Regular

Q1

Answer the following questions with justifications.

1) Consider the system of equations $\boldsymbol{AX} = \boldsymbol{b}$ where $\boldsymbol{A} = [\boldsymbol{A}_1, \cdots, \boldsymbol{A}_n]$ with $\boldsymbol{A}_i$s are columns of $\boldsymbol{A}$. Prove or disprove that if matrix $\boldsymbol{C}$ is obtained by interchanging $\boldsymbol{A}_i$ with $\boldsymbol{A}_j$ where $i < j \leq n$, the consistency of the new system of equations $\boldsymbol{CX} = \boldsymbol{b}$ is same as that of $\boldsymbol{AX} = \boldsymbol{b}$. (1.5 marks)

2) Define $f : \mathbb{R}^{3\times 2} \to \mathbb{R}^{3\times 3}$ as $f(\boldsymbol{B}) = \boldsymbol{BB}^T$. Compute $\frac{\partial f}{\partial \boldsymbol{B}}$. (2.5 marks)

3) Let $\boldsymbol{A}, \boldsymbol{B}$ be two square matrices of order n and $f(x_1, x_2) = x_2 \cos(x_1)$
   i. If rank of $\boldsymbol{A}$ is n and $\boldsymbol{B}$ is symmetric positive definite matrix then prove $\boldsymbol{A}^T\boldsymbol{BA}$ is symmetric positive definite. (1 mark)

   ii. If $\nabla f(x_1, x_2) = \text{grad } f(x_1, x_2)$ then define $\boldsymbol{A} = \nabla^2 f(x_1, x_2)$ as the Hessian of f. Then find the condition on $x_1, x_2 \in \mathbb{R}$ such that $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_C := \boldsymbol{x}^T\boldsymbol{Cy}$ is an inner product where $\boldsymbol{C} = \boldsymbol{A}^T\boldsymbol{BA}$ and $\boldsymbol{B}$ is symmetric positive definite matrix of order 2. (2 marks)

   iii. Find Taylor's 2nd degree polynomial approximation of f about $[\frac{\pi}{2}, 1]^T$ (1 mark)

Answer

1) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. Now $\{\boldsymbol{v} \in \mathbb{R}^m | \boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{A}_i,\ \alpha_i \in \mathbb{R},\ \forall i = 1 \cdots n\} = \{\boldsymbol{v} \in \mathbb{R}^m | \boldsymbol{v} = \sum_{i=1}^{n} \beta_i \boldsymbol{C}_i,\ \beta_i \in \mathbb{R},\ \forall i = 1 \cdots n\}$ as $\{\boldsymbol{C}_1, \cdots \boldsymbol{C}_n\} = \{\boldsymbol{A}_1, \cdots \boldsymbol{A}_n\}$.
Therefore, rank $(\boldsymbol{A})$ = rank $(\boldsymbol{C})$ and rank $(\boldsymbol{A}|\boldsymbol{b})$ = rank $(\boldsymbol{C}|\boldsymbol{b})$. (1 mark)
Hence the consistency of the new system of equations $\boldsymbol{CX} = \boldsymbol{b}$ is same as that of $\boldsymbol{AX} = \boldsymbol{b}$. (0.5 marks)

2) Let $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{r}_1^T \\ \boldsymbol{r}_2^T \\ \boldsymbol{r}_3^T \end{bmatrix}$ and so we have $\boldsymbol{B}^T = \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \boldsymbol{r}_3 \end{bmatrix}$.
Therefore $\boldsymbol{BB}^T(p,q) = \boldsymbol{r}_p^T\boldsymbol{r}_q = \boldsymbol{B}(p,1)\boldsymbol{B}(q,1) + \boldsymbol{B}(p,2)\boldsymbol{B}(q,2)$ (0.5 marks)

$$
\begin{aligned}
\Rightarrow \frac{\partial \boldsymbol{BB}^T(p,q)}{\partial \boldsymbol{B}(i,j)} &= \delta_{pqij} \text{ where} \\
\delta_{pqij} &= \boldsymbol{B}(q,j) \text{ if } i = p, p \neq q \\
\delta_{pqij} &= \boldsymbol{B}(p,j) \text{ if } i = q, p \neq q \\
\delta_{pqij} &= 2\boldsymbol{B}(q,j) \text{ if } i = p = q \\
\delta_{pqij} &= 0 \text{ otherwise.}
\end{aligned}
$$

where $p, q, i = 1, 2, 3$ and $j = 1, 2$ (2 marks)

