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6. Complex Variables

Natural Number $\mathbb{N} = \{1, 2, \ldots\}$

whole $W = \{0, 1, 2, \ldots\}$

Integers $I = \{\ldots -1, 0, 1, 2, \ldots\}$

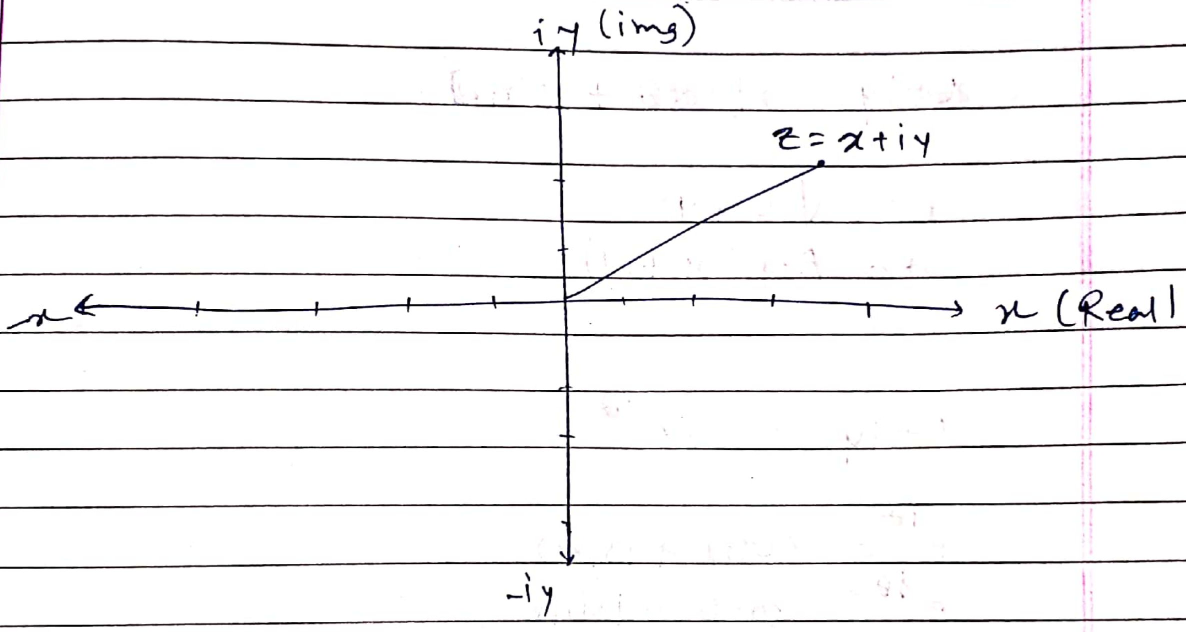
Rational $Q = \{ p/q ,\ p, q \in I,\ q \neq 0 \}$

Irrational $Q^c = \{$ no. which is not rational

e.g $\sqrt{2}$, $\sqrt{3}$, π, etc.

Real $= \{$ ~~∞~~ $-\infty$ to $\infty \}$

Complex $= \{ x+iy,\ x, y \in R,\ i = \sqrt{-1} \}$



$z = x+iy$

$|z| = \sqrt{x^2+y^2}$

$$\operatorname{Arg}(x+iy) = \begin{cases} \tan^{-1}\frac{y}{x}, & \cancel{x<0}\ x \geq 0 \\ \pi + \tan^{-1}\frac{y}{x}, & x<0 \end{cases}$$

complex conjugate : $\bar{z} = x - iy$

$$(x+iy)(x-iy) = x^2+y^2$$

$$|z|^2 = z \cdot \bar{z}$$

$$(x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i[x_1y_2+x_2y_1]$$

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$\boxed{\frac{1}{i} = -i}$$

$$x+iy = r[\cos\theta + i\sin\theta]$$

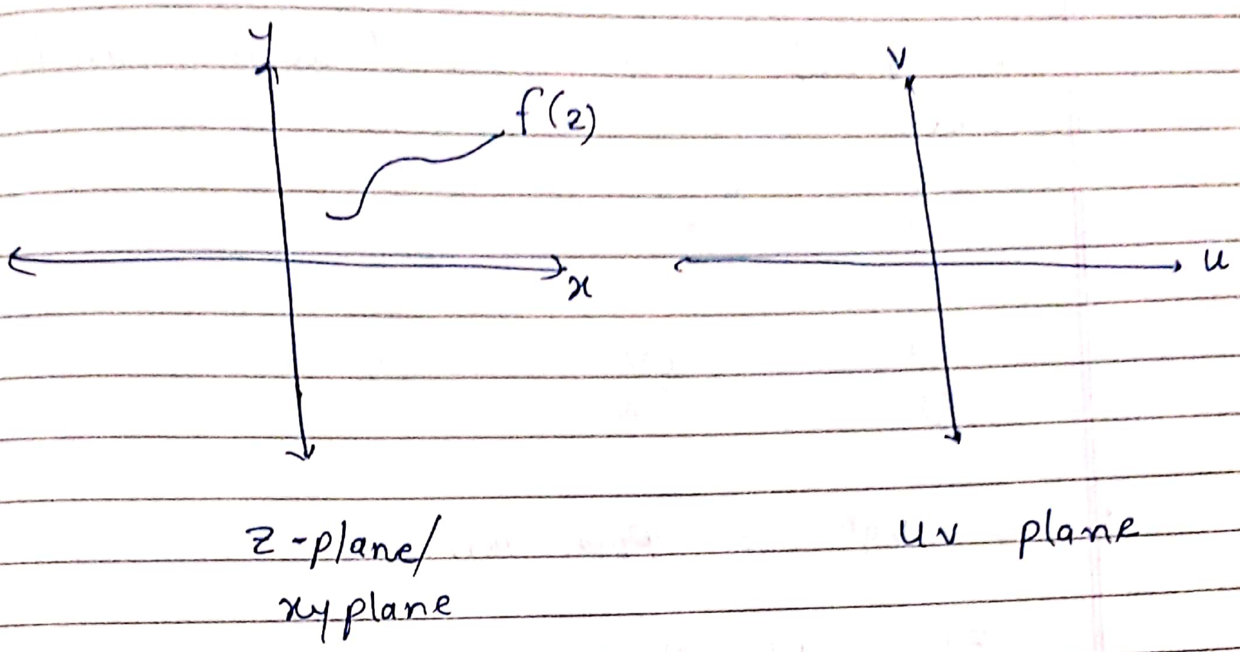
$$r = \sqrt{x^2+y^2}$$
$$\theta = \mathrm{Arg}(x+iy)$$

$$x+iy = re^{i\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

* Function of complex variables :

$f(z) = u + iv = u(x,y) + i v(x,y)$



If $f(z) = u + iv$
then,

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

* Analytic Function:-

Function which is continuously differentiable then it is analytic f^n.
(i.e. denominator does not become zero).

