BITS Pilani
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Mathematical Foundations

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Webinar#2

Agenda

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Problems on

- **Orthogonal Matrices**
- **Gram-Schmidt orthogonalization**
- **Characteristic polynomial**
- **Eigen values and Eigen vectors**
- **Spectral theorem**
- **Singular Value Decomposition**

Some Important Definitions

A square matrix $'A'$ is said to be orthogonal if its columns are orthonormal

(Orthonormal columns means dot product of any two columns is zero and each column is of length 1)

Equivalently, we can say $AA^T = A^TA = I$

$$A^{-1} = A^T$$

Remark: Rows of the orthogonal matrices are also orthonormal

Example: $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix for all value of θ.

Gram-Schmidt Orthonormalization Method

Given a linearly independent set $\{x_1, x_2, \ldots. x_n\}$ we have to find an orthonormal set $\{e_1, e_2, \ldots e_n\}$ such that span $\{x_1, x_2, \ldots. x_n\}$ = span $\{e_1, e_2, \ldots e_n\}$.

1st step. The first element of (e_k) *is*

$$e_1 = \frac{1}{\|x_1\|} x_1.$$

2nd step. x_2 can be written

$$x_2 = \langle x_2, e_1 \rangle e_1 + v_2.$$

Then (Fig. 32)

$$v_2 = x_2 - \langle x_2, e_1 \rangle e_1$$

is not the zero vector since (x_j) is linearly independent; also $v_2 \perp e_1$ since $\langle v_2, e_1 \rangle = 0$, so that we can take

$$e_2 = \frac{1}{\|v_2\|} v_2.$$

Gram-Schmidt Orthonormalization Method

3rd step. The vector

$$v_3 = x_3 - \langle x_3, e_1 \rangle e_1 - \langle x_3, e_2 \rangle e_2$$

is not the zero vector, and $v_3 \perp e_1$ as well as $v_3 \perp e_2$. We take

$$e_3 = \frac{1}{\|v_3\|} v_3.$$

***n*th step.** The vector (see Fig. 33)

(13) $$v_n = x_n - \sum_{k=1}^{n-1} \langle x_n, e_k \rangle e_k$$

is not the zero vector and is orthogonal to $e_1, \cdots, e_{n-1}$. From it we obtain

(14) $$e_n = \frac{1}{\|v_n\|} v_n.$$

Example

Considert the basis $\{\begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\3\end{bmatrix},\}$ of R^3. Using Gram-Schmidt Orthonormalization Method convert this basis to an orthonormal basis of R^3.

Let $x_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$, $x_2 = \begin{bmatrix}-2\\-1\\1\end{bmatrix}$, and $x_3 = \begin{bmatrix}2\\2\\3\end{bmatrix}$

Take $v_1 = x_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$. $||v_1|| = \sqrt{1+1+1} = \sqrt{3}$.

Normalizing we get $e_1 = \frac{v_1}{||v_1||} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}$

Example

$$v_2 = x_2 - < x_2, e_1 > e_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} . \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\right) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \text{ and } ||v_2|| = \sqrt{4+1+1} = \sqrt{6}.$$

Normalizing, $e_2 = \frac{v_2}{||v_2||} = \begin{bmatrix} \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$

Example

$v_3 = x_3 - < x_3, e_1 > e_1 - < x_3, e_2 > e_2 =$

$$\begin{bmatrix}2\\2\\3\end{bmatrix} - \left(\begin{bmatrix}2\\2\\3\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix} - \left(\begin{bmatrix}2\\2\\3\end{bmatrix} \cdot \begin{bmatrix}\frac{-2}{\sqrt{6}}\\ \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{bmatrix}\right)\begin{bmatrix}\frac{-2}{\sqrt{6}}\\ \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{bmatrix} =$$

$$\begin{bmatrix}2\\2\\3\end{bmatrix} - \sqrt{3}\begin{bmatrix}\frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix} + \frac{3}{\sqrt{6}}\begin{bmatrix}\frac{-2}{\sqrt{6}}\\ \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{bmatrix} = \begin{bmatrix}2\\2\\3\end{bmatrix} - \begin{bmatrix}1\\-1\\1\end{bmatrix} + \begin{bmatrix}-1\\ -\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} = \begin{bmatrix}0\\ \frac{5}{2}\\ \frac{5}{2}\end{bmatrix}$$

