

Maths

Date

Partial Differentiation

- * function of two variables

If any z depends upon any x & y , the z is called a function of x and if (x & y are variable) Denoted by $\therefore z = f(x, y)$

Ex → Area of Δ depends upon two variables base & altitude ($\frac{1}{2} \times \text{base} \times \text{altitude}$)

- * Partial derivatives -

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x \quad (\text{keeping } y \text{ constant})$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y \quad (\text{keeping } x \text{ constant})$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = f_{xy}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = f_{xx}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = f_{yy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = f_{yx}$$

⇒ In general $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}; f_{xy} = f_{yx}$

Date

Ques1- If $f(x, y) = x^3y - xy^3$, find $\left\{ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right\}_{x=1, y=2}$

Sol:-
 $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3y - xy^3) = 3x^2y - y^3$
 $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3y - xy^3) = x^3 - 3xy^2$ (Let $z = f(x, y)$)

$$\begin{aligned} & \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right]_{x=1, y=2} = \left[3x^2y - y^3 + x^3 - 3xy^2 \right]_{x=1, y=2} \\ & = \left[\frac{1}{6-8} + \frac{1}{1-12} \right] \\ & = \left[\frac{-1}{2} - \frac{1}{11} \right] = \left[\frac{-11-2}{22} \right] \\ & = \boxed{\frac{-13}{22}} \rightarrow \text{Ans.} \end{aligned}$$

Ques2- If $z(x+y) = x^2 + y^2$, show that-

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 = 4 \left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)$$

Sol-
 $z = \frac{x^2 + y^2}{x+y}$

LHS
 $\frac{\partial z}{\partial x} = \frac{(x+y)2x - (x^2 + y^2)}{(x+y)^2} = \frac{x^2 - y^2 + 2xy}{(x+y)^2}$

$$\frac{\partial z}{\partial y} = \frac{(x+y)2y - (x^2 + y^2)}{(x+y)^2} = \frac{y^2 - x^2 + 2xy}{(x+y)^2}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 = \left[\frac{x^2 - y^2 + 2xy}{(x+y)^2} - \frac{y^2 - x^2 + 2xy}{(x+y)^2} \right]^2$$

$$= \left[\frac{x^2 - y^2 + 2xy - y^2 + x^2 - 2xy}{(x+y)^2} \right]^2$$

$$= \left(\frac{2x^2 - 2y^2}{(x+y)^2} \right)^2$$

$$= \frac{2^2(x^2 - y^2)^2}{(x+y)^4} = \frac{4(x+y)^2(x-y)^2}{(x+y)^4}$$

~~$$= 8(x^2 + y^2 + 2xy)(x^2 - y^2 + 2xy)$$~~

$$= \frac{4(x-y)^2}{(x+y)^2}$$

RHS $4 \left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right) = 4 \left(1 - \left(\frac{x^2 - y^2 + 2xy}{(x+y)^2} \right) - \left(\frac{y^2 - x^2 + 2xy}{(x+y)^2} \right) \right)$

$$= \frac{4(x-y)^2}{(x+y)^2}$$

LHS = RHS

Proved

Ques 3- Prove that $f_{xy} = f_{yx}$ if $f(x, y) = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}}$

$$\text{Sol- } f_x = \frac{\partial z}{\partial x} = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}} \cdot \frac{(-2(x-a))}{2\sqrt{y}} \\ = -\frac{1}{2} y^{-3/2} (x-a) e^{-\frac{(x-a)^2}{4y}}$$

$$f_y = \frac{\partial z}{\partial y} = -\frac{1}{2} y^{-3/2} e^{-\frac{(x-a)^2}{4y}} + \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}} \cdot \frac{(x-a)^2}{4y^2}$$

$$= e^{-\frac{(x-a)^2}{4y}} \left[-\frac{1}{2} y^{-3/2} + y^{-1/2} \frac{(x-a)^2}{4y^2} \right]$$

$$= e^{-\frac{(x-a)^2}{4y}} \left[-\frac{1}{2} y^{-3/2} + \frac{1}{4} y^{-5/2} (x-a)^2 \right]$$

$$= \frac{1}{4} y^{-3/2} e^{-\frac{(x-a)^2}{4y}} \left[-2 + \frac{(x-a)^2}{y} \right]$$

$$f_{xy} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left[\frac{1}{4} y^{-3/2} e^{-\frac{(x-a)^2}{4y}} \left(-2 + \frac{(x-a)^2}{y} \right) \right]$$

~~$$\frac{1}{4} y^{-3/2} \frac{\partial}{\partial x} \left[e^{-\frac{(x-a)^2}{4y}} \left(-2 + \frac{(x-a)^2}{y} \right) \right]$$~~

$$= \frac{1}{4} y^{-3/2} \left[e^{-\frac{(x-a)^2}{4y}} (0 + \frac{2}{y} (x-a)) + \left(-2 + \frac{(x-a)^2}{y} \right) \left(e^{-\frac{(x-a)^2}{4y}} \right) \left(\frac{-2(x-a)}{2\sqrt{y}} \right) \right]$$

ques 4 find the first order partial derivatives:-

$$\textcircled{1} \quad u = y^x$$

$$\frac{\partial u}{\partial x} = y^x \log y \quad [a^x = a^x \log a]$$

$$\frac{\partial u}{\partial y} = x y^{x-1}$$

$$\textcircled{2} \quad u = \log(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$$

$$\textcircled{3} \quad u = x^2 \sin \frac{y}{x}$$

$$\frac{\partial u}{\partial x} = \text{uv rule} = 2x \sin \frac{y}{x} + x^2 (\cos \frac{y}{x}) \cdot \left(-\frac{1}{x^2}\right)$$

$$= 2x \sin \frac{y}{x} - \cos \frac{y}{x}$$

$$\frac{\partial u}{\partial y} = x^2 (\cos \frac{y}{x}) \cdot \frac{1}{x}$$

$$= x \cos \frac{y}{x}$$

$$\textcircled{4} \quad u = \frac{x}{y} \tan^{-1} \left(\frac{y}{x} \right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{y} \tan^{-1} \left(\frac{y}{x} \right) + \frac{x}{y} \cdot \frac{x^2}{x^2 + y^2} \cdot \left(-\frac{y}{x^2} \right)$$

$$= \frac{1}{y} \tan^{-1} \left(\frac{y}{x} \right) - \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y^2} \tan^{-1} \left(\frac{y}{x} \right) + \cancel{\frac{x^2}{y^2 + y^2}} \cancel{\frac{x^2}{y^2 + y^2}} \frac{x^2}{y^2 + y^2} \cdot \frac{1}{x}$$

$$= -\frac{x}{y^2} \tan^{-1} \left(\frac{y}{x} \right) + \frac{x^2}{y(x^2 + y^2)}$$

~~Very Very Important~~

Ques- If $\theta = t^n e^{-\frac{x^2}{4t}}$, find the value of n which will make $\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \theta}{\partial r}) = \frac{\partial \theta}{\partial t}$

Sol- LHS

$$\frac{\partial \theta}{\partial r} = t^{n-1} e^{-\frac{x^2}{4t}} \left(-\frac{2x}{24t} \right)$$

$$= -\frac{1}{2} rt t^{n-1} e^{-\frac{x^2}{4t}}$$

$$r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} r^3 t^{n-1} e^{-\frac{x^2}{4t}}$$

$$\frac{\partial}{\partial r} (r^2 \frac{\partial \theta}{\partial r}) = -\frac{1}{2} t^{n-1} \frac{\partial}{\partial r} \left[r^3 \cdot e^{-\frac{x^2}{4t}} \right] \quad \text{uv rule}$$

$$= -\frac{1}{2} t^{n-1} \left[3r^2 \cdot e^{-\frac{x^2}{4t}} + r^3 \cdot e^{-\frac{x^2}{4t}} \left(-\frac{2x}{24t} \right) \right]$$

$$= -\frac{1}{2} t^{n-1} \cdot e^{-\frac{x^2}{4t}} \cdot r^2 \left[3 - \frac{x^2}{2t} \right]$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \theta}{\partial r}) = -\frac{1}{2} t^{n-1} e^{-\frac{x^2}{4t}} \left(3 - \frac{x^2}{2t} \right)$$

~~Also~~ $\frac{\partial \theta}{\partial t} = nt^{n-1} \cdot e^{-\frac{x^2}{4t}} + t^n \cdot e^{-\frac{x^2}{4t}} \left(\frac{r^2}{4t^2} \right)$

$$= t^{n-1} e^{-\frac{x^2}{4t}} \left(n + t \left(\frac{r^2}{4t^2} \right) \right)$$

$$= t^{n-1} e^{-\frac{x^2}{4t}} \left(n + \frac{r^2}{4t} \right)$$

for LHS & RHS to be equal-

$$-\frac{1}{2} t^{n-1} \cdot e^{-\frac{x^2}{4t}} \left(3 - \frac{r^2}{2t} \right) = t^{n-1} e^{-\frac{x^2}{4t}} \left(n + \frac{r^2}{4t} \right)$$

