Geodesic Strips and Effective Conductance

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We are interested in comparing the conductance across the network with the conductance across a narrow tube containing the geodesic. So we start by considering the one-dimensional geodesic, and then think about how we might construct a narrow two-dimensional tube containing the geodesic, working in a 10×10 lattice with endpoints $(2, 2)$ and $(7, 7)$.

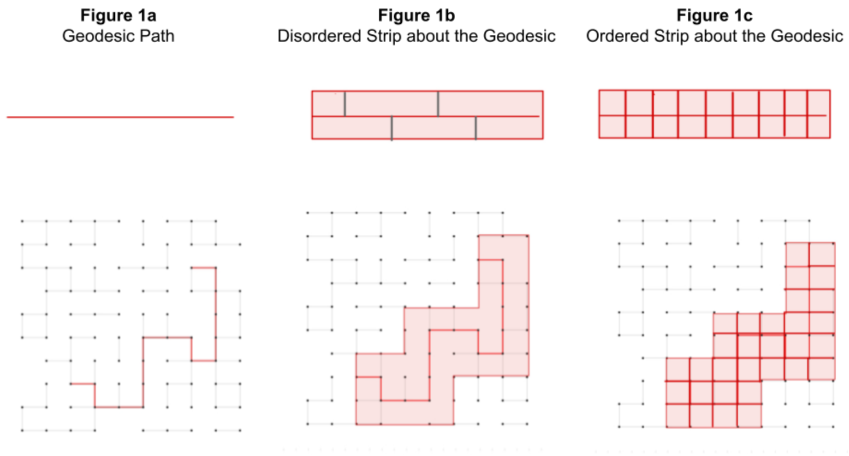

Figure 1: Construction of 2D Strip around Geodesic in 10×10 lattice with endpoints $(2, 2)$ and $(7, 7)$ and $p = 0.53$

Fig. 1a depicts a random resistor network with $p = 0.53$, with 'open edges' colored in red and gray, representing resistors with $R_{ij} = 1$. The red edges represent the geodesic path between the two endpoints $(2, 2)$ and $(7, 7)$. The missing edges represent the $R_{ij} = \infty$ resistors. The straightened red line at the top-left represents a 'stretched out' geodesic. In Fig. 1b, we trace out a 2D strip with width $d = 3$. Fig. 1b includes a stretched-out version of this strip, with some sparse vertical resistors representing the present edges. In Fig. 1c, we fully connect all of the edges with $R_{ij} = 1$ resistors, and its accompanying 'stretched out version' resembles a fully connected 2D strip.

Some preliminary results

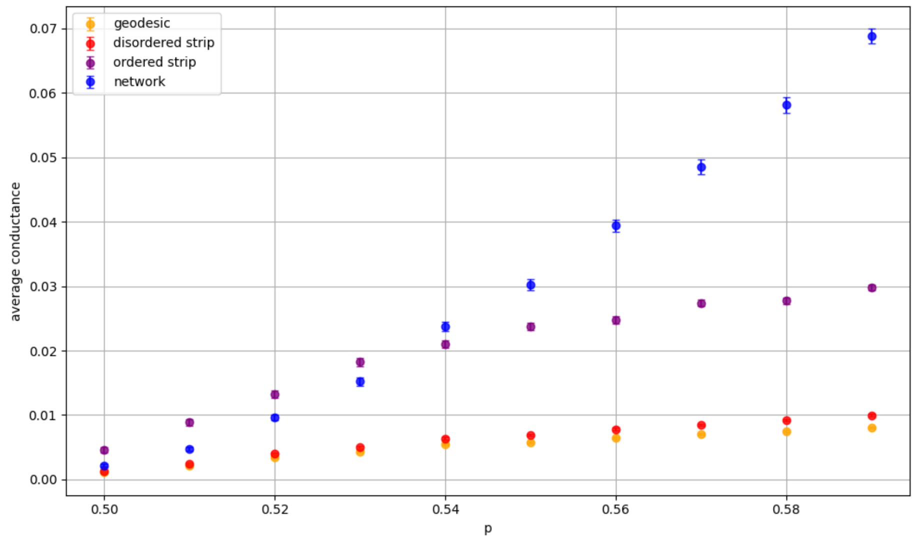

Figure 2: Average Conductance over 500 Configurations Across Different Sections of 100×100 network, over $p \in [0.50, 0.60)$ with spacing 0.01 and endpoints $(24, 24)$ and $(74, 74)$

