INDIAN INSTITUTE OF TECHNOLOGY ROPAR
Department of Mathematics
Second Semester of Academic Year 2015-2016
MAL 114 - Linear Algebra
Lab tutorial sheet II

---

(1) Solve the following system of equations $AX = B$ using **Jacobi's iterative method**. Consider the residual vector at the k-iteration is given by $r_k = B - AX_k$. Find the number of iterations when the norm of residual vectors is less than 10^{-5}.

(a) $27x + 6y - z = 85, 6x + 15y + 2z = 72, x + y + 54z = 110$

(b) $20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$

(c) $10x - y + 2z = 6, -x + 11y - z + 3w = 25, 2x - y + 10z - w = -11, 3y - z + 8w = 15$ with $x_0 = (0, 0, 0, 0)$

(2) Solve the following system of equations using **Gauss–Seidel method** accurate to four significant digits in the norm of the residual vector.

(a) $2x_1 - x_2 + 2x_3 = 3, x_1 + 3x_2 + 3x_3 = -1, x_1 + 2x_2 + 5x_3 = 1$ with initial approximation (0.3, -0.8, 0.3).

(b) $10x + y + 2z = 44, 2x + 10y + z = 51, x + 2y + 10z = 61$

(3) Apply Gauss–Siedel method to find the solution and find number of iterations for convergence to error tolerance 10^{-7}.

(a) $6x + y + z = 107$

(b) $x + 9y - 2z = 36$

(c) $2x - y + 8z = 121$ next consider equations in order (b),(c),(a) and apply same procedure. Compare progress of method in two cases.

(4) Determine which method converges faster by finding the number of iterations using Gauss–Seidel method as well as Jacobi's method. Use the convergence criterion as difference between successive iteration's residual vector is less than $\epsilon = 10^{-5}, 10^{-7}$ and 10^{-10}

$$\begin{aligned} 10x - 2y - z - w &= 3 \\ -2x + 10y - z - w &= 15 \\ -x - y + 10z - 2w &= 27 \\ -x - y - 2z + 10w &= -9 \end{aligned}$$

(5) Solve the following system of equations using a) Jacobi's iterative method b) Gauss–Seidel iterative method:

$$\begin{aligned} 2x - 3y &= -7 \\ x + 3y - 10z &= 9 \\ 3x + z &= 13 \end{aligned}$$

and explain why both methods diverges faster.

(6) Consider the system $AX = B$, where the $m \times m$ symmetric tridiagonal matrix & B are given by

$$A = \begin{pmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 \\ 2 \\ \vdots \\ 2 \\ 3 \end{pmatrix}$$

Solve the system using the Jacobi and the Gauss-Seidel method with stopping criteria $\frac{\|X^{k+1}-X^k\|}{\|X^k\|} \leq 10^{-7}$. Plot the number of iterations vs. m graph, for $m = 10, 20, \ldots, 100$.

(7) If $A \in \mathbb{R}^{n\times n}$, then 1-norm, ∞-norm and p-norm defined as $\|A\|_1 = \max\limits_{1\leq j\leq n} \sum\limits_{i=1}^{n} |A_{ij}|$,

$\|A\|_\infty = \max\limits_{1\leq i\leq n} \sum\limits_{j=1}^{n} |A_{ij}|$ and $\|A\|_p = \left(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} |A_{ij}|^p)\right)^{1/p}$ respectively

(a) Solve the above system with convergence condition that 2-norm relative error $\frac{\|AX^k-B\|_1}{\|X^k\|_1} \leq 10^{-7}$. Find the possible number of iterations.

(b) Find the number of iterations if we modify the convergence criterion $\frac{\|AX^k-B\|_2}{\|X^k\|_2} \leq 10^{-7}$ and $\frac{\|AX^k-B\|_\infty}{\|X^k\|_\infty} \leq 10^{-7}$

(8) Solve by **Successive over relaxation** (SOR) method with $\omega = 1.04, 1.1, 1.2$ for the following linear system. Take initial approximation as zero

(a) $4x + y + z + t = 6, -x - 3y + z + s = 6, 2x + y + 5z - s - t = 6,$
$-x - y - z + 4s = 6, 2y - z + s + 4t = 6$
(b) $3x - y + z = 1, 3x + 6y + 2z = 0, 3x + 3y + 7z = 4$
(c) Find optimal choice of ω for the above systems (a) and (b)?

