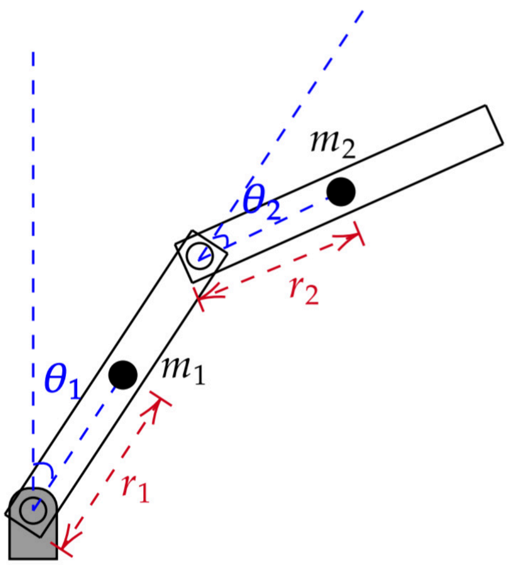
1.2



Generalized coordinates

$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$L = KE_1 + KE_2 - PE_1 - PE_2$$

$$KE_1 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} I_1 \omega_1^2$$

$$\omega_1 = \dot{\theta}_1 \quad , \quad v_1^2 = \dot{x}_{r_1}^2 + \dot{y}_{r_1}^2$$

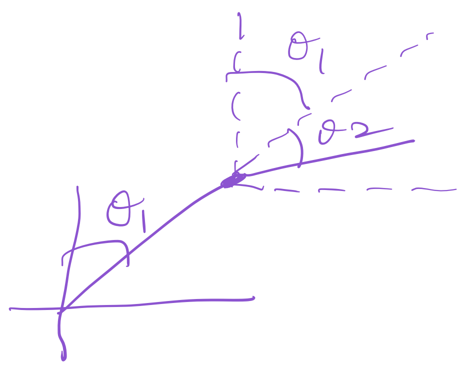
$$x_{r_1} = r_1 \sin\theta_1 \Rightarrow \dot{x}_{r_1} = r_1 \dot{\theta}_1 \cos\theta_1$$

$$y_{r_1} = r_1 \cos\theta_1 \Rightarrow \dot{y}_{r_1} = -r_1 \dot{\theta}_1 \sin\theta_1$$

$$v_1^2 = r_1^2 \dot{\theta}_1^2$$

$$KE_1 = \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_1 \dot{\theta}_1^2$$

$$PE_1 = m g r_1 \cos\theta_1$$



$$KE_2 = \frac{1}{2} m_2 v_2^2 + \frac{1}{2} I_2 \omega_2^2$$

$$\omega_2 = \dot{\theta}_1 + \dot{\theta}_2 \, , \quad v_2^2 = \dot{x}_{r_2}^2 + \dot{y}_{r_2}^2$$

$$x_{r_2} = l_1 \sin\theta_1 + r_2 \sin(\theta_1 + \theta_2)$$

$$\dot{x}_{r_2} = l_1 \dot{\theta}_1 \cos\theta_1 + r_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2)$$

$$y_{r2} = l_1 \cos\theta_1 + r_2 \cos(\theta_1+\theta_2)$$

$$\dot{y}_{r2} = -l_1 \dot{\theta}_1 \sin\theta_1 - r_2 (\dot{\theta}_1+\dot{\theta}_2) \sin(\theta_1+\theta_2)$$

$$v_2^2 = l_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + r_2^2 (\dot{\theta}_1+\dot{\theta}_2)^2 \cos^2(\theta_1+\theta_2) +$$
$$2 l_1 \dot{\theta}_1 r_2 (\dot{\theta}_1+\dot{\theta}_2) \cos\theta_1 \cos(\theta_1+\theta_2) + l_1^2 \dot{\theta}_1^2 \sin^2\theta_1 +$$
$$r_2^2 (\dot{\theta}_1+\dot{\theta}_2)^2 \sin^2(\theta_1+\theta_2) + 2 l_1 \dot{\theta}_1 \sin\theta_1 \, r_2 (\dot{\theta}_1+\dot{\theta}_2) \sin(\theta_1+\theta_2)$$

