Name of the Examination : First Semester End Semester Examination
Name of the Subject : Mathematics
Subject Code : MA1101
Date of Examination : 30th March, 2021
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Examination Roll Number : 2020ITB065
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Ans 1a) Let $f(x) = \sin x$

Therefore, $f(0) = 0$

$f'(x) = \cos x \quad ; \quad f'(0) = 1$

$f''(x) = -\sin x \quad ; \quad f''(0) = 0$

$f'''(x) = -\cos x \quad ; \quad f'''(\theta x) = -\cos\theta x$

Now, the Maclaurin's series for the function $f(x)$ with Lagrange's form of remainder after after three term is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x); \quad 0<\theta<1$$

$$\therefore \sin x = x - \frac{x^3}{3!}\cos\theta x = x - \frac{x^3}{6}\cos\theta x \quad \text{-----} (1)$$

Now, $0 < x < \frac{\pi}{2}$ and $0 < \theta < 1$

$$\Rightarrow 0 < \theta x < \frac{\pi}{2}$$

$$\Rightarrow 0 < \cos\theta x < 1$$

$$\therefore \frac{x^3}{6}\cos\theta x < \frac{x^3}{6}$$

$$\Rightarrow -\frac{x^3}{6}\cos\theta x > -\frac{x^3}{6}$$

$$\Rightarrow x - \frac{x^3}{6}\cos\theta x > x - \frac{x^3}{6}$$

$$\therefore \sin x > x - \frac{1}{6}x^3 \quad \ldots\ldots [\text{by } (1)]$$

if $0 < x < \frac{\pi}{2}$ (Hence proved)

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Ans 1b) Given , $x = a(\theta + \sin\theta)$ $\qquad y = a(1-\cos\theta)$

[Differentiating both sides w.r.t $\theta$]

$\Rightarrow x' = a(1+\cos\theta)$

[Differentiating both sides w.r.t $\theta$]

$x'' = -a\sin\theta$

[Differentiating both sides w.r.t $\theta$]

$y' = a\sin\theta$

[Differentiating both sides w.r.t $\theta$]

$y'' = a\cos\theta$

$\therefore$ Radius of curvature

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} \quad \text{(Parametric form)}$$

$$= \frac{\left[a^2(1+\cos\theta)^2 + a^2\sin 2\theta\right]^{3/2}}{a^2(\cos\theta + \cos^2\theta) + a^2\sin^2\theta}$$

$$= \frac{(2a^2 + 2a^2\cos\theta)^{3/2}}{a^2 + a^2\cos\theta}$$

$$= 2\sqrt{2}\,(a^2 + a^2\cos\theta)^{1/2}$$

$$= 2\sqrt{2}a\,(2\cos^2\theta/_2)^{1/2}$$

$$= 4a\cos\frac{\theta}{2} \quad \text{(Ans)}$$

Ans 1c) Equating the coefficient of highest power of $x$ i.e; $x^2$ equal to zero, horizontal asymptotes are given by

$y^2 - y = 0$ $\Rightarrow y(y-1) = 0$ i.e; $y = 0$, $y = 1$

Equating coefficient of highest power of $y$, i.e; $y^2$ equal to zero, vertical asymptotes are given by

$x^2 - x = 0$ $\Rightarrow x(x-1) = 0$ i.e; $x = 0$, $x = 1$

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Given equation is 4th degree equation in x and y and hence curve cannot have more than 4 asymptotes.

$\therefore$ All asymptotes are $y=0, y=1, x=0, x=1$.

Ans 2a) ii) $u_n = \dfrac{n^p}{n!}$ , $u_{n+1} = \dfrac{(n+1)^p}{(n+1)!}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p} = \frac{1}{(n+1)}\left(1+\frac{1}{n}\right)^p$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left\{\frac{1}{(n+1)}\cdot\left(1+\frac{1}{n}\right)^p\right\}$$

$$= 0 < 1$$

$\therefore$ The series is convergent by D'Alembert's Test.

Ans 2a) iii) $a_n = \dfrac{n^3+a}{2^n+a}$ ; $b_n = \dfrac{n^3}{2^n}$

$$\frac{a_n}{b_n} = \frac{n^3+a}{2^n+a} \times \frac{2^n}{n^3}$$

$$= \left(1+\frac{a}{n^3}\right)\left(\frac{2^n}{2^n+a}\right)$$

$$= \left(1+\frac{a}{n^3}\right)\left(\frac{1}{1+\frac{a}{2^n}}\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

$\therefore$ By comparison test the given series is convergent or divergent according as $\sum \frac{n^3}{2^n}$ is convergent or divergent

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respectively. By Cauchy's Root Test,

$$\lim_{n\to\infty} (b_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{3/n}}{2} = \frac{1}{2} < 1$$

$\sum_{n=0}^{\infty} b_n$ is convergent $\longrightarrow \sum_{n=0}^{\infty} a_n$ is also convergent.