It is easy to verify, as we do later in this report, that the network conductance Σ_{exact} is amenable to a linear model and admits a critical exponent of 1.24, close to accepted values around 1.28 found in the literature. However, we are trying to understand the primacy of the narrow strips in explaining the network dynamics. It is obvious that, in theory,

$$\Sigma_{\text{geodesic}} \leq \Sigma_{\text{disordered strip}} \leq \Sigma_{\text{exact}}$$

and that

$$\Sigma_{\text{disordered strip}} \leq \Sigma_{\text{ordered strip}}$$

and so, in the next few pages, I will discuss some insights and challenges involved in both the ordered strip and disordered strip models.

Network conductances represented by Fig. 1c (the ordered strip) have analytical solutions, but some drawbacks
We can use analytical tools to study the geodesic and fully-connected strip. See the Appendix for a discussion of how to do this. However, a 'fully connected strip' in a highly sparse lattice does not naturally correspond to a 2D rectangular strip, because geodesic paths in these lattices are highly tortuous. Take our example in Fig. 1:

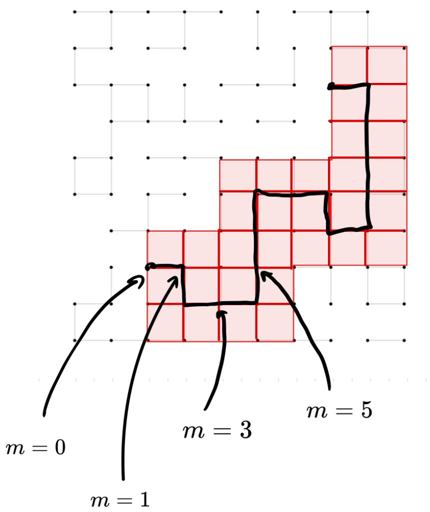

Figure 3: Fully-Connected 2D Strip around Geodesic in 10×10 lattice with endpoints $(2, 2)$ and $(7, 7)$ and $p = 0.53$

The strip contains all nodes within L_1-distance d from the geodesic, together with the 'boundary' formed by these nodes. The above figure has $d = 1$. Let $w_d(m)$ denote the number of nodes included in the strip that lie within L_1-distance d from the geodesic, in the direction orthogonal to the geodesic at point m, where m indexes positions along the geodesic path.

For example, in the figure, at $m = 0$ we have $w_1(0) = 4$, $w_1(3) = 4$, $w_1(5) = 5$. This gives a rough, imperfect shorthand to grasp the 'width' of the strip at different positions. At $m = 1$, where nodes appear in two orthogonal directions, we define the local width as the arithmetic mean of the number of nodes in each direction: in this case, $w_1(1) = \frac{5+3}{2} = 4$.

For an analytical solution for a $d = 3$ rectangular strip to be useful, we would want our tortuous strip to be roughly isometric to such a rectangular strip. To assess this rough isometry, we can examine whether the average width $\langle d_1(m) \rangle$ approximates a target width d, and use this mean as a rough measure of 'width'.

This problem lends itself to simulation because we are dealing with two competing effects. Near the percolation threshold, paths become much longer $\Rightarrow$ length increases. The tortuosity of the path sometimes leads to a greater width being traversed, as we see in Fig. 3.

As $p \to p_c^+$, geodesic paths become more tortuous, and one may ask for the behavior of $w = 3$ and $\langle w_1(m) \rangle$ at the limit for large n. We test this theory in a 100×100 lattice, generating a random resistor network, identifying the geodesic, forming a fully connected strip, counting the number of nodes in the strip to measure area, and dividing by the length of the geodesic to obtain an average width.