$$= l_1^2 \dot{\theta}_1^2 + r_2^2 (\dot{\theta}_1+\dot{\theta}_2)^2 + 2 l_1 \dot{\theta}_1 r_2 (\dot{\theta}_1+\dot{\theta}_2) \cos\theta_2$$

$$KE_2 = \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 r_2^2 (\dot{\theta}_1+\dot{\theta}_2)^2 + m_2 l_1 \dot{\theta}_1 r_2 (\dot{\theta}_1+\dot{\theta}_2) \cos\theta_2$$
$$+ \frac{1}{2} I_2 (\dot{\theta}_1+\dot{\theta}_2)^2$$

$$PE_2 = m_2 g (l_1 \cos\theta_1 + r_2 \cos(\theta_1+\theta_2))$$

$$PE_2 = m_2 g l_1 \cos\theta_1 + m_2 g r_2 \cos(\theta_1+\theta_2)$$

$$L = (KE_1 + KE_2) - (PE_1 + PE_2)$$

$$KE_1 = \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_1 \dot{\theta}_1^2$$

$$PE_1 = m_1 g r_1 \cos\theta_1$$

$$L = \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 r_2^2 (\dot{\theta}_1+\dot{\theta}_2)^2$$
$$+ m_2 l_1 \dot{\theta}_1 r_2 (\dot{\theta}_1+\dot{\theta}_2) \cos\theta_2 + \frac{1}{2} I_2 (\dot{\theta}_1+\dot{\theta}_2)^2 - m_1 g r_1 \cos\theta_1$$
$$- m_2 g l_1 \cos\theta_1 - m_2 g r_2 \cos(\theta_1+\theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 r_1^2 \dot{\theta}_1 + I_1 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 r_2^2 (\dot{\theta}_1+\dot{\theta}_2) + m_2 l_1 \dot{\theta}_1 r_2 \cos\theta_2$$
$$+ m_2 l_1 r_2 (\dot{\theta}_1+\dot{\theta}_2) \cos\theta_2 + I_2 (\dot{\theta}_1+\dot{\theta}_2)$$

$$\frac{\partial L}{\partial \theta_1} = m_1 g r_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1 + m_2 g r_2 \sin(\theta_1+\theta_2)$$

$$\frac{\partial L}{\partial \dot\theta_1} = m_1 r_1^2 \dot\theta_1 + I_1 \dot\theta_1 + m_2 l_1^2 \dot\theta_1 + m_2 r_2^2 \dot\theta_1 + m_2 r_2^2 \dot\theta_2 + m_2 l_1 \dot\theta_1 r_2 \cos\theta_2$$
$$+ m_2 l_1 r_2 \dot\theta_1 \cos\theta_2 + m_2 l_1 r_2 \dot\theta_2 \cos\theta_2 + I_2(\dot\theta_1+\dot\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = m_1 r_1^2 \ddot\theta_1 + I_1 \ddot\theta_1 + m_2 l_1^2 \ddot\theta_1 + m_2 r_2^2 \ddot\theta_1 + m_2 r_2^2 \ddot\theta_2 + m_2 l_1 r_2 \ddot\theta_1 \cos\theta_2$$
$$m_2 l_1 \ddot\theta_1 r_2 \cos\theta_2 - m_2 l_1 \dot\theta_1 \dot\theta_2 r_2 \sin\theta_2$$
$$- m_2 l_1 r_2 \dot\theta_1 \dot\theta_2 \sin\theta_2 + m_2 l_1 r_2 \ddot\theta_2 \cos\theta_2$$
$$- m_2 l_1 r_2 \dot\theta_2^2 \sin\theta_2 + I_2(\ddot\theta_1 + \ddot\theta_2)$$

