# Maths

Date

## Partial Differentiation

* function of two variables
  If any $z$ depends upon any $x$ & $y$ the $z$ is called a function of $x$ and $y$ ($x$ & $y$ are variables). Denoted by: $z = f(x, y)$

  Ex → Area of Δ depends upon two variables base & altitude ($\frac{1}{2} \times$ base $\times$ altitude)

* Partial derivatives -

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x \text{ (keeping } y \text{ constant)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y \text{ (keeping } x \text{ constant)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{xy}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{yx}$$

⇒ In general $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ ; $f_{xy} = f_{yx}$

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Ques1- If $f(x,y) = x^3y - xy^3$, find $\left\{\frac{1}{f_x} + \frac{1}{f_y}\right\}_{\substack{x=1\\y=2}}$

Sol: $f_x = \frac{\partial z}{\partial x} = 3x^2y - y^3$

$f_y = \frac{\partial z}{\partial y} = x^3 - 3xy^2$ (Let $z = f(x,y)$)

$$\left[\frac{1}{f_x} + \frac{1}{f_y}\right]_{\substack{x=1\\y=2}} = \left[\frac{1}{3x^2y - y^3} + \frac{1}{x^3 - 3xy^2}\right]_{\substack{x=1\\y=2}}$$

$$= \left[\frac{1}{6-8} + \frac{1}{1-12}\right]$$

$$= \left[-\frac{1}{2} - \frac{1}{11}\right] = \left[\frac{-11-2}{22}\right]$$

$$= \boxed{\frac{-13}{22}} \rightarrow \text{Ans}$$

Ques2- If $z(x+y) = x^2+y^2$, Show that-

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Sol- $z = \frac{x^2+y^2}{x+y}$

LHS

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x - (x^2+y^2)}{(x+y)^2} = \frac{x^2-y^2+2xy}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)2y - (x^2+y^2)}{(x+y)^2} = \frac{y^2-x^2+2xy}{(x+y)^2}$$

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$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{x^2-y^2+2xy}{(x+y)^2} - \frac{y^2-x^2+2xy}{(x+y)^2}\right]^2$$

$$= \left[\frac{x^2-y^2+\cancel{2xy}-y^2+x^2-\cancel{2xy}}{(x+y)^2}\right]^2$$

$$= \left(\frac{2x^2-2y^2}{(x+y)^2}\right)^2$$

$$= \frac{2^2(x^2-y^2)^2}{(x+y)^4} = \frac{4\cancel{(x+y)^2}(x-y)^2}{(x+y)^{\cancel{4}\,2}}$$

$$= \cancel{\frac{4(x^2+y^2+2xy)(x^2+y^2-2xy)}{(x+y)^4}}$$

$$= \frac{4(x-y)^2}{(x+y)^2}$$

RHS

$$4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left(1 - \left(\frac{x^2-y^2+2xy}{(x+y)^2}\right) - \left(\frac{y^2-x^2+2xy}{(x+y)^2}\right)\right)$$

$$= \frac{4(x-y)^2}{(x+y)^2}$$

LHS = RHS

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Ques 3- Prove that $f_{xy} = f_{yx}$ if $f(x,y) = \frac{1}{\sqrt{y}} e^{-\left(\frac{(x-a)^2}{4y}\right)}$

Sol- $f_x = \frac{\partial z}{\partial x} = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}} \cdot \left(\frac{-\cancel{2}(x-a)}{_2\cancel{4}y}\right)$

$= -\frac{1}{2} y^{-3/2} (x-a) e^{-\frac{(x-a)^2}{4y}}$.

$f_y = \frac{\partial z}{\partial y}$ u.v rule $= -\frac{1}{2} y^{-3/2} e^{-\frac{(x-a)^2}{4y}} + \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}} \cdot \frac{(x-a)^2}{4y^2}$

$= e^{-\frac{(x-a)^2}{4y}} \left[-\frac{1}{2} y^{-3/2} + y^{-1/2} \frac{(x-a)^2}{4y^2}\right]$

$= e^{-\frac{(x-a)^2}{4y}} \left[-\frac{1}{2} y^{-3/2} + \frac{1}{4} y^{-5/2} (x-a)^2\right]$

$= \frac{1}{4} y^{-3/2} e^{-\frac{(x-a)^2}{4y}} \left[-2 + \frac{(x-a)^2}{y}\right]$

$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[\frac{1}{4} y^{-3/2} e^{-\frac{(x-a)^2}{4y}} \left(-2 + \frac{(x-a)^2}{y}\right)\right]$

$\frac{1}{4} y^{-3/2} \frac{\partial}{\partial x}\left[e^{-\frac{(x-a)^2}{4y}} \left(-2 + \frac{(x-a)^2}{y}\right)\right]$

$\cancel{= \frac{\partial}{\partial x}\left[-\frac{1}{2} y^{-3/2} e^{-\frac{(x-a)^2}{4y}} (x-a)\right]}$

$= \frac{1}{4} y^{-3/2} \left[e^{-\frac{(x-a)^2}{4y}} \left(0 + \frac{2}{y}(x-a)\right) + \left(-2 + \frac{(x-a)^2}{y}\right)\left(e^{-\frac{(x-a)^2}{4y}}\right) \left(\frac{-\cancel{2}(x-a)}{_2\cancel{4}y}\right)\right]$

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Ques:- find the first order partial derivatives:-

(1.) $u = y^x$

$\frac{\partial u}{\partial x} = y^x \log y$ $\quad [a^x = a^x \log a]$

$\frac{\partial u}{\partial y} = x y^{x-1}$

(2) $u = \log(x^2+y^2)$

$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$

(3.) $u = x^2 \sin y/x$

$\frac{\partial u}{\partial x} = uv \text{ rule} = 2x \sin y/x + \cancel{x^2} \cos y/x \cdot \left(\frac{-1}{\cancel{x^2}}\right)$

$= 2x \sin y/x - \cos y/x$

$\frac{\partial u}{\partial y} = x^{\cancel{2}} \cos y/x \cdot \frac{1}{\cancel{x}}$

$= x \cos y/x$

(4.) $u = \frac{x}{y} \tan^{-1}\left(\frac{y}{x}\right)$

uv rule

$\frac{\partial u}{\partial x} = \frac{1}{y} \tan^{-1}\left(\frac{y}{x}\right) + \frac{x}{\cancel{y}} \cdot \frac{\cancel{x^2}}{x^2+y^2} \cdot \frac{(-\cancel{y})}{\cancel{x^2}}$

$= \frac{1}{y} \tan^{-1}\left(\frac{y}{x}\right) - \frac{x}{x^2+y^2}$

$\frac{\partial u}{\partial y} = -\frac{x}{y^2} \tan^{-1}\left(\frac{y}{x}\right) + \frac{\cancel{x}}{y} \cdot \frac{x^2}{x^2+y^2} \cdot \frac{1}{\cancel{x}}$

$= \frac{-x}{y^2} \tan^{-1}\left(\frac{y}{x}\right) + \frac{x^2}{y(x^2+y^2)}$

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Very Very Important

Ques- If $\theta = t^n e^{-r^2/4t}$, find the value of n which will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$

Sol- LHS $\frac{\partial \theta}{\partial r} = t^{n} e^{-r^2/4t}\left(\frac{-2r}{4t}\right)$

$$= -\frac{1}{2} r t^{n-1} e^{-r^2/4t}$$

$$r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} r^3 t^{n-1} e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1} \frac{\partial}{\partial r}\left[\underbrace{r^3 \cdot e^{-r^2/4t}}_{\text{uv rule}}\right]$$

$$= -\frac{1}{2} t^{n-1}\left[3r^2 \cdot e^{-r^2/4t} + r^3 \cdot e^{-r^2/4t}\left(\frac{-2r}{4t}\right)\right]$$

$$= -\frac{1}{2} t^{n-1} e^{-r^2/4t} \cdot r^2 \left[3 - \frac{r^2}{2t}\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1} e^{-r^2/4t}\left(3 - \frac{r^2}{2t}\right)$$

