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$\therefore d^2F > 0$

So, the given function is minimum & $u = 3a^2$ is its minimum value.

Ques- Use Lagrange's method to find the minimum value of $x^2+y^2+z^2$ subject to the conditions $x+y+z=1$ and $xyz+1=0$

Sol- Using Lagrange's method-

$$f = f + \lambda\phi$$

$$f = x^2+y^2+z^2+\lambda(x+y+z-1)+\mu(xyz+1) \quad \text{---} (1)$$

$$df = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

$$dF = (2x+\lambda+\mu yz)dx + (2y+\lambda+\mu xz)dy + (2z+\lambda+\mu xy)dz$$

Putting $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$ & $\frac{\partial F}{\partial z}=0$

$$2x+\lambda+\mu yz = 0 \quad \text{---} (2)$$

$$2y+\lambda+\mu xz = 0 \quad \text{---} (3)$$

$$2z+\lambda+\mu xy = 0 \quad \text{---} (4)$$

~~(2) - (3) ⇒ 2(x-y) + μ(yz-xz) = 0~~

Multiplying (2) by x, (3) by y & (4) by z and adding them, we get-

$$2x^2+\lambda x+\mu xyz = 0$$

$$2y^2+\lambda y+\mu xyz = 0$$

$$2z^2+\lambda z+\mu xyz = 0$$

$$2(x^2+y^2+z^2)+\lambda(x+y+z)+3\mu xyz = 0$$

$$2f+\lambda-3\mu = 0$$

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from ② $\Rightarrow$ $x = \frac{-\mu yz - \lambda}{2}$
from ③ $\Rightarrow$ $y = \frac{-\mu xz - \lambda}{2}$
from ④ $\Rightarrow$ $z = \frac{-\mu xy - \lambda}{2}$

Putting in $xyz + 1 = 0$

$$\left(\frac{-\mu yz - \lambda}{2}\right)\left(\frac{-\mu xz - \lambda}{2}\right)\left(\frac{-\mu xy - \lambda}{2}\right) + 1 = 0$$

$$-(\mu yz + \lambda)(+\mu xz + \lambda)(\mu xy + \lambda) + 8 = 0$$

After solving we get -

$x = y = z$ or $\mu = \frac{2}{x} = \frac{2}{y} = \frac{2}{z}$

given, $x + y + z = 1$

$\Rightarrow x = \frac{1}{3}, y = \frac{1}{3}, z = \frac{1}{3}$

& $\mu = \frac{2}{1/3} = 6$

$\therefore$ st. pt is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Also, $d^2f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{xz}\,dx\,dz$

$= 2(dx)^2 + 2(dy)^2 + 2(dz)^2 + 2\mu z\,dx\,dy + 2\mu x\,dy\,dz + 2\mu y\,dz\,dx$

$= 2[dx^2 + dy^2 + dz^2 + \mu z\,dx\,dy + \mu x\,dy\,dz + \mu y\,dx\,dz]$

$= 2[dx + dy + dz)$

$d^2f > 0$

$\therefore f$ is min at $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ &

min value of $x^2 + y^2 + z^2 = \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$

$= \frac{1}{3}$

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* Leibnitz's Linear Differential equation-
A function-
$$\frac{dy}{dx} + Py = Q$$
is said to be Leibnitz's linear differential equation where P & Q are constant or functions of x only.

⇒ Working rule :-

Step I - Convert given equation to the standard form that is $\frac{dy}{dx} + Py = Q$

Step II - find integrating factor (I.F) $I.f = e^{\int P dx}$

Step III Solution is $y(I.f.) = \int Q(I.f.)\,dx + C$

Ques - Solve $(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$

Sol - Divide by $(x+1)$

$$\frac{dy}{dx} - \frac{1}{x+1}\cdot y = e^x(x+1)$$

& this equation is of form-
$\frac{dy}{dx} + Py = Q$ where $P = -\frac{1}{x+1}$
$Q = e^x(x+1)$

Now,
$$I.f = e^{\int P dx} = e^{\int \frac{-1}{x+1}dx} = e^{-\log(x+1)}$$
$$= e^{\log(x+1)^{-1}}$$
$$= (x+1)^{-1}$$

Solution - $y(x+1)^{-1} = \int e^x\cancel{(x+1)}\cancel{(x+1)^{-1}}\,dx + c$

$$y(x+1)^{-1} = e^x + c$$

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$$y = e^x(x+1) + C(x+1)$$

Ques- $(x^2+1)\dfrac{dy}{dx} + y = e^{\tan^{-1}x}$

Sol- Divide by x^2+1

$$\frac{dy}{dx} + \frac{1}{x^2+1}\cdot y = e^{\tan^{-1}x}\cdot\frac{1}{x^2+1}$$

it is of the form -

$$\frac{dy}{dx} + Py = Q \quad \text{where } P = \frac{1}{x^2+1}$$
$$Q = e^{\tan^{-1}x}\,\frac{1}{x^2+1}$$

Now, $I.f = e^{\int P dx} = e^{\int \frac{1}{x^2+1} dx} = e^{\tan^{-1}x}$

Solution $\rightarrow$ $y\left(e^{\tan^{-1}x}\right) = \displaystyle\int \frac{e^{\tan^{-1}x}}{x^2+1}\cdot e^{\tan^{-1}x}\,dx + C$

Using Substitution method,

Put $\tan^{-1}x = t$

$$\frac{1}{1+x^2}dx = dt \quad \text{(only on RHS)}$$

$$y\left(e^{\tan^{-1}x}\right) = \int e^{2t}\,dt + C$$

$$y\,e^{\tan^{-1}x} = \frac{e^{2t}}{2} + C$$

$$2y\,e^{\tan^{-1}x} = e^{2\tan^{-1}x} + C$$

Ques- Solve $\sin x\dfrac{dy}{dx} + 2y = \tan^3\dfrac{x}{2}$

Sol- divide by $\sin x$

$$\frac{dy}{dx} = \frac{2y}{\sin x} = \tan^3\frac{x}{2}\cdot\frac{1}{\sin x}$$

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$* \sin 2x = 2\sin x \cos x \Rightarrow \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

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~~reducesole~~

$$\frac{dy}{dx} + 2y\operatorname{cosec}x = \frac{\sin^{3}\frac{x}{2}}{\cos^3\frac{x}{2}} \times \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$\frac{dy}{dx} + 2y\operatorname{cosec}x = \frac{\sin^2\frac{x}{2}}{2\cos^4\frac{x}{2}}$$

$$\frac{dy}{dx} + 2y\operatorname{cosec}x = \frac{1}{2}\tan^2\frac{x}{2}\cdot\sec^2\frac{x}{2}$$

here $P = 2\operatorname{cosec}x$ & $Q = \frac{1}{2}\tan^2\frac{x}{2}\cdot\sec^2\frac{x}{2}$

Now, $If = e^{\int P dx} = e^{\int 2\operatorname{cosec}x\,dx} = e^{2\log\tan\frac{x}{2}}$

$$If = e^{\log(\tan\frac{x}{2})^2}$$

$$If = \left(\tan\frac{x}{2}\right)^2$$

Solution $\rightarrow y(If) = \int Q(If)\,dx + C$

$$y\tan^2\frac{x}{2} = \int \frac{1}{2}\tan^2\frac{x}{2}\cdot\sec^2\frac{x}{2}\cdot\tan^2\frac{x}{2}\,dx + C$$

$$y\tan^2\frac{x}{2} = \frac{1}{2}\int \tan^4\frac{x}{2}\,\sec^2\frac{x}{2}\,dx + C$$

Using Substitution Method-

Put $\tan\frac{x}{2} = t$

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$$

$$= 2dt$$

$$y\tan^2\frac{x}{2} = \frac{1}{\cancel{2}}\int t^4 \cancel{2}dt + C$$

$$y\tan^2\frac{x}{2} = \frac{1}{\cancel{2}}\cdot\frac{t^5}{5} + C$$

~~$\log \tan^2\frac{x}{2} = \tan^3\frac{x}{2} + C$~~

~~$\log = \tan^3\frac{x}{2} + C$~~

$$\int u \cdot v = u\int v - \int\left(\frac{du}{dt} \cdot \int v\right) dt + C$$

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$$5y \tan^2\frac{x}{2} = \tan^5\frac{x}{2} + C$$

$$5y = \tan^3\frac{x}{2} + C\left(\tan^{-2}\frac{x}{2}\right)$$

Ques- $\sec x \frac{dy}{dx} = y + \sin x$ Solve this.

Sol- $\sec x \frac{dy}{dx} - y = \sin x$

Divide by $\sec x$

$$\frac{dy}{dx} - \frac{y}{\sec x} = \frac{\sin x}{\cos x} \cdot \cos x$$

here $P = \frac{-1}{\sec x}$ & $Q = \sin x \cos x$

Now, $If = e^{\int P dx} = e^{\int \cos x \, dx} = e^{-\sin x \, dx}$

$If = e^{-\sin x \, dx}$

Solution $\rightarrow y(If) = \int Q(If)\, dx + C$

$$y e^{\sin x} = \int \sin x \cos x \, e^{-\sin x} dx + c$$

By using Substitution method
put $\sin x = t$
$\cos x \, dx = dt$

$$y e^{\sin x} = \int t e^{-t} \cdot dt + c$$

using by parts method

$$y e^{\sin x} = t\int e^{-t} dt - \int\left(\frac{d(t)}{dt} \cdot \int e^{-t} dt\right) + C$$

$$= t e^{-t} - \int 1 \cdot e^{-t} + C$$

$$= -t e^{-t} - e^{-t} + C$$

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$$y e^{\sin x} = -\sin x\, e^{-\sin x} - e^{-\sin x} + C$$

$$= e^{-\sin x}(-\sin x - 1) + C$$

$$y e^{\sin x} \cdot e^{\sin x} = -\sin x - 1 + C \cdot e^{\sin x}$$

$$y e^{2\sin x} = -\sin x - 1 + C e^{\sin x}$$

☆ Bernoulli's Equation or Reducible to Leibnitz Linear Differential eq^n (LDE)

Any eq^n of the form

$$\frac{dy}{dx} + Py = Qy^n$$

is said to be Bernoulli's function where P & Q are constant or function of x only.

⇒ Working rule :-

$$\frac{dy}{dx} + Py = Qy^n \quad \text{—— (1)}$$

Step I Divide (1) by y^n

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Step II

Let $y^{1-n} = t$

$$(1-n)\, y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$$

Step III

$$\frac{1}{1-n}\frac{dt}{dx} + Pt = Q$$

$$\frac{dt}{dx} + P(1-n)t = Q(1-n)$$

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Ques- Solve - $x^2 dy + y(x+y)\,dx = 0$

Sol- $x^2 \frac{dy}{dx} + y(x+y) = 0$

$$\frac{dy}{dx} = -\frac{yx+y^2}{x^2}$$

$$\frac{dy}{dx} = -\left[\frac{y}{x} + \frac{y^2}{x^2}\right]$$

$$\frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2} \quad \text{--- (1)}$$

which is of the form-

$$\frac{dy}{dx} + Py = Qy^n \text{ where } P = \frac{1}{x},\ Q = -\frac{1}{x^2}$$

Divide (1) by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = -\frac{1}{x^2} \quad \text{--- (2)}$$

Let $y^{-1} = t$

$$-y^{-2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-2}\frac{dy}{dx} = -\frac{dt}{dx}$$

Put in (2).

$$-\frac{dt}{dx} + \frac{1}{x}t = -\frac{1}{x^2}$$

$$\frac{dt}{dx} - \frac{1}{x}t = +\frac{1}{x^2}$$

which is LDE where $P = -\frac{1}{x}$, $Q = \frac{1}{x^2}$

Now,

$$I.F = e^{\int P dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}}$$

$$I.F. = x^{-1}$$

Solution $\rightarrow$ $t(IF) = \int Q(IF)\,dx + C$

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$$t(x^{-1}) = \int \frac{1}{x^2}\cdot\frac{1}{x}\,dx + C$$

$$t(x^{-1}) = \int \frac{1}{x^3}\,dx + C$$

$$t\left(\frac{1}{x}\right) = \int x^{-3}\,dx + C$$

$$t\left(\frac{1}{x}\right) = \frac{x^{-2}}{-2} + C$$

$$\frac{1}{y}\cdot\frac{1}{x} = \frac{1}{-2x^2} + C$$

$$y^{-1} = \frac{1}{2}x^{-1} + C\cdot x$$

Ques- Solve - $x\frac{dy}{dx} + y\log y = xye^x$

Sol- Divide by x

$$\frac{dy}{dx} + \frac{y}{x}\log y = ye^x \quad \text{—— (1)}$$

Divide (1) by y

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^x$$

let $\log y = t$

$$\frac{1}{y}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + \frac{1}{x}\cdot t = e^x$$

which is LDE where $P = \frac{1}{x}$ & $Q = e^x$

Now, $IF = e^{\int P dx} = e^{\int \frac{1}{x}dx} = e^{\log x}$

$IF = x$

Solution = $t(x) = \int e^x(x)\,dx + C$

using by parts

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$$t(x) = x \cdot \int e^x - \int \left(\frac{d(x)}{dx} \cdot \int e^x dx \right) dx + c$$

$$t(x) = x e^x - \int e^x + c$$

$$t(x) = x e^x - e^x + c$$

$$x \log y = e^x (x-1) + c$$

Solve - $y \log y \, dx + (x - \log y) dy = 0$

$$y \log y \frac{dx}{dy} + x - \log y = 0$$

$$\frac{dx}{dy} = \frac{\log y - x}{y \log y}$$

$$\frac{dx}{dy} = \left[\frac{1}{y} - \frac{x}{y \log y} \right]$$

$$\frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$$

$$\frac{dx}{dy} + \frac{1}{y \log y} \cdot x = \frac{1}{y}$$

which is LDE where $P = \frac{1}{y \log y}$ & $Q = \frac{1}{y}$

Now,

$$If = e^{\int P dy} = e^{\int \frac{1}{y \log y} dy}$$

$$= \int \frac{1}{y} \cdot \frac{1}{\log y} dy$$

let $\log y = t$

$$\frac{1}{y} dy = dt$$

$$= \int \frac{1}{t} dt = \log t = \log(\log y)$$

$$\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1}$$

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$$I.f = e^{\log(\log y)}$$
$$If = \log y$$

Solution $\rightarrow$ $x(If) = \int Q\,(If) + C$

$$x(\log y) = \int \frac{1}{y} \log y\,dy + C$$

let $\log y = z$

$$\frac{1}{y}dy = dz$$

$$x \log y = \int z\,dz + C$$
$$x \log y = \frac{z^2}{2} + C$$
$$x \log y = \frac{(\log y)^2}{2} + C$$
$$2x = \log y + C$$

Ques-Sol- Solve - $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$
$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$
$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$$

which is LDE where $P = \frac{1}{1+y^2}$ & $Q = \frac{\tan^{-1}y}{1+y^2}$

Now,

$$If = e^{\int P\,dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$$
$$If = e^{\tan^{-1}y}$$

Solution $\rightarrow$ $x(If) = \int Q\,(If)\,dy + C$

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$$xe^{\tan^{-1}x} = \int \frac{\tan^{-1}x}{1+y^2} \cdot e^{\tan^{-1}x}\, dy + C$$

Put $\tan^{-1}y = t$

$$\frac{1}{1+y^2}\, dy = dt$$

$$xe^{\tan^{-1}x} = \int t \cdot e^{t}\, dt + c$$

By using by parts

$$xe^{\tan^{-1}x} = t \cdot e^t - \int 1 \cdot e^t\, dt + c$$

$$xe^{\tan^{-1}x} = te^t - e^t + c$$

$$xe^{\tan^{-1}x} = \tan^{-1}x\, e^{\tan^{-1}x} - e^{\tan^{-1}x} + c$$

$$x\cancel{e^{\tan^{-1}x}} = \cancel{e^{\tan^{-1}x}}(\tan^{-1}x - 1) + c$$

$$x = \tan^{-1}x - 1 + c\, e^{-\tan^{-1}x}$$

* Exact Differential Equation

[for a function, $U(x,y) = 0$
& $Mdx + Ndy = 0$ will be its exact derivative.
and the given equation will be exact differential equation.]

