Indian Institute of Engineering Science and Technology, Shibpur
B.Tech. 2$^{\text{nd}}$ Semester Final Examination, July 2021
Subject: Mathematics - II (MA 1201)

Time : 1 hour 30 minutes **Full Marks : 50**

Answer any FIVE questions.
Only the first five questions answered by the candidate will be evaluated.

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1. (a) Let $\{\alpha, \beta, \gamma\}$ be a basis of a real vector space V and c be a non-zero real number. Determine all admissble values of c, for which the set $\{\alpha + c\beta, \beta + c\gamma, \gamma + c\alpha\}$ forms a basis of V. [4]

   (b) Classify the nature of singularity at each of the given points of the following functions: [3+3=6]

   i. $f(z) = (z - \pi) \cot z$ at $z = \pi$

   ii. $f(z) = (z + 2) \sin(\frac{1}{z+1})$ at $z = -1$.

2. (a) Diagonalize the matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and hence find A^{50}. [6]

   (b) Determine the linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ which maps the basis vectors $\{(1, 1), (0, 1)\}$ of $\mathbb{R}^2$ to $\{(2, 1), (1, 2)\}$ of $\mathbb{R}^2$ respectively. [4]

3. Obtain the Fourier series expansion of

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ 0, & 0 \leq x < \pi \ . \end{cases}$$

   Hence prove the following: [5+2+3=10]

$$\text{(i)} \qquad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

$$\text{(ii)} \qquad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \cdots = \frac{\pi^2}{8}$$

4. (a) State Cauchy's Residue theorem. Using residue theorem, evaluate [1+5=6]

$$\int_c \frac{e^z}{z(z-1)^2} \, dz, \quad \text{where } C \text{ is the circle } |z| = 2.$$

   (b) Find the radius of convergence and the circle of convergence of the following power series, [3+1=4]

$$\sum_{n=0}^{\infty} \left(\frac{iz - 2}{5 + i} \right)^n.$$

5. (a) Obtain the Laurent's series which represent the function

$$f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)}$$

in the region $1 < |z| < 4$. [6]

(b) Let M_5 be the vector space of all 5×5 real matrices. Find the dimension of the following subspace [4]

$$W = \{A \in M_5 : A \text{ is upper triangular and } a_{11} = 0 = a_{44}\}.$$

6. (a) Let $\vec{\mathbf{r}} = x\,\vec{\mathbf{i}} + y\,\vec{\mathbf{j}} + z\,\vec{\mathbf{k}}$ and $r = |\vec{\mathbf{r}}|$. Prove that the vector $\vec{\mathbf{F}}$ is solenoidal, if $n + 3 = 0$, where $\vec{\mathbf{F}} = r^n\ \vec{\mathbf{r}}$. [5]

(b) Prove the following by using divergence theorem :

$$\iint_S (x^2\ dy\ dz + y^2\ dz\ dx + z^2\ dx\ dy) = 3,$$

where S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. [5]

7. (a) Show that if $f(z)$ is analytic and $Re[f(z)]$ is constant, then $f(z)$ is constant. [3]

(b) Verify Stokes' theorem for $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where, [7]

$$\vec{\mathbf{F}} = (2x - y)\,\vec{\mathbf{i}} - yz^2\,\vec{\mathbf{j}} - y^2 z\,\vec{\mathbf{k}},$$

over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.