Also $\Rightarrow$ $\frac{\partial \theta}{\partial t}$ (uv rule) $= n t^{n-1} \cdot e^{-r^2/4t} + t^n \cdot e^{-r^2/4t}\left(\frac{r^2}{4t^2}\right)$

$$= t^{n-1} e^{-r^2/4t}\left(n + t\left(\frac{r^2}{4t^2}\right)\right)$$

$$= t^{n-1} e^{-r^2/4t}\left(n + \frac{r^2}{4t}\right)$$

for LHS & RHS to be equal-

$$-\frac{1}{2} \cancel{t^{n-1}} \cdot \cancel{e^{-r^2/4t}} \left(3 - \frac{r^2}{2t}\right) = \cancel{t^{n-1}} \cancel{e^{-r^2/4t}}\left(n + \frac{r^2}{4t}\right)$$

$$\cancel{-\frac{r^2}{2}\left(3 - \frac{r^2}{2t}\right) = \left(n + \frac{r^2}{4t}\right)}$$

$$\cancel{\frac{r^2}{2}\left(\frac{r^2}{2t} - 3\right) = \left(n + \frac{r^2}{4t}\right)}$$

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$$\frac{r^2}{4t} - \frac{3}{2} = n + \frac{r^2}{4t}$$

$$\cancel{n \left(\frac{r^2}{4t} - \frac{r^4}{4t} + \frac{3r^2}{2}\right)}$$

$$\cancel{\left(\frac{r^2 - r^4 + 6tr^2}{4t}\right)}$$

$$\left(\frac{r^2}{4t} - \frac{3}{2}\right) = \left(n + \frac{r^2}{4t}\right)$$

$$\frac{\cancel{r^2}}{\cancel{4t}} - \frac{3}{2} = n + \frac{\cancel{r^2}}{\cancel{4t}}$$

$$\therefore \boxed{n = -\frac{3}{2}} \rightarrow \text{Ans}$$

Ques: If $u = \log(x^3+y^3+z^3-3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Sol- $$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2+3y^2+3z^2-3xy-3yz-3zx}{x^3+y^3+z^3-3xyz}$$

$$= 3\left(\frac{x^2+y^2+z^2-xy-yz-zx}{x^3+y^3+z^3-3xyz}\right)$$

We know that,

$$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

$$= \frac{3}{x+y+z}$$

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Now,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

Proved

Ques 7- If $x^x y^y z^z = c$, show that at $x=y=z$

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}.$$

Sol.- $x^x y^y z^z = c$

taking log both sides-

$\log(x^x y^y z^z) = \log c$

$x\log x + y\log y + z\log z = \log c$

Now, as z is dependent variable and x & y are independent variables-

first partially differentiating x w.r.t. z

$$\frac{\partial x}{\partial z} \Rightarrow x\cdot\frac{1}{x} + \log x + 0 + \left(z\cdot\frac{1}{z} + \log z\right)\frac{\partial z}{\partial x} = 0$$

Similarly, $\frac{\partial y}{\partial z} \Rightarrow \left(y\cdot\frac{1}{y} + \log y\right) + \left(z\cdot\frac{1}{z} + \log z\right)\frac{\partial z}{\partial y} = 0$

$$\frac{\partial z}{\partial x} = -\left(\frac{1+\log x}{1+\log z}\right) \quad \text{——— (1)}$$

$$\frac{\partial z}{\partial y} = -\left(\frac{1+\log y}{1+\log z}\right) \quad \text{——— (2)}$$

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Now, partially differenting (2) w.r.t $x$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{\partial}{\partial x}\left[\frac{1+\log y}{1+\log z}\right] \qquad \underbrace{(1+\log y)}_{\text{constant}} \cdot (1+\log z)^{-1}$$

$$= (1+\log y) \cdot \frac{1}{(1+\log z)^2} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1+\log y}{(1+\log z)^2} \cdot \frac{1}{z}\left[-\left(\frac{1+\log x}{1+\log z}\right)\right]$$

At $x = y = z$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\cancel{1+\log x}}{(1+\log x)^{\cancel{2}}} \cdot \frac{1}{x}\left[-\left(\frac{\cancel{1+\log x}}{\cancel{1+\log x}}\right)\right]$$

$$= -\frac{1}{x} \cdot \frac{1}{1+\log x}$$

Also $\log e = 1$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(\log e + \log x)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{x(\log ex)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -x(\log ex)^{-1}$$

<u>Proved</u>

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Ques- If $z = f(x+ay) + \phi(x-ay)$, prove that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Sol-

$$\frac{\partial z}{\partial x} = f' + \phi'$$

$$\frac{\partial^2 z}{\partial x^2} = f'' + \phi''$$

$$\frac{\partial z}{\partial y} = af' + a\phi' = a(f' + \phi')$$

$$\frac{\partial^2 z}{\partial y^2} = a^2(f'' + \phi'')$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Hence proved

Ques- If $u = e^{xyz}$, prove that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$$

Sol- LHS

$$\frac{\partial u}{\partial z} = xye^{xyz}$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left(xye^{xyz}\right)$$

$$= x\left(yxz\, e^{xyz} + e^{xyz}\right)$$

$$= x\left(xyz\, e^{xyz} + e^{xyz}\right)$$

$$= xe^{xyz}(xyz + 1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial}{\partial x}\left(x^2yz\, e^{xyz} + xe^{xyz}\right)$$

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$$= \left[yz\left(2xe^{xyz} + x^2yze^{xyz}\right) + \left(e^{xyz} + xyze^{xyz}\right)\right]$$

$$= e^{xyz}\left[2xyz + x^2y^2z^2 + 1 + xyz\right]$$

$$= \left(3xyz + x^2y^2z^2 + 1\right)e^{xyz} \rightarrow \underline{\underline{RHS}}$$

LHS = RHS

Proved

Ques- If $u = \log\sqrt{x^2+y^2+z^2}$, prove that

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$$

Sol- We have,

$$u = \log(x^2+y^2+z^2)^{1/2}$$

$$u = \frac{1}{2}\log(x^2+y^2+z^2)$$

$$2u = \log(x^2+y^2+z^2)$$

Partially Differentiating both sides w.r.t $x$

$$\cancel{2}\frac{\partial u}{\partial x} = \frac{\cancel{2}x}{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2+z} \qquad \left(\frac{u}{v}\text{ rule}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z) - x(2x)}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2}$

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

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LHS

$$(x^2+y^2+z^2)\left(\frac{\cancel{y^2}+\cancel{z^2}-\cancel{x^2}+\cancel{x^2}+z^2-\cancel{y^2}+x^2+y^2-\cancel{z^2}}{(x^2+y^2+z^2)^2}\right)$$

$$(x^2+y^2+z^2)\left(\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}\right) = 1 \rightarrow \text{RHS}$$

LHS = RHS

Proved

Ques- If $u = lx + my$, $v = mx - ly$, show that

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2} \text{ and}$$

$$\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2+m^2}{l^2}$$

Sol-

$$u = lx + my \quad \text{--- (1)}$$
$$v = mx - ly \quad \text{--- (2)}$$

Now,

$$\left(\frac{\partial u}{\partial x}\right)_y = l$$

from (1) & (2)

$$x = \frac{lu+mv}{l^2+m^2} \quad \text{--- (3)}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{l}{l^2+m^2}$$