Or

Any differential equation of the form
$$Mdx + Ndy = 0$$
is said to be Exact differential equation if it can be obtained by differenting equation
$$U(x,y) = c$$
i.e. if $dU = Mdx + Ndy$.

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The necessary and sufficient condition for a differential equation to be exact is:-

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

then its solution will be -

$$\int_{y\text{-constant}} M\,dx + \int (\text{Terms of } N \text{ not cont } x)\,dy = C$$

Ques Solve $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$

Sol- The given eq^n is of the form-

$$M\,dx + N\,dy = 0$$

where $M = 5x^4 + 3x^2y^2 - 2xy^3$.

$N = 2x^3y - 3x^2y^2 - 5y^4$

Now, $\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$.

$\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so we can say that the given eq^n is exact differential equation.

$\therefore$ its solution is :-

$$\int_{y\text{ constant}} M\,dx + \int (\text{Terms of } N \text{ not containing } x)\,dy = c$$

$$\int (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int -5y^4\,dy = c$$

$$\frac{\cancel{5}x^5}{\cancel{5}} + \frac{\cancel{3}x^3}{\cancel{3}}y^2 - \frac{\cancel{2}x^2}{\cancel{2}}y^3 - \frac{\cancel{5}y^5}{\cancel{5}} = C$$

$$x^5 + x^3y^2 - x^2y^3 - y^5 = C$$

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Ques:- Solve $(\cos x \tan y + \cos(x+y))dx + (\sin x \sec^2 y + \cos(x+y))dy = 0$

Sol- the given eq^n is of the form-

$$Mdx + Ndy = 0$$

where $M = \cos x \tan y + \cos(x+y)$

$N = \sin x \sec^2 y + \cos(x+y)$

Now,

$$\frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x+y)$$

$$\frac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x+y)$$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, So given eq^n is exact diff. eq^n.

$\therefore$ solution is $\rightarrow$

$$\int_{y\ \text{constant}} (\cos x \tan y + \cos(x+y))dx + \int 0\,dy = c$$

$$\sin x \tan y + \sin(x+y) + c = c$$

$$\sin x \tan y + \sin(x+y) = c$$

* Reducible to Exact differential equation

If the given eq^n is not exact differential eq^n then we will find an integrating factor and then multiply it with the given eq^n to make the equation $\rightarrow$ Exact differential eq^n.

$\Rightarrow$ By inspection-

Ques:- Solve $y\,dx - x\,dy +$ [illegible] $= 0$

Sol- $(y +$ [illegible]$)dx + (-x)dy = 0$

$M = y +$ [illegible]

$N = -x$

Now, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

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So, we will find Integrating factor

$$\text{If} = \frac{1}{y^2}$$

Now, multiplying $\frac{1}{y^2}$ both sides in eq^n $y\,dx - x\,dy = 0$

$$\frac{1}{y}dx - \frac{x}{y^2}dy = 0$$

here, $M = 1/y$ and $N = -x/y^2$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \quad \frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Now, solution is $\longrightarrow$

$$\int \frac{1}{y}dx - \int 0\,dy = c$$

$$\frac{x}{y} = c$$

Ques- Sol- Solve $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$

$$(y + 3x^2y^2e^{x^3})dx + (-x)dy = 0 \quad \text{—— (1)}$$

here, $M = y + 3x^2y^2e^{x^3}$ and $N = -x$

$$\frac{\partial M}{\partial y} = 1 + 6x^2ye^{x^3} \quad \& \quad \frac{\partial N}{\partial x} = -1$$

here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so we have to inspect If

$$\text{If} = 1/y^2$$

Multiplying both sides with $1/y^2$ to (1)

$$\left(\frac{1}{y} + 3x^2e^{x^3}\right)dx - \frac{x}{y^2}dy = 0$$

here, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

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∴ solution is →

$$\int\left(\frac{1}{y}+3x^2e^{x^3}\right)dx + \int 0\,dy = C$$

$$x^3 \cdot e^{x^3}$$

Rule-I

for any equation

$$Mdx + Ndy = 0$$

where M and N are homogeneous then integrating factor will be-

$$If = \frac{1}{Mx+Ny}$$

Ques:- Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$

Sol:- The given eq^n is homogenous in x & y.

$$(x^2y)dx - (x^3+y^3)dy = 0 \quad \text{--- (1)}$$

where $M = x^2y$

$$N = x^3 - y^3$$

$$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = 3x^2$$

$$\text{Now, } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

∴ given eq^n is not exact

We can find If acc. to the Rule I.

$$If = \frac{1}{(x^2y)x - (x^3+y^3)y}$$

$$If = \frac{1}{x^3y - x^3y + y^4} = \frac{-1}{y^4}$$

Now Multiplying If in (1)

$$\frac{-1}{y^4}(x^2y)\,dx + \frac{1}{y^4}(x^3+y^3)\,dy = 0$$

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$$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

where $M = -\frac{x^2}{y^3}$, $N = \frac{x^3}{y^4} + \frac{1}{y}$

$\frac{\partial M}{\partial y} = \frac{3x^2}{y^4}$ $\quad\frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

∴ solution is →

$$\int_{y\ \text{constant}} M\,dx - \int N(\text{not cont } x)\,dy = c$$

$$-\int \frac{x^2}{y^3}\,dx - \int \frac{1}{y}\,dy = c$$

$$-\frac{x^3}{3y^3} - \log y = c$$

Rule II

for any equation

$$M\,dx + N\,dy = 0$$

is in the form -

$$(f_1(x,y))\,y\,dx + (f_2(x,y))\,x\,dy = 0$$

then integrating factor will be -

$$If = \frac{1}{Mx - Ny}$$

Ques- Solve $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$ — ①

Sol- here, $M = x^2y^3 + xy^2 + y$, $N = x^3y^2 - x^2y + x$

$\frac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1$, $\frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

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Now, the given eq^n is of the form-

$$(f_1(x,y))\,y\,dx + (f_2(x,y))\,x\,dy = 0$$

So, $\text{If} = \dfrac{1}{Mx - Ny}$

$$\text{If} = \frac{1}{\cancel{x^3y^3} + x^2y^2 + \cancel{xy} - \cancel{x^3y^3} + x^2y^2 - \cancel{xy}}$$

$$\text{If} = \frac{1}{2x^2y^2}$$

Multiplying ① by $\dfrac{1}{2x^2y^2}$

$$\frac{1}{2x^2y^2}(x^2y^2+xy+1)\,y\,dx + \frac{1}{2x^2y^2}(x^2y^2 - xy + 1)\,x\,dy = 0 \quad \text{—②}$$

$$\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \left(\frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2}\right)dy = 0$$

Now, $M = \dfrac{y}{2} + \dfrac{1}{2x} + \dfrac{1}{2x^2y}$, $N = \dfrac{x}{2} - \dfrac{1}{2y} + \dfrac{1}{2xy^2}$

$$\frac{\partial M}{\partial y} = \frac{1}{2} - \frac{1}{2x^2y^2}, \quad \frac{\partial N}{\partial x} = \frac{1}{2} - \frac{1}{2x^2y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So eq^n ② is exact.

∴ solution is-

$$\cancel{\int\left(\frac{1}{2} - \frac{1}{2x^2y^2}\right)dx - \int \frac{x}{2}\,dy = c}$$

$$\int\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \int \frac{1}{2y}\,dy = c$$

$$\frac{y}{2}x + \frac{1}{2}\log x + \frac{1}{2y}\left(\frac{x^{-1}}{-1}\right) - \frac{1}{2}\log y = c$$

$$\frac{xy}{2} - \frac{x}{2y} + \frac{1}{2}(\log x - \log y) = c$$

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Ques. Solve - $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$ —(1)

Sol. $M = y^3 - 2yx^2$, $N = 2xy^2 - x^3$

$\frac{\partial M}{\partial y} = 3y^2 - 2x^2$, $\frac{\partial N}{\partial x} = 2y^2 - 3x^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Now, the given eq^n is not exact and of the form

$$(f_1(x,y))\,y\,dx + (f_2(x,y))\,x\,dy = 0$$

$\Rightarrow (y^2 - 2x^2)\,y\,dx + (2y^2 - x^2)\,x\,dy = 0$ —(2)

So, $IF = \frac{1}{Mx - Ny}$

$= \frac{1}{(y^3 - 2x^2y)x - (2y^2x - x^3)y}$

$= \frac{1}{xy^3 - 2x^3y - 2xy^3 + x^3y}$

$= \frac{1}{-xy^3 - x^3y} = \frac{-1}{(xy^3 + x^3y)}$

$= \frac{-1}{xy(y^2 + x^2)}$

Multiplying (2) by IF

$$\frac{-1}{xy(x^2+y^2)}(y^3 - 2x^2y)\,dx - \frac{1}{xy(x^2+y^2)}(2y^2 - x^2)x\,dy = 0$$

$$\frac{2x^2 - y^2}{x(x^2+y^2)}dx + \frac{(x^2 - 2y^2)}{y(x^2+y^2)}\,dy = 0$$

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Rule III

for any equation,

$$Mdx + Ndy = 0$$

where, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

then calculate - $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ = function of x $[f(x)]$

So, $If = e^{\int f(x)dx}$

Rule IV

Or

calculate - $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ = function of y $[f(y)]$

So, $If = e^{\int f(y)dy}$

Ques- Solve $\Rightarrow (x^4e^x - 2mxy^2)\,dx + 2mx^2y\,dy = 0$ — (I)

Sol- here $M = (x^4e^x - 2mxy^2)$, $N = 2mx^2y$

$$\frac{\partial M}{\partial y} = -4mxy \qquad \frac{\partial N}{\partial x} = 4mxy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\text{III} \Rightarrow \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4mxy - 4mxy}{N}$$

$$= \frac{-\cancel{8}^{4}\cancel{mxy}}{\cancel{2}\cancel{mx^2y}} = \frac{-4}{x} = f(x)$$

$$If = e^{\int f(x)dx} = e^{-4\int \frac{1}{x}dx} = e^{-4\log x} = e^{\log x^{-4}}$$

$$If = x^{-4} = \frac{1}{x^4}$$

Now, Multiplying (I) by If i.e. x^{-4}

$$\left(\frac{x^4e^x}{x^4} - \frac{2mxy^2}{x^{4\,3}}\right)dx + \left(\frac{2mx^2y}{x^{4\,2}}\right)dy = 0$$

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$$\left(e^x - \frac{2my^2}{x^3}\right)dx + \frac{2my}{x^2}dy = 0$$

here, $M = e^x - \frac{2my^2}{x^3}$, $N = \frac{2my}{x^2}$

$$\frac{\partial M}{\partial y} = -\frac{4my}{x^3} , \quad \frac{\partial N}{\partial x} = -\frac{4my}{x^3}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ its solution is $\rightarrow$

$$\int_{y\ \text{constant}} M dx + \int (\text{Terms of } N \text{ not containg } x)\, dy = C$$

$$\int \left(e^x - \frac{2my^2}{x^3}\right)dx + \int 0\, dy = C$$

$$e^x - 2my^2 \int x^{-3} dx + 0 = C$$

$$e^x - \cancel{2}my^2 \left[\frac{x^{-2}}{-\cancel{2}}\right] = C$$

$$e^x + \frac{my^2}{x^2} = C$$

Ques- Solve - $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

Sol- here $M = xy^3 + y$, $N = 2x^2y^2 + 2x + 2y^4$

$$\frac{\partial M}{\partial y} = 3xy^2 + 1 \qquad \frac{\partial N}{\partial x} = 4xy^2 + 2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Now, acc. to Rule IV -

$$f(x) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4xy^2 + 2 - 3xy^2 - 1}{xy^3 + y}$$

$$f(x) = \frac{xy^2 + 1}{y(xy^2 + 1)} = \frac{1}{y}$$

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$$If = e^{\int f(y)dy} = e^{\int \frac{1}{y} dy} = \cancel{e}^{\log y}$$

$$If = y$$

Multiplying ① by If

$$(xy^4 + y^2)\,dx + 2(x^2y^3 + xy + y^5)\,dy = 0$$

here, $M = xy^4 + y^2$, $N = 2x^2y^3 + 2xy + 2y^5$

$$\frac{\partial M}{\partial y} = 4xy^3 + 2y, \quad \frac{\partial N}{\partial x} = 4xy^3 + 2y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

∴ solution is →

$$\int_{y\ \text{constant}} M\,dx + \int (\text{Terms of } N \text{ not containing } x)\,dy = C$$

$$\int (xy^4 + y^2)\,dx + \int 2y^5\,dy = C$$

$$y^2 \int (xy^2 + 1)\,dx + \frac{2y^6}{\cancel{6}\,3} = C$$

$$y^2\left[\frac{x^2}{2}y^2 + x\right] + \frac{y^6}{3} = C$$

$$\frac{x^2y^4 + 2xy^2}{2} + \frac{y^6}{3} = C$$

$$\frac{3x^2y^4 + 6xy^2 + 2y^6}{6} = C$$

$$3x^2y^4 + 6xy^2 + 2y^6 = C$$

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Ques:- Solve- $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$ —— (1)

here $M = xy^2 - e^{1/x^3}$, $N = -x^2y$

$$\frac{\partial M}{\partial y} = 2xy \qquad \frac{\partial N}{\partial x} = -2xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Acc to Rule III -

$$f(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy + 2xy}{-x^2y}$$

$$= \frac{4xy}{-x^2y} = \frac{-4}{x}$$

$\therefore$ $If = e^{\int f(x)dx} = e^{-4\int \frac{1}{x}dx} = e^{-4\log x}$

$$If = x^{-4} = \frac{1}{x^4}$$

Multiplying (1) by If

$$\left(\frac{xy^2}{x^{4}} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{x^2y}{x^{4}}dy = 0$$

$$\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}dy = 0$$

here, $M = \frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}$, $N = \frac{-y}{x^2}$

$$\frac{\partial M}{\partial y} = \frac{2y}{x^3} \; ; \; \frac{\partial N}{\partial x} = \frac{2y}{x^3}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ solution is $\rightarrow$

$$\int\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx + \int 0\,dy = C$$

$$y^2\frac{x^{-2}}{-2} - \int e^{1/x^3}\cdot x^{-4}\,dx = C$$

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$$-\frac{y^2}{2x^2} - \frac{1}{3}\int e^{1/x^3} 3x^{-4}\,dx = c$$

