))$.

Proof. By means of Lemma 2.3, we label (u_i,v_1) first, $i=1,2$. After that, the order of the residual lists $L'(u_1,v_2)$, $L'(u_2,v_2)$, $L'(u_1,v_3)$ and $L'(u_2,v_3)$ are 3,3,7,7, respectively. For those of the other unlabeled vertices are unchanged. This situation is referred to the beginning situation. So we can label the whole graph. □

From the above proof since we only require the order of RLA of the path P_2 is $(2,3)$, the size of some prescribed color lists can be slightly reduced.

Lemma 2.4. *P_n is $(1,4,5^{[n-2]})$-$L(2,1)$-choosable, $n \geq 2$.*

Proof. We first label v_1. Then at the worst three labels in $L(v_2)$ will be eliminated, we can still label v_2 with one of the remaining label(s). After that there are at most four labels in $L(v_3)$ will be eliminated, we label v_3 with one of the remaining label(s). This process can be continued until v_n has been labeled. □

Theorem 2.1. $\lambda_\ell(K_{1,p}) = p+2$.

Proof. It suffices to show that $K_{1,p}$ is $(p+2)$-$L(2,1)$-choosable. Let c be the center of the star $K_{1,p}$ and $v_1,\ldots,v_p$ be neighbors of c. Suppose that an arbitrary list assignment $\mathscr{L} = \{L(v) : v \in V(K_{1,p}), |L(v)| = p+2\}$ has been given.

Suppose all lists $L(v_i)$ for $1 \leq i \leq p$ are the same as $L(c)$, then by $\lambda_(K_{1,p}) = p+1$ we can label $K_{1,p}$ properly.

Suppose $L(v_i) \neq L(c)$ for some i. Without loss of generality, we may assume $L(v_p) \neq L(c)$. Let $\alpha \in L(c) \setminus L(v_p)$. Label c by α. Then label v_1 by $\alpha_1 \in L(v_1) \setminus \{\alpha-1,\alpha,\alpha+1\}$. Suppose v_{i-1} has been labeled by α_{i-1} for $2 \leq i \leq p-1$. Then we label v_i by

$$\alpha_i \in L(v_i) \setminus (\{\alpha-1,\alpha,\alpha+1\} \cup \{\alpha_j \mid 1 \leq j \leq i-1\}).$$

Finally, we label v_p by

$$\alpha_p \in L(v_p) \setminus (\{\alpha-1,\alpha+1\} \cup \{\alpha_j \mid 1 \leq j \leq p-1\}).$$

So $K_{1,p}$ is labeled properly. □

Theorem 2.2. *$\lambda_\ell(B_1(m_1, m_2, \ldots, m_p)) = p+2$ for $p \geq 3$.*

Proof. The vertex $(1,0,0)$ with its neighbors induce $K_{1,p}$. By Theorem 2.1 we can label it properly. After labeling this star, the order the RLA of each path $P_{m_k} \setminus \{(1,0,0),(1,k,1)\}$ $(1 \leq k \leq p)$ becomes $(p-2, p+1, (p+2)^{[m_k-4]})$ and $(p-2)$ for $m_k \geq 4$ and $m_k = 3$, respectively. So we can label the whole P_{m_k} by means of Lemma 2.4. Note that when $m_k = 2$, there is nothing to do. $\square$

Let $L(v) = \{\ell^1, \ldots, \ell^n\}$ denote the sequence of labels available for vertex v, where all the labels are in descending order. Let $L^*(v) = \{\ell \;:\; \{\ell-1, \ell, \ell+1\} \subseteq L(v)\}$. When we label a neighbor of v with a label in $L^*(v)$, it will eliminate three labels $\ell-1$, ℓ, $\ell+1$ from $L(v)$. When we label a neighbor of v with a label not in $L^*(v)$, it will eliminate at most two labels from $L(v)$. Since $L^*(v) \subset L(v)$ and $L^*(v)$ does not contain ℓ^1 and ℓ^n, we have $|L^*(v)| \leq |L(v)| - 2$. If there exists $1 \leq i \leq n-1$ such that $\ell^i - \ell^{i+1} > 1$, we say there is a *gap* between ℓ^i and ℓ^{i+1} in $L(v)$. If there is a gap in $L(v)$, then $|L^*(v)| \leq |L(v)| - 3$.

For convenience, we use $B_2(2^{[p]})$ to denote the book $B_2(\overbrace{2,2,\ldots,2}^{p})$. For short, we let $u_0 = (1,0,0)$, $v_0 = (2,0,0)$, $u_k = (1,k,1)$ and $v_k = (2,k,1)$ for $1 \leq k \leq p$.

