6. Complex Numbers

$N = \{1, 2, 3, 4, \ldots\}$

$W = \{0, 1, 2, 3, 4, \ldots\}$

$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$Q = \{p/q \mid p, q \in I \ \& \ q \neq 0\}$.

$$N \subseteq W \subseteq I \subseteq Q \subseteq R \subseteq C.$$

$$Q \cup Q^c = R$$

$z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

$$\text{Arg } z = \text{Arg}(x+iy) = \begin{cases} \tan^{-1}(y/x) & , x \geq 0. \\ \pi + \tan^{-1}(y/x) & , x < 0. \end{cases}$$

$\bar{z} = x - iy.$

$(x+iy)(x-iy) = x^2 + y^2.$

$|z|^2 = z \cdot \bar{z}$

$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

$z = x + iy = r(\cos\theta + i\sin\theta)$

$r = |z|.$

$z = re^{i\theta}$ $\quad \theta = \text{arg}/\theta/.\ \text{arg}(z).$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta.$

- Function of complex variables →

$w = f(z)$

$f(z) = u + iv$

$f(z) = u(x,y) + iv(x,y)$

eg. $f(z) = z^2 = (x+iy)^2$

$= (x^2 - y^2) + i(2xy)$

$\therefore u = x^2 - y^2 \quad v = 2xy$

$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$|f(z)|^2 = f(z) \cdot \bar{f}(\bar{z})$ $\quad (|z|^2 = z \cdot \bar{z})$

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$

- Analytic funⁿ:

$f(z)$ is analytic funⁿ at z_0 if $f(z)$ has derivatives at all points in some neighbourhood of z_0.

- Singular pt:

The pt. at which z_0 is not analytic.

- C-R equations →

If $f(z) = u + iv$ analytic funⁿ then

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

C-R eqⁿs in polar form

$f(z) = u(r,\theta) + iv(r,\theta)$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}.$$

$\phi(x,y)$ is harmonic iff

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0.$$

If $f(z) = u + iv$ analytic funⁿ
then u & v are harmonic funⁿ.

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right) \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0\right).$$

~~It is~~ u is harmonic conjugate of v.
v is harmonic conjugate of u.

- Miline Thomson's method :
  To express $u+iv$ in terms of z.
  put $x = z$, $y = 0$.
  We get $f(z)$ in terms of z.

  OR

  $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

  $f(z) = z^2$.

  put $x = z$, $y = 0$.
  We get $f'(z)$ in z.
  integrate $f'(z)$ w.r.t. z, we get $f(z)$.

$f(z) \cdot \bar{f}(\bar{z}) = |f(z)|^2$

$f'(z) \cdot \bar{f}'(\bar{z}) = |f'(z)|^2$

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Ex. $f(z) = x^2 - y^2 + i2xy$.

$f(z) = u + iv$

$u = x^2 - y^2 \quad v = 2xy$

Let $x = z, y = 0$.

$f(z) = z^2 + i2z(0)$

$= z^2 + i0$.

If $f(z) = r(\cos\theta + i\sin\theta)$

then put $r = z$, $\theta = 0$.

Ex. If $f(z)$ is analytic, then show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2.$$

Ans. Let, $f(z) = u + iv$

$$LHS = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2$$

But $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}}$

& $|f(z)|^2 = f(z) \cdot \bar{f}(\bar{z})$.

$$LHS = 4 \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} f(z) \cdot \bar{f}(\bar{z})$$

$$= 4 \frac{\partial f(z)}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} \bar{f}(\bar{z})$$

$$= 4 f'(z) \cdot \bar{f}'(\bar{z})$$

$$= 4 |f'(z)|^2$$

$$= RHS.$$

Ex. If $f(z)$ is analytic then show that
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^n = n^2|f(z)|^{n-2}|f'(z)|^2$$

Ans. $LHS = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^n$

But $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \bar{z}}$ & $|f(z)|^n = |f(z)|^2|^{n/2}$
$= |f(z).\bar{f}(\bar{z})|^{n/2}$

$|f(z)|^n = f(z)^{n/2}.\bar{f}(\bar{z})^{n/2}$

$= 4\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \bar{z}} f(z)^{n/2}.\bar{f}(\bar{z})^{n/2}$

$= 4\frac{\partial}{\partial z} f(z)^{n/2}.\frac{\partial}{\partial \bar{z}}\bar{f}(\bar{z})^{n/2}$

$= 4.\frac{n}{2}(f(z))^{n/2-1}f'(z).\frac{n}{2}(\bar{f}(\bar{z}))^{n/2-1}.\bar{f}'(\bar{z})$

$= n^2\left[(f(z))^{n/2-1}.\bar{f}(\bar{z})^{n/2-1}.f'(z).\bar{f}'(\bar{z})\right]$

$= n^2\left[\left(f(z).\bar{f}(\bar{z})\right)^{n/2-1}.(f'(z).\bar{f}'(\bar{z}))\right]$

$= n^2\left[\left(|f(z)|^2\right)^{\frac{n-2}{2}}.|f'(z)|^2\right]$

$= n^2|f(z)|^{n-2}|f'(z)|^2$

$= RHS$

Harmonic conjugate ⇒

Harmonic function.

