Vector Differentiation.

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* Basic Formulae : Vector Algebra.

If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$,

$\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$,

$\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$

then,

1. $\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3 \longrightarrow$ component wise product of their addition.

2. $\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \longrightarrow$ its 3x3 determinant

Dot product $= \bar{a}\cdot\bar{b} \rightarrow$ Scalar quantity

Cross product $= \bar{a}\times\bar{b} \rightarrow$ vector quantity

3. Unit vector -

$\hat{a} = \dfrac{\bar{a}}{|\bar{a}|}$ where, $|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$|\bar{a}|^2 = \bar{a}\cdot\bar{a}$

4. Angle between $\bar{a}$ & $\bar{b}$, $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$

$\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|.|\bar{b}|}$

5. Scalar triple product :

$(\bar{a}\times\bar{b})\cdot\bar{c}$ or $\bar{a}\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(Vector are same then its value is ze

6. Vector Triple Product :

$\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

- **Properties** -
  1. $\bar{a}\times\bar{a} = 0$, $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$, $\bar{a}\times\bar{b} = -(\bar{b}\times\bar{a})$, $(\bar{a}\times\bar{b})\cdot\bar{a} =$
  2. If $\bar{a}$ is perpendicular to $\bar{b} \Rightarrow \bar{a}\cdot\bar{b} = 0$

- **Vector differentiation** -

1. Position vector or Radius vector $\bar{r}$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k};$$

$$d\bar{r} = \bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz.$$

2. If $\bar{u}, \bar{v}, \bar{w}$ are function of t and $\bar{u}, \bar{v}, \bar{w} \rightarrow t$

Then $$\frac{d}{dt}(\bar{u}\pm\bar{v}) = \frac{d\bar{u}}{dt} \pm \frac{d\bar{v}}{dt};$$

$$\frac{d}{dt}(\bar{u}\cdot\bar{v}) = \frac{d\bar{u}}{dt}\cdot\bar{v} + \bar{u}\cdot\frac{d\bar{v}}{dt};$$

$$\frac{d}{dt}(\bar{u}\times\bar{v}) = \frac{d\bar{u}}{dt}\times\bar{v} + \bar{u}\times\frac{d\bar{v}}{dt};$$

$$\frac{d}{dt}[\bar{u}\times\bar{v}\cdot\bar{w}] = \frac{d\bar{u}}{dt}\times\bar{v}\cdot\bar{w} + \bar{u}\times\frac{d\bar{v}}{dt}\cdot\bar{w} + \bar{u}\times\bar{v}\cdot\frac{d\bar{w}}{dt};$$

$$\frac{d}{dt}(s\bar{u}) = \frac{ds}{dt}\bar{u} + s\frac{d\bar{u}}{dt}$$; s is scalar.

$$\frac{d}{dt}\left(\frac{\bar{u}}{s}\right) = \frac{s\frac{d\bar{u}}{dt} - \bar{u}\frac{ds}{dt}}{s^2}$$

$$\frac{d}{dt}\left(\frac{\bar{u}}{\bar{v}}\right) \longrightarrow \text{not defined.}$$

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* Example - (Vector Differentiation)

8) Show that tangent at any point on the curve $x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$ makes constant angle with z-axis.

Soln.

$$x = e^t \cos t \quad , y = e^t \sin t \quad , z = e^t$$

We know,

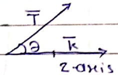
$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\text{Tangent vector} = \frac{d\bar{r}}{dt}$$

So,

$$\bar{T} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$$

To show: Angle between tangent vector $(\bar{T})$ and z-axis is constant.



$$\cos\theta = \frac{\bar{T}\cdot\bar{k}}{|\bar{T}|\,|\bar{k}|} \quad \text{---} \quad (1)$$

$$\bar{T} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$$

$$\bar{T} = \frac{d}{dt}(e^t\cos t)\bar{i} + \frac{d}{dt}(e^t \sin t)\bar{j} + \frac{d}{dt}(e^t)\bar{k}$$

$$\bar{T} = [e^t\cos t + e^t(-\sin t)]\bar{i} + [e^t \sin t + e^t\cos t]\bar{j} + e^t k$$

$$\bar{T} = e^t[(\cos t - \sin t)\bar{i} + (\sin t + \cos t)\bar{j} + \bar{k}]$$

$$|\bar{T}| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1}$$

$$= e^t\sqrt{\cos^2 t + \sin^2 t - 2\sin t\cos t + \sin^2 t + \cos^2 t + 2\sin t \cos t + 1}$$

$$= e^t\sqrt{1+1+1}$$

$= e^t \sqrt{3}$

$\bar{T} \cdot \bar{k} = 0 + 0 + e^t(1) = e^t$

$k(0,0,1)$ $\quad |\bar{k}| = \sqrt{0+0+1} = 1$

Putting above values in (1)

$$\cos\theta = \frac{\bar{T} \cdot \bar{k}}{|\bar{T}||\bar{k}|}$$

$$= \frac{e^t}{e^t \sqrt{3} \times 1}$$

$$= \frac{1}{\sqrt{3}}$$ $\Rightarrow$ Not in term of t.

$$\boxed{\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)} \rightarrow \text{constant.}$$

(11) The position vector of a particular at time t is,

$$\bar{r} = \cos(t-1)\,\bar{i} + \sinh(t-1)\,\bar{j} + mt^3\,\bar{k}$$

Find the condition imposed on m by requiring that at time $t = 1$, the acceleration is normal to the position vector.

Soln: To find m, such that acceleration is normal to the position vector.

i.e. acc^n $\perp$'ar position vector.

i.e. $$\frac{d^2 r}{dt^2} \cdot \bar{r} = 0 \longrightarrow (1)$$

at $t = 1$

$$\frac{d\bar{r}}{dt} = -\sin(t-1)\,\bar{i} + \cosh(t-1)\,\bar{j} + 3mt^2\,\bar{k}$$

$$\frac{d^2\bar{r}}{dt^2} = -\cos(t-1)\,\bar{i} + \sinh(t-1)\,\bar{j} + 6mt\,\bar{k}$$

at $t = 1$

$$\frac{d^2\bar{r}}{dt^2} = -\cos(0)\bar{i} + \sinh(0)\bar{j} + 6m\bar{k}$$

$$= -\bar{i} + 0\bar{j} + 6m\bar{k} \quad \text{--- (2)}$$

$$\bar{r} = \cos 0\,\bar{i} + \sinh(0)\bar{j} + m\bar{k}$$

$$\bar{r} = \bar{i} + 0\bar{j} + m\bar{k} \quad \text{--- (3)}$$

Put (2) & (3) in (1)

$$\frac{d^2\bar{r}}{dt} \cdot \bar{r} = 0$$

$$(-\bar{i} + 6m\bar{k}) \cdot (\bar{i} + m\bar{k}) = 0$$

$$(-1)(1) + 6m(m) = 0$$

$$6m^2 - 1$$

$$m^2 = \frac{1}{6}$$

$$\boxed{m = \pm \frac{1}{\sqrt{6}}}$$

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* Del operator (∇)

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

* Gradient of ϕ :-

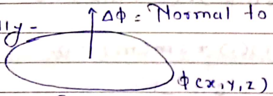
$$\nabla\phi = \text{grad of } \phi \quad (\phi \rightarrow x, y, z)$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$\nabla\phi$ = Normal to surface ϕ.
$\phi(x, y, z) = 0$.

<u>Note</u>: $\nabla\phi \cdot d\bar{r} = d\phi \longrightarrow$ total differential

<u>Geometrically</u> -



<u>(3D)</u>

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* Example :- (Gradient, Divergence and Curl)

e.g. $\phi = x^2 + 2xy^2 + z^3xy$

Find $\Delta\phi$ at $(1,1,1)$

Soln.

$$\frac{\partial\phi}{\partial x} = 2x + 2y^2 + z^3y$$

$$\frac{\partial\phi}{\partial y} = 0 + 4xy + z^3x(1)$$

$$\frac{\partial\phi}{\partial z} = 0 + 0 + 3z^2xy$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = i[2x + 2y^2 + z^3y] + \bar{j}[4xy + z^3x] + \bar{k}[3z^2xy]$$

at $(1,1,1)$

$$\nabla\phi = i(2+2+1) + j(4+1) + \bar{k}(3)$$

$$= 5\bar{i} + 5\bar{j} + 3\bar{k}$$

1 e.g. (B) $\phi = 2xz^4 - x^2y$, at $(2,-2,1)$

$$\frac{\partial\phi}{\partial x} = 2z^4 - y2x$$

$$\frac{\partial\phi}{\partial y} = 0 - x^2$$

$$\frac{\partial\phi}{\partial z} = 8xz^3 - 0$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \bar{i}\,[2z^4 - 2xy] + \bar{j}\,[-x^2] + \bar{k}\,[8xz^3]$$

at $(2,-2,1)$

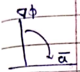
$$\nabla\phi = \bar{i}\,[2(1)^4 - 2(2)(-2)] + \bar{j}\,[-(2)^2] + \bar{k}\,[8(2)(1)^3]$$

$$= +10\,\bar{i} + (-4)\,\bar{j} + 16\,\bar{k}$$

$$\nabla\phi = 10\,\bar{i} - 4\,\bar{j} + 16\,\bar{k}$$

* Directional Derivative of ϕ along $\bar{a}$:

Directional Derivative of ϕ along $\bar{a} = \nabla\phi \cdot \hat{a}$



$\boxed{\text{The projection of } \nabla\phi \text{ on } \bar{a}}$ $= \nabla\phi \cdot \dfrac{\bar{a}}{|\bar{a}|}$

- To find $\bar{a}$:-

① at point $A(x_1, y_1, z_1)$ towards the point $B(x_2, y_2, z_2)$
Then, $\bar{a} = (x_2 - x_1)\bar{i} + (y_2 - y_1)\bar{j} + (z_2 - z_1)\bar{k}$

② Along the direction normal to the surface ϕ_1

$$\bar{a} = \nabla\phi_1$$

③ Along the tangent the curve $\bar{r}(t)$

$$\bar{a} = \frac{d\bar{r}}{dt}$$

④ Along the direction parallel to line

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}$$

$$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

⑤ Equally inclined with co-ordinate axes

$$\bar{a} = \bar{i} + \bar{j} + \bar{k}$$

Note: i) Magnitude of maximum directional derivative $= |\nabla\phi|$

ii) Maximum directional derivative of ϕ is in the direction of normal vector.