$lx + my = u \quad \times l$

$mx - ly = v \quad \times m$

$l^2x + mly = ul$

$m^2x - mly = mv$

$x(l^2+m^2) = ul + mv$

$x = \dfrac{ul+mv}{l^2+m^2}$, $y = \dfrac{mx-v}{l}$

~~and $y = \frac{mx-v}{l}$ (4)~~

So,

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = l\left(\frac{l}{l^2+m^2}\right) = \frac{l^2}{l^2+m^2}$$

and,

$$y = \frac{mx-v}{l} \quad \text{--- (4)}$$

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$$\left(\frac{\partial y}{\partial v}\right)_x = \frac{-1}{l}$$

$$v = \frac{mu - (l^2+m^2)y}{l}$$

$$\left(\frac{\partial v}{\partial y}\right)_u = -\frac{(l^2+m^2)}{l}$$

$$\therefore \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2+m^2}{l^2}$$

Proved

Eliminating $x$ from ① & ②

① $\times m$, ② $\times l$

$\Rightarrow mu = mlx + m^2y$

$lv = mlx - l^2y$

$(mu - lv) = (m^2+l^2)y$

$-lv = (m^2+l^2)y - mu$

$$v = \frac{mu - (m^2+l^2)y}{l}$$

## * Homogeneous functions -

If function $f(x,y)$ is said to be homogenous of degree $n$ in variable $x$ & $y$ if it can be expressed in the form $x^n f\left(\frac{y}{x}\right)$ or $y^n f\left(\frac{x}{y}\right)$.

or if

$$f(tx, ty) = t^n f(x,y)$$

eg - $f(x,y) = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

$$f(tx,ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t^1(x+y)}{t^{1/2}(\sqrt{x}+\sqrt{y})}$$

$$\text{Degree} = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

$$= t^{1/2} f(x,y)$$

or

$$f(x,y) = \frac{y\left(\frac{x}{y}+1\right)}{\sqrt{y}\left(\sqrt{\frac{x}{y}}+1\right)} = y^{1/2} f(x,y)$$

Degree $= \frac{1}{2}$

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⇒ Euler's Theorem on Homogeneous Functions:-
If $u$ is a homogeneous function of degree $n$ in $x$ & $y$, then

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

Proof→ Since $u$ is a homogeneous function of degree $n$ in $x$ and $y$, so it can be expressed as:-

$$u = x^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) + x^{n+1} f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right)$$

Now,

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$y\frac{\partial u}{\partial y} = yx^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = yx^{n-1} f'\left(\frac{y}{x}\right)$$

As per Euler's theorem-

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

LHS $nx^n f\left(\frac{y}{x}\right) - \cancel{yx^{n-1} f'\left(\frac{y}{x}\right)} + \cancel{yx^{n-1} f'\left(\frac{y}{x}\right)} = nx^n f\left(\frac{y}{x}\right)$

and $x^n f\left(\frac{y}{x}\right) = u$

So, $nx^n f\left(\frac{y}{x}\right) = nu \to$ RHS  LHS = RHS

Proved

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- Euler's theorem can be extended to any no of variables. If $u$ is homogeneous function of $n$ in $x, y$ & $z$, then -

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

$\Rightarrow$ If $u$ is a homogeneous function of degree $n$ in $x+y$, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Proof $\rightarrow$ Since $u$ is a homogeneous function of degree $n$, then it can be expressed as -

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu. \quad \text{—— (1)}$$

Partially differentiating (1) w.r.t $x$

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial u}{\partial x\partial y} = n\frac{\partial u}{\partial x}$$

Multiplying by $x$

$$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial u}{\partial x\partial y} = nx\frac{\partial u}{\partial x} \quad \text{—— (2)}$$

Partially differentiating (1) w.r.t. $y$

$$x\frac{\partial u}{\partial y\partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y}$$

Multiplying by $y$

$$y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial u}{\partial y\partial x} = ny\frac{\partial u}{\partial y} \quad \text{—— (3)}$$

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Adding (2) & (3)

$$xy\frac{\partial^2 u}{\partial x\partial y} + x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} = nx\frac{\partial u}{\partial x}$$

$$xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = ny\frac{\partial u}{\partial y}$$

$$xy\left(\underbrace{\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y\partial x}}_{=}\right) + x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$
$$= n\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} = \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)(n-1)$$
$$= nu(n-1)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Proved

Ques- Verify Euler's theorem for the following :-

(1) $u = (x^{1/2} + y^{1/2})(x^n + y^n)$

Sol- Put $x \to tx$, $y \to ty$

$$u(tx, ty) = \left((tx)^{1/2} + (ty)^{1/2}\right)\left(t^n(x^n + y^n)\right)$$
$$= t^{1/2}(x^{1/2} + y^{1/2})\, t^n(x^n + y^n)$$
$$= t^{n+1/2}\, u(x, y)$$

Degree $= n + \frac{1}{2}$

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right)u$$

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Verification:-

$$u = (x^{1/2}+y^{1/2})(x^n+y^n) \quad \text{[uv rule]}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}x^{-1/2}(x^n+y^n) + nx^{n-1}(x^{1/2}+y^{1/2})$$

$$x\frac{\partial u}{\partial x} = \frac{1}{2}x^{1/2}(x^n+y^n) + nx^n(x^{1/2}+y^{1/2})$$

$$y\frac{\partial u}{\partial y} = \frac{1}{2}y^{1/2}(x^n+y^n) + ny^n(x^{1/2}+y^{1/2})$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}(x^n+y^n)(x^{1/2}+y^{1/2}) + n(x^n+y^n)(x^{1/2}+y^{1/2})$$

$$= \left(n+\frac{1}{2}\right)(x^n+y^n)(x^{1/2}+y^{1/2}) = \left(n+\frac{1}{2}\right)u$$

Verified

(2) $f(x,y,z) = 3x^2yz + 5xy^2z + 4z^4$

Sol- $u = 3x^2yz + 5xy^2z + 4z^4$

Put $x \to tx$, $y \to ty$, $z \to tz$

$$u = 3(tx)^2(ty)(tz) + 5(tx)(ty)^2(tz) + 4(tz)^4$$
$$= 3t^4(x^2yz) + 5t^4(xy^2z) + 4t^4z^4$$
$$= t^4(3(x^2yz) + 5(xy^2z) + 4z^4)$$
$$= t^4\,u(x,y,z)$$

Degree = 4

∴ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4u$$

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Verification :-

$u = 3x^2yz + 5xy^2z + 4z^4$

$\frac{\partial u}{\partial x} = 6xyz + 5y^2z + 0$

$= 6xyz + 5y^2z$

$x\frac{\partial u}{\partial x} = 6x^2yz + 5xy^2z$ —— (1)

$\frac{\partial u}{\partial y} = 3x^2z + 10xyz + 0$

$y\frac{\partial u}{\partial y} = 3x^2yz + 10xy^2z$ —— (2)

$\frac{\partial u}{\partial z} = 3x^2y + 5xy^2 + 16z^3$

$z\frac{\partial u}{\partial z} = 3x^2yz + 5xy^2z + 16z^4$ —— (3)

Adding (1), (2) & (3)

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial x} \Rightarrow$

$= 6x^2yz + 5xy^2z + 3x^2yz + 10xy^2z + 3x^2yz + 5xy^2z +$

$= 12x^2yz + 20xy^2z + 16z^4$

$= 4(3x^2yz + 5xy^2z + 4z^4) = 4u$

Verified

(3) $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Sol - $u = \operatorname{cosec}^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right) = x^0 f\left(\frac{y}{x}\right)$

Degree = 0

$\therefore$ By Euler's theorem,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Date

Verification :-

$$u = \operatorname{cosec}^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2/y^2}}\cdot\frac{1}{y} + \frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}}\cdot\frac{\cancel{y}}{\cancel{y}} + \frac{1}{x^2+y^2}\left(\frac{-y}{\cancel{x^2}}\right)\cancel{x^2}$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$$= 0$$

<u>Verified</u>

<u>Ques</u>- If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$$
$$= 2\cos 3u \sin u$$

<u>Sol</u>- Here $u$ is not a homogeneous function.

So, $\tan u = \frac{x^3+y^3}{x-y}$

$$\tan u = \frac{x^3\left(\frac{y^3}{x^3}+1\right)}{x\left(1-\frac{y}{x}\right)}$$

Degree $= 2$

Date

So, tan u is a homogeneous function of degree 2.