If $u = u(x,y)$ harmonic so that

To find harmonic conjugate 'v' of u so that

$f(z) = u + iv$ analytic.

1. Find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

2. To find v by total derivative

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy \quad \text{— (A)}$$

3. By C.R. Eq^n

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{— (i)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{— (ii)}$$

$$A \Rightarrow dv = \left(-\frac{\partial u}{\partial y}\right) \cdot dx + \left(\frac{\partial u}{\partial x}\right) \cdot dy$$

Put values

$$dv = M \cdot dx + N \cdot dy$$

$$v = \int_{y\text{ const.}} M \cdot dx + \int \begin{bmatrix} \text{Those terms of } N \\ \text{which are free from } x \end{bmatrix} \cdot dy + c$$

OR $+c$

$$v = \int_{x\text{ const.}} N \cdot dy + \int \begin{bmatrix} \text{Terms of } M \\ \text{which are free from } y \end{bmatrix} \cdot dx + c$$

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Ex. S.T. $u = x^4 - 6x^2y^2 + y^2$ is harmonic, find harmonic conjugate 'v' of u such that $f(z) = u + iv$ analytic & determine $f(z)$ in terms of z.

Ans. $f(z) = u + iv$.

$u = x^4 - 6x^2y^2 + y^4$.

$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$ $\qquad$ $\frac{\partial u}{\partial y} = -12x^2y + 4y^3$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left[(4x^3 - 12xy^2)^2 + (-12x^2y + 2y)^2\right]$$

$$= \left[\begin{array}{l} 16x^6 + 144x^2y^4 - 96x^4y^2 + 144x^4y^2 \\ + 4y^2 - 48x^2y^2 \end{array}\right]$$

$\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$ $\qquad$ $\frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12x^2 - 12y^2 - 12x^2 + 12y^2 = 0.$$

$\therefore$ u is harmonic.

To find harmonic conjugate 'v' of u.

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy \quad \text{— (A)}$$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\qquad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial v}{\partial y} = 4x^3 - 12xy^2$ $\qquad$ $\frac{\partial v}{\partial x} = 12x^2y - 4y^3$.

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy$$

$$= \left(12x^2y - 4y^3\right) \cdot dx + \left(4x^3 - 12xy^2\right) \cdot dy$$

$M = 12x^2y - 4y^3$

$N = 4x^3 - 12xy^2$

$$v = \int_{y\ \text{const}} M \cdot dx + \int \left(\begin{array}{l}\text{Terms of } N \\ \text{free from } x\end{array}\right) \cdot dxy + c$$

$$= \int_{y\ \text{const}} 12x^2y - 4y^3 \cdot dx + \int 0 \cdot dy + c$$

$$= \left[\frac{12x^3}{3}y - 4xy^3\right] + c$$

$$v = \left[4x^3y - 4xy^3\right] + c$$

$f(z) = u + iv$

$= (x^4 - 6x^2y^2 + y^4) + i\,(4x^3y - 4xy^3) + ic$

By Miline Thomson's method,

put $x = z$ & $y = 0$.

$f(z) = z^4 + ic$

Ex. If $v = \frac{-y}{x^2+y^2}$ harmonic function, find harmonic conjugate 'u' of v so that $f(z) = u + iv$ analytic function & hence determine $f(z)$ in terms of z.

Friday.

Ans.
$$f(z) = u + iv$$
$$v = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(0) - (-y)2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2 - y^2 + 2y^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}.$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\cancel{\frac{\partial^2 v}{\partial x^2} = (x^2+y^2)^2 2y - 2xy\, 2(x^2+y^2) 2x}$$

To find harmonic conjugate of u of v.

$$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\therefore du = \left(\frac{y^2-x^2}{(x^2+y^2)^2}\right).dx - \frac{2xy}{(x^2+y^2)^2}.dy$$

Comparing eq^n with
$$du = M.dx + N.dy$$
$$\therefore M = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad N = \frac{-2xy}{(x^2+y^2)^2}$$