From Lemma 2.5 to Lemma 2.8, we consider the $L(2,1)$-choosability of $B_2(2^{[p]})$ when the two vertices on the spine, u_0 and v_0, have been prelabeled. Suppose u_0 and v_0 have been labeled. Let

$$\mathscr{L} = \{L(v) : v \in V(B_2(2^{[p]})) \setminus \{u_0, v_0\}\}$$

be a list assignment for the vertices of $B_2(2^{[p]})$ except u_0 and v_0. A list assignment $\mathscr{L}$ is of *order* $(a_{ik})_{2\times p}$ if $|L(i,k,1)| \geq a_{ik}$, for all $i = 1,2$ and $k = 1,2,\ldots,p$. $B_2(2^{[p]})$ is said to be $(a_{ik})_{2\times p}$-$L(2,1)$-*choosable* if $B_2(2^{[p]})$ is $\mathscr{L}$-$L(2,1)$-colorable for every list assignment $\mathscr{L}$ of order $(a_{ik})_{2\times p}$. Note that since the page numbers of the book can be permutated, the order of $\mathscr{L}$ may be $(a'_{ik})_{2\times p}$, where $(a'_{ik})_{2\times p}$ is obtained from $(a_{ik})_{2\times p}$ by permutating some columns. Similarly, since the two vertices on the spine of $B_2(2^{[p]})$ can be permutated, $(a'_{ik})_{2\times p}$ may be obtained from $(a_{ik})_{2\times p}$ by permutating two rows. Fig. 2 shows the graph of $B_2(2^{[p]})$.

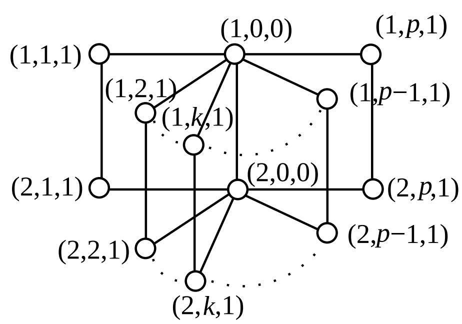

Figure 2. The graph of $B_2(2^{[p]})$.

Lemma 2.5. *$B_2(2^{[3]})$ is $\left(\begin{smallmatrix} 4 & 4 & 4 \\ 3 & 3 & 4 \end{smallmatrix}\right)$-$L(2,1)$-choosable when the two vertices on the spine are prelabeled.*

Proof. Suppose that an arbitrary list assignment

$$\mathscr{L} = \{L(v) : v \in V(B_2(2^{[3]})) \setminus \{u_0, v_0\}\}$$

with order $\left(\begin{smallmatrix} 4 & 4 & 4 \\ 3 & 3 & 4 \end{smallmatrix}\right)$ has been given.

We first label v_1 with a proper label such that at most two labels in $L(u_1)$ can be eliminated. This can be done since $|L^*(u_1)| \le |L(u_1)| - 2 = 2$ and $L(v_1) = 3$. Now at the worst the order of the RLA $\mathscr{L}'$ becomes $\left(\begin{smallmatrix} 2 & 4 & 4 \\ * & 2 & 3 \end{smallmatrix}\right)$ (entry recorded by $*$ means the corresponding vertex has been labeled). We denote $L'(v_2) = \{\ell_1^1, \ell_1^2\}$ and $L'(v_3) = \{\ell_2^1, \ell_2^2, \ell_2^3\}$ with the labels in decreasing order.

Case 1. $L'(v_2) \nsubseteq L'(v_3)$.

Without loss of generality (by reversing the order if necessary), we assume $\ell_1^1 \in L'(v_2) \setminus L'(v_3)$. We label v_2 with ℓ_1^1, then none of the three labels of $L'(v_3)$ will be eliminated, at most three labels in $L'(u_2)$ will be eliminated. Then we label u_2 with one of the remaining label(s), and this will eliminate both at most one label from $L'(u_1)$ and $L'(u_3)$. Now we label u_1 with a remaining label, then at most another label of $L'(u_3)$ will be eliminated. Now there are at least two labels for u_3 and three labels for v_3. We can label them properly according to Lemma 2.3.

Case 2. $L'(v_2) \subseteq L'(v_3)$.

Case 2.1. There is a label in $L'(v_2)$ equals to ℓ_2^2.

Without loss of generality, we assume $\ell_1^1 = \ell_2^2$. We label v_2 with ℓ_1^1, then ℓ_2^2 will be eliminated from $L'(v_3)$, at most three labels in $L'(u_2)$ will be eliminated. Then we label u_2 with one of the remaining label(s), and this will eliminate both at most one label from $L'(u_1)$ and $L'(u_3)$. Now we label u_1 with a remaining label, then at most another label of $L'(u_3)$ will be eliminated. Now we have at least two labels for u_3 and two labels for v_3 with a gap. We label u_3 with a label different from ℓ_2^2. Then at most another label in $L'(v_3)$ can be eliminated. Now we can label v_3 with a remaining label.

Case 2.2. $\ell_1^1 = \ell_2^1$ and $\ell_1^2 = \ell_2^3$.

We label v_3 with ℓ_2^2, then none of the two labels of $L'(v_2)$ will be eliminated and at most three labels of $L'(u_3)$ will be eliminated. Then we label u_3 and u_1, this will eliminate at most two label in $L'(u_2)$. Now we have at least two labels for u_2 and two labels for v_2 with a gap. We can use the similar method as Case 2.1 to label them. $\square$

Lemma 2.6. *For $p \ge 4$, $B_2(2^{[p]})$ is $\begin{pmatrix} p{+}1 & p{+}1 & p & p & \cdots & p \\ p{+}1 & p & p & p & \cdots & p \end{pmatrix}$-$L(2,1)$-choosable when the two vertices on the spine are prelabeled.*

Proof. We prove the lemma by induction on p. For $p = 4$, suppose that an arbitrary list assignment $\mathscr{L} = \{L(v) : v \in V(B_2(2^{[4]})) \setminus \{u_0, v_0\}\}$ with order $\left(\begin{smallmatrix} 5 & 5 & 4 & 4 \\ 5 & 4 & 4 & 4 \end{smallmatrix}\right)$ has been given.

