MAT 196 Laboratory Exercise No.2

Jeanne Marie B. Quiñanola

July 10, 2025

1 Introduction

In difference equations, the changes in states of a system are modeled over discrete intervals. In most cases, the discrete intervals represent time intervals, and therefore the letter t is used to denote the time variable. A form of the difference equation that will be encountered most often is

$$x_{t+k} + a_1 x_{t+k-1} + \ldots + a_{k-1} x_{t+1} + a_k x_t = b_t, \quad t = 0, 1, \ldots$$

The order of the difference equation is k, provided $a_k \neq 0$. The coefficients are assumed to be real and the functions are real-valued. The coefficients a_j, $j = 1, \ldots, k$, can be functions of t and state variables x_i for $i = t, \ldots, t + k - 1$. The function b_t on the right-hand side of the equation may depend on t but not on the state variables.

If the coefficients a_j, $j = 1, \ldots, k$ in the equation are constant or depend on t but do not depend on the state variables, then the difference equation is said to be linear. Otherwise, it is said to be nonlinear. In addition, if the difference equation is linear and $b_t = 0$ for all t, then it is said to be homogeneous; otherwise, it is said to be nonhomogeneous.

The method for finding the general solution to the k-th order, nonhomogeneous, linear difference equation involves the following steps.

(a) Find the general solution to the homogeneous difference equation ($b_t = 0$), a solution depending on k arbitrary constants and k linearly independent solutions. (Recall that the k solutions $x^1, \ldots, x^k$ are linearly independent if $\sum_{i=1}^{k} a_i x^i = 0$ implies $a_i = 0$ for $i = 1, \ldots, k$). Denote the general solution as x_h. Then

$$x_h = c_1 x^1 + \ldots + c_k x^k = \sum_{i=1}^{k} c_i x^i,$$

where $c_1, \ldots, c_k$ are constants, and $x^1, \ldots, x^k$ are k linearly independent solutions of the homogeneous equation. (Note: The notation x^i means the i-th solution and not x raised to the power i.) Any linear combination of solutions to the linear homogeneous difference equation is also a solution. This property is known as the superposition principle. The key to finding the general solution is to identify the k linearly independent solutions.

(b) Find a particular solution to the nonhomogeneous difference equation, a solution with no arbitrary constants. Denote this particular solution as x_p.

(c) The sum of the general solution x_h and the particular solution x_p gives the general solution to the nonhomogeneous linear difference equation:

$$x_t = x_h + x_p$$

If initial conditions $x_0, x_1, \ldots, x_{k-1}$ are prescribed, then the constants $c_1, c_2, \ldots, c_k$ can be uniquely determined. [1]

By understanding these concepts, this study explores a specific difference equation, investigating its long-term behavior and convergence through simulation and analysis.

2 Analysis and Discussion

Consider the second-order,nonhomogeneous,linear difference equation

$$6x_t + 5x_{t-1} + x_{t-2} = \left(\frac{1}{5}\right)^t$$

for $t > 2$ with initial conditions $x_0 = 0$ and $x_1 = 1$.

Solution

To find two linearly independent solutions, x^1 and x^2, assume the solution has the form $x_t = \lambda^t$ where $\lambda \neq 0$.
Solving for the general solution, we have:

$$6\lambda^t + 5\lambda^{t-1} + \lambda^{t-2} = 0$$
$$\lambda^{t-2}(6\lambda^2 + 5\lambda + 1) = 0$$
$$6\lambda^2 + 5\lambda + 1 = 0$$
$$(3\lambda + 1)(2\lambda + 1) = 0$$
$$\lambda_1 = -\frac{1}{3}, \quad \lambda_2 = -\frac{1}{2}$$

The general solution to the homogeneous equation is given by:

$$x_t^h = c_1 \lambda_1^t + c_2 \lambda_2^t$$
$$x_t^h = c_1 \left(-\frac{1}{3}\right)^t + c_2 \left(-\frac{1}{2}\right)^t$$

To solve for the particular solution, assume $x_t^p = A \left(\frac{1}{5}\right)^t$, and substitute into the given difference equation.

