Partial Differential Equations
Part 1

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The Heat equation in 1D

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$$a_1 u\left(0, t\right) + a_2 u_x\left(0, t\right) = g_1\left(t\right)$$

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In what follows we will solve the simpler case $q(x,t) = 0$

The finite difference scheme

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- We also need approximations for higher derivatives, e.g

$$u_{xx}\left(x_i, t_n\right) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\left(\Delta x\right)^2}$$

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- can be written in a matrix form $\mathbf{u}^{(n+1)} = A\mathbf{u}^{(n)}$ where $\mathbf{u}^{(n)} = \begin{pmatrix} u_1^n, & u_2^n, & \ldots, & u_{N-1}^n \end{pmatrix}^T$

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- The most general solution, then, is $D_n = Ar_1^n + Br_2^n$

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