19/04/2019

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VECTOR DIFFERENTIATION

~~$x^2 + (t+x)^2 = t^2$~~

Point function :-

A variable quantity which depends upon the co-ordinates of the point of a region of space is known as pt f^n.

If this function is scalar then it's called scalar pt. f^n. eg temp, density.

If this is vector quantity then its vector pt. f^n. eg. velocity.

if $\bar{f}(t)$ is vector f^n. then

$$\frac{d\bar{f}(t)}{dt} = \lim_{\delta t \to 0} \frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t}.$$

Vector operator del (nabla) ∇

It is defined as $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Then if ϕ is the scalar pt. f^n then $\nabla\phi$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\vec{r}$ is the position vector defined as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

the vector function is called
the gradient of scalar pt. fn.
grad $\phi = \nabla\phi$.

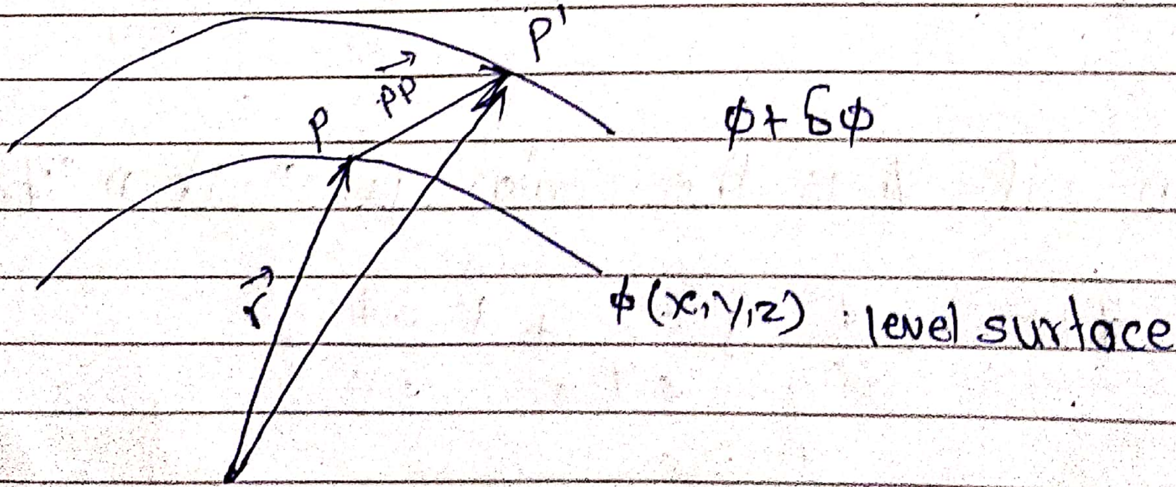
$$\nabla\phi \cdot d\vec{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right).$$

$$\nabla\phi \cdot d\vec{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz.$$

$$= d\phi$$

Directional derivative



level surface the scalar pt. function $\phi(x,y,z)$ is constant and is defined in a certain ~~poin~~ region of space the surface drawn in space containing all these points where $\phi(x,y,z)$ has same value is called a level surface.

The limiting value of $\lim_{\delta r \to 0} \frac{\delta \phi}{\delta r}$

is known as direction derivative of ϕ at P along the dirn $\overrightarrow{PP}$

thus & $\frac{\partial \phi}{\partial r}$ represent rate of change of ϕ in any other dirn & is called directional derivative.

- The dirnal derivative of ϕ along certain dirn $\vec{a}$ is given by $\nabla\phi \cdot \hat{a}$ where

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Grad of ϕ = $\nabla\phi$ repesent maximum rate of ϕ towards normal direction to the level surface ϕ.

Example obtain the dirn derivation of ϕ.

$$\phi = xy^2 + yz^3$$

i) at the point $(2,-1,1)$ in the dirn of the vector $i + 2j + 2k$

ii) at pt $(1,-1,1)$ towards pt $(2,1,-1)$

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25/4/19

Find the directional derivative of $\phi = xy^2 + yz^3$ at $(1,-1,1)$ along the direction $x^2+y^2+z^2=9$ at $(1,2,2)$

Ans:

Given ϕ is $xy^2 + yz^3$ therefore

$$\nabla\phi = \hat{i}(y^2+0) + \hat{j}(2xy + z^3) + \hat{k}(3yz^2)$$

$$\nabla\phi \text{ at } (1,-1,1) = \hat{i} - \hat{j} - 3\hat{k}$$

Given surface $\varphi = x^2+y^2+z^2-9$

then the normal vector normal to any surface φ is $\nabla\varphi$

then $\nabla\varphi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$$\nabla\varphi(1,2,2) = 2\hat{i} + 4\hat{j} + 4\hat{k} = \bar{a}$$

$$\bar{a} = 2\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\hat{a} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

$\therefore$ directional derivative dd is

$$= \nabla\phi \cdot \hat{a}$$

$$= (\hat{i} - \hat{j} - 3\hat{k})\left(\frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}\right) = \frac{-7}{3}.$$

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Q. find the directional derivative $\phi = 4xz^3 - 3x^2y^2$ at $(2,-1,2)$ along the tangent to the curve $x = e^t \cos t$
$y = e^t \sin t$
$z = e^t$

at $t=0$.

