28/3/2019.

Complex Variable :-

Complex $z = x + iy$

$= r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$. polar form Complex No.

if $w = f(z) = \cancel{z}(u+iv) = z^2$

then $w = (x+iy)^2 = x^2 - y^2 + i(2xy)$

$= u + iv$

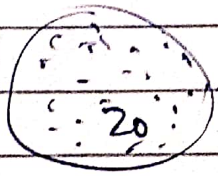
Derivative of $f(z)$ where u = Real Part of $f(z)$,
v = img " — " —

where $f(z)$ is a function of complex variable z

derivative $\dfrac{d\,f(z)}{dz} = \lim\limits_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$.

where h is $= h_1 + ih_2$

Analytic function : a function $f(z)$ is said to be analytic at a point $z = z_0$ if it is defined and has a derivative at every point in some neighbourhood of z_0



Singular point

The point $z = z_1$ where function ceases to be analytic is called the singular point of $f(z)$.

when $f(z) = \frac{1}{z}$ is differentiable everywhere except $z = 0$ is a singular pt. of $f(z)$

Necessary condition for analytic function :-

Condition :-

if $f(z) = u + iv$ is an analytic function at any pt z in region r.

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\}$$

Cauchy's - Riemann equation.

if

$f(z) = z^2 = r^2 e^{i2\theta}$ is in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{du}{\partial \theta}$$

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Harmonic function :-

if $f(z)$ is $= u + iv$.

A function $\phi(x,y)$ is said to be harmonic if it is continuous and it has continuous 1st and 2nd order partial derivative and satisfies Laplace equation.

$\phi(x,y)$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Note:- if $f(z) = u + iv$. is analytic both u, v are harmonic.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Milne Thomson's method :-

$f(z) = u + iv$.

Given function is analytic therefore using CR equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

integrate $\frac{\partial v}{\partial y}$ w.r.t. y.

$$v = \phi(x,y) + f(x).$$

differentiate w.r.t. x.

$$\frac{\partial v}{\partial x} = \phi(x,y) + f'(x).$$

equating $-\frac{\partial u}{\partial y}$ from CR equation.

$f'(x) = \psi(x)$. integrate w.r.t x.

$f(x) = \int \psi(x)\, dx$. hence

$f(z) = u + iv$.

put $x = z$ and $y = 0$,
we get. the $f(z)$ ~~is~~ interms of z only.

Que if $v = 3x^2y - y^3$ find its harmonic conjugate and analytic function $f(z)$

u, v are harmonic conjugate of eachother

Ans

$$\frac{\partial v}{\partial x} = 6xy - 0 = 6xy.$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\int \partial u = \int -6xy.\,\partial y.$$

$$u = -\frac{6xy^2}{2} + f(x).$$

$$\frac{\partial u}{\partial x} = -3y^2 + f'(x).$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 = \frac{\partial u}{\partial x}.$$

$$3x^2 - \cancel{3y^2} = \cancel{-3y^2} + f'(x)$$

$$f'(x) = 3x^2$$

$$\boxed{f(x) = x^3}$$

$$\boxed{u = x^3 - 3xy^2}$$

from milne thomsons method put $x = z$

$y = 0$.

$u = z^3$ $\quad$ $V = 3z^2\ 0$.

$$f(z) = u + iv.$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3).$$

$$\boxed{f(z) = z^3.}$$

ex

if $w = \phi + i\psi$ represent the complex potential for an electric field and ϕ

$\phi = -2xy + \dfrac{y}{x^2+y^2}$

find the function ψ.

Ans

$$\frac{\partial \phi}{\partial x} = -2y + \frac{-y}{(x^2+y^2)^2} \cdot 2x.$$

$$= -2y - \frac{2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

$$\int \partial \psi = \int -2y - \frac{2xy}{(x^2+y^2)^2} \partial y$$

$$\psi = \frac{-2y^2}{2} + \frac{\cancel{2}x \cancel{\log(x^2+y^2)}}{(x^2+y^2)} + f(x).$$

$$\psi = \frac{-y^2}{1} + \frac{x}{x^2+y^2} + f(x).$$

$$\frac{\partial \psi}{\partial x} = \frac{-\partial \phi}{\partial y}$$

$$\frac{\partial \psi}{\partial x} = \frac{-\partial \phi}{\partial y} = -\left[-2x + \frac{1}{x^2+y^2} + \frac{y(-2y)}{(x^2+y^2)^2}\right].$$