Ans 2 b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}(x-3)^n$

$$u_{n+1} = \frac{2^{n+1}}{(n+1)^2}(x-3)^{n+1}$$

$$u_n = \frac{2^n}{n^2}(x-3)^n$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} |x-3|\, 2 .\frac{n^2+2n+1}{n^2} \quad \text{[By D'Alembert's Test]}$$

$$= 2|x-3| \lim_{n\to\infty} \frac{n^2+2n+1}{n^2}$$

$$= 2|x-3|$$

So, the series converges,

for $|x-3| < \frac{1}{2}$ i.e; $\frac{5}{2} < x < \frac{7}{2}$

at $x = \frac{5}{2}$ $\quad \sum_{n=1}^{\infty} \frac{2^n}{n^2}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ (convergent)

So, the original series is convergent

when $x = \frac{7}{2}$, $\sum_{n=1}^{\infty} \frac{2^n}{n^2}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (convergent)

Hence, the interval of convergence is $[5/2, 7/2]$

radius of curvature, $r = \left|\frac{a_{n+1}}{a_n}\right|$

$$= 2 .\frac{n^2+2n+1}{n^2} = 2 \text{ (Ans)}$$

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Ans 3a) $u = f(x, y)$ ; $x = r\cos\theta$ $y = r\sin\theta$ (given)

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y} \times \sin\theta \quad \text{---} (1)$$

$$\boxed{\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}}$$

Also $$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \times \frac{\partial u}{\partial \theta}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = -r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}}$$

R.H.S

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y}\right)^2 + \frac{1}{r^2}\left(-r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}\right)^2$$

$$= (\cos^2\theta + \sin^2\theta)\left(\frac{\partial u}{\partial x}\right)^2 + (\sin^2\theta + \cos^2\theta)\left(\frac{\partial u}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \text{L.H.S} \quad \text{(Hence proved)}$$

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Ans 3b) $V = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Let $u = \cos V$

$\Rightarrow u = \sqrt{x}\left(\frac{1+\frac{y}{x}}{1+\sqrt{\frac{y}{x}}}\right)$

$\Rightarrow u = \sqrt{x}\,\phi\left(\frac{y}{x}\right)$ ---- Degree $n = \frac{1}{2}$ ; homogenous equation

Using Euler's Theorem,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}u \quad [\because n = \frac{1}{2}]$

$\Rightarrow x\frac{\partial}{\partial x}(\cos V) + y\frac{\partial}{\partial y}(\cos V) = \frac{1}{2}\cos V \quad [\because u = \cos V]$

$\Rightarrow -x\sin V\frac{\partial v}{\partial x} + -y\sin V\frac{\partial v}{\partial y} = \frac{1}{2}(\cos V)$

$\Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + \frac{1}{2}\cot V = 0$

Ans 4a) $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & ; x^2+y^2 \neq 0 \\ 0 & ; x^2+y^2 = 0 \end{cases}$ (given)

$f_{xy}(0,0) = \lim_{h\to 0}\frac{f_y(0+h,0) - f_y(0,0)}{h} \quad \left[\frac{\partial}{\partial x}f_y\Big|_{\substack{\text{at } x=0\\ y=0}}\right]$

$f_{xy}(0,0) = \lim_{h\to 0}\frac{f_y(h,0) - f_y(0,0)}{h}$ —— (1)

Now, $f_y(h,0) = \frac{\partial f}{\partial y}\Big|_{\text{at } x=h, y=0}$

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$$= \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{h\cancel{k}\,\frac{h^2-k^2}{h^2+k^2} - 0}{\cancel{k}}$$

$$= \lim_{k\to 0} h\left(\frac{h^2-k^2}{h^2+k^2}\right) = \frac{h^3}{h} = h \quad \text{—— (2)}$$

$$f_y(0,0) = \left.\frac{\partial f}{\partial y}\right|_{\text{at } x=0,\, y=0}$$

$$= \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0 \quad \text{--- (3)}$$

From (1),

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h-0}{h} \quad [\text{By (2) \& (3)}]$$

$$= 1$$

Again; $f_{yx}(0,0) = \left.\frac{\partial}{\partial y} f_x\right|_{\text{at } x=0,\, y=0}$

$$= \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \quad \text{—— (4)}$$

Now, $f_x(0,k) = \left.\frac{\partial f}{\partial x}\right|_{\text{at } x=0,\, y=k}$

$$= \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}k\,\frac{h^2-k^2}{h^2+k^2} - 0}{\cancel{h}}$$

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$$= \underset{h\to 0}{Lt}\ k\,\frac{h^2-k^2}{h^2+k^2} = \underset{h\to 0}{Lt}\ \frac{h^2k-k^3}{h^2+k^2} = \frac{-k^3}{k^2} = -k \quad \text{---} (5)$$

$$f_x(0,0) = \left.\frac{\partial f}{\partial x}\right|_{\text{at } x=0,\, y=0} = \underset{h\to 0}{Lt}\ \frac{f(h,0)-f(0,0)}{h}$$

$$= 0 \quad \text{----} (6)$$

Using (5) & (6) in (4), we get

$$f_{yx}(0,0) = \underset{k\to 0}{Lt}\ \frac{f_x(0,k)-f_x(0,0)}{k}$$

$$= \underset{k\to 0}{Lt}\ \frac{-k-0}{k} = -1$$

$\therefore f_{xy}(0,0) \neq f_{yx}(0,0)$ (Hence proved)

Ans 4b) $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{4x^4+25}$ $\qquad f(x) = \dfrac{x^2}{4x^4+25}$

Let $\mu = 2$

$$\underset{x\to \infty}{Lt}\ x^{\mu}\,\frac{x^2}{4x^4+25} = \underset{x\to\infty}{Lt}\ \frac{x^4}{4x^4+25} = \underset{x\to\infty}{Lt}\ \frac{1}{4+\frac{25}{x^4}}$$

$$= \frac{1}{4} \text{ (finite)}$$

here $\mu = 2 > 1$

$\therefore$ By $\mu$-test the integral is convergent.

Ans 4c) $\Gamma(n+1)$

$$= \int_0^{\infty} e^{-x}\, x^{n+1-1}\,dx = \int_0^{\infty} e^{-x} x^n\,dx$$

$$= \lim_{b\to\infty} \int_0^{b} e^{-x} x^n\,dx$$

$$= \lim_{b\to\infty} \left[ \left[\frac{e^{-x}}{-1}\,x^n\right]_0^b - \int_0^b \frac{e^{-x}}{-1}\cdot n x^{n-1}\,dx \right]$$

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$$= \lim_{b\to\infty}\left[e^{-b}b^{n} + n\int_0^b e^{-x}x^{n-1}dx\right]$$

$$= \lim_{b\to\infty}\frac{b^n}{e^b} + n\lim_{b\to\infty}\int_0^b e^{-x}x^{n-1}dx$$

$$= 0 + n\int_0^\infty e^{-x}x^{n-1}dx$$

$$= n\Gamma(n) \quad [\text{proved}] \quad \left[\because \int_0^\infty e^{-x}x^{n-1}dx = \Gamma(n)\right]$$

Ans 5a) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \dfrac{e^x}{1+e^x}$

$(D^2 - 3D + 2)y = \dfrac{e^x}{1+e^x}$

where $R = \dfrac{e^x}{1+e^x}$

Auxillary Equation is $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$ so

Complementary Function $= c_1 e^x + c_2 e^{2x}$

Hence parts of C.F. are $u = e^x$, $v = e^{2x}$

$$\Rightarrow W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = e^{3x}$$

$$A = -\int \frac{Rv}{W}dx = -\int \frac{e^x \cdot e^{2x}}{(1+e^x)e^{3x}}dx = -\int \frac{e^{-x}}{e^{-x}+1}dx = \log(e^{-x}+1)$$

$$B = \int \frac{Ru}{W}dx = \int\left(\frac{e^x}{1+e^x}\cdot\frac{e^x}{e^{3x}}\right)dx = \int \frac{1}{e^x(e^x+1)}dx$$

$$= \int\left(\frac{1}{e^x} - \frac{1}{e^x+1}\right)dx$$

$$= \int\left(e^{-x} - \frac{e^{-x}}{e^{-x}+1}\right)dx = -e^{-x} + \log(e^{-x}+1)$$

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So, by variation of parameters, general solution is

$$y = CF + PI\ (c_1 u + c_2 v) + (Au + Bv)$$

$$= c_1 e^{x} + c_2 e^{2x} + (\log(e^{-x}+1)).e^{x} + \{-e^{-x} + \log(e^{-x}+1)\}.e^{2x}$$

Ans 5b) $P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n}(x^2-1)^n$

Here let $v = (x^2-1)^n \Rightarrow v_1 = \frac{dv}{dx} = 2nx(x^2-1)^{n-1}$

i.e; $(1-x^2)v_1 + 2nxv = 0$ ---- (1)

Differentiating $(n+1)$ times using Leibnitz rule,

$$(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{1}{2!}(n+1)n(-2)v_n + 2n[xv_{n+1} + (n+1)v_n] = 0$$

$$\Rightarrow (1-x^2)\frac{d^2(v_n)}{dx^2} - 2x\frac{d(v_n)}{dx} + n(n+1)v_n = 0$$

It is the Legendre equation and $cv_n$ is its solution. Also its finite series solution is $P_n(x)$.

$$P_n(x) = cv_n = c\frac{d^n}{dx^n}(x^2-1)^n \quad \text{---- (2)}$$

For finding the constant $c$, putting $x=1$,

then $1 = c\left[\frac{d^n}{dx^n}\{(x-1)^n(x+1)^n\}\right]_{x=1}$

$$= c\,[n!\,(x+1)^n]$$

Also, it contains the terms of $(x-1)$ and its powers $|_{x=1}$

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$= c \cdot n! \, 2^n$ ; i.e; $c = \frac{1}{n!\,2^n}$

Substituting this value of $c$ in (2), we get (1) which proves that $P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2-1)^n$ ..

(hence proved)

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