

Adaptive-timesteps-v2

February 6, 2024

```
[39]: using Plots
using LaTeXStrings
using LinearAlgebra
using Printf
pyplot()
```

```
[39]: Plots.PyPlotBackend()
```

```
[40]: function rk4(f1, x, y, h)
    k1 = h*f1(x, y)
    k2 = h*f1(x+h/2, y+k1/2)
    k3 = h*f1(x+h/2, y+k2/2)
    k4 = h*f1(x+h, y+k3)
    return y + (k1+2k2+2k3+k4)/6
end

function caller(my_method, fn, y_ini, x0, xT, h)
    # generate x values from x0 to xT
    xs = range(x0, step=h, stop=xT);

    # calculate the number of steps
    N = length(xs);

    # initialize y
    y = y_ini;

    # initialize storage to keep y values
    ys = zeros(Float64, N, length(y_ini));

    # for each i
    for i=1:N
        x = xs[i] # get x
        ys[i,:] .= y # store y
        y = my_method(fn,x,y,h) # update y
    end

    # return y
    return ys
end
```

```
end
```

[40]: caller (generic function with 1 method)

```
[41]: # for  $dy/dx = f(x,y)$ 
# define  $f(x,y)$ 
f(x, y) = -y;

# Step size h
h = 1.0e-1;

# Time span
x0 = 0.0;
xT = 10.0;

# Generate x values (for plotting and comparison)
xs = range(x0, stop=xT, step=h);
N = length(xs);

# Initialize y
y_ini = 1.0;

# Run caller for small time (to compile)
caller(rk4, f, y_ini, 0.0, 2h, h);

# Final call
@time ysrk4 = caller(rk4, f, y_ini, x0, xT, h);
```

0.000144 seconds (709 allocations: 18.250 KiB)

```
[42]: # generate x values with small interval
xfine = range(0.0, stop=xs[end], length=1000);

# generate corresponding y values
yfine = exp.(-xfine)

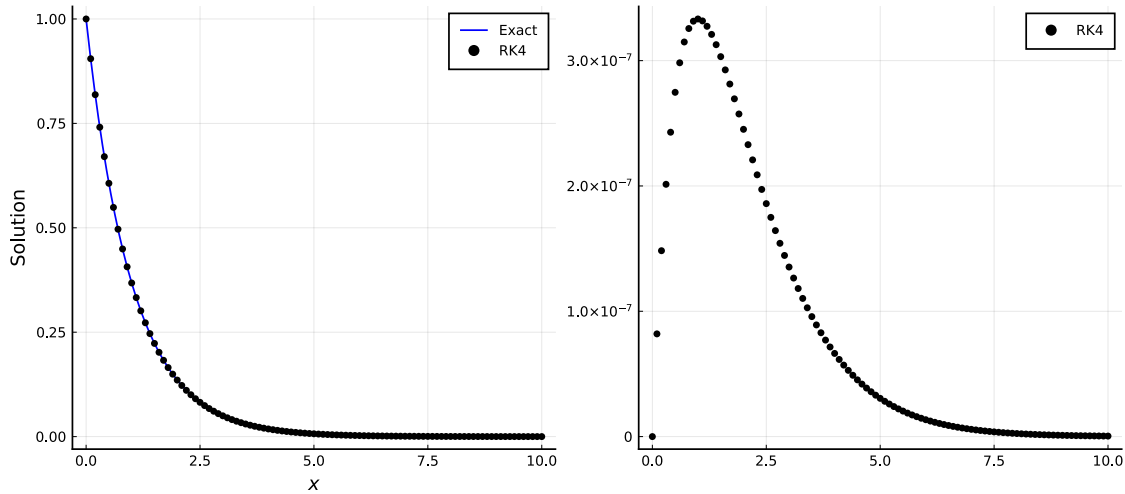
# Also calculate exact y at the solution point (xs)
yexact = exp.(-xs);

# Plot the line (exact solution)
a = plot(xfine, yfine, seriestype=:line, linecolor=:blue,
    ↪linewidth=1, label="Exact", xlabel=L"$x$", ylabel="Solution", size=(600,600));
# Scatter plot the computed solution
scatter!(xs, ysrk4, color=:black, markersize=4,
    ↪markerstrokewidth=0, label="RK4");

# Show the error
```

```
b = scatter(xs, ysrk4 .- yexact, color=:black, markersize=4,
    ↪markerstrokewidth=0,label="RK4");
plot(a, b, size=(900,400))
```