Put $\frac{1}{x^3} = t$

$$-3x^{-4}\,dx = dt$$

$$-\frac{y^2}{2x^2} + \frac{1}{3}\int e^t\,dt = c$$

$$-\frac{y^2}{2x^2} + \frac{1}{3}e^t = c$$

$$-\frac{y^2}{2x^2} + \frac{1}{3}e^{1/x^3} = c$$

Ques- Solve $(3x^2y^3e^y + y^3 + y^2)\,dx + (x^3y^3e^y - xy)\,dy = 0$

Sol- here, $M = 3x^2y^3e^y + y^3 + y^2$, $N = x^3y^3e^y - xy$ — (1)

$$\frac{\partial M}{\partial y} = 9x^2y^2e^y + 3y^2 + 2y + 3x^2y^3e^y, \quad \frac{\partial N}{\partial x} = 3x^2y^3e^y - y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Acc. to Rule III -

$$f(y) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \cancel{\frac{3x^2y^3e^y - y - 9x^2y^2e^y - 3y^2 - 2y}{3x^2y^3e^y + y^3 + y^2}}$$

$$= \cancel{\frac{-6x^2y^3e^y - 3y^2 - 3y}{3x^2y^3e^y + y^3 + y^2}}$$

$$= \frac{9x^2y^2e^y + \cancel{3x^2y^3e^y} + 3y^2 + 2y - \cancel{3x^2y^3e^y} + y}{x^3y^3e^y - xy}$$

$$= \frac{9x^2y^2e^y + 3y^2 + 3y}{x^3y^3e^y - xy}$$ Not Possible

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So, $f(y) = \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \dfrac{-3}{y}$, $If = e^{\int f(y)\,dy} = y^{-3} = \dfrac{1}{y^3}$

Multiplying ① by If $\rightarrow \underbrace{\left(3x^2e^y + 1 + \frac{1}{y}\right)}_{M} dx + \underbrace{\left(x^3e^y - \frac{x}{y^2}\right)}_{N} dy = 0$ —— ②

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ ② is exact

Solution $\rightarrow \int \left(3x^2e^y + 1 + \frac{1}{y}\right) dx + \int 0\, dy = C$

$\boxed{x^3e^y + x + \frac{x}{y} = C}$

Rule V

If the equation is of the form

$x^m y^n (ay\,dx + bx\,dy) + x^{m'} y^{n'} (a'y\,dx + b'x\,dy)$

then integrating factor will be -

$If = x^h y^k$

Calculation of h & k $\rightarrow$

$\dfrac{m+h+1}{a} = \dfrac{n+k+1}{b}$ —— ①

$\dfrac{m'+h+1}{a'} = \dfrac{n'+k+1}{b'}$ —— ②

Ques- Solve- $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$ —— ①

Sol- ~~$y^3dx + 2xy^2dy + (-2x^2y\,dx - x^3)\,dx$~~

$(y^3\,dx + 2xy^2\,dy) + (-2x^2y\,dx - x^3\,dy) = 0$

$y^2(y\,dx + 2x\,dy) + x^2(-2y\,dx - x\,dy) = 0$

Comparing it with-

$x^m y^n (ay\,dx + bx\,dy) + x^{m'} y^{n'} (a'y\,dx + b'x\,dy)$

where,

$m = 0,\ n = 2,\ a = 1,\ b = 2$

$m' = 2,\ n' = 0,\ a' = -2,\ b' = -1$

Now calculating h & k

$\dfrac{m+h+1}{a} = \dfrac{n+k+1}{b}$

$\dfrac{h+1}{1} = \dfrac{3+k}{2}$

$2h + 2 = 3 + k$

$2h = k + 1$

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$2h - K = 1$ ——— (2)

$\frac{m' + h + 1}{a'} = \frac{n' + K + 1}{b'}$

$\frac{2 + h + 1}{+2} = \frac{K + 1}{+1}$

$3 + h = 2K + 2$

$2K - h = 1$ ——— (3)

~~from (1), 2h = K~~

~~h = K/2~~

~~Put in (2)~~

~~$2\left(\frac{K}{2}\right) - K = 1$~~

~~4K - K = 2~~

from (2) & (3)

$2h - K = 1$

$2K - h = 1$

$$\begin{aligned} \cancel{2h} - K &= 1 \\ \cancel{2h} - 4K &= -2 \\ \hline +3K &= 3 \end{aligned}$$

$\boxed{K = 1}$, $\boxed{h = 1}$

$\therefore$ If $= x^h y^K = xy$

Now multiplying (1) by If

$(xy^4 - 2x^3y^2)\,dx + (2x^2y^3 - x^4y)\,dy = 0$

here, $M = xy^4 - 2x^3y^2$, $N = 2x^2y^3 - yx^4$

$\frac{\partial M}{\partial y} = 4xy^3 - 4x^3y$, $\frac{\partial N}{\partial x} = 4xy^3 - 4x^3y$

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hence, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ solution is $\rightarrow$

$$\int_{y\text{ constant}} M dx + \int(\text{Terms of } N \text{ not containing } x)\, dy = C$$

$$\int (xy^4 - 2x^3y^2) dx + \int 0\, dy = C$$

$$\frac{x^2}{2}y^4 - 2\frac{x^4}{4_2}y^2 = C$$

$$\frac{x^2y^4 - x^4y^2}{2} = C$$

$$x^2y^4 - x^4y^2 = C$$

Ques: Solve - $(3y - 2xy^3)\, dx + (4x - 3x^2y^2)\, dy = 0$ — (1)

Sol- $M = 3y - 2xy^3$, $N = 4x - 3x^2y^2$

$\frac{\partial M}{\partial y} = 3 - 6xy^2$, $\frac{\partial N}{\partial x} = 4 - 6xy^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

from (1)

$$(\boxed{3y\, dx} - 3x^2y^2 dy) + (-2xy^3 dx + 4x\, dy) = 0$$

3dx but we want ydx here

$$(3y\, dx + 4x\, dy) + (-2xy^3 dx - 3x^2y^3 dy) = 0$$

$$x^0y^0(3y\, dx + 4x\, dy) + xy^2(-2y\, dx - 3x\, dy) = 0$$

here, $m = 0$, $n = 0$, $a = 3$, $b = 4$

$m' = 1$, $n' = 2$, $a = -2$, $b = -3$

(after comparing with eq^n from Rule V)

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$$\frac{m+h+1}{a} = \frac{n+k+1}{b}$$

$$\frac{h+1}{3} = \frac{k+1}{4}$$

$$4h+4 = 3k+3$$

$$4h-3k = -1 \quad \text{——— (1)}$$

$$\frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

$$\frac{1+h+1}{2} = \frac{2+k+1}{3}$$

$$\frac{2+h}{2} = \frac{3+k}{3}$$

$$6+3h = 6+2k$$

$$3h-2k = 0 \quad \text{——— (2)}$$

from (1) & (2)

$$4h-3k = -1 \quad \times 3$$

$$3h-2k = 0 \quad \times 4$$

$$12h-9k = -3$$

$$12h-8k = 0$$

$$- \quad + \quad -$$

$$+k = +3$$

$$\boxed{k=3} \quad \boxed{h=2}$$

$\therefore$ If $= x^2y^3$

Multiplying If to (1) -

$$(\underbrace{3x^2y^4 - 2x^3y^6}_{M})dx + (\underbrace{4x^3y^3 - 3x^4y^5}_{N})dy = 0$$

$$\frac{\partial M}{\partial y} = 12x^2y^3 - 12x^3y^5, \quad \frac{\partial N}{\partial x} = 12x^2y^3 - 12x^3y^5$$

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$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

hence the eq^n is exact now :-

$\therefore$ the solution is -

$$\int (3x^2y^4 - 2x^3y^6)dx + \int 0\, dy = C$$

$$\cancel{3}\frac{x^3}{\cancel{3}}y^4 - \cancel{2}\frac{x^4}{\cancel{4}_2}y^6 = C$$

$$2x^3y^4 - x^4y^6 = C$$

Ques- Solve $(\overset{M}{2x^2y^2 + y})dx + (\overset{N}{3x - x^3y})dy = 0$ — (1)

Sol- $\frac{\partial M}{\partial y} = 4x^2y + 1 \qquad \frac{\partial N}{\partial x} = 3 - 3x^2y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So, the given eq^n is not exact.

from (1)

$$(y\,dx + 3x\,dy) + (2x^2y^2dx - x^3y\,dy) = 0$$

$$x^0y^0(ydx + 3xdy) + x^2y\,(2ydx - xdy) = 0$$

Comparing it with-

$$x^my^n(aydx + bxdy) + x^{m'}y^{n'}(a'ydx + b'xdy)$$

where

$m = 0,\ n = 0,\ a = 1,\ b = 3$

$m' = 2,\ n' = 1,\ a' = 2,\ b' = -1$

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Now,

$$\frac{m+h+1}{a} = \frac{n+k+1}{b}$$

$$\frac{h+1}{1} = \frac{k+1}{3}$$

$$3h+3 = k+1$$

$$3h-k = -2 \quad \text{—— (2)}$$

$$\frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

$$\frac{2+h+1}{2} = \frac{1+k+1}{-1}$$

$$-3-h = 4+2k$$

$$3+h = -4-2k$$

$$h+2k = -4-3$$

$$h+2k = -7 \quad \text{—— (3)}$$

from (2) & (3)

$$3h-k = -2 \quad \times 2$$

$$h+2k = -7$$

$$6h-2k = -4$$

$$h+2k = -7$$

$$7h = -11$$

$$\boxed{h = -11/7} \qquad \boxed{k = -\frac{19}{7}}$$

Now, $IF = x^h y^k = x^{-11/7} y^{-19/7}$

Multiplying IF to eqn (1)

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$$\cancel{\left(\underbrace{2x^{-1/5}y^{-13/5} + x^{-11/5}y^{-18/5}}_{M}\right)dx + \left(\underbrace{3x^{-6/5}y^{-23/5} - x^{4/5}y^{-18/5}}_{N}\right)dy = 0}$$

$$\cancel{\frac{\partial M}{\partial y} = \left(\frac{-13}{5}\right) \times 2x^{-1/5}y^{-18/5} + \left(\frac{-18}{5}\right)x^{-11/5}y^{-23/5}}$$

$$\cancel{\frac{\partial N}{\partial x} = \left(\frac{-6}{5}\right)3x^{-11/5}y^{-23/5} - \frac{4}{5}x^{-1/5}y^{-18/5}}$$

$$\left(\underbrace{2x^{3/7}y^{-5/7} + x^{-11/7}y^{-12/7}}_{M}\right)dx + \left(\underbrace{3x^{-4/7}y^{-19/7} - x^{10/7}y^{-12/7}}_{N}\right)dy = 0 \quad (2)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{-12}{7}x^{-11/7}y^{-19/7} - \frac{10}{7}x^{3/7}y^{-12/7}$$

Hence eq^n (2) is ~~equ~~ exact.

∴ its solution is:-

$$\int\left(2x^{3/7}y^{-5/7} + x^{-11/7}y^{-12/7}\right)dx + \int 0 \cdot dy = C$$

$$2\frac{x^{10/7}}{10/7}y^{-5/7} + \frac{x^{-4/7}}{-4/7}y^{-12/7} = C$$

$$\frac{7}{5}x^{10/7}y^{-5/7} - \frac{7}{4}x^{-4/7}y^{-12/7} = C$$

$$\frac{1}{5}x^{10/7}y^{-5/7} - \frac{1}{4}x^{-4/7}y^{-12/7} = C$$

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Differential equation of first and higher order

Order → highest derivative
degree → power of highest derivative

⇒ Any equation of the form $p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots - - P_n p^0 = 0$ where $p = \frac{dy}{dx}$ and $P_1, P_2 - - - P_n$ are functions of x and y

Working Rule to solve differential eqⁿ of first or higher order:-

* Solvable for $p = \frac{dy}{dx}$:-

Step1 Write eqⁿ in p.

Step2 find p

Step3 Now put $p = \frac{dy}{dx}$ and solve the differential equation (by using variable seperable or lebinitz).

Step4 Then find the solution for the equations obtained in step 3.

So let the solutions are-

$$a_1 x + b_2 = c$$
$$a_2 x + b_2 = c$$

Step5 Its general solution is-

$$(a_1 x + b_2 - c)(a_2 x + b_2 - c) = 0$$

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Ques- Solve $p^2 - 7p + 12 = 0$

Sol- $p^2 - 7p + 12 = 0$

$p^2 - 4p - 3p + 12 = 0$

$p(p-4) - 3(p-4) = 0$

$(p-3)(p-4) = 0$

$p = 3, 4$

Or

$$p = \frac{7 \pm \sqrt{49-48}}{2}$$

$$p = \frac{7 \pm 1}{2} = 3, 4$$

$p = 4$ $\qquad\qquad$ $p = 3$

According to step 3

putting $p = \frac{dy}{dx}$

$\frac{dy}{dx} = 4$	$\frac{dy}{dx} = 3$
$\int dy = \int 4\,dx$	$\int dy = \int 3\,dx$
$y = 4x + C$	$y = 3x + C$
$y - 4x - C = 0$	$y - 3x - C = 0$

$\therefore$ its general solution is -

$$(y - 4x - c)(y - 3x - c) = 0$$

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Ques- Solve- $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

Sol- The given eq^n is -

$$x^2p^2 + xyp - 6y^2 = 0 \quad \left(\because p = \frac{dy}{dx}\right)$$

$$p = \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2}$$

$$p = \frac{-xy \pm 5xy}{2x^2}$$

$$p = \frac{4xy}{2x^2} \quad \text{or} \quad \frac{-6xy}{2x^2}$$

$$p = \frac{2y}{x} \quad \text{or} \quad p = \frac{-3y}{x}$$

$\frac{dy}{dx} = \frac{2y}{x}$	$\frac{dy}{dx} = \frac{-3y}{x}$
$\int \frac{1}{2y}dy = \int \frac{1}{x}dx$	$-\int \frac{1}{3y}dy = \int \frac{1}{x}dx$
$\frac{1}{2}\log y = \log x + \log c$	$-\frac{1}{3}\log y = \log x + \log c$
$\log y = 2\log x + \log c$	$\log y = -3\log x + \log c$
$\log y - \log x^2 = \log c$	$\log y + \log x^3 = \log c$
$\log \frac{y}{x^2} = \log c$	$\log yx^3 = \log c$
$\frac{y}{x^2} = c$	$yx^3 = c$
$\frac{y}{x^2} - c = 0$	$yx^3 - c = 0$
$y - cx^2 = 0$	

