# Q1

Represent the standard inner product in $\mathbb{R}^3$ with respect to the basis $(1,0,1),(0,1,1),(1,1,0)$. Also find the inner product between $(-2,1,3)$ and $(-4,5,9)$. [4 Marks]

## Ans

The standard inner product in $R^3$ is defined as:

$<u, v> = u_1v_1 + u_2v_2 + u_3v_3$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$.

Given the basis {(1, 0, 1), (0, 1, 1), (1, 1, 0)}, we can represent any vector in $R^3$ as a linear combir vectors. For example, a vector x can be written as:

$x = a(1, 0, 1) + b(0, 1, 1) + c(1, 1, 0)$

where a, b, and c are scalars.

Then, the inner product of two vectors x and y in this basis can be computed as:

$<x, y> = (a, b, c) * A * (p, q, r)$

where A is the matrix whose columns are the basis vectors, and * denotes the matrix multipli

A = | (1, 0, 1) |
| (0, 1, 1) |
| (1, 1, 0) |

and

(p, q, r) = (a, b, c) * A

Therefore, the inner product of x and y can be computed as:

<x, y> = (a, b, c) * A * A^-1 * (p, q, r)

where A^-1 is the inverse of matrix A.

To find the inner product between (-2, 1, 3) and (-4, 5, 9), we can write these vectors as linea
basis vectors:

(-2, 1, 3) = -2(1, 0, 1) + 1(0, 1, 1) + 3(1, 1, 0)
(-4, 5, 9) = -4(1, 0, 1) + 5(0, 1, 1) + 9(1, 1, 0)

Then, using the formula above, we can compute the inner product as:

<(-2, 1, 3), (-4, 5, 9)> = (-2, 1, 3) * A * A^-1 * (-4, 5, 9)

which evaluates to 26.

Therefore, the inner product between (-2, 1, 3) and (-4, 5, 9) with respect to the given basis i

# Q2

Let $A$ and $B$ be two $n \times n$ matrices. A Linear Algebra Professor asked whether rank of $B$ and $AB$ are the same if $A$ has full rank. What should be the answer? Justify.

# Ans

No, the rank of B and AB are not necessarily the same, even if A has full rank. Here's a counte

Let:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then:

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The rank of A is 1, but the rank of AB is 0.

Therefore, the statement "If A has full rank, then the rank of B and AB are the same" is false.

# Q3

(b) If $A^2 = A$, $B^2 = B$ and $I - A - B$ is invertible then help the students to prove $A$ and $B$ are of same rank. [5 Marks]

# Ans

1. **Express B in terms of A and I:**
   Start by rewriting the equation $I - A - B = 0$ as $B = I - A$. Substitute this expression
   $B^2 = B$:

   $$(I - A)^2 = (I - A)$$

2. **Expand and simplify:**
   Expanding the left side of the equation, we get:

   $$I^2 - 2AI + A^2 = I - A$$

   Simplifying and rearranging, we get:

   $$AI = 0$$

3. **Analyze the implications:**
   The equation $AI = 0$ implies that at least one of the following must be true:

- A is the zero matrix.
- I is singular (not invertible).
- A and I have non-trivial common null spaces.

However, we are given that $I - A - B$ is invertible. This means that I cannot be singular, and
possibility of A being the zero matrix. Therefore, the only remaining possibility is that A and I h
common null spaces.

4. **Relate the ranks of A and B:** Since A and I have non-trivial common null spaces, this mea
of A is not just the zero vector. In other words, there exists a non-zero vector x such that

Now, consider the expression $(I - A)x$. We know that Ax = 0, so:

$$(I - A)x = Ix - Ax = x - 0 = x$$

This shows that $(I - A)x = x$ for any non-zero vector in the null space of A. Since $(I - A)$
the null space of B is also non-trivial.

In linear algebra, the rank of a matrix is defined as the dimension of its column space (or equi
The null space and the column space of a matrix are always orthogonal, meaning they have n
common. Therefore, if a matrix has a non-trivial null space, its column space cannot have full
cannot be equal to the number of columns).

Since we have shown that both A and B have non-trivial null spaces, we can conclude that the
than their number of columns. Furthermore, since they have the same null space (as shown in
have the same rank.

**In conclusion, we have proven that if A^2 = A, B^2 = B, and I - A - B is invertible, then A and
same rank.**