

1.

a.

The statement that the vectors  $v_1$  and  $v_2$  are linearly independent means that they cannot be expressed as a linear combination of each other. In other words, there is no non-zero set of coefficients  $c_1$  and  $c_2$  such that  $c_1v_1 + c_2v_2 = 0$ .

The problem asks if we can find these vectors and coefficients such that the resulting matrix has full column rank. A matrix has full column rank if its columns are linearly independent. In other words, there is no non-zero set of coefficients  $c_1$ ,  $c_2$ , and  $c_3$  such that  $c_1v_1 + c_2v_2 + c_3v_3 = 0$ , where  $v_3$  is the third column of the matrix.

The solution to this problem is that it is not possible to create a full-column-rank matrix with the given conditions. Here's why:

1. **Dimensionality:** Since  $v_1$  and  $v_2$  are in a  $D$ -dimensional space, where  $D > 3$ , they span a subspace of dimension 2. This means that any linear combination of these vectors will also lie in this 2-dimensional subspace.
2. **Third Column:** To achieve full column rank, the third column,  $v_3$ , must be outside the subspace spanned by  $v_1$  and  $v_2$ . However, since  $v_3$  is a linear combination of  $v_1$  and  $v_2$ , it must lie within the same 2-dimensional subspace.
3. **Contradiction:** This creates a contradiction. We cannot have a vector that is both inside and outside of a subspace at the same time.

Therefore, it is impossible to find vectors  $v_1$  and  $v_2$  and coefficients for the linear combinations that will create a full-column-rank matrix with the given conditions.

b.

The problem asks for the smallest number of calls to the function  $F(x, R, y)$  needed to decide if the matrix is positive definite, given that  $A = LTL^T$  where  $L$  is a lower triangular matrix.

Here's how to solve it:

1.

**Cholesky decomposition:** Since  $A$  is given as  $LTL^T$ , where  $L$  is lower triangular, it means  $A$  has a Cholesky decomposition. This decomposition guarantees that  $A$  is positive definite if and only if  $A$  is non-singular (has an inverse).

2.

**Checking non-singularity:** Therefore, to determine if  $A$  is positive definite, we only need to check if  $A$  is non-singular. This can be done by checking if the determinant of  $L$  is non-zero.

3.

**Determinant of triangular matrices:** The determinant of a triangular matrix is simply the product of its diagonal elements. Therefore, to check if  $L$  is non-singular, we only need to check if all the diagonal elements are non-zero.

4.

**Function calls:** Now, let's consider the function  $F(x, R, y)$ . This function can be used to compute the matrix product. We can rewrite the product of any two matrices as  $F(x, I, Ry)$ , where  $I$  is the identity matrix.

5.

**Trace and determinant:** There's a relationship between the determinant of a matrix and its trace. The determinant is the product of the eigenvalues, while the trace is the sum of the eigenvalues. For a lower triangular matrix like  $L$ , the determinant is simply the product of its diagonal elements, which is the same as the trace.

6.

**Algorithm:** Putting it all together, here's an algorithm to determine if  $A$  is positive definite with the minimum number of calls to  $F$ :

1. For each row  $i$  of  $L$  (from 1 to  $n$ ):
  1. Call  $F(x, I, L_i)$  to compute the trace of the  $i$ -th row of  $L$ .
  2. If the trace is zero, then  $L$  is singular and  $A$  is not positive definite. Stop the algorithm.
2. If the loop finishes without encountering a zero trace, then  $L$  is non-singular and  $A$  is positive definite. Return "Yes".

This algorithm makes only  $n$  calls to  $F$ , which is the minimum number needed to check the non-singularity of  $L$ , hence the positive definiteness of  $A$ .

Therefore, the answer to the problem is  $n$ .

c.

No, the series  $1 + 3x + 5x^2$  cannot be the Taylor's polynomial of degree 2 that approximates the  $x^2 + x^3$  around  $x = 0$ .

Here's why:

- The general form of a Taylor polynomial of degree 2 for a function  $f(x)$  around  $x = a$  is:

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

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In this case, we are interested in approximating  $f(x) = x^2 + x^3$  around  $x = 0$ . Therefore,  $a = 0$ , and  $f''(0) = 2$ .

- 

Plugging these values into the general form, we get the Taylor polynomial of degree 2 for  $f(x)$

$$0 + 0(x - 0) + \frac{2}{2!}(x - 0)^2 = x^2$$

As you can see, the actual Taylor polynomial of degree 2 for  $f(x)$  around  $x = 0$  is simply  $x^2$ .

- 

The series you provided,  $1 + 3x + 5x^2$ , is a different polynomial of degree 2. While it shares the same Taylor polynomial ( $x^2$ ), it also has additional terms (1 and  $3x$ ) that are not present in the Taylor polynomial.