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$\therefore$ its general solution is -

$$(y - cx^2)(yx^3 - c) = 0$$

Ques - $p^2 + 2py\cot x = y^2$

Sol - $p^2 + 2py\cot x - y^2 = 0$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\sqrt{(1+\cot^2 x)}}{2}$$

$$p = \frac{\cancel{2}(-y\cot x \pm y\operatorname{cosec} x)}{\cancel{2}}$$

$$p = -y\cot x \pm y\operatorname{cosec} x$$

$$p = y(-\cot x + \operatorname{cosec} x) \quad \text{or} \quad p = y(-\cot x - \operatorname{cosec} x)$$

$$\Rightarrow \frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$$

$$\int \frac{dy}{y} = \int (\operatorname{cosec} x - \cot x)\, dx$$

$$\log y = +\log \tan\frac{x}{2} - \log \sin x + \log c$$

$$\log y + \log \sin x - \log \tan\frac{x}{2} = \log c$$

$$\frac{y \sin x}{\tan x/2} = c$$

$$y = \frac{c \tan x/2}{\sin x}$$

$$y = \frac{c\,\frac{\sin x/2}{\cos x/2}}{2\sin x/2 \cos x/2}$$

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$$y = \frac{c}{2} \sec^2 \frac{x}{2}$$

$$y - \frac{c}{2} \sec^2 \frac{x}{2} = 0$$

or

$$y = \frac{c}{2\cos^2 x/2}$$

$$y = \frac{c}{1+\cos x}$$

$$\boxed{y - \frac{c}{1+\cos x} = 0}$$

$$\Rightarrow \frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$$

$$\int \frac{dy}{y} = -\int (\cot x + \operatorname{cosec} x)\, dx$$

$$\log y = -\left[\log \sin x + \log \tan \frac{x}{2}\right] + \log c$$

$$\log y + \log \sin x + \log \tan \frac{x}{2} = \log c$$

$$y \sin x \tan \frac{x}{2} = c$$

$$y = \frac{c}{2\sin \frac{x}{2} \cos x/2 \cdot \frac{\sin x/2}{\cos x/2}}$$

$$y = \frac{c}{2\sin^2 x/2}$$

$$y = \frac{c}{1-\cos x}$$

$$\boxed{y - \frac{c}{1-\cos x} = 0}$$

$\therefore$ its general solution is -

$$\left(y - \frac{c}{1+\cos x}\right)\left(y - \frac{c}{1-\cos x}\right) = 0$$

Ques- Solve- $p^2 - 2p\sinh x - 1 = 0$

$* \ \sinh x = \frac{e^x - e^{-x}}{2}$

$* \ \cosh x = \frac{e^x + e^{-x}}{2}$

Sol.- $p^2 - 2p\left(\frac{e^x - e^{-x}}{2}\right) - 1 = 0$

$p^2 - p\left(e^x - e^{-x}\right) - 1 = 0$

$p = \frac{(e^x - e^{-x}) \pm \sqrt{(e^x - e^{-x})^2 + 4}}{2}$

$p = \frac{(e^x - e^{-x}) \pm \sqrt{e^{2x} + e^{-2x} - 2e^x e^{-x} + 4}}{2}$

$p = \frac{e^x - e^{-x} \pm \sqrt{e^{2x} + e^{-2x} + 2}}{2}$

$p = \frac{e^x - e^{-x} \pm \sqrt{(e^x + e^{-x})^2}}{2}$

$p = \frac{(e^x - e^{-x}) \pm (e^x + e^{-x})}{2}$

$p = \frac{2e^x}{2}$ or $p = \frac{-2e^{-x}}{2}$

$p = e^x$ or $p = -e^{-x}$

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Now,

$\frac{dy}{dx} = e^{x}$	$\frac{dy}{dx} = -e^{-x}$
$\int dy = \int e^{x} dx$	$\int dy = -\int e^{-x} dx$
$y = e^{x} + c$	$y = e^{-x} + c$
$y - e^{x} - c = 0$	$y - e^{-x} - c = 0$

$\therefore$ its general solution is -

$$(y - e^{x} - c)(y - e^{-x} - c) = 0$$

Ques- Solve $\Rightarrow xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$

Sol-

$$p = \frac{-(3x^2 - 2y^2) \pm \sqrt{(3x^2-2y^2)^2 + 24x^2y^2}}{2xy}$$

$$p = \frac{2y^2 - 3x^2 \pm \sqrt{9x^4 + 4y^4 - 12x^2y^2 + 24x^2y^2}}{2xy}$$

$$p = \frac{2y^2 - 3x^2 \pm \sqrt{9x^4 + 4y^4 + 12x^2y^2}}{2xy}$$

$$p = \frac{2y^2 - 3x^2 \pm \sqrt{(3x^2 + 2y^2)^2}}{2xy}$$

$$p = \frac{(2y^2 - 3x^2) \pm (3x^2 + 2y^2)}{2xy}$$

$$p = \frac{\cancel{2y^2} - 3x^2 - 3x^2 - \cancel{2y^2}}{2xy} \quad \text{or} \quad \frac{2y^2 - \cancel{3x^2} + \cancel{3x^2} + 2y^2}{2xy}$$

$$p = \frac{-\cancel{6}^{3}x^2}{\cancel{2}xy} \quad \text{or} \quad \frac{\cancel{4}^{2}y^2}{\cancel{2}xy}$$

$$p = \frac{-3x}{y} \quad \text{or} \quad p = \frac{2y}{x}$$

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Now,

$$\frac{dy}{dx} = \frac{-3x}{y}$$

$$\int y\,dy = -3\int x\,dx$$

$$\frac{y^2}{2} = -3\frac{x^2}{2} + C$$

$$y^2 + 3x^2 - 2c = 0$$

$$y^2 + 3x^2 - 2c = 0$$

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\int \frac{1}{y}\,dy = 2\int \frac{1}{x}\,dx$$

$$\log y = 2\log x + \log c$$

$$\log y - \underbrace{2\log x}_{\log x^2} = \log c$$

$$y/x^2 = c$$

$$y = cx^2$$

$$y - cx^2 = 0$$

$\therefore$ its general solution is-

$$(y^2 + 3x^2 - 2c)(y - cx^2) = 0$$

★ Solvable for y :-

Step1 find y (y should be on left side)
Step2 Differentiate w.r.t x
Step3 Put $\frac{dy}{dx} = p$

Step4 Make the factors or form differential eq^n in p
Step5 find its solution.
Step6- Substitute the solution in given equation will give the general solution.

Date

Ques- Solve the differential equation-

$$y + px = x^4 p^2$$

Sol- $y = x^4 p^2 - px$

$$\frac{dy}{dx} = \left[x^4 \cdot 2p\frac{dp}{dx} + 4x^3 \cdot p^2\right] - \left[p \cdot 1 + x\frac{dp}{dx}\right]$$

$$p = 2x^4 p\frac{dp}{dx} + 4x^3 p^2 - p - x\frac{dp}{dx}$$

$$2p + x\frac{dp}{dx} - 2x^4 p\frac{dp}{dx} - 4x^3 p^2 = 0$$

$$2p(1 - 2x^3 p) + x\frac{dp}{dx}(1 - 2x^3 p) = 0$$

$$\left(2p + x\frac{dp}{dx}\right)\left(1 - 2x^3 p\right) = 0$$

neglecting this term as it is not having the derivative term

$$2p + x\frac{dp}{dx} = 0$$

$$x\frac{dp}{dx} = -2p$$

Using Variable Seperable :-

$$\int\frac{dp}{p} = -2\int\frac{dx}{x}$$

$$\log p = -2\log x + \log c$$

$$\log p + \log x^2 = \log c$$

$$px^2 = c$$

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$$\boxed{p = \frac{c}{x^2}}$$

Putting p in given eq^n
i.e.

$$y = -px + x^4 p^2$$

$$y = -\frac{c}{\cancel{x^2}} \cdot \cancel{x} + \cancel{x^4} \cdot \frac{c^2}{\cancel{x^4}}$$

$$y = c^2 - \frac{c}{x}$$

$$xy = xc^2 - c$$

$$xy - xc^2 - c = 0$$

Ques- Solve the differential equation-

$$y = 2px - p^2$$

Sol-

$$\frac{dy}{dx} = 2\left[p \cdot 1 + x\frac{dp}{dx}\right] - 2p\frac{dp}{dx}$$

$$p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$2p\frac{dp}{dx} - 2x\frac{dp}{dx} - p = 0$$

$$p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx} = 0$$

$$(2x - 2p)\frac{dp}{dx} = -p$$

$$\frac{dp}{dx} = \frac{-p}{2x - 2p}$$

$$\frac{dx}{dp} = \frac{2p - 2x}{p}$$

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$$\frac{dx}{dp} = \frac{2p}{\cancel{p}} - \frac{2x}{p}$$

$$\frac{dx}{dp} + \frac{2x}{p} = 2$$

which is of the form lebinitz linear i.e.

$$\frac{dy}{dx} + Py = Q$$

where $P = \frac{2}{p}$, $Q = 2$

$$If = e^{\int P dx} = e^{\int P dp} = e^{2\int \frac{1}{p} dp}$$

$$= e^{2\log p} = e^{\log p^2}$$

$$= p^2$$

$\therefore$ Its solution is -

$$x(If) = \int Q(If)\, dp + c$$

$$xp^2 = 2\int p^2\, dp + c$$

$$xp^2 = \frac{2}{3}p^3 + c$$

$$x = \frac{2}{3}p + cp^{-2} \quad \text{—— (1)}$$

Putting value of x in given eq^n i.e.

$$y = 2px - p^2$$

$$y = 2p\left(\frac{2}{3}p + cp^{-2}\right) - p^2 \quad \text{—— (2)}$$

(1) & (2) together constitutes the general solution.

Date

Ques- Solve the differential eq^n -

$$x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$$

Sol- $x^2p^4 + 2xp - y = 0$

$$y = x^2p^4 + 2xp$$

$$\frac{dy}{dx} = \left[x^2 \cdot 4p^3\frac{dp}{dx} + 2xp^4\right] + 2\left[x\frac{dp}{dx} + p\right]$$

$$p = 4p^3x^2\frac{dp}{dx} + 2xp^4 + 2x\frac{dp}{dx} + 2p$$

$$-p - 4p^3x^2\frac{dp}{dx} - 2x\frac{dp}{dx} - 2xp^4 = 0$$

$$p + 4p^3x^2\frac{dp}{dx} + 2x\frac{dp}{dx} + 2xp^4 = 0$$

$$2x\frac{dp}{dx}(1+2p^3x) + p(1+2p^3x) = 0$$

$$\left(2x\frac{dp}{dx} + p\right)\underbrace{\cancel{(1+2p^3x)}}_{\text{Discarding factor}} = 0$$

$$2x\frac{dp}{dx} + p = 0$$

$$2x\frac{dp}{dx} = -p$$

$$2\int\frac{1}{p}dp = -\int\frac{1}{x}dx$$

Using Variable seperable

$$2\log p = -\log x + \log c$$

$$\log p^2 + \log x = \log c$$

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$$p^2 x = c$$
$$p^2 = \frac{c}{x}$$
$$p = \left(\frac{c}{x}\right)^{1/2}$$

Putting p in given eq^n i.e.

$$y = x^2 p^4 + 2xp$$
$$y = x^2\left(\left(\frac{c}{x}\right)^{1/2}\right)^4 + 2x\left(\frac{c}{x}\right)^{1/2}$$
$$y = x^2 \frac{c^2}{x^2} + 2x^{1/2} c^{1/2}$$
$$y = c^2 + 2x^{1/2} c^{1/2}$$

* Solvable for x

Step 1 find x

Step 2 find $\frac{dx}{dy} = \frac{1}{p}$

Step 3 Solve the differential eq^n in p

Step 4 Substitute the solution obtained in the given equation

Ques:- Solve the differential equation
$$y = 2px + y^2 p^3$$

Sol:-
$$y - y^2 p^3 = 2px$$
$$x = \frac{1}{2}\left[\frac{y}{p} - y^2 p^2\right] \quad \text{—— (1)}$$

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Differentiating ① w.r.t y

$$\frac{dx}{dy} = \frac{1}{2}\left[\left(\frac{p(1) - y\frac{dp}{dx}}{p^2}\right) - \left(2yp^2 + 2y^2p\frac{dp}{dy}\right)\right]$$

$$\frac{1}{p} = \frac{1}{2}\left[\frac{1}{p} - \frac{y\frac{dp}{dx}}{p^2} - 2yp^2 - 2y^2p\frac{dp}{dy}\right]$$

Multiplying whole eqⁿ by p^2

$$p = \frac{1}{2}\left[p - y\frac{dp}{dy} - 2yp^4 - 2y^2p^3\frac{dp}{dy}\right]$$

$$2p = p - y\frac{dp}{dy} - 2yp^4 - 2y^2p^3\frac{dp}{dy} = 0$$

$$p + y\frac{dp}{dy} + 2yp^4 + 2y^2p^3\frac{dp}{dy} = 0$$

$$y\frac{dp}{dy}(1 + 2yp^3) + p(1 + 2yp^3) = 0$$

$$\left(y\frac{dp}{dy} + p\right)\cancel{(1 + 2yp^3)} = 0$$

Discarding factor

$$y\frac{dp}{dy} = -p$$

$$\int\frac{1}{p}dp = -\int\frac{1}{y}dy$$

$$\log p = -\log y + \log c$$

$$\log py = \log c$$

$$py = c$$

$$\boxed{p = \frac{c}{y}}$$

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Putting $p = \frac{c}{y}$ i.e in (2) given eq^n

$$y = 2px + y^2 p^3$$

$$y = 2\frac{c}{y}x + \cancel{y^2}\,\frac{c^3}{y^{\cancel{3}}}$$

$$y = 2\left(\frac{c}{y}\right)x + \frac{c^3}{y}$$

Ques- Solve the differential equation-

$$p = \tan\left(x - \frac{p}{1+p^2}\right)$$

Sol- $$\tan^{-1} p = x - \frac{p}{1+p^2}$$

$$x = \tan^{-1} p + \frac{p}{1+p^2} \quad \text{—— (1)}$$

differentiating (1) w.r.t y

$$\frac{dx}{dy} = \frac{1}{1+p^2}\frac{dp}{dy} + \left[\frac{(1+p^2)\frac{dp}{dy} - 2p^2\frac{dp}{dy}}{(1+p^2)^2}\right]$$

$$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{1}{1+p^2}\frac{dp}{dy} - \frac{2p^2}{(1+p^2)^2}\frac{dp}{dy}$$

Multiplying whole eq^n by $p(1+p^2)^2$

$$\frac{(1+p^2)^2}{p} = (1+p^2)\frac{dp}{dy} + (1+p^2)\frac{dp}{dy} - 2p^2\frac{dp}{dy}$$

$$\frac{(1+p^2)^2}{p} = \frac{dp}{dy}\left(1+\cancel{p^2}+1+\cancel{p^2}-\cancel{2p^2}\right)$$

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$$\frac{(1+p^2)^2}{p} = 2\frac{dp}{dy}$$

$$\int \frac{2p}{(1+p^2)^2}\,dp = \int dy$$

$$y = \int \frac{2p}{(1+p^2)^2}\,dp$$

Put $1+p^2 = t$

$2p\,dp = dt$

$$y = \int \frac{1}{t^2}\,dt$$

$$y = -t^{-1} + c$$

$$y = -\frac{1}{t} + c$$

$$y = -\frac{1}{1+p^2} + c \quad \text{—— (2)}$$

(1) & (2) together constitutes the solution.

* **Clairaut's Equation :-**

Any equation of the form

$$y = px + f(p)$$

is known as the Clairaut's equation.

