Week – 4

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1. Bipartite Graph : $G(V, E)$ if V can be partitioned into two disjoint subsets L and R so that any edge has one end in L and other end in R.
2. Matching : Subset of edges $M \subseteq E$ which have no common endpoints.
3. Maximum Matching : Matching M of maximum size. Hence $\forall M'$ matchings, $|M'| \leq |M|$.
4. Perfect Matching : For a Bipartite Graph $G(V, E)$, with the bipartition $V = L \bigcup R$ and $|L| = |R| = n$, then M is a perfect matching if it is a maximum matching of size n.
5. d – Regular Graph : A graph whose every vertex has degree ' d '. Hence exactly d edges originate from each vertex.

Hall's Theorem

It gives necessary and sufficient condition for Perfect Matching to exist. For a bipartite Graph $G(V, E)$ with $V = L \bigcup R$ and $|L| = |R| = n$, perfect matching exists if and only if $\forall$ subsets $S \subseteq L$, $|N(S)| \geqslant |S|$, where $N(S)$ denotes neighbourhood of S.

▹ Corollary : Every d – regular bipartite graph has Perfect Matching.

Problem – 1

1. Bipartite Maching

This is a classic bipartite matching problem which needs to be solved using the algorithmic proof of hall's theorem. You are given n workers and m jobs. Each worker is willing to do some jobs between 1 and m. Also, one job can be done by atmost one worker and each worker can do atmost one job. Can you assign a job to each worker?

Input : n m
< n lines with each line containing the number of jobs (k) that a worker is willing to do followed by k space separated integers >

Output : < n lines with each line having two space separated integers which are worker, job if assignment possible, just print – 1 if not possible >

Understanding

This problem can be modeled as a graph, with one set of vertices representing workers and other set representing jobs. An edge between two vertices of these two sets represents a job.
For any subset of workers, the number of jobs that are compatible with at least one of the workers in the subset is at least as large as the number of workers in the subset.
If this condition holds, then there exists a perfect matching in the graph, which means that each worker can be assigned to a job.
Let's say there are 3 workers and 4 jobs. The workers are named A, B, and C, and the jobs are named 1, 2, 3, and 4.
The graph would have 3 worker vertices and 4 job vertices. The edges would be as follows:
- A is willing to do jobs 1 and 2, so there would be edges between A and vertices 1 and 2.
- B is willing to do jobs 2 and 3, so there would be edges between B and vertices 2 and 3.
- C is willing to do jobs 3 and 4, so there would be edges between C and vertices 3 and 4.

Once the graph is created, we can check whether the condition in Hall's theorem holds. If it does, then there exists a perfect matching in the graph, and each worker can be assigned to a job.