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* Example (Divergence)

(11) Find the directional derivative of $\phi = xy^2 + yz^3$ at $(1,-1,1)$.

Soln :

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}[y^2+0] + \bar{j}[2xy+z^3] + \bar{k}[0+3yz^2]$$

at $(1,-1,1)$

$$\nabla\phi = \bar{i} - \bar{j} - 3\bar{k} = \bar{i} - \bar{j} - 3\bar{k}$$

Directional derivative of ϕ along $\bar{a} = \nabla\phi \cdot \hat{a}$ ——— (1)

i) Along the vector $\bar{i} + 2\bar{j} + 2\bar{k}$:

$$\bar{a} = \bar{i} + 2\bar{j} + 2\bar{k}$$

$$|\bar{a}| = \sqrt{1+4+4}$$

$$= \sqrt{9}$$

$$= 3$$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{3}$$

$$D.D = \nabla\phi \cdot \vec{a}$$

$$= (\bar{i} - \bar{j} - 3\bar{k}) \cdot \left(\frac{\bar{i}+2\bar{j}+2\bar{k}}{3}\right)$$

$$= \frac{1(1) - 1(2) - 3(2)}{3}$$

$$\boxed{D.D = \frac{-7}{3}}$$

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ii) Towards the point $(2,1,-1)$

$(1,-1,1)$ $\qquad\qquad$ $(2,1,-1)$

A $\qquad\qquad\qquad$ B

$$\bar{a} = (2-1)\bar{i} + (1-(-1))\bar{j} + (-1-1)\bar{k}$$

$$\bar{a} = \bar{i} + 2\bar{j} - 2\bar{k}$$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i}+2\bar{j}-2\bar{k}}{\sqrt{1+4+4}}$$

$$= \frac{\bar{i}+2\bar{j}-2\bar{k}}{3}$$

$$D.D = \nabla\phi \cdot \hat{a}$$

$$= (\bar{i}-\bar{j}-3\bar{k}) \cdot \left(\frac{\bar{i}+2\bar{j}-2\bar{k}}{3}\right)$$

$$= \frac{1}{3}(1-2+6)$$

$$\boxed{D.D. = \frac{5}{3}}$$

iii) Along the direction normal to the surface

$$x^2+y^2+z^2=9$$

at $(1,2,2)$

$$\bar{a} = \nabla\phi = \bar{i}\frac{\partial\phi_1}{\partial x} + \bar{j}\frac{\partial\phi_1}{\partial y} + \bar{k}\frac{\partial\phi_1}{\partial z}$$

$$\bar{a} = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

at $(1,2,2)$

$$\bar{a} = 2\bar{i} + 4\bar{j} + 4\bar{k}$$

$$= 2(\bar{i}+2\bar{j}+2\bar{k})$$

$$|\bar{a}| = 2\sqrt{1+4+4}$$
$$= 2 \times 3$$
$$= 6.$$

$$D.D = \nabla\phi \cdot \hat{a}$$
$$= (\bar{i} - \bar{j} - 3\bar{k}) \cdot \frac{\cancel{2}(\bar{i}+2\bar{j}+2\bar{k})}{\cancel{6}\,3}$$
$$= \frac{1-2-6}{3}$$
$$\boxed{D.D = -\frac{7}{3}}$$

(6) Find directional derivative of xy^2+yz^3 at $(2,-1,1)$ along the line.
$2(x-2) = (y+1) = (z-1)$.

Soln.

Let $\phi = xy^2+yz^3$

$$\frac{2(x-2)}{\cancel{2}} = \frac{y+1}{2} = \frac{z-1}{2}$$
$$(1,2,2) = (\bar{i},\bar{j},\bar{k})$$
$$\bar{a} = \bar{i} + 2\bar{j} + 2\bar{k}$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$
$$= \bar{i}[y^2] + \bar{j}[2xy+z^3] + \bar{k}[3yz^2]$$

at $(2,-1,1)$
$$\nabla\phi = \bar{i} - 3\bar{j} - 3\bar{k}$$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{1+4+4}}$$

$$= \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})$$

$$D.D = \nabla\phi \cdot \hat{a}$$
$$= (\bar{i}-3\bar{j}-3\bar{k}) \cdot \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})$$

$$= \frac{1}{3}[1(1)-3(2)-3(2)]$$

$$\boxed{D.D = \frac{-11}{3}.}$$

(7) Find the directional derivative of the function $\phi = e^{2x-y-z}$ at $(1,1,1)$ in the direction of the tangent to the curve $x = e^{-t}$, $y = 2\sin t+1$, $z = t-\cos t$ at $t=0$.

Soln:

$$\phi = e^{2x-y-z}$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$
$$= \bar{i}e^{2x-y-z}(2) + \bar{j}e^{2x-y-z}(-1) + \bar{k}e^{2x-y-z}(-1)$$

at $(1,1,1)$

$$\nabla\phi = \bar{i}e^{2-1-1}(2) + \bar{j}e^{2-1-1}(-1) + \bar{k}e^{2-2-2}(-1)$$

$$\nabla\phi = 2\bar{i}-\bar{j}-\bar{k}$$

<u>To find</u> $\bar{a} = \frac{d\bar{r}}{dt}$

$$= \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$$

$$\bar{a} = \frac{d}{dt}(e^{-t})\bar{i} + \frac{d}{dt}(2\sin t+1)\bar{j} + \frac{d}{dt}(t-\cos$$

$$\bar{a} = -e^{-t}\bar{i} + 2\cos t\,\bar{j} + (1+\sin t)\bar{k}$$

at $t=0$,

$$\bar{a} = -e^{-0}\bar{i} + 2\cos 0\,\bar{j} + (1+\sin 0)\bar{k}$$

$$\bar{a} = -\bar{i}+2\bar{j}+\bar{k}$$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{-\bar{i}+2\bar{j}+\bar{k}}{\sqrt{1+4+1}}$$

$$= \frac{1}{\sqrt{6}}(-\bar{i}+2\bar{j}+\bar{k})$$

$$DD = \nabla\phi\cdot\hat{a}$$

$$= (2\bar{i}-\bar{j}-\bar{k})\cdot\left(\frac{-\bar{i}+2\bar{j}+\bar{k}}{\sqrt{6}}\right)$$

$$= \frac{1}{\sqrt{6}}(-2-2-1)$$

$$= -\frac{5}{\sqrt{6}}$$

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(5.) Find the directional derivative of

$\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$

Soln:

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= i[4z^3 - 6xy^3z] + j[0 - 6x^2yz] + \bar{k}[12xz^2 - 3x^2y^2 \cdot]$$

$$= i[4(2)^3 - 6(2)(-1)^3(2)] + j[-6(2)^2(-1)(2)] + \bar{k}[12(2)(2)^2 - 3(2)^2(-1)^2]$$

at $(2,-1,2)$

$$\nabla\phi = \bar{i}(8) + 48\bar{j} + 84\bar{k}$$

$$\nabla\phi = 8\bar{i} + 48\bar{j} + 84\bar{k}$$

(i) In the direction $2\bar{i} - 3\bar{j} + 6\bar{k}$

$$\bar{a} = 2\bar{i} - 3\bar{j} + 6\bar{k}$$

$$|\bar{a}| = \sqrt{4+9+36}$$

$$= \sqrt{49}$$

$$|\bar{a}| = 7.$$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \left(\frac{2\bar{i} - 3\bar{j} + 6\bar{k}}{7.}\right)$$

$$DD = \nabla\phi \cdot \hat{a}$$

$$= (8\bar{i} + 48\bar{j} + 84\bar{k}) \cdot \left(\frac{(2\bar{i} - 3\bar{j} + 6\bar{k})}{7}\right)$$

$$= \frac{1}{7}(8(2) + 48(-3) + 84(6))$$

$$= \frac{376}{7.}$$

(ii) Towards the point $\bar{i}+\bar{j}-\bar{k}$

$(2,-1,2)$ A ———— B $(1,1,-1)$

$$\bar{a} = (1-2)\bar{i} + (1-(-1))\bar{j} + (-1-2)\bar{k}$$

$$\bar{a} = -\bar{i} + 2\bar{j} - 3\bar{k}$$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{-\bar{i}+2\bar{j}-3\bar{k}}{\sqrt{1+4+9}}$$

$$= \left(\frac{-\bar{i}+2\bar{j}-3\bar{k}}{\sqrt{14}}\right)$$

$$D.D = \nabla\phi \cdot \hat{a}$$

$$= (8\bar{i}+48\bar{j}+84\bar{k}) \cdot \left(\frac{-\bar{i}+2\bar{j}-3\bar{k}}{\sqrt{14}}\right)$$

$$= \frac{1}{\sqrt{14}}\left(8(-1)+48(2)+84(-3)\right)$$

$$= \frac{-164}{\sqrt{14}}.$$

(iii) Along a line equally inclined with co-ordinate axes.

$$\bar{a} = \bar{i}+\bar{j}+\bar{k}$$

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{1+1+1}}$$

$$= \left(\frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}\right)$$

$$D.D = \nabla\phi \cdot \hat{a}$$

$$= (8\bar{i} + 48\bar{j} + 84\bar{k}) \cdot \left(\frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} (8(1) + 48(1) + 84(1))$$

$$= \frac{140}{\sqrt{3}}.$$

(12) If the directional derivative of $\phi = axy + byz + czx$ at (1,1,1) has maximum magnitude 4 in a direction parallel to x-axis, find the values of a,b,c.