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$$\frac{\partial \psi}{\partial x} = 0 + \frac{x}{x^2+y^2} + \frac{x(-2x)}{(x^2+y^2)^2} + f'(x).$$

$$\frac{2y^2}{\cancel{x^2+y^2}} + \frac{-1}{\cancel{x^2+y^2}} + 2\cancel{x} = \frac{\cancel{x}}{x^2+y^2} + \frac{-2\cancel{x}^2}{\cancel{x^2+y^2}} + \cancel{f'(x)}.$$

$$2 + \cancel{2x} + \frac{-(x+y)}{x^2+y^2} = f'(\cancel{x}).$$

$$\frac{\partial \psi}{\partial x} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}.$$

$$f'(x) = 2x.$$

$$f(x) = x^2$$

$$\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$$

$$\phi = -2xy + \frac{y}{x^2+y^2}.$$

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Que

if $v = \dfrac{-y}{x^2+y^2}$ find u such that

$f(z) = u + iv$ is analytic and determine $f(z)$ in terms of z.

Ans

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}. \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$= \frac{(-y)}{x^2+y^2} + \frac{(-y)(-1)\,2y}{(x^2+y^2)^2}$$

$$= \frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$(x^2+y^2)^2 = t$

$2(x^2+y^2) \cdot 2x\,dx = dt$

$2x\,dx = dt.$

$$\int \partial u = \int \frac{y^2-x^2}{(x^2+y^2)^2}\,\partial x.$$

$$u = \int$$

$$\frac{\partial v}{\partial x} = \frac{(-y)\cancel{+}(-2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}.$$

$$-\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$$

Alternative

$$-\frac{\partial v}{\partial x} = -\left[\frac{-y\cdot(2x))}{(x^2+y^2)^2}\right] = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\int \partial u = \int \frac{-2xy}{(x^2+y^2)^2}\,\partial y$$

$$u = \int -x \cdot \frac{1}{t^2}\,\partial t$$

$$u = \frac{x}{t} + f(x).$$

$$u = \frac{x}{(x^2+y^2)} + f(x).$$

$$\frac{\partial u}{\partial x} = f'(x) + \frac{1}{(x^2+y^2)} + \frac{-(x)(2x)}{(x^2+y^2)^2}.$$

$$\frac{\partial u}{\partial x} = f'(x) + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow f'(x) = 0. \quad f(x) = c$$

$$f(z) = u + iv = \frac{x}{x^2+y^2} + c + i\left(\frac{-y}{x^2+y^2}\right)$$

$$\boxed{f(z) = \frac{1}{z} + c.}$$

eg

$u = \cosh x \cos y.$ $\qquad \cosh x = \frac{e^{x} + e^{-x}}{2}.$

Ans

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} = \cos y \left(\frac{e^{x} - e^{-x}}{2} \right).$$

$$\frac{\partial v}{\partial y} = \cos y \left(\frac{e^{x} - e^{-x}}{2} \right)$$

$$v = \left(\frac{e^{x} - e^{-x}}{2} \right) . + \sin y + f(x).$$

we know $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$

$$\frac{\partial v}{\partial x} = f'(x) + \sin y \left(\frac{e^{x} + e^{-x}}{2} \right).$$

$$-\frac{\partial u}{\partial y} = -\left[(-\sin y) \left(\frac{e^{x} + e^{-x}}{2} \right) \right].$$

$$= \sin y \left(\frac{e^{x} + e^{-x}}{2} \right)$$

$f'(x) = 0.$ $\qquad f(x) = C.$

$$f(z) = u + iv = \cosh x . \cos y + i \sin y \left(\frac{e^{x} - e^{-x}}{2} \right).$$

$x = z$
$y = 0.$

$$f(z) = \cosh z \cos y + i . 0 . (\quad).$$

$$= \cosh z \cos y.$$

Qu

$u = r^3 \cos 3\theta + r\sin\theta$

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

Ans

$\frac{\partial u}{\partial r} = 3r^2\cos 3\theta + \sin\theta .$

$\frac{1}{r}\frac{\partial v}{\partial \theta} = 3r^2\cos 3\theta + \sin\theta .$

$\int \partial v = \int (3r^3\cos 3\theta + r\sin\theta)\, \partial\theta .$

$v = r^3 \sin 3\theta - r\cos\theta + f(r) .$

$\frac{\partial v}{\partial r} = 3r^2 \sin 3\theta - \cos\theta + f'(r) .$

$\frac{-1}{r}\frac{\partial u}{\partial \theta} = \frac{-1}{r}\left[r^3(-\sin 3\theta) . 3 + r\cos\theta \right] .$

$= 3r^2 \sin 3\theta - \cos\theta .$

$f'(r) = 0$ $f(r) = C$

$\boxed{v = r^3 \sin 3\theta - r\cos\theta + C .}$

$f(z) = u + vi$

$f(z) = r^3\cos 3\theta + r\sin\theta + i(r^3\sin 3\theta - r\cos\theta + c)$

$r = z \qquad \theta = 0.$

$f(z) = z^3 \pm i(z-c).$

$$\boxed{f(z) = z^3 - i(z-c).}$$

$f(z) = r^3(\cos 3\theta + \sin\theta + i(r^3\sin 3\theta - r\cos\theta + c)$

$= r^3(\cos 3\theta + i\sin 3\theta) - ri(\cos\theta + i\sin\theta) + ic.$

$= r^3 e^{i3\theta} + ic - ire^{i\theta}$

$= z^3 - iz + c.$

Complex Integration

- if $u-v=(x-y)(x^2+4xy+y^2)$

find $f(z)$

- if $u+v=e^y(\cos y+\sin y)+\dfrac{x-y}{x^2+y^2}$

fin $f(z)=u+iv$.

CI

Draw the figure

$f(z)=u+iv$ where $z=x+iy$.

therefore $dz=dx+idy$.

$$I=\int_c f(z)\,dz.=\int_c (u+iv)(dx+idy)$$

$$=\int_c u\,dx.-v\,dy.+i\int_c u\,dy+v\,dx$$

As u and v are real evaluation of i is equivalent to evaluation of real integrals and the value of i will depend upon the path of integration.

or the equation of the curve c.

eg evaluate

① $\int_c f(z)dz$ where $f(z) = z^2$ & c is the path joining point A $(z=0)$

B $(z = 1+i)$.

i) c is a parabola $y = x^2$.
ii) c is a st. line $y = x$.
iii) c is the path $A = (0,0)$, $M(1,0)$ $B(1,1)$.

Ans

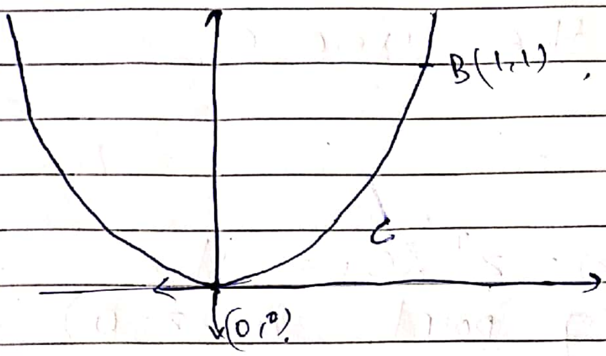
$$\int_c f(z)\,dz = \int z^2 dz.$$

$$= \int_c (x+iy)^2 (dx + i\,dy).$$

$$= \int_c (x^2 - y^2 + 2ixy)(dx + i\,dy)$$

$$I = \int_c (x^2 - y^2)\,dx - 2xy\,dy + i\int (x^2 - y^2)\,dy + 2xy\,dx.$$

(i) Path parabola $y = x^2$. $A - z = 0$ $B(1+i)$



$$I = \int_A^B f(z)\,dz. \qquad y = x^2. \quad dy = 2x\,dx.$$

$$I = \int_A^B (x^2 - x^4)\,dx - 4x^3\,dx + i\int (x^2 - x^4)\,2x\,dx + 2x^3\,dx.$$

$$I = \int_A^B (x^2 - 5x^4)\,dx + i\int_A^B (4x^3 - 2x^5)\,dx$$

$$= \left[\frac{x^3}{3} - x^5 + i\left(x^4 - \frac{2x^6}{6}\right)\right]_0^1$$

$$= \frac{1}{3} - 1 + i\left(1 - \frac{2}{6}\right) = -\frac{2}{3} + i\,\frac{2}{3}.$$

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(ii) $y = x.$ $\qquad dy = dx.$

$$I = \int_A^B \cancel{(x^2 - x^2)\,dx} - 2x.x\,dx + i\int_A^B \cancel{(x^2 - x^2)\,dy} + 2x.x\,dx$$

$$= \int_A^B -2x^2\,dx + i\int_A^B 2x^2\,dx.$$

$$= \left[-2\frac{x^3}{3} + i\,\frac{2}{3}x^2 \right]_0^1$$

$$= -\frac{2}{3} + \frac{2}{3}i.$$

(iii) $y = x^3$ $\qquad dy = 3x^2\,dx$

$$I = \int_C (x^2 - x^6)\,dx - 2x.x^3.3x^2\,dx$$

$$+ i\int_C (x^2 - x^6)\,3x^2\,dx + 2x.x^3\,dx.$$

$$= \int_C (x^2 - 7x^6)\,dx + i\int_C (5x^4 - 3x^8)\,dx.$$

$$= \left[\frac{x^3}{3} - x^7 + i\left(x^5 - \frac{x^9}{3} \right) \right]_0^1$$

$$= \frac{1}{3} - 1 + i\left(1 - \frac{1}{3}\right) = \boxed{-\frac{2}{3} + i\frac{2}{3}}$$

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4/4/2019

Cauchy's Theorem :-

if $f(z)$ is analytic on an within a closed curve c then integration within closed curve

$\oint f(z)\,dz = 0$ where c is called closed Contour & the corresponding

integration $\oint_c f(z)\,dz$ is called contour integral.

Subtheorem/Corollary

$\int_{z_1}^{z_1} f(z)\,dz$ is independent of path joining the points z_1 and z_2 if $f(z)$ is analytic function

Corollary 2

If $f(z)$ is analytic in region R between two simple closed contours c and c_1 then integration

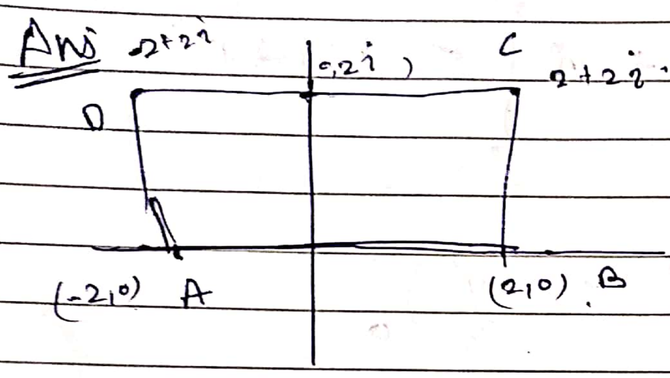
$\oint_c f(z)\,dz$

described in a same direction

Que

Verify cauchy's theorem for $f(z) = z^2+1$ over the path of rectangle whose vertices are -2, $+2$, $2+2i$, $-2+2i$.

Ans



$$f(z)dz = z^2+1 .dz.$$

$$= \left[(x+iy)^2+1\right]\left[dx + i\,dy\right]$$

$$= \int \left[(x^2-y^2+1)dx - 2xy\,dy\right] + i\int (x^2-y^2+1)dy + 2xy\,dx.$$

i) Along line AB

$y=0$ $x=-2$ to $x=2$

$dy=0$ $\displaystyle\int_{AB} f(z)dz = \int_{-2}^{2} (x^2+1)\,dx.$

$$= -\frac{2\cancel{8}}{3}. \quad \frac{28}{3}$$

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ii) Along BC

$x = 2 \quad dx = 0 \quad y = 0$ to $y = 2$.

$$\int_{BC} f(z)dz = \int_0^2 -4y\,dy + i\int_0^2 (5 - y^2)\,dy$$

$$= \left(-\frac{4y^2}{2}\right)_0^2 + i\left(5y - \frac{y^3}{3}\right)_0^2$$

$$= (-4) + i \times \frac{22}{3} = -4 + i\frac{22}{3}$$

iii) Along lin CD.

$y = 2 \quad dy = 0 \quad x = 2$ to $x = -2$.

$$\int_{CD} f(z)dz = \int_{+2}^{-2} (x^2 - 3)\,dx + i\int_2^{-2} 4x\,dx.$$

$$= \left[\frac{x^3}{3} - 3x\right]_2^{-2} + i\left[\frac{4x^2}{2}\right]_2^{-2}$$

$$= -\frac{16}{3} - 3\times(-2 - 2)$$

$$= -\frac{16}{3} + 12. = \boxed{\frac{20}{3}}$$

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iv) Along line DA.

$x = -2 \quad y = 2$ to $0 \quad dx = 0 \quad dy = dy$

$$\oint_{DA} f(z)\,dz = \int -4y\,dy + i\int 5 - y^2\,dy$$

$$= \left(-2y^2\right)_2^0 + i\left(5y - \frac{y^3}{3}\right)_{+2}^0$$

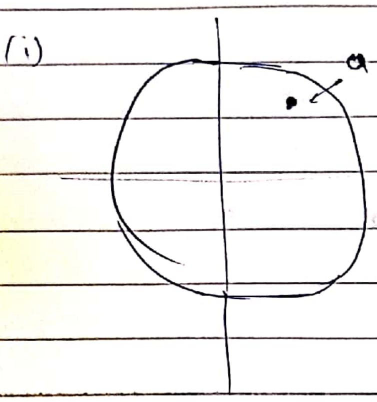
$$= +8 + i - \left(10 - \frac{8}{3}\right)$$

$$= \boxed{8 - \frac{22i}{3}}$$

(i) (+) (2) + (3) + (4) $= 0$.

Cauchy's Integral Formula :-

(i)



if $f(z)$ is analytic on and within closed contour C and if a is any point within c then $f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)} dz$

Corollory if $f(z)$ is analytic on an within closed contour C and a is any point within C then $f^{n}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$.

Que

Evaluate $\oint_C \frac{z^2+1}{z-2} dz$ where

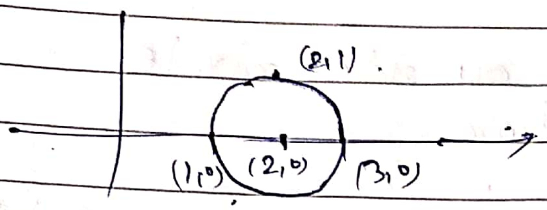
i) C is the circle with $|z-2| = 1$

ii) C is the circle $|z| = 1$

Ans the function $f(z) = z^2+1$ is analytic f^n in the region $|z-2| = 1$ which.

$$(x-2)^2 + y^2 = 1$$

Circle with centre $(2,0)$ & radius $= 1$



$$\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a).$$

the point $z = 2 \Rightarrow x + iy = 2$

$x = 2 \quad y = 0$

$z = (2,0)$. is in the given circle

therefore by the cauchy's integral formula

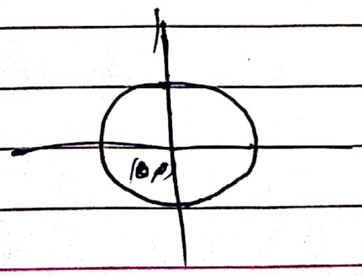
$$\int \frac{z^2+1}{z-2}\, dz = 2\pi i\, f(2).$$

$$f(2) = f(2) = (z^2+1), = 5.$$

so $$\oint \frac{z^2+1}{z-2}\, dz = \boxed{10 i\pi.}$$

(ii) $|z| = |x+iy| = x^2+y^2 = 1.$

centre $(0,0)$. radius 1.



the point $z_0 = 2$ outside circle $|z| = 1$ & therefore

we put $F(z) = \frac{z^2+1}{z-2}$ which is analytic in the closed circle c therefore by cauchy's there

$$\oint_C F(z)\, dz = \oint_C \frac{z^2+1}{z-2}\, dz = 0$$

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The point z_0 where the function $f(z)$ ceases to be analytic is called the singular point of the function $f(z)$ if in the small neighbourhood of z_0

there is no singular point of $f(z)$ then other than z_0 then z_0 is called isolated singular point

then $f(z)$ can be expanded in a series of the form $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2$

$- - - + a_{-1}(z-z_0)^{-1} + a_{-2}(z-z_0)^{-2} + \text{---} \; a_{-n}(z-z_0)^{-n}$

this series is called Laurent's series.

series 1) series consisting of positive powers of $(z-z_0)$ is called analytic part.

2) series consisting of -ve powers of $z-z_0$ is called -ve part of Laurent's series.

$$a_n = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

where c is this circle surrounding the singular point z_0

if the principle part of Laurent's series contains n terms then singular point z_0 is called pole of the order n.

Cauchy's Residue theorem :

if $f(z)$ is analytic on on within a closer curvel counter C except at finite Number of isolated singular points within C. then integration over

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = 2\pi i \left[r_1 + r_2 + \dots r_n \right]$$

where r_1, r_2, r_n are the residues of singular points within C where

$$r = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} (z-z_0)^{n+1} f(z) \right]$$

Que Evaluate using residue theorem integration over closed curve C

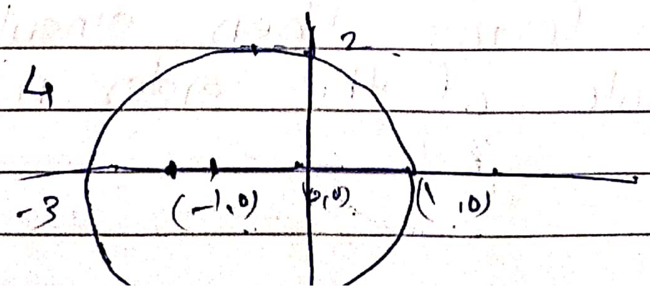
$$\oint \frac{2z^2+2z+1}{(z+1)^3 (z-3)} dz,$$ where c is the contour $|z+1| = 2$

Ans Given circle is $|z+1| = 2$.

$$|(x+1) + iy| = 2,$$

$$(x+1)^2 + y^2 = 4$$

Center $(-1, 0)$



$z+1=0$,
$z=-1$, $\Rightarrow$ $(-1,0)$ -1 is pole of
$x+iy=$ · order

Since $z=-1$ $\Rightarrow$ $x=-1$ $y=0$ & hence it
has a pole
the point lies in the circle
and $z=3$ $\Rightarrow$ $(3,0)$ is outside the circle
therefore $z=-1$ is the pole of order 3

residue theorem state that

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i\,[r_1+r_2+\cdots r_n].$$

here $f(z)=\dfrac{2z^2+2z+1}{z-3}$. $z-z_0=z+1$ $z_0=1$
$n+1=3$ $\Rightarrow$ $n=2$,

$$\oint \frac{f(z)}{(z-z_0)^{n+1}} = 2\pi i\,[r_1+r_2].$$ $r_1 \Rightarrow$ residue of at $z=-1$

$$r_1 = \frac{1}{(2-1)!}\left[\frac{d^2}{dz^2}(z+1)^3 f(z)\right]$$

$$= \frac{1}{2!}\left[\frac{d^2}{dz^2}\,\frac{(z+1)^3\,(2z^2+2z+1)}{(z+1)^3\,(z-3)}\right]$$

$$= \frac{1}{2}\left[\frac{d}{dz}\left(\frac{(z-3)(4z+2)-(2z^2+2z+1)}{(z-3)^2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{d}{dz}\left(\frac{2z^2-12z-7}{(z-3)^2}\right)\right]$$

$$= \frac{1}{2!}\left[\frac{(z-3)^2(4z-12) - (2z^2+2z+1)2(z-3)}{(z-3)^4}\right]$$

$$= \frac{1}{2!}\left[\frac{(z-3)(4z-12) - (2z^2+2z+1)}{(z-3)^3}\right]$$

$$\boxed{z = -1}$$

$$= \frac{-25}{64}$$

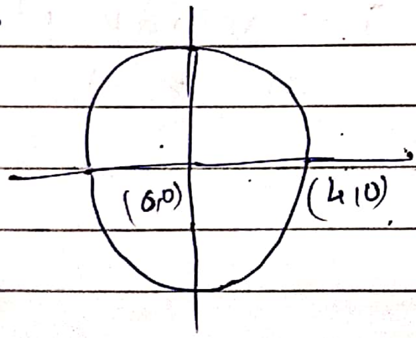
so

Qus Evaluate $\oint_C \frac{\sin(\pi \times z^2) + 2z}{(z-1)^2(z-2)} dz$.

wher c is circle $|z| = 4$.

$x^2 + y^2 = 4$ (x, y) $(0,0)$ $r = 4$.

Ans



$z - 1 = 0$ $z - 2 = 0$

$z = 1$. $z = 2$

$(1, 0)$ $(2, 0)$

both inside the circle.

order 2 order 1

$$\oint_C \frac{\sin \pi z^2 + 2z}{(z-1)^2(z-2)} dz = 2\pi i [r_1 + r_2]$$

$z^2 + \sin\pi = 0$

by residue theorem.

$$\oint_C f(z) \frac{\sin\pi z^2 + 2z}{(z-1)^2(z-2)} dz = 2\pi i(r_1 + r_2)$$

$r_1 =$ residue at $z = 1$

$$r_1 = \frac{1}{1!}\left[\frac{d}{dz} \cdot \cancel{(z-1)^2} \cdot \frac{(\sin\pi z^2 + 2z)}{\cancel{(z-1)^2}(z-2)}\right].$$

$$r_1 = \frac{(z-2)(2z\pi \cancel{\sin(\pi z^2)} \cdot \cos\pi z^2) + 2) - (\sin\pi z^2 + 2z)}{(z-2)^2}$$

$$r_1 = \frac{2z^2\pi \cos(\pi z^2) - 4z\pi \cos(\pi z^2) \cdot - \sin(\pi z^2) - 4}{(z-2)^2}.$$

$\boxed{z=1}$ $\quad r_1 = \frac{+2\pi - 4\pi(-1) - 4}{1} = \boxed{2\pi - 4}$

$$r_2 = \frac{1}{0!}\left[\frac{d^0}{dz^0}\left[\frac{\cancel{(z-2)} \cdot (\sin\pi z^2 + 2z)}{(z-1)^2 \cancel{(z-2)}}\right]\right]$$

$z = 2$.

$$\boxed{r_2 = 4 + \sin 4\pi}$$

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Que $\int_0^{2\pi} \frac{\sin 2\theta}{5+4\cos\theta}\, d\theta$. $|z|=1$

Ans $z = re^{i\theta}$
$= r(\cos\theta + i\sin\theta)$. $r=1$

~~$1+4(1+\cos\theta)$~~. ~~$\cos 2\theta =$~~

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$. $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

$= \frac{1}{2}\left[z+\frac{1}{z}\right]$ $= \frac{1}{2i}\left[z-\frac{1}{z}\right]$

$\int_0^{2\pi} \frac{\frac{1}{2i}\left(z^2-\frac{1}{z^2}\right)}{\frac{\cancel{4}^{2}}{\cancel{2}}\left(z+\frac{1}{z}\right)+5}\, d\theta$. $z = e^{i\theta}$.
$dz = ie^{i\theta}d\theta$.
$iz\,d\theta$.

$\int_0^{2\pi} \frac{z^4-1}{2i\cancel{z^2}\left(\frac{z^2+1+5z}{z}\right)} \frac{\cancel{dz}}{iz} \cancel{d\theta}$. $\theta \to 0 - \pi$.
$z \to 1 - +1$

$\int_1^1 \frac{z^4-1}{2(\cancel{z^2+5z+1})} \cancel{dz}$.

$\frac{z^4 \cancel{-1}}{\cancel{-1/2}}$ $\int \frac{\frac{1}{2i}\left(z^2-\frac{1}{z^2}\right)}{\left[5+\cancel{4}^{2}\times\frac{1}{\cancel{2}}\left(z+\frac{1}{z}\right)\right]} \frac{dz}{iz}$

$$= \frac{1}{2i} \frac{(z^4-1)}{\left(\frac{2z^2+2+5z}{z}\right) z^2} \frac{dz}{iz}.$$

$$= \oint \frac{-(z^4-1)}{2z^2(2z^2+2+5z)} dz.$$

$$= \oint \frac{(z^4-1)(-1/2)}{z^2(2z+1)(z+2)} dz.$$

Circle $|z|$.

$$\oint_0^{2\pi} \frac{(-1/2)(z^4-1)}{z^2(2z+1)(z+2)} = 2\pi i(r_1+r_2).$$

$r_1 \Rightarrow$ residue at $z=0$

$r_2 \Rightarrow$ residue at $z=-1/2$

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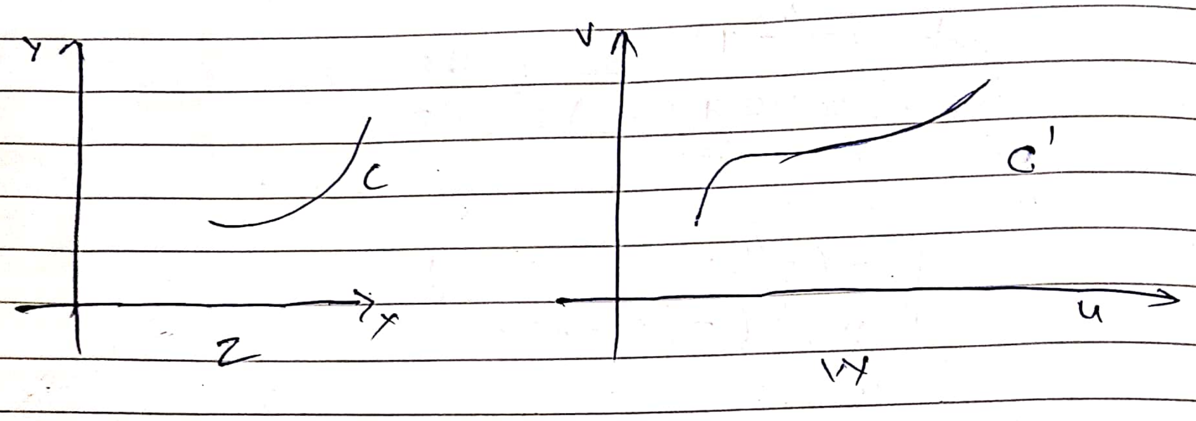
18/4/2019



if $w = f(z)$ is analytic function which transform the region R (curve C) in the z plane to the region R' (curve C') in the w plane then the transformation is called confirmal transformation.

provide $f'(z) \neq 0$.

Some General transformation :-

1) translation : if $w = z + h$ by this transformation figures in the z plane translated in the direction of h where $h = h_1 + i h_2$

2) Rotation :-

if $w = z e^{i\theta_0}$ by this transformation figures in z plane are rotated through an angle θ_0, if $\theta_0 > 0$ the rotation is anticlockwise while at $\theta_0 < 0$ the rotation is clockwise.

3) Stretching = $w = a * z$

~~by this~~ by this transformation figures in z plane are stretched or contracted in the direction of z if a is > 0.

4) Inversion : if $w = 1/z$ in this transformation figures in z plane are mapped upon the reciprocal figure in w plane

5) linear transform-ation :-

tran. of the form $w = az + b$ is called a linear transformation where a and b are complex constant.

