Deterministic chaos in a simple mathematical model of a leaky faucet

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Introduction

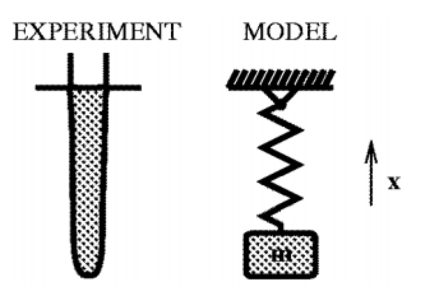
Simple deterministic models can sometimes produce what looks like random behavior. A leaky faucet may seem like a mundane and strange physical system to model, but it has evolved into a well known illustration of chaotic behavior in simple nonlinear systems since being introduced by Robert Shaw in 1985 [1].

The physical intuition originates from something all of us may have observed: dripping behavior of a faucet which may not be completely shut. Drops form at the head of faucet, and make a sound as they hit the sink. The sound produced may sometimes be rhythmic, and sometimes noisy, depending on the flow of the liquid. This behavior has been investigated by numerous scientists experimentally and theoretically employing complex fluid dynamics models.

In this project, I present a simple mathematical model of a leaky faucet along Shaw's original approach and inspired by [2]. By simulating the model, I explore and visualize different kinds of periodic and chaotic behaviors displayed by this simple mathematical system.

Modeling a leaky faucet

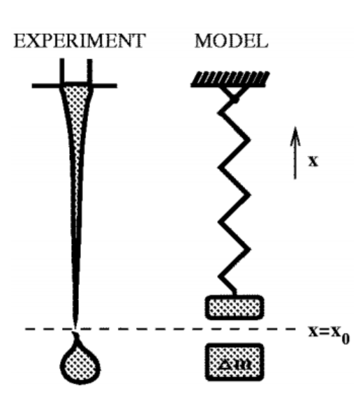
We treat the drop of water as a damped harmonic oscillator under gravity with its mass m increasing at a constant rate r, to simulate the flowing of liquid from faucet to droplet. The spring constant of the oscillator k represents the surface tension of the liquid, whereas the damping force b represents viscosity of the liquid.



Differential equations for the system:

$$\dot{v} = mg - kx - \dot{m}v - bv \qquad \dot{x} = v \qquad \dot{m} = r$$

To simulate the dropping, there are additional constraints. on the system. When the oscillator reaches a critical position $\boxed{x_c}$ we simulate the detachment of the water droplet from the bulk of the system by decreasing the mass of a system by $\boxed{\Delta m = \frac{v}{\beta + v}}$ and then start over with a new droplet.