Sol^n:

$$\phi = axy + byz + czx$$

at (1,1,1)

Max^m mgnitude = 4.

$$|\nabla\phi| = 4$$

Given Direction parallel to x-axis.

$$\therefore \nabla\phi = 4\bar{i} \quad \text{——— (1)}$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \bar{i}(ay+cz) + \bar{j}(ax+bz) + \bar{k}(by+cx)$$

at (1,1,1)

$$\nabla\phi = (a+c)\bar{i} + (a+b)\bar{j} + (b+c)\bar{k} \quad \text{——— (2)}$$

From ① and ②
(equating components)

$a + c = 4, \quad a + b = 0, \quad b + c = 0$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$a - c = 0 \qquad b = -c$

$a + a = 4 \qquad a = c$

$2a = 4$

$\boxed{a = 2}, \boxed{c = 2}, \boxed{b = -2.}$

Q(16) If directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at $(1,1,1)$ has maximum magnitude 15 in the direction parallel to

$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z}{1},$$

hence find the values of a, b, c.

Sol^n:

$$\phi = ax^2y + by^2z + cz^2x.$$

at $(1,1,1)$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \bar{i}\left[2axy + cz^2\right] + \bar{j}\left[ax^2 + 2byz\right] + \bar{k}\left[by^2 + 2czx\right]$$

at $(1,1,1)$ ∴

$$\nabla\phi = (2a+c)\bar{i} + (a+2b)\bar{j} + (b+2c)\bar{k} \longrightarrow ①$$

Given direction along the line.

$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z}{1}$$

Given direction $2\bar{i} - 2\bar{j} + \bar{k}$

$$\frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{b+2c}{1}$$

$$\Rightarrow \frac{2a+c}{2} = \frac{a+2b}{-2}$$

$\Rightarrow 2a+c = -a-2b$

$\Rightarrow 3a+2b+c = 0$ —— (2).

2nd & 3rd.

$$\frac{a+2b}{-2} = \frac{b+2c}{1}$$

$\Rightarrow a+2b = -2b-4c$

$\Rightarrow a+4b+4c = 0$ —— (3)

Solving (2) and (3) by Cramer's rule,

$$\frac{a}{\begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}}$$

$$\frac{a}{4} = \frac{-b}{11} = \frac{c}{10} = \lambda$$

$a = 4\lambda$, $b = -11\lambda$, $c = 10\lambda$

but, $|\nabla\phi| = 15$ --- (given).

$$\sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2} = 15$$

$$\sqrt{(8\lambda+10\lambda)^2 + (4\lambda-22\lambda)^2 + (-11\lambda+20\lambda)^2} = 15$$

$$\sqrt{(18\lambda)^2 + (-18\lambda)^2 + (9\lambda)^2} = 15$$

$$27\lambda = \pm 15$$

$$\boxed{\lambda = \pm \frac{5}{9}}$$

$$a = 4\lambda = 4\left(\pm \frac{5}{9}\right)$$

$$\boxed{a = \pm \frac{20}{9}}$$

$$b = -11\lambda = -11\left(\pm \frac{5}{9}\right)$$

$$\boxed{b = \pm \frac{55}{9}}$$

$$c = 10\lambda = 10\left(\pm \frac{5}{9}\right)$$

$$\boxed{c = \pm \frac{50}{9}}$$

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Very imp.

(17) Find the constants a and b, so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at point $(1,-1,2)$

Soln:

Let $\phi = ax^2 - byz - (a+2)x = 0$ —— (1)

$\psi = 4x^2y + z^3 - 4 = 0$ —— (2)

at point $(1,-1,2)$

Point $(1,-1,2)$ satisfies (1) & (2)

$a(1)^2 - b(-1)(2) - (a+2)(1) = 0$

$a + 2b - a - 2 = 0$

$\boxed{b = 1}$ put in (1).

$\phi = ax^2 - yz - (a+2)x = 0$ —— (1)

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$= [2ax - (a+2)]\bar{i} + \bar{j}[-z] + (-y)\bar{k}$

at $(1,-1,2)$

$\nabla\phi = (a-2)\bar{i} - 2\bar{j} + \bar{k}$ —— (3)

$$\nabla\psi = \bar{i}\frac{\partial\psi}{\partial x} + \bar{j}\frac{\partial\psi}{\partial y} + \bar{k}\frac{\partial\psi}{\partial z}$$

$\nabla\psi = \bar{i}[8xy] + \bar{j}(4x^2) + 3z^2\bar{k}$

at $(1,-1,2)$

$\nabla\psi = -8\bar{i} + 4\bar{j} + 12\bar{k}$

Given that ϕ is orthogonal with ψ

Normals are $\perp^{lar}$

$$\nabla\phi \cdot \nabla\psi = 0.$$

$$-8(a-2) + 4(-2) + 12(1) = 0$$

$$-\cancel{8}^{2}(a-2) = -\cancel{4}$$

$$2(a-2) = 1$$

$$a-2 = \frac{1}{2}$$

$$a = 2 + \frac{1}{2}$$

$$\boxed{a = \frac{5}{2}}$$

$$\boxed{a = \frac{5}{2}}, \quad \boxed{b = 1}$$

(8) Find the directional derivative of f at (1,2,-1) where $f(x,y,z) = x^2y + xyz + z^3$ along normal to the surface $x^3y^3 = 4xy + y^2z$ at the point (1,2,0).

Soln:

$$f = x^2y + xyz + z^3$$

$$\nabla f = \bar{i}[2xy + yz] + \bar{j}[x^2 + xz] + \bar{k}(xy + 3z^2)$$

at (1,2,-1).

$$\nabla f = \bar{i}[2(1)(2) + (2)(-1)] + \bar{j}[(1)^2 + (1)(-1)] + \bar{k}[(1)(2) + 3(-1)^2]$$

$$\nabla f = 2\bar{i} + 0\bar{j} + 5\bar{k}$$

Let, $\phi_1 = x^2y^3 - 4xy - y^2z = 0$

$\bar{a} = \nabla\phi_1$

$\nabla\phi = [2xy^3 - 4y)\bar{i} + (x^3 3y^2 - 4x - 2yz)\bar{j} + (-y^2\bar{k})$

at $(1,2,0)$

$= [2(1)(2)^3 - 4(2)]\bar{i} + [(1)^3 3(2)^2 - 4(1) - 2(2)(0)]\bar{j} + (-(2)^2)\bar{k}$

$= 8\bar{i} + 8\bar{j} - 4\bar{k}$

$D.D = \nabla f \cdot \frac{\bar{a}}{|a|}$

$= (2\bar{i} + 0\bar{j} + 5\bar{k}) \cdot \frac{(8\bar{i} + 8\bar{j} - 4\bar{k})}{\sqrt{64+64+16}}$

$= \frac{16 + 0 - 20}{\sqrt{144}}$

$= \frac{-4}{12}$

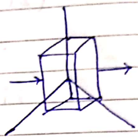
$\boxed{D.D. = -\frac{1}{3}}$

* Divergence of vector $(\bar{F})$:

$$\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

where, $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Results:



① If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$,

$\nabla \cdot \bar{r} = 1 + 1 + 1 = 3$

$$\boxed{\nabla \cdot \bar{r} = 3}$$

② $\nabla \cdot \bar{F} = 0 \Rightarrow$ ($\bar{F}$ is solenoidal vector field)

③ $\nabla \cdot \bar{F} \neq \bar{F} \cdot \nabla$

④ $(\bar{a} \cdot \nabla)\bar{r} = \bar{a}$

⑤ $\nabla \cdot \bar{a} = 0$, $\bar{a}$ constant vector.

* Curl of vector $(\bar{F})$:-

$$\text{Curl } \bar{F} = \nabla \times \bar{F}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Curl $\bar{F}$ = rotation of $\bar{F}$

= rot$(\bar{F})$

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Results:

1. $\nabla \times \bar{r} = \bar{0}$

2. $\nabla \times \bar{F} = \bar{0} \implies \bar{F}$ is irrotational

3. Curl $\bar{V} = 2\bar{\omega} \implies \bar{V}$ = velocity
   $\bar{\omega}$ = angular velocity

4. Curl grad $\phi = 0$
   i.e. $\nabla \times (\nabla\phi) = \bar{0}$

   Div curl $\bar{u} = 0$
   i.e. $\nabla \cdot (\nabla \times \bar{u}) = 0$

5. $\nabla \times \bar{a} = 0$,
   $\bar{a}$ = constant vector.