[42]:



```
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
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'vmin', 'vmax' will be ignored
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'vmin', 'vmax' will be ignored
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
```

[43]:

```
## Cash-Karp parameters
#a2 = 1/5; b21 = 1/5;
#a3 = 3/10; b31 = 3/40; b32 = 9/40;
#a4 = 3/5; b41 = 3/10; b42 = -9/10; b43 = 6/5;
#a5 = 1; b51 = -11/54; b52 = 5/2; b53 = -70/27; b54 = 35/27;
#a6 = 7/8; b61 = 1631/55296; b62 = 175/512; b63 = 575/13824; b64 = 44275/
    ↪110592; b65 = 253/4096;
#
#c1 = 37/378; c2 = 0.0; c3 = 250/621; c4 = 125/594; c5 = 0.0; c6 = 512/1771;
#c1s = 2825/27648; c2s = 0.0; c3s = 18575/48384; c4s = 13525/55296; c5s = 277/
    ↪14336; c6s = 1/4;
# Dormand-Prince in matrix form
a = [0.2, 0.3, 0.8, 8/9, 1.0, 1.0];
b = [
    [1/5, 0, 0, 0, 0, 0],
    [3/40, 9/40, 0, 0, 0, 0],
```

```

[44/45, -56/15, 32/9, 0, 0, 0],
[19372/6561, -25360/2187, 64448/6561, -212/729, 0, 0],
[9017/3168, -355/33, 46732/5247, 49/176, -5103/18656, 0],
[35/384, 0, 500/1113, 125/192, -2187/6784, 11/84]
]
c = [35/384, 0, 500/1113, 125/192, -2187/6784, 11/84, 0]

# Dormand-Prince
a2 = 1/5; b21 = 1/5;
a3 = 3/10; b31 = 3/40; b32 = 9/40;
a4 = 4/5; b41 = 44/45; b42 = -56/15; b43 = 32/9;
a5 = 8/9; b51 = 19372/6561; b52 = -25360/2187; b53 = 64448/6561; b54 = -212/729;
a6 = 1; b61 = 9017/3168; b62 = -355/33; b63 = 46732/5247; b64 = 49/176;
↪ b65 = -5103/18656;
a7 = 1; b71 = 35/384; b72 = 0; b73 = 500/1113; b74 = 125/192;
↪ b75 = -2187/6784; b76 = 11/84;

c1 = 35/384; c2 = 0; c3 = 500/1113; c4 = 125/192; c5 = -2187/6784; c6 = 11/84; c7
↪ = 0;
c1s = 5179/57600; c2s = 0; c3s = 7571/16695; c4s = 393/640; c5s = -92097/339200;
↪ c6s = 187/2100; c7s = 1/40;

```

```

[44]: #Embedded Runge-Kutta formulas
function erk54(f, x, y, h)
    k1 = h*f(x, y)
    k2 = h*f(x + a2*h, y + b21*k1)
    k3 = h*f(x + a3*h, y + b31*k1 + b32*k2)
    k4 = h*f(x + a4*h, y + b41*k1 + b42*k2 + b43*k3)
    k5 = h*f(x + a5*h, y + b51*k1 + b52*k2 + b53*k3 + b54*k4)
    k6 = h*f(x + a6*h, y + b61*k1 + b62*k2 + b63*k3 + b64*k4 + b65*k5)
    y5 = y + c1*k1 + c2*k2 + c3*k3 + c4*k4 + c5*k5 + c6*k6
    k7 = h*f(x + a7*h, y5)
    y4 = y + c1s*k1 + c2s*k2 + c3s*k3 + c4s*k4 + c5s*k5 + c6s*k6 + c7s*k7
    return y5, y4
end

function caller5(my_method, fn, y_ini, x0, xT, h)
    xs = range(x0, step=h, stop=xT);
    N = length(xs);
    y = y_ini;
    ys = zeros(Float64, N, length(y_ini));
    for i=1:N
        x = xs[i]
        ys[i,:] = y
        y, y4 = my_method(fn,x,y,h)
    end
    return ys
end