~~$$-\frac{1}{2} \left(3 - \frac{r^2}{2t} \right) = n + \frac{r^2}{4t}$$~~

~~$$\frac{r^2}{2t} (2t - 3) = n + \frac{r^2}{4t}$$~~

$$\frac{r^2}{4t} - \frac{3}{2} = n + \frac{r^2}{4t}$$

~~$$n = \frac{\frac{r^2}{4t} - \frac{3}{2}}{2}$$~~

~~$$n = \frac{\frac{r^2}{4t} - \frac{3}{2} + 6t^2}{2}$$~~

~~$$\textcircled{*} \left(\frac{r^2}{4t} - \frac{3}{2} \right) = \left(n + \frac{r^2}{4t} \right)$$~~

$$\frac{r^2}{4t} - \frac{3}{2} = n + \frac{r^2}{4t}$$

$$\therefore \boxed{n = -\frac{3}{2}} \rightarrow \underline{\text{Ans}}$$

Ques: If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that
 $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \frac{9}{(x+y+z)^2}$

Sol - $\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{3x^2 + 3y^2 + 3z^2 - 3xy - 3yz - 3zx}{x^3 + y^3 + z^3 - 3xyz} \\ &= 3 \left(\frac{x^2 + y^2 + z^2 - xy - yz - zx}{x^3 + y^3 + z^3 - 3xyz} \right) \end{aligned}$$

We know that,

$$\begin{aligned} x^3 + y^3 + z^3 - 3xyz &= (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) \\ &= \frac{3}{x+y+z} \end{aligned}$$

Now,

$$\begin{aligned}
 (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 u &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u \\
 &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left(\frac{3}{x+y+z}\right) \\
 &= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} \\
 &= -\frac{9}{(x+y+z)^2}
 \end{aligned}$$

Proved

Ques- If $x^x y^y z^z = c$, show that $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = z$ and $\frac{\partial^2 z}{\partial x^2} = -(\log x)^{-1}$.

Sol- $x^x y^y z^z = c$

taking log both sides -
 $\log(x^x y^y z^z) = \log c$

$$x \log x + y \log y + z \log z = \log c$$

Now, as z is dependent variable and x, y are independent variables -

first partially differentiating w.r.t. z

$$\frac{\partial z}{\partial z} \Rightarrow x \cdot \frac{1}{x} + \log x + 0 + (z \cdot \frac{1}{z} + \log z) \frac{\partial z}{\partial z} = 0$$

Similarly, $\frac{\partial y}{\partial z} \Rightarrow (y \cdot \frac{1}{y} + \log y) + (z \cdot \frac{1}{z} + \log z) \frac{\partial z}{\partial y} = 0$

~~①~~ $\frac{\partial z}{\partial x} = -\left(\frac{1+\log x}{1+\log z}\right) \quad \text{--- } ①$

~~②~~ $\frac{\partial z}{\partial y} = -\left(\frac{1+\log y}{1+\log z}\right) \quad \text{--- } ②$

Now, partially differentiating (2) w.r.t x

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x} \left[\frac{1 + \log y}{1 + \log z} \right] \quad \begin{matrix} (1 + \log y), (1 + \log z)^{-1} \\ \text{constant} \end{matrix}$$

$$= (1 + \log y) \cdot \frac{1}{(1 + \log z)^2} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1 + \log y}{(1 + \log z)^2} \cdot \frac{1}{z} \left[-\left(\frac{1 + \log x}{1 + \log z} \right) \right]$$

$$\text{At } x = y = z$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1 + \log x}{(1 + \log x)^2} \cdot \frac{1}{x} \left[-\left(\frac{1 + \log x}{1 + \log x} \right) \right]$$

$$= -\frac{1}{x} \cdot \frac{1}{1 + \log x}$$

$$\text{Also } \log e = 1$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(\log e + \log x)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(\log x)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -x(\log x)^{-1}$$

Proved

Ques- If $z = f(x+ay) + \phi(x-ay)$, prove that
 $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

Sol- $\frac{\partial z}{\partial x} = f' + \phi'$

$$\frac{\partial^2 z}{\partial x^2} = f'' + \phi''$$

$$\frac{\partial z}{\partial y} = af' + a\phi' = a(f' + \phi')$$

$$\frac{\partial^2 z}{\partial y^2} = a^2(f'' + \phi'')$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Hence proved

Ques- If $u = e^{xyz}$, prove that
 $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2) e^{xyz}$

Sol- LHS $\frac{\partial u}{\partial z} = xyz e^{xyz}$

$$\begin{aligned} \frac{\partial^2 u}{\partial y \partial z} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial z} \right) = \frac{\partial}{\partial y} (xyz e^{xyz}) \\ &= x \left(yxz e^{xyz} + e^{xyz} \right) \end{aligned}$$

$$= x \left(xyz e^{xyz} + e^{xyz} \right)$$

$$= xe^{xyz} (xyz + 1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x} \left(\frac{\partial^2 u}{\partial y \partial z} \right) = \frac{\partial}{\partial x} (xyz e^{xyz} + xe^{xyz})$$

$$\begin{aligned}
 &= \left[yz(2xe^{xyz} + x^2yze^{xyz}) + (e^{xyz} + xyz e^{xyz}) \right] \\
 &= e^{xyz} \left[2xyz + x^2y^2z^2 + 1 + xyz \right] \\
 &= (3xyz + x^2y^2z^2 + 1) e^{xyz} \rightarrow \underline{\text{RHS}}
 \end{aligned}$$

LHS = RHS

Proved

Ques- If $u = \log \sqrt{x^2 + y^2 + z^2}$, prove that

$$(x^2 + y^2 + z^2) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 1$$

Sol- we have,

$$u = \log (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$u = \frac{1}{2} \log (x^2 + y^2 + z^2)$$

$$2u = \log (x^2 + y^2 + z^2)$$

Partially Differentiating both sides w.r.t. x

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2 + z^2}$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2 + z^2} \quad (\text{u rule})$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2 + z^2) - x(2x)}{(x^2 + y^2 + z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2}$$

$$\text{Similarly, } \frac{\partial^2 u}{\partial y^2} = \frac{x^2 + z^2 - y^2}{(x^2 + y^2 + z^2)^2}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2}$$

LHS

$$\begin{aligned} & (x^2 + y^2 + z^2)(y^2 + z^2 - x^2 + x^2 + z^2 - y^2 + u^2 + y^2 - z^2) \\ & (x^2 + y^2 + z^2) \left(\frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} \right) = 1 \rightarrow \text{RHS} \\ & \text{LHS} = \text{RHS} \\ & \text{Proved} \end{aligned}$$

Ques- If $u = lx + my$, $v = mx - ly$, show that

$$\left(\frac{\partial u}{\partial x} \right)_y \left(\frac{\partial v}{\partial u} \right)_v = \frac{l^2}{l^2 + m^2} \text{ and}$$

$$\left(\frac{\partial v}{\partial y} \right)_x \left(\frac{\partial u}{\partial y} \right)_u = \frac{l^2 + m^2}{l^2}$$

Sol.

$$\begin{aligned} u &= lx + my \quad \text{--- } ① \\ v &= mx - ly \quad \text{--- } ② \end{aligned}$$

Now,

$$\left(\frac{\partial u}{\partial x} \right)_y = l$$

from ① & ②

$$u = \frac{lx + my}{l^2 + m^2} \quad ③$$

$$\left(\frac{\partial u}{\partial u} \right)_v = \frac{l}{l^2 + m^2}$$

$$\begin{aligned} lx + my &= u \times l \\ mx - ly &= v \times m \\ l^2 x + mly &= ul \\ m^2 x - mly &= mv \\ x(l^2 + m^2) &= \frac{ul + mv}{l^2 + m^2} \\ x^2 \frac{ul + mv}{l^2 + m^2}, \\ y &= \frac{mx - v}{l} \end{aligned}$$

and ~~$\frac{\partial v}{\partial y} = \frac{m}{l}$~~ ~~$\frac{\partial u}{\partial y} = \frac{m}{l}$~~

So,

$$\left(\frac{\partial u}{\partial x} \right)_y \left(\frac{\partial v}{\partial u} \right)_v = l \left(\frac{l}{l^2 + m^2} \right) = \frac{l^2}{l^2 + m^2}$$

and,

$$y = \frac{mx - v}{l} \quad \text{--- } ④$$

$$\left(\frac{\partial y}{\partial v}\right)_x = - \frac{f}{l}$$

$$v = \frac{mu - (l^2 + m^2)y}{l}$$

$$\left(\frac{\partial v}{\partial y}\right)_u = - \frac{(l^2 + m^2)}{l}$$

$$\therefore \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2 + m^2}{l^2}$$