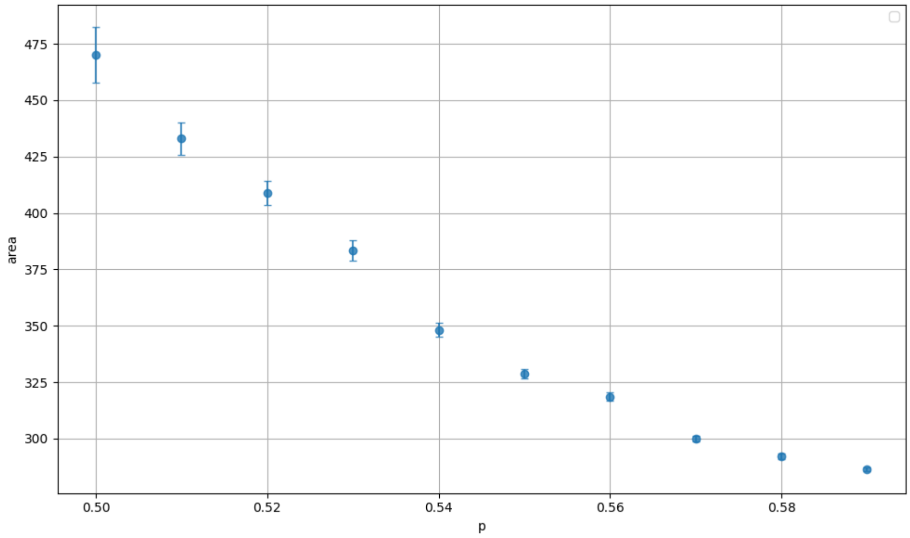

Figure 4: Area of 2D Strip with $w = 1$ for 100×100 lattice for $p \in [0.50, 0.60)$ with step size 0.01 and endpoints $(24, 24)$ and $(74, 74)$

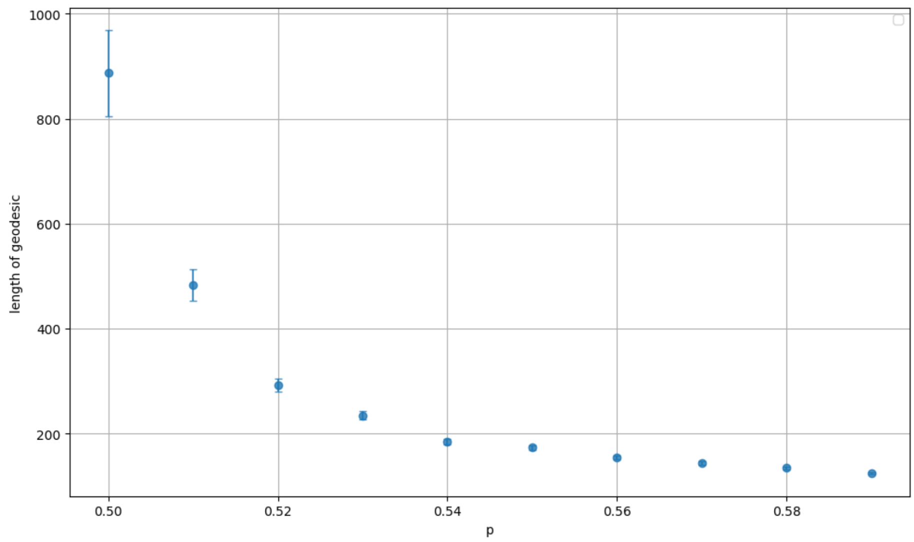

Figure 5: Length of 2D Strip with $w = 1$ for 100×100 lattice for $p \in [0.50, 0.60)$ with step size 0.01 and endpoints $(24, 24)$ and $(74, 74)$

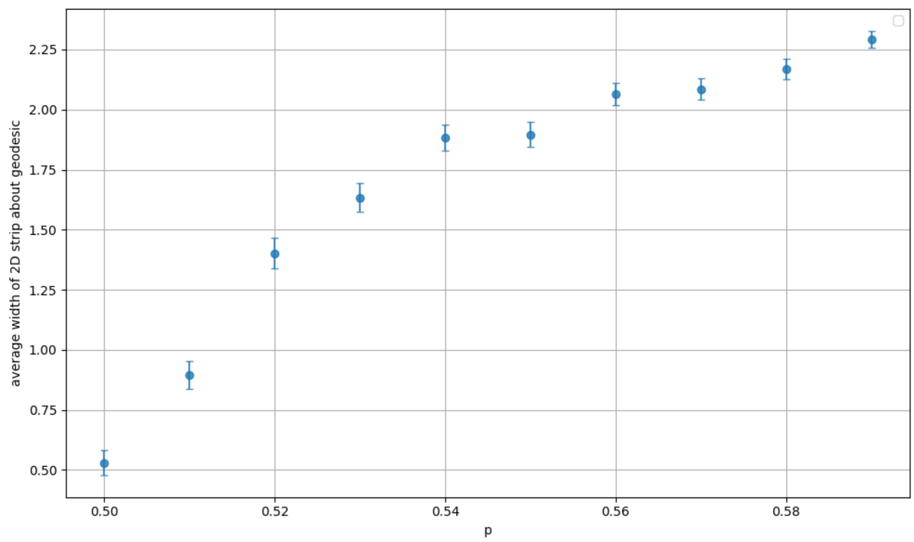

Figure 6: Average Width of 2D Strip with $w = 1$ for 100×100 lattice for $p \in [0.50, 0.60)$ with step size 0.01 and endpoints $(24, 24)$ and $(74, 74)$