```

```

end

function caller4(my_method, fn, y_ini, x0, xT, h)
    xs = range(x0, step=h, stop=xT);
    N = length(xs);
    y = y_ini;
    ys = zeros(Float64, N, length(y_ini));
    for i=1:N
        x = xs[i]
        ys[i,:] = y
        y5, y = my_method(fn,x,y,h)
    end
    return ys
end

```

[44]: caller4 (generic function with 1 method)

```

[45]: f(x, y) = -y;
max_iter = 500;
abstol = 1.0e-6;
reltol = 1.0e-6;
y_ini = 1.0;
x0 = 0.0;
xT = 10.0;
h = 0.1;
xs = range(x0, stop=xT, step=h);
yserk5 = caller5(erk54, f, y_ini, x0, xT, h);
yserk4 = caller4(erk54, f, y_ini, x0, xT, h);

```

```

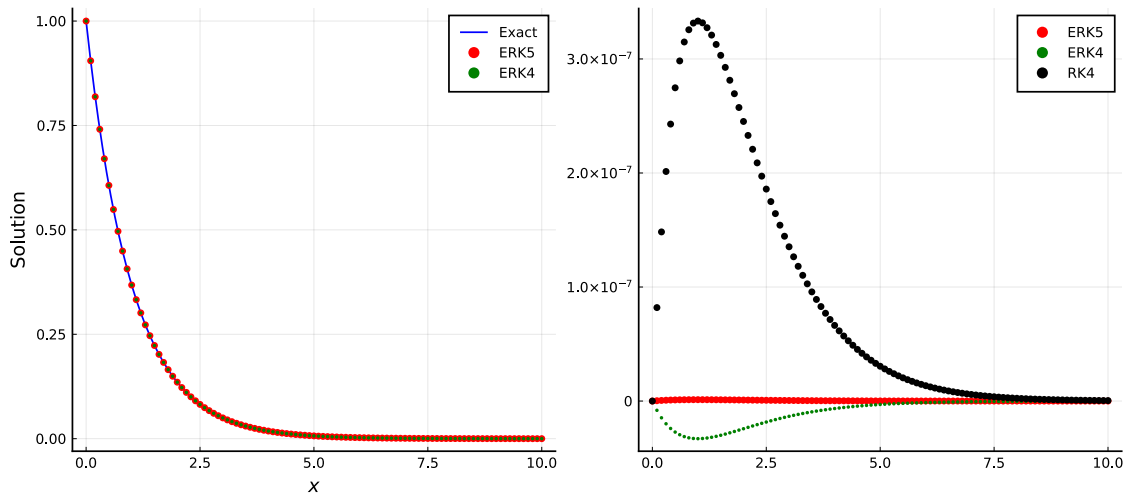
[46]: xfine = range(0.0, stop=xs[end], length=1000);
yfine = exp.(-xfine)
yexact = exp.(-xs);
#yfine = xfine/2 .- sin.(2xfine)/4;
#yexact = xs/2 .- sin.(2xs)/4;
a = plot(xfine, yfine, seriestype=:line, linecolor=:blue,
    ↪linewidth=1, label="Exact", xlabel=L"$x$", ylabel="Solution", size=(600,600));
scatter!(xs, yserk5, color=:red, markersize=4,
    ↪markerstrokewidth=0, label="ERK5");
scatter!(xs, yserk4, color=:green, markersize=2,
    ↪markerstrokewidth=0, label="ERK4");

b = scatter(xs, yserk5 .- yexact, color=:red, markersize=4,
    ↪markerstrokewidth=0, label="ERK5");
scatter!(xs, yserk4 .- yexact, color=:green, markersize=2,
    ↪markerstrokewidth=0, label="ERK4");
scatter!(xs, ysrk4 .- yexact, color=:black, markersize=4,
    ↪markerstrokewidth=0, label="RK4");