Also $||v_3|| = \sqrt{0 + \frac{25}{4} + \frac{25}{4}} = \frac{\sqrt{50}}{2} = \frac{5}{\sqrt{2}}$

Example

Normalizing we get $e_3 = \frac{v_3}{||v_3||} = \frac{\sqrt{2}}{5}\begin{bmatrix} 0 \\ \frac{5}{2} \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

The required orthonormal basis = $\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$

Problem on Diagonalization

Is the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$. Find all the Eigen Values of A. Is A Diagonalizable? Justify your answer. If it is diagonalizable, find the diagonal matrix D.

Since A is lower triangular matrix, the eigen values of A are the diagonal elements of A. That is 1, -1, -2, and 2

Note that the Eigen vectors corresponding to distinct eigen values are linearly independent. Since the Eigen values are distinct, there are four independent Eigen vectors each one is corresponding to one Eigen value.
An nxn matrix A is diagonalizable if and only if it has n linearly independent vectors.
Hence D is diagonalizable.

The diagonal matrix D = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Characteristics Polynomial

Characteristics polynomial : For $\lambda \in \mathbb{R}$ and Matrix $A \in \mathbb{R}^{n \times n}$ we can define a polynomial $\rho_A(\lambda) = \det(A - \lambda I)$.

This polynomial can be written as:

$\rho_A(\lambda) = c_0 + c_1\lambda + \ldots + c_{n-1}\lambda^{n-1} + (-1)^n c_n \lambda^n$

where $c_0, c_1, \ldots c_n \in \mathbb{R}$.

Example: Let the Matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

The Characteristics polynomial is $\det(A - \lambda I) = (1 - \lambda)^2 - 1$.

Eigenvalues and eigenvectors

Let A be $n \times n$ square matrix.

$\lambda \in R$ is called eigenvalue of A and $x \in R^n \setminus \{0\}$ is the corresponding eigenvector of λ if

$$Ax = \lambda x$$

This equation is called the eigenvalue equation.

Spectral Theorem

Theorem: If A is n x n symmetric matrix then there exists an orthonormal basis of the corresponding vector space V consisting of the eigenvectors of A, and each eigenvalue is real.

Singular Value Decomposition

Singular values:

The square root of eigen values of a symmetric matrix A^TA or AA^T are called singular values of the matrix A.

Singular Vectors:

The eigen vectors of A^TA corresponding to singular values of A are called singular vectors.

The singular vectors of A^TA are called right singular vectors and singular vectors of AA^T are called left singular vectors.

Definition of SVD:

Let A be any matrix $m \times n$ then this matrix can be decomposed in to product of three matrices given by $A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V^T_{n \times n}$ and it is called singular value decomposition of A.

Where U and V are called orthogonal matrices or unitary matrices and Σ is called singular matrix.

Working Rule:

Step-1: Compute A^TA or AA^T .

Step-2: Find the eigen values of A^TA or AA^T .

Step-3: Compute the square root of non-zero eigen values of above step called singular values of A.

Step-4: Construct a singular matrix Σ of order same as A having singular values on the diagonal in decreasing order.

Step-5: Find the eigen vector of A^TA or AA^T and construct a orthogonal matrix U or V consisting of orthonormal eigen vectors of A^TA.

Step-6: Construct the vectors $\sigma_i u_i = Av_i$ and the orthogonal matrix U.

Step-7: Finally $A = U\Sigma V^T$ gives the singular value decomposition of matrix A.

Example

Q5 Find Singular value decomposition of the matrix $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

Solution:

Required solution is $A_{3\times 2} = U_{3\times 3}\Sigma_{3\times 2}V_{2\times 2}^T$

Step-1: Compute A^TA

$$\Rightarrow A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} * \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

Step-2: Find the eigen values of A^TA: The characteristic equation(CE) of matrix A^TA.