3) Let $\boldsymbol{A}, \boldsymbol{B}$ be two square matrices of order n and $f(x_1, x_2) = x_2 \cos(x_1)$
   i. Now $(\boldsymbol{A}^T\boldsymbol{BA})^T = \boldsymbol{A}^T\boldsymbol{B}^T\boldsymbol{A} = \boldsymbol{A}^T\boldsymbol{BA}$ since $\boldsymbol{B}$ is symmetric.
   Let $\boldsymbol{x} \neq \boldsymbol{0}$

$$
\begin{aligned}
&\Rightarrow \quad \boldsymbol{Ax} \neq \boldsymbol{0} \text{ as } \boldsymbol{A} \text{ is a full rank square matrix.} \\
&\Rightarrow \quad (\boldsymbol{Ax})^T\boldsymbol{BAx} > 0 \text{ as } \boldsymbol{B} \text{ is a positive definite matrix.} \\
&\Rightarrow \quad \boldsymbol{x}^T(\boldsymbol{A}^T\boldsymbol{BA})\boldsymbol{x} > 0 \\
&\Rightarrow \quad \boldsymbol{A}^T\boldsymbol{BA} \text{ is a positive definite matrix.}
\end{aligned}
$$

(1 mark)

   ii. $\nabla f(x_1, x_2) = \begin{bmatrix} -x_2 \sin(x_1), & \cos(x_1) \end{bmatrix}$, (0.5 marks)
   $\boldsymbol{A} = \nabla^2 f(x_1, x_2) = \begin{bmatrix} -x_2\cos(x_1) & -\sin(x_1) \\ -\sin(x_1) & 0 \end{bmatrix}$. (0.5 marks)
   Now, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_C = \boldsymbol{x}^T\boldsymbol{Cy}$ is an inner product when $\boldsymbol{C}$ is symmetric, positive definite. But $\boldsymbol{C} = \boldsymbol{A}^T\boldsymbol{BA}$ and $\boldsymbol{B}$ is symmetric positive definite matrix of order 2. Clearly $\boldsymbol{C}$ is symmetric. From i. it is clear that $\boldsymbol{C}$ is positive definite if $\boldsymbol{A}$ is of rank 2.
   Now if $\sin(x_1) = 0$ then rank $(\boldsymbol{A}) < 2$ and if $\sin(x_1) \neq 0$ then rank$(\boldsymbol{A}) = 2$. Therefore, the condition for $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_C = \boldsymbol{x}^T\boldsymbol{Cy}$ to be an inner product is $\sin(x_1) \neq 0$ which means $x_1 \neq n\pi, \forall n \in \mathbb{Z}$.
   (1 mark)

   iii. Now Taylor's 2nd degree polynomial approximation of f at $[\frac{\pi}{2}, 1]^T$ is given by

$$
\begin{aligned}
& f(\frac{\pi}{2}, 1) + \nabla f(\frac{\pi}{2}, 1) \begin{bmatrix} x_1 - \frac{\pi}{2} \\ x_2 - 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 - \frac{\pi}{2} & x_2 - 1 \end{bmatrix} \nabla^2 f(\frac{\pi}{2}, 1) \begin{bmatrix} x_1 - \frac{\pi}{2} \\ x_2 - 1 \end{bmatrix} \\
= \quad & x_2(\frac{\pi}{2} - x_1)
\end{aligned}
$$

(1 mark)

Q2

1) Let $\mathbf{x} \in \mathbb{R}^n$. We define a function $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{Z}^+ \cup \{0\}$ as $f(\mathbf{x}) =$ *total number of non-zero entries in* $\mathbf{x}$. Assuming that $f(\mathbf{x})$ satisfies the triangle inequality property of norms, prove or disprove whether $f(\mathbf{x})$ satisfies the other two properties (i.e. absolutely homogeneous, positive definiteess) of norms. (1+1 marks)