```

```
plot(a, b, size=(900,400))
```

[46]:



```
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
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'vmin', 'vmax' will be ignored
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
```

```
[47]: # Predictor-Corrector
# Embedded Runge-Kutta formulas
function erk54h(f, x, y, h, k7bh)
    # Implement FSAL (first same as last)
    # The first
    # k1 = k7bh, unless k1 is zero (initially)
    sum(k7bh)==0 ? k1 = h*f(x, y) : k1 = k7bh*h;

    # Implement rest of the Butcher table
    k2 = h*f(x + a2*h, y + b21*k1)
    k3 = h*f(x + a3*h, y + b31*k1 + b32*k2)
    k4 = h*f(x + a4*h, y + b41*k1 + b42*k2 + b43*k3)
    k5 = h*f(x + a5*h, y + b51*k1 + b52*k2 + b53*k3 + b54*k4)
    k6 = h*f(x + a6*h, y + b61*k1 + b62*k2 + b63*k3 + b64*k4 + b65*k5)

    # get y fifth-order correct
    y5 = y + c1*k1 + c2*k2 + c3*k3 + c4*k4 + c5*k5 + c6*k6 # c7 = 0

    # FSAL -> the last
    k7bh = f(x + a7*h, y5)
```

```

    #@printf("%.6f %.6f %.6f %.6f\n",x, y[1], y5[1], k7bh[1]);

    # get y fourth-order correct
    y4 = y + c1s*k1 + c2s*k2 + c3s*k3 + c4s*k4 + c5s*k5 + c6s*k6 + c7s*k7bh*h

    # return y, error, k7bh
    err = @. abs(y5 - y4);
    return y5, err, k7bh
end

function caller54(fn, y_ini, x0, xT, h0, max_iter, abstol, reltol)
    y = y_ini; h = h0;
    ys = zeros(Float64, max_iter, length(y_ini));
    xs = zeros(Float64, max_iter, 1);
    xs[1] = x0; ys[1,:] .= y_ini;
    x = x0; i = 1; k = 1;
    k7bh = zeros(Float64, size(y_ini));
    while x<=xT && i<max_iter

        # calculate the tolerance
        if i == 1
            tol = abstol + reltol*norm(ys[i,:]);
        else
            tol = abstol + reltol*maximum([norm(ys[i,:]), norm(ys[i-1,:])]);
        end

        # get the next y and the error
        y, err, k7bh = erk54h(fn,xs[i],ys[i,:],h, k7bh);

        # get max error and the scale factor
        merr = maximum(err);
        #scale_factor = (tol/merr)^0.2;

        # for debugging
        #@printf("%2d %.4f %.6f %.6f %g\n",i,xs[i],ys[i],h,merr);

        merr == 0.0 ? merr = tol/100 : nothing;

        # If the step is valid (less error than tol)
        # then we proceed
        if merr<tol
            x += h;
            i += 1;
            xs[i] = x;
            ys[i,:] .= y;
        end
    end
end

```

```

        #k7 = k7t;
        #@printf("%2d %.4f %.6f %.6f\n",i,xs[i],ys[i],k7[1]);
    end

    # anyway we update h every time
    h = 0.9*h*(tol/merr)^0.2;
end

# if the maximum number of steps exceeded
if i>=max_iter
    println(i," Increase max_iter.")
    return Nothing
end

# return the usable portion of the array
return xs[1:i-1], ys[1:i-1,:]
end

```

[47]: caller54 (generic function with 1 method)

```

[48]: # Choose dy/dx = -y
f(x, y) = -y;

# Initialize parameters
max_iter = 2000;
abstol = 1.0e-6;
reltol = 1.0e-8;
y_ini = 1.0;
x0 = 0.0;
xT = 20.0;
h0 = 0.1;

# Call it with small delta to compile
caller54(f, y_ini, x0, 2h0, h0, max_iter, abstol, reltol)

# Final call
@time xs54, yserk54 = caller54(f, y_ini, x0, xT, h0, max_iter, abstol, reltol);

```

0.001310 seconds (4.38 k allocations: 205.125 KiB)

```

[49]: # Plot solution
scatter(xs54, yserk54, markercolor=:red, markerstrokewidth=0, label="Computed_
↪solution")

# Generate and plot exact solution
xfine = range(xs54[1], stop=xs54[end], length=1000);
yfine = exp.(-xfine)
yexact = exp.(-xs54);