$\therefore$ By Euler's theorem-

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = 2\frac{\sin u}{\cancel{\cos u}}\cos^{\cancel{2}} u$$

$$= 2\sin u\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \quad \text{——— (1)}$$

LHS Partially differentiating (1) w.r.t $x$

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x\partial y} = 2\cos 2u\frac{\partial u}{\partial x}$$

Multiplying by $x$

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = 2x\cos 2u\frac{\partial u}{\partial x} \quad \text{——— (2)}$$

Now,

Partially differentiating (1) w.r.t $y$

$$y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} + xy\frac{\partial^2 u}{\partial y\partial x} = 2y\cos 2u\frac{\partial u}{\partial y} \quad \text{——— (3)}$$

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Adding ② & ③, we get-

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+xy\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y\partial y}\right)$$
$$=2\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)(2\cos 2u-1)$$

$$\left[\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=(2\cos 2u-1)(\sin 2u)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\sin 2u-\sin 2u$$
$$=\sin 4u-\sin 2u$$
$$=2\cos 3u\sin u \;\; \rightarrow \text{RHS}$$

LHS = RHS

Proved

Ques- If $u=\tan^{-1}\frac{y^2}{x}$, Show that-

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\sin 2u\sin^2 u$$

Sol- $\tan u=\frac{y^2}{x}$ degree = 1

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$\therefore$ By Euler's theorem-

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u} = \sin u \cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u \cos u \quad \text{--- (1)}$$
$$= \frac{1}{2}\sin 2u$$

Ans Partially differentiating (1) w.r.t $x$

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x \partial y} = \frac{1}{x} \cdot x \cos 2u \frac{\partial u}{\partial x}$$

Multiplying by $x$

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial u}{\partial x \partial y} = \cos 2u \left(x\frac{\partial u}{\partial x}\right) \quad \text{--- (2)}$$

Now, Partially differentiating (1) w.r.t $y$

$$y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} + xy\frac{\partial u}{\partial x \partial y} = \cos 2u \left(y\frac{\partial u}{\partial y}\right) \quad \text{--- (3)}$$

Adding (2) & (3), we get-

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial u}{\partial x \partial y} = \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)(\cos 2u - 1)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial u}{\partial x \partial y} = \left(\frac{1}{2}\sin 2u\right)(\cos 2u - 1)$$

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$$\cancel{x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial u}{\partial x\partial y}=\frac{1}{2}\sin 2u\cos 2u-\frac{1}{2}\sin 2u}$$

$$=\frac{1}{\cancel{2}}\sin 2u\cdot(-\cancel{2}\sin^2 u)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial u}{\partial x\partial y}=-\sin 2u\sin^2 u \rightarrow RHS$$

$$LHS = RHS$$

Proved

Ques- If $u=\sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial u}{\partial x\partial y}=-\frac{\sin u\cos 2u}{4\cos^3 u}$$

Sol- $\sin u=\frac{x+y}{\sqrt{x}+\sqrt{y}}=\frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)}$

Degree $=\frac{1}{2}$

$\therefore$ By Euler's theorem-

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial x}(\sin u)=\frac{1}{2}\sin u$$

$$x\cdot\cos u\cdot\frac{\partial u}{\partial x}+y\cdot\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u$$

$$\cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\frac{1}{2}\sin u$$

$$x\frac{\partial y}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u \quad \text{—— (1)}$$

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Partially differentiating ① w.r.t $x$

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u\,\frac{\partial u}{\partial x}$$

Multiplying by $x$

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u\left(x\frac{\partial u}{\partial x}\right) \quad (2)$$

Now, Partially differentiating ① w.r.t $y$

$$y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}+xy\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u\left(y\frac{\partial u}{\partial x}\right) \quad (3)$$

Adding ② & ③, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}\right)\left(\frac{1}{2}\sec^2 u-1\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\tan u\left(\frac{1}{2}\sec^2 u-1\right)$$

$$=\frac{1}{4}\frac{\sin u}{\cos u}\cdot\frac{1}{\cos^2 u}-\frac{1}{2}\frac{\sin u}{\cos u}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\sin u}{4\cos^3 u}-\frac{\sin u}{2\cos u}$$

$$x^2\frac{\partial^2 u}{\partial x^2} - - - - - - = \frac{\sin u-2\cos^2 u\sin u}{4\cos^3 u}$$

$$'' \quad '' \quad '' \quad '' = \frac{\sin u\,(1-2\cos^2 u)}{4\cos^3 u}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}\rightarrow \underline{RHS}$$

LHS = RHS

<u>Proved</u>

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Ques: If $u = (x^2+y^2)^{1/3}$, show that
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{2u}{9}$$

Sol- let $x \to tx$, $y \to ty$

$$u = \left((tx)^2 + (ty)^2\right)^{1/3}$$

$$= \left(t^2(x^2+y^2)\right)^{1/3}$$

$$u = t^{2/3}(x^2+y^2)^{1/3}$$

So, u is a homogeneous function of x & y of degree $\frac{2}{3}$.

$\therefore$ By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2}{3}u \quad \text{—— (1)}$$

Partially differentiating (1) w.r.t $x$

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x\partial y} = \frac{2}{3}\frac{\partial u}{\partial x}$$

Multiplying by $x$

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial u}{\partial x\partial y} = \frac{2}{3}\left(x\frac{\partial u}{\partial x}\right) \quad \text{—— (2)}$$

Similarly,

$$y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} + xy\frac{\partial u}{\partial x\partial y} = \frac{2}{3}\left(y\frac{\partial u}{\partial y}\right) \quad \text{—— (3)}$$

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Adding ② & ③, we get-

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(\frac{2}{3}-1\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{2}{3}u\right)\left(\frac{-1}{3}\right) = \frac{-2u}{9}$$

RHS

LHS = RHS

Proved

* Composite functions / Total Derivative :-

⇒ If $u = f(x,y)$, where $x = \phi(t)$, $y = \psi(t)$ then $u$ is composite function of single variable $t$. Total derivative of $u$ is $\frac{du}{dt}$.

and

⇒ If $z = f(x,y)$, where $x = \phi(u,v)$, $y = \psi(u,v)$ then $z$ is composite function of two variables $u$ and $v$. so that we can find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

⊙ Differentiation of Composite functions-

If $u = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$ then-

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

Note ① If $z = f(x,y)$ where $x = \phi(u,v)$ and $y = \psi(u,v)$ then-

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}}$$

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$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}}$$

(2) If $z = f(x,y)$ where $x = \phi(u,v)$ and $y = \psi(u,v)$ then

$$\boxed{\frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}}$$

Ques:- find $\frac{du}{dt}$ when $u = x^2+y^2$, $x = at^2$, $y = 2at$. Also verify by direct substitution.

Sol:- $u = x^2+y^2$

Here $u$ is a composite function of $t$.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2x)(2at) + (2y)(2a)$$

$$= 4xat + 4ay$$

Substituting values of $x$ & $y$

$$= 4(at^2)at + 4a(2at)$$

$$= 4a^2t^3 + 8a^2t$$

$$\boxed{\frac{du}{dt} = 4a^2t\,(t^2+2)}$$

Verification by direct substitution-

$$u = x^2+y^2 = (at^2)^2 + (2at)^2$$

$$u = a^2t^4 + 4a^2t^2$$

$$\frac{du}{dt} = 4a^2t^3 + 8a^2t$$

$$\frac{du}{dt} = 4a^2t\,(t+2)$$

Verified

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Ques- If $u = \sin\frac{x}{y}$, $x = e^t$, $y = t^2$, find $\frac{du}{dt}$

Sol- here u is composite of t

$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$\frac{du}{dt} = \cos\frac{x}{y}\left(\frac{1}{y}\right) \cdot e^t + \cos\frac{x}{y}\left(\frac{-x}{y^2}\right) \cdot 2t$

$\frac{du}{dt} = \frac{e^t}{y}\cos\frac{x}{y} - \frac{2xt}{y^2}\cos\frac{x}{y}$

Subsituting values of x & y

$\frac{du}{dt} = \frac{e^t}{t^2}\cos\frac{e^t}{t^2} - \frac{2e^t \cancel{t}}{t^{\cancel{4}3}}\cos\frac{e^t}{t^2}$

