Recitation 8

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Brief Overview

- Gershgorin's disc theorems - **5 min**
- Examples of the disc theorems using MATLAB (see [7], [8]) - **10 min**
- Householder's method - **20 min**
- Quiz - **25 min**

Gershgorin's theorems

Theorem 1:

Every eigenvalue of matrix A_{nn} satisfies:

$$|\lambda - A_{ii}| \leq \sum_{j \neq i} |A_{ij}| \qquad i \in \{1, 2, ..., n\}$$

Every eigenvalue of a matrix A must lie in a Gershgorin disc corresponding to the columns of A.

Theorem 2:

A Subset G of the Gershgorin discs is called a disjoint group of discs if no disc in the group G intersects a disc which is not in G. If a disjoint group G contains r nonconcentric discs, then there are r eigenvalues.

Visualizing Gershgorin's discs

Examples:

1. A = [1 2; 1 -1]
2. A = [1 -1; 2 -1]

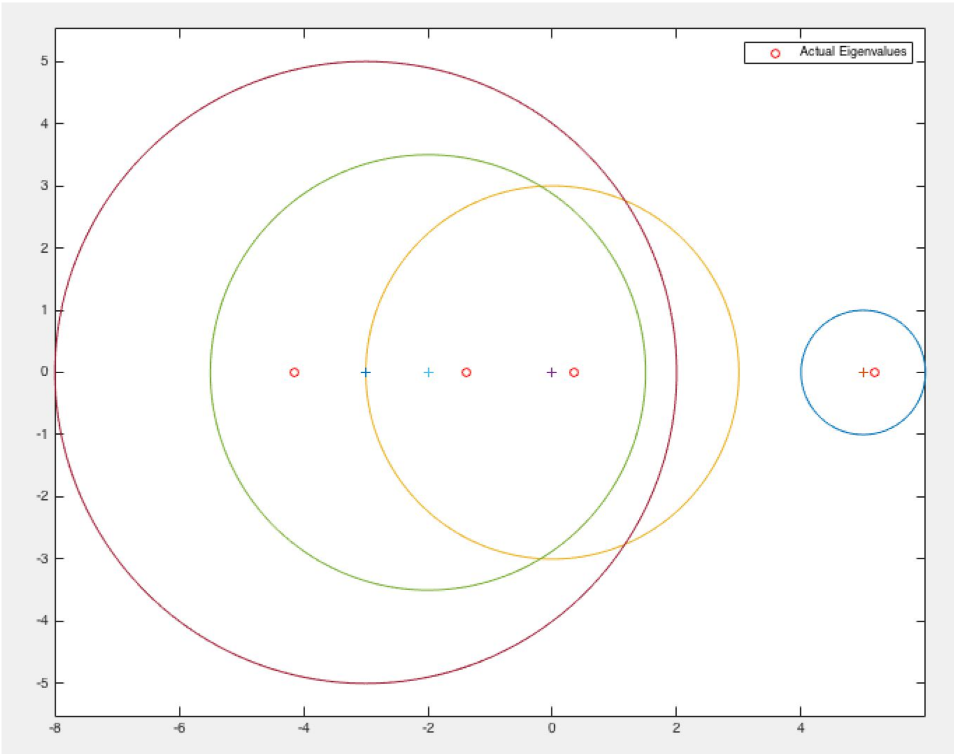
3. A = [5 0 0 -1; 1 0 -1 1; -1.5 1 -2 1; -1 1 3 -3]

You can visualize the Gershgorin discs for the above examples in MATLAB as follows:

1. Define a function gershdisc.m using the code below.
2. Define A = [...] from the example above.
3. Run `gershdisc(A)` in the folder where you saved this function.
4. Try to understand what the function is doing. Try some examples of your own!

```
% gershdisc.m
% This function plots the Gershgorin Discs for the matrix A passed as an argument.
% It will also plot the centers of such discs, and the actual eigenvalues
% of the matrix.
 function gershdisc(A)

error(nargchk(nargin,1,1));
if size(A,1) ~= size(A,2)
   error('Matrix should be square');
   return;
end

for i=1:size(A,1)
   h=real(A(i,i)); k=imag(A(i,i));
   r=0;
   for j=1:size(A,1)
     if i ~= j
        r=r+(norm(A(i,j)));
     end
   end
   N=256;
   t=(0:N)*2*pi/N;
   plot( r*cos(t)+h, r*sin(t)+k ,'-');
   hold on;
   plot( h, k,'+');
end
axis equal;
ev=eig(A);
for i=1:size(ev)
   rev=plot(real(ev(i)),imag(ev(i)),'ro');
end
hold off;
legend(rev,'Actual Eigenvalues');

end

%code source: http://www.mathworks.com/matlabcentral/fileexchange/13989-gershgorin-discs-plot/content/gershdisc.m
```

For eg. 3, you should get:



Householder's method

Recall that we defined the Householder matrix as:

$$H = I - 2vv^T/v^Tv$$

A few interesting properties of this matrix:

- It transforms a vector to its reflection
  - $v^T Hx = -v^T x \Rightarrow If\ \theta_{v,x} = \phi,\ then\ \theta_{v,Hx} = \pi + \phi$
- Symmetric and orthogonal
  - $H^T = H,\ H^T H = H^2 = I$.
- Hyperplane H consists of all vectors 'x' which are normal to 'v'
- For a given matrix $A_{n\times n} = [x_1\ x_2\ ...\ x_n]$, we choose $v = x + \alpha e_1 = x + sgn(\beta)\sqrt{x^T x} e_1$, depending on $\beta = e_1{}^T x$.
- Using a sequence of pre- and post-multiplications by Householder matrices, we can transform any matrix to a tridiagonal form.
  - $H^T{}_n H^T{}_{n-1} ... H^T{}_1 A H_1 ... H_{n-1} H_n = T_n$

We will consider the example from the book, and apply this method sequentially. Commands are in **bold**.

```
>> A = [4 1 2 1 2; 1 3 0 -3 4; 2 0 1 2 2; 1 -3 2 4 1; 2 4 2 1 1]

A =

     4     1     2     1     2
     1     3     0    -3     4
     2     0     1     2     2
     1    -3     2     4     1
     2     4     2     1     1

>> x1=A(2:5,1)

x1 =

     1
     2
     1
     2

>> v1=x1+[norm(x),0,0,0]'

v1 =

    4.1623
    2.0000
    1.0000
    2.0000

>> v1=[0,v1']'

v1 =

         0
    4.1623
    2.0000
    1.0000
    2.0000

>> H=eye(5)-2*v1*v1'./(v1'*v1)

H =

    1.0000         0         0         0         0
         0   -0.3162   -0.6325   -0.3162   -0.6325
         0   -0.6325    0.6961   -0.1519   -0.3039
         0   -0.3162   -0.1519    0.9240   -0.1519
```

```
         0   -0.6325   -0.3039   -0.1519    0.6961

>> H'*A*H

ans =

    4.0000   -3.1623    0.0000    0.0000         0
   -3.1623    5.3000    1.2325   -0.3325    0.2838
    0.0000    1.2325    1.6534    3.3124    0.2755
    0.0000   -0.3325    3.3124    5.1490    1.1234
    0.0000    0.2838    0.2755    1.1234   -3.1024
```

We can continue this till we get a tridiagonal matrix. Besides the textbook, you'll find a few nice references for proofs below.

In the next class, we'll see how to use this method in tandem with the QR method to obtain eigenvalues of general matrices.

Helpful links

1. Nice reference on Gerschgorin's theorems
2. Nice resource for Householder's method
3. Another resource for Householder's method