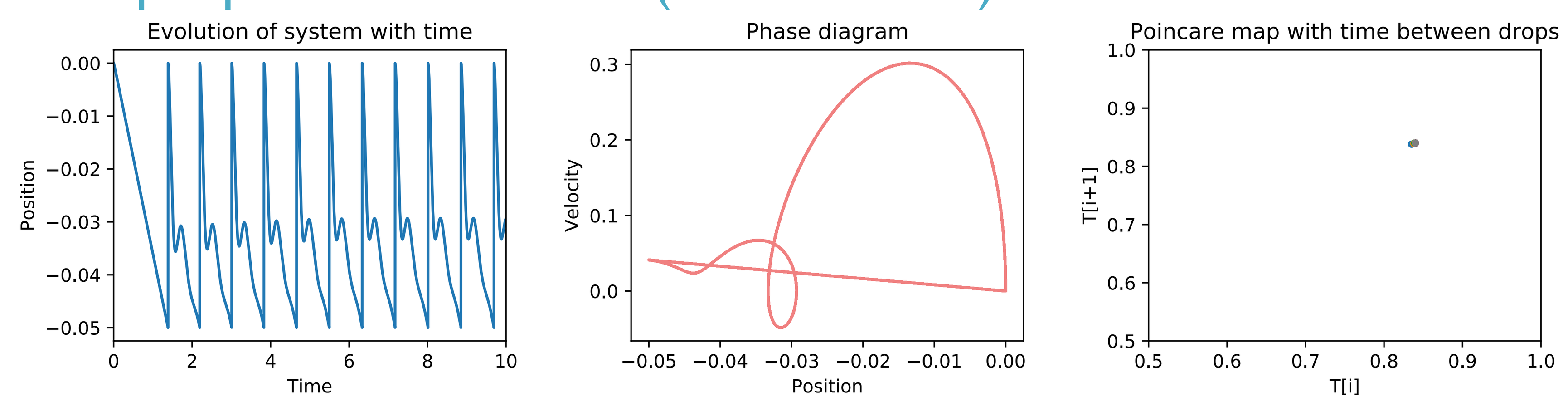


Note that this model does not rigorously consider the fluid mechanics involved in droplet formation, but is a useful heuristic which provides results which are qualitatively similar to real world phenomenon.

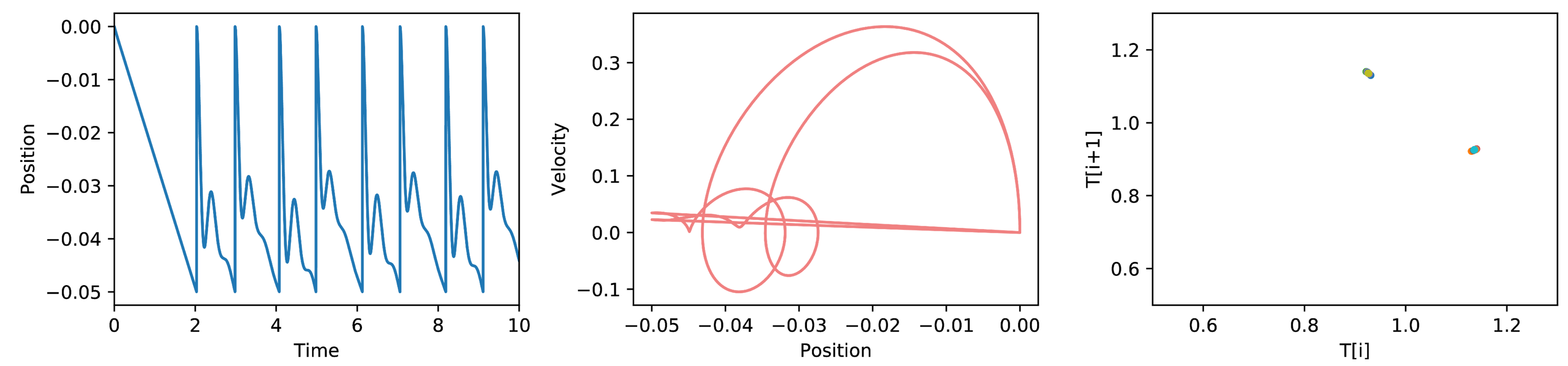
Simulations

The differential equations were solved numerically using a fourth order Runga Kutta method modified for the additional constraints. The following simulations was conducted over 800000 iterations with a timestep of 0.001.

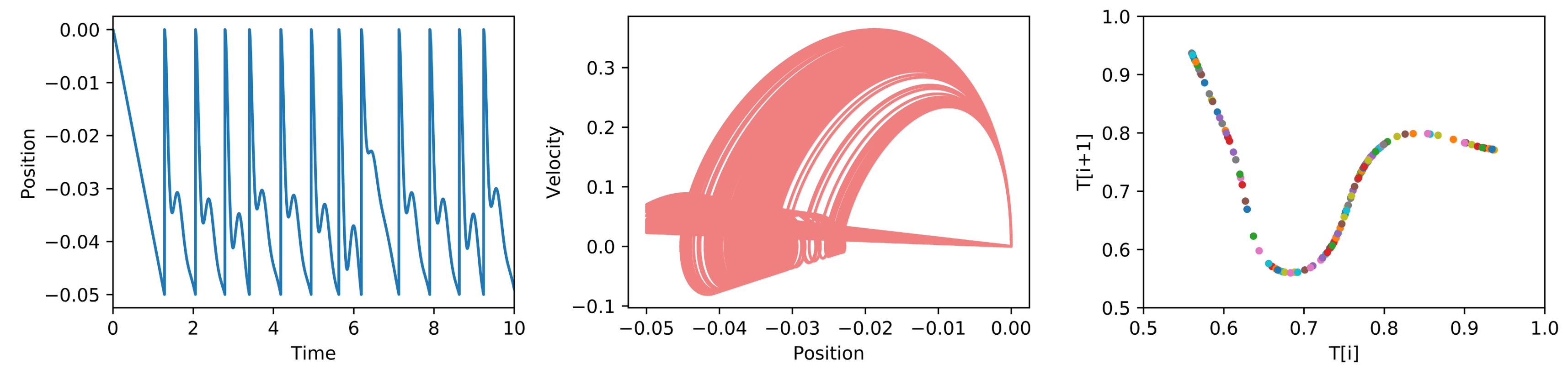
Simple periodic behavior (r = 0.000035)