$\frac{du}{dt} = \frac{e^t}{t^2}\cos\frac{e^t}{t^2}\left(1 - \frac{2}{t^3}\right)$

$\frac{du}{dt} = e^t\left(\frac{t^3-2}{t^3}\right)\cos\frac{e^t}{t^2}$

Ques- If $z = \log(u^2+v)$, $u = e^{x^2+y^2}$, $v = x^2+y$

find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Sol- $z = \log(u^2+v)$

here z is composite function of x & y

$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$

$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v} \cdot 2xe^{x^2+y^2} + \frac{2x}{u^2+v}$

$\frac{\partial z}{\partial x} = \frac{4xue^{x^2+y^2}}{u^2+v} + \frac{2x}{u^2+v}$

$\frac{\partial z}{\partial x} = \frac{2x}{u^2+v}\left(2ue^{x^2+y^2} + 1\right)$

$\frac{\partial z}{\partial x} = \frac{2x}{u^2+v}\left(2u^2+1\right)$ $\left[e^{x^2+y^2} = u\right]$

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and $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$

$= \frac{2u}{u^2+v} \cdot 2y e^{x^2+y^2} + \frac{1}{u^2+v}$

$= \frac{1}{u^2+v}\left(4uy e^{x^2+y^2} + 1\right)$

$\frac{\partial z}{\partial y} = \frac{1}{u^2+v}\left(4yu^2+1\right)$

Ques- If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Sol- let $x \to tx$, $y \to ty$, $z \to tz$

$u = f\left(\frac{tx}{ty}, \frac{ty}{tz}, \frac{tz}{tx}\right)$

$= t^0 u(x, y, z)$

$\therefore$ u is a homogeneous function of degree 0.

So, By Euler's theorem-

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Proved

Ques If $x = u+v+w$, $y = uv+vw+wu$, $z = uvw$ and f is a function of $x, y, z$. Show that -

$$u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} + w\frac{\partial f}{\partial w} = x\frac{\partial f}{\partial x} + 2y\frac{\partial f}{\partial y} + 3z\frac{\partial f}{\partial z}.$$

Sol. Here f is a composite function of $u, v$ & $w$

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$x = u + v + w$
$y = uv + vw + wu$
$z = uvw$

Now

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}(v+w) + \frac{\partial f}{\partial z}(vw) \quad \text{—— (1)}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}(u+w) + \frac{\partial f}{\partial z}\cdot(uw) \quad \text{—— (2)}$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial w}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}(u+v) + \frac{\partial f}{\partial z}(uv) \quad \text{—— (3)}$$

Now, Multiply (1) by u, (2) by v and (3) by w and add we get -

LHS $\quad u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} + w\frac{\partial f}{\partial w} = (u+v+w)\frac{\partial f}{\partial x} + 2(uv+vw+wu)\frac{\partial f}{\partial y} + 3(uvw)\frac{\partial f}{\partial z}$

$$= x\frac{\partial f}{\partial x} + 2y\frac{\partial f}{\partial y} + 3z\frac{\partial f}{\partial z} \rightarrow \text{RHS}$$

LHS = RHS

Proved

Date

~~Implicit function~~

Ques- If $u = xe^{y}z$, where $y = \sqrt{a^2-x^2}$, $z = \sin^2 x$
find $\frac{du}{dx}$.

Sol- $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx}$$

$$= e^{y}z + xe^{y}z\cdot\left(\frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}}\right) + xe^{y}\cdot \overset{2\sin x\cos x}{\sin 2x}$$

$$\frac{du}{dx} = e^{y}\left[z - \frac{x^2}{\sqrt{a^2-x^2}} + x\sin 2x\right]$$

⊙ Differentiation of Implicit functions-

If $f(x,y) = c$, then $u = f(x,y)$ where $u = c$ is an implicit function ($c \rightarrow$ any constant).

* By Note (2)

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\overset{1}{\cancel{\frac{dx}{dx}}} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

we know that,

$$\frac{du}{dx} = 0$$

So, $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial x}}$$

$$\boxed{\frac{dy}{dx} = \frac{-f_x}{f_y}} \quad \text{—— (1)}$$

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Differentiating (1) w.r.t $x$

$\left(\frac{u}{v} \text{ rule}\right)$

$$\frac{d^2y}{dx^2} = -\frac{f_y\left(\frac{d}{dx}f_x\right) - f_x\left(\frac{d}{dx}f_y\right)}{f_y^2}$$

$$= -\frac{f_y\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_x}{\partial y}\cdot\frac{dy}{dx}\right) - f_x\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_y}{\partial y}\cdot\frac{dy}{dx}\right)}{f_y^2}$$

$$= -\frac{f_y\left[f_{xx} + f_{yx}\left(\frac{-f_x}{f_y}\right)\right] - f_x\left[f_{xy} + f_{yy}\left(\frac{-f_x}{f_y}\right)\right]}{f_y^2}$$

$$= -\frac{\left[f_{xx}f_y - f_{yx}f_x - f_x f_{xy} + f_x f_{yy}\cdot\frac{f_x}{f_y}\right]}{f_y^2}$$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{\left[f_{xx}f_y^2 - 2f_x f_{xy}f_y + f_{yy}f_x^2\right]}{f_y^3}}$$

Ques- Find $\frac{dy}{dx}$, when (i) $x^y + y^x = c$

(ii) $(\cos x)^y = (\sin y)^x$

Sol- (i) $x^y + y^x = c$

So, $f(x,y) = x^y + y^x$ and $f(x,y) = c$

given is an implicit function,

and we know that-

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$f_x = \frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$f_y = \frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

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$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}}$$

(ii) $(\cos x)^y - (\sin y)^x = 0$

given is an implicit function in which,

$$f(x,y) = (\cos x)^y - (\sin y)^x$$

so, $f(x,y) = 0 = c$

and we know that -

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$f_x = \frac{\partial f}{\partial x} = y(\cos x)^{y-1}(-\sin x) - (\sin y)^x \log \sin y$$

$$= -y \sin x (\cos x)^{y-1} - (\sin y)^x \log \sin y$$

$$f_y = \frac{\partial f}{\partial y} = (\cos x)^y \log \cos x - x(\sin y)^{x-1} \cos y$$

$$\frac{dy}{dx} = \frac{y \sin x (\cos x)^{y-1} - (\sin y)^x \log \sin y}{(\cos x)^y \log \cos x - x(\sin y)^{x-1} \cos y}$$

Ques- If z is a function of x and y and u, v be two other variables such that $u = lx + my$, $v = ly - mx$ show that -

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

Sol- from above question, we get to know that - z is a composite function of x and y.

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$u = lx + my \qquad v = ly - mx$

~~$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$~~

* $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot l + \frac{\partial z}{\partial v}(-m)$

$\Rightarrow \frac{\partial z}{\partial x} = l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}$

$\Rightarrow \frac{\partial}{\partial x} = l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}$

Now,

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

$= \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}\right)\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)$

$= l^2\frac{\partial^2 z}{\partial u^2} - ml\frac{\partial^2 z}{\partial u\partial v} - ml\frac{\partial^2 z}{\partial v\partial u} + m^2\frac{\partial^2 z}{\partial v^2}$

$\frac{\partial^2 z}{\partial x^2} = l^2\frac{\partial^2 z}{\partial u^2} - 2ml\frac{\partial^2 z}{\partial u\partial v} + m^2\frac{\partial^2 z}{\partial v^2}$ ——— (1)

* $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot m + \frac{\partial z}{\partial v}\cdot l$

$\Rightarrow \frac{\partial z}{\partial y} = m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}$

$\Rightarrow \frac{\partial}{\partial y} = m\frac{\partial}{\partial u} + l\frac{\partial}{\partial v}$

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Now,
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$
$$= \left(m\frac{\partial}{\partial u} + l\frac{\partial}{\partial v}\right)\left(m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v}\right)$$
$$= m^2\frac{\partial^2 z}{\partial u^2} + 2ml\frac{\partial^2 z}{\partial u \partial v} + l^2\frac{\partial^2 z}{\partial v^2} \quad \text{——— (2)}$$

Adding (1) & (2) we get-

LHS
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = l^2\frac{\partial^2 z}{\partial u^2} + m^2\frac{\partial^2 z}{\partial u^2} + m^2\frac{\partial^2 z}{\partial v^2} + l^2\frac{\partial^2 z}{\partial v^2}$$
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right) \rightarrow \text{RHS}$$

LHS = RHS

Proved

Ques- If $x = e^r\cos\theta$, $y = e^r\sin\theta$. Show that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\right]$$ where $u = f(x,y)$

Sol- Here $u$ is composite function of $r$ and $\theta$.