We observe that, near the percolation threshold, the average width of the 2D strip is approximately 0.53, far from our expectation of 3. We observe this width monotonically increasing as p increases. To identify a 'rough isometry' between a the tortuous 2D strip and the rectangular 2D strip, we would hope for near-integer width averages with small standard deviation. The failure of this isometry in our results makes this correspondence less convincing.

Network conductance represented by Fig. 1b (disordered strip) displays critical behavior near threshold
Now consider forming a strip by collecting only those resistors that are already present in the disordered lattice. We experimented using our 100×100 model with endpoints $(24, 24)$ and $(74, 74)$ and $p \in [0.51, 0.60)$ with step size 0.01, and observed critical behavior in the conductance across the disordered strip $\Sigma_s(p)$.

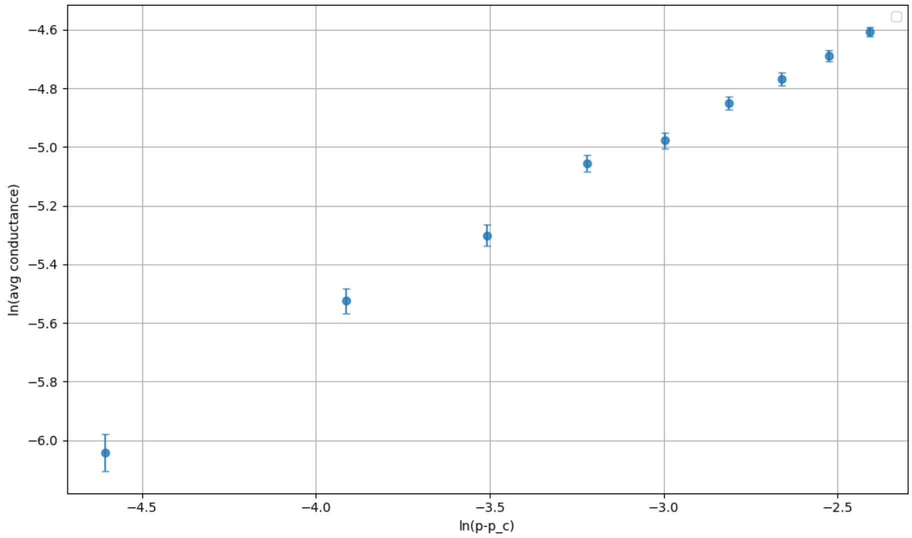

Figure 7: $\ln\langle\Sigma_s(p)\rangle$ for a disordered 2D Strip with $w = 1$ for 100×100 lattice vs. $\ln(p - p_c)$ for $p \in [0.51, 0.60)$ with step size 0.01 and endpoints $(24, 24)$ and $(74, 74)$

When fitting this log-log plot with a linear model, we get the slope $s_{\text{strip}} \approx 0.641$ and constant $c_{\text{strip}} \approx -3.05$, with $r^2 = 0.996$, so that

$$\ln\langle\Sigma_{\text{strip}}(p)\rangle \approx 0.641 \ln(p - p_c) - 3.05$$

Compare this with the critical behavior in the whole network:

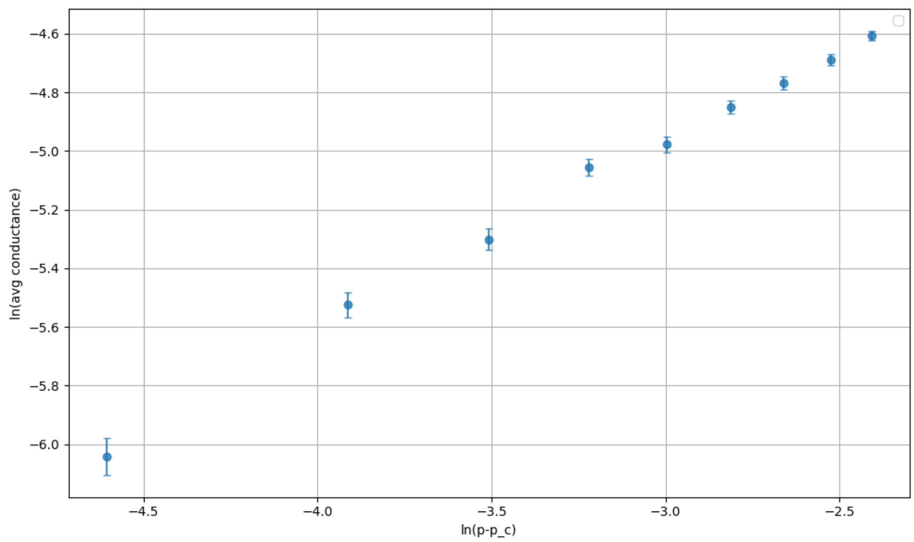

Figure 8: $\ln\langle\Sigma_{exact}(p)\rangle$ for disordered network for 100×100 lattice vs. $\ln(p - p_c)$ for $p \in [0.51, 0.60)$ with step size 0.01 and endpoints $(24, 24)$ and $(74, 74)$

where a log-log plot yields slope $s_{\text{exact}} \approx 1.24$ and constant $c_{\text{exact}} = 0.244$, with $r^2 = 0.996$, so that

$$\ln\langle\Sigma_{\text{exact}}(p)\rangle \approx 1.24 \ln(p - p_c) + 0.244$$

.
Now suppose we consider the resistance between each node in the disordered strip R and want to increase R for a fixed p so that $\Sigma_{\text{strip}}(R) = \Sigma_{\text{exact}}$. If we change resistances in the strip from 1 to R, the Laplacian will map from $L \mapsto \frac{1}{R}L$ and the Green's function $\mathcal{G}$ will scale so that $\mathcal{G} \mapsto R\mathcal{G}$. Thus the conductance

$$\Sigma_{\text{strip}}(R = 1) \mapsto \frac{\Sigma_{\text{strip}}(R = 1)}{R}$$

Since we want to choose R so that the conductance across the strip captures the conductance across the network,

$$\Sigma_{\text{strip}} = \Sigma_{\text{exact}}$$
$$\Rightarrow \frac{\Sigma_{\text{strip}}(R = 1)}{R} = \Sigma_{\text{exact}}$$
$$\Rightarrow R = \frac{\Sigma_{\text{strip}}(R = 1)}{\Sigma_{\text{exact}}}$$

Using our model of critical behavior, this gives us a natural way to measure R :

$$\ln R = \ln \frac{\Sigma_{\text{strip}}(R = 1)}{\Sigma_{\text{exact}}} = \ln \Sigma_{\text{strip}}(R = 1) - \ln \Sigma_{\text{exact}} = c_{\text{strip}} + s_{\text{strip}} \ln(p - p_c) - c_{\text{exact}} - s_{\text{exact}} \ln(p - p_c)$$

Assuming $c_{\text{strip}}, c_{\text{exact}} \sim O(\frac{1}{n})$, at the $n \to \infty$ limit we may observe

$$\ln R \approx (s_{\text{strip}} - s_{\text{exact}}) \ln(p - p_c)$$

And so the natural next step would be to understand the behavior of s_{strip} as a function of p.

Appendix: Analytical Solution to Geodesic and Fully-Connected Strip Network Conductances We build up the machinery by first considering the conductance across the 1D geodesic. A 1D lattice (imagined as a sequence of nodes interpolating a straight line) has Laplacian matrix.

$$L = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

Since the Laplacian matrix is symmetric, by the spectral theorem, L is diagonalizable in an orthonormal basis with distinct eigenvectors and real eigenvalues.

