Recitation 5

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Brief Overview

- Vector norms
- Matrix norms
- Discussion of Homework questions
- Solving exercises from book

Norms and inequalities (15 min)

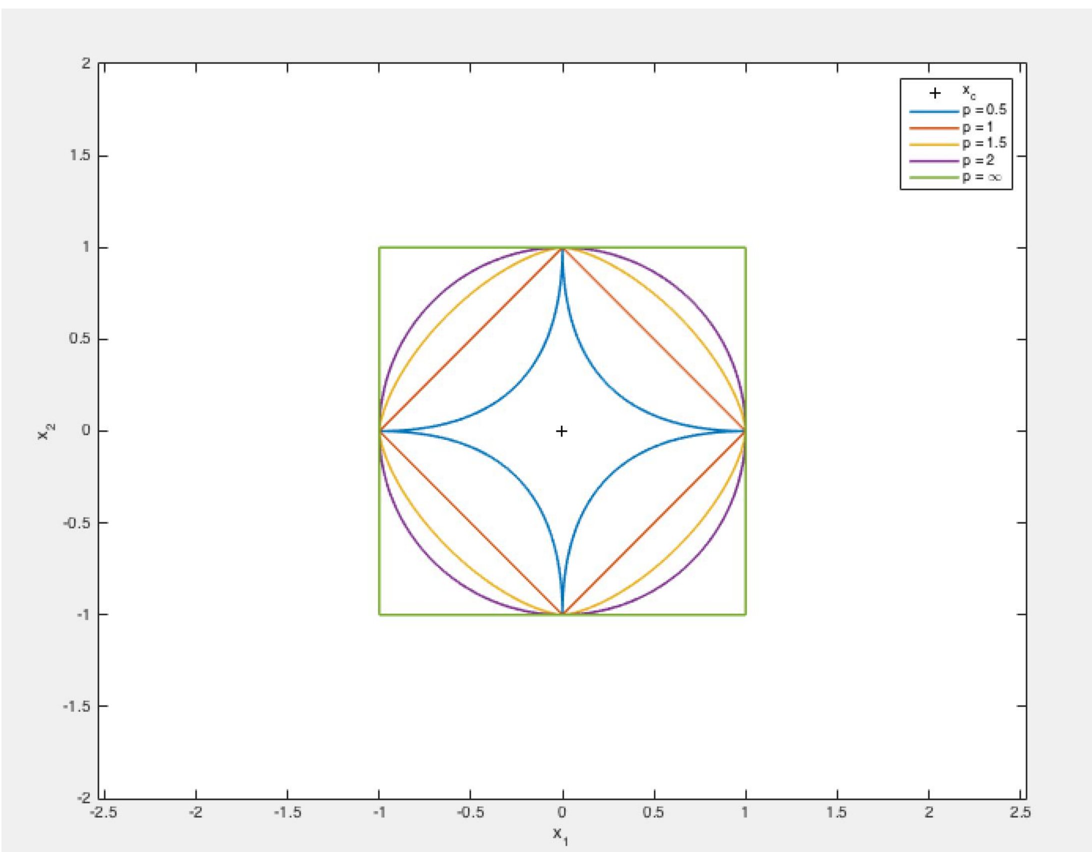
- What is a norm?
- An Lp norm is defined as $||x||_p = (\sum_{i=1}^{n} |x|^p)^{1/p}$
- 2-norm or Euclidean norm: $||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
- 1-norm: $||x||_2 = \sum_{i=1}^{n} |x_i|$
- $\infty -$norm: $||x||_\infty = max(|x_1|, |x_2|, ..., |x_n|)$

- Visualizing norms in MATLAB

```
function [] = VisualizeNorms(x_c,r,p)
%plots a p-norm ball with radius r, centred at x_c
    low = min(x_c);
    high = max(x_c);
    x = linspace(low-2*r,high+2*r);
    y = linspace(low-2*r,high+2*r);
    theta = 0:pi/100:3*pi;
    xp = r*sign(cos(theta)).*abs(cos(theta)).^(2/p);
    yp = r*sign(sin(theta)).*abs(sin(theta)).^(2/p);
    plot(xp,yp,'-','LineWidth',1.5)
end
```

```
%tester code
x_c = [0,0]; %center at origin
r = 1; %radius of ball is 1
plot(x_c(1),x_c(2),'k+') %plots the center
hold on
xlabel('x_1')
ylabel('x_2')
axis([-(x_c(1)+2*r) x_c(1)+2*r -(x_c(2)+2*r)
x_c(2)+2*r])
axis equal
VisualizeNorms(x_c,r,0.5) %p = 0.5
VisualizeNorms(x_c,r,1) %p = 1
VisualizeNorms(x_c,r,1.5) %p = 1.5
VisualizeNorms(x_c,r,2) %p = 2
VisualizeNorms(x_c,r,inf) %p = inf
legend('x_c','p = 0.5','p = 1','p = 1.5','p = 2', 'p = \infty')
```

You should get a plot which looks like this:



- Think about why this happens. Read the references to know more about norms.
- Can also define matrix norms

- Subordinate norm: $sup_{v \in R^n} \frac{||Av||}{||v||}$

Condition number

- Absolute condition number
    - $sup_{\delta x}(||\delta f||)/(||\delta x||)$, where $||\delta f|| = ||f(x + \delta x) - f(x)||$