Proved

Eliminating x from ① &
① $\times m$, ② $\times l$

$$\Rightarrow ml = mlxy + my$$

$$lv = mlx - l^2y$$

$$(ml + l^2)x = (m^2 + l^2)y$$

$$-lv = (m^2 + l^2)y - mu$$

$$v = \frac{mu - (m^2 + l^2)y}{l}$$

* Homogeneous functions -

If function $f(x, y)$ is said to be homogeneous of degree n in variable x & y if it can be expressed in the form $x^n f\left(\frac{y}{x}\right)$ or $y^n f\left(\frac{x}{y}\right)$.

or if

$$f(tx, ty) = t^n f(x, y)$$

$$\text{eg- } f(x, y) = \frac{x+y}{\sqrt{x+y}}$$

$$f(tx, ty) = \frac{tx+ty}{\sqrt{tx+ty}} = \frac{t(x+y)}{\sqrt{t^2(x+y)}} = t^{\frac{1}{2}} \cancel{\frac{(x+y)}{\sqrt{(x+y)}}}$$

$$\text{Degree} = 1 - \frac{1}{2} = \frac{1}{2} = t^{\frac{1}{2}} f(x, y)$$

or

$$f(x, y) = \frac{y(x+y+1)}{\sqrt{y}(\sqrt{y}+1)} = y^{\frac{1}{2}} f(x, y)$$

$$\text{Degree} = \frac{1}{2}$$

→ Euler's Theorem on Homogeneous functions:-
 If u is a homogeneous function of degree n
 in x & y , then

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu}$$

Proof Since u is a homogeneous function of
 degree n in x and y , so it can be
 expressed as:-

$$u = x^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right)$$

$$x \frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) + x^{n+1} f'\left(\frac{y}{x}\right) \left(-\frac{y}{x^2}\right)$$

$$x \frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right)$$

Now,

$$\frac{\partial u}{\partial y} = nx^n f'\left(\frac{y}{x}\right) \left(\frac{1}{x}\right)$$

$$y \frac{\partial u}{\partial y} = yx^n f'\left(\frac{y}{x}\right) \left(\frac{1}{x}\right) = yx^{n-1} f'\left(\frac{y}{x}\right)$$

As per Euler's theorem-

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

$$\text{LHS } nx^n f\left(\frac{y}{x}\right) - yx^{n-1} f'\left(\frac{y}{x}\right) + yx^{n-1} f'\left(\frac{y}{x}\right) = nx^n f\left(\frac{y}{x}\right)$$

$$\text{and } x^n f\left(\frac{y}{x}\right) = u$$

$$\text{so, } nx^n f\left(\frac{y}{x}\right) = nu \rightarrow \text{RHS} \quad \text{LHS} = \text{RHS}$$

Proved

- Euler's theorem can be extended to any no of variables. If u is homogeneous function of n in $x, y \& z$, then -

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu.$$

\Rightarrow If u is a homogeneous function of degree n in x, y , then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Proof Since u is a homogeneous function of degree n , then it can be expressed as -

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu. \quad \textcircled{1}$$

Partially differentiating $\textcircled{1}$ w.r.t x

$$\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = n \frac{\partial u}{\partial x}$$

Multiplying by x

$$x \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = nx \frac{\partial u}{\partial x} \quad \textcircled{2}$$

Partially differentiating $\textcircled{1}$ w.r.t y

$$x \frac{\partial u}{\partial y \partial x} + y \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = n \frac{\partial u}{\partial y}$$

Multiplying by y

$$y \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial y \partial x} = ny \frac{\partial u}{\partial y} \quad \textcircled{3}$$

Adding ② & ③

$$xy \frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} = n \cancel{xu} \frac{\partial u}{\partial x}$$

$$xy \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} = n \cancel{yu} \frac{\partial u}{\partial y}$$

$$\begin{aligned} xy \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} \right) + x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \\ = n \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) \end{aligned}$$

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} - \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) (n-1) \\ = nu(n-1) \end{aligned}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Proved

Ques- Verify Euler's theorem for the following :-

① $u = (x^{1/2} + y^{1/2})(x^n + y^n)$

Sol. - put $x \rightarrow tx$, $y \rightarrow ty$

$$u(tx, ty) = ((tx)^{1/2} + (ty)^{1/2})(t^n(x^n + y^n))$$

$$= t^{1/2}(x^{1/2} + y^{1/2}) t^n(x^n + y^n)$$

$$= t^{n+1/2} u(x, y)$$

Degree = $n + \frac{1}{2}$

∴ By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \left(n + \frac{1}{2} \right) u$$

Verification :-

$$u = (x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n) \quad [\text{uv rule}]$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} x^{-\frac{1}{2}} (x^n + y^n) + n x^{n-1} (x^{\frac{1}{2}} + y^{\frac{1}{2}})$$

$$x \frac{\partial u}{\partial x} = \frac{1}{2} x^{\frac{1}{2}} (x^n + y^n) + n x^n (x^{\frac{1}{2}} + y^{\frac{1}{2}})$$

$$y \frac{\partial u}{\partial y} = \frac{1}{2} y^{\frac{1}{2}} (x^n + y^n) + n y^n (x^{\frac{1}{2}} + y^{\frac{1}{2}})$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} (x^n + y^n) (x^{\frac{1}{2}} + y^{\frac{1}{2}}) + n (x^n + y^n) (\cancel{x^{\frac{1}{2}}} + \cancel{y^{\frac{1}{2}}})$$

~~$$(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^n + y^n)$$~~

$$= \left(n + \frac{1}{2}\right) (x^n + y^n) (x^{\frac{1}{2}} + y^{\frac{1}{2}}) = \left(\frac{n+1}{2}\right) u$$

Verified

② $f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4$

Sol - $u = 3x^2yz + 5xy^2z + 4z^4$

Put $x \rightarrow tx, y \rightarrow ty, z \rightarrow tz$

$$\begin{aligned} u &= 3(tx)^2(ty)(tz) + 5(tx)(ty)^2(tz) + 4(tz)^4 \\ &= 3t^4(x^2yz) + 5t^4(xy^2z) + 4t^4z^4 \\ &= t^4(3(x^2yz) + 5(xy^2z) + 4z^4) \\ &= t^4 u(x, y, z) \end{aligned}$$

Degree = 4

∴ By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 4u$$

Verification :-

$$u = 3x^2yz + 5xy^2z + 4z^4$$

$$\frac{\partial u}{\partial x} = 6xyz + 5y^2z + 0$$

$$\frac{\partial u}{\partial x} = 6xyz + 5y^2z$$

$$x \frac{\partial u}{\partial x} = 6x^2yz + 5xy^2z \quad \text{--- } ①$$

$$\frac{\partial u}{\partial y} = 3x^2z + 10xyz + 0$$

$$y \frac{\partial u}{\partial y} = 3x^2yz + 10xy^2z \quad \text{--- } ②$$

$$\frac{\partial u}{\partial z} = 3x^2y + 5xy^2 + 16z^3$$

$$z \frac{\partial u}{\partial z} = 3x^2yz + 5xy^2z + 16z^4 \quad \text{--- } ③$$

Adding ①, ② & ③

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} \Rightarrow$$

$$\begin{aligned} &= 6x^2yz + 5xy^2z + 3x^2yz + 10xyz + 3x^2y + 5xy^2 + 16z^4 \\ &= 12x^2yz + 20xyz + 16z^4 \\ &= 4(3x^2yz + 5xy^2z + 4z^4) = 4u \end{aligned}$$

Verified

$$③ u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{Sol - } u = \sec^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right) = x^\circ f\left(\frac{y}{x}\right)$$

Degree = 0

∴ By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

Verification :-

$$u = \operatorname{cosec}^{-1} \frac{y}{x} + \tan^{-1} \frac{y}{x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2/y^2}} \cdot \frac{1}{y} + \frac{1}{1+y^2/x^2} \left(-\frac{y}{x^2} \right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} \cdot \frac{y}{y} + \frac{1}{x^2+y^2} \left(-\frac{y}{x^2} \right) x^2$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\frac{x \frac{\partial u}{\partial x}}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$$

$$\frac{y \frac{\partial u}{\partial y}}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

Now,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \\ = 0$$

Verified

Ques- If $u = \tan^{-1} \frac{x^3+y^3}{x-y}$, prove that

$$\frac{x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}}{\partial x^2} = \sin 4u - \sin 2u \\ = 2 \cos 3u \sin u$$

Sol- Here u is not a homogeneous function.

$$\text{So, } \tan u = \frac{x^3+y^3}{x-y}$$

$$\tan u = \frac{x^3 \left(\frac{y^3}{x^3} + 1 \right)}{x \left(1 - \frac{y}{x} \right)}$$

$$\text{Degree} = 2$$

∴ $\tan u$ is a homogeneous function of degree 2.