2) Let $\beta \in \mathbb{R}$ be an unknown constant and let $i = \sqrt{-1}$, a symbol commonly used in definition of complex numbers. Now consider a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. It is given to you that $\mathbf{A}$ can be factorized as $\mathbf{A} = \mathbf{B^T B}$ where $\mathbf{B} \in \mathbb{R}^{n \times n}$. Let $\mathbf{x} \in \mathbb{R}^n$ be an eigenvector of $\mathbf{A}$ corresponding to an eigenvalue λ i.e. $\mathbf{Ax} = \lambda\mathbf{x}$. It is also known that this eigenvalue λ is of the form $\lambda = 7 + i\beta$.
   i) Is it possible to find the exact value of λ by deriving the constant β based on the given information? If yes, derive β. If no, provide proper reason(s) as to why it is not possible to finf β with the given information. (1 mark)
   ii) Is the matrix $\mathbf{A}$ a positive semi-definite matrix ? If yes prove it. If no, give proper reason why it cannot be a positive semi-definite matrix. (1 mark)

3) Consider a square matrix $\mathbf{B} \in \mathbb{R}^{2 \times 2}$ defined as follows:

$$\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$$

   Assume that $\mathbf{B} = \mathbf{U \Sigma V^T}$ is the singular value decomposition of $\mathbf{B}$.
   i) Derive the largest singular value σ_1 of this matrix $\mathbf{B}$. (1 mark)
   ii) Derive left singular vector $\mathbf{u_1}$ of this matrix $\mathbf{B}$ corresponding to largest singular value σ_1. (1 mark)
   iii) Derive right singular vector $\mathbf{v_1}$ of this matrix $\mathbf{B}$ corresponding to largest singular value σ_1. (1 mark)
   iv) Derive the rank-1 approximation matrix $\mathbf{B_1} = \sigma_1(\mathbf{u_1 v_1^T})$. (1 mark)

Answer

Q2 Answer

1) Considering the positive definite property first. Let $\mathbf{x} \in \mathbb{R}^n$, then $f(\mathbf{x})$ will always be a positive number as long as $\mathbf{x} \neq 0$ as it simply counts the number of non-zero entries. (0.5 marks).
   When $\mathbf{x} = 0$, then $f(\mathbf{x}) = 0$ and in no other case can $f(\mathbf{x})$ take the value zero. Hence $f(\mathbf{x})$ **satisfy both the components of positive definite property**. (0.5 marks).
   Now consider, absolutely homegeneous property. It says $g(\alpha.\mathbf{x}) = |\alpha|.g(\mathbf{x})$. This cannot be true in the case of given $f(\mathbf{x})$ as scaling by $\alpha \neq 0$, will not change the count of non-zero entries. Hence $f(\mathbf{x})$ **do not satisfy absolutely homogeneous property**. (0.5 marks)
   As an example consider, $x = [1 \;\; 0 \;\; 3]^T$ and $\alpha = 10$. $f(\alpha.\mathbf{x}) = 2$ but $\alpha.f(\mathbf{x}) = 20$. (0.5 marks)

2) i) Yes, It is possible to find the exact value of λ by deriving the constant β. This is because $\mathbf{A^T} = \mathbf{B^T B} = \mathbf{A}$. (0.5 marks)
   Hence $\mathbf{A}$ is symmetric matrix and hence has only real eigenvalues. Hence $\beta = 0$ is the only possible value of β. Hence $\lambda = 7$. (0.5 marks)
   ii) Observe that $\mathbf{A} = \mathbf{B^T B}$. Hence

$$\mathbf{x^T A x} = \mathbf{x^T B^T B x} = \mathbf{(Bx)^T (Bx)} \quad \mathbf{(0.5 marks)}$$

   Now recall that $\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|_2^2$. Hence $\mathbf{(Bx)^T (Bx)} = \langle \mathbf{Bx}, \mathbf{Bx} \rangle = \|\mathbf{Bx}\|_2^2$. But by positive definite property of norms $\|\mathbf{Bx}\|_2 \geq \mathbf{0}$. This implies that $\mathbf{x^T A x} \geq \mathbf{0}$. Hence the matrix $\mathbf{A}$ a positive semi-definite matrix. (0.5 marks)

3) i) To derive σ_1, we construct $\mathbf{A^T A}$ as $\mathbf{A^T A} = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$
   Now, we derive the eigenvalues of $A^T A$ by constructing the characteristic equation $|\mathbf{A^T A} - \lambda \mathbf{I}| = \mathbf{0}$. By solving the characteristic equation we get $\lambda_1 = 25$ and $\lambda_2 = 0$. Hence, the largest singular value $\sigma_1 = 5$ (1 marks)
   ii) Now we derive the eigenvalue corresponding to λ_1 by solving $\mathbf{A^T A x} = \mathbf{25.x}$ to obtain a representative eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
   Hence the right singular vector $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (1 marks)
   iii) Now, recall that $u_1 = \frac{1}{\sigma_1} A v_1$.

