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INDIAN INSTITUTE OF ENGINEERING SCIENCE AND TECHNOLOGY, SHIBPUR

FIVE YEAR INTEGRATED DUAL DEGREE (B. Tech. – M. Tech.) FIRST SEMESTER EXAMINATION, NOV. 2016

Sub. : Mathematics-I (MA-101) (For all Engineering Branches)

Full Marks : 70 Time: 3 hours

(Use separate answer-script for each half)

FIRST HALF

Answer question no. 1 and any TWO questions from the rest in this half.

1. **Answer any three questions.** **(3x5=15)**

(a) (i) Find the n-th partial sum of the infinite series

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\ldots\ldots\ldots\text{to }\infty.$$

Hence find the sum of the series, if it is convergent.

(ii) If the infinite series $\sum_1^\infty u_n$ is convergent, then prove that $\lim_{n\to\infty} u_n = 0$. Is the converse true ? Give an example in support of your answer.

(3+2)

(b) If P(x) is a polynomial and c is a real constant, apply Rolle's theorem to prove that there is a root of the equation $\frac{d}{dx}$ P(x) + c P(x) = 0 between any pair of roots of P(x) = 0. (5)

(c) If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$. (5)

(d) Apply $\epsilon - \delta$ definition to verify that

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{1+x^2+y^2} = 0.$$ (5)

(e) (i) Determine the region, if any, in which the following function is defined:

$$F(x,y) = \frac{\sqrt{1-x^2-y^2}}{x^2+y^2}.$$

(ii) Show that u = f(x-ct) + g(x-ct), where f, g are arbitrary functions, is a solution of the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$ (2+3)

2. (a) Show that

$$\frac{d^n}{dx^n}\left(\frac{1}{x^2+a^2}\right) = \frac{(-1)^n\, n!}{a^{n+2}}\, sin^{n+1}\left\{tan^{-1}\frac{a}{x}\right\} \sin\left\{(n+1)\, tan^{-1}\frac{a}{x}\right\}.$$

Hence find an expression for the nth derivative of $tan^{-1}x$. (4+2)

(b) Show that the double limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

does not exist. (4)

3. (a) State the Lagrange's Mean Value theorem with its geometrical interpretation. (2+1)

(b) Prove that under certain conditions (to be stated by you) imposed on the function f(x), it can be expanded in power series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where $a_n = f^{(n)}(0)/n!$, n = 0,1,2,............ (3)

(c) Show that f(x) = log (1 + x) can be expanded in powers of x in infinite series in the interval $-1 < x \le 1$ and

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots\ldots \text{ to } \infty, \quad -1 < x \le 1.$$ (4)

4. (a) Find the interval of convergence of the power series $\sum_1^{\infty} \frac{(x-3)^n}{n\, 3^n}$. (3)

(b) Find the asymptotes of the curve

$$y^3 - 6xy^2 + 11x^2y - 6x^3 + y^2 - x^2 + 2x - 3y - 1 = 0.$$ (3)

(c) If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$. (4)

5. (a) Explain the concept of differentiability of a function z = f(x, y). Show that

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Is not differentiable at (0, 0) though both the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (0, 0).

(1+2+2)

(b) If u = f(x, y) is a homogeneous function of x and y of degree n having continuous partial derivatives, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\, f(x, y). \qquad (2)$$

(c) Determine all stationary points , extreme points and saddle points (if any) of the function

$$f(x, y) = x^3 + y^3 + 3xy. \qquad (3)$$

Second Half
Answer any three questions
Two marks are reserved for general proficiency

6. a) Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\, dydx$ and hence evaluate the same.

b) Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

[5+6]

7. a) Prove that $\Gamma(n+1) = n\,\Gamma(n)$, where $n > 0$.

Hence show that $\Gamma(n+1) = n!$, when n is a positive integer.

b) Prove that $\int_0^\infty e^{-x^4} x^2\, dx \times \int_0^\infty e^{-x^4}\, dx = \frac{\pi}{8\sqrt{2}}$. [5+6]

8. Solve the following differential equations:

a) $\frac{d^2y}{dx^2} - y = \cosh x$.

b) $(x^3 D^3 - x^2 D^2 + 2x D - 2)\, y = x^2$, where $D \equiv \frac{d}{dx}$. [5+6]

9. a) Apply the method of variation of parameters to solve the following differential equation

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x.$$

b) Show that $x\,J_n'(x) = x\,J_{n-1}(x) - n\,J_n(x)$, where $J_n(x)$ is the Bessel function of first kind of order n. [7+4]

10. a) Solve the differential equation

$$\frac{d^2y}{dx^2} + (x-1)^2 \frac{dy}{dx} - 4(x-1)y = 0$$

in series about the ordinary point x = 1.

b) Discuss the convergence of the integral $\int_0^\pi \frac{\sqrt{x}}{sinx}\, dx$. [7+4]