$\Rightarrow \lambda^2 - Trace(A^TA)\lambda + |A^TA| = 0$

$Trace = 90,\ |A^TA| = 0$

$\Rightarrow CE : \lambda^2 - 90\lambda + 0 = 0$

$\Rightarrow \lambda(\lambda - 90) = 0$

$\Rightarrow \lambda_1 = 90, \lambda_2 = 0$ are called eigen values of matrix A^TA.

Note: You can also form the characteristic equation: $\det(A - \lambda I) = 0$.

Step-3: Next find the singular values of A: $\sigma_1 = \sqrt{\lambda_1} = \sqrt{90}$ and $\sigma_2 = 0$.

Step-4: Construct the singular matrix of size same as A and with its diagonal elements being singular values in decreasing order and all other entries are zeros(equivalent diagonal matrix)

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Step-5: Find the eigen vectors of $A^T A$ called right singular vector of A with respect to each eigen value;

For $\lambda = 90$

$$\Rightarrow A^T A - 90I = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} - 90 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix}$$

$$\Rightarrow = \begin{bmatrix} -9 & -27 \\ 0 & 0 \end{bmatrix} \Rightarrow y = k \text{ is a free variable.}$$

Hence from first row we get $-9x - 27y = 0 \Rightarrow x = -3y = -3k$

Therefore eigen vectors w.r.t $\lambda = 90$ is

$$X_1 = k \begin{bmatrix} -3 \\ 1 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

For $\lambda = 0$

$$\Rightarrow A^T A - 0I = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} - 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

$$\Rightarrow = \begin{bmatrix} 81 & -27 \\ 0 & 0 \end{bmatrix} \Rightarrow y = k \text{ is a free variable.}$$

Hence from first row we get $81x - 27y = 0 \Rightarrow x = \frac{1}{3}y = \frac{1}{3}k$

Therefore eigen vectors w.r.t $\lambda = 0$ is

$$X_2 = \frac{k}{3}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Step-6: Write the orthogonal matrix or right singular vectors matrix $V = [\hat{v_1} || \hat{v_2}]$

$$V = \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Step-7: Find the left eigen vectors or left singular vectors $\hat{u}_i$ using

$$\hat{u}_i = \frac{Av_i}{\sigma_i}$$

$i = 1$

$$u_1 = \frac{Av_i}{\sigma_i} = \frac{1}{\sqrt{90}} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} * \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{10}{\sqrt{900}} \\ -\frac{20}{\sqrt{900}} \\ -\frac{20}{\sqrt{900}} \end{bmatrix}$$

Since the second eigen value is zero $lambda_2 = \sigma_2 = 0$, to find the other two vector let us construct the orthogonal vectors using orthogonality conditions

Let the next vector be $w = (x, y, z)$ then $w \cdot u_1 = 0$

$\Rightarrow 10x - 20y - 20z = 0$

$\Rightarrow y = k_1$ and $z = k_2$ are free variables hence

$x = 2y + 2z = 2k_1 + 2k_2$

$$\Rightarrow w = k_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Hence the vectors orthogonal to u_1 are

$$w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } w_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$\text{Let } u_2 = w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \hat{u}_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$u_3 = w_2 - \frac{\langle w_2, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$$

$$\Rightarrow u_3 = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} \Rightarrow \hat{u}_3 = \frac{u_3}{\|u_3\|} = \begin{bmatrix} \frac{2}{\sqrt{45}} \\ -\frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{bmatrix}$$

Hence the orthogonal matrix U or matrix of left singular vectors is given by

$$U = \begin{bmatrix} \frac{10}{\sqrt{900}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ -\frac{20}{\sqrt{900}} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{45}} \\ -\frac{20}{\sqrt{900}} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix}$$

Step-8:

Hence the singular value decomposition of given matrix A is

$A = U\Sigma V^T$

$$= \begin{bmatrix} \frac{10}{\sqrt{900}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ -\frac{20}{\sqrt{900}} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{45}} \\ -\frac{20}{\sqrt{900}} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Problem 1

Q Answer the following questions with justifications.