* Cauchy - Reimann Eqns.

C-R- Eqn

If $f(z) = u+iv$ is Analytic function, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—— ①}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—— ②}$$

* C-R Eqns in Polar form :-

$$f(z) = u(r,\theta) + iv(r,\theta)$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

* Harmonic Function:-

$\phi(x,y)$ is harmonic iff

$$\cancel{\frac{\partial^2 \phi}{\partial^2 x}} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Thm: If $f(z) = u+iv$ is analytic function then u & v are harmonic function.

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e.g. i) $f(z) = x^2 - y^2 + i\,2xy$

$u = x^2 - y^2$, $v = 2xy$

$\because f(z)$ is analytic f^n $\therefore$
Hence, u & v are harmonic function

ii) $f(z) = \cos z$
$= \cos(x+iy)$
$= \cancel{\cos x \cosh x}$
$= \cos x \cosh y - i \sin x \sinh y$

$u = \cos x \cosh y$, $v = \sin x \sinh y$

u & v are harmonic

u is harmonic conjugate of v. & vice versa.
OR u is orthogonal trajectory of v & vice versa

$|f(z)|^2 = f(z) \cdot \bar{f}(\bar{z})$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}}$$

singular point : Point $z = z_0$ is singular point iff $f(z)$ is not analytic at z_0

* Miline Thomsons Theorem :

To express $f(z) = u(x,y) + iv(x,y)$
in terms of z
put $x = z$, $y = 0$,
we get $f(z)$ in terms of z.

e.g. i) $f(z) = \cos x \cosh y - i \sin x \sinh y$

Put $x = z$ $y = 0$

$\therefore f(z) = \cos z \cosh 0 - i \sin z \sin 0$
$= \cos z \, 1 - i 0$
$= \cos z$

ii) $f(z) = x^2 - y^2 + i\,2xy$

$x = z$, $y = 0$

$= z^2 - 0 + i\,2z0$
$\therefore f(z) = z^2$

* Note:- If $f(z) = u + iv$ & is analytic f^n
then
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4 |f'(z)|^2$$

proof:
L.H.S. $= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2$ —— ①

$\because \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}}$ &
$|f(z)|^2 = f(z) \cdot \bar{f}(\bar{z})$

$$\therefore \text{L.H.S.} = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} f(z) \cdot \bar{f}(\bar{z})$$

$$= 4 \frac{\partial}{\partial z} f(z) \cdot \frac{\partial}{\partial \bar{z}} \bar{f}(\bar{z})$$

$$= 4 f'(z) \cdot \bar{f}'(\bar{z})$$

$$= 4 |f'(z)|^2 \quad \ldots \left(\cancel{f'(z) = f} \quad (w\bar{w} = |w|^2) \right)$$

$$= \text{R.H.S.}$$

Q. If $f(z) = u + iv$ is analytic f^n then S.T

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^4 = 16 \, |f(z)|^2 \, |f'(z)|^2$$

$$\text{L.H.S.} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^4$$

$$= 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \left[f(z) \cdot \bar{f}(\bar{z}) \right]^2$$

$$= 4 \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} (f(z))^2 \, (\bar{f}(\bar{z}))^2$$

$$= \cancel{\frac{4}{\partial}} \; 4 \frac{\partial}{\partial z} (f(z))^2 \; \frac{\partial}{\partial \bar{z}} (\bar{f}(\bar{z}))^2$$

$$= 16 \; f(z) \cdot \bar{f}(\bar{z}) \; f'(z) \; \bar{f}'(\bar{z})$$

$$= 16 \; |f(z)|^2 \cdot |f'(\bar{z})|^2$$

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Show that analytic function with constant magnitude is constant.

/3/ Harmonic conju

* If harmonic function 'u' is given to find harmonic conjugate 'v' of u such that

$f(z) = u + iv$ is analytic

step 1]:

Given $u = u(x,y)$

find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

step 2]: To find v

By total derivative

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \quad \text{--- (A)}$$

By C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (2)}$$

Put $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ in (A)

$$\therefore dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

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$\therefore dv = M dx + N dy$ — exact diff eqⁿ.

~~$\therefore v = \int_{y\text{-const}} M dx + \int_{x\text{-const}} N dy$ + terms of N free from x] dy + c~~

$$\Rightarrow v = \int_{y\text{-const}} M\,dx + \int \begin{bmatrix} \text{terms of } N \\ \text{free from } x \end{bmatrix} dy + c$$

Q. Show that $u = x^4 - 6x^2y^2 + y^4$ is harmonic function, hence find harmonic conjugate v of u such that $f(z) = u + iv$ is analytic function. Determine $f(z)$ in terms of z.

Solⁿ: Given: $u = x^4 - 6x^2y^2 + y^4$

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2 + \cancel{y^4}\,0$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2 + \cancel{y^4}\,0$$

$$\frac{\partial u}{\partial y} = 0 - 12x^2y + 4y^3$$

$$\frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12x^2 - 12y^2 - 12x^2 + 12y^2$$

$$= 0.$$

$\Rightarrow$ u is harmonic function.

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To find harmonic conjugate v

By total derivative

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

By C-R eqns

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{--- (i)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (ii)}$$

$$\therefore dv = -\frac{\partial u}{\partial y} . dx + \frac{\partial u}{\partial x} . dy$$

$$dv = -(-12x^2y + 4y^3) . dx + (4x^3 - 12xy^2) dy$$

$$= (12x^2y - 4y^3) dx + (4x^3 - 12xy^2) dy$$

— exact diff eqn

$$\therefore dv = M\,dx + N . dy$$

$$v = \int_{y\text{-const}} (12x^2y - 4y^3)\,dx + \int 0 . dy + c$$

$$= \cancel{3x^3}$$

$$\boxed{v = 4x^3y - 4xy^3 + c}$$

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$\therefore \quad f(z) = u + iv$

$f(z) = x^4 - 6x^2y^2 + y^4 + i\,(4x^3y - 4xy^3 + c)$

By Milline Thomson's method,

put $x = z$ & $y = 0$.