Then, $p = c$ is the solution of this eq^n.

$$\Rightarrow \boxed{y = cx + f(c)}$$

Solution of the given eq^n

Date

Proof ⟹ $y = px + f(p)$ —— (1)

differentiating (1) w.r.t x

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\cancel{p} - \cancel{p} + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} = 0$$

$$\cancel{(x + f'(p))}\frac{dp}{dx} = 0$$

↓

Neglecting this term bcz it doesnot contains differentiation term of p ie $\frac{dp}{dx}$

$$\frac{dp}{dx} = 0$$

$$\int dp = \int 0 \, dx$$

$$p = 0 + c$$

$$\boxed{p = c}$$

⇓

is the required solution of the clairaut's eq^n

Proved

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Ques- Solve the differential equation-

$$(y - px)(p-1) = p$$

Sol- $y = px + \frac{p}{p-1}$

~~$\frac{dy}{dx} = p + x\frac{dp}{dx}$~~

which is in the form of Clairaut's eq^n

$$y = px + f(p)$$

Sol, put $p = c$

$$y = cx + \frac{c}{c-1}$$

Ques- Solve - $e^{4x}(p-1) + e^{2y}p^2 = 0$ —— (1)

Sol- HCF of 4 & 2 = 2

Now Put $X = e^{2x}$ & $Y = e^{2y}$

$dX = 2e^{2x}dx$ & $dY = 2e^{2y}dy$

$$\frac{2e^{2y}}{2e^{2x}}\frac{dy}{dx} = \frac{dY}{dX}$$

$$\frac{Y}{X}\frac{dy}{dx} = \frac{dY}{dX}$$

$$\frac{dy}{dx} = \frac{X}{Y}\frac{dY}{dX}$$

$$p = \frac{X}{Y}P$$

Now substite value of p in (1)

we get-

$$X^2\left(\frac{X}{Y}P - 1\right) + Y\left(\frac{X}{Y}\cdot P\right)^2 = 0$$

$$X^2\left(\frac{XP - Y}{Y}\right) + \frac{X^2P^2}{Y} = 0$$

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$$\frac{X^2}{Y}\left(XP - Y + P^2\right) = 0$$

$$XP - Y + P^2 = 0$$

$$Y = XP + P^2$$

$$Y = PX + P^2$$

which is clairaut's eq^n

. So put $P = C$

$$Y = XC + C^2$$

$$e^{2y} = e^{2x}C + C^2$$

Ques - Solve - $(px - y)(py + x) = 2p$ —— ①

Sol - After multiplying, highest power of x & y will be 2 [as HCF = 2]

So put $X = x^2$, $Y = y^2$

$$dX = 2x\,dx, \quad dY = 2y\,dy$$

$$\frac{dY}{dX} = \frac{2y\,dy}{2x\,dx}$$

$$\frac{dy}{dx} = \frac{x}{y}\cdot\frac{dY}{dX}$$

$$p = \frac{x}{y}P$$

~~Put value of p in ①~~

$$p = \sqrt{\frac{X}{Y}}\,P$$

Put value of p in ①

$$\left(\sqrt{\frac{X}{Y}}P\cdot\sqrt{X} - \sqrt{Y}\right)\left(\sqrt{\frac{X}{Y}}P\cdot\sqrt{Y} + \sqrt{X}\right) = 2\sqrt{\frac{X}{Y}}P$$

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$$\left(\frac{xP}{\sqrt{y}} - \sqrt{y}\right)\left(\sqrt{x}P + \sqrt{x}\right) = 2\sqrt{\frac{x}{y}}\,P$$

$$\left(\frac{xP - y}{\sqrt{y}}\right)\left(P\sqrt{x} + \sqrt{x}\right) = 2\sqrt{\frac{x}{y}}\,P$$

$$\cancel{\sqrt{\frac{x}{y}}}\,(xP - y)(P+1) = 2\cancel{\sqrt{\frac{x}{y}}}\,P$$

$$(xP - y)(P+1) = 2P$$

$$xP - y = \frac{2P}{P+1}$$

$$y = xP - \frac{2P}{P+1}$$

which is clairaut's eq^n

$\therefore$ Put $P = C$

$$y = xC - \frac{2C}{C+1}$$

$$y^2 = x^2 C - \frac{2C}{C+1}$$

Operator :-

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$

$\downarrow$

operator

$$\frac{d}{dx} = D$$

→ higher order differential eq^n

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \ldots$$

Symbols - $Dy,\ D^2y,\ D^3y\ \ldots$

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Linear Differential Eqn with Second or higher orders

The general differential equation of n^{th} order is of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{------} P_n y = X$$

here, $P_1, P_2, \text{-----} P_n$ & X are functions of x only and for equation of the form-

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{-------} a_n y = X$$

here $a_1, a_2, \text{----} a_n$ are constants & X is function of x only. this is called Linear diff. eqn with constant coefficient

→ Working rule to solve linear diff. eq^n with constant coefficient-

- Homogeneous equation-

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 6 = 0$$

Step 1 Convert the equation in symbolic form.

Step 2: Write Auxiliary equation (AE) i.e equate to zero. the coefficient of y in Step 1.

Step 3. find zeroes of D.

Step 4 Write complementary function (C.F.)

Step 5 $y = Cf$ will be its solution.

* Rule to find cf.

Case I If all the roots of AE are real & distinct say m_1 and m_2 are roots of D ($m_1, m_2 \in$ Real)

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then $Cf = C_1e^{m_1x} + C_2e^{m_2x}$

Case II- If the roots are real and equal
say m, m are roots of D ($m, m \in$ Real)
then $Cf = (C_1 + C_2x)e^{mx}$
for $D = m, m, m$
$Cf = (C_1 + C_2x + C_3x^2)e^{mx}$

Case III If the roots are imaginary
say $\alpha \pm i\beta$ are roots of D
then $Cf = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

Case IV If the roots are imaginary & equal
say $\alpha \pm i\beta$, $\alpha \pm i\beta$ are roots of D
then $CF = e^{\alpha x}[(C_1 + C_2x)\cos \beta x + (C_3 + C_4x)\sin \beta x]$

Ques- Solve $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$

Sol- $D^3y - 7Dy - 6y = 0$

$(D^3 - 7D - 6)y = 0$

Its Auxiliary eqn is
$D^3 - 7D - 6 = 0$
by using hit & trial
$D = -1$ is one of the roots

$$
\begin{array}{r|l}
 & D^2 - D - 6 \\
\hline
D+1 & D^3 - 7D - 6 \\
 & D^3 + D^2 \\
\hline
 & -D^2 - 7D - 6 \\
 & -D^2 - D \\
\hline
 & -6D - 6 \\
 & -6D - 6 \\
\hline
 & 0
\end{array}
$$

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$$(D+1)(D^2-D-6)=0$$

$$D=\frac{1\pm\sqrt{1+24}}{2}=\frac{1\pm5}{2}$$

$$D=-2,3$$

$\therefore$ all the roots of D are $-1,-2,3$ which are real & distinct

So its $Cf = C_1e^{-x}+C_2e^{-2x}+C_3e^{3x}$

its solution is :$\longrightarrow$

$$y=C_1e^{-x}+C_2e^{-2x}+C_3e^{3x}$$

Ques- Solve- $\frac{d^2y}{dx^2}-3\frac{dy}{dx}-4y=0$

Sol- $D^2y-3Dy-4y=0$

$$(D^2-3D-4)y=0$$

its auxiliary eq^n is-

$$D^2-3D-4=0$$

$$D=\frac{3\pm\sqrt{9+16}}{2}=\frac{3\pm5}{2}=4,-1$$

$$(D-4)(D+1)=0$$

$\therefore$ roots of D are 4 & -1 which are real and distinct.

So its $Cf = C_1e^{4x}+C_2e^{-x}$

its solution is :$\longrightarrow$

$$y=C_1e^{4x}+C_2e^{-x}$$

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Ques- Solve $\frac{d^4y}{dx^4} + 13\frac{d^2y}{dx^2} + 36y = 0$

Sol- $D^4y + 13D^2y + 36y = 0$

$(D^4 + 13D^2 + 36)y = 0$

its AE $\Rightarrow D^4 + 13D^2 + 36 = 0$

~~$D^4 + 9D^2 + 4D^2 + 36$~~

Put $D^2 = t$

$t^2 + 13t + 36 = 0$

$(t+9)(t+4) = 0$

$t = -9, -4$

$D^2 = -9, -4$

$D = \pm\sqrt{-9}, \pm\sqrt{4}$

$D = \pm 3i, \pm 2i$

$D = 0 \pm 3i, 0 \pm 2i$

$\therefore$ roots are imaginary

$Cf = e^{0x}(C_1 \cos 3x + C_2 \sin 3x) + e^{0x}(C_3 \cos 2x + C_4 \sin 2x)$

its solution is-

$y = (C_1 \cos 3x + C_2 \sin 3x) + (C_3 \cos 2x + C_4 \sin 2x)$

$\Downarrow$

$\boxed{y = Cf}$

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Theorem - The necessary and sufficient condition for a given equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proof - The condition is necessary.

Let The equation $Mdx + Ndy = 0$ is exact

Then, by definition -

$$dU = Mdx + Ndy \quad \text{--- (1)} \qquad U = f(x, y)$$

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

Put dU in (1)

$$\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = Mdx + Ndy \quad \text{--- (2)}$$

from (2), we get -

$$\frac{\partial U}{\partial x} = M \quad \& \quad \frac{\partial U}{\partial y} = N$$

↓ partially differentiating ↓

w.r.t y w.r.t x

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \& \quad \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

hence, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proved

⇒ The condition is sufficient.

Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

We shall prove that the $Mdx + Ndx = 0$ is exact.

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Let $U = \int_{y\ cons.} M dx$

$$\frac{\partial U}{\partial x} = M \text{ and } \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial M}{\partial y}$$

$$\text{If } \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y}$$

$$\text{So, } \frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x \partial y} \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$$

Integrating both sides:

$$N = \frac{\partial U}{\partial y} + C \rightarrow f(y)$$

$$\text{Now, } Mdx + Ndy = \frac{\partial U}{\partial x} dx + \left[\frac{\partial U}{\partial y} + f(y)\right] dy$$

$$Mdx + Ndy = \left(\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy\right) + f(y) dy$$

$$Mdx + Ndy = dU + d\left(\int f(y) dy\right)$$

$$Mdx + Ndy = d\left(\underline{U + \int f(y) dy}\right)$$

The given eq^n is exact.

Hence proved

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⇒ Continuing Linear diff. eq^n with second or higher order :-

$f(D) = X$ [Some function of X] → different case

★ Inverse operator $\left(\frac{1}{f(D)}\right)$

$\frac{1}{f(D)} X$ is that function of x free from arbitrary constants which when operated upon by $f(D)$ gives X.

$$f(D)\left[\frac{1}{f(D)} X\right] = X$$

★ Theorem 1 → $\frac{1}{f(D)} X$ is the particular integral (PI) of $f(D)y = X$.

Proof → The given equation is

$$f(D)y = X \quad \text{——— (1)}$$

$$y = \frac{X}{f(D)} = \frac{1}{f(D)} X$$

Put y in (1)

$$f(D)\left[\frac{1}{f(D)} X\right] = X$$

$$X = X$$

which is true.

$\therefore y = \frac{1}{f(D)} X$ is the solution of eq^n (1).

Since it is free from all arbitrary constants so, this is the particular integral (PI).

$$\boxed{PI = \frac{1}{f(D)} X}$$

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Theorem 2 → $\frac{1}{D}X = \int X\,dx$ where $D = \frac{d}{dx}$ (Derivative)

Proof → Let $\frac{1}{D}X = y$ —— (1)

Operating both sides by D

$$D\left(\frac{1}{D}X\right) = Dy$$

$$X = \frac{dy}{dx} \qquad \left(\because D = \frac{d}{dx}\right)$$

$$X\,dx = dy$$

Integrating both sides

$$\int X\,dx = y$$

from (1)

$$y = \frac{1}{D}X$$

$$\therefore \boxed{\frac{1}{D}X = \int X\,dx}$$

Proved

Theorem 3 → $\frac{1}{D-a}X = e^{ax}\int X e^{-ax}\,dx$

Proof → Let $\frac{1}{D-a}X = y$ —— (1)

Operating both sides by $D-a$

$$(D-a)\left(\frac{1}{D-a}X\right) = (D-a)y$$

$$X = Dy - ay$$

$$X = \frac{dy}{dx} - ay$$

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which is Leibnitz differential eqⁿ
where, $P = -a$ & $Q = X$
Now,
$$If = e^{\int P dx} = e^{-\int a dx} = e^{-ax}$$
Solution :-
$$y(If) = \int Q(If)\,dx$$
$$y e^{-ax} = \int X e^{-ax}\,dx$$
$$\boxed{\frac{1}{D-a}X = e^{ax}\int X e^{-ax}\,dx} \quad \text{(from (1))}$$
Proved

⇒ Prove $\frac{1}{D+a}X = e^{-ax}\int X e^{ax}\,dx$

Let $\frac{1}{D+a}X = y$

Operating both sides by $D+a$
$$D+a\left(\frac{1}{D+a}X\right) = (D+a)y$$
$$X = Dy + ay$$
$$\frac{dy}{dx} + ay = X$$
which is LDE
where,
$$P = a, \; Q = X$$
$$If = e^{\int P dx}$$
$$= e^{\int a dx}$$
$$= e^{ax}$$

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Solution :-

$$ye^{ax} = \int x e^{ax} dx$$

$$\boxed{\frac{1}{D+a} x = e^{-ax} \int x e^{ax} dx} \quad \text{Proved}$$

⇒ Working Rule to solve $f(D)y = X$.

Step1 Write in ~~standard~~ symbolic form

Step2 Write Auxiliary equation (AE)

Step3 find roots of AE

Step4 Write Complimentry function (Cf)

Step5 Write Particular Integral (PI)

Step6 $y = Cf + PI$ will be the solution.

★ To find Particuler Integral :-

Case 1 - $X = e^{ax+b}$

then $PI = \frac{1}{f(D)} e^{ax+b}$

Put $D = a$

$$PI = \frac{1}{f(a)} e^{ax+b} \quad (f(a) \neq 0)$$

if $f(D) = 0$ when $D = a$

then $PI = x \cdot \frac{1}{f'(a)} e^{ax+b} \quad (f'(a) \neq 0)$

if $f'(a) = 0$

then $PI = x^2 \frac{1}{f''(a)} e^{ax+b} \quad (f''(a) \neq 0)$

and so on...

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Q. Solve the differential equation-
$(4D^2+4D-3)y = e^{2x}$

Sol:- Its AE is -
$4D^2+4D-3=0$
$D = \frac{-4 \pm \sqrt{16+48}}{8} = \frac{-4 \pm 8}{8} = -\frac{3}{2}, \frac{1}{2}$
$D = -\frac{3}{2}, \frac{1}{2}$

So, Cf for real & distinct roots
is $\longrightarrow$ $C_1 e^{-\frac{3}{2}x} + C_2 e^{\frac{1}{2}x}$

$PI = \frac{1}{4D^2+4D-3} e^{2x}$

Now, put $D=2$
$= \frac{1}{16+8-3} e^{2x} = \frac{e^{2x}}{21}$

Now, solution is: $\rightarrow$
$y = Cf + PI$

$y = C_1 e^{-3/2 x} + C_2 e^{1/2 x} + \frac{e^{2x}}{21}$

Ques- Solve - $(D^3-3D^2+4)y = e^{2x}$

Sol- AE $\Longrightarrow$ $D^3-3D^2+4=0$

$D = -1$

$$\begin{array}{r|l} & D^2-4D+4 \\ \hline D+1 & D^3-3D^2+4 \\ & D^3+D^2 \\ \hline & -4D^2+4 \\ & \mp 4D^2 \quad -4D \\ \hline \end{array}$$

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$(D+1)(D^2-4D+4) = 0$

$(D+1)(D-2)^2 = 0$

$D = -1, 2, 2$

Its CF is $\rightarrow$ $\underline{C_1 e^{-x} + (C_2 + C_3 x)e^{2x}}$

Now, $PI = \dfrac{1}{D^3 - 3D^2 + 4} e^{2x}$

Put $D = 2$

$PI = \dfrac{1}{8 - 12 + 4} = \dfrac{1}{0} e^{2x}$

Not possible.