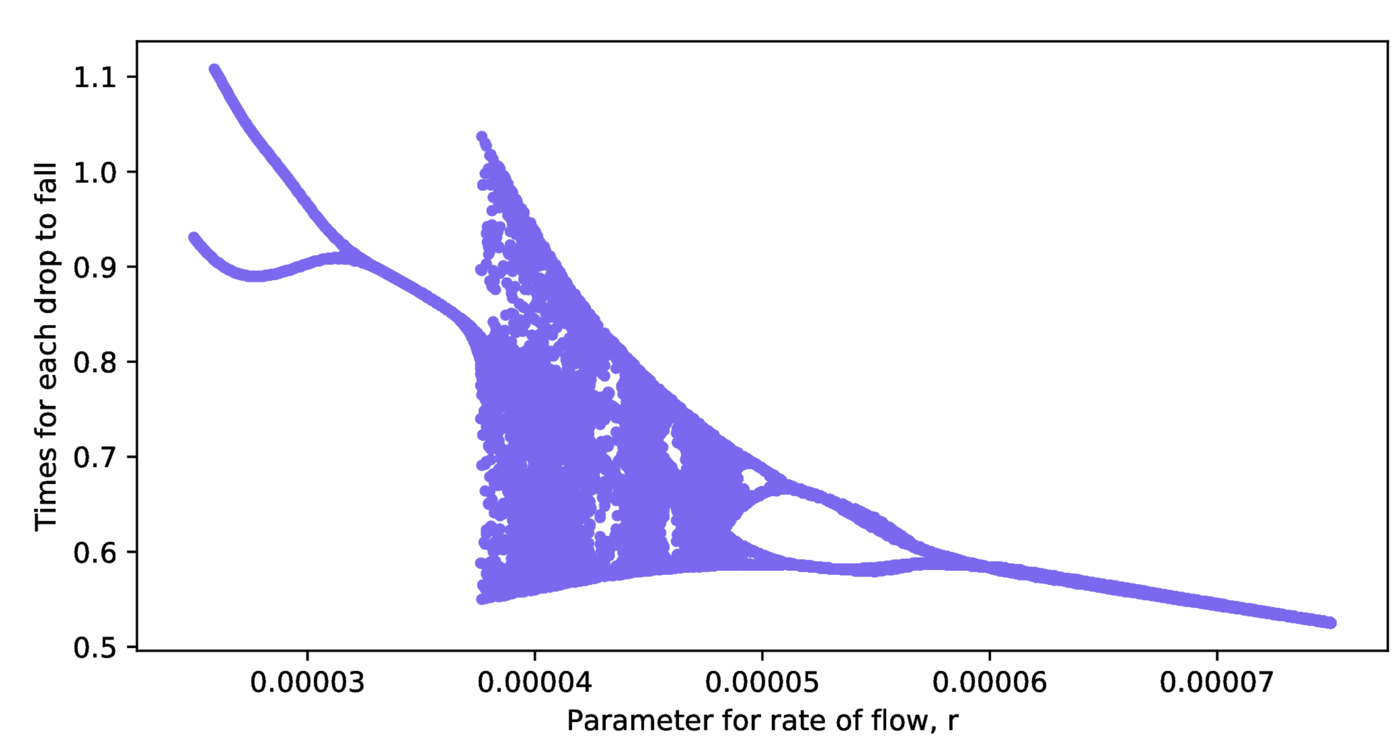
Period two behavior (r = 0.000025)



