Understanding Stabilizers and Parity in Surface Codes

1 Introduction

In the surface code for quantum error correction, stabilizers play a crucial role in detecting and correcting errors without disturbing the logical quantum state. This document explains key concepts such as plaquettes, stabilizers, and parity measurement, with examples for better understanding.

2 What is a Plaquette?

A **plaquette** is a small region in the grid of data qubits, typically represented as a square in the 2D lattice. In the surface code, each plaquette is associated with a **stabilizer** that checks the parity of the qubits within that plaquette.

For a 3×3 grid, there are 4 plaquettes:

1. Top-left
2. Top-right
3. Bottom-left
4. Bottom-right

Each plaquette involves 4 data qubits, and the stabilizer ensures that the parity (even or odd number of bit-flip errors) of these 4 qubits is preserved.

3 What Do $(0, 0)$, $(0, 1)$, etc., Represent?

These are the **row and column indices** of the qubits in the 3×3 grid of data qubits. For example:

- $(0, 0)$ refers to the qubit at the top-left corner of the grid (row 0, column 0).
- $(0, 1)$ refers to the qubit in row 0 and column 1 (the second qubit in the top row).
- $(1, 1)$ refers to the qubit in the middle of the grid.

Thus, the stabilizer:

```
[(0, 0), (0, 1), (1, 0), (1, 1)]  % Top-left plaquette
```

is responsible for checking the parity of the top-left 4 qubits in the grid.

4 What Does 0 or 2 Mean?

The numbers 0, 1, and 2 are **indices** of the qubits in the grid. They do not represent parity directly.

- 0 is the **row or column index** of the first row/column in the grid.
- 2 is the **row or column index** of the last row/column in the grid.

For example:

- $(0, 2)$ refers to the top-right qubit (row 0, column 2).
- $(2, 1)$ refers to the qubit in the bottom row, second column (row 2, column 1).

5 Error Identification in Quantum Error Correction

5.1 How Does This Relate to Parity?

The **stabilizer measurement** calculates the **parity** (even or odd) of the qubits in a plaquette:

- The stabilizer combines the values of the qubits (using XOR, or addition modulo 2).
- If the parity is even, the stabilizer outputs $+1$.
- If the parity is odd, the stabilizer outputs -1.

For example, in the top-left plaquette:

- If $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$ have states $[0,1,0,1]$, their parity is $0 \oplus 1 \oplus 0 \oplus 1 = 0$ (even parity).

It depends on how much memory you want to use to store parity measurement. I have recorded local parity (4 bits per stabilizer) as it is simpler for simulation but less hardware like.

5.2 Parity Measurement

Parity measurement involves determining whether a group of qubits has an even or odd number of bit-flip errors. This is achieved using ancillary qubits (also called measure qubits) that interact with the data qubits to record parity information. The parity is encoded into the ancillary qubits, which are subsequently measured in the computational basis.

5.3 Detailed Example

We explain parity measurement with a simple 3-qubit repetition code, which protects against bit-flip errors:

$$|0\rangle_L = |000\rangle,$$
$$|1\rangle_L = |111\rangle.$$

Step 1: Initialization of Ancillary Qubits Ancillary qubits are initialized in the state $|0\rangle$.

Step 2: Interaction with Data Qubits Controlled-Z (CZ) gates are applied between the ancillary qubit and the data qubits. For instance, to measure the parity of qubits 1 and 2, a CZ gate is applied between the ancillary qubit and qubit 1, followed by another CZ gate between the ancillary qubit and qubit 2.

Step 3: Measurement of Ancillary Qubit The ancillary qubit is measured in the computational basis. The measurement outcome indicates the parity:

- $+1$: Even parity (qubits 1 and 2 are in the same state).
- -1: Odd parity (qubits 1 and 2 are in different states).

5.4 Example: Measuring Z_1Z_2

Consider data qubits in the state $|q_1q_2q_3\rangle = |010\rangle$ (with a bit-flip error on qubit 2). The process is as follows:

1. **Initial State:** The ancillary qubit is in state $|0\rangle$.
2. **First CZ Gate:** A CZ gate is applied between qubit 1 and the ancillary qubit. Since qubit 1 is in $|0\rangle$, the ancillary qubit remains unchanged.
3. **Second CZ Gate:** A CZ gate is applied between qubit 2 and the ancillary qubit. Since qubit 2 is in $|1\rangle$, the CZ gate flips the phase of the ancillary qubit, changing its state to $|1\rangle$.
4. **Measurement:** The ancillary qubit is measured, and the result is -1, indicating odd parity (an error occurred between qubits 1 and 2).

6 Summary

- **Plaquettes** divide the surface code grid into regions, each associated with a stabilizer that checks parity.
- Indices such as $(0, 0)$ and $(1, 1)$ denote specific qubits in the grid.
- **Parity measurement** detects errors without collapsing the logical state by using ancillary qubits and stabilizer measurements.