---

Answer any FIVE questions.

---

1.  (a) Check whether Rolle's Theorem is applicable to the function

    $f(x) = \sin |x|$    for     $x \in [-1, 1]$. [1]

    (b) Show that the equation $x^3 + x - 1 = 0$ has exactly one real root on $\mathbb{R}$. [2]

    (c) Prove that between two real roots of the equation $e^x \cos x - 1 = 0$, there is at least one real root of the equation $e^x \sin x - 1 = 0$. [3]

2.  (a) A function $f$ is thrice differentiable on $[a, b]$ and $f(a) = f(b) = 0$, $f'(a) = f'(b) = 0$. Prove that $f'''(c) = 0$ for some $c \in (a, b)$. [3]

    (b) State Lagrange's Mean Value Theorem. [1]

    (c) Prove that

    $$\frac{b - a}{\sqrt{1 - a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b - a}{\sqrt{1 - b^2}}$$

    for $0 < a < b < 1$. [2]

3. If $y(x) = e^{a \sin^{-1} x}$, then prove that

    $$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + a^2)y_n = 0.$$

    Hence show that

    $$y_n(0) = \begin{cases} a(1^2 + a^2) \cdots ((n - 2)^2 + a^2), & \text{if } n \text{ is odd} \\ a^2(2^2 + a^2) \cdots ((n - 2)^2 + a^2), & \text{if } n \text{ is even.} \end{cases}$$ [3 + 3]

4.  (a) Let $f$ be a real valued function on $\mathbb{R}$ such that $f'$ is continuous. Then show that

    $$\lim_{h \to 0} \frac{f(c + h) - f(c - h)}{2h} = f'(c)$$

    for any point $c \in \mathbb{R}$. [3]

    (b) Use Taylor's theorem to show that

    $$1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1 + x} < 1 + \frac{x}{2}$$

    for $x > 0$. [3]

5.  (a) Show that the function $f(x) = \frac{\sin x}{x}$, $x \in (0, \frac{\pi}{2})$ is a strictly decreasing function on $(0, \frac{\pi}{2})$. [3]

(b) By Cauchy's Mean Value Theorem, show that

$$\frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta$$

where $0 < \alpha < \theta < \beta < \frac{\pi}{2}$. [2]

(c) Prove or disprove: If $\sum u_n^2$ is a convergent series, then $\sum u_n$ is also a convergent series. [1]

6. (a) Find a particular integral of $(D^2 - 2D + 4)y = e^x \cos x$. [2]

(b) Find the general solution of the following differential equation: [4]

$$\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$$

7. (a) Discuss the convergence of the following series: [1+2]

i. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{n!\, 2^n}{n^n}$

(b) Show that the series

$$\sum_{n=1}^{\infty} \frac{3.6.9 \ \dots \ 3n}{7.10.13 \ \dots \ (3n+4)} x^n, \quad x > 0$$

converges for $x \leq 1$ and diverges for $x > 1$. [3]