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$$u = \int_{y\,const} M\cdot dx + \int \begin{pmatrix}\text{Terms of } N \\ \text{free from } x\end{pmatrix}\cdot dy + c$$

$$= \int_{y\,const} \left(\frac{y^2 - x^2}{(x^2+y^2)^2}\right)\cdot dx + \int 0\cdot dy + c$$

$$= \int_{y\,const} \frac{y^2}{(x^2+y^2)^2} - \frac{x^2}{(x^2+y^2)^2}\cdot dx$$

$$u = \int_{x\,const.} N\cdot dy + \int \begin{pmatrix}\text{Terms of } M \\ \text{free from } y\end{pmatrix}\cdot dx + c$$

$$= -\int_{x\,const.} \frac{2xy}{(x^2+y^2)^2}\cdot dy + \int \frac{-x^2}{(x^2+y^2)^2}\cdot dx = 0$$

$$= -x\int \frac{2y}{(x^2+y^2)^2}\cdot dy - \int \frac{x^2}{(x^2+y^2)^2}\cdot dx$$

$$= -x\left[\frac{1}{-\cancel{3}(x^2+y^2)^{\cancel{3}}}\right] - 0 + c$$

$$= \frac{x}{\cancel{3}(x^2+y^2)^{\cancel{3}}} + c$$

$$u = \frac{x}{(x^2+y^2)} + c$$

$$f(z) = u + iv$$

$$= \frac{x}{(x^2+y^2)} + c + i\left(\frac{-y}{x^2+y^2}\right)$$

By Miline Thomson method,

Put $x = z,\ y = 0.$

$$f(z) = \frac{z}{z^2} + c + i\left(\frac{-0}{z^2}\right)$$

Ex.

Ans.

$$\therefore \quad f(z) = \frac{1}{z} + c$$

Ex. $u = \frac{1}{2}\log(x^2+y^2)$

Ans. $$\frac{\partial u}{\partial x} = \frac{2x}{2(x^2+y^2)} = \frac{x}{(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$\frac{\partial u}{\partial y} = \frac{1 \cdot 2y}{2(x^2+y^2)} = \frac{y}{(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0.$$

$\therefore$ u is harmonic.

$$\frac{\partial u}{\partial x} = \frac{x}{(x^2+y^2)} \qquad \frac{\partial u}{\partial y} = \frac{y}{(x^2+y^2)}$$

To find harmonic conjugate v of u.

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$$

$$\therefore \quad dv = \cancel{x}\ \overset{M}{\frac{-y}{x^2+y^2}} \cdot dx + \overset{N}{\frac{x}{x^2+y^2}} \cdot dy$$

$$\cancel{v = \int_{x\ \text{const.}} \frac{-y}{x^2+y^2} \cdot dy + \int 0 \cdot dx}$$

$$v = \int_{y\ \text{const.}} \frac{-y}{x^2+y^2} \cdot dx + \int 0 \cdot dy + c$$

$$= \frac{-y}{y} \tan^{-1}\left(\frac{x}{y}\right) + c.$$

$$v = -\tan^{-1}\frac{x}{y} + c$$

$$f(z) = \frac{1}{2}\log(x^2+y^2) + i\left(-\tan^{-1}\left(\frac{x}{y}\right) + c\right).$$

put $x = z,\ y = 0.$

$$f(z) = \frac{1}{2}\log z^2 + i\left(-\tan^{-1}\left(\frac{z}{0}\right) + c\right).$$

$$= \frac{2}{2}\log z + i\left(-\frac{\pi}{2} + c\right).$$

$$f(z) = \log z + i\left(\frac{-\pi}{2} + c\right).$$

Ex. If $u \pm v = f(x,y)$ to find analytic funⁿ $f(z) = u + iv$ in terms of z.

Ans. Given: $u + v = f(x,y)$ (A)

Diff. (A) with w.r.t. to x

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = \frac{\partial}{\partial x} f(x,y) \quad — (1)$$

$$\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} f(x,y) \quad — (2)$$

By CR eqⁿ

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad — (i)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad — (ii)$$

② ⇒ $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{\partial}{\partial y} f(x,y)$

Adding ① & ③,

$$2\frac{\partial u}{\partial x} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) f(x,y)$$

complex integration →

$|z-z_0|=a$ where centre is z_0 & radius is a.

Cauchy's theorem

'c' is any closed curve

If f(z) is analytic & within closed curve 'c'

then $\int_c f(z)\,.dz=0$

Cauchy's integral formula

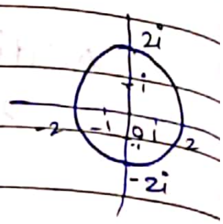
If 'c' is any closed curve, f(z) is analytic on & within closed curve 'c' And pt. $z=a$ is inside curve 'c' then

$$\int_c \frac{f(z)}{z-a} = 2\pi i f(a)$$

$$\int \frac{f(z)}{(z-a)^{n+1}}\,.dz = \frac{2\pi i\, f^n(a)}{n!}$$

Ex. Evaluate $\int_c \frac{e^z}{z+1} \cdot dz$ where 'c' is the circle |z

Ans. $|z| = 2$



$I = \int \frac{e^z}{(z-(-1))} \cdot dz$

Let $f(z) = e^z$.

