Recitation 12

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Brief Overview

- Recall Euler's method & Trapezoid method
- Recall Runge-Kutta method
- Linear multi-step method
- Exercise problems from Ch. 12
- Zero-stability and consistency
- Quiz

Exercise problems

- 12.1 (a), (b)
  - (a): $|f(x,u)-f(x,v)| = 2x^{-4}|u-v| \le 2|u-v|,\ x \in [1,\infty) \Rightarrow L = 2$
  - (b): Use MVT, $|f(x,u)-f(x,v)| = |\frac{\partial f}{\partial y}(x,\eta)|.|u-v|$ where $\eta \in [u,v]$.
    $\frac{\partial f}{\partial y}(x,\eta) = e^{-x^2}\frac{1}{1+\eta^2} \le \frac{1}{e}\ \ \forall\ x \in [1,\infty) \Rightarrow |f(x,u)-f(x,v)| \le \frac{1}{e}|u-v| \Rightarrow L = \frac{1}{e}$
- 12.5
  - $y_{n+1} = y_n + h(x_n e^{-5x_n} - 5y_n),\ y_0 = 0.$
  - This is similar to Q1 & 2 in HW.
- 12.6
  - (a): $T_n = \frac{y(x_{n+1})-y(x_n)}{h} - f(x_n, y(x_n)) = \frac{1}{2}hy^{//}(\eta_n) \Rightarrow |T_n| \le T = \frac{h}{4}$
  - (b): $y_{n+1} = y_n + hf(x_n,y_n),\ y(x_{n+1}) = y(x_n) + hf(x_n,y(x_n)) + hT_n$
    $\Rightarrow |e_{n+1}| \le (1+hL)|e_n| + h|T_n|$. Use MVT to get L.
  - (c): Express $|e_n|$ in terms of T and L using result from (b).
- 12.9
  - Use definition of fourth-order Runge-Kutta, and set $f = \lambda y$.
- 12.14(e)
  - Use Root Condition. See sec. 12.6 and ref. [10-12].
  - $y_{n+1} - y_n = \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1})$
  - Charac. Polynomial is: $z - 1 \Rightarrow root\ is\ z = 1 \Rightarrow$ since root lies on unit circle in complex plane and is simple (multiplicity = 1), it satisfies root condition and is zero-stable.

Zero-Stability, Root Condition and Consistency

Zero-stability

- A multi-step method is zero-stable if it satisfies the root condition.
- The root condition says that "**a linear multistep method is 0-stable for any initial value problem of the form (12.1), (12.2), where f satisfies the hypotheses of Picard's Theorem, if, and only if, all roots of the first characteristic polynomial of the method are inside the closed unit disc in the complex plane, with any which lie on the unit circle being simple.**"
- Eg. Adams-Moulton method given by: $y_{n+3} - y_{n+2} = \frac{h}{24}(9f_{n+3} + 19f_{n+2} - 5f_{n+1} - 9f_n)$
- 1st charac. Polynomial $\equiv z^3 - z^2 = z^2(z-1) = 0$ [We construct the first charac. Polynomial from the coefficients of the y_{n+k} terms, by setting $y_k = z^k$ in LHS, and setting it to zero.]
- Since it has roots $z_{1/2} = 0 \textit{ and } z_3 = 1$, it satisfies the Root Condition and is 0-stable.

Consistency

- See sec. 12.7 and ref. [12-14].
- The truncation error is given by:

$$T_n = \frac{1}{h\sigma(1)}\left[C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \cdots\right]$$

Where

$$\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j \, , \\
C_1 &= \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j \, , \\
C_2 &= \sum_{j=1}^{k} \frac{j^2}{2!}\alpha_j - \sum_{j=1}^{k} j\beta_j \, , \\
&\cdots \\
C_q &= \sum_{j=1}^{k} \frac{j^q}{q!}\alpha_j - \sum_{j=1}^{k} \frac{j^{q-1}}{(q-1)!}\beta_j \, .
\end{aligned}$$

A method is consistent if
$\textit{as } h \to 0 \textit{ and } n \to \infty \textit{ with } x_n \to x \in [x_0, X_M], \textit{ the truncation error } T_n \textit{ tends to } 0$. This requires that $C_1 = C_2 = \ldots = C_k = 0, \; C_{k+1} \neq 0$ for accuracy of order-k.
See [15] for example.

Dahlquist's theorem: $stability + consistency \rightarrow convergence$

References

1. http://www.uio.no/studier/emner/matnat/math/MAT-INF1100/h10/kompendiet/kap13.pdf
2. http://faculty.olin.edu/bstorey/Notes/DiffEq.pdf
3. http://www.damtp.cam.ac.uk/user/examples/3N2a.pdf
4. https://na.uni-tuebingen.de/~lubich/pcam-ode.pdf
5. http://www.cs.elte.hu/~faragois/ODE_angol.pdf
6. https://people.maths.ox.ac.uk/suli/nsodes.pdf
7. http://lpsa.swarthmore.edu/NumInt/NumIntFirst.html
8. http://people.math.sfu.ca/~ralfw/math467w03/matlab/euler_matlab.pdf
9. http://www.mathworks.com/help/symbolic/solve-a-single-differential-equation.html
10. http://web.mit.edu/16.90/BackUp/www/pdfs/Chapter5.pdf
11. https://www.win.tue.nl/casa/meetings/seminar/previous/_abstract081001_files/multistepmethods_LeiLiu.pdf
12. http://minitorn.tlu.ee/~jaagup/uk/dynsys/ds2/num/Zero/Zero.html
13. http://www.math.ubc.ca/~peirce/M405_607E_Lecture%2017.pdf
14. https://www.siam.org/books/ot98/sample/OT98Chapter7.pdf
15. http://www.math.iit.edu/~fass/478578_Chapter_2.pdf