$$6A \left(\frac{1}{5}\right)^t + 5A \left(\frac{1}{5}\right)^{t-1} + A \left(\frac{1}{5}\right)^{t-2} = \left(\frac{1}{5}\right)^t$$
$$6A + 25A + 25A = 1$$
$$56A = 1$$
$$A = \frac{1}{56}$$

Thus, the particular solution is:

$$x_t^p = \frac{1}{56} \left(\frac{1}{5}\right)^t$$

Hence,

$$x_t = x_t^h + x_t^p$$
$$x_t = c_1 \left(-\frac{1}{3}\right)^t + c_2 \left(-\frac{1}{2}\right)^t + \frac{1}{56} \left(\frac{1}{5}\right)^t.$$

To solve for the solution of x_t satisfying the initial conditions $x_0 = 0$ and $x_1 = 1$, we have:

$$x_0 = c_1 \left(-\frac{1}{3} \right)^0 + c_2 \left(-\frac{1}{2} \right)^0 + \frac{1}{56} \left(\frac{1}{5} \right)^0 = 0$$
$$c_1 + c_2 + \frac{1}{56} = 0$$
$$c_1 + c_2 = -\frac{1}{56}$$

$$x_1 = c_1 \left(-\frac{1}{3} \right) + c_2 \left(-\frac{1}{2} \right) + \frac{1}{56} \left(\frac{1}{5} \right) = 1$$
$$-\frac{1}{3}c_1 - \frac{1}{2}c_2 + \frac{1}{280} = 1$$
$$-\frac{1}{3}c_1 - \frac{1}{2}c_2 = \frac{279}{280}$$

Hence, we have the following system of equations.

$$\begin{cases} c_1 + c_2 = -\frac{1}{56} \\ -\frac{1}{3}c_1 - \frac{1}{2}c_2 = \frac{279}{280} \end{cases}$$

Solving for c_1 and c_2, we have:

$$-\frac{1}{3}c_1 - \frac{1}{2} \left(-\frac{1}{56} - c_1 \right) = \frac{279}{280}$$
$$-\frac{1}{3}c_1 + \frac{1}{112} + \frac{1}{2}c_1 = \frac{279}{280}$$
$$\frac{-2c_1 + 3c_1}{6} = \frac{279}{280} - \frac{1}{112}$$
$$c_1 = \left(\frac{79}{80} \right)(6)$$
$$c_1 = \frac{237}{40}$$
$$c_2 = -\frac{1}{56} - \frac{237}{40}$$
$$c_2 = -\frac{208}{35}$$

Thus,

$$x_t = \frac{237}{40} \left(-\frac{1}{3} \right)^t - \frac{208}{35} \left(-\frac{1}{2} \right)^t + \frac{1}{56} \left(\frac{1}{5} \right)^t.$$
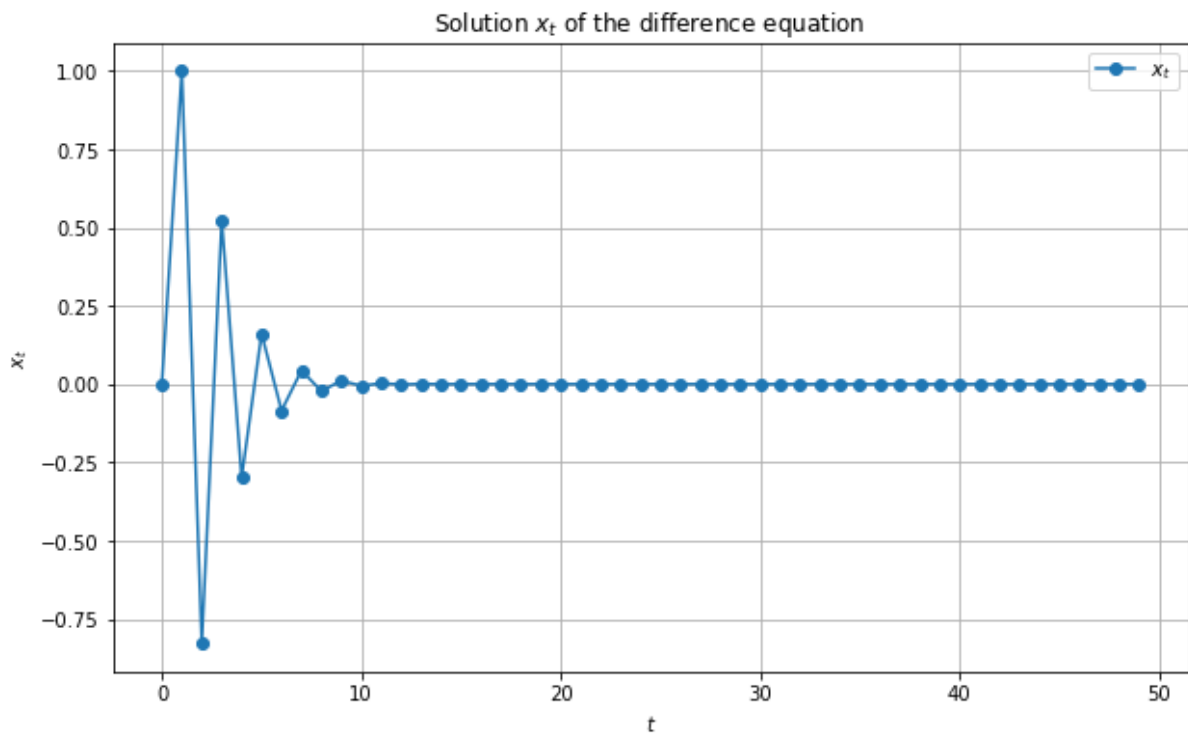
2.1 Convergence Analysis