Consider a mapping f from a subset D of a normed linear space $\mathcal{V}$ with norm $\|\cdot\|_{\mathcal{V}}$ into another normed linear space $\mathcal{W}$ with norm $\|\cdot\|_{\mathcal{W}}$, depicted in Figure 2.3, where $x \in D \subset \mathcal{V}$ is regarded as the 'input' for f and $f(x) \in \mathcal{W}$ is the 'output'. We shall be concerned with the sensitivity of the output to perturbations in the input; therefore, as a measure of sensitivity, we define the **absolute condition number** of f by

$$\mathrm{Cond}(f) = \sup_{\substack{x,y \in D \subset \mathcal{V} \\ x \neq y}} \frac{\|f(y) - f(x)\|_{\mathcal{W}}}{\|y - x\|_{\mathcal{V}}}. \tag{2.42}$$

If $\mathrm{Cond}(f) = +\infty$ or if $1 \ll \mathrm{Cond}(f) < +\infty$, we say that the mapping f is **ill-conditioned.**

- Relative condition number
    - $sup_{\delta x}(||\delta f||/||f(x)||)/(||\delta x||/||x||)$
    - For a matrix (image from Trefethen et al., Numerical Linear Algebra, p. 93)

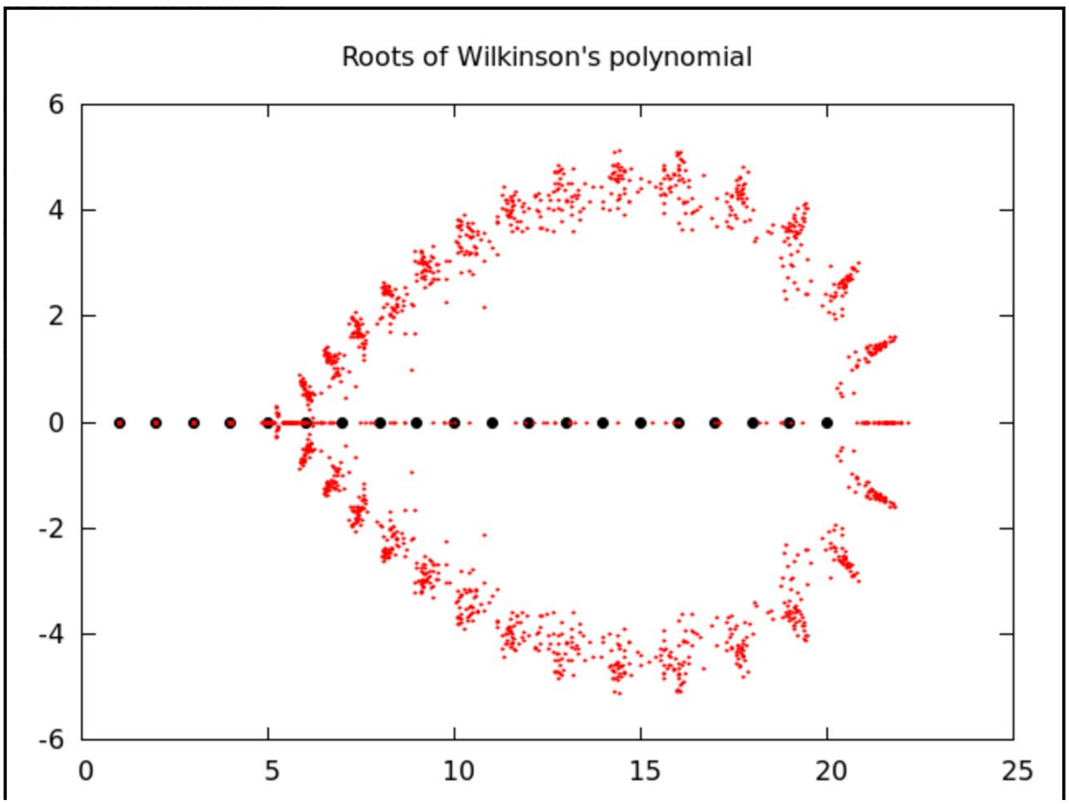
$$\kappa = \sup_{\delta x} \left(\frac{\|A(x + \delta x) - Ax\|}{\|Ax\|} \bigg/ \frac{\|\delta x\|}{\|x\|} \right) = \sup_{\delta x} \frac{\|A\delta x\|}{\|\delta x\|} \bigg/ \frac{\|Ax\|}{\|x\|}$$

that is,

$$\kappa = \|A\| \frac{\|x\|}{\|Ax\|}$$

- Relative condition number more important in numerical analysis, as floating point system introduces relative errors
- Small condition number means well-conditioned. Large means ill-conditioned.
- Condition number of A:
    - $\kappa(A) = ||A||.||A^{-1}||$

- Eg.
    - Finding the eigenvalues of a matrix
        - Take A = [1 1000; 0 1], eig(A) = [1 1]
        - B = [1 1000; 0.001 1], eig(B) = [0 2]
        - Problem is ill-conditioned for non-symmetric matrices, well-conditioned for symmetric matrices
    - Determining roots of a polynomial is ill-conditioned. If you. See [6] for famous example on Wilkinson's polynomial. You can also try checking this in MATLAB (see [7] for instructions).



- See [5] for more info. on examples.

Homework discussion and exercises

Homework 3 problems:

- Ex. 2.3
- Ex. 2.8
- 2.12, 13

Let's try a few exercises:

- Ex. 2.4
- Ex. 2.7
- Ex. 2.9
- Ex. 2.14

Helpful resources

1. Notes on vector norms
2. Notes on matrix norms
3. Intuition behind norms
4. *Numerical Mathematics (Quarteroni et al)* is a very good resource for LU/norms/condition numbers/etc. It's also freely available for NYU students via Springer!
5. *Numerical Linear Algebra, Trefethen & Bau* also has a nice discussion on condition numbers (see Lec. 12).
6. https://en.wikipedia.org/wiki/Wilkinson%27s_polynomial
7. http://blogs.mathworks.com/cleve/2013/03/04/wilkinsons-polynomials/