```



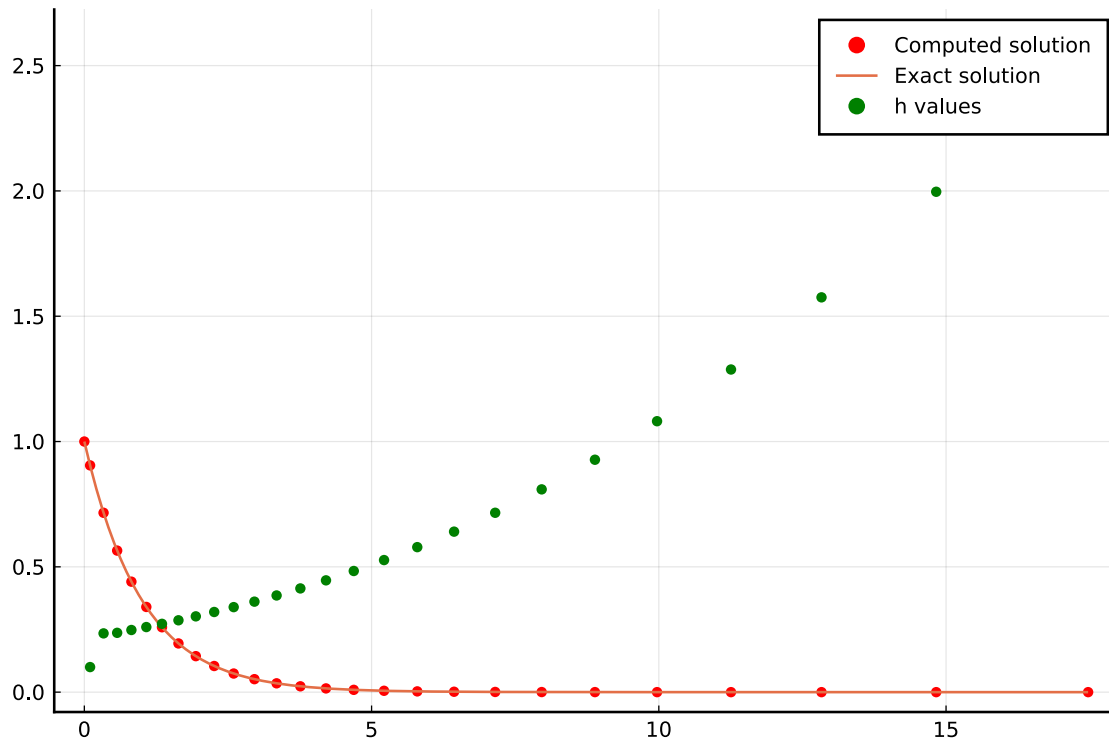
```

plot!(xfine, yfine[:,1], label="Exact solution")

# Also plot h values
plot!(xs54[2:end],diff(xs54),seriestype=:scatter,markercolor=:
↪green,markerstrokewidth=0, label="h values")

```

[49]:



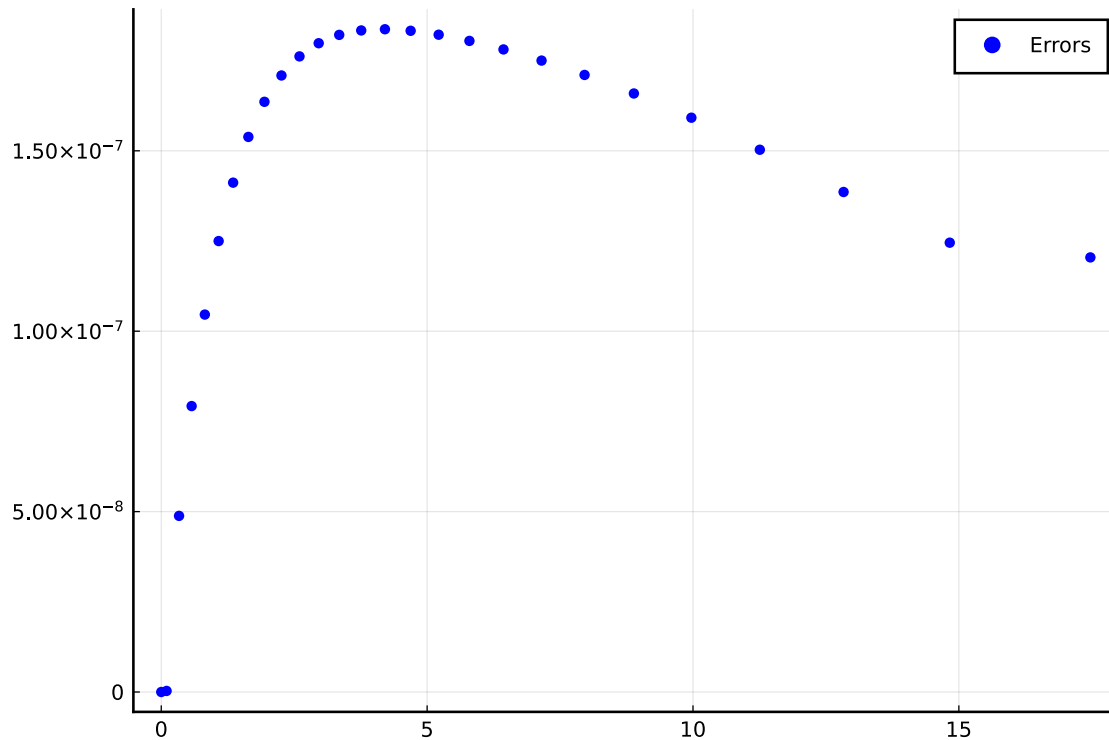
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters 'vmin', 'vmax' will be ignored
 sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters 'vmin', 'vmax' will be ignored
 sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters 'vmin', 'vmax' will be ignored
 sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters 'vmin', 'vmax' will be ignored

```

[50]: scatter(xs54, yserk54 .- exp.(-xs54), markercolor=:blue, markerstrokewidth=0, ↵
↪label="Errors")

```

[50]:



```

sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
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'vmin', 'vmax' will be ignored
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'vmin', 'vmax' will be ignored
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored

```

```

[51]: # d2y/dx2 = -y
f(x, y) = [y[2], -y[1]];

# initialize parameters
max_iter = 2000;
abstol = 1.0e-6;
reltol = 1.0e-8;
y_ini = [1.0, 0.0];
x0 = 0.0;
xT = 20.0;
h0 = 0.1;

# compile
caller54(f, y_ini, x0, 2h0, h0, max_iter, abstol, reltol)

```

```

@time xs54, yserk54 = caller54(f, y_ini, x0, xT, h0, max_iter, abstol, reltol);

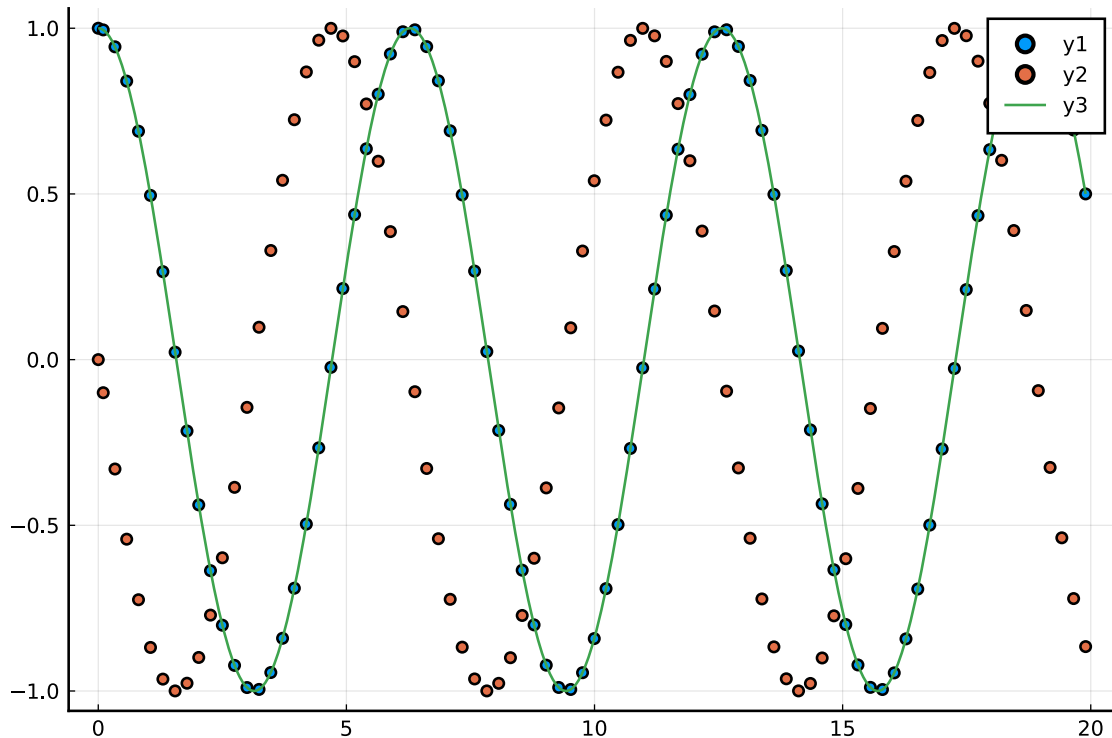
# plot the computed solution
scatter(xs54, yserk54)

# plot the exact solution
xfine = range(xs54[1], stop=xs54[end], length=1000);
yfine = @. [cos(xfine) -sin(xfine)];
plot!(xfine, yfine[:,1])