$$\mathbf{u_1} = \mathbf{\frac{1}{5}} \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

   (1 marks)
   iv)

$$\mathbf{B_1} = \sigma_1(\mathbf{u_1 v_1^T}) = \mathbf{5}. \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} . \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$$

   (1 marks)

Q3

(1) Following matrix $\boldsymbol{Q}$, where α, β, γ and ω are real numbers and solutions of the polynomial equation $\sum_{i=0}^{2024} c_i x^{2024-i} = 0$, has 3 linearly independent eigenvectors (as columns) in $\boldsymbol{P}$. Find the sum of all the elements of $\mathbf{P^{-1}QP}$. (1 mark)

$$\mathbf{Q} = \begin{bmatrix} 3 & \gamma & \alpha+\beta \\ \gamma & 0 & \omega \\ \alpha+\beta & \omega & 3 \end{bmatrix}$$

(2) Compute $\boldsymbol{A}^{\mathbf{9}}$ without using the conventional matrix multiplication. Show all the computations. (2 marks)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$$

(3) Determine which of the following are subspaces of $\mathbb{R}^3$? Justify your answer. If it is a subspace find its basis and dimension. (2 marks)
  (a) All vectors of the form (a, b, c) where $b = a + c$
  (b) All vectors of the form (a, b, c) where $c = a + b + 1$

(4) Find dimension of a vector space spanned by the following vectors $\{(1, 1, -2, 0, 1), (1, 2, 0, -4, 1), (0, 1, 3, -3, 2), (2, 3, 0, -2, 0)\}$. (1 mark)

(5) For the matrix below construct an elementary matrix for every elementary row operation that is performed on $\mathbf{A}$ and write the decomposition of $\mathbf{A}$ as $\mathbf{LU}$, where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is an upper triangular matrix (2 Marks)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Answer

Q3

(1) $\mathbf{P^{-1}QP} = \mathbf{D}$ (diagonal matrix) (0.5)
Trace($\mathbf{Q}$)=sum of eigen values of $\mathbf{Q}$ = sum of elements of $\mathbf{D}$=6 (0.5)

(2) $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$.
$det(A - \lambda * I)=0 \Rightarrow \lambda^2 - 10=0$
Every matrix satisfies characteristic equation
$\mathbf{A^2} = 10\mathbf{I} \Rightarrow \mathbf{A^8} = 10^4\mathbf{I}$ (1mark)
Multiplying A on both the sides $\mathbf{A^9} = \begin{bmatrix} 30000 & 10000 \\ 10000 & -30000 \end{bmatrix}$. (1 mark)
Alternate If they use eigendecomposition and obtain the result, give them the full marks.

(3) (a) $W = \{(a, b, c) | b = a + c\}$
$(0, 0, 0) \in W$ as $0 = 0 + 0$ (0.25 Mark)
Let $X = (x, y, z) \in W$ and $A = (a, b, c) \in W$, then let $\alpha \in R$ and $\beta \in R$
then consider $\alpha X + \beta A = \alpha(x, y, z) + \beta(a, b, c) = (\alpha x + \beta a, \alpha y + \beta b, \alpha z + \beta c)$
Consider $\alpha y + \beta b = \alpha(x + z) + \beta(a + c) = \alpha x + \alpha z + \beta a + \beta c = \alpha x + \beta a + \alpha z + \beta c$ (0.5 Mark)
Hence W s the subspace.
Basis(W) = $\{(1, 1, 0), (0, 1, 1)\}$ dim(W)=2 (0.5 Mark)
(b) Its not subspace since $(0, 0, 0)$ is not in subspace (0.5 Mark)

(4) $\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 1 & 2 & 0 & -4 & 1 \\ 0 & 1 & 3 & -3 & 2 \\ 2 & 3 & 0 & -2 & 0 \end{bmatrix}$ Compute Rank (A) as 4 using REF Alternate students can take the transpose of above matrix and find the rank. So dimension = 4. (2 Marks)

(5) $\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}$ (1.25 Marks)

$\mathbf{U} = \begin{bmatrix} 2 & 1 \\ 0 & 3.5 \end{bmatrix}$ (0.75 Mark)