Case 1. $L(u_3) \neq L(u_4)$.

Suppose $\ell \in L(u_3) \setminus L(u_4)$. We label u_3 with ℓ, then none of the four labels in $L(u_4)$ will be eliminated, at most one label in both $L(u_1)$ and $L(u_2)$ will be eliminated, at most three labels in $L(v_3)$ will be eliminated. We label v_3 with one remaining label, then at most one label in each of $L(v_1)$, $L(v_2)$ and $L(v_4)$ will be eliminated. Now at the worst the order of the RLA becomes $\mathscr{L}' = \left(\begin{smallmatrix} 4 & 4 & * & 4 \\ 4 & 3 & * & 3 \end{smallmatrix}\right)$. We can label the remaining vertices according to Lemma 2.5.

Case 2. $L(u_3) = L(u_4)$.

We can choose $\ell' \in L(u_2) \setminus L(u_3)$. We label u_2 with ℓ', then none of the four labels in both $L(u_3)$ and $L(u_4)$ will be eliminated, at most one label in $L(u_1)$ and three labels in $L(v_2)$ will be eliminated. We label v_2 with one remaining label, then at most one label in each of $L(v_1)$, $L(v_3)$ and $L(v_4)$ will be eliminated. Now at the worst the order of the RLA becomes $\mathscr{L}' = \left(\begin{smallmatrix} 4 & * & 4 & 4 \\ 4 & * & 3 & 3 \end{smallmatrix}\right)$. We can label the remaining vertices according to Lemma 2.5.

Suppose the lemma is true when the number of pages is less than p for $p \geq 5$. We consider the case when the number of pages is p. Suppose that an arbitrary list assignment $\mathscr{L} = \{L(v) : v \in V(B_2(2^{[p]})) \setminus \{u_0, v_0\}\}$ with order $\left(\begin{smallmatrix} p+1 & p+1 & p & p & \cdots & p \\ p+1 & p & p & p & \cdots & p \end{smallmatrix}\right)$ has been given.

We label u_3 and v_3 properly first. Then at the worst the order of the RLA becomes $\mathscr{L}' = \left(\begin{smallmatrix} p & p & * & p-1 & \cdots & p-1 \\ p & p-1 & * & p-1 & \cdots & p-1 \end{smallmatrix}\right)$. We can label the remaining vertices by induction. $\square$

We can use a similar proof as in Lemma 2.6 to get the following:

Lemma 2.7. *For $p \geq 4$, $B_2(2^{[p]})$ is $\left(\begin{smallmatrix} p+1 & p+1 & p & p & \cdots & p \\ p & p & p & p & \cdots & p+1 \end{smallmatrix}\right)$-$L(2,1)$-choosable when the two vertices on the spine are prelabeled.*

Lemma 2.8. *For $p \geq 4$, $B_2(2^{[p]})$ is $\left(\begin{smallmatrix} p+2 & p+2 & \cdots & p+2 \\ p & p & \cdots & p \end{smallmatrix}\right)$-$L(2,1)$-choosable when the two vertices on the spine are prelabeled.*

Proof. We prove the lemma by induction on p. For $p = 4$, suppose that an arbitrary list assignment $\mathscr{L} = \{L(v) : v \in V(B_2(2^{[4]})) \setminus \{u_0, v_0\}\}$ with order $\left(\begin{smallmatrix} 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 \end{smallmatrix}\right)$ has been given.

Case 1. There exist $1 \leq k_1 < k_2 \leq 4$ such that $L(v_{k_1}) \neq L(v_{k_2})$.

Without loss of generality, suppose $k_1 = 1$, $k_2 = 2$. We choose $\ell \in L(v_1) \setminus L(v_2)$ and label v_1 with ℓ. Then no label of $L(v_2)$; at most one label of $L(v_k)$, $3 \leq k \leq 4$; and at most three labels of $L(u_1)$ will be eliminated. We label u_1 with one of the remaining labels. This will eliminate at most one label of $L(u_k)$, $2 \leq k \leq 4$. Now at the worst the order of the RLA becomes $\mathscr{L}' = \left(\begin{smallmatrix} * & 5 & 5 & 5 \\ * & 4 & 3 & 3 \end{smallmatrix}\right)$. We can label the remaining vertices according to Lemma 2.5.

Case 2. All the $L(v_k), 1 \leq k \leq 4$ are identical.

Case 2.1. There exists $1 \leq k \leq 4$ such that $L(v_k) \nsubseteq L^*(u_k)$.

Without loss of generality, suppose $k = 1$. We choose $\ell' \in L(v_1) \setminus L^*(u_1)$ and label v_1 with ℓ'. Then at most one label of $L(v_k)$, $2 \leq k \leq 4$; and two labels of $L(u_1)$ will be eliminated. Now we have $|L'(u_1)| \geq 4$, $|L'(u_k)| \geq 6$, $2 \leq k \leq 4$; and $|L'(v_k)| \geq 3$, $2 \leq k \leq 4$. We label u_k and v_k, $2 \leq k \leq 4$ properly. Now there is still at least one label left for u_1 to complete the labeling.