Ans
Given curve $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{r} = e^t \cos t\,\hat{i} + e^t \sin t\,\hat{j} + e^t\,\hat{k}.$$

therefore tangent to the curve is $\frac{d\vec{r}}{dt}$

$$\frac{d\vec{r}}{dt} = (-e^t \sin t + e^t \cos t)\,\hat{i} + (e^t \sin t + e^t \cos t)\,\hat{j} + e^t\,\hat{k}. = T.$$

at $t=0$.

$$\vec{T}_{t=0} = \vec{a} = \hat{i} + \hat{j} + \hat{k}.$$

$$\hat{a} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

$$\nabla\phi = (4z^3 - 6xy^2)\,\hat{i} + (-6x^2y)\,\hat{j} + (12xz^2)\,\hat{k}$$

$$\nabla\phi_{(2,-1,2)} = 20\,\hat{i} + 24\,\hat{j} + 96\,\hat{k}$$

$$\nabla\phi \cdot \hat{a} = (20\hat{i} + 24\hat{j} + 96\hat{k}) \cdot \frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}} = \frac{140}{\sqrt{3}}$$

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Qu) Find the directional derivative of $\phi = xy^2 + yz^3$ at $(2,-1,1)$ along the line $2(x-2) = y+1 = z-1$

Ans from given

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-1}{2}$$

$l=1$ $m=2$ $n=2$.

direction of this line is $\vec{a} = 1\hat{i} + 2\hat{j} + 2\hat{k}$.

$$\hat{a} = \frac{1}{3}(1\hat{i} + 2\hat{j} + 2\hat{k})$$

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Diversion

diversion $\bar{F} = \nabla . \vec{F}$

if $\nabla . \bar{F} = 0$ then the vector field $\bar{F}$ is called solenoidal

curl of $\vec{F}$

$\nabla \times \vec{F} \rightarrow$ curl of $\vec{F}$

$\vec{\omega} = \frac{1}{2}$ curl of $\bar{F}$

diversion $\vec{\nabla} = \nabla . \vec{\nabla}$ is rate of flow of fluid passing through unit cube

if $\nabla \times \vec{F} = 0$ the $\vec{F}$ is a e rotational or consecutive field.

Formulae:

1> $\nabla . \vec{r} = 3$.

2> $\nabla \times \vec{r} = 0$

3> $\nabla f(0) = f'(0) . \nabla$

4> $\nabla f(r) = f'(r) \frac{\vec{r}}{r}$

5> $\nabla (\vec{a} . \vec{r}) = \vec{a}$

6> $\nabla (\phi \bar{u}) =$

$\nabla \phi . u + \phi \nabla . \vec{u}$

7>

8>

Qu

Qu

7) $\nabla \times (\phi \bar{U}) = \nabla \phi \times \bar{U} + \phi \nabla \times \vec{U}$

8) $\vec{r} \cdot d\vec{r} = r\, dr$.

<u>Qu</u> Prove that.

$$\nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^n}\right) = \frac{\vec{a}}{r^n} - \frac{n(\vec{a}\cdot\vec{r})}{r^{n+2}}$$

$$LHS = \nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^n}\right)$$

$$= (\vec{a}\cdot\vec{r})\,\nabla r^{-n} + r^{-n}\,\nabla(\vec{a}\cdot\vec{r})$$

$$= (\vec{a}\cdot\vec{r})\left((-n)\, r^{-n-1}\right)\frac{\vec{r}}{r} + r^{-n}\,\vec{a}$$

<u>Qu</u> prove that.

$$\nabla^2\left(\nabla \cdot \vec{r}/r^2\right) = 2/r^4$$

$$\nabla \cdot \frac{\vec{r}}{r^2} = \nabla r^{-2} \cdot \vec{r} + r^2 \cdot \nabla \cdot \vec{r}$$

$$= (-2)\, r^{-3} \cdot \frac{\vec{r}\cdot\vec{r}}{r} + 3r^2.$$

$$\nabla \cdot \frac{\vec{r}}{r^2} = 1/r^2$$

$$\nabla^2\left(\frac{1}{r^2}\right) = \nabla^2\left(r^{-2}\right)$$

$$= \nabla\left(-2r^{-3} \cdot \frac{\vec{r}}{r}\right)$$

$$= \nabla\left(-2r^{-4} \cdot \vec{r}\right)$$

$$= \nabla\left(-2r^{-4}\right) \cdot r \quad + -2r^{-4}\left(\nabla\left(\vec{r}\right)\right)$$

$$= -2\left(-4 \cdot r^{-5} \frac{\vec{r}}{r}\right) \cdot \vec{r} + -2r^{-4} \quad (3)$$

$$= 8r^{-4} + (-6)\, r^{-4}$$

$$= 2r^{-4} = \frac{2}{r^4}$$

Qu $\vec{F} = \left(6xy + z^3\right)\hat{i} + \left(3x^2 - 3\right)\hat{j} + \left(3xz^2 - y\right)\hat{k}$