Q4

a) You are building an application to scan documents using a mobile phone camera. When a user captures a document, the edges of the document are not aligned with the horizontal and vertical axis of the phone screen. Let us assume that user takes precaution to ensure that the bottom edge of the document is parallel to the x-axis when photo is taken (as shown in the figure **P0** below, A,B,C and D are the corners of the document in the captured image)

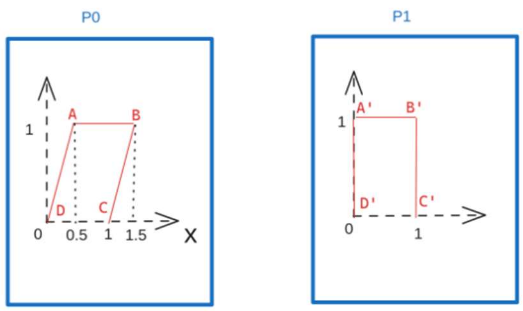

FIGURE 1. Document Scanner

You need to construct a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that, when $\mathbf{A}$ is applied on the vectors representing the document corners in diagram **P0**, the result is as shown in diagram **P1**. (2 marks)

b) Find the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$. (2 marks)

c) Find the algebraic and geometric multiplicities for eigenvalues found in part (b) of this question? (2 marks)

Answer

Q4

(1) (a) alternative 1:
$D'A'$ and $D'C'$ in $P1$ are the standard basis unit vectors. So, we can find the matrix $\mathbf{A}^{-1}$ that can be applied to these to get DA and DC in $P0$ as follows:
$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$.

So, the required matrix $\mathbf{A} = (\mathbf{A}^{-1})^{-1} = \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}$. (2 Marks)

(b) alternative 2:
Let $A = \begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix}$. be the matrix which transforms the corners in $P0$ to $P1$.
$\begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix} * \begin{bmatrix} 0.5 & 0 & 1 & 1.5 \\ 1 & 0 & 0 & 1 \end{bmatrix} . = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$
$\Rightarrow a11 * 0.5 + a12 * 1 = 0$
$\Rightarrow a21 * 0.5 + a22 * 1 = 1$
$\Rightarrow a11 * 1 = 1$
$\Rightarrow a21 * 1 = 0$
$\Rightarrow a11 * 1.5 + a12 * 1 = 1$
$\Rightarrow a21 * 1.5 + a22 * 1 = 1$
solving: $a11$=1, $a12$=−0.5, $a21$=0, $a22$=1 (2 Marks)

(2) $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$.
$det(A - \lambda * I)$=0 $\Rightarrow (1-\lambda)^2$=0 $\Rightarrow \lambda$=1, 1
$nullspace(A-\lambda_i * I) \Rightarrow$ (RREF of augmented matrix) $\Rightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$.
eigenvector $= \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$, where t is the free variable. Thus, eigenvector is in the direction of x-axis. (2 Marks)

(3) algebraic multiplicity of $\lambda = 1$ is 2
geometric multiplicity of the eigenspace corresponding to eivenvalue of $\lambda = 1$ is dim of eigenspace (= 1) (2 Marks)

Makeup

Q1

1) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We define a function $f(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |\mathbf{x}_i - \mathbf{y}_i|$. Prove or disprove whether $f(\mathbf{x}, \mathbf{y})$ satisfies all the properties of a distance (metric) function. (2 marks)

2) Consider a square matrix $\mathbf{A} \in \mathbb{R}^{\mathbf{3}\times\mathbf{3}}$ defined as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 4 & 6 \end{bmatrix}$$

Consider two vectors $\mathbf{x_1} = [1 \;\; \alpha \;\; 1]^T$ and $\mathbf{x_2} = [2 \;\; 2 \;\; 3]^T$. Consider an inner product space where the inner product in that space is defined as $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{y}$. Find the value of α if it is known that the angle between $\mathbf{x_1}$ and $\mathbf{x_2}$ in the inner product space is (i) 45° (ii) 0°. (1+1 mark)

3) Consider two square matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$.
   (a) Define two new matrices $\mathbf{C_1} = \mathbf{AB}$ and $\mathbf{C_2} = \mathbf{BA}$. Assume that $\lambda = 0$ is one of the eigenvalues of $\mathbf{C_1}$. Prove or disprove whether 0 is one of the eigenvalues of $\mathbf{C_2}$. (1 mark)
   (b) Let the singular value decompostion (SVD) of $\mathbf{A}$ be given as $\mathbf{A} = \mathbf{U\Sigma V^T}$. Let $\mathbf{P} \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$ be an orthogonal matrix. Derive SVD of $\mathbf{G}$ where $\mathbf{G} = \mathbf{PA}$ in terms of $\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}$ and $\mathbf{P}$. Derive SVD of $\mathbf{E}$ where $\mathbf{E} = \mathbf{AP}$ in terms of $\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}$ and $\mathbf{P}$ (1 mark)