6. Scalar potential

$$\bar{F} = \nabla\phi,$$

$$d\bar{F} \cdot d\bar{r} = \int_{y,z\ \text{const}} F_1\,dx + \int_{z\text{-const},\ \text{No } x} F_2\,dx + \int_{\text{No } x,y} F_3\,dz + c$$

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* Examples :- (Divergence and Curl)

(18) Show that the vector field given by $\bar{F} = (y^2\cos x + z^2)\bar{i} + (2y\sin x)\bar{j} + 2xz\bar{k}$ is conservative and find scalar field such that $\bar{F} = \nabla\phi$.

Soln:

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^2 & 2y\sin x & 2xz \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(2xz) - \frac{\partial}{\partial z}(2y\sin x)\right] - \bar{j}\left[\frac{\partial}{\partial x}(2xz) - \frac{\partial}{\partial z}(y^2\cos x + z^2)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}(2y\sin x) - \frac{\partial}{\partial y}(y^2\cos x + z^2)\right]$$

$$= \bar{i}[0-0] - \bar{j}[2z-2z] + \bar{k}[2y\cos x - 2y\cos x]$$

$$\nabla \times \bar{F} = \bar{0}$$

$\therefore \bar{F}$ is irrotational vector field.

To find ϕ -

$$d\phi = \bar{F}\cdot d\bar{r}$$

$$\phi = \underset{y,z\ \text{const}}{\int F_1\,dx} + \underset{\text{No } x,\ z\text{-const}}{\int F_2\,dy} + \underset{\text{No } x,y}{\int F_3\,dz} + c$$

$$\phi = \int (y^2\cos x + z^2)\,dx + \int 2y\sin x\,dy + \int 2xz\,dz + c$$

$$= y^2 \int \cos x \, dx + z^2 \int 1 \, dx + 0 + c$$

$$\phi = y^2 \sin x + z^2 x + c.$$

(10) Show that $\bar{F} = (2xz^3 + 6y)\bar{i} + (6x - 2yz)\bar{j} + \cancel{(4x+6y+} (3x^2z^2 - y^2)\bar{k}$ is irrotational. Find scalar potential ϕ such that $\bar{F} = \nabla\phi$.

Sol^n:

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz^3+6y & 6x-2yz & 3x^2z^2-y^2 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(3x^2z^2 - y^2) - \frac{\partial}{\partial z}(6x - 2yz)\right]$$

$$- \bar{j}\left[\frac{\partial}{\partial x}(3x^2z^2 - y^2) - \frac{\partial}{\partial z}(2xz^3 + 6y)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}(6x - 2yz) - \frac{\partial}{\partial y}(2xz^3 + 6y)\right]$$

$$= \bar{i}[-2y - (-2y)] - \bar{j}[6xz^2 - 6xz^2] + \bar{k}[6 - 6]$$

$$\nabla . \bar{F} = \bar{0}$$

$\therefore$ $\bar{F}$ is irrotational.

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To find ϕ :-

$$d\phi = \bar{F}\cdot d\bar{r}$$

$$\phi = \int_{y,z\text{-const}} F_1\,dx + \int_{\substack{\text{No }x\\ z\text{-const.}}} F_2\,dy + \int_{\text{No }x,y} F_3\,dz + C$$

$$= \int (2xz^3+6y)\,dx + \int (6x-2yz)\,dy + \int (3x^2z^2-y^2)$$

$$= \frac{\cancel{2}x^2}{\cancel{2}}z^3 + 6xy - \frac{\cancel{2}y^2}{\cancel{2}}z + 0 + C.$$

$$\phi = x^2z^3 + 6xy - y^2z + C.$$

(19) If the vector field $\bar{F} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is irrotational, find a, b, c and determine ϕ such that $\bar{F} = \nabla\phi$.

Soln:

$$\bar{F} = (x+2y+az)\,\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$$

$$\bar{F}.\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\therefore\ \bar{i}\,[c-(-1)] - \bar{j}\,[4-a] + \bar{k}\,[b-2] = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

compairing, both sides

$\bar{i}$: $c+1=0$, $4-a=0$, $b-2=0$

$\boxed{c=-1}$, $\boxed{a=4}$, $\boxed{b=2}$

$\bar{F} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$

$$\phi = \underset{y,z\text{ const}}{\int F_1\,dx} + \underset{\substack{\text{No } x\\ z\text{-const}}}{\int F_2\,dy} + \underset{\text{No } x,y}{\int F_3\,dz} + c$$

$$\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + \frac{2z^2}{2} + c.$$

(21) Show that vector field $\bar{F}\,(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is irrotational. Find scalar potential ϕ such that $\bar{F} = \nabla\phi$.

Sol.

$$\Delta\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= \bar{i}\left((-x)-(-x)\right) - \bar{j}\left((-y)-(-y)\right) + \bar{k}\left(-z-(-z)\right)$$

$\therefore \Delta\times\bar{F} = 0$

$\therefore \bar{F}$ is irrotational.

To find ϕ: $d\phi = \bar{F}\cdot d\bar{r}$

$$\phi = \underset{y,z\text{-const}}{\int F_1\,dx} + \underset{\substack{\text{No } x\\ z\text{-const}}}{\int F_2\,dy} + \underset{\substack{\text{No}\\ x,y}}{\int F_3\,dz} + c$$

$$= \underset{y,z\text{ const}}{\int (x^2-yz)\,dx} + \underset{\substack{\text{No } x\\ z\text{-const}}}{\int (y^2-zx)\,dy} + \underset{\substack{\text{No}\\ x,y}}{\int (z^2-xy)\,dz} + c$$

$$= \int (x^2-yz)\,dx + \int y^2\,dy + \int z^2\,dz + c$$

$$= \frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3} + c$$

* Vector Identities :-

1) $\text{Div}(\bar{u}\times\bar{v}) = \text{curl}\,\bar{u}\cdot\bar{v} - \bar{u}\cdot\text{curl}\,\bar{v}$

$\nabla\cdot(\bar{u}\times\bar{v}) = (\nabla\times\bar{u})\cdot\bar{v} - \bar{u}\cdot(\nabla\times\bar{v})$

Ex: If $\bar{u}$ and $\bar{v}$ are irrotational vectors then show that $\bar{u}\times\bar{v}$ is solenoidal vector.

Sol^n:

Given,

$\bar{u}$ & $\bar{v}$ are irrotational

$\nabla\times\bar{u} = \bar{0}$, $\nabla\times\bar{v} = 0$

To show,

$\bar{u}\times\bar{v}$ is solenoidal

$\nabla\cdot(\bar{u}\times\bar{v}) = 0$

We have,

$$\nabla\cdot(\bar{u}\times\bar{v}) = (\nabla\times\bar{u})\cdot\bar{v} - \bar{u}\cdot(\nabla\times\bar{v})$$

$$= \bar{0}\cdot\bar{v} - \bar{u}\cdot\bar{0} \quad \text{---}(\nabla\times\bar{F} = \bar{0})$$

$$\nabla\cdot(\bar{u}\times\bar{v}) = \bar{0}$$

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② $\nabla \times \nabla \times \bar{u} = \nabla(\nabla \cdot \bar{u}) - (\nabla \cdot \nabla)\bar{u}$

$= \nabla(\nabla \cdot \bar{u}) - \nabla^2 \bar{u}$

Ex: If $\bar{E}$ is solenoidal vector then show that curl curl curl curl $\bar{E} = \nabla^4 E$

Soln: E is solenoidal vector

$\nabla \cdot \bar{E} = 0$

To show:

$$\nabla \times \nabla \times \underbrace{\nabla \times \nabla \times \bar{E}}_{\bar{F}} = \nabla^4 E$$

Let $\bar{F} = \nabla \times \nabla \times \bar{E}$

$= \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E}$

$\nabla \cdot \bar{E} \to 0$ -- (given)

$= -\nabla^2 \bar{E}$.

$$L.H.S = \nabla \times \nabla \times \underbrace{\nabla \times \nabla \times \bar{F}}_{\bar{F}}$$

$= \nabla \times \nabla \times \bar{F}$

$= \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$

$= \nabla[\nabla \cdot (-\nabla^2 F)] - \nabla^2[-\nabla^2 \bar{E})$

$= \nabla[-\nabla^2(\nabla \cdot \bar{E})] + \nabla^4 \bar{E}$

$\nabla^2 \to$ scalar, $(\nabla \cdot \bar{E}) \to 0$ --- given,

$= \nabla^4 \bar{E}$

(3) $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$2r\dfrac{\partial r}{\partial x} = 2x \quad , \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$\dfrac{\partial r}{\partial y} = \dfrac{y}{r} \quad , \quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