Now,

$PI = x \cdot \dfrac{1}{f'(2)} e^{2x}$ $\quad (f'(D) = 3D^2 - 6D)$

$PI = x \cdot \dfrac{1}{12 - 12} e^{2x}$

$PI = \dfrac{xe^{2x}}{0} \rightarrow$ Not possible

~~∴ solution is →~~

$\cancel{y = CF + PI}$

$\cancel{y = C_1 e^{-x} + (C_2 + C_3 x)e^{2x} + \dfrac{xe^{2x}}{8}}$

$PI = x^2 \cdot \dfrac{1}{f''(2)} e^{2x}$ $\quad (f''(D) = 6D - 6)$

$PI = x^2 \cdot \dfrac{1}{12 - 6} e^{2x} = \dfrac{x^2 e^{2x}}{6}$

∴ solution is $\rightarrow$

$y = C_1 e^{-x} + (C_2 + C_3 x)e^{2x} + \dfrac{x^2 e^{2x}}{6}$

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Case 2 - $X = \sin(ax+b)$ or $\cos(ax+b)$

$$PI = \frac{1}{f(D)} \sin(ax+b)$$

put $D^2 = -(a^2)$

Ques: Solve the differential eqn

$$(D^3+1)y = \sin(2x+3)$$

Sol: AE is $\Rightarrow$

$$D^3+1 = 0$$

as we know, $a^3+b^3 = (a+b)(a^2-ab+b^2)$

$$(D+1)(D^2-D+1) = 0$$

$$D = -1,\ D = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$D = -1,\ \frac{1}{2}+i\frac{\sqrt{3}}{2},\ \frac{1}{2}-i\frac{\sqrt{3}}{2}$$

Its CF is $\Rightarrow$ $C_1e^{-x} + e^{\frac{1}{2}x}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right)$

$$PI \Rightarrow \frac{1}{D^3+1}\sin(2x+3)$$

Put $D^2 = -(2)^2$

$$PI = \frac{1}{-4D+1}\sin(2x+3)$$

$$= \frac{1}{1-4D} \times \frac{1+4D}{1+4D}\sin(2x+3)$$

$$= \frac{1+4D}{1-16D^2}\sin(2x+3)$$

Put $D^2 = -(2)^2 = -4$

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$$PI = \frac{1+4D}{65} \sin(2x+3)$$

$$PI = \frac{1}{65}\left[\sin(2x+3) + 4D\sin(2x+3)\right]$$

$$PI = \frac{1}{65}\left[\sin(2x+3) + 8\cos(2x+3)\right]$$

∴ solution is :-

$$y = Cf + PI$$

$$y = C_1 e^{-x} + e^{\frac{1}{2}x}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{65}\left[\sin(2x+3) + 8\cos(2x+3)\right]$$

Solve - $(D^2+4)y = \cos 2x$

- AE ⇒ $D^2+4 = 0$

$$D^2 = -4$$

$$D = \pm 2i = 0 \pm 2i$$

- $Cf = e^{0x}\left(C_1 \cos 2x + C_2 \sin 2x\right)$

$$Cf = C_1 \cos 2x + C_2 \sin 2x$$

- $PI = \frac{1}{D^2+4} \cos 2x$

Put $D^2 = -(2)^2 = -4$

$$= \frac{1}{\cancel{-4}+\cancel{4}} \cos 2x$$

which is the case of faliure.

Now, $PI = x \cdot \frac{1}{2D} \cos 2x$

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$$PI = \frac{x}{2}\int \cos 2x \, dx$$

$$PI = \frac{x}{2}\frac{\sin 2x}{2}$$

$$PI = \frac{x}{4}\sin 2x$$

$\therefore$ solution is :-

$$y = Cf + PI$$

$$y = C_1\cos 2x + C_2\sin 2x + \frac{x}{4}\sin 2x$$

Ques- Solve the differential eqn :-

$$(D+2)(D-1)^2 y = 3\sinh x = 3\left(\frac{e^{x}-e^{-x}}{2}\right)$$

Sol- • AE $\Rightarrow (D+2)(D-1)^2 = 0$

$$(D+2)(D^2-2D+1) = 0$$

$$D = -2,\ D = \frac{2 \pm \sqrt{4-4}}{2}$$

$$D = -2, 1, 1$$

• $Cf = C_1 e^{-2x} + (C_2 + C_3 x)e^{x}$

• $PI = \frac{1}{(D+2)(D-1)^2} \cdot \frac{3}{2}\left(\frac{e^{x}-e^{-x}}{2}\right)$

$$PI = \frac{3}{2}\frac{1}{(D+2)(D-1)^2}\, e^{x}-e^{-x}$$

$$PI = \frac{3}{2}\left[e^{x}\cdot\frac{1}{(D+2)(D-1)^2} - e^{-x}\frac{1}{(D+2)(D-1)^2}\right]$$

$$PI = \frac{3}{2}[I_1 - I_2]$$

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$$I_1 = \frac{1}{(D+2)(D-1)^2} e^x$$

Put $D = 1$

$$I_1 = \frac{1}{(3)(0)} e^x$$

which is the case of faliur

$$\therefore I_1 = x \cdot \frac{1}{(D+2)\,2(D-1) + (D-1)^2 \cdot 1} e^x$$

Put $D = 1$

$$I_1 = x \frac{1}{0} e^x$$

Again case of faliur

$$I_1 = x^2 \frac{1}{2(2D+1) + 2(D-1)} e^x$$

Put $D = 1$

$$I_1 = x^2 \cdot \frac{1}{6} e^x = \frac{x^2 e^x}{6}$$

Now, $I_2 = \frac{1}{(D+2)(D-1)^2} e^{-x}$

Put $D = -1$

$$I_2 = \frac{1}{4} e^{-x} = \frac{1}{4} e^{-x}$$

$$PI = \frac{3}{2}\left[I_1 - I_2\right] = \frac{3}{2}\left[\frac{x^2 e^x}{6} - \frac{e^{-x}}{4}\right]$$

$\therefore$ solution is :-

$$y = Cf + PI$$

$$y = C_1 e^{-2x} + (C_2 + C_3 x)e^x + \frac{3}{2}\left[\frac{x^2 e^x}{6} - \frac{e^{-x}}{4}\right]$$

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Solve- $\frac{d^3y}{dx^3} + y = \sin 3x - \cos^2\frac{x}{2}$

SE $\Rightarrow (D^3+1)y = \sin 3x - \cos^2\frac{x}{2}$

AE $\Rightarrow D^3+1 = 0$

$(D+1)(D^2-D+1) = 0$ $\left(\because a^3+b^3 = (a+b)(a^2-ab+b^2)\right)$

$D = -1,\ D = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$

$D = -1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$

$Cf = C_1e^{-x} + e^{x/2}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right)$

$PI = \frac{1}{D^3+1}\left(\sin 3x - \cos^2\frac{x}{2}\right)$

Put $D^2 = -(3)^2 = -9$

$= \frac{1}{-9D+1}\left(\sin 3x - \cos^2\frac{x}{2}\right)$

$= \left(\frac{1}{1-9D} \times \frac{1+9D}{1+9D}\right)\left(\sin 3x - \cos^2\frac{x}{2}\right)$

$= \frac{1+9D}{1-81D^2}\left(\sin 3x - \cos^2\frac{x}{2}\right)$

Again Put $D^2 = -(3)^2 = 9$

$= \frac{1+9D}{730}\left(\sin 3x - \cos^2\frac{x}{2}\right)$

$= \frac{1}{730}\left[\left(\sin 3x - \cos^2\frac{x}{2}\right) + 9D\left(\sin 3x - \cos^2\frac{x}{2}\right)\right]$

$PI = \frac{1}{730}\left[\left(\sin 3x - \cos^2\frac{x}{2}\right) + 9\left(3\cos 3x + \cos\frac{x}{2}\right)\right]$

$\therefore$ solution is-

$y = C_1e^{-x} + e^{x/2}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{730}\left[\sin 3x + 27\cos 3x - \cos^2\frac{x}{2} + 9\cos\frac{x}{2}\right]$

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Case 3 - $X = e^{ax} V$

$$PI = \frac{1}{f(D)} e^{ax} V$$

Put $D = D + a$

$$PI = e^{ax}\left[\frac{1}{f(D+a)} V\right]$$

Ques:- Solve the differential eq^n -

$$(D^2 - 4D + 3)y = e^x \cos 2x \to V$$

Sol- Its AE is $\Rightarrow$

$$D^2 - 4D + 3 = 0$$

$$D = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2}$$

$$D = 1, 3$$

Its Cf is $\Rightarrow$ $C_1 e^x + C_2 e^{3x}$

$$PI \Rightarrow \frac{1}{D^2 - 4D + 3} e^x \cos 2x$$

Put $D = D + 1$

$$= e^x\left[\frac{\cos 2x}{(D+1)^2 - 4(D+1) + 3}\right]$$

$$= e^x . \frac{1}{D^2 + 2D + 1 - 4D - 4 + 3} . \cos 2x$$

$$= e^x\left[\frac{1}{D^2 - 2D} \cos 2x\right]$$

Put $D^2 = -4$

$$= e^x\left[\frac{1}{-4 - 2D} \cos 2x\right]$$

$$= e^x . \left[\frac{1}{-2(D+2)} \cos 2x\right]$$

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$$PI = \frac{e^x}{-2}\left[\frac{1}{D+2} \times \frac{D-2}{D-2} \cos 2x\right]$$

$$PI = \frac{e^x}{-2}\left[\frac{D-2}{D^2-4} \cos 2x\right]$$

Put $D^2 = -(2)^2$

$$PI = -\frac{e^x}{2}\left[\frac{D-2}{-8} \cos 2x\right]$$

$$PI = \frac{e^x}{16}\left[D(\cos 2x) - 2\cos 2x\right]$$

$$PI = \frac{e^x}{16}\left[-2\sin 2x - 2\cos 2x\right]$$

Solution :-

$$y = CF + PI$$

$$y = C_1 e^x + C_2 e^{3x} + \frac{e^x}{16}\left[-2\sin 2x - 2\cos 2x\right]$$

Ques- Solve- $\frac{d^4y}{dx^4} - y = \cos x \cosh x$

Sol-

$$\frac{d^4y}{dx^4} - y = \cos x\left(\frac{e^x + e^{-x}}{2}\right)$$

Its Symbolic form is :-

$$D^4y - y = \cos x\left(\frac{e^x + e^{-x}}{2}\right)$$

$$(D^4 - 1)y = \frac{e^x}{2}\cos x + \frac{e^{-x}}{2}\cos x$$

Its AE is :- $D^4 - 1 = 0$

$$(D^2)^2 - (1)^2 = 0$$

$$(D^2+1)(D^2-1) = 0$$

$$D = 1, -1, i, -i$$

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Its CF is :- $C_1 e^{x} + C_2 e^{-x} + e^{0x}(C_3 \cos x + C_4 \sin x)$

Its PI is :- $\frac{1}{D^4-1} \cdot \frac{e^x}{2} \cos x + \frac{e^{-x}}{2} \cos x$

$$PI = \frac{1}{2}\left[\frac{1}{D^4-1} e^x \cos x + \frac{1}{D^4-1} e^{-x} \cos x\right]$$

$$PI = \frac{1}{2}\left[I_1 + I_2\right]$$

$$I_1 = \frac{1}{D^4-1} e^x \cos x$$

$$I_1 = e^x \frac{1}{D^4-1} \cos x$$

Put $D = D+1$

$$I_1 = e^x \frac{1}{(D+1)^4-1} \cos x$$

$$I_1 = e^x \frac{1}{D^4+4D^3+6D^2+4D+\cancel{1}-\cancel{1}} \cos x$$

Put $D^2 = -(1)^2$

$$I_1 = e^x \frac{1}{1-\cancel{4D}-6+\cancel{4D}} \cos x$$

$$I_1 = e^x \frac{1}{-5} \cos x = -\frac{e^x}{5} \cos x$$

Now,

$$I_2 = e^{-x} \frac{1}{D^4-1} \cos x$$

Put $D = D-1$

$$I_2 = e^{-x} \frac{1}{(D-1)^4-1} \cos x$$

$$I_2 = e^{-x} \frac{1}{D^4-4D^3+6D^2-4D+1-1} \cos x$$

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$$I_2 = e^{-x} \frac{1}{D^4 - 4D^3 + 6D^2 - 4D} \cos x$$

Put $D^2 = -(1)^2 = -1$

$$I_2 = e^{-x} \frac{1}{(-1)^2 - 4(-1)D + 6(-1) - 4D} \cos x$$

$$I_2 = e^{-x} \frac{1}{1 + \cancel{4D} - 6 - \cancel{4D}} \cos x$$

$$I_2 = -\frac{e^{-x}}{5} \cos x$$

So, $$PI = \frac{1}{2}\left[\frac{-e^{x}}{5}\cos x - \frac{e^{-x}}{5}\cos x\right]$$

$$= -\frac{1}{10}\left[e^{x}\cos x + e^{-x}\cos x\right]$$

$$PI = -\frac{1}{10}\cos x\left[e^{x} + e^{-x}\right]$$

$\therefore$ Its solution is :-

$$y = Cf + P.I$$

$$y = C_1e^{x} + C_2e^{-x} + C_3\cos x + C_4\sin x - \frac{1}{10}\cos x\left[e^{x} + e^{-x}\right]$$

Ques- Solve - $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin 2x$

Sol- $SE = D^2y - 3Dy + 2y = \sin 2x$

$$(D^2 - 3D + 2)y = \sin 2x$$

$AE \Rightarrow D^2 - 3D + 2 = 0$

$$D = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2}$$

$$D = 1, 2$$

$Cf = C_1e^{x} + C_2e^{2x}$

Date

$$PI = \frac{1}{D^2-3D+2}\sin 2x$$

Put $D^2 = -(2)^2$

$$= \frac{1}{-4-3D+2}\sin 2x$$

$$= \frac{1}{-2-3D}\sin 2x$$

$$= -\left[\frac{1}{2+3D} \times \frac{2-3D}{2-3D}\sin 2x\right]$$

$$= -\left[\frac{2-3D}{4-9D^2}\right]\sin 2x$$

Put $D^2 = -(2)^2 = -4$

$$= -\left[\frac{2-3D}{4-9(-4)}\sin 2x\right]$$

$$= -\left[\frac{(2-3D)\sin 2x}{40}\right]$$

$$= \frac{-2}{40}\sin 2x + \frac{3D}{40}\sin 2x$$

$$= -\frac{1}{20}\sin 2x + \frac{3\times 2\cos 2x}{\cancel{40}\,20}$$

$$PI = \frac{1}{20}\left[-\sin 2x + 3\cos 2x\right]$$

$\therefore$ solution is :-

$$y = CF + PI$$

$$y = C_1 e^{x} + C_2 e^{2x} + \frac{1}{20}\left[-\sin 2x + 3\cos 2x\right]$$

Case 4 $\longrightarrow$ $X = x^m \rightarrow$ quad, linear, biquad etc...