$$u_1 = \ddot\theta_2 \left[m_2 r_2^2 + m_2 r_2 l_1 \cos\theta_2 + I_2 \right] - \dot\theta_2 \left[l_1 m_2 r_2 \dot\theta_1 \sin\theta_2 \right.$$
$$\left. + l_1 m_2 r_2 \sin\theta_2 (\dot\theta_1 + \dot\theta_2) \right]$$
$$\ddot\theta_1 \left[m_2 l_1^2 + 2 m_2 \cos\theta_2 \, l_1 r_2 + m_1 r_1^2 + m_2 r_2^2 + I_1 + I_2 \right]$$
$$- g l_1 m_2 \sin\theta_1 - g m_1 r_1 \sin\theta_1 - m_2 g r_2 \sin(\theta_1+\theta_2)$$

$$L = \frac{1}{2} m_1 r_1^2 \dot\theta_1^2 + \frac{1}{2} I_1 \dot\theta_1^2 + \frac{1}{2} m_2 l_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 r_2^2 (\dot\theta_1 + \dot\theta_2)^2$$
$$+ m_2 l_1 \dot\theta_1 r_2 (\dot\theta_1 + \dot\theta_2) \cos\theta_2 + \frac{1}{2} I_2 (\dot\theta_1+\dot\theta_2)^2 - m_1 g r_1 \cos\theta_1$$
$$- m_2 g l_1 \cos\theta_1 - m_2 g r_2 \cos(\theta_1 + \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 r_2^2(\dot{\theta}_1+\dot{\theta}_2) + m_2 l_1 \dot{\theta}_1 r_2 \cos\theta_2 + I_2(\dot{\theta}_1+\dot{\theta}_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 r_2^2(\ddot{\theta}_1+\ddot{\theta}_2) + m_2 l_1 \ddot{\theta}_1 r_2 \cos\theta_2 - m_2 l_1 \dot{\theta}_1 \dot{\theta}_2 r_2 \sin\theta_2 + I_2(\ddot{\theta}_1+\ddot{\theta}_2)$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 \dot{\theta}_1 r_2 (\dot{\theta}_1+\dot{\theta}_2)\sin\theta_2 + m_2 g r_2 \sin(\theta_1+\theta_2)$$

$$u_2 = m_2 r_2^2(\ddot{\theta}_1+\ddot{\theta}_2) + m_2 l_1 \ddot{\theta}_1 r_2 \cos\theta_2 - m_2 l_1 \dot{\theta}_1 \dot{\theta}_2 r_2 \sin\theta_2$$
$$I_2(\ddot{\theta}_1+\ddot{\theta}_2) + m_2 l_1 \dot{\theta}_1 r_2 (\dot{\theta}_1+\dot{\theta}_2)\sin\theta_2 - m_2 g r_2 \sin(\theta_1+\theta_2)$$

$$u_2 = \ddot{\theta}_2\left[m_2 r_2^2 + I_2\right] + \ddot{\theta}_1\left[m_2 r_2^2 + m_2 l_1 r_2 \cos\theta_2 + I_2\right] - g m_2 r_2 \sin(\theta_1+\theta_2) + l_1 m_2 r_2 \dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\sin\theta_2 - l_1 m_2 r_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2$$

Manipulator equation:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

$$M = \begin{bmatrix} m_2 l_1^2 + 2m_2\cos\theta_2 l_1 r_2 + m_1 r_1^2 + m_2 r_2^2 + I_1 + I_2 & m_2 r_2^2 + m_2 r_2 l_1 \cos\theta_2 + I_2 \\ m_2 r_2^2 + m_2 l_1 r_2 \cos\theta_2 + I_2 & m_2 r_2^2 + I_2 \end{bmatrix}$$

$$g = \begin{bmatrix} -gl_1m_2\sin\theta_1 - gm_1r_1\sin\theta_1 - m_2gr_2\sin(\theta_1+\theta_2) \\ -gm_2r_2\sin(\theta_1+\theta_2) \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & l_1m_2r_2\dot{\theta}_1\sin\theta_2 + l_1m_2r_2\sin\theta_2(\dot{\theta}_1+\dot{\theta}_2) \\ l_1m_2r_2(\dot{\theta}_1+\dot{\theta}_2)\sin\theta_2 - l_1m_2r_2\dot{\theta}_2\sin\theta_2 & 0 \end{bmatrix}$$

M should be symmetric and positive definite matrix.

$$M = M^T \quad (\because \text{symmetric})$$

A symmetric matrix with all positive eigen values is positive definite matrix

This is verified with attached matlab code b)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}$$

$$\dot x = \begin{bmatrix} \dot x_1 \\ \dot x_2 \\ \dot x_3 \\ \dot x_4 \end{bmatrix} = \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} = F(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2, u_1, u_2)$$

$$\ddot\theta_1 =$$

(I2*u1 - I2*u2 + m2*r2^2*u1 - m2*r2^2*u2 + l1*m2^2*r2^3*theta1_dot^2*sin(theta2) + l1*m2^2*r2^3*theta2_dot^2*sin(theta2) + g*l1*m2^2*r2^2*sin(theta1) + I2*g*l1*m2*sin(theta1) + I2*g*m1*r1*sin(theta1) - l1*m2*r2*u2*cos(theta2) + 2*l1*m2^2*r2^3*theta1_dot*theta2_dot*sin(theta2) + l1^2*m2^2*r2^2*theta1_dot^2*cos(theta2)*sin(theta2) - g*l1*m2^2*r2^2*sin(theta1 + theta2)*cos(theta2) + I2*l1*m2*r2*theta1_dot^2*sin(theta2) + I2*l1*m2*r2*theta2_dot^2*sin(theta2) + g*m1*m2*r1*r2^2*sin(theta1) + 2*I2*l1*m2*r2*theta1_dot*theta2_dot*sin(theta2))/(- l1^2*m2^2*r2^2*cos(theta2)^2 + l1^2*m2^2*r2^2 + I2*l1^2*m2 + m1*m2*r1^2*r2^2 + I1*m2*r2^2 + I2*m1*r1^2 + I1*I2)

$$\ddot\theta_2 =$$

-(I2*u1 - I1*u2 - I2*u2 - l1^2*m2*u2 - m1*r1^2*u2 + m2*r2^2*u1 - m2*r2^2*u2 + l1*m2^2*r2^3*theta1_dot^2*sin(theta2) + l1^3*m2^2*r2*theta1_dot^2*sin(theta2) + l1*m2^2*r2^3*theta2_dot^2*sin(theta2) - g*l1^2*m2^2*r2*sin(theta1 + theta2) - I1*g*m2*r2*sin(theta1 + theta2) + g*l1*m2^2*r2^2*sin(theta1) + I2*g*l1*m2*sin(theta1) + I2*g*m1*r1*sin(theta1) + l1*m2*r2*u1*cos(theta2) - 2*l1*m2*r2*u2*cos(theta2) + 2*l1*m2^2*r2^3*theta1_dot*theta2_dot*sin(theta2) + 2*l1^2*m2^2*r2^2*theta1_dot^2*cos(theta2)*sin(theta2) + l1^2*m2^2*r2^2*theta2_dot^2*cos(theta2)*sin(theta2) - g*l1*m2^2*r2^2*sin(theta1 + theta2)*cos(theta2) + g*l1^2*m2^2*r2*cos(theta2)*sin(theta1) - g*m1*m2*r1^2*r2*sin(theta1 + theta2) + I1*l1*m2*r2*theta1_dot^2*sin(theta2) + I2*l1*m2*r2*theta1_dot^2*sin(theta2) + I2*l1*m2*r2*theta2_dot^2*sin(theta2) + g*m1*m2*r1*r2^2*sin(theta1) +

c) Simulation plot time vs $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$