Case 2.2. $L(v_k) \subseteq L^*(u_k)$, $1 \leq k \leq 4$.

This means all the $L(u_k)$ are identical and without gaps for $1 \le k \le 4$. Without loss of generality, suppose $L(v_k) = \{1, 2, 3, 4\}$, $L(u_k) = \{0, 1, 2, 3, 4, 5\}$, $1 \le k \le 4$.

We label u_1, u_2, u_3, u_4 with $3, 5, 4, 2$ and v_1, v_2, v_3, v_4 with $1, 3, 2, 4$, respectively.

Suppose the lemma is true when the number of pages is less than p for $p \ge 5$. We consider the case when the number of pages is p. Suppose that an arbitrary list assignment $\mathscr{L} = \{L(v) : v \in V(B_2(2^{[p]})) \setminus \{u_0, v_0\}\}$ with order $\left(\begin{smallmatrix} p+2 & p+2 & \cdots & p+2 \\ p & p & \cdots & p \end{smallmatrix}\right)$ has been given.

We label u_1 and v_1 properly first. Then at the worst the order of the RLA becomes $\mathscr{L}' = \left(\begin{smallmatrix} * & p+1 & \cdots & p+1 \\ * & p-1 & \cdots & p-1 \end{smallmatrix}\right)$. We can label the remaining vertices by induction. $\square$

Lemma 2.9. *$B_2(2^{[4]})$ is 8-$L(2,1)$-choosable.*

Proof. Suppose that an arbitrary list assignment $\mathscr{L} = \{L(v) : v \in V(B_2(2^{[4]})), |L(v)| = 8\}$ has been given.

Case 1. At least one of $L(u_k)$ has a gap, $1 \le k \le 4$, say $L(u_a)$.

Then $|L^*(u_a)| \le |L(u_a)| - 3 = 5$. So $|L(u_0) \setminus L^*(u_a)| \ge 3$. We also know that $|L^*(u_k)| \le |L(u_k)| - 2 = 6$, $1 \le k \le 4$ and $k \ne a$. So $|L(u_0) \setminus L^*(u_k)| \ge 2$, $1 \le k \le 4$ and $k \ne a$. Since $\sum_{k=1}^{4} |L(u_0) \setminus L^*(u_k)| \ge 9$ and $|\bigcup_{k=1}^{4} \{L(u_0) \setminus L^*(u_k)\}| \le 8$, by pigeon hole principle there exist $1 \le k_1 < k_2 \le 4$ such that $(L(u_0) \setminus L^*(u_{k_1})) \cap (L(u_0) \setminus L^*(u_{k_2})) \ne \varnothing$. Without loss of generality, assume $k_1 = 1$, $k_2 = 2$. Let $\ell \in (L(u_0) \setminus L^*(u_1)) \cap (L(u_0) \setminus L^*(u_2))$. We label u_0 with ℓ. Then at most two labels in both $L(u_1)$ and $L(u_2)$; three labels in $L(u_3)$, $L(u_4)$, $L(v_0)$; and one label in $L(v_k)$, $1 \le k \le 4$, will be eliminated. It suffices to consider the worst case: $|L'(u_1)| = 6$, $|L'(u_2)| = 6$; $|L'(u_3)| = 5$, $|L'(u_4)| = 5$, $|L'(v_0)| = 5$; and $|L'(v_k)| = 7$, $1 \le k \le 4$, because we may throw out some label(s) from each list so that it becomes the worst case.

Case 1.1. There is a gap in one of $L'(v_k)$, $1 \le k \le 4$, say $L'(v_b)$.

Then $|L'^*(v_b)| \le |L'(v_b)| - 3 = 4$. Since $|L'(v_0)| = 5$, there is an $\ell' \in L'(v_0) \setminus L'^*(v_b)$. We label v_0 with ℓ'. Then at most two labels in $L'(v_b)$; three labels in $L'(v_k)$, $1 \le k \le 4$ and $k \ne b$; one label in $L'(u_k), 1 \le k \le 4$, will be eliminated. Now at the worst the order of the RLA $\mathscr{L}'$ for $V(B_2(2^{[4]})) \setminus \{u_0, v_0\}\}$ is $\left(\begin{smallmatrix} 5 & 5 & 4 & 4 \\ 5 & 4 & 4 & 4 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 5 & 5 & 4 & 4 \\ 4 & 4 & 4 & 5 \end{smallmatrix}\right)$.

We can label it properly according to Lemma 2.6 or 2.7.

Case 1.2. There is no gap in each of $L'(v_k)$, $1 \le k \le 4$, but there exist $1 \le k_1 \ne k_2 \le 4$ such that $L'(v_{k_1}) \ne L'(v_{k_2})$.