So,
$$* \ \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot e^r\cos\theta + \frac{\partial u}{\partial y}\cdot e^r\sin\theta$$
$$\Rightarrow \frac{\partial u}{\partial r} = e^r\cos\theta\frac{\partial u}{\partial x} + e^r\sin\theta\frac{\partial u}{\partial y}$$
$$\Rightarrow \frac{\partial}{\partial r} = \underbrace{e^r\cos\theta}_{x}\frac{\partial}{\partial x} + \underbrace{e^r\sin\theta}_{y}\frac{\partial}{\partial y}$$

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Now, $\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)$

$$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial r^2} = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} \quad \text{—— (1)}$$

* $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$

$$= -e^r \sin\theta \frac{\partial u}{\partial x} + e^r \cos\theta \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \theta} = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}$$

Now, $\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)$

$$= \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}\right)$$

$$= x^2\frac{\partial^2 u}{\partial y^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial x^2} \quad \text{—— (2)}$$

Adding (1) & (2), we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} = (x^2+y^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} = e^{2r}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\boxed{\begin{aligned} &x^2 = e^{2r}\sin^2\theta \\ &y^2 = e^{2r}\cos^2\theta \\ &x^2+y^2 = \\ &e^{2r}(\sin^2\theta + \cos^2\theta) \\ &= e^{2r} \end{aligned}}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\right)$$

Proved

Date

**Ques-** If $z = \sqrt{x^2+y^2}$ and $x^3+y^3+3axy = 5a^2$, find the value of $\frac{dz}{dx}$ when $x=y=a$.

**Sol-**

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} \quad \text{—— (1)}$$

Now, $\frac{\partial z}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial z}{\partial y} = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \qquad \left(f(x,y) = x^3+y^3+3axy = 5a^2\right)$$

$$f_x = \frac{\partial f}{\partial x} = 3x^2+3ay$$

$$f_y = \frac{\partial f}{\partial y} = 3y^2+3ax$$

$$\frac{dy}{dx} = -\left(\frac{3x^2+3ay}{3y^2+3ax}\right) = -\left(\frac{x^2+ay}{y^2+ax}\right)$$

→ Putting values of $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ & $\frac{dy}{dx}$ in (1)

$$\frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}\left(\frac{x^2+ay}{y^2+ax}\right)$$

→ Putting $x=y=a$

$$\frac{dz}{dx} = \frac{a}{a\sqrt{2}} - \frac{a}{a\sqrt{2}}\left(\frac{2a^2}{2a^2}\right)$$

$$\frac{dz}{dx} = 0$$

Date

Ques- Prove that if $y^3 - 3ax^2 + x^3 = 0$, then

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0 \quad \text{—— (1)}$$

Sol- $f(x,y) = y^3 - 3ax^2 + x^3 = 0$

$f_x = 3x^2 - 6ax \qquad f_y = 3y^2$

$f_{xx} = 6x - 6a \qquad f_{yy} = 6y$

$f_{xy} = 0$

We know that,

$$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{\underbrace{f_y^3}_{9y^4}}$$

$$= -\frac{(6x-6a)(3y^2)^2 - 2(0) + (6y)(3x^2-6ax)^2}{(3y^2)^3}$$

$$= -\frac{6(9xy^4 - 9ay^4) + 6y(9x^4 + 36a^2x^2 - 36ax^3)}{27y^6}$$

$$= -\left[\frac{54xy^4 - 54ay^4 + 54yx^4 + 216a^2x^2y - 216ax^3y}{27y^6}\right]$$

$$= -\frac{\cancel{27}y}{\cancel{27}y^{\cancel{6}\,5}}\left[2xy^3 - 2ay^3 + 2x^4 + 8a^2x^2 - 8ax^3\right]$$

$$= -\frac{1}{y^5}\left[2xy^3 - 2ay^3 + 2x^4 + 8a^2x^2 - 6ax^3 - 2ax^3\right]$$

$$= -\frac{1}{y^5}\left(2x(\underbrace{y^3 + x^3 - 3ax^2}_{0}) - 2ay^3 + 8a^2x^2 - 2ax^3\right)$$

$$= -\frac{1}{y^5}\left(8a^2x^2 - 2a(\underbrace{x^3 + y^3}_{3ax^2})\right)$$

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$$\frac{d^2y}{dx^2} = \frac{-1}{y^5}\left(8a^2x^2 - 2a(3ax^2)\right) = \frac{-1}{y^5}\left(2a^2x^2\right)$$

Putting value of $\frac{d^2y}{dx^2}$ in ①

LHS $-\frac{2a^2x^2}{y^5} + \frac{2a^2x^2}{y^5} = 0 \rightarrow$ RHS

LHS = RHS

Proved

* Jacobians :-

If $u$ and $v$ are functions of $x$ and $y$, then
$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of $u$ and $v$ w.r.t $x$ and $y$ and is denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$.

◉ Properties of Jacobians -

(i) If $u$ & $v$ are functions of $r, s$ and $r, s$ are functions of $x, y$, then -

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)}$$

$$\boxed{J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}}$$

(ii) If $J_1$ is Jacobian of $u,v$ w.r.t $x,y$ and $J_2$ is Jacobian of $x,y$ w.r.t $u,v$ then -

$$\boxed{J_1 J_2 = 1}$$

Date

Ques- If $x = r\cos\theta$, $y = r\sin\theta$, Verify
$$\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

Sol- We know that,
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Now, ~~Adding~~ Squaring & adding $x$ and $y$

$$x^2 = r^2\cos^2\theta$$
$$y^2 = r^2\sin^2\theta$$
$$x^2+y^2 = r^2$$

So, $r^2 = x^2+y^2$, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$$
$$= \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2+y^2}{r^3} = \frac{\cancel{r^2}}{r^{\cancel{3}}} = \frac{1}{r}$$

So,

LHS $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \times \frac{1}{r} = 1 \rightarrow$ RHS

LHS = RHS

Proved

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Ques- If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$

Sol- We know that,

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta(r\cos\theta) + r\sin\theta(\sin\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

Ques- If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

Sol- $\frac{\partial x}{\partial r} = \sin\theta\cos\phi$, $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$, $\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$, $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$, $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$

$\frac{\partial z}{\partial r} = \cos\theta$, $\frac{\partial z}{\partial \theta} = -r\sin\theta$, $\frac{\partial z}{\partial \phi} = 0$

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Ans $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial x} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \Big[\sin\theta\cos\phi\,(+r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,(-r\sin\theta\cos\phi\,\cos\theta) - r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)\Big]$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = +r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin\theta\sin^2\phi\,(\underbrace{\cos^2\theta+\sin^2\theta}_{1})$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta\,(+\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \underbrace{\sin^2\phi}_{1})$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta\,(+\cos^2\phi\,(\underbrace{\cos^2\theta+\sin^2\theta}_{1}) + \sin^2\phi)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta \rightarrow \text{RHS}$$

LHS = RHS

Proved

Date

Ques- If $u = x+y+z$, $uv = y+z$, $uvw = z$, Show that

$x = u - uv$

$z = uvw$

$y = uv - uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$$

Sol- LHS

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= 1-v\,(u^2v - u^2\cancel{v}w + u^2\cancel{v}w) + u(v^2u - v^2\cancel{uw} + \cancel{v^2uw})$$

$$= u^2v - u^2v^2 + v^2u^2$$

$$= u^2 v \longrightarrow \text{RHS}$$

LHS = RHS

Proved

Ques- If $u = x^2 - 2y$, $v = x+y+z$, $w = x-2y+3z$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Sol-