∴ By Euler's theorem-

$$x \frac{\partial}{\partial x} (\tan u) + y \frac{\partial}{\partial y} (\tan u) = 2 \tan u$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

$$\sec^2 u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 2 \tan u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = \frac{2 \sin u}{\cos^2 u} \cdot \frac{\cos^2 u}{\cos u}$$

$$= 2 \sin u \cos u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u \quad \text{--- } ①$$

~~Ans~~ Partially differentiating ① w.r.t x

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \cos 2u \frac{\partial u}{\partial x}$$

Multiplying by x

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \partial y} = 2x \cos 2u \frac{\partial u}{\partial x} \quad \text{--- } ②$$

Now,

Partially differentiating ① w.r.t y

$$y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} + xy \frac{\partial^2 u}{\partial y \partial x} = 2y \cos 2u \frac{\partial u}{\partial y} \quad \text{--- } ③$$

Adding ② & ③, we get-

~~(2cos2u)~~ + ~~(sin2u)~~

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + xy \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} \right) \\ = 2 \cos 2u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \end{aligned}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) (2 \cos 2u)$$

$$\left[\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = (2 \cos 2u - 1) (\sin 2u)$$

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} &= 2 \cos 2u \sin 2u - \sin 2u \\ &= \sin 4u - \sin 2u \\ &= 2 \cos 3u \sin u \end{aligned} \rightarrow \text{RHS}$$

LHS = RHS

Proved

Ques- If $u = \tan^{-1} \frac{y^2}{x}$, show that-

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -8 \sin 2u \sin^2 u$$

Sol- $\tan u = \frac{y^2}{x}$ degree = 1

\therefore By Euler's theorem-

$$x \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = \tan u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u} = \frac{\tan u}{1 + \tan^2 u}$$

$$\textcircled{1} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \sin u \cos u \quad \textcircled{1}$$

$$= \frac{1}{2} \sin 2u$$

LHS Partially differentiating $\textcircled{1}$ w.r.t x

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{1 - \cos 2u}{x} \frac{\partial u}{\partial x}$$

Multiplying by x

RHS

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \partial y} = \cos 2u \left(x \frac{\partial u}{\partial x} \right) \textcircled{2}$$

Now, Partially differentiating $\textcircled{1}$ w.r.t y

$$y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} + xy \frac{\partial^2 u}{\partial x \partial y} = \cos 2u \left(y \frac{\partial u}{\partial y} \right) \textcircled{3}$$

Adding $\textcircled{2}$ & $\textcircled{3}$, we get-

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) (\cos 2u)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2} \sin 2u (\cos 2u)$$

$$= \left(\frac{1}{2} \sin 2u \right) (\cos 2u - 1)$$

$$\frac{x^2}{\sin^2 u} \frac{\partial^2 u}{\partial x^2} + \frac{y^2}{\sin^2 u} \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2} \sin 2u \cos 2u - \frac{1}{2} \sin 2u$$

$$= \frac{1}{2} \sin 2u \cdot (-2 \sin^2 u)$$

$$\frac{x^2}{\sin^2 u} \frac{\partial^2 u}{\partial x^2} + \frac{y^2}{\sin^2 u} \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = -\sin 2u \sin^2 u \rightarrow \text{RHS}$$

LHS = RHS

Proved

Ques- If $u = \sin^{-1} \frac{x+y}{\sqrt{x+y}}$, prove that

$$\frac{x^2}{\sin^2 u} \frac{\partial^2 u}{\partial x^2} + \frac{y^2}{\sin^2 u} \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = -\frac{\sin u \cos 2u}{4 \cos^3 u}$$

Sol- $\sin u = \frac{x+y}{\sqrt{x+y}} = \frac{x(1 + \frac{y}{x})}{\sqrt{x}(1 + \frac{y}{x})}$

$$\text{Degree} = \frac{1}{2}$$

∴ By Euler's theorem-

$$x \frac{\partial (\sin u)}{\partial x} + y \frac{\partial (\sin u)}{\partial y} = \frac{1}{2} \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u$$

$$\cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = \frac{1}{2} \sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u \quad \text{--- } ①$$

Partially differentiating ① w.r.t x

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial xy} = \frac{1}{2} \sec^2 u \frac{\partial u}{\partial x}$$

Multiplying by x

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial u}{\partial xy} = \frac{1}{2} \sec^2 u \left(x \frac{\partial u}{\partial x} \right) \quad ②$$

Now, Partially differentiating ① w.r.t y

$$y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} + xy \frac{\partial u}{\partial xy} = \frac{1}{2} \sec^2 u \left(y \frac{\partial u}{\partial x} \right) \quad ③$$

Adding ② & ③, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial xy} + y^2 \frac{\partial^2 u}{\partial y^2} = \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \left(\frac{1}{2} \sec^2 u - 1 \right)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial xy} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{2} \tan u \left(\frac{1}{2} \sec^2 u - 1 \right)$$

$$= \frac{1}{4} \frac{\sin u}{\cos u} \cdot \frac{1}{\cos^2 u} - \frac{1}{2} \frac{\sin u}{\cos u}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial xy} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\sin u}{4 \cos^3 u} - \frac{\sin u}{2 \cos u}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} - \dots = \frac{\sin u - 2 \cos^2 u \sin u}{4 \cos^3 u}$$

$$\text{, , , , } = \frac{\sin u (1 - 2 \cos^2 u)}{4 \cos^3 u}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial xy} + y^2 \frac{\partial^2 u}{\partial y^2} = - \frac{\sin u \cos 2u}{4 \cos^3 u} \rightarrow \text{RHS}$$

LHS = RHS

Proved

Ques- If $u = (x^2 + y^2)^{\frac{1}{3}}$, show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial xy} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{2u}{9}$$

Sol- let $x \rightarrow tx, y \rightarrow ty$

$$u = ((tx)^2 + (ty)^2)^{\frac{1}{3}}$$

$$u = t^{\frac{2}{3}} (x^2 + y^2)^{\frac{1}{3}}$$

so, u is a homogeneous function of x & y of degree $\frac{2}{3}$.

∴ By Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2}{3} u \quad \textcircled{1}$$

Ans Partially differentiating $\textcircled{1}$ w.r.t x

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial xy} = \frac{2}{3} \frac{\partial u}{\partial x}$$

Multiplying by x

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial xy} = \frac{2}{3} \left(u \frac{\partial u}{\partial x} \right) \quad \textcircled{2}$$

Similarly,

$$y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} + xy \frac{\partial^2 u}{\partial xy} = \frac{2}{3} \left(y \frac{\partial u}{\partial y} \right) \quad \textcircled{3}$$

Adding ② & ③, we get -

$$\frac{x^2 \partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \left(\frac{2}{3} - \frac{1}{3} \right)$$

$$\frac{x^2 \partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{2}{3} u \right) \left(-\frac{1}{3} \right) = -\frac{2}{9} u$$

RHS

LHS = RHS

Proved

- * Composite functions / Total Derivative :-
- ⇒ If $u = f(x, y)$, where $x = \phi(t)$, $y = \psi(t)$ then u is composite function of single variable t . Total derivative of u is $\frac{du}{dt}$.
- ⇒ If $z = f(x, y)$, where $x = \phi(u, v)$, $y = \psi(u, v)$ then z is composite function of two variables u and v . So that we can find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

④ Differentiation of composite functions -

If $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$ then -

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}}$$

Note ① If $z = f(x, y)$ where $x = \phi(u, v)$ and $y = \psi(u, v)$ then -

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}}$$

② If $z = f(x, y)$ where $x = \phi(u, v)$ and $y = \psi(u, v)$
then

$$\boxed{\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

Ques- find du when $u = x^2 + y^2$, $x = at^2$, $y = 2at$. Also verify by direct substitution.

Sol- $u = x^2 + y^2$

Here u is a composite function of t .

$$\begin{aligned}\therefore \frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \\ &= (2x)(2at) + (2y)(2a) \\ &= 4xat + 4ay\end{aligned}$$

Substituting values of x & y

$$\begin{aligned}&= 4(at^2)at + 4a(2at) \\ &= 4a^2t^3 + 8a^2t\end{aligned}$$

$$\boxed{\frac{du}{dt} = 4a^2t(t^2 + 2)}$$

Verification by direct substitution-

$$u = x^2 + y^2 = (at^2)^2 + (2at)^2$$

$$u = a^2t^4 + 4a^2t^2$$

$$\frac{du}{dt} = 4a^2t^3 + 8a^2t$$

$$\frac{du}{dt} = 4a^2t(t+2)$$

Verified

Ques- If $u = \sin \frac{x}{y}$, $x = e^t$, $y = t^2$, find $\frac{du}{dt}$

Sol- here u is composite of t

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dy}{dt} = \cos \frac{x}{y} \left(\frac{1}{y} \right) \cdot e^t + \cos \frac{x}{y} \left(-\frac{x}{y^2} \right) \cdot 2t$$

$$\frac{du}{dt} = \frac{e^t}{y} \cos \frac{x}{y} - \frac{2xt}{y^2} \cos \frac{x}{y}$$