Let $L'(v_{k_1}) = (\ell_1^1, \ell_1^2, \dots, \ell_1^7)$ and $L'(v_{k_2}) = (\ell_2^1, \ell_2^2, \dots, \ell_2^7)$. Since $L'(v_{k_1}) \ne L'(v_{k_2})$ and they have no gap, either $\ell_1^1 > \ell_2^1$ or $\ell_1^1 < \ell_2^1$. Without loss of generality, assume $\ell_1^1 > \ell_2^1$, then $\ell_1^2 \ge \ell_2^1$. Hence $\ell_1^2 \notin L'^*(v_{k_2})$ and we can conclude that $|L'^*(v_{k_1}) \cap L'^*(v_{k_2})| \le 4$. Since $|L'(v_0)| = 5$, we can choose $\ell'' \in L'(v_0) \setminus (L'^*(v_{k_1}) \cap L'^*(v_{k_2}))$. We label v_0 with ℓ''. Then at most two labels from either $L'(v_{k_1})$ or $L'(v_{k_2})$, say $L'(v_{k_1})$, will be eliminated. Also at most three labels from $L'(v_k)$, $1 \le k \le 4$ and $k \ne k_1$; and one label from $L'(u_k)$, $1 \le k \le 4$ will be eliminated. Now at the worst the order of the RLA $\mathscr{L}'$ for $V(B_2(2^{[4]})) \setminus \{u_0, v_0\}\}$ is $\left(\begin{smallmatrix} 5 & 5 & 4 & 4 \\ 5 & 4 & 4 & 4 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 5 & 5 & 4 & 4 \\ 4 & 4 & 4 & 5 \end{smallmatrix}\right)$. We can label it properly according to

Lemma 2.6 or Lemma 2.7.

Case 1.3. There is no gap in each of $L'(v_k)$, $1 \le k \le 4$, and $L'(v_1) = L'(v_2) = L'(v_3) = L'(v_4)$. This implies that $L(v_1) = L(v_2) = L(v_3) = L(v_4)$.

So $L^*(v_1) = L^*(v_2) = L^*(v_3) = L^*(v_4)$ and $|L^*(v_k)| \le 6$, $1 \le k \le 4$. Instead of labeling u_0 first, we will choose a proper label $\ell^3 \in L(v_0) \setminus L^*(v_1)$ and label v_0 with ℓ^3 first. Then at most two labels from $L(v_k), 1 \le k \le 4$; three labels from $L(u_0)$; and one label from $L(u_k), 1 \le k \le 4$ will be eliminated. Now at the worst we have $|L'(v_k)| = 6, 1 \le k \le 4$, $|L'(u_0)| = 5$; and $|L'(u_k)| = 7, 1 \le k \le 4$.

Since $L(u_a)$ has a gap, after eliminating at most one label from each of $L(u_k)$, $1 \le k \le 4$, (a) there is at least one of $L'(u_k)$, $1 \le k \le 4$, with a gap; (b) there is no gap in each of $L'(u_k)$, $1 \le k \le 4$, but there exist $1 \le k_1 \ne k_2 \le 4$ such that $L'(u_{k_1}) \ne L'(u_{k_2})$; or (c) $L'(u_k)$ are identical without gaps, $1 \le k \le 4$. For cases (a) and (b), similar to the discuss in Case 1.1 and 1.2, we can choose $\ell^4 \in L'(u_0)$ such that after labeling u_0 with ℓ^4, at least one of $L'(u_k)$, $1 \le k \le 4$, say $L'(u_c)$, can be eliminated at most two labels. Also at most three labels of $L'(u_k)$, $1 \le k \le 4$ and $k \ne c$; one label of $L'(v_k)$, $1 \le k \le 4$ will be eliminated. Now at the worst the order of the RLA $\mathscr{L}'$ for $V(B_2(2^{[4]})) \setminus \{u_0, v_0\}$ is $\left(\begin{smallmatrix} 5 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{smallmatrix}\right)$. We can label it properly according to Lemma 2.7.

For case (c), after labeling v_0 by ℓ^3, it is similarly to show that $L(u_k)$ are identical, $1 \le k \le 4$. Since $L(u_a)$ has a gap, $L(u_a) = \{\ell^3, y, y-1, \ldots, y-6\}$ with $\ell^3 \ge y+2$ or $L(u_a) = \{y+6, y+5, \ldots, y, \ell^3\}$ with $\ell^3 \le y-2$ for some y.

Now we will choose a proper label $\ell' \in L(v_0) \setminus (L^*(v_1) \cup \ell^3)$ and label v_0 with ℓ' instead of ℓ^3 first. Then at most two labels from $L(v_k), 1 \le k \le 4$; three labels from $L(u_0)$; and one label from $L(u_k), 1 \le k \le 4$ will be eliminated. Now at the worst we have $|L'(v_k)| = 6, 1 \le k \le 4$, $|L'(u_0)| = 5$; and $|L'(u_k)| = 7, 1 \le k \le 4$. Moreover, all $L'(u_k)$ are identical with a gap. We can use the similar way as in case (a) to label it properly.

Case 2. There is no gap in each of $L(u_k)$, $1 \le k \le 4$, but there exist $1 \le k_1 < k_2 \le 4$ such that $L(u_{k_1}) \ne L(u_{k_2})$.

Case 2.1. There exist $1 \le k_1' < k_2' \le 4$ such that $L(u_{k_1'}) = L(u_{k_2'})$.