$\therefore \quad f(z) = z^4 + 0 + 0 + i\,(0 - 0 + c)$

$\therefore \boxed{f(z) = z^4 + ic}$

Q. If $v = \dfrac{-y}{x^2+y^2}$ is harmonic function. Find harmonic conjugate ~~of~~ u of v such that $f(z) = u + iv$ is analytic function & determine $f(z)$ in terms of z.

Sol^n: $\quad v = \dfrac{-y}{x^2+y^2}$ $\quad \Big| \quad \dfrac{(x^2+y^2)(0) - (-y)(2x)}{x^2+y^2} = \dfrac{2xy}{(x^2+y^2)^2}$

$\cancel{(-y)(2x) - (x^2+y^2)(0)}$

$$\frac{\partial v}{\partial x} = \cancel{\frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2}} = \cancel{\frac{-2xy}{(x^2+y^2)^2}}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

To find harmonic conjugate of v. i.e u

By total derivative,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

By CR eqns

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{--- (i)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (ii)}$$

$$\therefore \partial du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

$$\therefore du = \frac{y^2 - x^2}{(x^2+y^2)^2} dx - \frac{2xy}{(x^2+y^2)^2} dy$$

$M\,dx + N\,dy$ — exact diff. eq.

By

$$\Rightarrow \cancel{u = \int \frac{y^2-x^2}{(x^2+y^2)^2} dx - \int 0 \cdot dy + c}$$

$$\therefore u = \int 0 \cdot dx - \int \frac{2xy}{(x^2+y^2)^2} dy + c$$

$$u = -x \int \frac{2y \cdot dy}{(x^2+y^2)^2} + c$$

$$u = +\frac{x}{x^2+y^2} + c$$

$$\therefore f(z) = u + iv$$

$$= \frac{x}{x^2+y^2} + c + i\left(\frac{-y}{x^2+y^2}\right)$$

By Miline Thomson's Method,

put $x = z$, $y = 0$.

$$\therefore f(z) = \frac{z}{z^2+0} + c + i(0)$$

$$\cancel{f(z) = \frac{c}{z}}$$

$$\boxed{f(z) = c + \frac{1}{z}}$$

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Q. If $u = \frac{1}{2}\log(x^2+y^2)$ Find v.

Q. If $w = \phi + i\psi$ represents the complex potential for an electric field & $\phi = -2xy + \frac{y}{x^2+y^2}$. Determine the function ψ.

Sol^n: $\because$ $\phi = -2xy + \frac{y}{x^2+y^2}$

$$\frac{\partial \phi}{\partial x} = -2y + \frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial \phi}{\partial x} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial \phi}{\partial y} = -2x + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial \phi}{\partial y} = -2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

To find harmonic conjugate ψ of ϕ

By Total derivative

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \quad \text{---(I)}$$

By C-R eqn.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

substituting in eqn ①

$$\therefore \partial d\psi = -\frac{\partial \phi}{\partial y} dx + \frac{\partial \phi}{\partial x} dy$$

$$= -\left(-2x + \frac{x^2-y^2}{(x^2+y^2)^2}\right) dx + \left(-2y - \frac{2xy}{(x^2+y^2)^2}\right) dy$$

$$= \cancel{2x - y^2 +}$$

$$= \left[2x - \frac{x^2-y^2}{(x^2+y^2)^2}\right] dx - 2\left[y + \frac{xy}{(x^2+y^2)^2}\right] dy$$

— Exact diff eqn

$$= M dx + N dy$$

By Sol

$$\Rightarrow \phi = \int 2x \, . dx - \int 2y + \frac{xy}{x^2+y^2} . dy + c$$

$$\phi = x - \left[\frac{y^2}{2} + x \log\frac{1}{(x^2+y^2)}\right] + c$$

$$\phi = x - y^2 + \frac{x}{x^2+y^2} + c$$

* To find analytic function, $f(z) = u+iv$ in terms of z where $u \pm v = f(x,y)$ given

~~u+v =~~ $u+v = f(x,y)$ — (A)

diff. (A) w.r.t. x

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = \frac{\partial f''(x,y)}{\partial x} \quad \text{— (1)}$$

diff (A) w.r.t. y

$$\therefore \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial y} f(x,y) \quad \text{— (2)}$$

By C-R eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{— (i)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{— ii)}$$

Substituting $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ in $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ in eqn (2)

$$\therefore \frac{-\partial v}{\partial x} + \frac{\partial u}{\partial x} = \frac{\partial}{\partial y} f(x,y) \quad \text{— (3)}$$

By ~~Add~~ (1) + (3), we get $\frac{\partial u}{\partial x}$

By (1) − (3), we get $\frac{\partial v}{\partial x}$

Put values of $\frac{\partial u}{\partial x}$ & $\frac{\partial v}{\partial x}$ in

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

By Miline Thomson's ~~th^m~~ method

Put $x = z$ & $y = 0$

we get $f'(z)$ in terms of z.
integrate w.r.t. z
we get ~~z~~ $f(z)$.

Q. If $u+v = e^{-x}(\cos y - \sin y)$, then find analytic function $f(z) = u+iv$ in terms of z

sol $u+v = e^{-x}(\cos y - \sin y)$ — (A)

diff (A) w.r.t. x.

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = -e^{-x}(\cos y - \sin y)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = -e^{-x}\cos y + e^{-x}\sin y \quad \text{— (1)}$$

diff (A) w.r.t. y.

$$\therefore \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = e^{-x}(-\sin y - \cos y)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -e^{-x}\sin y - e^{-x}\cos y \quad \text{— (2)}$$

By C-R eqn.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{— (i)} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{— (ii)}$$

put $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ in eqn (2)

$$\therefore \quad -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = -e^{-x}\sin y - e^{-x}\cos y \quad (3)$$

Add (1) & (3)

$$\therefore \quad \frac{2\partial u}{\partial x} = -2e^{-x}\cos y$$

$$\boxed{\frac{\partial u}{\partial x} = -e^{-x}\cos y}$$

(1) − (3)

$$\therefore \quad \frac{2\partial v}{\partial x} = +2e^{-x}\sin y$$

$$\therefore \boxed{\frac{\partial v}{\partial x} = +e^{-x}\sin y}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = -e^{-x}\cos y + ie^{-x}\sin y$$

By Miline Thomsan's method put $x = z$, $y = 0$

$$\therefore f'(z) = -e^{-z}\cos 0 + ie^{-z}\sin 0.$$

$$\boxed{f'(z) = -e^{-z}}$$

Integrating w.r.t z

$$\int f'(z).dz = -\int e^{-z}.dz$$

$$\boxed{f(z) = e^{-z} + c}$$

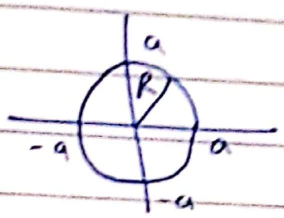
* Complex Integration :-

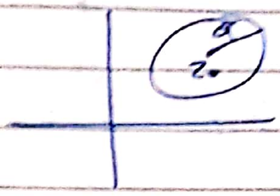
Cauchy's th^{m} :-

If $f(z)$ is analytic on and within closed curve 'c' then $\oint_c f(z) \cdot dz = 0$

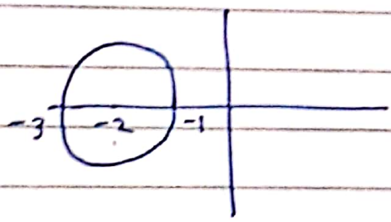
$|z| = a$

$|z - z_0| = a$





$|z+2| = 1$



Cauchy's Integral Formula :-
'c' is any closed curve.
If $f(z)$ is analytic function on & within closed curve 'c'. Point $z=a$ is inside curve 'c', then

$$\boxed{\oint_c \frac{f(z)}{z-a} \, dz = 2\pi i \, f(a)}$$

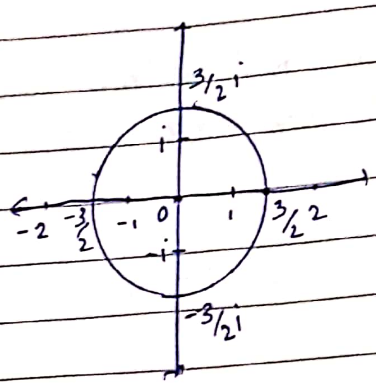
derivative

$$\boxed{\int \frac{f(z)}{(z-a)^{n+1}} \, dz = \frac{2\pi i}{n!} f^{n}(a)}$$

Q. Evaluate $\oint_c \frac{e^z}{z-1} \cdot dz$

where c is the circle $|z| = \frac{3}{2}$

Sol^n:



$|z - z_0| = a$

Here $f(z) = e^z$

$$I = \int \frac{f(z)}{z-1} dz$$

By cauchy's Integral ~~th^m~~, formula

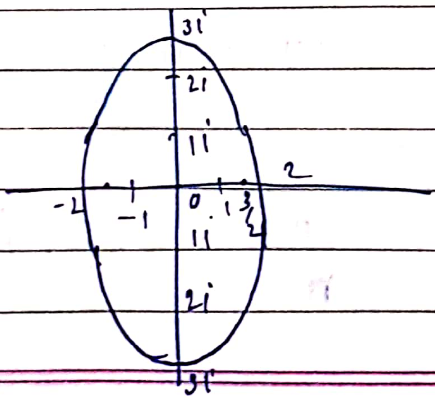
$$I = 2\pi i f(1)$$
$$= 2\pi i e^{(1)}$$
$$\boxed{I = 2\pi i e}$$

Q. Using Cauchy's Integral formula.

evaluate $\int \frac{2z^2 + z + 5}{(z - 3/2)^2} dz$

where 'C' is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Sol^n:



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$a = 3/2$ is inside ellipse

$f(z) = 2z^2 + z + 5$

$\therefore$ By Cauchy's Integral formula.

$$\cancel{I = 2\pi i \, f(a)}$$

$$\cancel{= 2\pi}$$

$$I = \frac{2\pi i}{1!} f^{1}(3/2)$$

$$= 2\pi i \left[2(3/2) + 3/2\right]$$

$$\boxed{I = \frac{9\pi i}{2}}$$

$$I = \int \frac{f(z)}{(z - 3/2)^{1+1}} \, dz$$

By Cauchy's Integral formula.

$$I = \frac{2\pi i}{1!} f'(3/2)$$

$$= 2\pi i \; f'(2z^2 + z + 5)$$

$$= 2\pi i \left(4 \times 3/2 + 1\right)$$

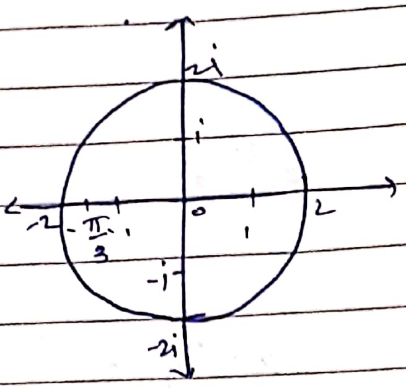
$$= 2\pi i \times 7$$

$$\boxed{I = 14\pi i}$$

Q. Evaluate $\displaystyle\int \frac{\sin 2z}{(z+\pi/3)^4}\,dz$

where 'C' is the circle $|z|=2$

Sol:



$a = -\pi/3$ is inside the circle

$f(z) = \cancel{z}\sin 2z$

$$I = \oint_C \frac{f(z)}{[z-(-\pi/3)]^{3+1}}\,dz$$

By Cauchy's integral ~~fro~~ formula

$$I = \frac{2\pi i}{3!}\, f'''(-\pi/3)$$

$f(z) = \sin 2z$

$f'(z) = 2\cos 2z$

$f''(z) = -4\sin 2z$

$f'''(z) = -8\ \cancel{\sin}\ \cos 2z$

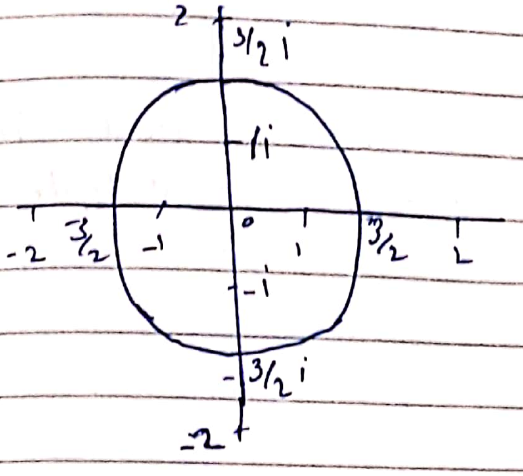
$f'''(-\pi/3) = -8\cos\left(-2 \times \frac{\pi}{3}\right)$

$f'''(-\pi/3) = -8 \times \left(-\frac{1}{2}\right) = 4$

$\therefore \boxed{I = \frac{\cancel{2}4\pi i}{3}}$

Q. $\oint_C \frac{e^z}{(z-1)(z-2)}$ where 'c' is circle $|z| = 3/2$

Sol^n:



$z=1$ inside circle & $z=2$ Outside circle

$\therefore a = 1$

Let $f(z) = \frac{e^z}{z-2}$

analytic on & within curve $|z| = 3/2$

$\therefore$ $I = \oint_C \frac{f(z)}{z-1} \, dz$

By cauchy's Integral method.

$\cancel{I =}$

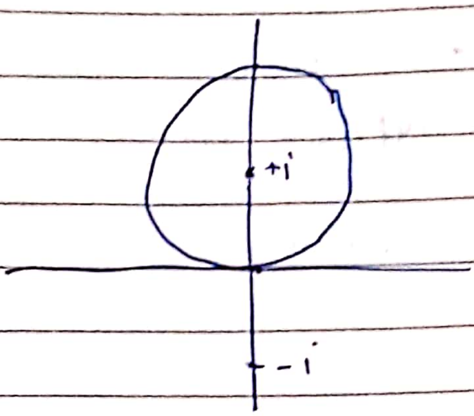
$I = 2\pi i \; f(1)$

$I = 2\pi i \left(\frac{e^1}{1-2} \right)$

$\boxed{I = -2\pi i e}$

Q. $I = \oint e \frac{z+2}{(z+i)(z-i)}$ over $|z-i| = 1$

Sol:



$z = i$ is inside circle
$z = -i$ is outside circle

$f(z) = \frac{z+2}{z+i}$

$I = \oint \frac{f(z)}{z-i}$

$\therefore \; I = 2\pi i \; f(i)$

$= 2\pi i \times \frac{i+2}{i+i}$

$\boxed{I = \pi(i+2)}$

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* Residue

$z = z_0$ simple pole — pole of order 1.

Residue of $f(z)$ at z_0

$$\text{Res}\,[f(z_0)] = \lim_{z \to z_0} (z - z_0) f(z)$$

Pole of order 2.

If z_0 is pole of $f(z)$ of order 2 then

$$\text{Res}\, f(z_0) = \frac{1}{1!} \lim_{z \to z_0} \frac{d}{dz} (z - z_0)^2 f(z).$$

Residue of $f(z)$ at pole of order 3.

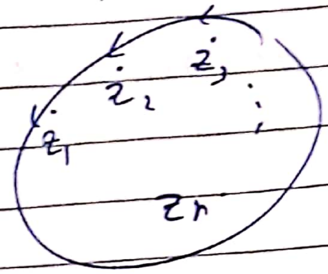
$z = z_0$ pole of order 3 then

$$\text{res}\, f(z_0) = \frac{1}{2!} \left[\lim_{z \to z_0} \frac{d^2}{dz^2} (z - z_0)^3 f(z) \right]_{z = z_0}$$

z_0 pole of order 'n' then

$$\text{Resi}\, f(z_0) = \frac{1}{(n-1)!} \left[\lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \right]_{z = z_0}$$

Cauchy Residue th^m:



'c' any closed curve.

$z_1, z_2 \ldots z_n$ are poles of $f(z)$ which are inside curve 'c' then

$$\int_c f(z).dz = 2\pi i \sum_{i=1}^{n} Res\, f(z_i)$$

z_i poles of $f(z)$ inside curve.

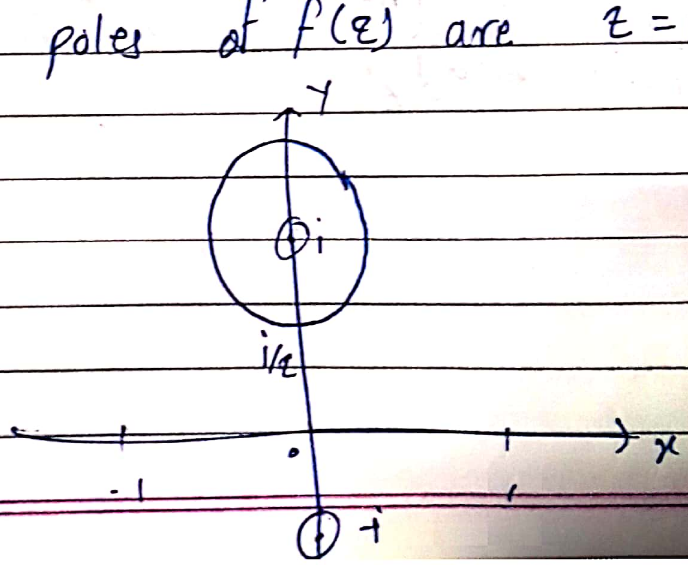
$$= 2\pi i \left[Res\, f(z_1) + Res\, f(z_2) + \cdots Res\, f(z_n) \right.$$

Q. Apply Residue th^m to evaluate $\int_c \frac{z+2}{z^2+1} dz$ where 'c' is the circle $|z-i| = \frac{1}{2}$

Sol^n: $f(z) = \frac{z+2}{z^2+1}$

$$= \frac{z+2}{(z+i)(z-i)}$$

poles of $f(z)$ are $z = i$ & $z = -i$



$z = i$ is inside circle

$\therefore$ By Cauchy Residue ~~theo~~ th^m,

$$I = \int f(z)\,dz = 2\pi i\,[\text{Res } f(i)]$$

$z = i$ is simple pole.

$$\text{Res}[f(i)] = \lim_{z \to i} (z-i) f(z)$$

$$= \lim_{z \to i} \cancel{(z-i)} \frac{(z+2)}{(z+i)\cancel{(z-i)}}$$

$$= \cancel{z} \frac{i+2}{2i}$$

$$\therefore I = 2\pi i \left[\frac{(i+2)}{2i}\right]$$

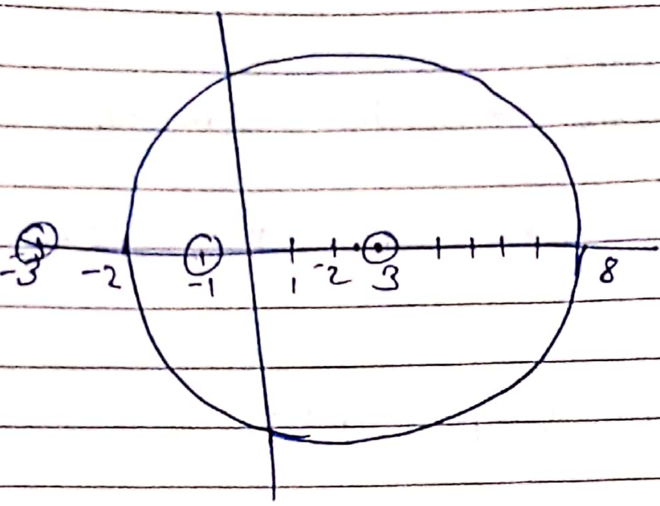
$$I = \cancel{2} \pi (2+i)$$

Q. Evaluate $\int_C \frac{z^2+2z}{(z+1)^3 (z^2-9)}$ where 'c' is the circle $|z-3| = 5$.

Sol^n: $f(z) = \dfrac{z^2+2z}{(z+1)^3 (z^2-9)}$

$$= \frac{z^2+2z}{(z+1)^3 \quad (z+3)(z-3)}$$

$\therefore$ poles of $f(z)$ are $z = -1$, $z = +3$, $z =$



$z = -1$, $z = 3$ are inside circle.

$\therefore$ By Cauchy Residue th^m.

$$I = \int f(z).dz = 2\pi i \left[\text{Res} f(-1) + \text{Res} f(3) \right]$$

$z = -1$ is a pole of order 3.

$$\therefore \text{Res}[f(-1)] = \frac{1}{2!} \lim_{z \to -1} \left[\frac{d^2}{dz^2} (z+1)^3 f(z) \right]_{z=-1}$$

$$= \frac{1}{2!} \lim_{z \to -1} \frac{d^2}{dz^2} \cancel{(z+1)^3} \frac{z^2 + 2z}{\cancel{(z+1)^3} (z-3)(z+3)}$$

$$= \frac{1}{2!} \lim_{z \to -1} \frac{d^2}{dz^2} \frac{z^2 + 2z}{(z^2 - 9)}$$

$$= \frac{1}{2!} \lim_{z \to -1} \frac{d}{dz} \frac{(z^2 - 9)(2z+2) - (z^2+2z)(2z)}{(z^2-9)^2}$$

$$= \frac{1}{2} \lim_{z \to -1} \frac{d}{dz} \left(\frac{2z^3 + 2z^2 - 18z - 18 - 2z^3 - 4z^2}{(z^2-9)^2} \right)$$

$$= -\frac{18}{2} \lim_{z \to -1} \frac{d}{dz}\left[\frac{z+1}{(z^2-9)^2}\right]$$

$$= -9 \lim_{z \to -1} \frac{(z^2-9)^2[1] - (z+1)2(z^2-9)\cdot 2z}{(z^2-9)^4}$$

$$= -9 \times \lim_{z \to -1} \frac{(z^2-9)^2 - 4(z^2+z)(z^2-9)}{(z^2-9)^4}$$

$$= -9\left[\frac{(1-9)^2 - 4(1+2)(1-9)}{(1-9)^4}\right]$$

$$= -9 \times \left[\frac{64+96}{64\times 64}\right]$$

$$= -9 \times \frac{160}{\cancel{9}\, 64\times 64} = -0.3515625$$

$z = 3$ is a simple pole

$$Res[f(3)] = \lim_{z \to 3} (z-3)\, f(z)$$

$$= \lim_{z \to 3} \cancel{(z-3)} \times \frac{z^2+2z}{\cancel{(z^2-9)}(z+3)(z+1)^3}$$

$$= \frac{3^2+2\times 3}{6\cdot(4)^3} = \frac{9+6}{6\times 64} = \frac{15}{6\times 64}$$

$$= \frac{5}{128}$$

$$= \frac{1}{2}\frac{d}{dz}\left[\frac{-18-18z-2z^2}{(z^2-9)^2}\right]_{z=-1}$$

$$= -\frac{d}{dz}\left[\frac{z^2+9z+9}{(z^2-9)^2}\right]_{z=-1}$$

$$= \left[\frac{(z^2-9)^2 [2z+9] - [z^2+9z+9] (2(z^2-9)(2z))}{(z^2-9)^4} \right]$$

$$= -\left[\frac{(1-9)^2 (-2+9) - (1-9+9) [2(1-9)(-2)]}{(1-9)^4} \right]$$

$$= -\left[\frac{64 \times 7 - [-40]}{\cancel{10000}\, 64 \times 64} \right] = -\left[\frac{448+40}{\cancel{10000}\, 64 \times 64} \right]$$

$$= \frac{-488}{\cancel{10000}\, 64 \times 64}$$

$$= \frac{-61}{8 \times 64}$$

$$= -\left[\frac{64 \times 7 - [32]}{64 \times 64} \right]$$

$$= -\cancel{64} \left[\frac{7 - 2}{\cancel{64} \times 64} \right]$$

$$= -\frac{13}{2 \times 64} = \frac{-13}{128}$$

$$\therefore I = 2\pi i \left[\frac{5}{128} - \frac{13}{128} \right]$$

$$= \cancel{2\pi} \; \frac{-16 \pi i}{128}$$

$$\boxed{I = \frac{-\pi i}{8}}$$

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If $f(z) = \dfrac{f_1(z)}{f_2(z)}$

$z = z_0$ is simple pole of $f(z)$

$$\text{Res} f(z_0) = \lim_{z \to z_0} \frac{f_1(z)}{\frac{d}{dz} f_2(z)}$$

eg. $f(z) = \tan z$

$$= \frac{\sin z}{\cos z}$$

$$= \frac{\sin z}{-\sin z}\Big|_{z=\pi/2}$$

Q. $\int_c \cot z \cdot dz$ where 'c' is the circle $|z| = 4$

Sol^n

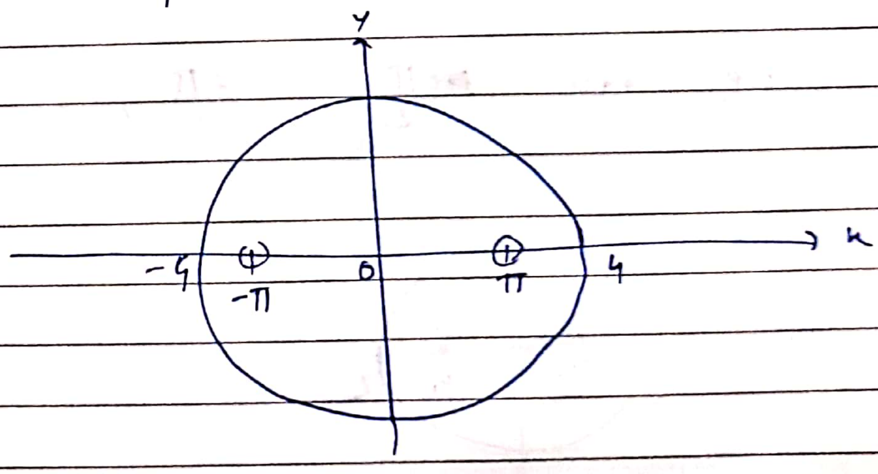
Q2 $\int_c e^z \cdot \sec z \cdot dz$ where 'c' is the circle $|z| = 2$

Sol^n:

$$f(z) = \cot z$$

$$= \frac{\cos z}{\sin z}$$

poles are $0, \pm\pi, \pm 2\pi, \ldots$



$z = 0$, $z = \pi$, $z = -\pi$ are inside circle

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$\therefore$ By Cauchy Residue th^m,

$I = 2\pi i \left[Res\, f(0) + Res\, f(-\pi) + Res\, f(\pi) \right]$

$Res f(0) = \frac{\cos z}{\cos z} = 1$

$Res f(\pi) = \frac{\cos z}{\cos z} = 1$

$Res\, f(-\pi) = \frac{\cos z}{\cos z} = 1$

$\therefore I = 2\pi i \left[1+1+1\right]$

$= 6\pi i$

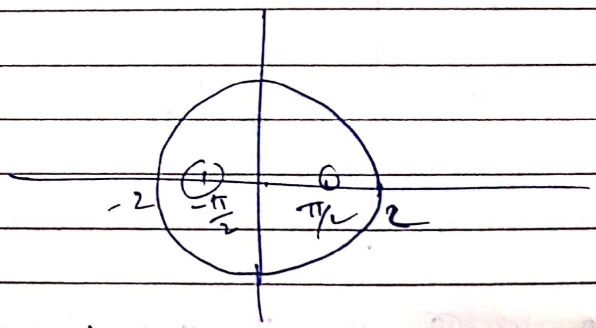
Q.2 $\int_c e^z \sec z \cdot dz$ 'C' is $|z| = 2$

$|z - 0| = 2$

$f(z) = e^z \cancel{sec} \sec z$

$= \frac{e^z}{\cos z}$.

poles are $\frac{\pm\pi}{2}, \frac{\pm 3\pi}{2}, \ldots$



$\therefore z = \frac{-\pi}{2}, \frac{\pi}{2}$ are inside circle

$\therefore$ By ~~Cauch~~ Cauchy's thm,

$$I = 2\pi i \left[\cancel{\Sigma} \ Res\, f\left(-\frac{\pi}{2}\right) + Res\, f\left(\frac{\pi}{2}\right) \right]$$

$$\cancel{I = 2\pi i \left[\right.}$$

$$Res\, f\left(-\frac{\pi}{2}\right) = \left. \frac{e^z}{-\sin z} \right|_{-\frac{\pi}{2}}$$

$$= \frac{e^{-\pi/2}}{-(-1)} = e^{-\pi/2}$$

$$Res\, f(\pi/2) = \left. \frac{e^z}{-\sin z} \right|_{\pi/2}$$

$$= \frac{e^{\pi/2}}{-1} = -e^{\frac{\pi}{2}}$$

$$\therefore I = 2\pi i \left(e^{-\pi/2} + (-e^{\pi/2}) \right)$$

$$= \cancel{e} \ 2\pi i \left[e^{-\pi/2} - e^{\pi/2} \right]$$

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5. Bilinear Transform

$$\boxed{w = \frac{az+b}{cz+d}}$$

Q. Find the Bilinear Transform which maps the point $-i, 0, 2+i$ of z plane onto the points $0, -2i, 4$ of the w plane respectively.

Sol^n

z plane	w-plane
$z_1 = -i$	0
$z_2 = 0$	$-2i$
$z_3 = 2+i$	4

$$\therefore w = \frac{az+b}{cz+d} \quad \text{——} \quad (A)$$

be required Bilinear transform,

at $z = -i$, $w = 0$.

$$\therefore 0 = \frac{a(-i)+b}{c(-i)+d}$$

$$\therefore -ai + b = 0.$$

$$\therefore \boxed{b = ai} \quad \text{——} \quad (1)$$

at $z = 0$, $w = -2i$

$$\therefore -2i = \frac{a(0)+b}{c(0)+d}$$

$$\therefore -2id = b$$

$$\therefore -2id = ai \quad \text{-- from (1)}$$

$$\therefore \boxed{d = -\frac{1}{2}a} \quad \text{——} \quad (2)$$

at $z = 2+i$, $w = 4$.

$$\therefore 4 = \frac{a(2+i)+b}{c(2+i)+d}$$

$$\therefore 4 = \frac{2a+ai+b}{2c+ci+d}$$

$$\therefore 8c+4ci+4d = 2a+ai+b$$

$$8c+4ci+4\left(-\frac{1}{2}\right)a = 2a+ai+ai$$

$$\therefore 8c+4ci-2a = 2a+2ai$$

$$8c+4ci = 4a+2ai$$

$$\therefore c = \frac{4a+2ai}{8+4i}$$

$$\boxed{c = \left(\frac{2+i}{4+2i}\right)a}$$

$$\boxed{c = \frac{1}{2}a} \quad \text{—— (3)}$$

(A) $\Rightarrow$ $$w = \frac{az+ai}{\frac{1}{2}az-\frac{1}{2}a}$$

$$w = \frac{z+i}{\frac{1}{2}z-\frac{1}{2}}$$

$$\boxed{w = \frac{2z+2i}{z-1}}$$

Q. Find the Bilinear Transform which maps the points $1, 0, i$ of z plane onto the points $\infty, -2, -\frac{1}{2}(1+i)$ of w-plane respectively.