Substituting values of x & y

$$\frac{du}{dt} = \frac{e^t}{t^2} \cos \frac{e^t}{t^2} - \frac{2et}{t^3} \cos \frac{e^t}{t^2}$$

$$\frac{du}{dt} = \frac{e^t}{t^2} \cos \frac{e^t}{t^2} \left(1 - \frac{2}{t^3} \right)$$

$$\frac{du}{dt} = e^t \left(\frac{t^3 - 2}{t^3} \right) \cos \frac{e^t}{t^2}$$

Ques- If $z = \log(u^2 + v)$, $u = e^{x^2+y^2}$, $v = x^2+y$
find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Sol- $z = \log(u^2 + v)$

here z is composite function of x & y

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2 + v} \cdot 2xe^{x^2+y^2} + \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{4xue^{x^2+y^2}}{u^2 + v} + \frac{2u}{u^2 + v}$$

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2 + v} \left(2ue^{x^2+y^2} + 1 \right)$$

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2 + v} \left(2u^2 + 1 \right) \quad [e^{x^2+y^2} = u]$$

$$\text{and } \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

$$= \frac{\partial u}{u^2+v} \cdot 2y e^{x^2+y^2} + \frac{1}{u^2+v}$$

$$= \frac{1}{u^2+v} (4uy e^{x^2+y^2} + 1)$$

$$\frac{\partial z}{\partial y} = \frac{1}{u^2+v} (4uy e^{x^2+y^2} + 1) \quad \cancel{=}$$

Ques: If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

Sol- let $x \rightarrow tx, y \rightarrow ty, z \rightarrow tz$
 $u = f\left(\frac{tx}{ty}, \frac{ty}{tz}, \frac{tz}{tx}\right)$

$\therefore u$ is a homogeneous function of degree 0.
 So, By Euler's theorem-

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

Proved

Ques 9) If $x = u+v+w, y = uv+vw+wu, z = uwv$ and
 f is a function of x, y, z . Show that -

$$u \frac{\partial f}{\partial u} + v \frac{\partial f}{\partial v} + w \frac{\partial f}{\partial w} = x \frac{\partial f}{\partial x} + 2y \frac{\partial f}{\partial y} + 3z \frac{\partial f}{\partial z}$$

Sol- Here f is a composite function of u, v & w

$$x = u + v + w$$

$$y = uv + vw + uw$$

$$z = uvw$$

Now

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial u} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} (v+w) + \frac{\partial f}{\partial z} (vw) \quad \text{--- } ①\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial v} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} (u+w) + \frac{\partial f}{\partial z} (uw) \quad \text{--- } ②\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial w} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial w} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} (u+v) + \frac{\partial f}{\partial z} (uv) \quad \text{--- } ③\end{aligned}$$

Now, Multiply ① by u , ② by v and ③ by w
we get -

$$\begin{aligned}\text{LHS} \quad u \frac{\partial f}{\partial u} + v \frac{\partial f}{\partial v} + w \frac{\partial f}{\partial w} &= (u+v+w) \frac{\partial f}{\partial x} + 2(uv+vw+wu) \frac{\partial f}{\partial y} + 3(uvw) \frac{\partial f}{\partial z} \\ &= x \frac{\partial f}{\partial x} + 2y \frac{\partial f}{\partial y} + 3z \frac{\partial f}{\partial z} \rightarrow \text{RHS}\end{aligned}$$

$$\text{LHS} = \text{RHS}$$

Proved

~~QUESTION~~
Ques- If $u = xe^y z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$
find $\frac{du}{dx}$.

Sol-
$$\begin{aligned}\frac{du}{dx} &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dx} \\ &= e^y z + xe^y z \cdot \frac{-2x}{\sqrt{a^2 - x^2}} + xe^y \cdot 2 \sin x \cos x \\ \frac{du}{dx} &= e^y \left[z - \frac{x^2}{\sqrt{a^2 - x^2}} + x \sin 2x \right]\end{aligned}$$

① Differentiation of Implicit functions-
If $f(x, y) = c$, then $u = f(x, y)$ where $u = c$ is
an implicit function ($c \rightarrow$ any constant).

* By Note ②

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

we know that,

$$\frac{du}{dx} = 0$$

so, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

$\frac{dy}{dx} = -\frac{f_x}{f_y}$	— ①
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Differentiating ① w.r.t x

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= - f_y \left(\frac{df_x}{dx} \right) - f_{xx} \left(\frac{df_y}{dy} \right) && (\text{u/v rule}) \\
 &= - \frac{f_y}{f_y^2} \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_x}{\partial y} \cdot \frac{dy}{dx} \right) - f_{xx} \left(\frac{\partial f_y}{\partial x} + \frac{\partial f_y}{\partial y} \cdot \frac{dy}{dx} \right) \\
 &= - \frac{f_y}{f_y^2} \left[f_{xx} + f_{xy} \left(\frac{-f_x}{f_y} \right) \right] - f_{xx} \left[f_{xy} + f_{yy} \left(\frac{-f_x}{f_y} \right) \right] \\
 &= - \frac{f_{xx}f_y - f_{xy}f_x - f_xf_{yy} + f_xf_{yy} \cdot \frac{f_x}{f_y}}{f_y^2}
 \end{aligned}$$

$$\frac{d^2y}{dx^2} = - \frac{f_{xx}f_y^2 - 2f_xf_{xy}f_y + f_{yy}f_{x^2}}{f_y^3}$$

Ques- find $\frac{dy}{dx}$, when (i) $x^y + y^x = c$
(ii) $(\cos x)^y = (\sin y)^x$

Sol- (i) $x^y + y^x = c$

so, $f(x, y) = x^y + y^x$ and $f(x, y) = c$

given is an implicit function,
and we know that -

$$\frac{dy}{dx} = - \frac{f_x}{f_y}$$

$$f_x = \frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$f_y = \frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{y x^{y-1} + y^x \log y}{x^y \log x + x y^{x-1}}}$$

(ii) $(\cos x)^y - (\sin y)^x = 0$
 given is an implicit function in which,

$$f(x, y) = (\cos x)^y - (\sin y)^x$$

~~$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$~~

$$\text{so, } f(x, y) = 0 = c$$

and we know that -

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$f_x = \frac{\partial f}{\partial x} = \cancel{y(\cos x)^{y-1}} - y(\cos x)^{y-1}(-\sin x) - (\sin y)^x \log \sin y$$

$$= -y \sin x (\cos x)^{y-1} - (\sin y)^x \log \sin y$$

$$f_y = \frac{\partial f}{\partial y} = (\cos x)^y \log \cos x - x (\sin y)^{x-1} \cos y$$

$$\frac{dy}{dx} = \frac{y \sin x (\cos x)^{y-1} - (\sin y)^x \log \sin y}{(\cos x)^y \log \cos x - x (\sin y)^{x-1} \cos y}$$

~~Ques-~~ * If z is a function of x and y and u, v are two other variables such that $u = lx + my$, $v = ly - mx$ show that -

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2) \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right)$$

~~Sol-~~ from above question, we get to ~~know~~ know that -
 z is a composite function of x and y .

$$u = lx + my \quad v = ly - mx$$

~~$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$~~

$$*\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot l + \frac{\partial z}{\partial v} \cdot (-m)$$

$$\Rightarrow \frac{\partial z}{\partial x} = l \frac{\partial z}{\partial u} - m \frac{\partial z}{\partial v}$$

$$\Rightarrow \frac{\partial}{\partial x} = l \frac{\partial}{\partial u} - m \frac{\partial}{\partial v}$$

Now,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial u} \left(\frac{\partial z}{\partial x} \right)$$

$$= \left(l \frac{\partial}{\partial u} - m \frac{\partial}{\partial v} \right) \left(l \frac{\partial z}{\partial u} - m \frac{\partial z}{\partial v} \right)$$

$$= l^2 \frac{\partial^2 z}{\partial u^2} - ml \frac{\partial^2 z}{\partial u \partial v} - ml \frac{\partial^2 z}{\partial v \partial u} + m^2 \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x^2} = l^2 \frac{\partial^2 z}{\partial u^2} - 2ml \frac{\partial^2 z}{\partial u \partial v} + m^2 \frac{\partial^2 z}{\partial v^2}$$

————— ①

$$*\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot m + \frac{\partial z}{\partial v} \cdot l$$

$$\Rightarrow \frac{\partial z}{\partial y} = m \frac{\partial z}{\partial u} + l \frac{\partial z}{\partial v}$$

$$\Rightarrow \frac{\partial}{\partial y} = m \frac{\partial}{\partial u} + l \frac{\partial}{\partial v}$$

$$\begin{aligned}
 \text{Now, } \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) \\
 &= \left(m \frac{\partial}{\partial u} + l \frac{\partial}{\partial v} \right) \left(m \frac{\partial z}{\partial u} + l \frac{\partial z}{\partial v} \right) \\
 &= m^2 \frac{\partial^2 z}{\partial u^2} + 2ml \frac{\partial^2 z}{\partial u \partial v} + l^2 \frac{\partial^2 z}{\partial v^2}
 \end{aligned}
 \quad (2)$$

Adding ① & ② we get-

$$\begin{aligned}
 \text{LHS } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= l^2 \frac{\partial^2 z}{\partial u^2} + m^2 \frac{\partial^2 z}{\partial v^2} + m^2 \frac{\partial^2 z}{\partial v^2} + l^2 \frac{\partial^2 z}{\partial v^2} \\
 \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= (l^2 + m^2) \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right) \rightarrow \text{RHS} \\
 \text{LHS} &= \text{RHS} \\
 \text{Proved}
 \end{aligned}$$

Ques- If $x = e^r \cos \theta$, $y = e^r \sin \theta$. Show that
 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r} \left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} \right]$ where $u = f(x, y)$

Sol- Here u is composite function of r and θ .