Show that $\vec{F}$ is irrotational find scalar ϕ such that. $\vec{F} = \nabla\phi$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ 6xy + z^3 & 3x^2 - 3 & 3xz^2 - y \end{vmatrix} = 0$$

Given $\vec{F}$ irrotational.

27/4/2019

let $f(x,y,z)$ is vector function define some region of space &let's see C be any curve in that region then the line integral of $\vec{F}$ along a path C is defined as integral of tangential component of $\vec{F}$.

line integral $= \int_C \vec{F}\cdot\hat{T}\, ds.$

where $\hat{T}$ is a unique vector tangent to the curve and ds is element of Ar of C.

if $\vec{r}$ is a position vector of any point P of C then $\frac{d\vec{r}}{ds}$ is a unique vector tangent to the curve at any point P.

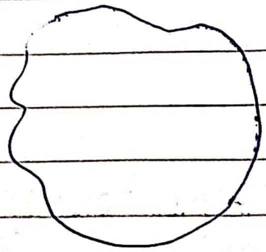
$$= \int_C \vec{F}\cdot\frac{d\vec{r}}{ds}\, ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int F_1\, dx + \int F_2\, dy + \int F_3\, dz.$$

the line integral over C $\int \vec{F}\cdot d\vec{r}$ represent the work done on moving object in the field of $\vec{F}$ along curve C of the field.

$$\vec{F} = u\hat{i} + v\hat{j}.$$



* Consider the closed curve C enclosing an Area A let u of u(x, y) v, (x, y) and their first partials $\frac{du}{dx}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ be continuous

and single value over the region bounded by curve C

then $$\oint_C u\,dx + v\,dy = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_A \hat{k} \cdot (\nabla \times \vec{F})\, ds$$

vector form of green's lemma is.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_A \hat{k} \cdot \text{curl}\, \vec{F}\, ds.$$

where $ds = dx \times dy$.

Que evaluate $\oint_C \vec{F} \cdot d\vec{r}$

$\vec{F} = (2x+3)\hat{i} + xy\hat{j} + (yz-x)\hat{k}$.

i) $x^2 = 2t^2$ $y = t$ $z = t^3$ from $t=0$ to $t=1$

ii) the st. line from point $(0,0,2)$ to $(0,0,1)$

then $(0,1,1)$ then to $(2,1,1)$

Ans line int. $= \int_C \vec{F} \cdot d\vec{r}$.

$$= \int_C (2x+3)\,dx + xy\,dy + (yz-x)\,dz$$

$x = \sqrt{2}t$ $\quad dx = \sqrt{2}dt$.

$y = t$ $\quad dy = dt$

$z = t^3$ $\quad dz = 3t^2dt$. $\quad t = 0$ to 1

$$= \int_0^1 (2\sqrt{2}t+3)\sqrt{2}\,dt + \sqrt{2}t^2\,dt + (t^4 - \sqrt{2}t)\,3t^2\,dt$$

$$= \frac{483\sqrt{2}+60}{140}.$$

eq^n of line

ii) $\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} = \frac{z-z_1}{z_1-z_2} = t$

$\frac{x-0}{0-0} = \frac{y-y_1}{0-0} = \frac{z-0}{0-1} = t$

$x=0 \quad y=0 \quad z=-t$

$dx=0 \quad dy=0 \quad dz=-1.dt$

$\int_C \vec{F}\cdot d\vec{r} = \int (2x+3)dx + xy\,dy + (yz-x)\,dz$

Answer = 0

From the pt (0,1,1) to (2,1,1)