Chaotic behavior (r = 0.000045)

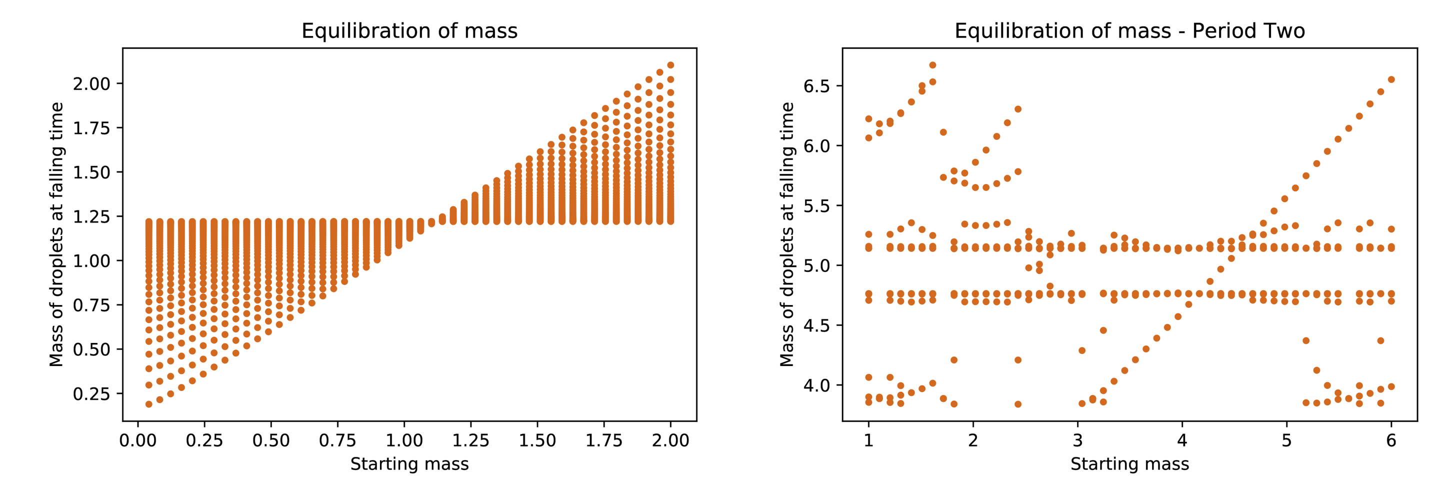


Bifurcation diagram- From order to chaos



Insensitivity to initial mass

A surprising result achieved while playing around with the parameters was system would always reach the same state after some time, irrespective of the mass of water that we started with. This equilibrating mass represents a balance between the rate of increasing of the mass due to r and decreasing of the mass as the more droplets form.



What gives rise to chaotic behavior?

Once the system reaches its equilibrium state, the mass of the droplet at each pinch-off event uniquely determines the mass of the next droplet, making this model a one-to-one mapping, the simplest map that can exhibit chaos. Changing parameters like the rate or spring constant alter the nature of this mapping due to the non-linearity of the system, which gives rise to different kinds of behaviors observed.

Deterministic non-periodic systems

The leaky faucet model is just one illustration of a deeper concept, simple mathematical models showing seemingly chaotic behavior. Such deterministic chaos has also been observed in a range of disciplines like fluid dynamics and turbulence, measles epidemics in New York City, fluctuations of populations of Canadian lynx and patterns in weather.

"Not only in research, but also in the everyday world of politics and economics, we would be better off if more people realized that simple non-linear systems do not necessarily possess simple dynamical properties" [4]

References

1. Shaw, R.S. (1984). The dripping faucet as a model chaotic system. *Aerial Press*.
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3. Gleick, James. (1988). Chaos: making a new science. New York, N.Y., U.S.A, Penguin
4. May, Robert. (1976). Simple Mathematical Models With Very Complicated Dynamics. *Nature*. 26. 457.

This project was part of the course PHYS 25000 - Computational Physics taught at the University of Chicago in the Fall of 2018. For more information visit: https://github.com/shiv-agr/PHSY250FinalProject

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