(A) Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix},$$

a professor asks two of his best students enrolled in Linear Algebra class to find the maximum value of $\mathbf{x^T A x}$, subject to the fact that $\|\mathbf{x}\|^2 = 1$, where $\|\|$ is the Euclidean norm. Given that the students have not studied Calculus earlier, the first student says that this is impossible, whereas the second one is optimistic in estimating the value. Who is correct and why? Give adequate justifications.
HINT: Find a symmetric matrix $\mathbf{B}$ such that $\mathbf{x^T A x = x^T B x}$

(4 marks)

(B) Is $\lambda = 4$ an eigenvalue of

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}?$$

If yes, find the corresponding eigenvector.

(3 Marks)

NOTE: 1 additional mark if you can do it **without explicitly** finding the eigenvalues and checking if 4 is one of the eigenvalues.

Solution

Answers

(A) expanding

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and rearranging terms gives us the off diagonal terms to be $(5+3)/2 = 4$ for the symmetric matrix. Thus,

$$\mathrm{B} = \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$$

Let λ_1, λ_2 be eigenvalues of B with $\mathbf{e_1}$, $\mathbf{e_2}$ being the corresponding eigenvectors. Any $\mathbf{x}$ in $\mathbb{R}^2$ can be written as:

$$\mathbf{x} = p * \mathbf{e_1} + q * \mathbf{e_2}, p^2 + q^2 = 1$$

Now,

$$\mathbf{x}^{\mathbf{T}}\mathrm{B}\mathbf{x} = \begin{bmatrix} p\mathbf{e_1} & q\mathbf{e_2} \end{bmatrix} \begin{bmatrix} p\mathbf{e_1}\lambda_1 \\ \mathbf{e_2}\lambda_2 \end{bmatrix} = p^2\lambda_1 + q^2\lambda_2 \leq \max(\lambda_1, \lambda_2)$$

$$\lambda_{max} = \frac{3+\sqrt{65}}{2}$$

Solution

(B) If $\lambda = 4$ is an eigenvalue, then the eigenvector is in the null space of $\mathbf{A} - \lambda\mathbf{I_3}$. For this, the null space should have dimension > 0. Which can be verified as:

$$\left[\begin{array}{ccc|c} 3-4 & 0 & -1 & 0 \\ 2 & 3-4 & 1 & 0 \\ -3 & 4 & 5-4 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 * -1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2*R_1, R_3 \leftarrow R_3 + 3*R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Corresponding eigenvector is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} t$$

Problem 2

Q Answer the following questions with justifications.

(1) Given the characteristic equation of a matrix A, can we compute the characteristic equation of cA where c is a non-zero scalar without knowing the entries of A? If so, show how to do it using detailed calculations. Otherwise explain why it is not possible. Clearly state all your assumptions

Solution:

(1) The characteristic equation of the matrix cA is $det(cA - \lambda I) = 0$. We can rewrite this as $det(c(A - \frac{\lambda}{c}I)) = 0$ which can then be written as $c^n det(A - \frac{\lambda}{c}I) = 0$ or $det(A - \frac{\lambda}{c}I) = 0$. Let $\mu = \frac{\lambda}{c}$, and assume that the characteristic equation $det(A - \mu I) = 0$ can be written in terms of the polynomial $\mu^n + a_{n-1}\mu^{n-1} + \ldots a_1\mu + a_0 = 0$. Substituting $\lambda/c = \mu$ in this polynomial we get $(\frac{\lambda}{c})^n + a_{n-1}(\frac{\lambda}{c})^{n-1} + \ldots a_1(\frac{\lambda}{c}) + a_0 = 0$. Multiplying throught by c^n we finally get $\lambda^n + ca_{n-1}\lambda^{n-1} + \ldots a_1c^{n-1}\lambda + a_0c^n = 0$.

Thus we see that the characteristic equation of cA can be obtained by taking the coefficient a_k of the kth term of the characteristic equation of A and multiplying it by c^{n-k}. Thus there is no need to look at the entries of the matrix A.