$$L\vec{x}(\alpha) = \lambda_\alpha \vec{x}(\alpha), \alpha = 0, 1, ..., N-1$$

Since the lattice conductance model discretizes a continuous diffusion operator, it inherits the structure that allows diagonalization by Fourier modes. Therefore, we proceed with the ansatz that our eigenvectors (or eigenfunctions) can be written as

$$x_j(\alpha) = c\cos(aj + b)$$

Substituting this into $-x_{j-1}(\alpha) + 2x_j(\alpha) - x_{j+1}(\alpha)$, we recover the form of the eigenvalues

$$\lambda_\alpha = 2(1 - \cos a)$$

The boundary condition

$$x_1(\alpha) - x_2(\alpha) = \lambda_\alpha x_1(\alpha)$$

$$\Rightarrow \cos(a + b) - \cos(2a + b) = 2(1 - \cos a)\cos(a + b)$$

yields the relationship $b = -\frac{a}{2}$. The other boundary condition

$$-x_{N-1}(\alpha) + x_N(\alpha) = \lambda_\alpha x_N(\alpha)$$

$$\Rightarrow -\cos(a(N-1) - \frac{a}{2}) + \cos(aN - \frac{a}{2}) = 2(1 - \cos a)\cos(aN - \frac{a}{2})$$

yields the solution

$$a = \frac{\alpha\pi}{N}, b = -\frac{\alpha\pi}{2N}$$

The normalization constant $c = c(\alpha)$ can be derived by normalizing the eigenvectors, which yields the result

$$c(\alpha) = \begin{cases} \frac{1}{\sqrt{N}}, & \alpha = 0 \\ \frac{2}{\sqrt{N}}, & \text{else} \end{cases}$$

This provides a closed-form analytical solution to computing global conductance on a 1D lattice, with eigenvectors

$$\vec{x}(\alpha) = \begin{bmatrix} x_1(\alpha) \\ x_2(\alpha) \\ \vdots \\ x_N(\alpha) \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^\top, & \alpha = 0 \\ \begin{bmatrix} \frac{2}{\sqrt{N}}\cos(\frac{\alpha\pi}{N} - \frac{\alpha\pi}{2N}) \\ \frac{2}{\sqrt{N}}\cos(\frac{2\alpha\pi}{N} - \frac{\alpha\pi}{2N}) \\ \vdots \\ \frac{2}{\sqrt{N}}\cos(\frac{\alpha\pi(N-1)}{N} - \frac{\alpha\pi}{2N}) \end{bmatrix}, & \text{else} \end{cases}$$

and eigenvalues

$$\lambda_\alpha = 2(1 - \cos(\frac{\alpha\pi}{N}))$$

To extend this to a 2D lattice, and even to lattices of arbitrary dimension, we use the Kronecker sum. Let L_N be the Laplacian in the x direction with length N, and L_M be the Laplacian in the y direction with length M. Define

$$L_{M,N} = L_x \oplus L_y = L_x \otimes I_w + I_n \otimes L_y$$

to be the 2D Laplacian of the fully connected $M \times N$ strip. The eigenvalues of this Kronecker product are simply the sum of the eigenvalues and the eigenvectors of the Kronecker sum are the tensor product of eigenvectors of of L_x and L_y. To illustrate, consider $N = 3, M = 2$. We have

$$L_N = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and

$$L_M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

L_N has orthonormal eigenvectors $\vec{x}(\alpha)$ and eigenvalues λ_α for $\alpha = 0, 1, 2$:

$$\vec{x}(0) = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \lambda_0 = 0; \vec{x}(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \lambda_1 = 1, \vec{x}(2) = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \lambda_2 = 3$$

and L_M has orthonormal eigenvectors $\vec{y}(\beta)$ and eigenvalues μ_β for $\beta = 0, 1$:

$$\vec{y}(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mu_0 = 0; \vec{y}(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mu_1 = 2$$

In the Kronecker sum $L_N \oplus L_M \in \mathcal{M}(\mathbb{R}^6)$ the six eigenvectors $\vec{z}(\alpha, \beta), \alpha = 0, 1, 2; \beta = 0, 1$ form an orthonormal basis of six eigenvectors

$$\vec{z}(0,0);\ \vec{z}(0,1);\ \vec{z}(1,0);\ \vec{z}(1,0);\ \vec{z}(2,0);\ \vec{z}(2,1)$$

with the corresponding six eigenvalues

$$\gamma_{0,0} = \lambda_0 + \mu_0;\ \gamma_{0,1} = \lambda_0 + \mu_1,\ \cdots$$

At the long strip limit, some trigonometric and asymptotic manipulation yields the ratio of differences in conductivities as

$$\eta(d) = \frac{\Delta\Sigma_{\text{strip}}}{\Delta\Sigma_{\text{network}}} = \frac{\Sigma_{\text{strip}} - \Sigma_{\text{geodesic}}}{\Sigma_{\text{network}} - \Sigma_{\text{geodesic}}} = \frac{d}{d_{\max}} + O(n^2)$$