Without loss of generality, we may assume $k_1' = 1$, $k_2' = 2$. Then $L^*(u_1) = L^*(u_2)$ and $|L^*(u_1)| = 6$. So $|L(u_0) \setminus L^*(u_1)| \ge 2$. We choose $\ell^5 \in L(u_0) \setminus L^*(u_1)$ to label u_0. Then at most two labels in both $L(u_1)$ and $L(u_2)$; three labels in $L(u_k)$, $3 \le k \le 4$ and $L(v_0)$; and one label in $L(v_k)$, $1 \le k \le 4$ will be eliminated. Now at the worst we have $|L'(u_1)| = 6$, $|L'(u_2)| = 6$; $|L'(u_3)| = 5$, $|L'(u_4)| = 5$, $|L'(v_0)| = 5$; and $|L'(v_k)| = 7$, $1 \le k \le 4$. Then we can use a similar proof of the three subcases of Case 1 to complete the labeling of the remaining part of $B_2(2^{[4]})$.

Case 2.2. All of the $L(u_k), 1 \le k \le 4$ are different.

Let $L(u_0) = \{\ell_0^1, \ell_0^2, \ldots, \ell_0^8\}$ and $L(u_k) = \{\ell_k^1, \ell_k^2, \ldots, \ell_k^8\}$, $1 \le k \le 4$. Without loss of generality, we may assume $\ell_1^1 > \ell_2^1 > \ell_3^1 > \ell_4^1$. Then we have $\ell_1^8 > \ell_2^8 > \ell_3^8 > \ell_4^8$.

If $\ell_0^1 \ge \ell_2^1$, then $\ell_0^1 \ge \ell_2^1 > \ell_3^1 > \ell_4^1$. We label u_0 with ℓ_0^1. Then at most two labels in $L(u_k)$, $2 \le k \le 4$; three labels in $L(u_1)$ and $L(v_0)$; and one label in $L(v_k)$, $1 \le k \le 4$ will be eliminated. Now at the worst we have $|L'(u_1)| = 5$, $|L'(u_2)| = 6$; $|L'(u_3)| = 6$,

$|L'(u_4)| = 6$, $|L'(v_0)| = 5$; and $|L'(v_k)| = 7$, $1 \leq k \leq 4$. Then we can use a similar proof of the three subcases of Case 1 to complete the labeling of the remaining part of $B_2(2^{[4]})$.

If $\ell_0^1 < \ell_2^1$, then $\ell_0^8 < \ell_2^8 < \ell_1^8$. We label u_0 with ℓ_0^8. Then at most two labels in $L(u_k), 1 \leq k \leq 2$; three labels in $L(u_k)$, $3 \leq k \leq 4$ and $L(v_0)$; and one label in $L(v_k)$, $1 \leq k \leq 4$ will be eliminated. Now at the worst we have $|L'(u_1)| = 6$, $|L'(u_2)| = 6$; $|L'(u_3)| = 5$, $|L'(u_4)| = 5$, $|L'(v_0)| = 5$; and $|L'(v_k)| = 7$, $1 \leq k \leq 4$. Then we can use a similar proof of the three subcases of Case 1 to complete the labeling of the remaining part of $B_2(2^{[4]})$.

Case 3. All of the $L(u_k)$, $1 \leq k \leq 4$, are identical and without gaps.

Let $L(u_0) = \{\ell_0^1, \ell_0^2, \ldots, \ell_0^8\}$ and $L(u_k) = \{\ell_1^1, \ell_1^2, \ldots, \ell_1^8\}$, $1 \leq k \leq 4$.

Suppose $L(u_0) \neq L(u_1)$. Then either $\ell_0^1 > \ell_1^1$ or $\ell_0^8 < \ell_1^8$. Without loss of generality, we may assume $\ell_0^1 > \ell_1^1$. We label u_0 with ℓ_0^1. Then at most one label in $L(u_k)$, $1 \leq k \leq 4$; three labels in $L(v_0)$; and one label in $L(v_k)$, $1 \leq k \leq 4$ will be eliminated. Now at the worst we have $|L'(u_k)| = 7$, $1 \leq k \leq 4$, $|L'(v_0)| = 5$; and $|L'(v_k)| = 7$, $1 \leq k \leq 4$. Then we label v_0 with any one of the remaining labels. Hence at most one label in $L'(u_k)$, $1 \leq k \leq 4$; and three labels in $L'(v_k)$, $1 \leq k \leq 4$ will be eliminated. Now at the worst the order of the RLA $\mathscr{L}'$ for $V(B_2(2^{[4]})) \setminus \{u_0, v_0\}$ is $\left(\begin{smallmatrix} 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 \end{smallmatrix}\right)$. We can label it properly according to Lemma 2.8.

Now the only unsolved case is when $L(u_0) = L(u_k)$, $1 \leq k \leq 4$ and there is no gap in $L(u_0)$. By symmetry, we can also assume that $L(v_0) = L(v_k)$, $1 \leq k \leq 4$ and there is no gap in $L(v_0) = \{r^1, r^2, \ldots, r^8\}$.