Sol:

z-plane	w-plane.
1	∞
0	-2
i	$-\frac{1}{2}(1+i)$

at $z = 1,$ $w = \infty$

$$\therefore \infty = \frac{a(1)+b}{c(1)+d}$$

$$\Rightarrow c+d = 0. \quad \therefore \boxed{c = -d.} \quad \text{—— (1)}$$

at $z = 0,$ $w = -2$

$$-2 = \frac{a(0)+b}{c(0)+d}$$

$$\therefore -2d = b$$

$$\therefore \boxed{b = -2d} \quad \text{—— (2)}$$

at $z = i$, $w = -\frac{1}{2}(1+i)$

$$-\frac{1}{2}(1+i) = \frac{a(i)+b}{c(i)+d}$$

$$\therefore ci+d\left[-\frac{1}{2}(1+i)\right] = ai+b.$$

$$\therefore -di+d\left[-\frac{1}{2}(1+i)\right] = ai-2d.$$

$\therefore \quad -di + 2d - \frac{1}{2}d(1+i) = ai$

$\therefore \quad -di - \frac{1}{2}di + 2d - \frac{1}{2}d = ai$

$\therefore \quad -\frac{3i}{2}d + \frac{3}{2}d = ai$

$\therefore \quad a = \frac{3}{2}\left[\frac{1-i}{i}\right]d$

$\boxed{a = \frac{3}{2}[i-1]d}$ —— (3)

$\therefore \quad w = \frac{az+b}{cz+d}$

$\therefore \quad w = \frac{3}{2}$

$\therefore \quad -\frac{1}{2}(1+i) = \frac{ai+b}{ci+di}$

$\therefore \quad -(1+i)(ci+d) = 2ai + 2b$

$\therefore \quad -(1+i)(-di+d) = 2ai + 2(-2d)$

$\therefore \quad (1+i)(di-d) = 2ai - 4d$

$\therefore \quad \cancel{di} - d^2 - d - \cancel{di} = 2ai - 4d$

$+2d = 2ai$

$\boxed{a = -di}$

$\therefore \quad w = \frac{az+b}{cz+d} = \frac{-diz - 2d}{-dz+d}$ $\quad \boxed{w = \frac{-iz-2}{-z+1}}$

* Cross Ratio Property

If points z_1, z_2, z_3 of z-plane are mapped into points w_1, w_2, w_3 of w-plane respectively then

$$\frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_1)(z_1-z_2)}{(z-z_3)(z_1-z_2)}$$

Q. Find the bilinear transformation which maps the points $-1, 0, 1$ of z-plane onto the points $0, i, 3i$ of w-plane respectively

Sol: z-plane $\qquad$ w-plane

z-plane	w-plane
$z_1 = -1$	$w_1 = 0$
$z_2 = 0$	$w_2 = i$
$z_3 = 1$	$w_3 = 3i$

$$\therefore \frac{(w-i)(0-3i)}{(w-3i)(0-i)} = \frac{[z-(0)][-1-1]}{[z-1][-1-0]}$$

$$\frac{-3\cancel{i}(w-i)}{-\cancel{i}(w-3i)} = \frac{(z-0)(-2)}{(z-1)(-1)}$$

$$\therefore \frac{-3w+3i}{w-3i} = \frac{-2z}{-z+1}$$

By componendo-dividendo

$$\frac{-3w+3i+w-3i}{-3w+3i-w+3i} = \frac{-2z-z+1}{-2z+z-1}$$

$$\frac{-2w}{-2w+6i} = \frac{-3z+1}{-z-1}$$

By regular method

let $w = \dfrac{az+b}{cz+d}$ —— (A)

B.be required Bilinear Transform.

at $z=-1,\ w=0.$

$\therefore A \Rightarrow \quad 0 = \dfrac{a(-1)+b}{c(-1)+d} \Rightarrow \boxed{b=a}$ —— (1)

at $z=0,\ w=i$

$\therefore i = \dfrac{a(0)+b}{c(0)+d} \Rightarrow \boxed{d=-bi}$ —— (2)

at $z=+1,\ w=3i$

$\therefore 3i = \dfrac{a(1)+b}{c(1)+d} \Rightarrow 3i(c+d) = a+b$

$3i[c-bi(i)] = a+a.$

$\therefore 3i(c+b) = 2a.$

$3i(c+a) = 2a.$

$c = \dfrac{2a}{3i} - a.$

$\boxed{c = \left[\dfrac{2-3i}{3i}\right] a.}$

$\therefore w = \dfrac{az+a}{\frac{2-3i}{3i}a - ai}$

$w = \dfrac{(z+1)\,3i}{2-3i+3i}$

$\boxed{w = \dfrac{3i(z+1)}{2}}$

1

$3i(c-ib) = a+b$

$3i(c-ib) = 2b$

$3ic + 3b = 2b$

$3ic = -b$

$c = \frac{1}{3} i b$

$$\boxed{c = \frac{i}{3} b}$$

$\therefore w = \frac{az+b}{cz+d}$

$w = \frac{bz+b}{\frac{i}{3}bz - ib}$

$w = \frac{3(z+1)}{(z-3)i}$

$$\boxed{w = \frac{3z+3}{zi-3i}}$$

Q. Find the map of the straight line $y = x$ under the transformation $w = \frac{z-1}{z+1}$

Sol: $w = \frac{z-1}{z+1} \longleftarrow (i)$

$\cancel{a=1,\ b=-1}$
$\cancel{c=1,\ d=1}$

$\frac{w+1}{w-1} = \frac{z-1+z+1}{\cancel{z}-1-\cancel{z}-1}$

$$\frac{-2(w+1)}{w-1} = 2z$$

$$\therefore z = \frac{-w-1}{w-1} \Rightarrow \boxed{z = \frac{1+w}{1-w}} \quad \text{— (2)}$$

$z = x+iy$ & $w = u+iv$

$$\therefore x+iy = \frac{1+u+iv}{1-u-iv}$$

$$x+iy = \frac{(1+u)+iv}{(1-u)-iv}$$

$$\therefore x+iy = \frac{[(1+u)+iv]\,[(1-u)+iv]}{[(1-u)-iv]\,[(1-u)+iv]}$$

$$= \frac{(1+u)(1-u) + v(1+u)i + v(1-u)i + i^2v^2}{(1-u)^2-(iv)^2}$$

$$x+iy = \frac{(1-u^2)+2vi-v^2}{(1-u)^2+v^2}$$

$$\therefore x+iy = \frac{1-u^2 \mp v^2+2vi}{(1-u)^2+v^2}$$

$$\therefore x = \frac{1-u^2-v^2}{(1-u)^2+v^2} \qquad y = \frac{2v}{(1-u)^2+v^2}$$

$\therefore x = y.$

$$\therefore \frac{1-u^2-v^2}{(1-u)^2+v^2} = \frac{2v}{(1-u)^2+v^2} \Rightarrow 1-u^2-v^2 = 2v$$

$$1-u^2 = \cancel{2v}\ v^2+2v$$

$$\cancel{u^2 = 1 - 3v}$$

$\therefore u^2 + v^2 + 2v - 1 = 0$

$\cancel{u^2 + (v-1)^2 = 0}$

$u^2 + v^2 + 2v - 2 + 1 = 0$

$\cancel{u^2 + (v+1)}$

$\boxed{u^2 + (v+1)^2 = 2}$

Centre $= (0, -1)$ $\boxed{\text{radius} = \sqrt{2}}$