$\frac{x-0}{-2} = \frac{y-1}{0} = \frac{z-1}{0} = t$

$x=0 \quad t=0$

$x=2 \quad t=-1$

$x=-2t \quad y=1 \quad z=1$

$dx=-2dt \quad dy=0 \quad dz=0$

$\int \vec{F}\cdot d\vec{r} = \int (2(-2t)+3)\,-2dt$

$= \int (8t-6)dt = \left(4t^2-6t\right)_0^{-1}$

$4+6-0 = \boxed{10}$

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Qu

Evaluate. Find the work done for $\vec{F} = (2x+y)\hat{i} + (3y-x)\hat{j}$ and C is the curve st. line (0,0) and (3,2)

$$\vec{F}\,d\vec{r} = \int (2x+y)\,dx + (3y-x)\,dy.$$

$$\frac{x-0}{-3} = \frac{y-0}{-2} \Rightarrow \frac{x}{3} = \frac{y}{2} \qquad x = \frac{3}{2}y.$$

$$y = \tfrac{2}{3}x \Rightarrow dy = \tfrac{2}{3}dx.$$

$$\text{work done} = \int_0^2 \left(2x + \frac{2}{3}x\right) dx + \left(3\times\frac{2}{3}x - x\right)\frac{2}{3}dx.$$

$$= \int_0^3 \frac{8}{3}x + \frac{2}{3}x \; dx.$$

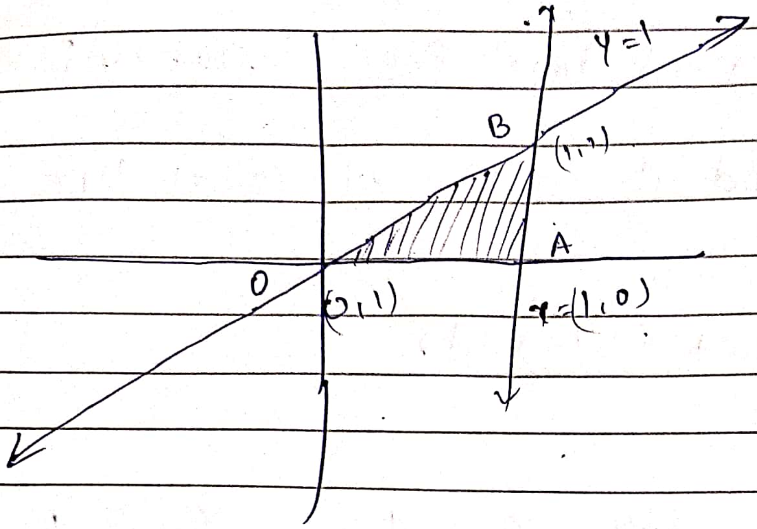
$$= \frac{10}{3}\left(\frac{x^2}{2}\right)_0^3 = \frac{5}{3}\times 9 = \boxed{15}$$

Qu

Using greens lemma evaluate integration over closed curve C.

$$\oint (xy - x^2)\,dx + x^2y\,dy.$$

along the closed curve C. $x=1$ $y=0$, $y=x$.



double integral

$$= \int_{x=a}^{x=b} dx \int_{y_1(x)}^{y_2(x)} dy$$

$$\oint_C = \int_{OA} + \int_{AB} + \int_{OC}.$$

$u = xy - x^2$

$v = x^2 y.$

$$\text{RHS} = \int_{x=0}^{1}\int_{y=0}^{x} (2xy - x)\, dx\, dy.$$

$$= \int_0^1 \left(\frac{2xy^2}{2} - xy\right)_0^x dx = \int_0^1 (x^3 - x^2)\, dx.$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3}\right)_0^1 = \boxed{\frac{-1}{12}.}$$

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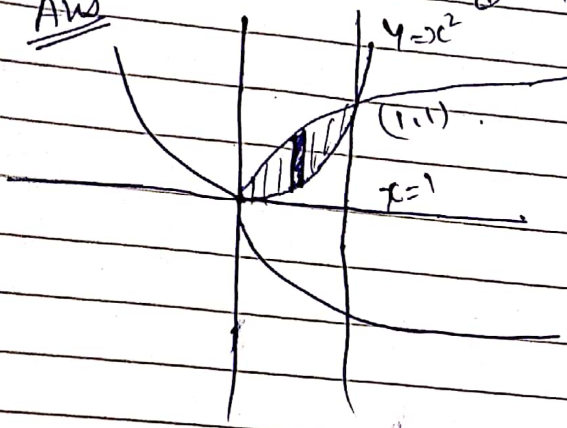
Qu Find value of $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the boundary of the region $y = \sqrt{x}$ & $y = x^2$. using green lemma.

Ans



$u = 3x^2 - 8y^2$

$v = 4y - 6xy$.

$\frac{\partial u}{\partial y} = -16y$

$\frac{\partial v}{\partial x} = -6y$.

$x \to 0$ to 1

$y \to \cancel{0}\ \cancel{\text{to}}\ x^2$ to $\sqrt{x}$.

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (-16y - 6y)\,dy\,dx.$$

$$= \int_0^1 \left(-\frac{22}{2} y^2\right)_{x^2}^{\sqrt{x}} dx.$$

$$= \int_0^1 -11(x^4 - x)\,dx. = -11\left(\frac{x^5}{5} - \frac{x^2}{2}\right)$$

$$= 11\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3 \times 11}{10} = \boxed{3.3}$$

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