If $L(u_0) \neq L(v_0)$, then without loss of generality, suppose $\ell_0^1 > r^1$. We label u_0 with ℓ_0^1. Then at most two labels in $L(u_k)$, $1 \leq k \leq 4$; one label in $L(v_0)$; and no label in $L(v_k)$, $1 \leq k \leq 4$ will be eliminated. Now at the worst we have $|L'(u_k)| = 6$, $1 \leq k \leq 4$, $|L'(v_0)| = 7$; and $|L'(v_k)| = 8$, $1 \leq k \leq 4$. Then we label v_0 with a label from $L'(v_0)$. Hence at most one label in $L'(u_k)$, $1 \leq k \leq 4$; and three labels in $L'(v_k)$, $1 \leq k \leq 4$ will be eliminated. Now at the worst the order of the RLA $\mathscr{L}'$ for $V(B_2(2^{[4]})) \setminus \{u_0, v_0\}$ is $\left(\begin{smallmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{smallmatrix}\right)$. We can label it properly according to Lemma 2.6.

If $L(u_0) = L(v_0)$, then all label lists of $V(B_2(2^{[4]}))$ are the same as $L(u_0)$. We label u_0 with ℓ_0^1, u_1 with ℓ_0^4, u_2 with ℓ_0^6, u_3 with ℓ_0^5, u_4 with ℓ_0^3, v_1 with ℓ_0^2, v_2 with ℓ_0^4, v_3 with ℓ_0^3, v_4 with ℓ_0^5, v_0 with ℓ_0^7. $\square$

Lemma 2.10. *$B_2(2^{[p]})$ is $(p+4)$-$L(2,1)$-choosable for $p \geq 5$.*

Proof. Suppose that an arbitrary list assignment $\mathscr{L} = \{L(v) : v \in V(B_2(2^{[p]})), |L(v)| = p+4\}$ has been given.

For $1 \leq k \leq p$, since $|L^*(u_k)| \leq |L(u_k)| - 2 = p + 2$, $|L(u_0) \setminus L^*(u_k)| \geq 2$. We have $\sum_{k=1}^{p} |L(u_0) \setminus L^*(u_k)| \geq 2p$ and $|\bigcup_{k=1}^{p} \{L(u_0) \setminus L^*(u_k)\}| \leq p + 4$. As $2p > p + 4$ for $p \geq 5$, by pigeon hole principle there exist $1 \leq a < b \leq p$ such that $(L(u_0) \setminus L^*(u_a)) \cap (L(u_0) \setminus L^*(u_b)) \neq \varnothing$. Without loss of generality, we may assume $a = 1$, $b = 2$. Let $\ell \in (L(u_0) \setminus L^*(u_1)) \cap (L(u_0) \setminus L^*(u_2))$. We label u_0 with ℓ. Then at most two labels in $L(u_1)$, $L(u_2)$; three labels in $L(u_k)$, $3 \leq k \leq p$ and $L(v_0)$; and one label in $L(v_k)$, $1 \leq k \leq p$ will be eliminated. Similar to the proof of Lemma 2.9, it suffices to consider the worst case: $|L'(u_1)| = p + 2$, $|L'(u_2)| = p + 2$; $|L'(u_k)| = p + 1$,

$3 \le k \le p$, $|L'(v_0)| = p+1$; and $|L'(v_k)| = p+3$, $1 \le k \le p$.

Case 1. There is a gap in one of $L'(v_k)$, $1 \le k \le p$, say $L'(v_d)$.

Then $|L'^*(v_d)| \le |L'(v_d)| - 3 = p$. Since $|L'(v_0)| = p+1$, there must be a label $\ell' \in L'(v_0) \setminus L'^*(v_d)$. We label v_0 with ℓ', then at most two labels in $L'(v_d)$; three labels in $L'(v_k), 1 \le k \le p$ and $k \ne d$; one label in $L'(u_k), 1 \le k \le p$ will be eliminated. Now at the worst the order of the RLA $\mathscr{L}'$ for $V(B_2(2^{[p]})) \setminus \{u_0, v_0\}\}$ is $\begin{pmatrix} p+1 & p+1 & p & \cdots & p \\ p+1 & p & p & \cdots & p \end{pmatrix}$ or $\begin{pmatrix} p+1 & p+1 & p & \cdots & p \\ p & p & p & \cdots & p+1 \end{pmatrix}$. We can label it properly according to Lemma 2.6 or Lemma 2.7.

Case 2. There is no gap in each of $L'(v_k), 1 \le k \le p$, but there exist $1 \le k_1 \ne k_2 \le p$ such that $L'(v_{k_1}) \ne L'(v_{k_2})$.

Let $L'(v_{k_1}) = \{\ell_1^1, \ell_1^2, \ldots, \ell_1^{p+3}\}$ and $L'(v_{k_2}) = \{\ell_2^1, \ell_2^2, \ldots, \ell_2^{p+3}\}$. Since $L'(v_{k_1}) \ne L'(v_{k_2})$ and they have no gap, either $\ell_1^1 > \ell_2^1$ or $\ell_1^1 < \ell_2^1$. Without loss of generality, we may assume $\ell_1^1 > \ell_2^1$, then $\ell_1^2 \ge \ell_2^1$. Hence $\ell_1^2 \notin L'^*(v_{k_2})$ and we can conclude that $|L'^*(v_{k_1}) \cap L'^*(v_{k_2})| \le p$. Since $|L'(v_0)| = p+1$, we can choose $\ell'' \in L'(v_0) \setminus (L'^*(v_{k_1}) \cap L'^*(v_{k_2}))$. We label v_0 with ℓ''. Then at most two labels from either $L'(v_{k_1})$ or $L'(v_{k_2})$, say $L'(v_{k_1})$, will be eliminated. The rest of the proof is similar to Case 1.