By defination,

$$\nabla r = \bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}$$

$$= \frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}$$

$$\nabla r = \frac{x\bar{i} + y\bar{j} + z\bar{k}}{r}$$

$$\boxed{\nabla r = \frac{\bar{r}}{r}}$$

$$\nabla f(r) = f'(r)\nabla r$$

$$= f'(r)\frac{\bar{r}}{r}$$

$$\nabla r^n = n r^{n-1}\frac{\bar{r}}{r}$$

$$= n r^{n-2}\bar{r}$$

$$\nabla \log r = \frac{1}{r}\frac{\bar{r}}{r}$$

$$= \frac{\bar{r}}{r^2}$$

$\nabla \log$

$$\nabla(r^2 e^{3r}) = \left[r^2 3e^{3r} + 2r e^{3r}\right]\frac{\bar{r}}{r}$$

$$= \left(3r e^{3r} + 2e^{3r}\right)\bar{r}$$

$$= (3r+2)e^{3r}\cdot\bar{r}$$

$$\nabla\left(\frac{r^2}{\log r}\right) = \left[\frac{(\log r)2r - r^2(1/r)}{(\log r)^2}\right]\frac{\bar{r}}{r}$$

Result :-

$\nabla\cdot\bar{r} = 3$, $\nabla\times\bar{r} = \bar{0}$

$\nabla\cdot\bar{a} = 0$, $\nabla\times\bar{a} = \bar{0}$

$\bar{r}\cdot\bar{r} = r^2$, $\bar{r}\times\bar{r} = \bar{0}$

Ex f $\nabla r^2 = 2r\nabla r$

$$= 2\cancel{r}\frac{\bar{r}}{\cancel{r}}$$

$$= 2\bar{r}$$

- $\nabla r^n = n r^{n-1}\dfrac{\bar{r}}{r}$, $\nabla r^n = n r^{n-2}\,\bar{r}$

Ex $\nabla(r^2 e^{-3r}) = \left[r^2(-3e^{-3r}) + 2re^{-3r}\right]\dfrac{\bar{r}}{r}$

(4) $\nabla\cdot(\phi\bar{u}) = \phi(\nabla\cdot\bar{u}) + (\nabla\phi)\cdot\bar{u}$

$\nabla\times(\phi\bar{u}) = \phi(\nabla\times\bar{u}) + (\nabla\phi)\times\bar{u}$

Where,

ϕ = Scalar

$\left(\phi \longrightarrow r, r^2, r^n, f(r), f'(r), \psi, \bar{a}\cdot\bar{r} \right)$

$\bar{u}$: Vector

$\bar{u} \longrightarrow \bar{r}, \bar{a}\times\bar{r}, \nabla\phi, \nabla\psi$

- Examples :- (show that)

(1) $\nabla\cdot[r^3\bar{r}] = 6r^3$

Soln:

$$L.H.S = \nabla\cdot[\underbrace{r^3}_{\phi}\underbrace{\bar{r}}_{u}]$$

$$= \phi(\nabla\cdot\bar{u}) + (\nabla\phi)\cdot\bar{u}$$

$$= r^3(\nabla\cdot\bar{r}) + (\nabla r^3)\cdot\bar{r}$$

$$= r^3(3) + 3r^2\frac{\bar{r}}{r}\cdot\bar{r} \quad ---(\bar{r}\cdot\bar{r} = r^2)$$

$$= 3r^3 + 3r^3$$

$$= \underline{6r^3}$$

② $\nabla \cdot \left[r \nabla \left(\frac{1}{r^3} \right) \right] = \frac{3}{r^4}$

Sol^n:

$$= \nabla \cdot \left(r \left(-3r^{-4} \frac{\bar{r}}{r} \right) \right]$$

$$= -3 \nabla \cdot [\underbrace{r^{-4}}_{\phi} \underbrace{\bar{r}}_{\bar{u}}]$$

$$= -3 [\phi (\nabla \cdot \bar{u}) + (\nabla \phi) \cdot \bar{u}]$$

$$= -3 [r^{-4} (\nabla \cdot \bar{r}) + (\nabla r^{-4}) \cdot \bar{r}]$$

$$= -3 [r^{-4} (3) + (-4r^{-5} \frac{\bar{r}}{r} \cdot \bar{r})] \quad -- (\bar{r} \cdot \bar{r} = r^2)$$

$$= -3 [3r^{-4} - 4r^{-4}]$$

$$= -3 [-r^{-4}]$$

$$= +3r^{-4}$$

$$= \frac{3}{r^4}$$

③ $\nabla \times (\bar{a} \times \bar{r}) = 2\bar{a} \implies$ Result

Sol^n:

$$\bar{a} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cancel{\partial/\partial x} & \cancel{\partial/\partial y} & \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= (a_2 z - a_3 y) i - (a_1 z - a_3 x) j + (a_1 y - a_2 x) \bar{k}$$

$$\nabla \times (\bar{a} \times \bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2 z - a_3 y & -a_1 z - a_3 x & a_1 y - a_2 x \end{vmatrix}$$

$$= \bar{i}[a_1 - (-a_1)] - \bar{j}[-a_2 - a_2] + \bar{k}[a_3 - (-a_3)]$$

$$= 2a_1 \bar{i} + 2a_2 \bar{j} + 2a_3 \bar{k}$$

$$\boxed{\nabla \times (\bar{a} \times \bar{r}) = 2\bar{a}}$$

(4) $\nabla(\bar{a} \cdot \bar{r}) = \bar{a}$

Soln:

$$\bar{a} \cdot \bar{r} = a_1 x + a_2 y + a_3 z$$

$$\nabla(\bar{a} \cdot \bar{r}) = \bar{i} \frac{\partial}{\partial x}(a_1 x + a_2 y + a_3 z) + \bar{j} \frac{\partial}{\partial y}(a_1 x + a_2 y + a_3 z)$$

$$+ \bar{k} \frac{\partial}{\partial z}(a_1 x + a_2 y + a_3 z)$$

$$= a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}$$

$$= \bar{a}$$

(5) $\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \frac{(2-n)}{r^n} \bar{a} + \frac{n}{r^{n+2}} (\bar{a} \cdot \bar{r}) \bar{r}$

Soln:

$$\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \nabla \times [\underbrace{r^{-n}}_{\phi} \underbrace{(\bar{a} \times \bar{r})}_{\bar{u}}]$$

$$= \phi(\nabla \times \bar{u}) + (\nabla \phi) \times \bar{u}$$

$$= r^{-n}\,\underbrace{(\nabla \times \bar{a} \times \bar{r})}_{2\bar{a}} + (\nabla r^{-n}) \times (\bar{a} \times \bar{r})$$

$$= r^{-n}\, 2\bar{a} + \left(-n r^{-n-1}\, \frac{\bar{r}}{r} \times \bar{a} \times \bar{r}\right)$$

$$= \frac{2\bar{a}}{r^{n}} - \frac{n}{r^{n+2}}\,(\bar{r} \times \bar{a} \times \bar{r})$$

$$\left(\bar{a} \times \bar{b} \times \bar{c} = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}\right)$$

$$= \frac{2\bar{a}}{r^{n}} - \frac{n}{r^{n+2}}\left[\underbrace{(\bar{r}\cdot\bar{r})}_{r^2}\bar{a} - (\bar{r}\cdot\bar{a})\bar{r}\right]$$

$$= \frac{2\bar{a}}{r^{n}} - \frac{n}{r^{n+2}}\left[r^2\bar{a} - (\bar{a}\cdot\bar{r})\bar{r}\right]$$

$$= \frac{2\bar{a}}{r^{n}} - \frac{n}{r^{n+2}}\,r^2\bar{a} + \frac{n}{r^{n+2}}\,(\bar{a}\cdot\bar{r})\bar{r}$$

$$= \frac{2\bar{a}}{r^{n}} - \frac{n}{r^{n}}\,\bar{a} + \frac{n}{r^{n+2}}\,(\bar{a}\cdot\bar{r})\,\bar{r}$$

$$= (2-n)\,\frac{\bar{a}}{r^{n}} + \frac{n}{r^{n+2}}\,(\bar{a}\cdot\bar{r})\bar{r}$$

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(6) $\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^3} \right) = \frac{-\bar{a}}{r^3} + \frac{3(\bar{a}\cdot\bar{r})\bar{r}}{r^5}$

Solⁿ:

$$\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^3} \right) = \nabla \Big(\underset{\phi}{\underset{\Downarrow}{r^{-3}}} \underset{\bar{u}}{\underset{\Downarrow}{(\bar{a} \times \bar{r})}} \Big)$$

$$= \phi(\nabla \times \bar{u}) + (\nabla \phi) \times \bar{u}$$

$$= r^{-3} (\underset{2\bar{a}}{\underset{\Downarrow}{\nabla \times \bar{a} \times \bar{r}}}) + (\nabla r^{-3}) \times (\bar{a} \times \bar{r})$$

$$= r^{-3} 2\bar{a} + \left(-3 r^{-4} \frac{\bar{r}}{r} \times \bar{a} \times \bar{r} \right)$$

$$= \frac{2\bar{a}}{r^3} - \frac{3}{r^4 \cdot r} (\bar{r} \times \bar{a} \times \bar{r})$$

$(\bar{a} \times \bar{b} \times \bar{c}) = [(\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}]$

$$= \frac{2\bar{a}}{r^3} - \frac{3}{r^5} [\bar{r}^2 \bar{a} - (\bar{a}\cdot\bar{r})\bar{r}]$$

$$= \frac{2\bar{a}}{r^3} - \frac{3}{r^5} r^2 \bar{a} + \frac{3}{r^5} (\bar{a}\cdot\bar{r})\bar{r}$$

$$= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3}{r^5} (\bar{a}\cdot\bar{r})\bar{r}$$

$$= (2-3)\frac{\bar{a}}{r^3} + \frac{3}{r^5} (\bar{a}\cdot\bar{r})\bar{r}$$

$$= \frac{-\bar{a}}{r^3} + \frac{3}{r^5} (\bar{a}\cdot\bar{r})\bar{r}$$

$$\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^3} \right) = \frac{-\bar{a}}{r^3} + \frac{3}{r^5} (\bar{a}\cdot\bar{r})\bar{r}$$

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(7) $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$

Solⁿ:

$$\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \frac{r^n\nabla(\bar{a}\cdot\bar{r}) - (\bar{a}\cdot\bar{r})\nabla r^n}{(r^n)^2}$$

$$= \frac{r^n\bar{a}}{(r^n)^2} - \frac{(\bar{a}\cdot\bar{r})\left[nr^{n-1}\frac{\bar{r}}{r}\right]}{(r^n)^2}$$

$$= \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$$

(8) $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r) \Longrightarrow$ Result

Solⁿ:

$$\nabla^2 f(r) = \nabla\cdot\nabla f(r)$$

$$= \nabla\cdot\left[f'(r)\frac{\bar{r}}{r}\right]$$

$$= \nabla\cdot\Big[\underbrace{\left(\frac{f'(r)}{r}\right)}_{\phi}\underbrace{\bar{r}}_{\bar{u}}\Big]$$

$$= \phi(\nabla\cdot\bar{u}) + (\nabla\phi)\cdot\bar{u}$$

$$= \left(\frac{f'(r)}{r}\right)\underbrace{(\nabla\cdot\bar{r})}_{3} + \left(\nabla\frac{f'(r)}{r}\right)\cdot\bar{r}$$

$$= \frac{3f'(r)}{r} + \left(\frac{rf''(r) - f'(r)}{r^2}\right)\frac{\bar{r}}{r}\cdot\bar{r}$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$\boxed{\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)}$$

9) $\nabla^4 e^r = e^r + \frac{4}{r} e^r$

Sol^n:

$$\nabla^4 e^r = \nabla^2 \nabla^2 e^r$$

$$= \nabla^2 [\underbrace{\nabla^2 e^r}_{\Downarrow}]$$

$$= \nabla^2 f(r)$$

$$= \nabla^2 \left[f''(r) + \frac{2}{r} f'(r) \right]$$

but $f(r) = e^r$, $f'(r) = e^r$, $f''(r) = e^r$

$$= \nabla^2 \Big[\underbrace{e^r + \frac{2}{r} e^r}_{F(r)} \Big]$$

$$= F''(r) + \frac{2}{r} F'(r) \quad \text{—— } (*)$$

But $F(r) = e^r + \frac{2}{r} e^r$

$$F'(r) = e^r + \frac{2}{r} e^r - \frac{2}{r^2} e^r$$

$$F''(r) = e^r + \frac{2}{r} e^r - \frac{2}{r^2} e^r - \frac{2}{r^2} e^r - 2(-2r^{-3}) e^r$$

$$F''(r) = e^r + \frac{2}{r} e^r - \frac{4}{r^2} e^r + \frac{4}{r^3} e^r$$

Put in (*)

$$\nabla^4 e^r = F''(r) + \frac{2}{r} F'(r)$$

$$= \left[e^r + \frac{2}{r} e^r - \cancel{\frac{4}{r^2}} e^r + \cancel{\frac{4}{r^3}} e^r + \frac{2}{r} e^r + \cancel{\frac{4}{r^2}} e^r - \cancel{\frac{4}{r^3}} e^r \right]$$

$$\nabla^4 e^r = e^r + \frac{4}{r} e^r$$

V.Imp.

(10) $\nabla^4 (r^2 \log r) = \dfrac{6}{r^2}$

Soln:

$$= \nabla^4 (\underbrace{r^2 \log r}_{f(r)})$$

$$f'(r) = r^{\cancel{2}} \times \frac{1}{\cancel{r}} + 2r \log r$$

$$f'(r) = r + 2r \log r$$

$$f''(r) = 1 + 2\cancel{r} \times \frac{1}{\cancel{r}} + 2 \log r$$

$$= 3 + 2 \log r$$

$$\nabla^2 [\nabla^2 f(r)] = \nabla^2 \left[f''(r) + \frac{2}{r} f'(r) \right]$$

$$= \nabla^2 \left[(3 + 2 \log r) + \frac{2}{r} (r + 2r \log r) \right]$$

$$= \nabla^2 \left[3 + 2\log r + 2 + 4\log r \right]$$

$$= \nabla^2 \left[5 + 6\log r \right] \rightarrow \underline{\underline{F}}.$$

$$= F'' + \frac{2}{r} F'$$

$$= \frac{-6}{r^2} + \frac{2}{r}\left(\frac{6}{r}\right)$$

$$= \underline{\underline{\frac{6}{r^2}}}.$$

- **Formula -**

(1) $\bar{r}\cdot d\bar{r} = r\,dr$

(2) $\bar{a}\cdot d\bar{r} = d(\bar{a}\cdot\bar{r})$

Ex.- To show $\bar{F}$ is irrotional (Conservative)

i.e. $\nabla\times\bar{F} = \bar{0}$

- Show that $\bar{F}$ is irrotational and scalar potential ϕ such that $\bar{F} = \nabla\phi$

(1) $\bar{F} = \dfrac{\bar{r}}{r^2}$

$\Rightarrow$ $\nabla\times\bar{F} = \nabla\times\dfrac{\bar{r}}{r^2}$

$$= \nabla\times(\underbrace{r^{-2}}_{\phi}\,\underbrace{\bar{r}}_{\bar{u}})$$

$$= \phi(\nabla\times\bar{u}) + (\nabla\phi)\times\bar{u}$$

$$= r^{-2}\underbrace{(\nabla\times\bar{r})}_{\bar{0}} + (\nabla r^{-2})\times\bar{r}$$

$$= 0 + (-2)r^{-3}\frac{\bar{r}}{r}\times\bar{r}$$

--- $(\bar{r}\times\bar{r} = \bar{0})$

$$= \bar{0}$$

$\bar{F}$ is irrotational.

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To find ϕ -

$$d\phi = \bar{F} \cdot d\bar{r}$$

$$d\phi = \frac{\bar{r}}{r^2} \cdot d\bar{r} \text{ —— } r\,dr.$$

$$= \frac{r\,dr}{r^2}$$

$$d\phi = \frac{dr}{r}.$$

On integrating,

$$\boxed{\phi = \log r + c}$$

(11) $\bar{F} = \bar{r} r^2$

$$\nabla \times \bar{F} = \nabla \times \bar{r} r^2$$

$$= \phi(\nabla \times \bar{u}) + (\nabla \phi) \times \bar{u}$$

$$= r^2(\nabla \times \bar{r}) + (\nabla r^2) \times \bar{r}$$

$$= 0 + 2r \cdot \frac{\bar{r}}{r} \times \bar{r} \quad \text{—} (\bar{r} \times \bar{r} = \bar{0})$$

$$= \bar{0}$$

F is irrotational.

To find ϕ -

$$d\phi = \bar{F} \cdot d\bar{r}$$

$$= \bar{r} r^2 \cdot d\bar{r}$$

$$d\phi = r^3 \cdot dr$$

$$\phi = \frac{r^4}{3} + c$$

(iii) $\nabla \times \bar{F} = (\bar{a}\cdot\bar{r})\bar{a}$

$$\Rightarrow \nabla\times\bar{F} = \nabla\times(\underset{\phi}{\underset{\downarrow}{\bar{a}\cdot\bar{r}}})\underset{\bar{u}}{\underset{\downarrow}{\bar{a}}}$$

$$= (\bar{a}\cdot\bar{r})\underbrace{(\nabla\times\bar{a})}_{0} + \underbrace{\nabla(\bar{a}\cdot\bar{r})}_{\bar{a}}\times\bar{a}$$

$$= \bar{0} + \underbrace{\bar{a}\times\bar{a}}_{\bar{0}}$$

$$= \bar{0} \quad \cdots (F \text{ is irrotational})$$

To find ϕ :-

$$d\phi = \bar{F}\cdot d\bar{r}$$

$$= (\bar{a}\cdot\bar{r})\,\bar{a}\cdot d\bar{r}$$

$$= (\bar{a}\cdot\bar{r})\,d(\bar{a}\cdot\bar{r})$$

$$d\phi = x\,dx$$

$$\phi = \frac{x^2}{2} + c$$

$$\phi = \frac{(\bar{a}\cdot\bar{r})^2}{2} + c.$$

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Vector Integration.

Line Integral = Work done.

$$= \int_C \bar{F} \cdot d\bar{r}$$

- **Conservative vector field :-**

$\bar{F}$ is said to be conservative field if $\bar{F} = \nabla\phi$ ($\Rightarrow \nabla \times \bar{F} = \bar{0}$)

Note :

If $\bar{F} = \nabla\phi$ (conservative) then.

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \nabla\phi \cdot d\bar{r} = \int_C^B d\phi = [\phi]_A^B = \phi_B - \phi_A$$

For closed curve,

$$A = B \ , \int_C \bar{F} \cdot d\bar{r} = \phi_A - \phi_A$$
$$= 0.$$

i.e. $\bar{F}$ conservative $\Leftrightarrow \int_C \bar{F} \cdot d\bar{r} = 0$ if c is closed curve

$\Leftrightarrow$ Workdone = line integral = 0.

$$\boxed{\text{Workdone} = \bar{F} \cdot d\bar{r}}$$

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Type-1

* Examples -

(4) Find the work done in moving a particle from $(0,1,-1)$ to $(\frac{\pi}{2},-1,2)$ in a force field
$\bar{F} = (y^2\cos x + z^3)\bar{i} + (2y\sin x - 4)\bar{j} + (3xz^2+2)\bar{k}$.
is the field conservative?