```

0.003650 seconds (13.62 k allocations: 655.734 KiB)

[51]:



sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters 'vmin', 'vmax' will be ignored

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sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters 'vmin', 'vmax' will be ignored

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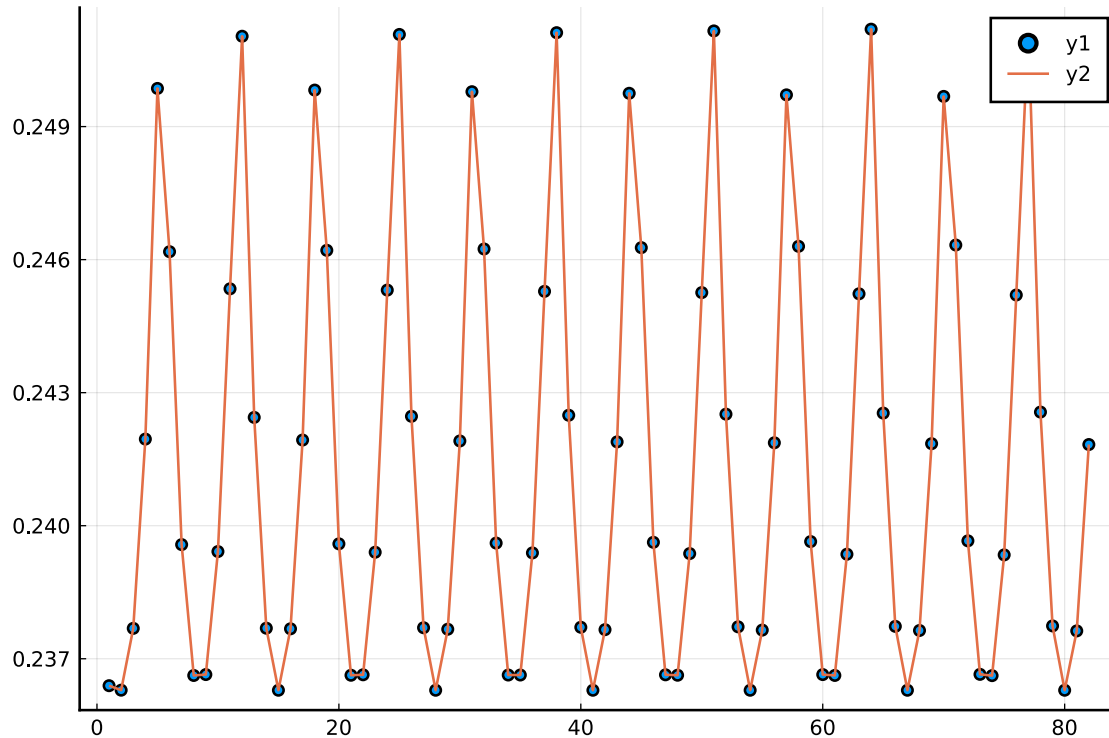
[52]:

```

scatter(diff(xs54)[2:end])
plot!(diff(xs54)[2:end])

