Recitation 10

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Brief Overview

- Rayleigh coefficient
- Perturbation analysis
- Problems from textbook
- Quiz

Rayleigh coefficient

$R(x) = \frac{x^T A x}{x^T x}$ where A is symmetric.

- If x is eigenvector, R(x) is corresponding eigenvalue
- Otherwise, if

$$\boldsymbol{x} = \sum_{j=1}^{n} \alpha_j \, \boldsymbol{x}^{(j)} ,$$

$$R(\boldsymbol{x}) = \frac{\sum_{j=1}^{n} \lambda_j \, \alpha_j^2}{\sum_{j=1}^{n} \alpha_j^2}$$

- Th. 5.12: $\lambda_{min} \leq R(x) \leq \lambda_{max}$
- If we have a fairly close approximation of the eigenvector, R(x) gives us a good approximate of the corresponding eigenvalue!

Perturbation Analysis

The following theorems (see p. 172 of textbook) give an idea of how a matrix's eigenvalues are affected under perturbation.

Theorem 5.14 *Let $M \in \mathbb{R}^{n\times n}_{\text{sym}}$, with eigenvalues λ_i and corresponding orthonormal eigenvectors $\boldsymbol{v}_i, i = 1, 2, \ldots, n$, and suppose that $\boldsymbol{u} \neq \boldsymbol{0}$ and $\boldsymbol{w}$ are vectors in $\mathbb{R}^n$ and μ is a real number such that*

$$(M - \mu I)\boldsymbol{u} = \boldsymbol{w}\,. \tag{5.44}$$

Then, at least one eigenvalue λ_j of M satisfies

$$|\lambda_j - \mu| \leq \|\boldsymbol{w}\|_2 / \|\boldsymbol{u}\|_2\,.$$

Theorem 5.15 (Bauer–Fike Theorem (symmetric case)) *Suppose that $A, E \in \mathbb{R}^{n\times n}_{\text{sym}}$ and $B = A - E$. Assume, further, that the eigenvalues of A are denoted by $\lambda_j, j = 1, 2, \ldots, n$, and μ is an eigenvalue of B. Then, at least one eigenvalue λ_j of A satisfies*

$$|\lambda_j - \mu| \leq \|E\|_2\,.$$

For a general matrix, this result can be extended using the condition number of the diagonalizing/similarity transformation matrix:

$|\lambda_j - \mu| \leq k(X)\|E\|_2$ where $X^{-1}AX = D$, and $k(X) = \|X^{-1}\|_2\|X\|_2$ is the condition number of X.

eg.

A = [1 1/2 1/3;
1/2 1/3 1/4 ;
1/3 1/4 1/5]

B = [1 0.5 0.3333;
0.5 0.3333 0.25;
0.3333 0.25 0.2]

B is a perturbation (approximation) of A.

$|\lambda_j - \mu| \leq \|E\|_2 = \|B - A\|_2 = 3.3 \times 10^{-5}$

In general, symmetric matrices are well-conditioned (robust to perturbation).
[**Exercise problems from textbook**]

See hints below to recall how we solved these problems in class.

- 5.3
  - Evaluate Sturm sequence at x = 0 and x = 1. Compare the sequences when $5\alpha^2 < 8$ and $5\alpha^2 > 8$.
- 5.7
  - Express R in terms of A and Q. Set this expression in $B = RQ + \mu I$ to get $B = Q^T AQ$. B is symmetric as A is symmetric.
  - Note that A is symmetric and **tridiagonal**. B is tridiagonal since $Q = \Pi_{p=1\ to\ n-1} R^{p,p+1}(\varphi)$, and each multiplication by $R^{p,p+1}(\varphi)^T$ on the left sets $A(p+1,p) = 0$.
  - Next step is RQ, which involves taking matrix R and applying same sequence of plane rotations on the right, but with each rotation transposed. Transpose means inverse rotation (as $Q^T = Q^{-1}$), i.e. setting the element which was made 0 in previous step to non-zero.
  - Thus, B has non-zero elements below diagonal only on first sub-diagonal (Hessenberg).
  - Since B is symmetric, it is tridiagonal.
  - Above steps are explained in sec. 5.7.1 in book.
- 5.8
  - $A^{(1)} = [0\ 1;\ 1\ 0].\ G_{(2,1)} = [c,\ s;\ -s,\ c];\ we\ want\ [a,b] = [0,1] \rightarrow [\alpha\ 0].$
  - $Choose\ r = \sqrt{a^2+b^2} = 1,\ c = a/r = 0,\ s = -b/r = -1.\ G_{(2,1)} = [0,-1;\ 1,\ 0].$
  - $Q = G_{(2,1)}{}^T \Rightarrow R = Q^T A = [-1,0;\ 0,1]$
  - $A^{(2)} = RQ = [-1,0;\ 0,1][0,1;-1,\ 0] = [0,1;1,0] = A^{(1)}$
  - Thus, $A^{(k)} = \ldots = A^{(1)}$, and A won't converge to solution. This is because our choice of $\mu = 0$. Try a different choice to see if we converge.
- 5.9
  - Follow same steps as previous question. Check your solution using MATLAB's `qr` function (see sample code from last recitation).
- 5.10
  - Use inverse iteration step rules to evaluate $\upsilon^{(1)}$ and $\upsilon^{(2)}$ at given value of $A, v, \upsilon^{(0)}$. Compare $\upsilon^{(2)}$ with corresponding eig-vector of A - should be within 5% error. Evaluate Rayleigh coefficient at $\upsilon^{(2)}$.