4) Consider the function $f(x, y) = x^2 + 3xy + y^3$ where $x = \cos(r)$ and $y = \sin(r)$. Derive the expression for $\frac{df}{dr}$ (1 mark)

Answer

Q1 Answer

(1) We need to prove 3 properties of distance metric

Property 1: $d(x, y) \geq 0$ as $d(x, y)$ involves taking sum of non-negative numbers due to the use of $|\cdot|$, $\sum_{i=1}^{n} |x_i - y_i|$ can be zero only when each component of form $x_i - y_i = 0$. This only happens when $x = y$. Hence $d(x, y) = 0$ only for $x = y$. (0.5 marks)

Property 2: Since $|x_i - y_i| = |y_i - x_i|$, it can be concluded that $d(x, y) = d(y, x)$. (0.5 marks)

Property 3: Note $d(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x_i - z_i + z_i - y_i|$. Now using triangle inequality of ℓ_1 norm.

$$\sum_{i=1}^{n} |x_i - z_i + z_i - y_i| \leq \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| = d(x, z) + d(z, y)$$

Hence $d(x, y) \leq d(x, z) + d(z, y)$. (1 marks)

In summary d(x,y) is a distance metric.

(2) $\cos^2(\theta) = \frac{(x_1^T A x_2)^2}{(x_1^T A x_1).(x_2^T A x_2)}$. Substitute values given in question to get $\cos^2(\theta) = \frac{(37+26\alpha)^2}{(5\alpha^2+12\alpha+9).(154)}$ (1 marks)

(i) $cos(45°) = \frac{1}{\sqrt{2}}$. The previous equation after substitution of θ can be rearranged to get $291\alpha^2 + 1000\alpha + 676 = 0$ (0.5 marks)

Finally solution by quadratic formula is $\alpha = \frac{-1000 \pm \sqrt{1000^2 - 4*676*291}}{2*291}$

(ii) $cos(0°) = 1$. The previous equation after substitution of θ can be rearranged to get $94\alpha^2 - 76\alpha + 17 = 0$ (0.5 marks)

Finally solution by quadratic formula is $\alpha = \frac{76 \pm \sqrt{76^2 - 4*17*94}}{2*94}$. Observe that here α do not have real solution.

NOTE: Give 0.5 marks in the previous 2 steps if the student has written correctly upto the quadratic equation in each case. The final step in two subparts using quadratic formula is not compulsory

(3) a) Recall that $det(C_1)$ = Product of its eigenvalues. Since 0 is one of its eigenvalues, hence $det(C_1) = 0$ (0.5 marks).
Since $det(C_1) = det(AB) = det(A).det(B) = det(B).det(A) = det(BA) = det(C_2)$. Hence $det(C_2) = 0$. This means 0 is one of eigenvalues of C_2 as determinant is product of eigenvalues. (0.5 marks)

b) $G = PA$. Recall that $A = U\Sigma V^T$. So $G = PU\Sigma V^T$.
Now since P, U are orthogonal matrices $P^{-1} = P^T$ and $U^{-1} = U^T$. This means $(PU)^T = U^T P^T = U^{-1}P^{-1} = (PU)^{-1}$. Hence PU is orthogonal matrix. (0.25 marks)

Hence SVD of G is $(PU)\Sigma V^T$ (0.25 marks)

$E = AP$. Recall that $A = U\Sigma V^T$. So $E = U\Sigma V^T P$.
Now since P, V are orthogonal matrices $P^{-1} = P^T$ and $V^{-1} = V^T$. This means $(V^T P)^T = P^T V = P^{-1}(V^{-1})^T = P^{-1}(V^T)^{-1} = (V^T P)^{-1}$. Hence $V^T P$ is orthogonal matrix. (0.25 marks)

Hence SVD of E is $U\Sigma(V^T P)$ (0.25 marks)

