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Applied Finite Element Method (ME7230)

Assignment-3

Due Date: 2 May 2023 Maximum Marks: 50

Problem 1: 2D Poisson's problem - structured vs. trapezoidal mesh

The governing equations is

$$-\nabla^2 u = f \quad \text{on } \Omega \in [0,1]^2 \quad \text{where } f = -6x - 10 + 60y - 12y^2$$

The exact solution is

$$u(x,y) = x^3 + 5y^2 - 10y^3 + y^4$$

The boundary conditions are

$$\text{on } \Gamma, \bar{u} = u \quad \text{(only dirichlet B.C.)}$$

The weak form

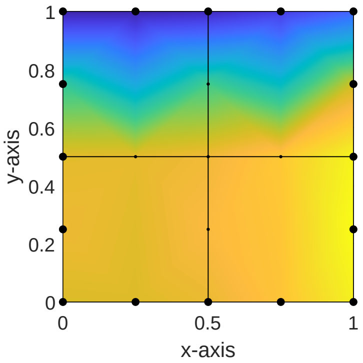
$$\int_\Omega \nabla N^T \nabla N \; d\Omega \;\; \{u\} = \int_\Omega N^T f d\Omega$$

The domain is discretised with the 8-noded quadrilateral serendipity element as shown in Figure 1. The L_2-norm and H_1 semi-norms are calculated for mesh sizes h = {1/2, 1/4, 1/8, 1/16, 1/32, 1/64}. Figure 2 shows the logarithmic plot of L_2-norm and H_1 semi-norms vs. h. The calculated convergence rates are tabulated in Table 1. From

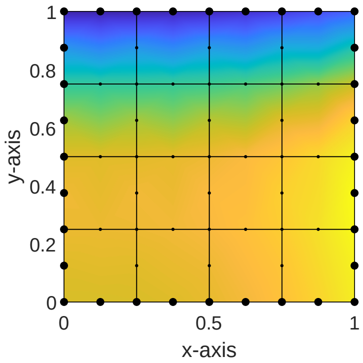
	Structured	Trapezoid
L2	2.9994	2.4936
H1	1.9994	1.3941

Table 1: convergence rates: structured vs. trapezoidal meshes

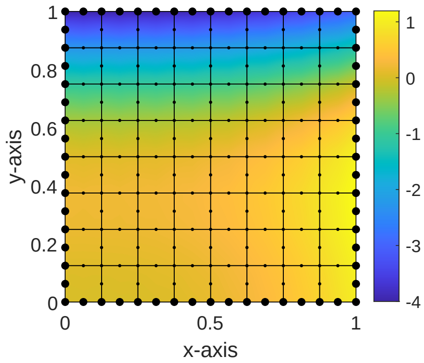
the Table 1, the calculated convergence rates for structured mesh are 2.9994 and 1.9994, which are close to expected values 3 and 2, respectively. Whereas for the trapezoidal mesh, the calculated convergence rates are 2.4936 and 1.3941, which are significantly lower than the expected values 3 and 2, respectively. From this, it can be concluded that distorted elements have a lower convergence rate than rectangular elements. And also, the error in the case of trapezoidal mesh is always higher than the structured mesh.



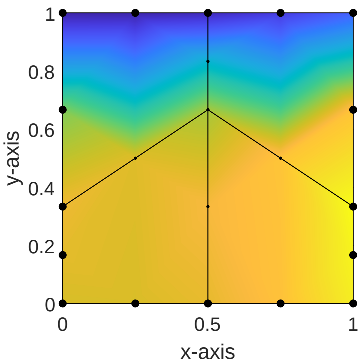
(a) h=0.5



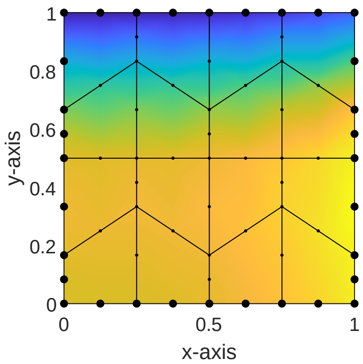
(b) h=0.25



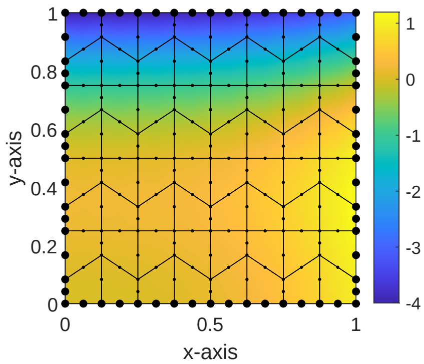
(c) h=0.125



(d) h=0.5

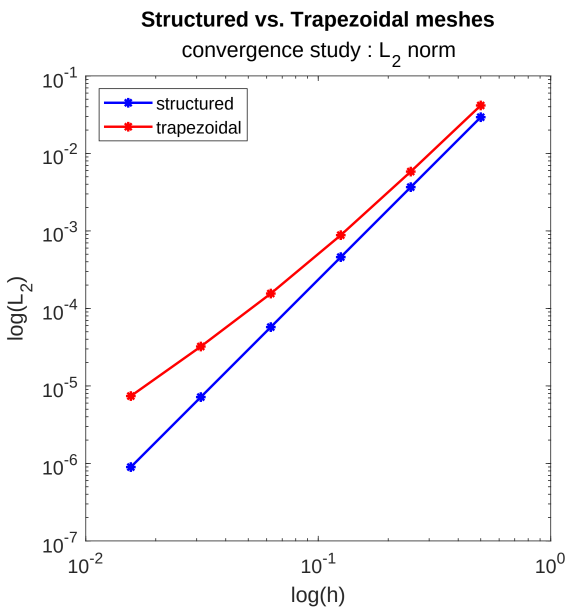


(e) h=0.25

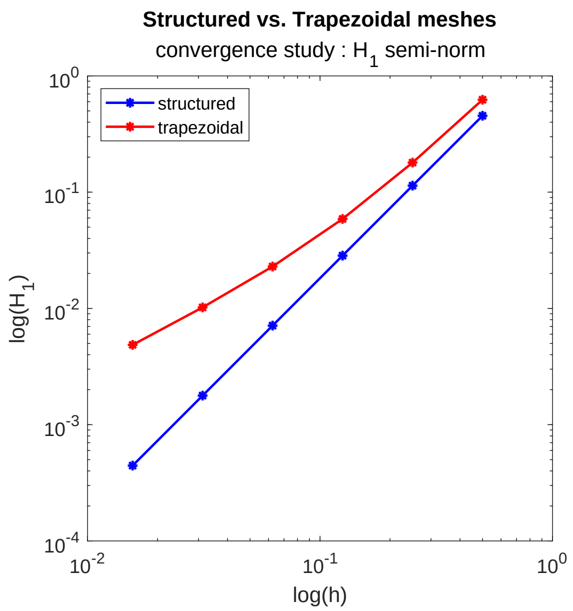


(f) h=0.125

Figure 1: Structured(a,b,c) and trapezoidal(d,e,f) sample meshes



(a) L_2 norm



(b) H_1 semi-norm

Figure 2: logarithmic plot of (a) L_2 norm and (b) H_1 semi-norm for different mesh discretisations (structured vs. trapezoidal elements)

Problem 2: 2D Acoustic Problem

The acoustic wave equation is

$$\Delta p' - \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} = 0 \tag{1}$$

by assuming a harmonic solution,

$$p' = pe^{i\omega t} \implies \frac{\partial^2 p'}{\partial t^2} = -p\omega^2 e^{i\omega t}$$

from Equation 1

$$\begin{aligned}
&\implies e^{i\omega t}\nabla^2 p - \frac{1}{c^2}\left(-p\omega^2 e^{i\omega t}\right) = 0 \\
&\implies \nabla^2 p + k^2 p = 0 \quad \text{where } k^2 = \frac{\omega^2}{c^2} \quad \text{(the Helmholtz equation)}
\end{aligned}$$

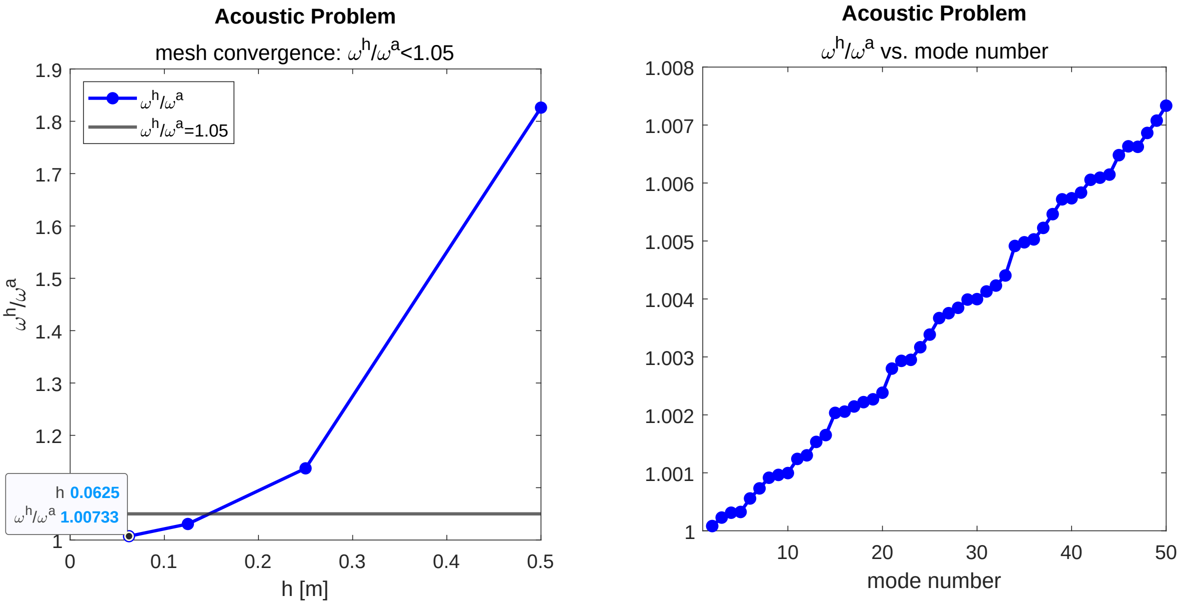
The weak form

$$\begin{aligned}
&\implies \left[\int_\Omega \nabla N^T.\nabla N d\Omega - \int_\Omega N^T k^2 N d\Omega\right]\{p\} = \underbrace{\int_{\partial\Omega} N^T \nabla p.\hat{n}\ d\Omega}_{\text{natural B.C.}} \\
&\implies \int_\Omega \nabla N^T.\nabla N d\Omega - \int_\Omega N^T k^2 N d\Omega = 0 \quad \because \text{zero Neumann B.C.} \\
&\implies K - \omega^2 M = 0 \quad \text{where } K = \int_\Omega \nabla N^T.\nabla N d\Omega,\ M = \int_\Omega N^T \frac{1}{c^2} N d\Omega
\end{aligned}$$

for a rectangular domain of length L and width D, the exact eigenmodes and eigenvalues are

$$\begin{aligned}
\phi_{mn} &= \cos\frac{m\pi x}{L}\cos\frac{n\pi y}{D} \\
f_{mn} &= \frac{c}{2}\sqrt{\left(\frac{m}{L}\right)^2 + \left(\frac{n}{D}\right)^2} \quad m, n = \{0, 1, 2, 3....\} \quad \text{where } f = \frac{\omega}{2\pi}
\end{aligned}$$

for L = 5 m, D = 3 m and c = 340 m/s, the ratio of numerically computed frequency to the analytical frequency $\left(\omega^h/\omega_{ana}\right)$ as a function of mode number is plotted in Figure 3



(a) mesh convergence

(b) first 50 frequencies

Figure 3: 2D acoustic problem

Problem 3: Cylindrical pipe subjected to internal pressure