Problem 3

Q Answer the following

(A) Given that the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & b \end{bmatrix}, a, b \in \mathbb{R}$$

has 2 and -1 as its eigenvalues with algebraic multiplicity of 2 and 1 respectively. Further, the eigenvector corresponding to eigenvalue -1 is

$$\mathbf{e_3} = \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

(a) Find the value of b

(1 mark)

(b) Write all the eigenvalues and their geometric multiplicity

(1 mark)

(c) By observation, and using the properties of a symmetric matrix, find the other two eigenvectors ($\mathbf{e_1}$ and $\mathbf{e_2}$) of $\mathbf{A}$. (2 mark)

(d) Write the spectral decomposition of the matrix $\mathbf{A}$

(1 mark)

(e) find the value of a

(1 mark)

Problem 3 – Solution

Answers

(A) (a) The eigenvalues of **A** are 2,2,-1. Hence, 1+2+b = 2+2-1. Thus, b = 0

(b) since A is symmetric, eigenvalue 2 has GM 2 and eigenvalue -1 has GM 1

(c) The eigenvector corresponding to eigenvalue -1 is given

$$\mathbf{e_3} = \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

The other two eigenvectors correspond to eigenvalue 2 and both should lie in a plane orthogonal to $\mathbf{e_3}$ Thus, $\mathbf{e_1}$ and $\mathbf{e_2}$ are:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

Problem 3 - Solution

(d) The spectral decomposition of $\mathbf{A}$ is:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & b \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

(e) multiplying only the relevant terms:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & b \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & & 0 \\ & & \frac{2}{\sqrt{3}} \\ & & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

$$\mathrm{a} = \sqrt{2}$$

Problem 3 B

(B) Given that the Singular Value Decomposition of $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & e_{22} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & \lambda_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

(a) Find the value of λ_2

(1 mark)

(b) Find the value of e_{22}

Problem 3 B solution

(B) (a)

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{2}} & & \\ \frac{\lambda_2}{\sqrt{18}} & -\frac{\lambda_2}{\sqrt{18}} & \frac{4\lambda_2}{\sqrt{18}} \end{bmatrix}$$

$$\frac{5}{2} + \frac{\lambda_2}{\sqrt{2}\sqrt{18}} = 3$$

$$\lambda_2 = 3$$

(b) $e_{22} = \frac{1}{\sqrt{2}}$

Problem 4

Q Answer the following for the given matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(A) Obtain the left-singular vectors of $\mathbf{A}$.

(3 marks)

(B) Obtain the right-singular vectors of $\mathbf{A}$.

(3 marks)

(C) Obtain the singular value matrix Σ. What is the spectral norm of $\mathbf{A}$?

(2 marks)

Problem 4 solution

A Answers

(A)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{A^T} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{AA^T} = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

eigenvalues and eigenvectors of AA^T :

$$\lambda_1 = \sigma_1^2 = 6, u_1 = \begin{bmatrix} 5 & 2 & 1 \end{bmatrix}^T, \|u_1\| = \sqrt{30}$$

$$\lambda_2 = \sigma_2^2 = 1, u_2 = \begin{bmatrix} 0 & \frac{-1}{2} & 1 \end{bmatrix}^T, \|u_2\| = \frac{\sqrt{5}}{2}$$

$$\lambda_3 = \sigma_3^2 = 0, u_3 = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}^T, \|u_3\| = \sqrt{6}$$

Singular value matrix:

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{bmatrix}$$

Left singular vectors

$$\mathbf{U} = \begin{bmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \frac{u_3}{\|u_3\|} \end{bmatrix}$$

solution

(B) Right Singular vectors:

$$v_1 = \frac{1}{\sigma_1}\mathbf{A}^T u_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}\frac{1}{\sqrt{30}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\frac{6}{\sqrt{6}\sqrt{30}}$$

$$v_2 = \frac{1}{\sigma_2}\mathbf{A}^T u_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}\frac{2}{\sqrt{5}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}\frac{1}{\sqrt{5}}$$

(C) Singular value matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{bmatrix}$$

Spectral norm $= \sqrt{6}$ Marking Scheme: 1 mark for matrix, 1 mark for spectral norm

THANK YOU