Sol^n:

$$\bar{F} = (y^2\cos x + z^3)\bar{i} + (2y\sin x - 4)\bar{j} + (3xz^2+2)\bar{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2+2 \end{vmatrix}$$

$$= \bar{i}[0-0] - \bar{j}[3z^2 - 3z^2] + \bar{k}[2y\cos x - 2y\cos x]$$

$$= 0.$$

$\therefore \nabla \times \bar{F} = \bar{0}$

$\therefore \bar{F}$ is conservative.

To find work done :-

$$W.D = \int \bar{F}\cdot d\bar{r}$$

$$= \underset{y,z\ \text{const}}{\int F_1 dx} + \underset{\substack{\text{No } x \\ z\text{-const}}}{\int F_2 dy} + \underset{\substack{\text{No} \\ x,y}}{\int F_3 dz}.$$

$$= y^2\int \cos x\, dx + z^3\int 1 dx + \int -4\, dy + \int 2dz$$

$$= \left[y^2\sin x + z^3x - 4y + 2z \right]_{(0,1,-1)}^{(\pi/2,-1,2)}$$

$$= \left[(-1)^2 \sin\frac{\pi}{2} + (2)^3\frac{\pi}{2} - 4(-1) + 2(2) \right] -$$

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$= 1 + 4\pi + 4 + 4 + 4 + 2$

W.D $= \underline{\underline{15 + 4\pi}}$

(6) Find work done by the Force $(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ in taking a particle $(1,1,1)$ to $(3,-5,7)$.

Soln.

$\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$= i[-x - (-x)] - j[-y - (-y)] + \bar{k}[-z - (-z)]$

$\therefore \nabla \times \bar{F} = 0$

$\therefore \bar{F}$ is conservative

To find workdone :-

$W.D = \bar{F} \cdot d\bar{r}$

$$= \int_{y,z\text{-const}} F_1\,dx + \int_{\substack{\text{Now}\\ z\text{-const}}} F_2\,dy + \int_{\substack{\text{No}\\ x,y}} F_3\,dz$$

$$= \int (x^2 - yz)\,dx + \int y^2\,dy + \int z^2\,dz$$

$$= \left[\frac{x^3}{3} - xyz + \frac{y^3}{3} + \frac{z^3}{3}\right]_{(1,1,1)}^{(3,-5,7)}$$

$$= \left[\frac{(3)^3}{3} - (3)(-5)(7) + \frac{(-5)^3}{3} + \frac{(7)^3}{3}\right]$$

$$- \left[\frac{1}{3} - (1)(1)(1) + \frac{1}{3} + \frac{1}{3}\right]$$

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$$= \left[\frac{\cancel{9}^3}{\cancel{27}} - (-105) + \frac{(\cancel{-125})^{-125}}{\cancel{3}73} + \frac{343}{3}\right] - \left[\frac{1}{3} - 1 + \frac{1}{3} + \frac{1}{3}\right]$$

$$I.D. = \frac{560}{3}$$

(7) If $\bar{F} = (2xy + 3z^2)\bar{i} + (x^2 + 4yz)\bar{j} + (2y^2 + 6xz)\bar{k}$, evaluate $\int_C \bar{F} \cdot d\bar{r}$ where C is the curve $x = t$, $y = t^2$, $z = t^3$ joining the points $(0,0,0)$ and $(1,1,1)$

Soln.

$$\bar{F} = (2xy + 3z^2)\bar{i} + (x^2 + 4yz)\bar{j} + (2y^2 + 6xz)\bar{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (2xy + 3z^2) & (x^2 + 4yz) & (2y^2 + 6xz) \end{vmatrix}$$

$$\nabla \times \bar{F} = 0$$

$\therefore \bar{F}$ is conservative.

$$I.D = \underset{y,z\text{ const}}{\int F_1\,dx} + \underset{\substack{\text{No } x \\ z\text{-const}}}{\int F_2\,dy} + \underset{\substack{\text{No} \\ x,y}}{\int F_3\,dz}$$

$$= \int (2xy + 3z^2)\,dx + \int 4yz\,dy + \int 0\,dz$$

$$= \frac{\cancel{2}x^2}{\cancel{2}}y + 3z^2 + \frac{\cancel{4}^2 y^2}{\cancel{2}}z + 0$$

$$= x^2y + 3z^2 + 2y^2z$$

$$= (t)^2t^2 + 3(t^3)^2 + 2(t^2)^2(t^3)$$

$$= \left[t^4 + 3t^6 + 2t^7\right]_{(0,0,0)}^{(1,1,1)}$$

$$= \left((1)^4 + 3(1)^6 + 2(1)\right) - \left((0)^4 + 0 + 0\right)$$

$$= 6$$

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- ## Green's theorem :-

For (xy - plane , z = 0)

Let $u, v, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ be continuous and single valued over the region bounded by closed curve c,

then,

$$\boxed{\oint_c u\,dx + v\,dy = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy}$$

Basic :

(1) Double integration)

1. Limits are not given

(i) Take vertical Strip : $\int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx.$

(ii) For Horizontal Strip : $\int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$

(2) For circle $x^2 + y^2 = a^2$

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$r \to 0$ to a , $\theta \to 0$ to 2π

(3) For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x = ar\cos\theta$, $y = br\sin\theta$

$dx\,dy = ab\,r\,dr\,d\theta$

$r \to 0$ to 1 , $\theta \to 0$ to 2π

- Examples : (Type 1) Green's theorem.

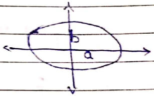
1) A vector field is given by $\bar{F} = \sin y\,\bar{i} + x(1+\cos y)\,\bar{j}$, Evaluate the integral $\int \bar{F}\cdot d\bar{r}$ where c is ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$.

Sol^n: $\bar{F} = \sin y\,\bar{i} + x(1+\cos y)\,\bar{j}$

To find $\int_c \bar{F}\cdot d\bar{r}$

C is ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$

$$\int_c \bar{F}\cdot d\bar{r} = \int_c F_1\,dx + \int F_2\,dy$$



$$= \int_c (\sin y + x(1+\cos y))\,dx\,dy$$

$$= \int_c u\,dx + v\,dy$$

$u = \sin y$	$v = x(1+\cos y)$
$\frac{du}{dy} = \cos y$	$\frac{dv}{dx} = (1+\cos y)$

By Green's theorem

$$\int_c u\,dx + v\,dy = \iint \left(\frac{dv}{dx} - \frac{du}{dy}\right) dx\,dy$$

$$= \iint (1 + \cancel{\cos y} - \cancel{\cos y})\,dx\,dy$$

$$= \iint dx\,dy$$

= Area under the curve

$= \pi ab$

(11) Find the workdone in moving a particle once round the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1, z = 0$ under the field force given by
$\bar{F} = (2x - y + z)\bar{i} + (x + y - z^2)\bar{j} + (3x - 2y + 4z)\bar{k}$ is the field conservative?

Soln we know,

$$\text{Workdone} = \int_C \bar{F} \cdot d\bar{r}$$

$$C \text{ is ellipse } \frac{x^2}{(5)^2} + \frac{y^2}{(4)^2} = 1, \quad z = 0$$

put $z = 0$

$$\bar{F} = (2x - y)\bar{i} + (x + y)\bar{j} + (3x - 2y)\bar{k}$$

$$\int \bar{F} \cdot d\bar{r} = \int F_1 dx + F_2 dy$$

$$= \int (2x - y)\, dx + (x + y)\, dy$$

$$= \int u\, dx + v\, dy.$$

$u = 2x - y$	$v = x + y$
$\frac{du}{dy} = \cancel{2x} - 1$	$\frac{dv}{dx} = 1 \cancel{+ y}$

By Greens theorem

$$= \iint \left(\frac{dv}{dx} - \frac{du}{dy} \right) dx\, dy$$

$$= \iint (1 - (-1))\, dx\, dy$$

$$= \iint 2\, dx\, dy$$

$$= 2 \iint dx\,dy$$

$$= 2 \times \pi a b$$

Put a & b. $\left(\frac{x^2}{(5)^2} + \frac{y^2}{(4)^2} = 1 \right)$

$$= 2 \times \pi \times 5 \times 4$$

$$= 40\pi$$

W.D $\neq 0$

$\therefore$ Field is not conservative.

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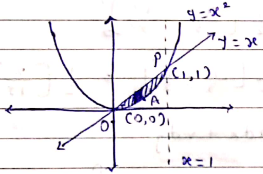
(12) Verify green's theorem for the field $\bar{F} = x^2 \hat{i} + xy\hat{j}$ over the region R enclosed by $y = x^2$ and the line $y = x$.