Case 3. All of the $L'(v_k)$, $1 \le k \le p$, are identical and without gaps.

Suppose $L'(v_0) \nsubseteq L'^*(v_1)$. Then we choose $\ell^3 \in L'(v_0) \setminus L'^*(v_1)$. We label v_0 with ℓ^3. Then at most two labels of $L'(v_k), 1 \le k \le p$; and one label of $L'(u_k), 1 \le k \le p$ will be eliminated. Now at the worst the order of the RLA $\mathscr{L}'$ for $V(B_2(2^{[p]})) \setminus \{u_0, v_0\}\}$ is $\begin{pmatrix} p+1 & p+1 & p & \cdots & p \\ p+1 & p+1 & p+1 & \cdots & p+1 \end{pmatrix}$. We can label it properly according to Lemma 2.6.

Suppose $L'(v_0) \subseteq L'^*(v_1)$. Let $L'(v_0) = \{\ell_0^1, \ell_0^2, \ldots, \ell_0^{p+1}\}$. Since $|L'(v_1)| = p+3$,

$$L'(v_1) = \{\ell_0^1 + 1, \ell_0^1, \ell_0^2, \ldots, \ell_0^{p+1}, \ell_0^{p+1} - 1\}.$$

By the property of elements in $L'^*(v_1)$, we have $L'(v_1) = \{i \;:\; \ell_0^1 - p - 1 \le i \le \ell_0^1 + 1\}$. Hence we know that

$$L(v_0) = \{i \;:\; \ell_0^1 - p \le i \le \ell_0^1 + 3\} \text{ and } L(v_1) = \{i \;:\; \ell_0^1 - p - 1 \le i \le \ell_0^1 + 2\};$$
$$\text{or } L(v_0) = \{i \;:\; \ell_0^1 - p - 3 \le i \le \ell_0^1\} \text{ and } L(v_1) = \{i \;:\; \ell_0^1 - p - 2 \le i \le \ell_0^1 + 1\}.$$

Actually $\ell = \ell_0^1 + 2$ for the former case and $\ell = \ell_0^1 - p - 2$ for the latter case.

For the former case, instead of labeling u_0 first, we label v_0 with $\ell_0^1 + 3$ first. Then one label of $L(v_k)$, $1 \le k \le p$; at most three labels of $L(u_0)$ and one label of $L(u_k)$, $1 \le k \le p$ will be eliminated. Now we have $|L'(u_k)| \ge p+3$, $1 \le k \le p$, $|L'(u_0)| \ge p+1$; and $|L'(v_k)| = p+3$, $1 \le k \le p$. Then we label u_0 with a label from $L'(u_0)$, at most three labels of $L'(u_k)$, $1 \le k \le p$; and one label of $L'(v_k)$, $1 \le k \le p$ will be eliminated. Now at the worst the order of the RLA $\mathscr{L}'$ for $V(B_2(2^{[p]})) \setminus \{u_0, v_0\}\}$ is $\begin{pmatrix} p & p & p & \cdots & p \\ p+2 & p+2 & p+2 & \cdots & p+2 \end{pmatrix}$. We can label it properly according to Lemma 2.8. For the latter case, instead of labeling u_0 first, we label v_0 with $\ell_0^1 - p - 3$ first. Then the labeling method is similar to the former case. $\square$

Theorem 2.3. $\lambda_\ell(B_2(m_1, m_2, \ldots, m_p)) \le p + 4$ *for* $p \ge 4$; $\lambda_\ell(B_2(m_1, m_2, m_3)) \le 8$.

Proof. Given an arbitrary list assignment $\mathscr{L} = \{L(v) : v \in B_2(m_1, m_2, \ldots, m_p), |L(v)| = p + 4\}$ for $p \ge 4$. The subgraph induced by $\{(1, 0, 0), (2, 0, 0)\} \cup \{(i, k, 1) : 1 \le i \le 2, 1 \le k \le p\}$ is isomorphic to $B_2(2^{[p]})$. First we label this induced subgraph according to Lemma 2.9 or 2.10. After that, at most five labels of $L(1, 1, 2)$ will be eliminated since $(1, 1, 2)$ is adjacent to $(1, 1, 1)$ and of distance two to $(1, 0, 0)$ and $(2, 1, 1)$. There are also at most five labels of $L(2, 1, 2)$ will be eliminated. We have $|L'(i, 1, 2)| \ge p - 1 \ge 3$, $|L'(i, 1, 3)| \ge p + 3 \ge 7$, $|L'(i, 1, j)| = p + 4 \ge 8$, $1 \le i \le 2$, $4 \le j \le m_1$. By means of Corollary 2.1 we can label the first page. For the remaining $p - 1$ pages, the labeling are similar.

It is straightforward to get $\lambda_\ell(B_2(m_1, m_2, m_3)) \le 8$ from $\lambda_\ell(B_2(m_1, m_2, m_3, m_4)) \le 8$. $\square$

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