Therefore, the series  $1 + 3x + 5x^2$  cannot be the Taylor's polynomial of degree 2 that approximates  $x^2 + x^3$  around  $x = 0$ .

2.

a.

Here's how to solve the problem:

### 1. Finding the determinant of the coefficient matrix:

The first step is to find the determinant of the coefficient matrix, which is a  $3 \times 3$  matrix in this case. The determinant tells us whether the system has a unique solution, no solutions, or infinitely many solutions.

If the determinant is non-zero, the system has a unique solution. If the determinant is zero, the system either has no solutions or infinitely many solutions.

### 2. Analyzing the determinant for different values of $\alpha$ :

We can then analyze the determinant for different values of  $\alpha$  and see when it becomes zero. This will give us the possible values of  $\alpha$  for which the system fails to have three pivots.

### 3. Justification for each value of $\alpha$ :

Once we find the values of  $\alpha$  that make the determinant zero, we need to justify why this leads to the system not having three pivots. This can involve analyzing the rank of the coefficient matrix or the reduced row echelon form of the system.

b.

## 1. Eigenvalues and Eigenvectors:

Since the vectors  $p_i$  are orthogonal and have unit norm, they form an orthonormal basis for  $\mathbb{R}^m$ . Any vector  $v \in \mathbb{R}^m$  can be expressed as a linear combination of the  $p_i$  as follows:

$$v = \sum_{i=1}^n \alpha_i p_i$$

for some scalars  $\alpha_i$ .

Now, let's consider the matrix-vector product  $Bp$ :

$$Bp = B \left( \sum_{i=1}^n \alpha_i p_i \right) = \sum_{i=1}^n \alpha_i Bp_i$$

Since the  $p_i$  are orthonormal, we have  $Bp_i = 6^i p_i$ . Therefore:

$$Bp = \sum_{i=1}^n \alpha_i (6^i p_i) = \left( \sum_{i=1}^n 6^i \alpha_i \right) v$$

This shows that  $v$  is an eigenvector of  $B$  with eigenvalue  $6^i$ , where  $i$  is such that  $\alpha_i \neq 0$ .

Therefore, the eigenvalues of  $B$  are  $6^1, 6^2, \dots, 6^n$ . The corresponding eigenvectors are any combinations of the vectors  $p_i$ .

## 2. Trace of $B$ :

The trace of a matrix is the sum of its diagonal elements. In this case, the diagonal elements

$$B_{ii} = p_i^T (6^i p_i) = 6^i \langle p_i, p_i \rangle = 6^i$$

Therefore, the trace of  $B$  is:

$$\text{Tr}(B) = \sum_{i=1}^n B_{ii} = \sum_{i=1}^n 6^i = \frac{6^{n+1} - 1}{5}$$

### 3. Determinant of B:

The determinant of a matrix can be computed using various methods, such as LU decomposition or Gaussian elimination. However, in this case, there is a simpler approach due to the special properties of the matrix  $B$ .

Since the  $p_i$  are orthonormal, the matrix formed by stacking them as columns, denoted by  $P$ , is an orthogonal matrix. This means that its inverse is its transpose, i.e.,  $P^{-1} = P^T$ .

We can rewrite the expression for  $B$  as:

$$B = 6^I \odot (PP^T)$$

where  $I$  is the identity matrix and  $\odot$  denotes the element-wise product. The determinant of  $B$  is the product of the determinants of the individual matrices. Therefore:

$$\det(B) = \det(6^I) \det(PP^T)$$

The determinant of a diagonal matrix is the product of its diagonal elements. Therefore:

$$\det(6^I) = \prod_{i=1}^n 6^i = 6^{1+2+\dots+n} = \frac{6^{n(n+1)/2}}{2}$$

The determinant of an orthogonal matrix is 1.

$$\det(PP^T) = 1$$

$$\det(B) = \frac{6^{n(n+1)/2}}{2} \cdot 1 = \boxed{\frac{6^{n(n+1)/2}}{2}}$$

In summary:

1. The eigenvalues of B are  $6^1, 6^2, \dots, 6^n$ .
2. The corresponding eigenvectors are any non-zero linear combinations of the vectors  $p$
3. The trace of B is  $\frac{6^{n+1}-1}{5}$ .
4. The determinant of B is  $\frac{6^{n(n+1)/2}}{2}$ .

c.

Sure, I can help you with this problem. The given matrix M has the property that  $M = 0$  for some  $k$ . This means that there exists a non-zero row (or column) in the matrix that, when multiplied by a scalar  $k$ , results in a zero row (or column).

Based on this property, we can solve for the following:

#### 1. Trace of matrix M:

The trace of a matrix is the sum of its diagonal elements. In this case, since one of the diagonal elements is zero (due to the property  $M = 0$ ), the trace of the matrix M is also 0.