Soln:



Points of intersection

$$y = x \quad , \quad y = x^2$$

$$x^2 = x \implies x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0, 1$$

$$x = 0 \implies y = x = 0$$

$$x = 1 \implies y = x = 1$$

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$$\int \bar{F}\cdot d\bar{r} = \int_C x^2 dx + xy\,dy$$

$u = x^2$, $v = xy$

$\frac{du}{dy} = 0$, $\frac{dv}{dx} = y$

By green's theorem.

$$\int u\,dx + v\,dy = \iint \left(\frac{dv}{dx} - \frac{du}{dy}\right) dx\,dy$$

$$R.H.S = \iint (y-0)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} y\,dy\,dx$$

$$= \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{x} dx$$

$$= \frac{1}{2}\int_0^1 (x^2 - x^4)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{1}{2}\left[\frac{2}{15}\right] = \frac{1}{15}$$

$$L.H.S = \int u\,dx + v\,dy$$

$$= \int_{OAP} x^2 dx + xy\,dy + \int_{PO} x^2 dx + xy\,dy$$

OAP	PO
$y = x^2$	$y = x$
$dy = 2x\,dx$	$dy = dx$
$x \to 0$ to 1	$x \to 1$ to 0

$$= \int_{x=0}^{1} x^2 dx + x x^2 (2x) dx + \int_{x=1}^{0} x^2 dx + x x\,dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^5}{5}\right]_0^1 + \left[\frac{x^3}{3} + \frac{x^3}{3}\right]_1^0$$

$$= \left(\frac{1}{3} + \frac{2}{5} - 0\right) + \left(0 - \frac{1}{3} - \frac{1}{3}\right) = \left(\frac{2}{5} - \frac{1}{3}\right) = \frac{1}{15}$$

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* Divergence of Vector theorem :-

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

Example :-

(13) Evaluate Divergence theorem for $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$ and S, the surface of the cube bounded by the planes: $x=0, x=2, y=0, y=2, z=0, z=2$.

Soln:

$$\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$\nabla\cdot\bar{F} = 4z - y$$

By divergence theorem,

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} (4z-y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} \left[\frac{4z^2}{2} - yz\right] dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} \left[2(2)^2 - y(2)\right] dy\,dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (8-2y)\,dy\,dx$$

$$= \int_{x=0}^{2} \left[8y - \frac{2y^2}{2}\right] dx$$

$$= \int_{x=0}^{2} \left[8(2) - (2)^2\right] dx$$

$$= 12 \int_{x=0}^{2} 1\,dx$$

$$= 12 [x]_0^2$$

$$= 12 \times 2$$

$$= \underline{\underline{24}}.$$

② Evaluate $\iint_S (2xy\bar{i} + yz^2\bar{j} + xz\bar{k}).d\bar{s}$ over the surface of the region bounded by $x=0, y=0, y=3$ $z=0$ and $x+2z=6$.

Sol^n: $\bar{F} = 2xy\bar{i} + yz^2\bar{j} + xz\bar{k}$

$$\nabla.\bar{F} = \frac{\partial}{\partial x}(2yx) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz)$$

$$= 2y + z^2 + x$$

$$\nabla.\bar{F} = (x + 2y + z^2)$$

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limits $x+2z=6$

put $z=0 \Rightarrow x=6$

$x \to 0$ to 6

$y \to 0$ to 3

$z \to 0$ to $\frac{6-x}{2}$

By divergence theorem,

$$\iint_S \bar{F}\cdot d\bar{s} = \iiint_V \nabla\cdot\bar{F}\, dv$$

$$= \int_{x=0}^{6}\int_{y=0}^{3}\int_{z=0}^{\frac{6-x}{2}} (x+2y+z^2)\,dz\,dy\,dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{3}\left[(x+2y)(z)+\frac{z^3}{3}\right]_0^{\frac{6-x}{2}} dy\,dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{3}(x+2y)\left(\frac{6-x}{2}\right)+\frac{1}{3}\left(\frac{6-x}{2}\right)^3 dy\,dx$$

$$= \int_{x=0}^{6}\left(\frac{6-x}{2}\right)\left[xy+\frac{2y^2}{2}\right]_0^3+\frac{1}{24}(6-x)^3[y]_0^3\,dx$$

$$= \int_{x=0}^{6}\left(\frac{6-x}{2}\right)[3x+9]+\frac{1}{\cancel{24}_{8}}(6-x)^3\times\cancel{3}\,dx$$

$$= \int_{x=0}^{6}\left(\frac{6-x}{2}\right)(3x+9)+\frac{1}{8}(6-x)^3\,dx$$

$$= \frac{351}{2}$$ $\Rightarrow$ Directly put it on calci.

Type using:

$$\nabla\cdot(\phi\bar{u}) = \phi(\nabla\cdot\bar{u}) + (\nabla\phi)\cdot\bar{u}$$

$\phi \to$ Scalar, $\bar{u} \to$ vector.

Examples :-

(8) Evaluate $\iint_S \bar{r}\cdot\hat{n}\,ds$ over the surface of a sphere of radius 1 with center at origin

Sol^n:

$$\bar{F} = \bar{r}$$
$$= x\hat{i} + y\hat{j} + z\hat{k}$$
$$\nabla\cdot\bar{r} = 1+1+1 = 3$$

$$\therefore \nabla\cdot\bar{F} = \nabla\cdot\bar{r}$$
$$= 3$$

By divergence theorem,

$$\iint_S \bar{F}\cdot d\bar{s} = \iiint_V \nabla\cdot\bar{F}\,dv$$
$$= \iiint_V 3\,dv$$
$$= 3\iiint_V dv$$
$$= 3\times \text{volume of sphere}$$
$$= \cancel{3}\times\frac{4}{\cancel{3}}\pi r^3$$
$$= 4\pi r^3 \quad (\text{radius} = 1)$$
$$= 4\pi(1)^3$$
$$= 4\pi$$

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① To show $\iint_S \frac{\bar{r}}{r^3} \cdot \vec{n}\, ds = 0$

Sol^n:

$$\bar{F} = \frac{\bar{r}}{r^3}$$

$$\nabla \cdot \bar{F} = \nabla \cdot \left(\frac{\bar{r}}{r^3}\right)$$

$$= \nabla \cdot [\underbrace{r^{-3}}_{\phi} \underbrace{\bar{r}}_{\bar{u}}] \longrightarrow \text{vector identities.}$$

$$= \phi(\nabla \cdot \bar{u}) + (\nabla \phi) \cdot \bar{u}$$

$$= r^{-3}(\nabla \cdot \bar{r}) + (\nabla r^{-3}) \cdot \bar{r}$$

$$= 3r^{-3} + \left(-3r^{-4} \frac{\bar{r}}{r} \cdot \bar{r}\right) \longrightarrow \text{Diff.}$$

$$= 3r^{-3} - 3r^{-3} \ldots (\bar{r} \cdot \bar{r} = r^2)$$

$$= 0$$

$$\iint_S \bar{F} \cdot \vec{n}\, ds = \iiint_V \nabla \cdot \bar{F}\, dv$$

$$= 0$$

* Stokes theorem :-

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla \times \bar{F})\cdot d\bar{s}$$

$S \longrightarrow$ open surface.

where,

$$\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$d\bar{s} = \hat{n}\, ds$$

$\hat{n}$ = Unit outward normal to surface

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

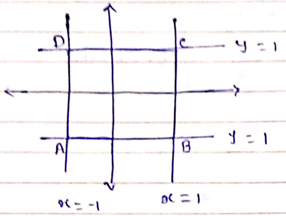
$$\bar{F}\cdot d\bar{r} = F_1 dx + F_2 dy + F_3 dz.$$

Type 1 :

- Examples.

(1) Verify Stoke's theorem for $\bar{F} = x^2\bar{i} + xy\bar{j}$ for the surface of a square lamina bounded by $x = -1$, $x = 1$, $y = -1$, $y = 1$.

Sol^n :-



$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \bar{i}[0-0] - \bar{j}[0-0] + \bar{k}[y-0]$$

$$\nabla \times \bar{F} = 0\bar{i} + 0\bar{j} + y\bar{k}$$

In xy-plane $\hat{n} = \bar{F}$

$$(\nabla \times \bar{F}) \cdot \hat{n} = (y\bar{k}) \cdot \bar{k}$$

$$= y.$$

$$R.H.S = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$$

$$= \int_{x=-1}^{1} \int_{y=-1}^{1} y\, dy\, dx.$$

$$= \int_{x=-1}^{1} \left(\frac{y^2}{2}\right)_{-1}^{1} dx$$

$$= \frac{1}{2} \int_{x=-1}^{1} (1-1)\, dx$$

$$= \frac{1}{2} \int_{x=1}^{1} 0$$

$$= \underline{\underline{0}}.$$

$$L.H.S = \int_C \bar{F} \cdot d\bar{r}$$

$$= \int_C x^2 dx + xy\, dy$$

$$= \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

AB	BC	CD	DA
$y=-1$	$x=1$	$y=1$	$x=-1$
$dy=0$	$dx=0$	$dy=0$	$dx=0$
$x=-1$ to 1	$y \to -1$ to 1	$x \to 1$ to -1	$y \to 1$ to -1

$$= \int_{-1}^{1} x^2\,dx + \int_{-1}^{1} xy\,dy + \int_{1}^{-1} x^2\,dx + \int_{-1}^{1} xy\,dy$$

$$= \left[\frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{y^2}{2}\right]_{-1}^{1} + \left[\frac{x^3}{3}\right]_{1}^{-1} - \left[\frac{y^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{3}(1+1) + \frac{1}{2}\underbrace{(1-1)}_{0} + \frac{1}{3}(-1-1) - \frac{1}{2}\underbrace{(1-1)}_{0}$$

$$= \frac{2}{3} - \frac{2}{3}$$

$$= 0$$

L.H.S = R.H.S.