```

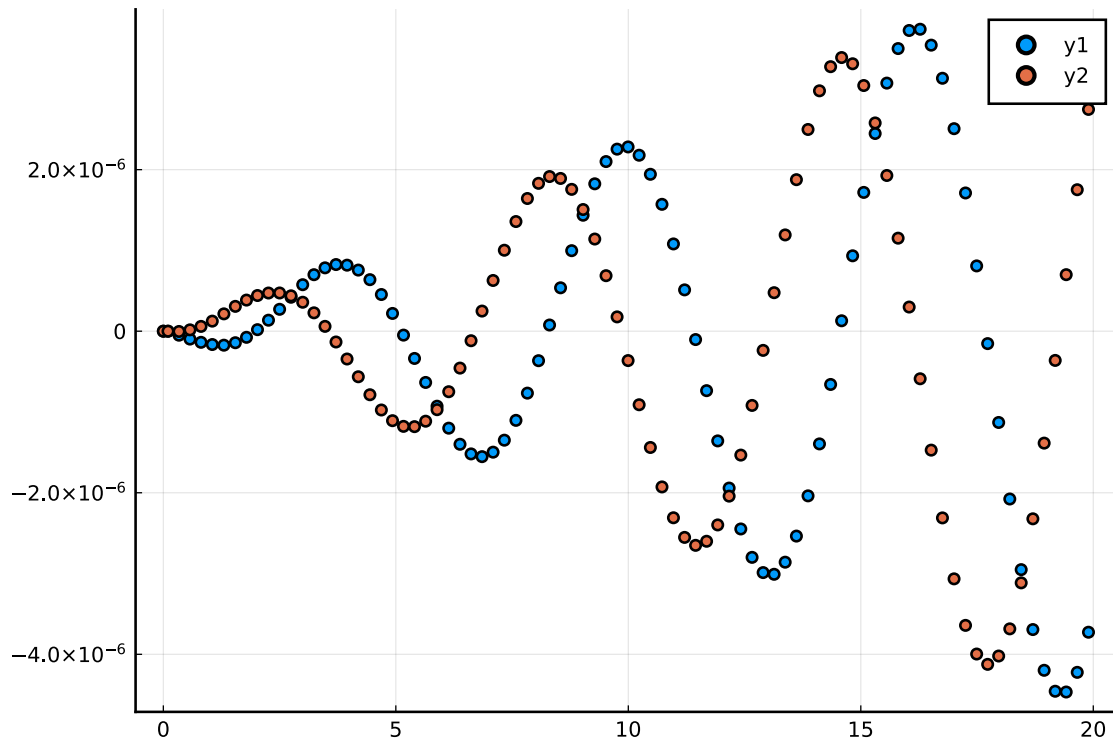
[52]:



```
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
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'vmin', 'vmax' will be ignored
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
```

```
[56]: scatter(xs54, @. yserk54 - [cos(xs54) -sin(xs54)])
```

[56]:



```

sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored
sys:1: UserWarning: No data for colormapping provided via 'c'. Parameters
'vmin', 'vmax' will be ignored

```

```
[54]: yserk54
```

```

[54]: 84x2 Matrix{Float64}:
 1.0      0.0
0.995004 -0.0998334
0.943952 -0.330084
0.84045  -0.541889
0.689231 -0.724541
0.495555 -0.868576
0.265395 -0.964139
0.0224333 -0.999748
-0.215436 -0.976518
-0.438349 -0.898804
-0.636849 -0.770988
-0.801513 -0.597976

```

-0.922753	-0.385391
-0.026787	0.999637
0.211264	0.977425
0.434529	0.900654
0.63355	0.773697
0.798905	0.601451
0.921012	0.389524
0.988934	0.148332
0.995611	-0.0935435
0.945577	-0.325386
0.84313	-0.537702
0.692864	-0.721063
0.500022	-0.866008

[]: