6 Vector Differential Calculus

$$\bar{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$$

$$\bar{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

$$|\bar{a}| = \sqrt{a_1^2 + b_1^2 + c_1^2} = a$$

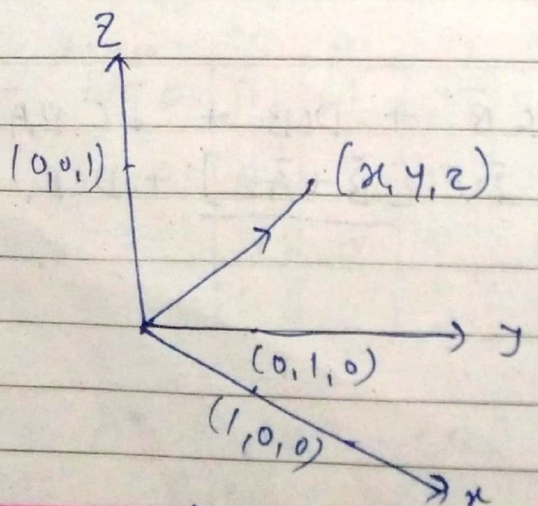
$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{(a_1\hat{i} + b_1\hat{j} + c_1\hat{k})}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

$$\bar{a} \cdot \bar{b} = ab\cos\theta$$

$\bar{a} \times \bar{b} = ab\sin\theta\, \hat{n}$, where $\hat{n}$ is unit normal to plane of $\bar{a}$,

$$\boxed{\bar{a} \cdot \bar{b} = a_1 a_2 + b_1 b_2 + c_1 c_2}$$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \qquad \bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$$



$$(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$$

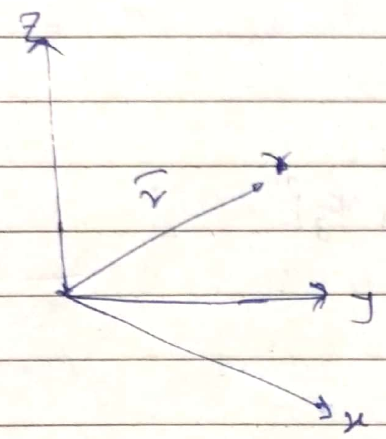
Angle θ betn $\bar{a}$ & $\bar{b}$ is given by

$$\theta = \cos^{-1}\left[\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2} \cdot \sqrt{a_2^2+b_2^2+c_2^2}}\right]$$

$\bar{a}$ & $\bar{b}$ are orthogonal iff
$a_1a_2 + b_1b_2 + c_1c_2 = 0$.

$\bar{a} \parallel \bar{b}$ iff

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$



$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Tangent vector to curve ~~at~~ $\bar{r}$ is

$$\bar{T} = \frac{d\bar{r}}{dt}$$

$$\boxed{\bar{T} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}}$$

velocity vector :-

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Accl^n vector :-

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

* Scalar Field :-
$\phi(x, y, z)$

* Vector Field :-

$$F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

* Surface :-

$$\phi(x, y, z) = c$$

* $$\boxed{\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}}$$ ← derivative operator

* Gradient of a scalar field $\phi(x, y, z)$
= Grad ϕ = $\nabla\phi$

$$\boxed{\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}}$$

$$\nabla (u \pm v) = \nabla u \pm \nabla v$$

$$\nabla (u \cdot v) = u \nabla v + v \nabla u$$

$$\nabla \frac{u}{v} = \frac{v \nabla u - u \nabla v}{v^2}$$

$$\nabla f(u) = f'(u) \nabla u$$

* $$\boxed{\nabla f(r) = \frac{f'(r)}{r} \bar{r}}$$

Normal vector $\hat{n}$ to the surface $f(x, y, z) = 0$ is

$$\bar{n} = \nabla f$$

$$\boxed{\bar{n} = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}}$$

e.g. If $f: 2x + 3y + 4z - 7 = 0$

$\bar{n} = 2\hat{i} + 3\hat{j} + 3\hat{k}$

Note:-

1) vector at point (x_1, y_1, z_1) towards point (x_2, y_2, z_2) is

$$\bar{a} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

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2) Unit vector equally inclined with the axes is $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$.

3) Vector along line

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

is $l\hat{i} + m\hat{j} + n\hat{k}$.

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* Directional ~~Dir~~ Derivative (D.D.)

D.D. of scalar field $\phi(x,y,z)$ along the vector $\bar{a} = \nabla\phi \cdot \hat{a}$

Q.1 Find the D.D. of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along tangent to the curve

$x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$, at $t=0$

Sol^n Required D.D is

$$\bar{D}\cdot\bar{D} = \nabla\phi \cdot \hat{d}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$\therefore \phi = 4xz^3 - 3x^2y^2z$

$$\frac{\partial \phi}{\partial x} = 4z^3 - 6xy^2z$$

$$\frac{\partial \phi}{\partial y} = 0 - \cancel{12x} \; 6x^2yz$$

$$\frac{\partial \phi}{\partial z} = 12xz^2 - 3x^2y^2$$

$$\nabla\phi = (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}.$$

at $(2, -1, 2)$

$$\boxed{\nabla\phi = 8\hat{i} + 48\hat{j} + 84\hat{k}}$$

$\frac{dy}{dt}$ ~~$\bar{T} = \frac{dx}{dt}\hat{i} + \frac{dy}{dz}\hat{j} + \frac{dz}{dt}\hat{k}.$~~

$$\bar{T} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}.$$

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Q. Find the D.D of f at $(1,2,-1)$ for $f = x^2y + xyz + z^3$ along normal to the surface $x^2y^2 = 4xy + y^2z$ at the point $(1,2,0)$

Sol^n $\because$ $D.D. = \nabla f \cdot \hat{n}$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$\because$ $f = x^2y + xyz + z^3$

$\therefore$ $\cancel{\frac{\partial f}{\partial x} = 2xy + yz + 3z}$

$$\nabla f = [2xy + yz]\hat{i} + [x^2 + xz]\hat{j} + [xy + 3z^2]$$

At $(1,2,-1)$

$$\nabla f = [2\hat{i} + 0\hat{j} + 5\hat{k}]$$

$\bar{n}$ is normal to the surface

$$g(x,y,z) = x^2y^2 - 4xy - y^2z$$

$$\bar{n} = \nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}$$

$$\bar{n} = [2xy^2 - 4y]\hat{i} + [2yx^2 - 4x - 2yz]\hat{j} + [-y^2]\hat{k}$$

at $(1,2,0)$

$\bar{n} = 0\hat{i} + 0\hat{j} - 4\hat{k}$

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|} = \frac{-4\hat{k}}{\sqrt{16}}$$

~~$\hat{n} = 0\hat{i} + 0\hat{j} - 4\hat{k}$~~

$\hat{n} = 0\hat{i} + 0\hat{j} - \hat{k}$

$\because$ D.D $= \nabla f \cdot \hat{n}$

$= [2\hat{i} + 0\hat{j} + 5\hat{k}] \cdot [0\hat{i} + 0\hat{j} - \hat{k}]$

$= 2 \cdot 0 + 0 \cdot 0 + 5 \cdot (-1)$

D.D. $= -5$

Q. Find the directional derivative of ~~of~~ $xy^2 + yz^3$ at $(2, -1, 1)$ along the line ~~to~~ $2(x-2) = y+1 = z-1$

Sol^n: $f = xy^2 + yz^3$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$\nabla f = y^2\hat{i} + [2xy + z^3]\hat{j} + 3z^2y\,\hat{k}$

At $(2, -1, 1)$

$\nabla f = \hat{i} - 3\hat{j} - 3\hat{k}$

$\hat{n}$ is along the line

$2(x-2) = y+1 = z-1$

$$\frac{x-2}{\frac{1}{2}} = \frac{y+1}{1} = \frac{z-1}{1}$$

Drs are $\frac{1}{2}, 1, 1$ or $1, 2, 2$

$\bar{n} = \cancel{2\hat{i} + \hat{j} + \hat{k}}$ $\hat{i} + 2\hat{j} + 2\hat{k}$

$$\hat{n} = \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$$

$$\hat{n} = \frac{2}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$$

$D.D. = \nabla f \cdot \hat{n}$

$$= [\hat{i} - 3\hat{j} - 3\hat{k}] \cdot \left[\frac{2}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}\right]$$

$$= \frac{2}{\sqrt{6}} - \frac{3}{\sqrt{6}} - \frac{3}{\sqrt{6}} = \frac{-4}{\sqrt{6}}$$

$$\hat{n} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$D.D. = [\hat{i} - 3\hat{j} - 3\hat{k}] \cdot \left[\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right]$$

$$= \frac{1}{3} - 2 - 2 = \frac{1}{3} - 4 = \frac{1-12}{3}$$

$$= \frac{-11}{3}$$

Note:-

D.D. of ϕ is max^m along $\nabla\phi$

Magnitude of maximum D.D of ϕ = $|\nabla\phi|$

Q.* If the directional derivative of $\phi = axy + byz + czx$ at $(1,1,1)$ has maximum magnitude 4 in the direction parallel to x-axis. Find the values of a, b, c.

Sol^n: Given $\nabla\phi$ is max. along x-axis — ①

$$|\nabla\phi| = 4 \quad — ②$$

$$\phi = axy + byz + czx$$

$$\nabla\phi = ay\,\hat{i} + bz\,\hat{j} + cx\,\hat{k}$$

at $(1,1,1)$ $= a\hat{i} + b\hat{j} + c\hat{k}$

$$|\nabla\phi| = \sqrt{a^2y^2 + b^2z^2 + c^2x^2}$$

$$16 = a^2y^2 + b^2z^2 + c^2x^2$$

~~at (1,1,1)~~

From ①

$$D.D. = (a\hat{i} + b\hat{j} + c\hat{k})(\hat{i} + 0\hat{j} + 0\hat{k})$$

$$D.D. = a$$

$\phi = axy + byz + cxz$

$\nabla\phi = (ay + cz)\hat{i} + (ax + bz)\hat{j} + (by + cx)\hat{k}$

at $(1, 1, 1)$,

$$\nabla\phi = (a+c)\hat{i} + (a+b)\hat{j} + (b+c)\hat{k}$$

Given D.D. is max. along x-axis with magnitude 4.

But D.D. ~~sh~~ is max. along $\nabla\phi$.

$\therefore \quad \nabla\phi = 4\hat{i} + 0\hat{j} + 0\hat{k}$.

$\therefore \quad a + c = 4, \qquad a + b = 0 \qquad b + c = 0$

$\qquad\qquad\qquad\quad a = -b \qquad\quad c = -b$

$-b - b = 4$

$\boxed{b = -2} \quad \boxed{a = 2} \quad \boxed{c = 2}$

Q.

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* Divergence of vector Field :-

$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Divergence of $\bar{F}$ = div $\bar{F}$

div $\bar{F} = \nabla.\bar{F}$

$$\boxed{\nabla.\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

* Solenoidal vector field :-

vector field $\bar{F}$ is solenoidal iff $\nabla.\bar{F} = 0$

If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\boxed{\nabla.\bar{r} = 3}$$

Q Show that

$\bar{F} = \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}$ is solenoidal vector field.

Sol: $\nabla.\bar{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$

$\nabla.\bar{F} = 0$

$\bar{F}$ is solenoidal & vector field

* Curl of a vector field

$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

curl $\bar{F} = \nabla \times \bar{F}$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

* Irrotational vector field :-
$\bar{F}$ is irrotational iff $\nabla \times F = 0$.

* Conservative vector field :-
$\bar{F}$ is conservative iff $\bar{F}$ is continuous
& ~~$\Delta\times$~~ $\nabla \times \bar{F} = 0$.

Note:- If $\nabla \times \bar{F} = 0$, then there exist scalar potential ϕ such that $\bar{F} = \nabla\phi$

If $\nabla \times \bar{F} = 0$
To find scalar field ~~of~~ ϕ such that

$\bar{F} = \nabla\phi$

$d\phi = \bar{F} \cdot d\bar{r}$

$d\phi = F_1 dx + F_2 dy + F_3 dz$

$$\phi = \int_{y,z \text{ const}} F_1\,dx + \int_{z\text{-const}} \begin{bmatrix} \text{Terms of } F_2 \\ \text{free from } x \end{bmatrix} dy + \int \begin{bmatrix} \text{Terms of } F_3 \\ \text{free from } x,y \end{bmatrix} dz + c$$

Q. Show that vector field $\bar{F} =$
$\bar{F} = (2xz^3 + 6y)\hat{i} + (6x - 2yz)\hat{j} + (3x^2z^2 - y^2)\hat{k}$
is irrotational, hence find corresponding scalar field ϕ such that $\bar{F} = \nabla\phi$.

Sol^n

$$\cancel{\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2z^3 + 6 + 6xz^2 & 6 - 2z - 2y & 6xz^2 - 2y + 6x^2z \\ 2xz^3 + 6y & 6x - 2yz & 3x^2z^2 - y^2 \end{vmatrix}}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^3 + 6y & 6x - 2yz & 3x^2z^2 - y^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3x^2z^2 - y^2) - \frac{\partial}{\partial z}(6x - 2yz)\right]$$
$$- \hat{j}\left[\frac{\partial}{\partial x}(3x^2z^2 - y^2) - \frac{\partial}{\partial z}(2xz^3 + 6y)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(6x - 2yz) - \frac{\partial}{\partial y}(2xz^3 + 6y)\right]$$

$= \hat{i}(-2y+2y) - \hat{j}(6xz^2 - 6xz^2) +$

$\hat{k}(6-6)$

$= 0$

$\Rightarrow \bar{F}$ is irrotational

To find scalar potential ϕ such that

$\bar{F} = \nabla\phi$

$d\phi = \bar{F}\cdot d\bar{r}$

$d\phi = (2xz^3+6y)\,dx + (6x-2yz)\,dy +$
$(3x^2z^2-y^2)\,dz$ — exact diff. eqn

$$\phi = \int_{y,z\ \text{const}} F_1\,dx + \int_{z\text{-const}} \left[\begin{matrix}\text{terms of } F_2\\ \text{free from } x\end{matrix}\right] dy + \int \left[\begin{matrix}\text{terms of } F_3\\ \text{free from } x,y\end{matrix}\right] dz + c$$

$$= \int_{y-z\ \text{const}} (2xz^3+6y)\,dx + \int_{z\ \text{const}} -2yz\,dy + \int 0\,dz + c$$

$$= \frac{2z^3x^2}{2} + 6xy - \frac{2zy^2}{2} + c$$

$$\boxed{\phi = x^2z^3 + 6xy - y^2z + c}$$

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Q. If $\bar{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational then find values of a, b, c and determine scalar field ϕ such that $\bar{F} = \nabla\phi$

Sol^n Given that $\bar{F}$ is irrotational

$\Rightarrow$

$$\nabla \times \bar{F} = 0$$

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\therefore \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0$$

$\Rightarrow$ $c+1=0$ $\quad \boxed{c=-1}$

$4-a=0$ $\quad \boxed{a=4}$

$b-2=0$ $\quad \boxed{b=2}$

$$\therefore \bar{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

To find the scalar field ϕ such that $\bar{F} = \nabla\phi$

$$d\phi = \bar{F}\cdot d\bar{r}$$

$$\therefore d\phi = (x+2y+4z)dx + (2x-3y-z)dy + (4x-ey+2z)dz$$

$$\therefore \phi = \int_{y,z\ const} (x+2y+4z)\,dx + \int_{z\ const} (\cancel{2x}-3y-z)\,dy$$

$$+ \int 2z\,dz + c$$

$$\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - zy + z^2$$

Q. Show that $\bar{F} = ye^{xy}\cos z\,\hat{i} + xe^{xy}\cos z\,\hat{j} - e^{xy}\sin z\,\hat{k}$ is conservative & hence find corresponding scalar potential ϕ such that $\bar{F} = \nabla\phi$

Sol^n: $\bar{F} = ye^{xy}\cos z\,\hat{i} + xe^{xy}\cos z\,\hat{j} - e^{xy}\sin z\,\hat{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xy}\cos z & xe^{xy}\cos z & -e^{xy}\sin z \end{vmatrix}$$

$$= \hat{i}\left(-xe^{xy}\sin z + xe^{xy}\sin z\right) - \hat{j}\left(-ye^{xy}\sin z - ye^{xy}\sin z\right)$$

$$\cancel{+\hat{k}\left(-xe^{xy}\sin z\right.}$$

$$+\hat{k}\left(x\cdot ye^{xy}\cos z + e^{xy}\cos z - yxe^{xy}\cos x - e^{xy}\cos z\right)$$

$$= 0$$

$\therefore$ $\bar{F}$ is conservative as $\bar{F}$ is continuous everywhere

To find the scalar field ϕ such that $\bar{F} = \nabla \phi$

$$d\phi = \bar{F} \cdot dr$$

$$d\phi = (ye^{xy} \cos z)\, dx + (xe^{xy} \cos z) dy - (e^{xy} \sin z)\, dz$$

$$\therefore \phi = \int_{y,z\ \text{const}} ye^{xy} \cos z \, dx + \int_{z-\text{const}} 0\, dy - \int 0\, dz + c$$

$$= \frac{y \cos z\, e^{xy}}{y} + 0 \cancel{-0} + c$$

$$\boxed{\phi = e^{xy} \cos z + c}$$

* Formulae

$\nabla \cdot \bar{r} = 3$ $\qquad$ $\nabla \times \bar{r} = 0$

$$\nabla f(r) = \frac{f'(r)}{r} \bar{r} = \frac{1}{r} \left[\frac{d}{dr} f(r) \right] \bar{r}$$

$\bar{r} \cdot \bar{r} = r^2$ $\qquad$ $\bar{r} \times \bar{r} = 0$

$$\bar{a} \times \bar{b} \times \bar{c} = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) . \bar{c}$$

$$\nabla (\bar{a} \cdot \bar{r}) = \bar{a}$$

$$\cancel{\nabla f} \ \nabla \times (\bar{a} \times \bar{r}) = 2\bar{a}$$

$\nabla r^n = n r^{n-2} \bar{r}$

* Vector Identity
  ϕ is scalar field
  $\bar{u}, \bar{v}$ are vector field.

1) $\nabla \cdot (\phi \bar{u}) = \nabla \phi \cdot \bar{u} + \phi (\nabla \cdot \bar{u})$

2) $\nabla \times (\phi \bar{u}) = \nabla \phi \times \bar{u} + \phi (\nabla \times \bar{u})$

3) $\nabla \cdot (\bar{u} \times \bar{v}) = (\nabla \times \bar{u}) \cdot \bar{v} - (\nabla \times \bar{v}) \cdot \bar{u}$

Q. Evaluate

1) $\nabla \cdot r^3 \bar{r}$

Sol^n: As $\nabla \cdot (\phi \bar{u}) = \nabla \phi \cdot \bar{u} + \phi (\nabla \cdot \bar{u})$

Here, $\phi = r^3$, $\bar{u} = \bar{r}$.

$\therefore \nabla \cdot (r^3 \bar{r}) = \nabla r^3 \cdot \bar{r} + r^3 (\nabla \cdot \bar{r})$

But

$\nabla \cdot \bar{r} = 3$

$\nabla r^3 = \frac{1}{r} \left[\frac{d}{dr} r^3 \right] \bar{r} = \frac{1}{r} 3r^2 \cdot \bar{r}$

$= 3 r \bar{r}$

$\therefore \nabla \cdot (r^3 \bar{r}) = 3 r \bar{r} \cdot \bar{r} + r^3 3$

$= 3 r^3 + 3 r^3$

$= 6 r^3$

$$\nabla \frac{1}{r^n} = \frac{-n}{r^{n+2}} \bar{r}$$

$\bar{r}$

Q. $\nabla \cdot \left[r \nabla \frac{1}{r^2} \right]$ $\frac{-2}{r^3}$

Sol.

$$\nabla \frac{1}{r^2} = \cancel{\frac{1}{r} f'\left(\frac{1}{r^2}\right)} \quad \frac{1}{r}\left(\frac{d}{dr}\frac{1}{r^2}\right)\bar{r}$$

$$= \frac{1}{r}\left(-\frac{2}{r^3}\right)\bar{r}$$

$$= \cancel{\frac{r}{r^2}} \quad \cancel{-\frac{2}{r^4}} \quad -\frac{2}{r^4}\bar{r}$$

$$\therefore \nabla \cdot \left[r \left(\frac{-2}{r^4}\right)\bar{r} \right] = \nabla \cdot \left[\frac{-2\bar{r}}{r^3} \right]$$

As $\nabla(\phi \bar{u}) = \nabla\phi \cdot \bar{u} + \phi(\nabla \cdot \bar{u})$

$$\phi = \frac{-2}{r^3} \qquad \bar{u} = \bar{r}$$

$$\therefore \nabla \cdot \left[\frac{-2}{r^3}\bar{r} \right] = \nabla\left(\frac{-2}{r^3}\right) \cdot \bar{r} + \left(\frac{-2}{r^3}\right)(\nabla \cdot \bar{r})$$

$$= -2 \cdot \left[\nabla\left(\frac{1}{r^3}\right) \cdot \bar{r} + \frac{3}{r^3} \right]$$

$$= -2\left[-\frac{3}{r^5}\bar{r} \cdot \bar{r} + \frac{3}{r^3} \right]$$

$$= -2\left[-\frac{3}{r^3} + \frac{3}{r^3} \right] = 0$$

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

or

$$\nabla^2 f(r) = \frac{d^2}{dr^2} f(r) + \frac{2}{r} \frac{d}{dr} f(r)$$

Q. show that:

$$\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$$

Sol:

$$\text{L.H.S.} = \nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right)$$

$$= \frac{r^n \nabla(\bar{a}\cdot\bar{r}) - (\bar{a}\cdot\bar{r})\nabla r^n}{(r^n)^2}$$

but $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$

$$\nabla r^n = n r^{n-2}\bar{r}$$

$$\text{L.H.S.} = \frac{r^n\bar{a} - (\bar{a}\cdot\bar{r}) n r^{n-2}\bar{r}}{r^{2n}}$$

$$= \frac{r^n\bar{a}}{r^{2n}} - \frac{(\bar{a}\cdot\bar{r}) n r^{n-2}\bar{r}}{r^{2n}}$$

$$= \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$$

$= \text{R.H.S}$

Q. S.T. $\nabla \times \frac{\bar{a} \times \bar{r}}{r^n} = \frac{2-n}{r^n}\,\bar{a} + \frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$

Sol: L.H.S. $= \nabla \times \left[\frac{1}{r^n}(\bar{a} \times \bar{r})\right]$

As $\nabla \times \phi\bar{u} = \nabla\phi \times \bar{u} + \phi(\nabla \times \bar{u})$

L.H.S. $= \nabla \frac{1}{r^n} \times (\bar{a} \times \bar{r}) + \frac{1}{r^n}(\nabla \times (\bar{a} \times \bar{r}))$

$= \left[\frac{-n\,\bar{r}}{r^{n+2}}\right] \times (\bar{a} \times \bar{r}) + \frac{1}{r^n}\,2\bar{a}$

$= \frac{-n}{r^{n+2}}[\bar{r} \times \bar{a} \times \bar{r}] + \frac{2\bar{a}}{r^n}$

as $\bar{a} \times (\bar{b} \times \bar{c}) = [(\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}]$

$\therefore$ L.H.S. $= \frac{-n}{r^{n+2}}[(\bar{r}\cdot\bar{r})\,\bar{a} - (\bar{r}\cdot\bar{a})\bar{r}] + \frac{2\bar{a}}{r^n}$

$= \frac{-n}{r^{n+2}}[r^2\bar{a} - (\bar{r}\cdot\bar{a})\,\bar{r}] + \frac{2\bar{a}}{r^n}$

$= \frac{-n}{r^{n+2}}[r^2\bar{a}] + \frac{n}{r^{n+2}}(\bar{r}\cdot\bar{a})\bar{r} + \frac{2\bar{a}}{r^n}$

$= \frac{-n\bar{a}}{r^{n+2-2}}$

$= \frac{2-n}{r^n}\,\bar{a} + \frac{n}{r^{n+2}}(\bar{a}\cdot\bar{r})\bar{r}$

$=$ RHS.

Q S.T. $\nabla^4 r^2 \log r = \frac{6}{r^2}$

Sol: L.H.S. $= \nabla^4 r^2 \log r$

$= \nabla^2 \nabla^2 r^2 \log r$

$= \nabla^2 \left[\nabla^2 (r^2 \log r) \right]$ —— (1)

$= \nabla^2 \left[\frac{d^2}{dr^2} r^2 \log r + \frac{2}{r} \frac{d}{dr} r^2 \log r \right]$

$= \nabla^2 \left[\frac{d}{dr} \left[\frac{r^2}{r} + 2r \log r \right] + \frac{2}{r} \left(\frac{r^2}{r} + \log r . 2r \right) \right]$

$= \nabla^2 \left[\frac{d}{dr} r(1 + 2\log r) + \frac{2}{r} [1 + 2 \log r] \right]$

$= \nabla^2 \left[r\left(\frac{2}{r}\right) + 1 + 2\log r + 2(1 + 2\log r) \right]$

$= \nabla^2 [5 + 6 \log r]$

$= \frac{d^2}{dr^2} (5 + 6 \log r) + \frac{2}{r} \frac{d}{dr} 5 + 6 \log r$

$= \frac{d}{dr} \frac{6}{r} + \frac{2}{r} \times \frac{6}{r} = -\frac{6}{r^2} + \frac{12}{r^2} = \frac{6}{r^2}$

$=$ R.H.S.

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Q. S.T. vector field $f(r)\bar{r}$ is always irrotational. Hence i) Find $f(r)$ so that vector field $f(r)\bar{r}$ is solenoidal.

ii) find $f(r)$ so that $\nabla^2 f(r) = 0$.

Sol^n: To show $f(r)\bar{r}$ irrotational.

consider curl $f(r)\bar{r} = \nabla \times f(r)\bar{r}$

$$= \nabla f(r) \times \bar{r} + f(r)(\nabla \times \bar{r})$$

But $\nabla \times \bar{r} = 0$.

$$= \nabla f(r) \times \bar{r}$$

$$= \frac{f'(r)}{r} \bar{r} \times \bar{r}$$

$$= \frac{f'(r)}{r} (\bar{r} \times \bar{r})$$

$$= 0 \quad \dots (\text{as } \bar{r} \times \bar{r} = 0)$$

Hence, $f(r)\bar{r}$ is always irrotational.

To find $f(r)$ so that $f(r)\bar{r}$ is solenoidal

$f(r).\bar{r}$ is solenoidal if

$$\nabla . f(r). \bar{r} = 0.$$

$$\Rightarrow \nabla f(r) . \bar{r} + f(r)(\nabla . \bar{r}) = 0.$$

$$\Rightarrow \frac{f'(r)}{r} \bar{r} . \bar{r} + f(r)\, 3 = 0.$$

$\Rightarrow f'(r)\, r + 3f(r) = 0$

$\Rightarrow f'(r) = \dfrac{-3f(r)}{r}$

$\cancel{f(r) = \int -\dfrac{3f(r)}{r}\, dr}$

$\dfrac{f'(r)}{f(r)} = \dfrac{-3}{r}$

$\therefore \int \dfrac{f'(r)}{f(r)}\, dr = -\int \dfrac{3}{r}\, dr$

$\log[f(r)] = -3\log r + \log c_1$

$= \log r^{-3} + \log c_1$

$\log\left(\dfrac{c_1}{r^3}\right)$

$\boxed{f(r) = \dfrac{c_1}{r^3}}$

ii) To find $f(r)$ so that $\nabla^2 f(r) = 0$.

$\nabla^2 f(r) = 0.$

$\Rightarrow \cancel{\dfrac{d^2}{dr^2} f(r) + \dfrac{2}{r}}$

$f''(r) + \dfrac{2}{r} f'(r) = 0$

$\dfrac{f''(r)}{f'(r)} = \dfrac{-2}{r}$

$\int \dfrac{f''(r)}{f'(r)}\, dr = \int -\dfrac{2}{r}\, dr$

$$\Rightarrow \log[f'(r)] = -2\log r + \log c_2$$

$$f'(r) = \frac{c_2}{r^2}$$

$$\int f'(r).\,dr = \int \frac{c_2}{r^2}.\,dr$$

$$\boxed{f(r) = \frac{-c_2}{r} + c_3}$$

25/04/19

* Line Integral:

$\bar{F} = \cancel{F_1} F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$L = W = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \bar{F}\cdot d\bar{r}$$

C

$$= \int_C F_1\,dx + F_2\,dy + F_3\,dz.$$

If C is st. line,

then put $x = x_1 + (x_2 - x_1)t$

$y = y_1 + (y_2 - y_1)t$

$z = z_1 + (z_2 - z_1)t.$

$0 \leq x/y/z \leq 1$

Q. Evaluate $\int_C \bar{F}.d\bar{r}$

for $\bar{F} = 3x^2\,\bar{i} + (2xz - y)\,\bar{j} + z\,\bar{k}$
where C is the st. line joining the points (0,0,0) & (2,1,3).

Soln $x_1 = 0,\quad y_1 = 0,\quad z_1 = 0$
$x_2 = 2,\quad y_2 = 1\quad z_2 = 3$

$$\int \bar{F}\cdot d\bar{r} = \int_{(0,0,0)}^{(2,1,3)} F_1 dx + F_2 dy + F_3 dz$$

$$= \int_{(0,0,0)}^{(2,1,3)} 3x^2\,dx + (2xz - y)\,dy + z\,dz$$

$x = 0 + (2-0)t$
$\Rightarrow x = 2t \quad \Rightarrow dx = 2dt$

$y = 0 + (1-0)t$
$y = t \quad \Rightarrow \quad dy = dt$

$z = 0 + (3-0)t$
$z = 3t \quad \Rightarrow \quad dz = 3dt.$

$$\therefore \int \bar{F}\cdot d\bar{r} = \int_0^1 3(2t)^2\,2\,dt + [2(2t)(3t) - t]\,dt + (3t)\cdot 3dt.$$

$$= \int_0^1 24\,t^2 dt + (12t^2 - t)\,dt + 9t\,.dt$$

$$= \int_0^1 (36t^2 + 8t) \cdot dt$$

$$= 4 \int_0^1 (9t^2 + 2t) dt$$

$$= 4 \left[3t^3 + t^2 \right]_0^1$$

$$= 4 [3 + 1 - 0] = 16$$

Q. Evaluate $\int \bar{F} \cdot d\bar{r}$
for $\bar{F} = (2x + y^2)\bar{i} + \cancel{(3y - 4x)}\ (3y - 4x)\bar{j}$
where 'c' is the parabolic arc $y = x^2$
joining $(0,0)$ & $(1,1)$.

Sol^n: $I = \int_{(0,0)}^{(1,1)} (2x + y^2) dx + (3y - 4x) dy$

$c : \quad y = x^2$

$\cancel{x}$ $\quad \therefore dy = 2x\, dx$

$0 \leq x \leq 1$ (limit of x).

$$I = \int_0^1 [2x + (x^2)^2] dx + [3(x^2) - 4x] 2x dx$$

$$= \int_0^1 x^4 + 6x^3 - 8x^2 + 2x \; dx$$

$$= \left[\frac{x^5}{5} + \frac{6x^4}{4} - \frac{8x^3}{3} + x^2 \right]_0^1$$

$$= \frac{1}{5} + \frac{3}{2} - \frac{8}{3} + 1$$

$$= \frac{6 + 45 - 80 + 30}{30} = \frac{1}{30}$$

Q. Evaluate $\int \bar{F} \cdot d\bar{r}$

for $\bar{F} = (2x+3y)\, i + xz\, j + (yz - x)\, k$

along the path

$x^2 = 2t^2 \quad y = t \quad z = t^3 \quad t = 0$ to $t = 1$

Sol^n $I = \int \bar{F} \cdot d\bar{r}$

$$= \int (2x+3y)\,dx + xz\,dy + (yz - x)\,dz$$

$\therefore x^2 = 2t^2 \quad \Rightarrow x = \sqrt{2}\,t$

$2x \cdot dx = 4t\, dt$

$$dx = \frac{2t}{\sqrt{2}t}\, dt$$

$dx = \sqrt{2} \cdot dt$

$y = t$

$dy = dt$

$z = t^3$

$dz = 3t^2\, dt$

$$\therefore I = \int_0^1 (2\sqrt{2}\,t + 3t)\sqrt{2}\,dt + \sqrt{2}\,t \cdot t^3\, dt + (t \cdot t^3 - \sqrt{2}\,t)\, 3t^2\, dt$$

$$= \int_0^1 (4t + 3\sqrt{2}\,t + \sqrt{2}\,t^4 + 3t^6 - 3\sqrt{2}\,t^3$$

$$= \int_0^1 3t^6 + \sqrt{2}\,t^4 - 3\sqrt{2}\,t^3 + (4 + 3\sqrt{2})t$$

$$\frac{60 + 28\sqrt{2} - 105\sqrt{2}}{140} + 280 + 210\sqrt{2}$$

$$= \left[\frac{3t^7}{7} + \frac{\sqrt{2}\,t^5}{5} - \frac{3\sqrt{2}\,t^4}{4} + \frac{(4+3\sqrt{2})\,t^2}{2} \right]_0^1$$

$$= \left[\frac{3}{7} + \frac{\sqrt{2}}{5} - \frac{3\sqrt{2}}{4} + \frac{4+3\sqrt{2}}{2} \right] - 0$$

$$= \frac{15 + 7\sqrt{2}}{35} - \frac{3\sqrt{2} + 8 + 6\sqrt{2}}{4} \Big]$$

$$= \left(\frac{15 + 7\sqrt{2}}{35} - \frac{8 + 9\sqrt{2}}{4} \right)$$

$$= \frac{60 + 28\sqrt{2} - 140 - 315\sqrt{2}}{140}$$

$$= \frac{60 + 28\sqrt{2} - 280 - 315\sqrt{2}}{140}$$

$$= \frac{15 + 7\sqrt{2}}{35} - \frac{3\sqrt{2}}{4} + \frac{4}{2} + \frac{3\sqrt{2}}{2}$$

$$= \frac{30 + 14\sqrt{2} + 105\sqrt{2}}{70} + \frac{4}{2} - \frac{3\sqrt{2}}{4}$$

$$= \frac{30 + 119\sqrt{2}}{70} + 2 - \frac{3\sqrt{2}}{4}$$

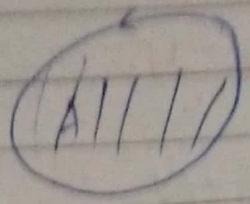
$$= \frac{30 + 119\sqrt{2} + 140}{70} - \frac{3\sqrt{2}}{4}$$

$$= \frac{170 + 119\sqrt{2}}{70} - \frac{3\sqrt{2}}{4}$$

$$= \frac{680 + 476\sqrt{2} - 210\sqrt{2}}{280}$$

$$= \frac{680 + 266\sqrt{2}}{280} = \frac{340 + 133\sqrt{2}}{140}$$

* Green's Theorem:-



'C' is any closed curve

A is area enclosed by closed curve 'C'.

~~∫xdx +~~

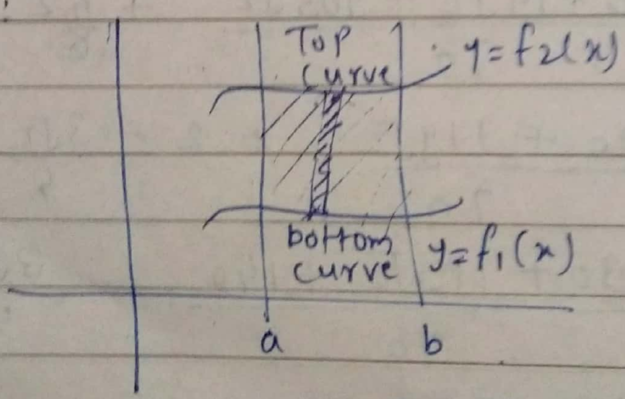
$$\oint_C u\,dx + v\,dy = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Q.1 Evaluate using Green's thm $\oint_C \bar{F}\cdot d\bar{r}$ fo

$F = x^2 i + xy j$ where A is the region bounded by curve $y = x^2$ & then $y = x$

Sol^n:

Note:



$$a \le x \le b$$

$$A = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right] dx$$

$$f_1(x) \le y \le f_2(y)$$

$\iint_R dx\,dy$ = Area bounded by R.

Area of ~~unit~~ circle $x^2+y^2=a^2$ is πa^2

Area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab

Sol. $I = \int_c \bar{F}\cdot d\bar{r}$

$= \oint_c x^2+y^2 \; x^2\,dx + xy\,dy$

$= \int u\,dx + v\,dy$

$u = x^2 \qquad v = xy$

points of intersection of $y=x^2$ & $y=x$

put $y=x$ in $y=x^2$

$\therefore x = x^2$

$\therefore x^2 - x = 0$

$x(x-1)=0.$

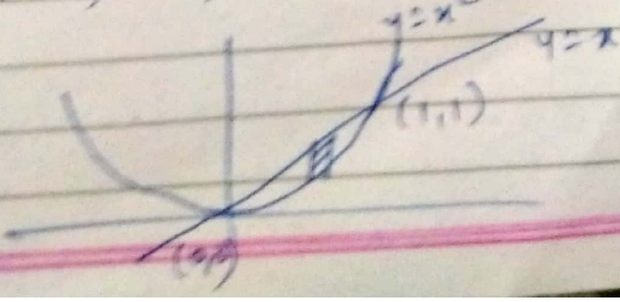
$x=1, \; x=0$

$y=x$

$x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1$

$\therefore (0,0)$ & $(1,1)$ are pts of intersection.



$0 \le x \le 1$

$x^2 \le y \le x$

By Green's th^m

$$I = \oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$v = xy \Rightarrow \frac{\partial v}{\partial x} = y$$

$$u = x^2 \Rightarrow \frac{\partial u}{\partial y} = 0.$$

$$I = \iint_R (y - 0)\cdot dx\,dy$$

$$= \int_0^1 \left[\int_{x^2}^{x} y\cdot dy\right] dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} \cdot dx$$

$$= \frac{1}{2}\int_0^1 x^2 - x^4 \cdot dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{1}{2} \times \frac{2}{15}$$

$$\therefore \boxed{I = \frac{1}{15}}$$

Q. Use Green's th^m to evaluate $\oint_C \bar{F}\cdot d\bar{r}$ for the vector field $\bar{F} = \sin y\,\bar{i} + x(1+\cos y)\,\bar{j}$ along the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol:

$$I = \oint \bar{F}\cdot d\bar{r}$$

$$= \int \sin y\,dx + \cancel{x(1+}\ x(1+\cos y)\,dy$$

$$= \int u\,dx + v\,dy$$

By Green's th^m,

$$I = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$= \iint (1 + \cos y - \cos y)\,dx\,dy$$

$$= \iint 1\cdot dx\,dy$$

$$= \text{Area of ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$= \pi ab$$

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