So,

$$* \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot e^r \cos \theta + \frac{\partial u}{\partial y} \cdot e^r \sin \theta$$

$$\Rightarrow \frac{\partial u}{\partial r} = e^r \cos \theta \frac{\partial u}{\partial x} + e^r \sin \theta \frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial r} = \underbrace{e^r \cos \theta}_{u} \frac{\partial}{\partial x} + \underbrace{e^r \sin \theta}_{y} \frac{\partial}{\partial y}$$

$$\text{Now, } \frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial r} \right)$$

$$= \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

$$\frac{\partial^2 u}{\partial r^2} = x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

1

$$* \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta}$$

$$= -e^r \sin \theta \frac{\partial u}{\partial x} + e^r \cos \theta \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \theta} = -y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y}$$

$$\text{Now, } \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial \theta} \right)$$

$$= \left(x \frac{\partial u}{\partial y} - y \frac{\partial u}{\partial x} \right) \left(x \frac{\partial u}{\partial y} - y \frac{\partial u}{\partial x} \right)$$

$$= x^2 \frac{\partial^2 u}{\partial y^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial x^2}$$

2

Adding ① & ②, we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} = (x^2 + y^2) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} = e^{2r} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r} \left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} \right)$$

$$\begin{aligned} x^2 &= e^{2r} \sin^2 \theta \\ y^2 &= e^{2r} \cos^2 \theta \\ x^2 + y^2 &= e^{2r} (\sin^2 \theta + \cos^2 \theta) \\ &= e^{2r} \end{aligned}$$

Proued

Ques- If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, find the value of $\frac{dz}{dx}$ when $x=y=a$.

Sol-

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \quad \textcircled{1}$$

$$\text{Now, } \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} = \frac{x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} = \frac{y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$(f(x,y) = x^3 + y^3 + 3axy = 5a^2)$$

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 3ay$$

$$f_y = \frac{\partial f}{\partial y} = 3y^2 + 3ax$$

$$\frac{dy}{dx} = -\left(\frac{3x^2 + 3ay}{3y^2 + 3ax}\right) = -\left(\frac{x^2 + ay}{y^2 + ax}\right)$$

→ Putting values of $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ & $\frac{dy}{dx}$ in $\textcircled{1}$

$$\frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}} \left(\frac{x^2+ay}{y^2+ax} \right)$$

→ Putting $x=y=a$

$$\frac{dz}{dx} = \frac{a}{a\sqrt{2}} - \frac{a}{a\sqrt{2}} \left(\frac{2a^2}{2a^2} \right)$$

$$\frac{dz}{dx} = 0$$

Ques- Prove that if $y^3 - 3ax^2 + x^3 = 0$, then

$$\frac{d^2y}{dx^2} + \frac{8a^2x^2}{y^5} = 0 \quad \textcircled{1}$$

Sol- $f(x, y) = y^3 - 3ax^2 + x^3 = 0$

$$\begin{aligned} f_x &= 3x^2 - 6ax \\ f_{xx} &= 6x - 6a \end{aligned} \quad \begin{aligned} f_y &= 3y^2 \\ f_{yy} &= 6y \end{aligned}$$

We know that, $f_{xy} = 0$

$$\begin{aligned} \frac{d^2y}{dx^2} &= f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2 \\ &= -\frac{(6x-6a)(3y^2)^2 - 2(0) + (6y)(3x^2-6ax)^2}{(3y^2)^3} \\ &= -\frac{6(9xy^4 - 9ay^4) + 6y(9x^4 + 36a^2x^2 - 36ax^2)}{27y^6} \\ &= -\left[\frac{54xy^4 - 54ay^4 + 54yx^4 + 216a^2x^2y - 216a^2y^2}{27y^6} \right] \\ &= -\frac{27y^4}{27y^6} \left[2xy^3 - 2ay^3 + 2x^4 + 8a^2x^2 - 8ax^3 \right] \\ &\quad \text{cancel } 27y^4 \\ &= -\frac{1}{y^2} \left[2xy^3 - 2ay^3 + 2x^4 + 8a^2x^2 - 6ax^3 - 2ax^3 \right] \\ &= -\frac{1}{y^2} \left(2x(y^3 + x^3 - 3ax^2) - 2ay^3 + 8a^2x^2 - 2ax^3 \right) \\ &= -\frac{1}{y^2} \left(8a^2x^2 - 2a(x^3 + y^3) \right) \end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{y^5} (8a^2x^2 - 2a(3ax^2)) = \frac{-1}{y^5} (2a^2x^2)$$

Putting value of $\frac{d^2y}{dx^2}$ in ①

LHS $= \frac{-2a^2x^2}{y^5} + \frac{2a^2x^2}{y^5} = 0 \rightarrow \text{RHS}$

LHS = RHS

Proved

* Jacobians :-
If u and v are functions of x and y , then

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u and v w.r.t x and y and is denoted by $J_{(x,y)}(u,v)$ or $\frac{\partial(u,v)}{\partial(x,y)}$.

○ Properties of Jacobians -

(i) If u & v are functions of r , s and r , s are functions of x , y , then

$$J_{(x,y)}(u,v) = \frac{\partial(u,v)}{\partial(x,y)}$$

$$J_{(x,y)}(u,v) = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

(ii) If J_1 is Jacobian of u, v w.r.t x, y and J_2 is Jacobian of x, y w.r.t u, v then -

$$[J_1 J_2 = 1]$$

Ques- If $x = r \cos \theta, y = r \sin \theta$, Verify
 $\frac{\partial(x, y)}{\partial(r, \theta)} \times \frac{\partial(r, \theta)}{\partial(x, y)} = 1$

Sol- We know that,

$$\begin{aligned} \frac{\partial(x, y)}{\partial(r, \theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= r(\cos^2 \theta + \sin^2 \theta) = r \end{aligned}$$

Now, Adding Squaring of adding x and y

$$x^2 = r^2 \cos^2 \theta$$

$$y^2 = r^2 \sin^2 \theta$$

$$x^2 + y^2 = r^2$$

$$\text{So, } r^2 = x^2 + y^2, \quad r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x}$$

$$\begin{aligned} \frac{\partial(r, \theta)}{\partial(x, y)} &= \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix} \end{aligned}$$

$$= \frac{x^2}{(x^2 + y^2)^{3/2}} + \frac{y^2}{(x^2 + y^2)^{3/2}}$$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2 + y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

So,
LHS $\frac{\partial(x, y)}{\partial(r, \theta)} \times \frac{\partial(r, \theta)}{\partial(x, y)} = r \times \frac{1}{r} = 1 \rightarrow \text{RHS}$

$$\text{LHS} = \text{RHS}$$

Proved

Ques- If $x = r \cos\theta$, $y = r \sin\theta$, $z = z$
find $\frac{\partial(x, y, z)}{\partial(r, \theta, z)}$

Sol- we know that,

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r \sin\theta & 0 \\ \sin\theta & r \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta(r \cos\theta) + r \sin\theta(\sin\theta)$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)}$$

Ques- If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$
Show that $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$.