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 2x(3+2)+2(3-1) = 10x+4$$

Ques- If $u = xyz$, $v = xy+yz+zx$, $w = x+y+z$. Show that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{(x-y)(y-z)(z-x)}$$

Sol- As we know -

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1 \quad \text{— (1)}$$

Now,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} yz & xz & xy \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(\cancel{x}+z-\cancel{x}-y) - xz(y+z-x-\cancel{y}) + xy(y+\cancel{z}-x-\cancel{z})$$

$$= yz(z-y) - xz(z-x) + xy(y-x)$$

$$= yz^2 - zy^2 - xz^2 + zx^2 + xy^2 - yx^2$$

$$= (x-y)(y-z)(z-x)$$

Putting in (1) -

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{(x-y)(y-z)(z-x)}$$

Proved

Date

## Jacobian of Implicit functions

If $x, y, u, v$ are connected by implicit functions $f_1(x, y, u, v) = 0$, $f_2(x, y, u, v) = 0$ where $u, v$ are implicit functions of $x$ & $y$, then -

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}}$$

**Ques-** If $\overset{f_1}{x^2+y^2+u^2-v^2=0}$ and $\overset{f_2}{uv+xy=0}$. Prove that -

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2-y^2}{u^2+v^2}$$

**Sol-**

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2x^2 - 2y^2$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2$$

**LHS**

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \left[\frac{2x^2-2y^2}{2u^2+2v^2}\right]$$

$$= \frac{x^2-y^2}{u^2+v^2} \rightarrow \text{RHS}$$

LHS = RHS

**Proved**

Date

Ques:- If $u, v, w$ are the roots of the equation $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol:- $\lambda^3-x^3-3\lambda x(\lambda-x)+\lambda^3-y^3-3\lambda y(\lambda-y)+\lambda^3-z^3-3\lambda z(\lambda-z$

$3\lambda^3-3\lambda^2x-3\lambda^2y-3\lambda^2z+3\lambda x^2+3\lambda y^2+3\lambda z^2-x^3-$
$y^3-z^3=0$

$\Rightarrow 3\lambda^3-3(x+y+z)\lambda^2+3(x^2+y^2+z^2)\lambda-(x^3+y^3+z^3)=0$

$$u+v+w=\frac{3(x+y+z)}{3}=x+y+z$$

$$uv+vw+wu=\frac{3(x^2+y^2+z^2)}{3}=x^2+y^2+z^2$$

$$uvw=\frac{1}{3}(x^3+y^3+z^3)$$

$$f_1=u+v+w-x-y-z$$
$$f_2=uv+vw+wu-x^2-y^2-z^2$$
$$f_3=uvw-\frac{1}{3}(x^3+y^3+z^3)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}=\begin{vmatrix}1 & 1 & 1\\ v+w & u+w & u+v\\ vw & uw & uv\end{vmatrix}$$

$$=(u^2v+\cancel{uvw}-u^2w-\cancel{uvw})-(uv^2+\cancel{uvw}-\cancel{uvw}-v^2w)$$
$$+(\cancel{uvw}+uw^2-\cancel{uvw}-vw^2)$$

$$=u^2(v-w)-v^2(u-w)+w^2(u-v)$$

$$=-(u-v)(v-w)(w-u)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}=\begin{vmatrix}-1 & -1 & -1\\ -2x & -2y & -2z\\ -x^2 & -y^2 & -z^2\end{vmatrix}=-2(x-y)(y-z)(z-x)$$

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LHS $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \dfrac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$

$= -2\dfrac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)} \rightarrow$ RHS

LHS = RHS

Proved

- Partial derivative of Implicit functions by Jacobian -

Given $f_1(x,y,u,v) = 0$ and $f_2(x,y,u,v) = 0$ then,

$\rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial(f_1,f_2)}{\partial(x,v)} \Big/ \dfrac{\partial(f_1,f_2)}{\partial(u,v)}$

$\rightarrow \dfrac{\partial u}{\partial y} = \dfrac{\partial(f_1,f_2)}{\partial(y,v)} \Big/ \dfrac{\partial(f_1,f_2)}{\partial(u,v)}$

$\rightarrow \dfrac{\partial v}{\partial x} = \dfrac{\partial(f_1,f_2)}{\partial(x,u)} \Big/ \dfrac{\partial(f_1,f_2)}{\partial(u,v)}$

$\rightarrow \dfrac{\partial v}{\partial y} = \dfrac{\partial(f_1,f_2)}{\partial(y,u)} \Big/ \dfrac{\partial(f_1,f_2)}{\partial(u,v)}$

Ques- Use Jacobian to find $\dfrac{\partial u}{\partial x}$ if $u^2 + xv^2 - xy = 0 \rightarrow f_1$ and $u^2 + xyv + v^2 = 0 \rightarrow f_2$

Sol- $\dfrac{\partial(f_1,f_2)}{\partial(x,v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v^2-y & 2vx \\ yv & xy+2v \end{vmatrix}$

$= (v^2-y)(xy+2v) - 2xyv^2$

$= xyv^2 + 2v^3 - xy^2 - 2vy - 2xyv^2$

$= 2v^3 - xy^2 - 2vy - xyv^2$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} 2u & 2vx \\ 2u & xy+2v \end{vmatrix}$$

$$= 2uxy + 4uv - 4uvx$$

Putting $\frac{\partial(f_1,f_2)}{\partial(x,v)}$ and $\frac{\partial(f_1,f_2)}{\partial(u,v)}$ in formula

$$\frac{\partial u}{\partial x} = \frac{\partial(f_1,f_2)}{\partial(x,v)} \Big/ \frac{\partial(f_1,f_2)}{\partial(u,v)}$$

$$\frac{\partial u}{\partial x} = (-1)^3 \frac{2v^3 - xy^2 - 2vy - xyv^2}{2uxy + 4uv + 4uvx}$$

$$\frac{\partial u}{\partial x} = \frac{xyv^3 + xy^2 + 2vy - 2v^3}{2xyu + 4uv - 4uvx}$$

Ques- If $x = \overset{f_1}{u+v+w}$, $y = \overset{f_2}{u^2+v^2+w^2}$, $z = \overset{f_3}{u^3+v^3+w^3}$. Show that $\frac{\partial u}{\partial x} = \frac{vw}{(u-v)(u-w)}$

Sol- We know that -

LHS $\frac{\partial u}{\partial x} = (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)} \Big/ \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)} = \begin{vmatrix} 1 & -1 & -1 \\ 0 & -2v & -2w \\ 0 & -3v^2 & -3w^2 \end{vmatrix}$$

$$= 6vw^2 - 6wv^2$$

$$= 6vw(w-v)$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} -1 & -1 & -1 \\ -2u & -2v & -2w \\ -3u^2 & -3v^2 & -3w^2 \end{vmatrix}$$

$$= -(6vw^2 - 6wv^2) + (6uw^2 - 6wu^2) - (6uv^2 - 6vu^2)$$

$$= 6vw(v-w) + 6uw(w-u) + 6uv(u-v)$$

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$$= 6(u-v)(v-w)(w-u)$$

$$\therefore \frac{\partial u}{\partial x} = -\frac{\cancel{6}vw\cancel{(v-w)}}{\cancel{6}(u-v)\cancel{(v-w)}(w-u)}$$

$$\frac{\partial u}{\partial x} = \frac{vw}{(u-v)(u-w)} \rightarrow RHS$$

$$LHS = RHS$$

Proved

## Applications of Partial Differentiation

1. Maxima and Minima

→ Working rule to find Maxima and Minima-

let $z = f(x,y)$

Step I

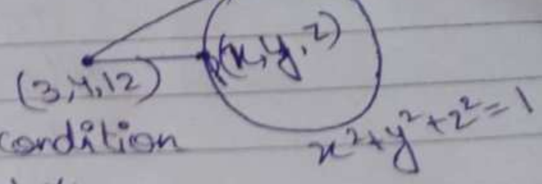
Date

Ques- find Maximum and Minimum distance from the sphere $x^2+y^2+z^2=1$ to the point $(3,4,12)$

Sol- Using Lagrange's Method.