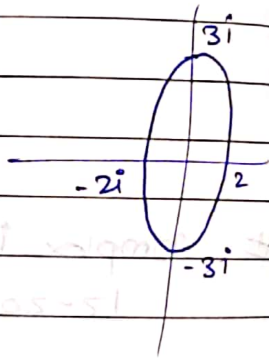
$I = \int \frac{f(z)}{z-(-1)} \cdot dz$ by Cauchy's formula

$= 2\pi i f(-1) = 2\pi i e^{-1}$

$= \frac{2\pi i}{e}$

Ex. $\int_c \frac{2z^2+z+5}{(z-3/2)^2} \cdot dz$

where 'c' is ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.



$z = \frac{3}{2}$ inside ellipse

Let $f(z) = 2z^2 + z + 5$

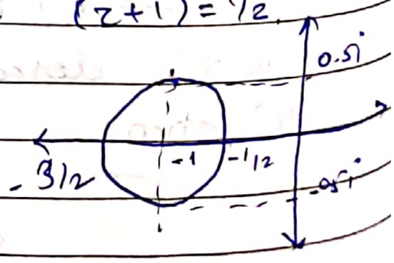
$I = \int \frac{f(z)}{(z-3/2)^2}$ $\therefore n = 1$

$f(3/2) = 2\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right) + 5 = 11.$

$f'(3/2) = 4 \cdot 3/2 + 1 = 7.$

$I = \frac{2\pi i f'(a)}{n!} = \frac{2\pi i\, 7}{1!} = 14\pi i$

Ex. $\int \frac{e^z}{(z+1)(z+2)} \cdot dz$ 'c' is circle $(z+1) = 1/2$.



$$\int \frac{e^z}{(z-(-1))(z-(-2))} = \frac{1}{-1-(-2)} \int \left[\frac{1}{(z+1)} - \frac{1}{(z+2)}\right] e^z$$

$$= \int_c \frac{e^z}{(z+1)} - \underbrace{\int \frac{e^z}{(z+2)}}_{0} \quad \text{as } -2 \text{ is outside}$$

$$= 2\pi i e^{-1} = \frac{2\pi i}{e}$$

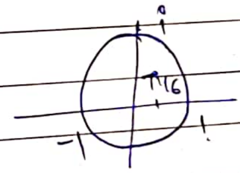
Alternate & correct method

$$\int \frac{e^z/z+2}{z-(-1)} \cdot dz \quad f(z) = \frac{e^z}{z+2}$$

$$I = \int \frac{f(z)}{z-(-1)} \cdot dz$$

$$I = 2\pi i f(-1) = 2\pi i \frac{e^{-1}}{-1+2} = \frac{2\pi i}{e}$$

Ex. $\int_c \frac{\sin^2 z}{(z-\pi/6)^3} \cdot dz \quad |z|=1$



$f(z) = \sin^2 z$

$a = \pi/6$ inside circle

$$I = \int_c \frac{f(z)}{(z-\pi/6)^{2+1}} \quad n=2$$

$f(z) = \sin^2 z$

$f'(z) = 2\sin z\cos z$

$f''(z) = 2\cos 2z$

∴ By cauchy's integral thm.

$$\int_c \frac{f(z)}{z-a} = \frac{2\pi i f^n(a)}{n!}$$

$$\frac{2\pi i f^2(a)}{2!} = \pi i f''(\pi/6) = \underline{\pi i}$$

$f(z)$ analytic then within closed curve 'c' then

$$\int \frac{f(z)}{z-a} \cdot dz = 2\pi i f(a)$$

$$\int \frac{f(z)}{(z-a)^{n+1}} \cdot dz = \frac{2\pi i}{n!} f^{n}(a)$$

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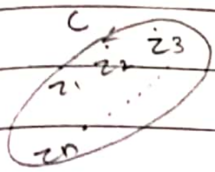
Wednesday

Cauchy

Ex. $\int \frac{z+1}{(z-2)(z-1)} \cdot dz$ when $|z|=3$.

cauchy's can be applied when partial fraction is proper i.e. numerator's degree is strictly less than denominator's i.e. if numerator was e^{2}, you wouldn't have been able to apply cauchy's.