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$$PI = \frac{1}{f(D)} x^m$$

$$= \frac{1}{1-\phi(D)} x^m$$

$$= (1-\phi(D))^{-1} x^m$$

or

After that, $= (1+\phi(D))^{-1} x^m$

Expand $(1+\phi(D))$ in ascending power of D

formulae-

$$(1+x)^{-1} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots$$

$$(1-x)^{-1} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

Ques- Solve the differential equation-

$$(D^2+5D+4)y = x^2+7x+9$$

Sol- $AE \Rightarrow D^2+5D+4=0$

$$D = \frac{-5 \pm \sqrt{25-16}}{2} = \frac{-5 \pm 3}{2}$$

$$D = -4, -1$$

$$Cf = C_1 e^{-4x} + C_2 e^{-x}$$

$$PI = \frac{1}{f(D)} x^m = \frac{1}{D^2+5D+4} x^2+7x+9$$

$$= \frac{1}{4\left[1+\frac{D^2+5D}{4}\right]} x^2+7x+9$$

$$= \frac{1}{4}\left[1+\frac{D^2+5D}{4}\right]^{-1} x^2+7x+9$$

which is of the form-

$$(1+\phi(D))^{-1} x^m$$

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Applying formula $((1+x)^{-1})$

$$= \frac{1}{4}\left[1 - \frac{D^2+5D}{4} + \left(\frac{D^2+5D}{4}\right)^2 + \cdots\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\left(\frac{D^2}{4}+\frac{5D}{4}\right)+\left(\frac{D^4}{16}+\frac{25D^2}{16}+\frac{10D^3}{16}\right)+\cdots\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\frac{5D}{4}-\frac{D^2}{4}+\frac{25D^2}{16}+\frac{10D^3}{16}+\frac{D^4}{16}+\cdots\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\frac{5D}{4}+\frac{21D^2}{16}+\cdots\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[x^2+7x+9-\frac{5}{4}(2x+7)+\frac{21}{16}(2)\right]$$ → rest all terms will become 0.

$$= \frac{1}{4}\left[x^2+7x+9-\frac{5x}{2}-\frac{35}{4}+\frac{21}{8}\right]$$

$$PI = \frac{1}{4}\left[x^2+\frac{9x}{2}+\frac{23}{8}\right]$$

Now, solution is:-

$$y = Cf + PI$$

$$y = C_1e^{-4x} + C_2e^{-x} + \frac{1}{4}\left[x^2+\frac{9x}{2}+\frac{23}{8}\right]$$

Ques- Solve- $\frac{d^2y}{dx^2} - 4y = x^2$

Sol- $SE = (D^2-4)y = x^2$

$AE \Rightarrow D^2-4 = 0$

$D = \pm 2$

$Cf = C_1e^{2x} + C_2e^{-2x}$

$$PI = \frac{1}{D^2-4}\cdot x^2 = \frac{1}{-4\left(1-\frac{D^2}{4}\right)}x^2$$

$$= \frac{1}{-4}\left[\left(1-\frac{D^2}{4}\right)^{-1}x^2\right]$$

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Applying formula $(1-x)^{-1}$

$$= -\frac{1}{4}\left[1+\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+\left(\frac{D^2}{4}\right)^3+\cdots\right]x^2$$

$$= -\frac{1}{4}\left[x^2+\frac{1}{4}D^2(x^2)+\frac{1}{16}D^4(x^2)+\cdots\right]$$

$$= -\frac{1}{4}\left[x^2+\frac{1}{\cancel{4}_2}(\cancel{2})+\frac{1}{16}D^4(0)\right]$$

$$= -\frac{1}{4}\left(x^2+\frac{1}{2}\right)$$

$$PI = -\frac{x^2}{4}-\frac{1}{8}$$

$\therefore$ its solution is :-

$$y = CF + PI$$

$$y = C_1e^{2x}+C_2e^{-2x} - \frac{x^2}{4}-\frac{1}{8}.$$

further situations where none of the above cases will be valid, then it would be solved by -

$$\rightarrow \frac{1}{f(D)}X$$

$$\frac{1}{f(D)} = \frac{A}{D-m_1}+\frac{B}{D-m_2} = \frac{1}{D-a}$$

$$\Rightarrow \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}\,dx$$

This type of questions can be solved by the method of Partial factorisation.

$D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$

$\frac{1}{D} = \int$, $\frac{1}{D^2} = \iint$

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Ques: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Sol: SF $= (D^2 - 2D + 1)y = xe^x \sin x$

AE $\Rightarrow D^2 - 2D + 1 = 0$

$D = \frac{2 \pm \sqrt{4-4}}{2} = 1, 1$

$D = 1, 1$

$Cf = (C_1 + C_2x)e^x$

$PI = \frac{1}{D^2 - 2D + 1} e^x (x \sin x)$

Put $D = D+1$

$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x \sin x$

$= e^x \frac{1}{D^2 + \cancel{2D} + 1 - \cancel{2D} - \cancel{2} + 1} x \sin x$

$= e^x \frac{1}{D^2} x \sin x$

$= e^x \frac{1}{D} \int \overset{I}{x} \overset{II}{\sin x}\, dx$

$= e^x \frac{1}{D} \left[x(-\cos x) - \int 1 \cdot (-\cos x)\, dx \right]$

$= e^x \frac{1}{D} \left[-x\cos x + \sin x \right]$

$= e^x \int (-x\cos x + \sin x)\, dx$

$= e^x \int -\overset{I}{x} \overset{II}{\cos x}\, dx + \int \sin x\, dx$

$= e^x \left[-x \sin x + \int 1 \cdot \sin x\, dx - \cos x \right.$

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$= e^x[-x\sin x - \cos x - \cos x]$

$= -e^x[x\sin x + 2\cos x]$

Now, solution is:-

$y = CF + PI$

$y = (C_1 + C_2 x)e^x - (x\sin x + 2\cos x)e^x$

$y = e^x[C_1 + C_2 x - x\sin x - 2\cos x]$

Ques- Solve $\frac{d^2y}{dx^2} + a^2y = \tan ax$

Sol- SE = $(D^2 + a^2)y = \tan ax$

AE ⇒ $D^2 + a^2 = 0$

$D^2 = -a^2$

$D = \pm ia$

$CF = e^{0x}(C_1\cos ax + C_2\sin ax)$

$CF = (C_1\cos ax + C_2\sin ax)$

$PI = \frac{1}{D^2+a^2}\tan ax$

$= \frac{1}{(D+ia)(D-ia)}\tan ax$

Now,

$\frac{1}{(D+ia)(D-ia)} = \frac{A}{D+ia} + \frac{B}{(D-ia)}$ —— (1)

$1 = A(D-ia) + B(D+ia)$

Put $D = ia$

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$e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$

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So, $1 = A(0) + B(2ia)$

$B = \frac{1}{2ia}$ & $A = \frac{-1}{2ia}$

Putting in (1), we get-

$$\frac{1}{(D+ia)(D-ia)} = \frac{-1}{2ia(D+ia)} + \frac{1}{2ia(D-ia)}$$

$$\frac{1}{(D+ia)(D-ia)} = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]$$

Now, Put in PI

$$PI = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\tan ax$$

$$= \frac{1}{2ia}\left[\underbrace{\frac{1}{D-ia}\tan ax}_{I_1} - \underbrace{\frac{1}{D+ia}\tan ax}_{I_2}\right]$$

$$I_1 = \frac{1}{D-ia}\tan ax$$

Acc. to theorm 3

$$\frac{1}{D-a}x = e^{ax}\int xe^{-ax}\,dx$$

So,

$$I_1 = e^{iax}\int \tan ax\, e^{-iax}\,dx$$

$$= e^{iax}\int \tan ax\,(\cos ax - i\sin ax)\,dx$$

$$= e^{iax}\int \frac{\sin ax}{\cos ax}\cdot\cos ax - \frac{\sin ax}{\cos ax}\cdot i\sin ax\,dx$$

$$= e^{iax}\int \sin ax - \frac{i\sin^2 ax}{\cos ax}\,dx$$

$$= e^{iax}\int \sin ax - i\frac{(1-\cos^2 ax)}{\cos ax}\,dx$$

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$$I_1 = e^{iax}\int \sin ax - i(\sec ax - \cos ax)\,dx$$

$$I_1 = e^{iax}\left[-\frac{\cos ax}{a} - \frac{i}{a}\left(\log|\sec ax + \tan ax| - \sin ax\right)\right]$$

$$I_1 = \frac{e^{iax}}{a}\left[-\cos ax - i\log|\sec ax + \tan ax| + i\sin ax\right]$$

Similarly,

$$I_2 = \frac{e^{-iax}}{a}\left[-\cos ax + i\log|\sec ax + \tan ax| - i\sin ax\right]$$

Now

$$PI = \frac{1}{2ia}\left[\frac{1}{I_1} - \frac{1}{I_2}\right]$$

$$= \frac{1}{2ia}\left[-\cos ax - i\log|\sec ax + \tan ax| + i\sin ax + \cos ax - i\log|\sec ax + \tan ax| + i\sin ax\right]$$

$$= \frac{1}{2ia}\left[2i\sin ax - 2i\log|\sec ax + \tan ax|\right]$$

$$PI = \frac{1}{a}\left[\sin ax - \log|\sec ax + \tan ax|\right]$$

$\therefore$ solution is :-

$$y = CF + PI$$

$$y = (C_1\cos ax + C_2\sin ax) + \frac{1}{a}\left(\sin ax - \log|\sec ax + \tan ax|\right)$$

Ques- Solve - $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

Sol- SE $\Rightarrow (D^2+1)y = \operatorname{cosec} x$

AE $\Rightarrow D^2 + 1 = 0$

$$D^2 = -1$$

$$D = \pm i$$

$$CF = e^{0x}(C_1\cos 1x + C_2\sin 1x)$$

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$$Cf = C_1 \cos x + C_2 \sin x$$

$$PI = \frac{1}{D^2+1} \operatorname{cosec} x$$

$$= \frac{1}{(D+i)(D-i)} \operatorname{cosec} x$$

Now, let -

$$\frac{1}{(D+i)(D-i)} = \frac{A}{(D+i)} + \frac{B}{(D-i)} \Rightarrow \frac{-1}{2i(D+i)} + \frac{1}{2i(D-i)}$$

Put in PI

$$1 = A(D-i) + B(D+i)$$

put $D = i$

$$A(0) + B(D+i) = 1$$

$$B = \frac{1}{2i}, \quad A = \frac{-1}{2i}$$

$$PI = \left[\frac{-1}{2i(D+i)} + \frac{1}{2i(D-i)}\right] \operatorname{cosec} x$$

$$= \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] \operatorname{cosec} x$$

$$= \frac{1}{2i}\left[\underbrace{\frac{\operatorname{cosec} x}{D-i}}_{I_1} - \underbrace{\frac{\operatorname{cosec} x}{D+i}}_{I_2}\right] = \frac{1}{2i}\left[I_1 - I_2\right]$$

$$I_1 = \frac{1}{D-i} \operatorname{cosec} x$$

Using theorem 3

$$I_1 = e^{ix} \int \operatorname{cosec} x \, e^{-ix} dx$$

$$= e^{ix} \int \frac{1}{\sin x} (\cos x - i \sin x) dx$$

$$= e^{ix} \int (\cot x - i) \, dx$$

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$$I_1 = e^{ix}\left[\log|\sin x| - ix\right]$$

Similarly,

$$I_2 = e^{-ix}\left[\log|\sin x| + ix\right]$$

$$PI = \frac{1}{2i}\left[I_1 - I_2\right]$$

$$= \frac{e^{ix}}{2i}\left[\cancel{\log|\sin x|} - ix - \cancel{\log|\sin x|} - ix\right]$$

$$PI = \frac{1}{2i}(-2ix) = -xe^{ix}$$

$\therefore$ Solution is:-

$$y = Cf + PI$$

$$y = C_1\cos x + C_2\sin x - xe^{ix}$$

☆ Variation of Parameters:-

Working rule to solve the differential equation by using variation of parameters:-

Step 1 Write Complimentary function (Cf)

$$C_1y_1 + C_2y_2 + \cdots$$

Step 2 find Wronskian (W) = $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Step 3 find $u = -\int \frac{y_2 X}{W}$ and $v = \int \frac{y_1 X}{W}$

Step 4 ~~then the solution is:-~~ find Particular integral-

$$PI = uy_1 + vy_2$$

Step 5- final solution is:- $y = Cf + PI$

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Ques:- Solve the given differential equation by using Variation of parameters-

$$\frac{d^2y}{dx^2} + 4y = 4\sec^2 2x$$

Sol- $SE = (D^2+4)y = 4\sec^2 2x$

$AE \Rightarrow D^2+4 = 0$

$D^2 = -4$

$D = \pm 2i$

$Cf = e^{0x}(C_1\cos 2x + C_2 \sin 2x)$

$Cf = C_1 \underbrace{\cos 2x}_{y_1} + C_2 \underbrace{\sin 2x}_{y_2}$, $y_1' = -2\sin 2x$
$y_2' = 2\cos 2x$

Now, Wronskian $(W) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$= 2\cos^2 2x + 2\sin^2 2x$

$= 2(\cos^2 2x + \sin^2 2x) = 2$

So, $W = 2$

Now, $u = -\int \frac{y_2 X}{W} dx = -\int \frac{\sin 2x \, 4\sec^2 2x dx}{2} = -2\int \sin 2x \sec^2 2x dx$

$u = -2\int \sin 2x \sec^2 2x \, dx$

$u = -2\int \tan 2x \sec 2x \, dx$

$u = -2 \cdot \frac{\sec 2x}{2} = -\sec 2x$

and $v = \int \frac{y_1 X}{W} dx = \int \frac{\cos 2x \cdot 4\sec^2 2x dx}{2} = 2\int \cos 2x \sec^2 2x dx$

$v = 2\int \sec 2x \, dx$

$= \frac{2\log|\sec 2x + \tan 2x|}{2} = \log|\sec 2x + \tan 2x|$

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$PI = uy_1 + vy_2$

$PI = -\sec 2x(\cos 2x) + \log|\sec 2x + \tan 2x| \sin 2x$

$PI = -1 + \sin 2x \log|\sec 2x + \tan 2x|$

So, its solution is :-

$y = CF + PI$

$y = C_1 \cos 2x + C_2 \sin 2x + \sin 2x \log|\sec 2x + \tan 2x| - 1$

Ques- Solve the diff. eq^n by using Variation of Parameters - $y'' - 2y' + 2y = e^x \tan x$

Sol - $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x$

$SE \Rightarrow (D^2 - 2D + 2)y = e^x \tan x$

$AE \Rightarrow D^2 - 2D + 2 = 0$

$D = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2}$

$D = 1 \pm i$

$CF = e^x(\underbrace{C_1 \cos x}_{y_1} + C_2 \underbrace{\sin x}_{y_2})$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x$