Step I- Write the function

$$F = f\left(\begin{matrix}\text{which has to}\\ \text{be min or max}\end{matrix}\right) + \lambda(\underbrace{\phi}_{\text{condition}})$$




In case of two conditions.

$$* \quad F = f + \lambda_1\phi_1 + \lambda_2\phi_2$$

Step II- $dF = \dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy + \dfrac{\partial F}{\partial z}dz$

Step III Put $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0$

Step IV $d^2F = F_{xx}(dx)^2 + F_{yy}(dy)^2 + F_{zz}(dz)^2 + 2\left(F_{xy}\,dx\,dy + F_{yz}\,dy\,dz + F_{zx}\,dz\,dx\right)$

then if $d^2F > 0$ then it would be a minima

& if $d^2F < 0$ then it would be a maxima

→ let a point $P(x,y,z)$ on the sphere is having min or max distance with $(3,4,12)$ ——— ①

$$d = \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$$

(if dis. is min, its square would also be min & same with max)

$$f = d^2 = (x-3)^2+(y-4)^2+(z-12)^2$$

$$F = f + \lambda\phi$$

$$F = (x-3)^2+(y-4)^2+(z-12)^2+\lambda(x^2+y^2+z^2-1)$$

$$\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x$$

$$\frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y$$

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$$\frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$$

Now, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

$2(x-3) + 2\lambda x = 0 \Rightarrow (x-3) + \lambda x = 0$ — (2)

$2(y-4) + 2\lambda y = 0 \Rightarrow (y-4) + \lambda y = 0$ — (3)

$2(z-12) + 2\lambda z = 0 \Rightarrow (z-12) + \lambda z = 0$ — (4)

Multiplying (2) by $x$, (3) by $y$ and (4) by $z$ and adding, we get-

$$(x^2+y^2+z^2) - (3x+4y+12z) + \lambda(x^2+y^2+z^2) = 0$$

[Condition $\Rightarrow x^2+y^2+z^2 = 1$]

$$\underbrace{\cancel{(x^2+y^2+z^2)}}^{1} - (3x+4y+12z) + \lambda = 0$$

$$\boxed{3x+4y+12z = 1+\lambda}$$ — (5)

from (2), $x = \frac{3}{1+\lambda}$

from (3), $y = \frac{4}{1+\lambda}$

from (4), $z = \frac{12}{1+\lambda}$

Putting values of $x, y, z$ in (5)

$$\frac{9}{1+\lambda} + \frac{16}{1+\lambda} + \frac{144}{1+\lambda} = 1+\lambda$$

$$9+16+144 = (1+\lambda)^2$$

$$(1+\lambda)^2 = 169$$

$$1+\lambda = \pm 13$$

$$\lambda = 12 \text{ or } -14$$

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when $\lambda = 12 \Rightarrow x = 3/13, y = 4/13, z = 12/13$
when $\lambda = -14 \Rightarrow x = 3/-13, y = 4/-13, z = 12/-13$

Now,
$$d = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$$

• when $\lambda = 12$, At $(3/13, 4/13, 12/13)$

$$d = \sqrt{(3/13 - 3)^2 + (4/13 - 4)^2 + (12/13 - 12)^2}$$

$$d = \sqrt{\left(\frac{3-39}{13}\right)^2 + \left(\frac{4-52}{13}\right)^2 + \left(\frac{12-156}{13}\right)^2}$$

$$d = \sqrt{\left(\frac{-36}{13}\right)^2 + \left(\frac{-48}{13}\right)^2 + \left(\frac{-144}{13}\right)^2}$$

$$d = \sqrt{\frac{1296}{169} + \frac{2304}{169} + \frac{20736}{169}}$$

$\boxed{d = 12}$ $\longrightarrow$ minimum distance from sphere

• when $\lambda = -14$, At $(-3/13, -4/13, -12/13)$

$$d = \sqrt{(-3/13 - 3)^2 + (-4/13 - 4)^2 + (-12/13 - 12)^2}$$

$$d = \sqrt{\left(\frac{-3-39}{13}\right)^2 + \left(\frac{-4-52}{13}\right)^2 + \left(\frac{-12-156}{13}\right)^2}$$

$$d = \sqrt{\left(\frac{-42}{13}\right)^2 + \left(\frac{-56}{13}\right)^2 + \left(\frac{-168}{13}\right)^2}$$

$$d = \sqrt{\frac{1764}{169} + \frac{3136}{169} + \frac{28224}{169}}$$

$\boxed{d = 14}$ $\longrightarrow$ maximum distance from sphere

Ques: find minimum value of function $u = x^2+y^2+z^2$ subject to the condition $xyz - a^3$.

Sol: Using Langrange's Method

$F = f + \lambda\phi$, $\quad \phi = xyz - a^3$ —— (1)

$F = u + \lambda\phi$

$F = x^2+y^2+z^2+\lambda(xyz-a^3)$

$\frac{\partial F}{\partial x} = 2x + \lambda yz \qquad \frac{\partial F}{\partial y} = 2y + \lambda xz$

$\frac{\partial F}{\partial z} = 2z + \lambda xy$

Now Putting $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ & $\frac{\partial F}{\partial z} = 0$

$2x + \lambda yz = 0$ —— (2)

$2y + \lambda xz = 0$ —— (3)

$2z + \lambda xy = 0$ —— (4)

Multiplying (2) by $x$, (3) by $y$ & (4) by $z$ and adding, we get-

$2(x^2+y^2+z^2) + 3\lambda xyz = 0$

$2u + 3\lambda a^3 = 0$ —— (5)

from (2), $x = -\frac{\lambda yz}{2}$

from (3), $y = -\frac{\lambda xz}{2}$

from (4), $z = -\frac{\lambda xy}{2}$

Putting values of $x, y, z$ in (1)

$xyz = a^3$

$-\frac{\lambda^3}{8}(x^2y^2z^2) = a^3$

$-\frac{\lambda^3}{8}(a^6) = a^3$

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$$-\lambda^3 = \frac{8a^3}{a^2}$$

$$\lambda = -\frac{2}{a}$$

Putting $\lambda$ in ⑤

$$2u + 3\lambda a^3 = 0$$

$$2u - 3\left(\frac{2}{a}\right)a^{3^2} = 0$$

$$2u = 6a^2$$

$$\boxed{u = 3a^2}$$

Now, we cannot depict whether it is minima or maxima so we will further calculate by finding $d^2F$.

$$d^2F = F_{xx}(dx)^2 + F_{yy}(dy)^2 + F_{zz}(dz)^2 + 2\big(F_{xy}(dx\,dy) + F_{yz}(dy\,dz) + F_{zx}(dz\,dx)\big)$$

$$d^2F = 2(dx)^2 + 2(dy)^2 + 2(dz)^2 + 2(\lambda z\,dx\,dy + \lambda x\,dy\,dz + \lambda y\,dz\,dx)$$

Now $2(dx)^2 + 2(dy)^2 + 2(dz)^2$ will always by greater than zero

$\rightarrow$ $\phi = xyz - a^3$

$$d^2\phi = \phi_{xx}(dx)^2 + \phi_{yy}(dy)^2 + \phi_{zz}(dz)^2 + 2(\phi_{xy}\,dx\,dy + \phi_{yz}\,dy\,dz + \phi_{zx}\,dz\,dx)$$

$$0 = 0 + 0 + 0 + 2(z\,dx\,dy + x\,dy\,dz + y\,dz\,dx)$$

Multiplying by $\lambda$

$$2(\lambda z\,dx\,dy + \lambda x\,dy\,dz + \lambda y\,dz\,dx) = 0$$

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$\therefore d^2F > 0$

So, the given function is minimum & $u = 3a^2$ is its minimum value.