In order to determine the convergence of divergence of x_t we have the following.

$$\lim_{t \to \infty} x_t = \lim_{t \to \infty} \left[\frac{237}{40} \left(-\frac{1}{3} \right)^t - \frac{208}{35} \left(-\frac{1}{2} \right)^t + \frac{1}{56} \left(\frac{1}{5} \right)^t \right]$$
$$= \frac{237}{40} \lim_{t \to \infty} (-1)^t \left(\frac{1}{3} \right)^t - \frac{208}{35} \lim_{t \to \infty} (-1)^t \left(\frac{1}{2} \right)^t + \frac{1}{56} \lim_{t \to \infty} \left(\frac{1}{5} \right)^t$$
$$= \frac{237}{40} (0) - \frac{208}{35} (0) + \frac{1}{56} (0)$$
$$= 0$$

Thus, x_t converges to 0 as $t \to \infty$.

3 Numerical Simulations

The following is the graph of the solution of $6x_t + 5x_{t-1} + x_{t-2} = \left(\frac{1}{5}\right)^t$ with the given initial conditions $x_0 = 0$ and $x_1 = 1$.

Here is the Python code for solving and plotting the difference equation:

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters of the equation
a = 6    # coefficient of x_t
b = 5    # coefficient of x_{t-1}
c = 1    # coefficient of x_{t-2}
initial_conditions = [0, 1]    # x_0 = 0, x_1 = 1
T = 50   # number of time steps to simulate

# Initialize the array to store the solution
x = np.zeros(T)
x[0], x[1] = initial_conditions

# Solve the difference equation iteratively
for t in range(2, T):
    x[t] = ((1/5)**t - b * x[t-1] - c * x[t-2]) / a
# Plotting the results
plt.figure(figsize=(10, 6))
plt.plot(np.arange(T), x, label=r'$x_t$', marker='o')
plt.title(r'Solution $x_t$ of the difference equation')
plt.xlabel(r'$t$')
plt.ylabel(r'$x_t$')
plt.grid(True)
plt.legend()
plt.show()
```

4 Summary and Conclusions

In this study, we examined the solution of a difference equation represented by x_t. We derived both the homogeneous and particular solutions, ultimately combining them to formulate the general solution. We then implemented a Python code to simulate the behavior of x_t over time, utilizing specific parameters and initial conditions. The results were plotted to visualize the progression of x_t across a defined number of time steps.

The analysis and computational simulation indicate that as t approaches infinity, the values of x_t converge towards zero. This conclusion is visually supported by the graph generated through the Python code, which illustrates the diminishing behavior of x_t over time. Thus, our findings confirm that indeed, x_t converges to 0, reinforcing the theoretical predictions made throughout this study. This convergence is essential for understanding the long-term behavior of the system described by the difference equation.

References

[1] Linda J. S. Allen. *An Introduction to Mathematical Biology*. Pearson Education, Inc., 2007.