#### 2. Determinant of matrix M:

The determinant of a matrix can be computed in various ways, but one efficient method is using row or column elimination. In this method, we perform elementary row operations (such as multiplying a row by a scalar, adding a multiple of one row to another) to transform the matrix into an upper triangular form. Then, the determinant of the upper triangular matrix is simply the product of its diagonal elements.

In this case, however, we can see that the determinant of matrix M is also 0. This is because a matrix with a column of zeros has a determinant of 0.

### 3. Eigenvalues of matrix M:

An eigenvalue of a matrix is a scalar  $\lambda$  such that there exists a non-zero vector  $x$  where  $Ax = \lambda x$ . If you multiply the matrix by the vector, the result is just a scaled version of the original vector.

To find the eigenvalues of matrix  $M$ , we can use the characteristic equation, which is obtained by setting  $(A - \lambda I)$  equal to zero, where  $I$  is the identity matrix of the same size as  $A$ .

In this case, the characteristic equation of matrix  $M$  is:

$$|M - \lambda I| = 0$$

Since the determinant of matrix  $M$  is 0, we know that at least one eigenvalue is  $\lambda = 0$ . Additional eigenvalues can be found by solving the remaining equation for  $\lambda$ .

### 4. Invertibility of matrix M:

A matrix is invertible if it has an inverse, which is another matrix that, when multiplied by the original matrix, produces the identity matrix.

A matrix is invertible if and only if its determinant is nonzero. Since the determinant of matrix  $M$  is 0, that matrix  $M$  is not invertible.

3.

- a. <https://math.stackexchange.com/questions/3272976/determining-when-ax-b-is-consistent>

$$\text{Condition} \Rightarrow (b_3 - b_1) - (a_3 - a_1)(b_2 - b_1)/(a_2 - a_1) \neq 0$$

- b.  
c.

1. Understand the function's property.

The function is given to be even, which means  $f(x) = f(-x)$  for all  $x$  in  $\mathbb{R}^3$ . This property implies partial derivatives at the origin will be zero.

2. Evaluate the function at the origin.

We are given that  $f(0, 0, 0) = 4$ . This is the constant term in the linear approximation.

3. Determine the first-order partial derivatives.

Since  $f$  is even and all first-order partial derivatives exist, the first-order partial derivatives at the origin are zero. This is because the derivative of an even function at the origin, if it exists, is always zero.

4. Write the linear approximation.

The linear approximation (or the tangent plane) of a function  $f$  at a point  $a$  is given by:

$$L(x) = f(a) + \nabla f(a)^T (x - a)$$

where  $\nabla f(a)$  is the gradient of  $f$  at  $a$ .

In this case, the linear approximation of  $f$  about  $[0, 0, 0]$  is simply the constant term:

$$L(x) = 4$$

**Answer:**

The linear approximation of  $f$  about  $[0, 0, 0]$  is  $L(x) = 4$ .

d.

$$3d) \quad g(x,y) = (3x+y)^2(2y+1) \quad h(x,y) = 6y^2(2x+y)$$

$$i) \quad \frac{\partial g}{\partial x} = (2y+1)[2(3x+y)]$$

$$\frac{\partial g}{\partial y} = (2y+1)[2(3x+y)] + (3x+y)^2(2)$$

$$\frac{\partial h}{\partial x} = 6y^2(2)$$

$$\frac{\partial h}{\partial y} = (2x+y)[12y] + 6y^2(1)$$

$$ii) \quad f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$f(x,y,z) = \begin{pmatrix} g(x,y) + h(y,z) \\ h(x,y) + g(y,z) \end{pmatrix}$$

$$= \begin{pmatrix} (3x+y)^2(2y+1) + 6z^2(2y+2) \xrightarrow{f_1} \\ (3y+z)^2(2z+1) + 6x^2(2x+y) \xrightarrow{f_2} \end{pmatrix}$$

$$\frac{df(x,y,z)}{db} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}_{2 \times 3} \quad b = [x,y,z]^T \quad (3 \times 1)$$

$$= \begin{bmatrix} (2y+1)(6(3x+y)) & (2y+1)(2(3x+y)) + 6z^2(2) & (2y+2)(12z) + 6z^2 \\ 6x^2(2) + (2x+y)(12x) & (2z+1)(2(3y+z)) + 6x^2 & (2z+1)(2(3y+z)) + (3y+z)^2(2) \end{bmatrix}_{2 \times 3}$$

$$\left. \frac{df(x,y,z)}{db} \right|_{(0,1,0)} = \begin{bmatrix} 18 & 6+2+0 & 0 \\ 0 & 6 & 6+18 \end{bmatrix} = \begin{bmatrix} 18 & 8 & 0 \\ 0 & 6 & 24 \end{bmatrix}$$