(4) Taking partial derivatives we get $\frac{\partial f}{\partial x} = 2x + 3y$, $\frac{\partial f}{\partial y} = 3x + 3y^2$.
Now taking derivatives of fuctions $x(r)$ and $y(r)$ with respect to r we get $\frac{dx}{dr} = -\sin(r)$ and $\frac{dy}{dr} = \cos(r)$. (0.5 marks)
Now using chain rule

$$\frac{df}{dr} = \frac{\partial f}{\partial x}\frac{dx}{dr} + \frac{\partial f}{\partial y}\frac{dy}{dr} = (2x + 3y)(-sin(r)) + (3x + 3y^2)(cos(r))$$

(0.25 marks)

Now substituting definition of $x(r)$ and $y(r)$, we get

$$\frac{df}{dr} = (2cos(r) + 3sin(r))(-sin(r)) + (3cos(r) + 3(sin(r))^2)(cos(r))$$

Finally, simplifying we get

$$\frac{df}{dr} = -2sin(r)cos(r) - 3sin^2(r) + 3cos^2(r) + 3sin^2(r)cos(r)$$

(0.25 marks)

Q2

1) The data matrix $\boldsymbol{X}$ is of dimension 1000×1000 which serves as an input for a function f. The function f is defined as $f(\boldsymbol{X}) = \boldsymbol{a}^T \boldsymbol{X} \boldsymbol{a}$ where $\boldsymbol{a}$ is a constant vector in $\mathbb{R}^{1000}$.

   i. Find $\dfrac{\partial f}{\partial \boldsymbol{X}}$. (1 mark)

   ii. Comment on the existence of the rank 1 approximation of $\dfrac{\partial f}{\partial \boldsymbol{X}}$ and find if it exists. (4 marks)

2) Define $f : \mathbb{R}^4 \to \mathbb{R}$ as

$$f(\boldsymbol{x}) = [x_1, x_2, x_3, x_4] \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 6 & 4 & 2 \\ 3 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{when } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

   i. Find $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \dfrac{\partial f}{\partial \boldsymbol{x}}$. (1 mark)

   ii. Find all values of $\boldsymbol{x}$ such that $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \boldsymbol{0}^T$. (2 marks)

Answer

Q2

1) $f(\boldsymbol{X}) = \boldsymbol{a}^T \boldsymbol{X} \boldsymbol{a}$

   i. Clearly, using identity, $\dfrac{\partial f}{\partial \boldsymbol{X}} = \boldsymbol{a}\boldsymbol{a}^T$. (1 mark)

   ii. If $\boldsymbol{a}$ is a zero vector, then $\dfrac{\partial f}{\partial \boldsymbol{X}}$ is zero matrix and hence of rank 0. Therefore rank 1 approximation doesnot exist. (1 mark)

   If $\boldsymbol{a}$ is a nonzero vector, then there exist $a_i \neq 0$.

   Then $$\frac{\partial f}{\partial \boldsymbol{X}} = \boldsymbol{a}\boldsymbol{a}^T = \begin{bmatrix} a_1 \boldsymbol{a}^T \\ \vdots \\ a_i \boldsymbol{a}^T \\ \vdots \\ a_{1000} \boldsymbol{a}^T \end{bmatrix}.$$

   Now $R_j - \frac{a_j}{a_i} R_i$, $\forall j = 1, \cdots, 1000$ and $j \neq i$, will give $\begin{bmatrix} \boldsymbol{0}^T \\ \vdots \\ a_i \boldsymbol{a}^T \\ \vdots \\ \boldsymbol{0}^T \end{bmatrix}$.

   By interchanging i^{th} and 1^{st} row we get $\begin{bmatrix} a_i \boldsymbol{a}^T \\ \boldsymbol{0}^T \\ \vdots \\ \boldsymbol{0}^T \end{bmatrix}$, which is REF and hence $\dfrac{\partial f}{\partial \boldsymbol{X}} = \boldsymbol{a}\boldsymbol{a}^T$ is rank 1 matrix. (2 marks)

   Therefore, the rank 1 approximation exists and is the same as the actual matrix. (1 mark)