6) bilinear transformation (Mobius) :-

a transformation of the type

$$w = \frac{az+b}{cz+d} \quad \text{or} \quad w = \frac{a+bz}{c+dz}$$

is called a bilinear transformation where a, b, c, d are complex constant and $ad - bc \neq 0$.

Que find a bilinear transformation which maps the point $0, -1, i$ of the z plane onto the points $2, \infty, \frac{(5+i)}{2}$ in w plane.

let bilinear transformation $\omega = \dfrac{a+bz}{c+dz}$

$z=0$ & $\omega = 2$

$$2 = \frac{a}{c} \Rightarrow \boxed{a = 2c}$$

$z=-1$ & $\omega=\infty$

$$\infty = \frac{a-b}{c-d} \qquad \boxed{c=d}$$

$\&(a \neq b)$

$z = i$ & $\omega = \frac{1}{2}(5+i)$

$$\frac{1}{2}(5+i) = \frac{a+bi}{c+di}$$

$$\frac{5+i}{2} = \frac{2c+bi}{(c+ci)}$$

$$2c+bi = \frac{5c+5ci+ci\,\cancel{}-c}{2}$$

$$bi+2c = \frac{4c+6ci}{2} = 2c+3ci$$

$$bi = 3ci \Rightarrow \boxed{b = 3c}$$

$$\omega = \frac{a+bz}{c+dz} = \frac{2c+3cz}{c+cz} = \frac{2+3z}{1+z}$$

$$\boxed{\omega = \frac{2+3z}{1+z}}$$

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Qu

find a bilinear transformation which maps the point $z = 1, i, 2i$ on the points $w = -2i, \cancel{0}, 0, 1$ respectively.

$$w = \frac{a + bz}{c + dz}$$

$z = 1$ $\quad w = -2i = \dfrac{a+b}{c+d}$ $\quad a = -2i(c+d) - b$

$z = i$ $\quad w = 0 = \dfrac{a + bi}{c + di}$ $\quad c \neq d.$

$$\boxed{a = -bi}$$

$z = 2i$ $\quad w = 1 = \dfrac{a + 2bi}{c + d2i}$

$c + 2di = a + 2bi.$

$\cancel{c + 2di = bi}$

$\boxed{\cancel{a} = \cancel{c + 2di} - 2bi}$

$\cancel{w = a+}$ $\quad c - a = 2i(d - b)$

$b(1 - i) = -2i(c + d).$

$b\cancel{(1-i)} = -\cancel{2i}(a + b)$

$\cancel{-2i}.$

$$w = \frac{(2 - 4i) + (2i + 4)z}{2i + (1 + i)z}$$

Show that $w = \frac{2z+3}{z-4}$ by this transformation transform circle $x^2+y^2-4x=0$ into the st. line $4u+3=0$

Given bilinear transformation

$$w = u + iv = \frac{2z+3}{z-4} \qquad z = x+iv.$$

$$= \frac{2(x+iy)+3}{x+iy-4}.$$

$$= \frac{(2x+3)+(2y)i}{x-4+yi}$$

$$= \frac{((2x+3)+2yi)((x-4)-iy)}{(x-4)^2+y^2}.$$

$$= \frac{(2x+3)(x-4)+2y^2}{(x-4)^2+y^2} + \frac{(2y(x-4)-y(2x+3))i}{(x-4)^2+y^2}.$$

$$= \frac{\cancel{(2x^2+3)}}{u+iv}.$$

$$u = \frac{(2x+3)(x-4)+2y^2}{(x-4)^2+y^2} = \frac{2x^2-5x-12+2y^2}{(x-4)^2+y^2}.$$

$$v = \frac{-11yi}{(x-4)^2+y^2}$$

Given that in w plane st. line

L

$4u+3=0$.

$$\Rightarrow \frac{4(2x^2+2y^2-5x-12)}{(x-4)^2+y^2}+3=0$$

$$8x^2+8y^2-20x-48+3x^2+\cancel{48}-24x+3y^2=0$$

$$11x^2+11y^2-44x=0$$

$$x^2+y^2-4x=0$$

$$(x-2)^2+y^2=4.$$

which is the circle in XOY z plane.

Q. Find a map of the st. line $y=x$. under transformtn

$$w=\frac{z-1}{z+1} \qquad z=x+yi.$$

$$=\frac{x-1+yi}{x+1+yi}. \quad (x-1$$

$$w=\frac{x^2-1}{(x+1)^2+y^2}$$

$$\frac{w}{1}=\frac{z-1}{z+1} \quad \text{using componendo. dividendo}$$

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}. \qquad \frac{w-1}{w+1}=\frac{z-1-z+1}{z-1+z+1}$$

$$\frac{w+1}{w-1}=\frac{2z}{-2}=-z$$

$u^2+(v+1)^2=2$