Row echelon form (REF)

1. All non-zero rows are above any zero rows
2. Each pivot of a row is to the right of pivot of the row above it.
3. Entries below each pivot is zero.

Example:-

$$\begin{pmatrix} 2 & 5 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 1 \end{pmatrix}$$

Non-example:-

1. $$\begin{pmatrix} 2 & 5 & 2 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

2. $$\begin{pmatrix} 2 & 5 & 3 \\ 0 & 0 & 0 \\ 5 & 2 & 3 \end{pmatrix}$$

3. $$\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reduced Row Echelon form (RREF)

1. Each pivot is equal to 1.
2. Each pivot is only Non-zero entry in its column
3. <u>RREF</u> satisfies all conditions of <u>REF</u>

Example:-

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>Non-example</u> (1)

$$\begin{pmatrix} 0 & 0 & 3 \\ 5 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(3)

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark:- (1) Each Matrix has unique/exactly one RREF but it may have many REF forms.

Rank of Matrix —

The Rank of a matrix is equal No. of pivot elements in REF form of matrix A.

<u>Example:</u>

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank is $\underline{\underline{3}}$

Remark: 1 If 'A' is a square matrix of order n.
and $\det(A) \neq 0$
then Rank $A = n$.

2 If Rank $(A) = n$, then $\det(A) \neq 0$.
where A is $n \times n$ square matrix.

Subspace

A subspace is a subset $W \subseteq V$ that is itself a vector space under the same operations and same field.

Subspace Test —

A Non-empty subset W is a subspace of 'V' iff

1) identity Element: The zero vector '0' of the vector space V is in W.

2) ~~Closed~~ closed Under Addition: For all ~~vector~~ $\alpha, \beta \in W$
$\alpha + \beta \in W$.

3) Closed under scalar Multiplication:
For all scalar $c \in F$ and $\alpha \in W$, $c\alpha \in W$.

Example:- $W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 0\}$

Sol^n: W is a subset of vector space $V = \mathbb{R}^3$ over field $\mathbb{R}$.

(1) $(0,0,0) \in W$ since $0+2\cdot 0+3\cdot 0=0$. (4)

(2) Let $w_1 \in W$ and $w_2 \in W$

if $w_1=(x_1, y_1, z_1)$ and $w_2=(x_2, y_2, z_2)$

$\Rightarrow \quad x_1+2y_1+3z_1=0 \quad -(i)$

and

$x_2+2y_2+3z_2=0 \quad -(ii)$

add i) and ii) $\Rightarrow$

$(x_1+x_2)+2(y_1+y_2)+3(z_1+z_2)=0$

$\Rightarrow \quad (x_1+x_2, y_1+y_2, z_1+z_2) \in W$

by definition of the set W

$\Rightarrow \quad w_1+w_2=(x_1, y_1, z_1)+(x_2, y_2, z_2) \in W$

(3) Let $c \in F$ and $w_1 \in W$

if $w_1=(x_1, y_1, z_1)$

$\Rightarrow \quad x_1+2y_1+3z_1=0$

multiply both sides by 'c'

$\Rightarrow \quad cx_1+2(cy_1)+3(cz_1)=0$

$\Rightarrow \quad (cx_1, cy_1, cz_1) \in W$ (By definition of W)

$\Rightarrow \quad c\cdot w_1 \in W$

proved

So W is subspace of V.

Non-example:

$$W = \{(x, y, z) : x + 2y + 3z = 1\}$$

W is Not space since $(0,0,0) \notin W$

if $(0,0,0) \in W$

$\Rightarrow \quad 0 + 2\cdot 0 + 3\cdot 0 = 1$

$\Rightarrow \quad 0 = 1$ which is contradication

$\Rightarrow (0,0,0) \notin W$

$\Rightarrow$ W is Not a subspace.

Remark: (1) Union of two subspaces may not be a subspace

i.e. If W_1 and W_2 are subspaces then $W_1 \cup W_2$ may not be ~~space~~ subspace.

(2) Intersection of two subspaces is always a subspace.

Linearly Independent ~~Sets~~ Vectors

A set of $\{v_1, v_2, \ldots v_n\}$ vectors in vector space $V(\mathbb{F})$ is said to be linearly independent if $c_1v_1 + c_2v_2 \ldots + c_nv_n = 0 \Rightarrow c_1 = 0, c_2 = 0, \ldots, c_n = 0,$

Basis of a vector space

A non-empty subset 'B' of a vector space 'V' is called a basis of V if

(1) 'B' is a linearly independent set.

(2) vectors in 'B' generates the whole vector space.
i.e. span $B = V$.

Example:-

(1) $B = \{e_1, e_2, e_3\}$ is a basis of $R^3(R)$

where $e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

(2) $B = \{1, x, x^2\}$ is a basis of vector space of all polynomials of degree less than 2.

Dimension:-

Number of vectors in a basis of vector space 'V' is called dimension of vector space V.

Remark:- If $|B| = n$, then $\dim V = n$.

Example:- $B \subseteq R^3(R)$ $|B| = 3 \Rightarrow \dim R^3(R) = 3$.