(9) Use SOR method with optimal choice of ω and initial approximation as zero to solve the following system

(a) $-10x - y = 9, -x + 10y - 2z = 7, -2y + 10z = 6$
(b) $10x + 5y = 6, 5x + 10y - 4z = 25, -4y + 8z - s = -11, -z + 5s = -11$

(10) Solve the following system of equations using **Conjugate gradient method**. Use initial conditions are zero and convergence tolerance of 3×10^{-7}.

(a) $4x + 3y = 24, 3x + 4y - z = 30, -y + 4z = -24$
(b) $7x_1 - 3x_2 = 4, -3x_1 + 9x_2 + x_3 = -6, x_2 + 3x_3 - x_4 = 3, -x_3 + 10x_4 + 4x_5 = 7,$
$4x_4 + 6x_5 = 2$
(c) $3x_1 + x_2 - x_3 = 2, x_1 + 4x_2 + 2x_3 = 7, -x_1 + 2x_2 + 5x_3 = 6$

(11) In 10 (a) and 10 (b), compare the number of iterations required to achieve the convergence with number of iterations of Gauss–Seidel and SOR method with optimal ω. Here optimal $\omega = 1.25$ for 10 (a) and $\omega = 1.1128$ for 10 (b).

(12) Using **Cholesky decomposition** solve the following system of equations :

(a) $x+y+z=3, x+2y+3z=6, x+3y+6z=10$

(b) $x+2y+6z=5, 2x+5y+15z=12, 6x+15y+46z=37$

(13) Solve the above systems $AX=B$ by finding the inverse of A using the upper triangular matrix (U) such that $A=U'U$.

(14) Find the inverse of the following matrices using Cholesky method :

(a) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 10 \\ 4 & 10 & 21 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 4 & 8 & 12 & -8 \\ 2 & 3 & 2 & 1 \\ -3 & -1 & 1 & -4 \end{bmatrix}$

(15) Use Cholesky matrix for the following matrix to see $U'U - A\sim O$ where U is upper triangular matrix and O is zero matrix. $A = \begin{bmatrix} 1 & 2 & 4 & 7 \\ 2 & 13 & 23 & 38 \\ 2 & 23 & 77 & 122 \\ 7 & 38 & 122 & 294 \end{bmatrix}$

(16) Using **Jacobi's method**, find all the eigenvalues and the corresponding eigenvectors of the matrices given below:

(a) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & -3 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

(17) (a) Reduce the following symmetric matrices to tridiagonal form using **Householder's method**

i. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$ ii. $\begin{bmatrix} 4 & 6 & 242 & 12 \\ 6 & 225 & 3 & 18 \\ 242 & 3 & 25 & 6 \\ 12 & 18 & 6 & 0 \end{bmatrix}$

iii. $\begin{bmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{bmatrix}$

(b) Calculate the eigenvalues from the computed tridiagonal system as accurately as possible.

(18) Find find all eigenvalues of the following matrices using **QR method**. Iterate until all the off-diagonal elements have magnitude less than 10^{-7}.

(a) $\begin{bmatrix} 6 & -3 & 4 & 1 \\ 4 & 2 & 4 & 0 \\ 4 & -2 & 3 & 1 \\ 4 & 2 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 4 \\ 1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 3 & -1 & 3 & -2 & 1 \\ 2 & -2 & 4 & 0 & 0 \\ -5 & 5 & -5 & 8 & -3 \\ -4 & 4 & -4 & 4 & 0 \end{bmatrix}$

(19) Obtain **Singular value decomposition** of

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & -2 \\ 1 & 4 & 3 \\ 2 & 1 & 4 \end{bmatrix}$

(20) Use iterative method to find the largest eigenvalue and corresponding eigenvector of the matrix $\begin{bmatrix} 5 & 2 & 1 & -2 \\ 2 & 6 & 3 & -4 \\ 1 & 3 & 19 & 2 \\ -2 & -4 & 2 & 1 \end{bmatrix}$