Cauchy Residue Theorem →



Let 'c' be any closed curve
$z_1, z_2, z_3, \ldots z_n$ are poles of $f(z)$ which
are inside closed curve 'c'

$$\int f(z).dz = 2\pi i \left[\sum_{i=i}^{n} \text{Res } f(z_i) \right].$$

$$= 2\pi i \left[\text{Res } f(z) + \text{Res } f(z_2) + \text{Res } f(z_3) + \ldots \ldots + \text{Res } f(z_n) \right].$$

Residue:

1. Simple pole

$$\text{Res } f(z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

2. Pole of order two

$$\text{Res } f(z_0) = \frac{1}{2!} \lim_{z \to z_0} \frac{d}{dz} (z - z_0)^2 f(z).$$

3. Pole of order three

$$\text{Res } f(z_0) = \frac{1}{3!} \lim_{z \to z_0} \frac{d^2}{dz^2} (z - z_0)^3 f(z).$$

4. Pole of order 'n'

$$\text{Res } f(z_0) = \frac{1}{n!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z).$$

5. Pole of order 'n+1'

$$\text{Res } f(z_0) = \frac{1}{(n+1)!} \lim_{z \to z_0} \frac{d^n}{dz^n} (z - z_0)^{n+1} f(z).$$

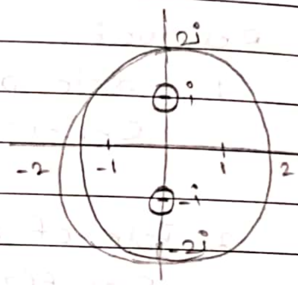
If $f(z) = \dfrac{f_1(z)}{f_2(z)}$

at $z = z_0$ simple pole

$$\text{Res } f(z_0) = \left.\frac{f_1(z)}{\frac{d}{dz} f_2(z)}\right|_{z=z_0}$$

Ex. Evaluate $\displaystyle\int_c \frac{z+2}{z^2+1} \cdot dz$, where c is the circle $|z| = 2$.

Ans. $f(z) = \dfrac{z+2}{z^2+1} = \dfrac{z+2}{(z+i)(z-i)}$



Poles of $f(z)$ are $\pm i$.

$z = i$ & $z = -i$, both poles are inside the circle.

By Residue theorem,

$$I = \int f(z) \cdot dz = 2\pi i [r_1 + r_2]$$

$$= 2\pi i [\text{Res } f(i) + \text{Res } f(-i)]$$

$r_1 = \text{Res } f(i)$

$z = i$ is simple pole

$$r_1 = \text{Res } f(i) = \lim_{z \to i} (z-i) f(z)$$

$$= \lim_{z \to i} \cancel{(z-i)} \frac{(z+2)}{(z+i)\cancel{(z-i)}}$$

$$r_1 = \frac{2+i}{2i}$$

$$r_2 = \text{Res } f(-i) = \lim_{z \to -i} \cancel{(z+i)} \frac{(z+2)}{\cancel{(z+i)}(z-i)}$$

$$r_2 = \frac{2-i}{-2i}$$

$I = \int f(z).dz = 2\pi i\,[r_1 + r_2]$

$= 2\pi i \left[\frac{2+i}{2i} + \frac{2-i}{-2i}\right]$

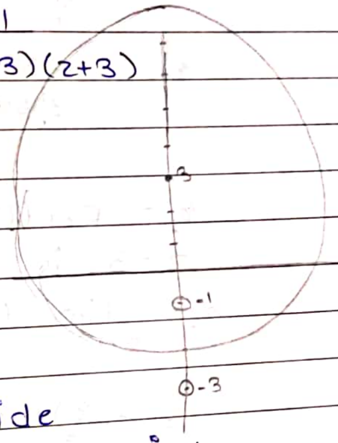
$= \frac{2\pi i}{2i}\left[2+i-2+i\right]$

$= \pi\,[2i]$

$I = 2\pi i.$

Ex. Evaluate $\int_c \frac{2z^2+2z+1}{(z+1)^3(z^2-9)}.dz$, where c is circle $|z-3|=5$.

Ans. $f(z) = \frac{2z^2+2z+1}{(z+1)^3(z^2-9)} = \frac{2z^2+2z+1}{(z+1)^3(z-3)(z+3)}$



Poles are $-1, 3, -3$.

$|z-3| = 5.$

$z-3 = 5$, $z = 8$

$z-3 = -5$, $z = -2$.

$z = 3, -1$ these poles are inside the circle, while -3 is outside the circle.

By residue theorem,

$I = \int f(z).dz = 2\pi i\,[r_1 + r_2]$

$= 2\pi i\,[\text{Res } f(-1) + \text{Res } f(3)].$

$r_1 = \text{Res } f(-1)$

$z=-1$ is pole of order three.

$r_1 = \text{Res } f(-1) = \frac{1}{3!}\lim_{z\to -1}\frac{d^2}{dz^2}\,\cancel{(z+1)^3}\cdot\frac{2z^2+2z+1}{\cancel{(z+1)^3}(z^2-9)}$

$= \frac{1}{3!}\lim_{z\to -1}\frac{d}{dz}\left[\frac{(z^2-9)(4z+2) - (2z^2+2z+1)2z}{(z^2-9)^2}\right]$

$= \frac{1}{3!}\lim_{z\to -1}\frac{d}{dz}\left[\frac{4z^3+2z^2-36z-18-4z^3-4z^2-2z}{(z^2-9)^2}\right]$

$= \frac{1}{3!}\lim_{z\to -1}\frac{d}{dz}\left[\frac{4z^3+2z^2}{(z^2-9)^2}\right.$

$$r_1 = \frac{1}{3!} \lim_{z \to -1} \frac{d}{dz}\left[\frac{-2z^2-38z-18}{(z^2-9)}\right]$$

$$= \frac{-2}{3!} \lim_{z \to -1} \frac{d}{dz}\left[\frac{z^2+19z+9}{(z^2-9)^2}\right]$$

$$= \frac{-2}{3!} \lim_{z \to -1} \left[\frac{(z^2-9)^2(2z+19)-(z^2+19z+9)(2z)(z^2-9)}{(z^2-9)^4}\right]$$

$$= \frac{-2}{3!}\left[\frac{(-8)^2(17)-(1-19+9)(-2)(-8)}{(-8)^4}\right]$$

$$= \frac{-2}{3!}\left[\frac{1088+160}{64\times 64}\right]$$

$$r_1 = \frac{-39}{3!\,64} = \frac{-\cancel{39}^{\,13}}{\cancel{3}\times 2\times 64} = \frac{-13}{128}$$

$r_2 = \text{Res}\, f(3)$

$z = 3$ is a pole of order 1.

$$r_2 = \lim_{z \to 3} \cancel{(z-3)} \frac{2z^2+2z+1}{(z+1)^3 \cancel{(z-3)} (z+3)}$$

$$= \frac{2(3)^2+6+1}{4^3\, 6}$$

$$= \frac{25}{6\times 64}$$

$$r_2 = \cancel{\frac{25}{256}} \quad \frac{25}{384}$$

$$I = 2\pi i\,[r_1 + r_2]$$

$$= 2\pi i\left[\frac{-13}{128} + \frac{25}{384}\right]$$

$$I = \frac{-\pi i}{128} \quad \frac{-14\pi i}{192} = \frac{-7\pi i}{96}$$

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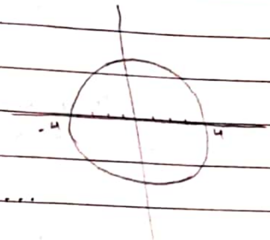
Thursday.

Ex. $\int_C \cot z \cdot dz \qquad |z|=4.$



Ans. $f(z) = \cot z = \frac{\cos z}{\sin z}$

Poles of $\cot z$ are $0, \pi, -\pi, 2\pi, -2\pi, \ldots$

~~All poles are~~

$0, \pi, -\pi$ are inside the circle.

$I = \int f(z) \cdot dz = 2\pi i \,(\text{Res } f(0) + \text{Res } f(-\pi) + \text{Res } f(\pi)).$

$= 2\pi i \,(r_1 + r_2 + r_3).$

$r_1 = \text{Res } f(0)$

$z = 0$ is simple pole.

$r_1 = \lim_{z\to 0} \frac{\cos z}{\frac{d}{dz}\sin z} = \lim_{z \to 0} \frac{\cos z}{\cos z} = 1$

$r_1 = 1$

$r_2 = \text{Res } f(-\pi)$

$= \lim_{z \to \pi} \frac{\cos z}{\cos z} = 1.$

$r_3 = \text{Res } f(\pi)$

$= \lim_{z \to -\pi} \frac{\cos z}{\cos z} = 1.$

$I = 2\pi i \,(r_1 + r_2 + r_3)$

$I = 6\pi i$

Ex. $\int_C e^z \sec z \cdot dz \qquad |z|=2.$

Ans. $f(z) = \frac{e^z}{\cos z}$

Poles of $f(z)$ are $\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{-3\pi}{2}, \ldots$

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$z = \frac{-\pi}{2}$ & $\frac{\pi}{2}$ are inside the circle.

By Residue theorem,

$I = \int f(z).dz = \oint 2\pi i\,[Res(-\pi/2) + Res(\pi/2)].$

$= 2\pi i\,[r_1 + r_2].$

$r_1 = Res\, f(-\pi/2)$

$= \lim_{z \to -\pi/2} \frac{e^z}{\frac{d}{dz}\cos z} = \lim_{z \to -\pi/2} \frac{e^z}{-\sin z}$

$r_1 = +e^{-\pi/2}$

$r_2 = -e^{\pi/2}$

$I = 2\pi i\,[-e^{\pi/2} + e^{-\pi/2}].$

$= 2\pi i\,[-23.1039 + 0.0433].$

$= -4\pi i \left[\frac{e^{\pi/2} - e^{-\pi/2}}{2}\right]$

$I = -4\pi i \sinh \frac{\pi}{2}$

Ex. $\int \frac{e^z}{(z+1)^2(z+2)^2}.dz \qquad |z+1| = 1/2.$

Ans. $f(z) = \frac{e^z}{(z+1)^2(z+2)^2}.$

Poles of of $f(z)$ are $-1, -2$.

-1 is inside the circle.

$I = \int f(z).dz$

$= 2\pi i\,[Res\, f(-1)].$

$= 2\pi i \left[\lim_{z \to -1} \frac{1}{2!}\frac{d}{dz}\cancel{(z+1)^2}.\frac{e^z}{\cancel{(z+1)^2}(z+2)^2}\right]$

$= 2\pi i \left[\frac{1}{2!} \lim_{z \to -1} \frac{(z+2)^2 e^z - 2(z+2)e^z}{(z+2)^4}\right]$

$I = \cancel{2}\pi i \left[\frac{1}{\cancel{2}} \quad \frac{e^{-1} - 2e^{-1}}{1}\right]$

$I = -e^{-1}\pi i$

Bilinear transformation →

$w = az+b/cz+d$

Ex. Find bilinear transformations which maps the pt. 1, 0, i of z-plane onto the pts. $\infty, -2, \frac{-1}{2}(i+1)$ of the w-plane.

Ans. Let $w = \frac{az+b}{cz+d}$ — (A) be given by bilinear trans.

z-plane	1	0	i
	↓	↓	↓
w-plane	$+\infty$	-2	$\frac{-1}{2}(1+i)$

Taking reciprocal of eq^n (A).

$\frac{1}{w} = \frac{cz+d}{az+b}$ — (B)

at $z=1, w=\infty$.

B ⇒ $\frac{1}{\infty} = \frac{c(1)+d}{a(1)+b}$

$0 = \frac{c+d}{a+b}$ $\qquad c = -d$ — (1).

at $z=0, w=-2$.

A ⇒ $-2 = \frac{a(0)+b}{c(0)+d}$

$b = -2d$ — (2)

at $z=i, w = -1/2\,(1+i)$

A → $\frac{-1}{2}(1+i) = \frac{a(i)+b}{c(i)+d} \times \frac{-ci+d}{-ci+d}$

$\frac{-1}{2}(1+i) = \frac{ac + (ad-cb)i + bd}{c^2+d^2}$

$$\frac{ac+bd}{c^2+d^2} = \frac{-1}{2} \qquad \frac{ad-cb}{c^2+d^2} = \frac{-1}{2}.$$

putting $c=-d$ & $b=-2d$.

$$\frac{-ad-2d^2}{2d^2} = \frac{-1}{2}. \qquad \frac{ad-2d^2}{2d^2} = \frac{-1}{2}.$$

$2ad + 2d^2 = 2d^2$. $\qquad 2ad - 4d^2 = -2d^2$

$2ad = +2d^{\cancel{2}}$ $\qquad \cancel{2}a\cancel{d} = \cancel{2}d^{\cancel{2}}$

$a = +d$ — ③. $\qquad a = d$.

$$w = \frac{az+b}{cz+d} = \frac{dz-2d}{-dz+d}$$

$$w = \frac{-iz-2}{-z+1} = \frac{iz+2}{z-1}$$

$$\frac{-1}{2}(1+i)(1-i)d = ai-2d$$

$a = -id$. — ③.

Ex. $-i, 0, 2+i$ of z plane onto the pts. $0, -2i, 4$. of w-plane.

Ans. $w = \frac{az+b}{cz+d}$ — Ⓐ

z	$-i$	0	$2+i$
w	0	$-2i$	4.

at $z=-i$, $w=0$.

$A \Rightarrow \quad 0 = \frac{-ai+b}{-ci+d}$ $\qquad b = +ai$ — ①.

$\cancel{a = bi}$.

at $z=0$, $w=-2i$.

$A \Rightarrow \quad -2i = \frac{b}{d}$ $\qquad b = -2id$

$d = \frac{1}{2}bi$ —②

$d = \frac{-1}{2}a$

at $z = 2+i$, $w = 4$.

$A \rightarrow \quad 4 = \dfrac{a(2+i)+b}{c(2+i)+d}$ $\qquad 4c(2+i)+4d = 2a+ai+b.$

$8c+4ic+4d = 2a+ai+b.$

$\cancel{4 = \dfrac{bi(2+i)+b}{c(2+i)+\frac{1}{2}bi}}$ $\qquad c(8+4i)+2bi = 2bi-b+b$

$4c(2+i) + \cancel{2bi} = \cancel{2bi} - \cancel{b} + \cancel{b}.$

$\cancel{8c+4ci}$

$A \rightarrow \quad 4 = \dfrac{a(2+i)+b}{c(2+i)+d} \rightarrow 4c(2+i)-2a = a(2+i)+ai.$

$8c+4ci-2a = 2a+ai+ai.$

$4c(2+i) = 2(2a+ai)$

$4c\cancel{(2+i)} = 2a\cancel{(2+i)}$

$c = \frac{1}{2}a. \quad — \ 3.$

$w = \dfrac{az+b}{cz+d} = \dfrac{az+ai}{\frac{1}{2}az-\frac{1}{2}a}$

$= \dfrac{2z+2i}{z-1} = \dfrac{2(z+i)}{z-1}$

$\therefore w = \dfrac{2(z+i)}{z-1}$

Cross ratio properties $\Rightarrow$

If z_1, z_2, z_3 point from z-plane are mapped onto the points w_1, w_2, w_3 of w-plane then

$$\frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$$

Ex. Find bilinear transform which maps the pts. $-1, 0, 1$ of z-plane onto the pts. $0, i, 3i$ of w-plane.

Ans. $w = \frac{az+b}{cz+d}$

at $z=-1$, $w=0$.

$0 = \frac{-a+b}{-c+d}$

$\boxed{a=b}$

at $z=0$, $w=i$

$i = \frac{b}{d}$

$b = id \rightarrow \boxed{-bi = d}$

at $z=1$, $w=3i$

$3i = \frac{a+b}{c+d}$

$3i = \frac{b+b}{c-bi}$

$3ic + 3b = 2b$.

$c = \frac{-b}{3i}$

$\boxed{c = \frac{bi}{3}}$

$w = \frac{az+b}{cz+d} \rightarrow \frac{bz+b}{bi/3 - bi} \rightarrow \frac{3b(z+1)}{bi-3bi}$

$w = \frac{3(z+1)}{-2i}$

Ex. Find the map of st. line $y=x$ under the transformation $w = \frac{z-1}{z+1}$ — (1).

Ans.

$w(z+1) = z-1$

$wz + w - z + 1 = 0$.

$z(w-1) + (w+1) = 0$.

$z = \frac{w+1}{1-w}$ — (2)

$z = x+iy$ $\quad w = u+iv$

$$x+iy = \frac{1+u+iv}{1-u-iv}$$

$$x+iy = \frac{(1+u)+iv}{(1-u)-iv} \times \frac{(1+u)+iv}{(1-u)+iv}$$

$$x+iy = \frac{1-u^2-v^2+2iv}{(1-u)^2+v^2}$$

$$x = \frac{1-u^2-v^2}{(1-u)^2+v^2} \qquad y = \frac{2v}{(1-u)^2+v^2}$$

$x=y$.

$1-u^2-v^2 = 2v$.

$u^2+v^2+2v=1$.

$u^2+(v+1)^2=2$.

Ex. S.T. analytic funⁿ with constant magnitude is constant

Given: $f(z) = u+iv$

& $|f(z)| = c$

$\sqrt{u^2+v^2} = c$

$\cancel{u \cdot \frac{\partial v}{\partial x} + v \partial}$

$u^2+v^2 = c^2$.

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0.$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0. \quad \text{--- (1)}$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0. \quad \text{--- (2)}$$

By CR eqⁿ

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

Eq^n 1 $\Rightarrow$

$$u \cdot \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} = 0 \quad \text{--- (3)}$$

$$u \cdot \frac{\partial u}{\partial y} + v \cdot \frac{\partial u}{\partial x} = 0 \quad \text{--- (4)}$$

$(3)^2 + (4)^2$.

$$u^2 \frac{\partial u^2}{\partial x^2} + v^2 \frac{\partial u^2}{\partial y^2} \cancel{- 2uv \frac{\partial u}{\partial x} \frac{\partial u}{\partial y}} = 0.$$

$$u^2 \frac{\partial u^2}{\partial y^2} + v^2 \frac{\partial u^2}{\partial x^2} \cancel{+ 2uv \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial x}} = 0.$$

$$(v^2 + u^2) \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) = 0.$$

but $u^2 + v^2 = c^2 \neq 0$

$$\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = 0.$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0.$$

$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 0.$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$= 0 + i0$$

$$= 0$$

$\therefore$ $f(z) =$ constant