Sol-

$$\frac{\partial x}{\partial r} = \sin\theta \cos\phi, \quad \frac{\partial x}{\partial \theta} = r \cos\theta \cos\phi, \quad \frac{\partial x}{\partial \phi} = -r \sin\theta \sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta \sin\phi, \quad \frac{\partial y}{\partial \theta} = r \cos\theta \sin\phi, \quad \frac{\partial y}{\partial \phi} = r \sin\theta \cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta, \quad \frac{\partial z}{\partial \theta} = -r \sin\theta, \quad \frac{\partial z}{\partial \phi} = 0$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & -r \sin \theta & 0 \end{vmatrix}$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \left[\begin{array}{l} \sin \theta \cos \phi (r^2 \sin^2 \theta \cos \phi) - r \cos \theta \cos \phi \\ (-r \sin \theta \cos \phi \sin \theta) - r \sin \theta \sin \phi \\ (-r \sin^2 \theta \sin \phi - r \cos^2 \theta \sin \phi) \end{array} \right]$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = +r^2 \sin^3 \theta \cos^2 \phi + r^2 \sin \theta \cos^2 \theta \cos^2 \phi + r^2 \sin \theta \sin^2 \phi (\underbrace{\cos^2 \theta + \sin^2 \theta}_1)$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta \left(+ \sin^2 \theta \cos^2 \phi + \underbrace{\cos^2 \theta \cos^2 \phi + \sin^2 \phi}_1 \right)$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta \left(+ \cos^2 \phi \left(\underbrace{\cos^2 \theta + \sin^2 \theta}_1 \right) + \sin^2 \phi \right)$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta \rightarrow \text{LHS}$$

$$\text{LHS} = \text{RHS}$$

Proved

Ques- If $u = x+y+z$, $uv = y+z$, $uvw = z$, show that
 $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

$$\begin{aligned} z &= uvw \\ y &= uv = uwz \end{aligned}$$

$$\begin{aligned} \text{Sol- LHS } \frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} 1-v & -u & 0 \\ v-w & u-uw & -uw \\ vw & uw & uv \end{vmatrix} \\ &= 1-v(u^2v - u^2vw + uwvw) + u(v^2u - v^2vw + v^2uw) \\ &= u^2v \rightarrow \text{RHS} \\ \text{LHS} &= \text{RHS} \end{aligned}$$

Proved

Ques- If $u = x^2 - 2y$, $v = x+y+z$, $w = x-2y+3z$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

$$\text{Sol- } \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 2x(3+2) + 2(3-1) = 10x+4$$

Ques- If $u = xyz$, $v = xy + yz + zx$, $w = x+y+z$. Show that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{(x-y)(y-z)(z-x)}$$

Sol- As we know - $\frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1 \quad \text{--- (1)}$

$$\begin{aligned} \text{Now, } \frac{\partial(u,v,w)}{\partial(x,y,z)} &= \begin{vmatrix} yz & xz & xy \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix} \\ &= yz(x+z-x-y) - xz(y+z-x-y) + xy(y+z-x) \\ &= yz(z-y) - xz(z-x) + xy(y-x) \\ &= yz^2 - xy^2 - xz^2 + zx^2 + xy^2 - yx^2 \\ &= (x-y)(y-z)(z-x) \end{aligned}$$

$$\text{Putting in (1) } - \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{(x-y)(y-z)(z-x)}$$

Proved

① Jacobian of Implicit functions

If x, y, u, v are connected by implicit functions $f_1(x, y, u, v) = 0, f_2(x, y, u, v) = 0$ where u, v are implicit functions of x, y , then -

$$\boxed{\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \cdot \frac{\partial(f_1, f_2)}{\partial(u, v)}} \quad \text{where } \frac{\partial(f_1, f_2)}{\partial(u, v)} = \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

Ques- If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$. Then find -

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{x^2 - y^2}{u^2 + v^2}$$

Sol-

$$\frac{\partial(f_1, f_2)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2x^2 - 2y^2$$

$$\frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2$$

LHS

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \left[\frac{2x^2 - 2y^2}{2u^2 + 2v^2} \right] = \frac{x^2 - y^2}{u^2 + v^2} \rightarrow \underline{\text{RHS}}$$

LHS = RHS

Proved

Ques- If u, v, w are the roots of the equation $(\lambda-x)^3 + (\lambda-y)^3 + (\lambda-z)^3 = 0$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$

Sol-

$$\begin{aligned} & \lambda^3 - x^3 - 3\lambda x(\lambda-x) + \lambda^3 - y^3 - 3\lambda y(\lambda-y) + \lambda^3 - z^3 - 3\lambda z(\lambda-z) \\ & 3\lambda^3 - 3\lambda^2 x - 3\lambda^2 y - 3\lambda^2 z + 3\lambda x^2 + 3\lambda y^2 + 3\lambda z^2 - x^3 - \\ & y^3 - z^3 = 0 \\ \Rightarrow & 3\lambda^3 - 3(x+y+z)\lambda^2 + 3(x^2+y^2+z^2)\lambda - (x^3+y^3+z^3) = 0 \\ u+v+w &= \frac{3(x+y+z)}{3} = x+y+z \end{aligned}$$

$$uv+vw+wu = \frac{3(x^2+y^2+z^2)}{3} = x^2+y^2+z^2$$

$$uvw = \frac{1}{3}(x^3+y^3+z^3)$$

~~$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$$~~

$$\begin{aligned} f_1 &= u+v+w-x-y-z \\ f_2 &= uv+vw+wu-x^2-y^2-z^2 \\ f_3 &= uvw - \frac{1}{3}(x^3+y^3+z^3) \end{aligned}$$

$$\begin{aligned} \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} &= \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix} \\ &= (u^2v+uvw-u^2w-uvw) - (uv^2+uwv-uw^2-vw^2) \\ &\quad + (uvw+uw^2-uvw-vw^2) \\ &= u^2(v-w) - v^2(u-w) + w^2(u-v) \\ &= -(u-v)(v-w)(w-u) \end{aligned}$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix} = -2(x-y)(y-z)(z-x)$$

$$\begin{aligned}
 \text{LHS} \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} &= (-1)^3 \frac{12(x-y)(y-z)(z-x)}{x(u-v)(v-w)(w-u)} \\
 &= -2 \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)} \rightarrow \text{RHS}
 \end{aligned}$$

LHS = RHS

Proved

④ Partial derivative of Implicit functions by
Jacobiyan -

Given $f_1(x, y, u, v) = 0$ and $f_2(x, y, u, v) = 0$
then,

$$\rightarrow \frac{\partial u}{\partial x} = \frac{\partial(f_1, f_2)}{\partial(x, v)} / \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$\rightarrow \frac{\partial u}{\partial y} = \frac{\partial(f_1, f_2)}{\partial(y, v)} / \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$\rightarrow \frac{\partial v}{\partial x} = \frac{\partial(f_1, f_2)}{\partial(u, u)} / \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$\rightarrow \frac{\partial v}{\partial y} = \frac{\partial(f_1, f_2)}{\partial(y, u)} / \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

Ques- Use Jacobiyan to find $\frac{\partial u}{\partial x}$ if $u^2 + xv^2 - xy = 0$ and $u^2 + xuv + v^2 = 0$

$$\begin{aligned}
 \text{Sol- } \frac{\partial(f_1, f_2)}{\partial(x, v)} &= \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v^2 - y & 2vx \\ yv & xy + 2v \end{vmatrix} \\
 &= (v^2 - y)(xy + 2v) - 2xv^2 \\
 &= xyv^2 + 2v^3 - xy^2 - 2vy \\
 &= 2v^3 - 2xy^2 - 2vy = xyv^2(2xyv^2 - 1)
 \end{aligned}$$

$$\begin{aligned}\frac{\partial(f_1, f_2)}{\partial(u, v)} &= \begin{vmatrix} 2u & 2uv \\ 2u & xy + 2v \end{vmatrix} \\ &= 2u^2xy + 4uv^2 - 4u^2v^2\end{aligned}$$

Putting $\frac{\partial(f_1, f_2)}{\partial(x, v)}$ and $\frac{\partial(f_1, f_2)}{\partial(u, v)}$ in formula

$$\frac{\partial u}{\partial x} = \frac{\frac{\partial(f_1, f_2)}{\partial(x, v)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}}$$

$$\frac{\partial u}{\partial x} = (-1)^3 \frac{2v^3 - xy^2 - 2vy - xyy^2}{2u^2xy + 4uv^2 + 4u^2v^2}$$

$$\frac{\partial u}{\partial x} = \frac{xyv^3 + xy^2 + 2vy - 2v^3}{2u^2xy + 4uv^2 + 4u^2v^2}$$

Ques If $x = u+v+w$, $y = u^2+v^2+w^2$, $z = u^3+v^3+w^3$. Show
that $\frac{\partial u}{\partial x} = \frac{v+w}{(u-v)(u-w)}$

Sol- We know that -

$$\text{LHS} \quad \frac{\partial u}{\partial x} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)} / \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)} = \begin{vmatrix} 1 & -1 & -1 \\ 0 & -2v & -2w \\ 0 & -3v^2 & -3w^2 \end{vmatrix}$$

$$\begin{aligned}&= \cancel{1} \cancel{(-1)} \cancel{(-1)} + \cancel{1} \cancel{(-2v)} \cancel{(-2w)} \\ &= 6vw^2 - 6wv^2\end{aligned}$$

$$= 6vw(v-w)$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} -1 & -1 & -1 \\ -2u & -2v & -2w \\ -3u^2 & -3v^2 & -3w^2 \end{vmatrix}$$

$$\begin{aligned}&= -(6vw^2 - 6wv^2) + (6uw^2 - 6wu^2) - (6uv^2 - 6vu^2) \\ &= 6vw(v-w) + 6uw(w-u) + 6uv(u-v)\end{aligned}$$

Date

$$= 6(u-v)(v-w)(w-u)$$

$$\therefore \frac{\partial u}{\partial x} = - \frac{6vw(v-w)}{k(u-v)(v-w)(w-u)}$$
$$\frac{\partial u}{\partial x} = \frac{vw}{(u-v)(u-w)} \rightarrow \text{RHS}$$

LHS = RHS

Proved

Applications of Partial Differentiation

- ① Maxima and Minima
Working rule to find Maxima and Minima-

let $z = f(x, y)$

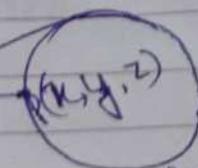
Step I

~~ques-~~ find Maximum and Minimum distance from the sphere $x^2 + y^2 + z^2 = 1$ to the point $(3, 4, 12)$

~~Sol-~~ Using Lagrange's Method.

~~Step I~~ Write the function

$$F = f \text{ (which has to be min or max)} + \lambda(\phi) \rightarrow \text{condition}$$



$$x^2 + y^2 + z^2 = 1$$

In case of two conditions.

$$\star F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2$$

$$\text{Step II} \quad dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$$

$$\text{Step III} \quad \text{Put } \frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$$

$$\text{Step IV} \quad d^2 F = F_{xx}(dx)^2 + F_{yy}(dy)^2 + F_{zz}(dz)^2 + 2(F_{xy}dx dy + F_{yz}dy dz + F_{zx}dz dx)$$

then if $d^2 F > 0$ then it would be a minima

if $d^2 F < 0$ then it would be a maxima

→ let a point $P(x, y, z)$ on the sphere is having min or max. distance with $(3, 4, 12)$ ①

$$d = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2} \quad (\text{if dis. is min, its square would also be min f same with max})$$

$$f = d^2 = (x-3)^2 + (y-4)^2 + (z-12)^2$$

$$F = f + \lambda \phi$$

$$F = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

$$\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x$$

$$\frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y$$

$$\frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$$

Now, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

$$2(x-3) + 2\lambda x = 0 \Rightarrow (x-3) + \lambda x = 0 \quad \textcircled{2}$$

$$2(y-4) + 2\lambda y = 0 \Rightarrow (y-4) + \lambda y = 0 \quad \textcircled{3}$$

$$2(z-12) + 2\lambda z = 0 \Rightarrow (z-12) + \lambda z = 0 \quad \textcircled{4}$$

Multiplying $\textcircled{2}$ by x , $\textcircled{3}$ by y and $\textcircled{4}$ by z
and adding, we get -

$$(x^2 + y^2 + z^2) - (3x + 4y + 12z) + \lambda(x^2 + y^2 + z^2) = 0$$

$$\left[\text{Condition } \Rightarrow x^2 + y^2 + z^2 = 1 \right]$$

$$\cancel{\textcircled{2}} - (3x + 4y + 12z) + \lambda = 0$$

$$\boxed{3x + 4y + 12z = 1 + \lambda} \quad \textcircled{5}$$

from $\textcircled{2}$, $x = \frac{3}{1+\lambda}$
 from $\textcircled{3}$, $y = \frac{4}{1+\lambda}$
 from $\textcircled{4}$, $z = \frac{12}{1+\lambda}$

Putting values of x, y, z in $\textcircled{5}$

$$\frac{9}{1+\lambda} + \frac{16}{1+\lambda} + \frac{144}{1+\lambda} = 1 + \lambda$$

$$9 + 16 + 144 = (1 + \lambda)^2$$

$$(1 + \lambda)^2 = 169$$

$$1 + \lambda = \pm 13$$

$$\lambda = 12 \text{ or } -14$$

when $\lambda = 12 \Rightarrow x = \frac{3}{13}, y = \frac{4}{13}, z = \frac{12}{13}$
 when $\lambda = -14 \Rightarrow x = \frac{-3}{13}, y = \frac{-4}{13}, z = \frac{-12}{13}$

Now,

$$d = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$$

• when $\lambda = 12$, At $(\frac{3}{13}, \frac{4}{13}, \frac{12}{13})$

$$d = \sqrt{\left(\frac{3}{13} - 3\right)^2 + \left(\frac{4}{13} - 4\right)^2 + \left(\frac{12}{13} - 12\right)^2}$$

$$d = \sqrt{\left(\frac{3-39}{13}\right)^2 + \left(\frac{4-52}{13}\right)^2 + \left(\frac{12-156}{13}\right)^2}$$

$$d = \sqrt{\left(\frac{-36}{13}\right)^2 + \left(\frac{-48}{13}\right)^2 + \left(\frac{-144}{13}\right)^2}$$

$$d = \sqrt{\frac{1296}{169} + \frac{2304}{169} + \frac{20736}{169}}$$

$$\boxed{d = 12} \rightarrow \text{minimum distance from sphere.}$$

• when $\lambda = -14$, At $(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13})$

$$d = \sqrt{\left(-\frac{3}{13} - 3\right)^2 + \left(-\frac{4}{13} - 4\right)^2 + \left(-\frac{12}{13} - 12\right)^2}$$

$$d = \sqrt{\left(\frac{-3-39}{13}\right)^2 + \left(\frac{-4-52}{13}\right)^2 + \left(\frac{-12-156}{13}\right)^2}$$

$$d = \sqrt{\left(\frac{-42}{13}\right)^2 + \left(\frac{-56}{13}\right)^2 + \left(\frac{-168}{13}\right)^2}$$

$$d = \sqrt{\frac{1764}{169} + \frac{3136}{169} + \frac{28224}{169}}$$

$$\boxed{d = 14} \rightarrow \text{maximum distance from sphere}$$

Ques - find minimum value of function $u = x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$.

Sol - Using Lagrange's Method

$$f = f + \lambda \phi, \quad \phi = xyz - a^3 \quad \text{--- (1)}$$

$$\Gamma = u + \lambda \phi$$

$$F = x^2 + y^2 + z^2 + \lambda(xyz - a^3)$$

$$\frac{\partial F}{\partial x} = 2x + \lambda yz \quad \frac{\partial F}{\partial y} = 2y + \lambda xz$$

$$\frac{\partial F}{\partial z} = 2z + \lambda xy$$

Now Putting $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ & $\frac{\partial F}{\partial z} = 0$

$$2x + \lambda yz = 0 \quad \text{--- (2)}$$

$$2y + \lambda xz = 0 \quad \text{--- (3)}$$

$$2z + \lambda xy = 0 \quad \text{--- (4)}$$

Multiplying (2) by x , (3) by y & (4) by z and adding, we get -

$$2(x^2 + y^2 + z^2) + 3\lambda xyz = 0$$

$$2u + 3\lambda a^3 = 0 \quad \text{--- (5)}$$

$$\text{from (2), } x = -\frac{\lambda yz}{2}$$

$$\text{from (3), } y = -\frac{\lambda xz}{2}$$

$$\text{from (4), } z = -\frac{\lambda xy}{2}$$

Putting values of x, y, z in (1)

$$xyz = a^3$$

$$-\frac{\lambda^3}{8}(x^2 y^2 z^2) = a^3$$

$$-\frac{\lambda^3}{8}(a^6) = a^3$$

$$\cancel{-\frac{\lambda^3}{8} a^6} = \cancel{a^3}$$

$$\cancel{-\frac{\lambda^3}{8}} = \cancel{a^3}$$

$$-\lambda^3 = \frac{8a^3}{a^2}$$

$$\lambda = -\frac{2}{a}$$

Putting λ in ⑤

$$2u + 3\lambda a^3 = 0$$

$$2u - 3\left(\frac{2}{a}\right)a^3 = 0$$

$$2u = 8a^2$$

$$u = 4a^2$$

Now, we cannot depict whether it is minima or maxima. So we will further calculate by finding d^2F .

$$d^2F = F_{xx}(dx)^2 + F_{yy}(dy)^2 + F_{zz}(dz)^2 + 2(F_{xy}(dx dy) + F_{yz}(dy dz) + F_{zx}(dz dx))$$

$$d^2F = 2(dx)^2 + 2(dy)^2 + 2(dz)^2 + 2(\lambda z dx dy + \lambda y dz dx)$$

Now $2(dx)^2 + 2(dy)^2 + 2(dz)^2$ will always be greater than zero

$$\rightarrow \phi = xyz - a^3$$

$$d^2\phi = \phi_{xx}(dx)^2 + \phi_{yy}(dy)^2 + \phi_{zz}(dz)^2 + 2(\phi_{xy} dx dy + \phi_{yz} dy dz + \phi_{zx} dz dx)$$

$$0 = 0 + 0 + 0 + 2(z dx dy + y dy dz + y dz dx)$$

Multiplying by λ

$$2(\lambda z dx dy + \lambda y dy dz + \lambda y dz dx) = 0$$

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$$\therefore d^2F > 0$$

So, the given function is minimum
if $u = 3a^2$ is its minimum value.