2) Now $f : \mathbb{R}^4 \to \mathbb{R}$ is defined as

$$f(\boldsymbol{x}) = [x_1, x_2, x_3, x_4] \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 6 & 4 & 2 \\ 3 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boldsymbol{x}^T \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 6 & 4 & 2 \\ 3 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \boldsymbol{x} \text{ when } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

   i. $$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \frac{\partial f}{\partial \boldsymbol{x}} = \boldsymbol{x}^T \left(\begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 6 & 4 & 2 \\ 3 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 6 & 3 & 1 \\ 2 & 4 & 0 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} \right)$$

   $$\Rightarrow \nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \boldsymbol{x}^T \left(\begin{bmatrix} 2 & 5 & 5 & 2 \\ 5 & 12 & 7 & 3 \\ 5 & 7 & 0 & 3 \\ 2 & 3 & 3 & 4 \end{bmatrix} \right)$$ (1 mark)

   ii. $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \boldsymbol{0}^T$

   $$\Rightarrow \quad \boldsymbol{x}^T \left(\begin{bmatrix} 2 & 5 & 5 & 2 \\ 5 & 12 & 7 & 3 \\ 5 & 7 & 0 & 3 \\ 2 & 3 & 3 & 4 \end{bmatrix} \right) = \boldsymbol{0}^T$$

   $$\Rightarrow \quad \begin{bmatrix} 2 & 5 & 5 & 2 \\ 5 & 12 & 7 & 3 \\ 5 & 7 & 0 & 3 \\ 2 & 3 & 3 & 4 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$$

   To solve this, we will convert coefficient matrix into REF which is equal to

   $$\begin{bmatrix} 2 & 5 & 5 & 2 \\ 0 & -0.5 & -5.5 & -2 \\ 0 & 0 & 48 & 20 \\ 0 & 0 & 0 & 5/3 \end{bmatrix}$$

   The rank of the matrix is 4 and the number of variables is also 4. Therefore, the solution is unique and the unique solution is $[0, 0, 0, 0]^T$. (2 marks)

Q3

a Find the matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$, where, $\mathbf{A} \neq \mathbf{I}$, $\mathbf{A}^2 \neq \mathbf{I}$, $\mathbf{A}^3 \neq \mathbf{I}$, $\mathbf{A}^4 = \mathbf{I}$ and justify the computations. (4 marks)

b Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$ found in part a. above. (2 marks)

c Find the the matrix $\mathbf{A}^{-1}$. Will the matrix $(\mathbf{A}^{-1})^4$ also be identity matrix $(\mathbf{I})$? Give reasons, you need not evaluate $(\mathbf{A}^{-1})^4$. The matrix here refers to the one mentioned in sub-part a. of this question. (2 marks)

Answer

a We see that when the transform $\mathbf{A}$ is applied four times, it results in the input vector itself. So, this can be rotation by 90° in the counter clockwise direction. (2 marks)

So, $\mathbf{A} = \begin{bmatrix} cos(90°) & -sin(90°) \\ sin(90°) & cos(90°) \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (2 marks)

b For the matrix $\mathbf{A}$, eigen values are:

$\lambda_1 = i$, $\lambda_2 = -i$ (1 mark)

eigen vector corresponding to eigen value $\lambda_1 = i$ is $[i, 1]^T$

eigen vector corresponding to eigen value $\lambda_2 = -i$ is $[-i, 1]^T$ (1 mark)

c $\mathbf{A}$ is rotation in counter clockwise. Hence, $\mathbf{A}^{-1}$ will be rotation in clockwise direction. (1 mark)

Yes, $(\mathbf{A}^{-1})^4$ will also be I, as applying 90° clockwise rotation successively to a vector 4 times gives back the same vector. (1 mark)

Q4

1) Find the dimension of following: (1.5 + 1.5 marks)
   (a) $W = \{\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n\times n} \quad | \quad a_{ij} = 0 \text{ for } i > j\}$
   (b) $V = \{\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n\times n} \quad | \quad a_{ij} = a_{ji}\}$

2) By applying row elementary operations, the system of equations $\mathbf{Ax} = \mathbf{b}$ is reduced to $\mathbf{Rx} = \mathbf{k}$, where $\mathbf{R}$ is RREF of $\mathbf{A}$. The general solution is given by: (2 + 2 marks)

$$x = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

   (a) Find $\mathbf{R}$ and k.
   (b) Find, if possible, the pivotal and non pivotal columns of $\mathbf{R}$.

Answer

Q4

(1) (a) $dim(V) = \frac{n(n+1)}{2}$ (1.5 marks)

(b) $dim(W) = \frac{n(n+1)}{2}$ (1.5 marks)

(2) $\boldsymbol{k} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ (1.25 marks)

$R = \begin{bmatrix} 1 & -0.5 & 2.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (2.75 marks)