$= 1$

$\therefore W = 1$

Now, $u = -\int \frac{y_2 X}{W}$

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$$Cf = C_1 \underbrace{e^x \cos x}_{y_1} + C_2 \underbrace{e^x \sin x}_{y_2}$$

$$y_1' = -e^x \sin x + \cos x \, e^x$$
$$y_2' = e^x \cos x + \sin x \, e^x$$

Now, $W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \end{vmatrix}$

$$W = e^{2x} \cos x(\cos x + \sin x) - e^{2x} \sin x(\cos x - \sin x)$$
$$W = e^{2x}(\cos^2 x + \sin x \cos x) - e^{2x}(\cos x \sin x - \sin^2 x)$$
$$W = e^{2x}\left[\cos^2 x + \sin^2 x + \cancel{\sin x \cos x} - \cancel{\sin x \cos x}\right]$$

$$W = e^{2x}$$

Now, $u = -\int \frac{y_2 X}{W} dx = -\int \frac{e^x \sin x \cdot e^x \tan x}{e^{2x}} dx$

$$u = -\int \sin x \tan x \, dx = -\int \frac{\sin^2 x}{\cos x} dx$$

$$u = -\int \frac{1 - \cos^2 x}{\cos x} = -\int (\sec x - \cos x) \, dx$$

$$u = -\left[\log|\sec x + \tan x| - \sin x\right]$$

$$u = \sin x - \log|\sec x + \tan x|$$

and $v = \int \frac{y_1 X}{W} dx = \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}}$

$$v = \int \sin x \, dx$$

$$v = -\cos x$$

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$PI = uy_1 + vy_2$

$PI = (\sin x - \log|\sec x + \tan x|) e^x \cos x + e^x \frac{\sin x}{\cos x}$

So, its solution is:-

$y = CF + PI$

$y = e^x(C_1 \cos x + C_2 \sin x) + (\sin x - \log|\sec x + \tan x|) e^x - (\sin x \cos x) e^x$

$y = e^x \left[C_1 \cos x + C_2 \sin x + \sin x - \log|\sec x + \tan x| - \sin x \cos x \right]$

Ques- Solve the differential equation by using Variation of parameters - $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

Sol- $SE \Rightarrow (D^2 + 3D + 2)y = e^{e^x}$

$AE \Rightarrow D^2 + 3D + 2 = 0$

$D = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$

$D = -2, -1$

$CF = C_1 e^{-x} + C_2 e^{-2x}$

here, $y_1 = e^{-x}$, $y_2 = e^{-2x}$

$y_1' = -e^{-x}$, $y_2' = -2e^{-2x}$

$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-x}e^{-2x} + e^{-x}e^{-2x}$
$= -e^{-x}e^{-2x} = -e^{-3x}$

$W = -e^{-3x}$

Now, $u = -\int \frac{y_2 X}{W} dx = +\int \frac{e^{-2x} e^{e^x}}{e^{-3x}} dx = \int e^x e^{e^x} dx$

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Put $e^x = t$

$e^x dx = t\,dt$

$u = \int e^t dt = e^t = e^{e^x}$

$u = e^{e^x}$

and $v = \int \frac{y_1 X dx}{W} = -\int \frac{e^{-x} e^{e^x}}{e^{-3x}} dx = -\int e^{2x} e^{e^x} dx$

Put $e^x = t$

$e^x dx = dt$

$v = -\int \underset{I}{t}\ \underset{II}{e^t}\, dt$

By using by parts

$v = -\left[te^t - \int 1 \cdot e^t dt \right]$

$v = -\left[te^t - e^t \right]$

$v = -\left[e^x e^{e^x} - e^{e^x} \right]$

$v = e^{e^x} - e^x e^{e^x}$

$v = e^{e^x}(1 - e^x)$

$PI = uy_1 + vy_2$

$= e^{e^x} \cdot e^{-x} + e^{e^x}(1-e^x) e^{-2x}$

$= e^{e^x}\left[\cancel{e^{-x}} + e^{-2x} - \cancel{e^{-x}} \right]$

$PI = e^{e^x} \cdot e^{-2x}$

∴ solution is :-

$y = Cf + PI$

$y = C_1 e^{-x} + C_2 e^{-2x} + e^{e^x} \cdot e^{-2x}$

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Ques- Solve - $\frac{d^2y}{dx^2} + 4y = x\sin x$

Sol- SE $\Rightarrow (D^2+4)y = x\sin x$

AE $\Rightarrow D^2+4 = 0$

$D^2 = -4$

$D = \pm 2i$

$Cf = \underset{1}{e^{0x}}(C_1\cos 2x + C_2\sin 2x)$

★ If $X = xV$ ($V \rightarrow$ function of x)

then $PI = \frac{1}{f(D)} xV$

$$PI = \left[x - \frac{f'(D)}{f(D)}\right] \cdot \frac{1}{f(D)} V$$

Method I

$$PI = \frac{1}{D^2+4} x\sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \cdot \frac{1}{D^2+4}\sin x \longrightarrow \text{Put } D^2 = -(1)^2$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{-1+4}\sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{3}\sin x$$

$$= \frac{1}{3}\left[x - \frac{2D}{D^2+4}\right]\sin x$$

$$= \frac{1}{3}\left[x\sin x - \frac{2D\sin x}{D^2+4}\right]$$

Put $D^2 = -(1)^2$

$$= \frac{1}{3}\left[x\sin x - \frac{2D\sin x}{3}\right]$$

$$PI = \frac{1}{3}\left[x\sin x - \frac{2\cos x}{3}\right]$$

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$\therefore$ solution is:-

$$y = CF + PI$$

$$y = C_1\cos 2x + C_2 \sin 2x + \frac{1}{3}\left[x\sin x - \frac{2}{3}\cos x\right]$$

Method 2

$$CF = C_1\cos 2x + C_2 \sin 2x$$

let $y_1 = \cos 2x$, $y_2 = \sin 2x$

$y_1' = -2\sin 2x$, $y_2' = 2\cos 2x$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$W = 2$$

Now, $u = -\int \frac{y_2 X}{W}\,dx = -\int \cancel{\frac{2\sin 2x \, x\sin x}{2}} - \int \frac{\sin 2x \cdot x\sin x}{2}\,dx$

$$= -\frac{1}{2}\int x\sin 2x \,\sin x\,dx = -\frac{1}{2}\int x\cdot 2\sin x\cos x\cdot \sin x\,dx$$

$$= -\frac{2}{2}\int x\sin^2 x\cos x\,dx = -\int x(1-\cos^2 x)\cos x\,dx$$

$$= -\int (x\cos x - x\cos^3 x)\,dx$$

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Ques:- Solve - $(D^2+3D+2)y = \sin e^x$

Sol:- AE $\Rightarrow D^2+3D+2=0$

$D = \frac{-3 \pm \sqrt{9-8}}{2}$

$D = -1, -2$

~~CF = $e^{0x}(C_1\cos 2x + C_2\sin 2x)$~~

~~CF = $C_1\cos 2x + C_2 \sin 2x$~~

~~here $y_1 =$~~

$Cf = C_1 e^{-x} + C_2 e^{-2x}$

here $y_1 = e^{-x}$, $y_2 = e^{-2x}$

$y_1' = -e^{-x}$, $y_2' = -2e^{-2x}$

$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-x}e^{-x} + e^{-x}e^{-2x}$

$W = -e^{-3x}$

Now, $u = -\int \frac{y_2 X}{W} dx = +\int \frac{e^{-2x} \sin e^x}{e^{-3x}} dx = \int e^x \sin e^x dx$

Put $e^x = t$

$e^x dx = dt$

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$$u = \int \sin t \, dt = -\cos t$$

$$u = -\cos e^{x}$$

and $v = \int \frac{y_1 X}{W} = -\int \frac{e^{-x} \sin e^{x} dx}{e^{-3x}}$

$$= -\int e^{2x} \sin e^{x} dx$$

Put $e^{x} = t$

$e^{x} dx = dt$

$$= -\int \underset{I}{t} \underset{II}{\sin t} \, dt$$

By using By Parts

$$= +\left[t \cos t + \int 1 \cdot \cos t \, dt \right]$$

$$v = \left[t \cos t + \sin t \right] = \left[e^{x} \cos e^{x} + \sin e^{x} \right]$$

$$PI = uy_1 + vy_2$$

$$= -\cos e^{x} e^{-x} + \left[e^{x} \cos e^{x} + \sin e^{x} \right] e^{-2x}$$

$$= -\cancel{\cos e^{x} \cdot e^{-x}} + \cancel{e^{-x} \cos e^{x}} + e^{-2x} \sin e^{x}$$

$$PI = e^{-2x} \sin e^{x}$$

$\therefore$ solution is:-

$$y = CF + PI$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x} \sin e^{x}$$

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Q- Solve - $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$ by using Variation of Parameters

Sol- $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

$SE \Rightarrow (D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

$AE \Rightarrow D^2 - 6D + 9 = 0$

$D = \frac{6 \pm \sqrt{36-36}}{2} = \frac{6}{2}$

$D = 3, 3$

$Cf \Rightarrow (C_1 + C_2 x)e^{3x}$

$= C_1 e^{3x} \; C_2 x e^{3x}$

here $y_1 = e^{3x}, \; y_2 = xe^{3x}$

$y_1' = 3e^{3x}, \; y_2' = 3xe^{3x} + e^{3x}$

So, $W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}(3x+1) \end{vmatrix}$

$= e^{6x}(3x+1) - 3e^{6x}x$

$= e^{6x}(\cancel{3x} + 1 - \cancel{3x})$

$W = e^{6x}$

Now, $u = -\int \frac{y_2 X}{W} dx = -\int \cancel{x e^{3x}} \cdot \frac{e^{3x}}{x^{\cancel{2}}} \cdot \frac{1}{\cancel{e^{6x}}} dx$

$= -\int \frac{1}{x} dx = \boxed{-\log x}$

and $v = \int \frac{y_1 X}{W} dx = \int e^{3x} \cdot \frac{e^{3x}}{x^2} \cdot \frac{1}{e^{6x}} dx$

$= \int \frac{1}{x^2} dx = \int x^{-2} dx = \boxed{\frac{-1}{x}}$

So, $PI = uy_1 + vy_2$

$= -\log x \cdot e^{3x} - \frac{1}{\cancel{x}} \cdot \cancel{x} e^{3x}$

$= -e^{3x}(\log x + 1)$

$\therefore$ solution is :-

$y = e^{3x}(C_1 + C_2 x) - e^{3x}(\log x + 1)$

$y = e^{3x}(C_1 + C_2 x - \log x - 1)$

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Ques:- Solve the differential equation-

$$(D^2-6D+13)y = 8e^{3x}\sin 4x + 2^x$$

Sol: AE $\Rightarrow$ $D^2-6D+13=0$

$$D = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$$

$$D = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

$$Cf = e^{3x}(C_1\cos 2x + C_2 \sin 2x)$$

$$PI = \frac{1}{f(D)} 8e^{3x}\sin 4x + 2^x$$

$$= \frac{1}{D^2-6D+13} 8e^{3x}\sin 4x + \frac{1}{D^2-6D+13} 2^x$$

$$= \underbrace{8 \cdot \frac{1}{D^2-6D+13} e^{3x}\sin 4x}_{8I_1} + \underbrace{\frac{1}{D^2-6D+13} 2^x}_{I_2} = 8I_1 + I_2$$

$$I_1 = \cancel{8}\frac{1}{D^2-6D+13} \cdot e^{3x}\sin 4x$$

Acc ~~to~~ Case 3 i.e. $X = e^{ax}V$

So, Put $D = D+3$ Put $D = D+a$

$$I_1 = e^{3x}\frac{1}{(D+3)^2-6(D+3)+13}\sin 4x$$

$$= e^{3x}\frac{1}{D^2+9+\cancel{6D}-\cancel{6D}-18+13}\sin 4x$$

$$= e^{3x}\frac{1}{D^2+4}\sin 4x$$

Put $D^2 = -(4)^2 = -16$

$$= e^{3x}\cdot\frac{1}{-16+4}\sin 4x = \frac{-1}{12}e^{3x}\sin 4x$$

$$= \boxed{-\frac{1}{12}e^{3x}\sin 4x}$$

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$$I_2 = \frac{1}{D^2-6D+13} \cdot 2^x$$

$$= \frac{1}{D^2-6D+13} e^{\log 2^x}$$

$$= \frac{1}{D^2-6D+13} e^{x\log 2}$$

Put $D = \log 2$

$$I_2 = \frac{1}{(\log 2)^2 - 6\log 2 + 13} e^{x\log 2} = \frac{1}{(\log 2)^2 - 6\log 2+13} 2^x$$

So, $PI = 8I_1 + I_2$

$$= {}^2\cancel{8}\left(-\frac{1}{\cancel{12}_3} e^{3x} \sin 4x\right) + 2^x \frac{1}{(\log 2)^2 - 6\log 2 + 13}$$

$$PI = -\frac{2}{3}e^{3x} \sin 4x + 2^x \frac{1}{(\log 2)^2 - 6\log 2+13}$$

$\therefore$ solution is :-

$$y = Cf + PI$$

$$y = e^{3x}(C_1\cos 2x + C_2 \sin 2x) - \frac{2}{3}e^{3x}\sin 4x + \frac{2^x}{(\log 2)^2 - 6\log 2 + 13}$$

Ques- Solve the differential eq^n-

$$(D^2+2D+2)y = e^{-x}\sec x$$

Sol- AE $\Rightarrow D^2+2D+2 = 0$

$$D = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2}$$

$$D = -1 \pm i$$

$$Cf = e^{-x}(C_1 \cos x + C_2 \sin x)$$

$$PI = \frac{1}{D^2+2D+2} \cdot e^{-x}\sec x$$

Put $D = D-1$

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$$= e^{-x}\frac{1}{(D-1)^2+2(D-1)+2}\sec x$$

$$= e^{-x}\frac{1}{D^2+1-\cancel{2D}+\cancel{2D}-\cancel{2}+\cancel{2}}\sec x$$

$$= e^{-x}\left[\frac{1}{D^2+1}\sec x\right] \rightarrow \text{By Partial factorisation}$$

$$= e^{-x}\left[\frac{1}{(D+i)(D-i)}\sec x\right]$$

let $\frac{1}{(D+i)(D-i)} = \frac{A}{(D+i)} + \frac{B}{(D-i)}$ —— (1)

$$1 = A(D-i) + B(D+i)$$

Put $D = i$

$$1 = A(0) + B(2i)$$

$\boxed{B = \frac{1}{2i}}$, and put $D = -i$

$$1 = A(-2i) + B(0)$$

$$\boxed{A = \frac{-1}{2i}}$$

Put in (1)

$$\frac{1}{(D+i)(D-i)} = \frac{-1}{2i(D+i)} + \frac{1}{2i(D-i)}$$

Putting in PI

$$PI = e^{-x}\left[\frac{-1}{2i(D+i)} + \frac{1}{2i(D-i)}\right]\sec x.$$

$$= e^{-x}\frac{1}{2i}\left[\underbrace{\frac{1}{(D-i)}\sec x}_{I_1} - \underbrace{\frac{1}{(D+i)}\sec x}_{I_2}\right]$$

$I_1 = \frac{1}{(D-i)}\sec x$ Acc. to Theorem 3

$$\frac{1}{\ }X = e^{ax}\int X e^{-ax}\,dx$$

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$$I_1 = e^{ix}\int \sec x\, e^{-ix}\,dx$$

$$= e^{ix}\int \sec x\,(\cos x - i\sin x)\,dx$$

$$= e^{ix}\int \frac{1}{\cos x}(\cos x - i\sin x)\,dx$$

$$\cancel{= -e^{ix}\int i\tan x\,dx} = e^{ix}\int 1 - i\tan x\,dx$$

$$\cancel{= -ie^{ix}\int \tan x\,dx} = e^{ix}\int 1 - i\tan x\,dx$$

$$\cancel{= -ie^{ix}\left[-\log|\cos x|\right]} = e^{ix}\left[x + i\log|\cos x|\right]$$

$$\cancel{I_1 = ie^{ix}\log|\cos x|}\quad \boxed{I_1 = e^{ix}\left[x + i\log|\cos x|\right]}$$

and $I_2 = \frac{1}{D+i}\sec x$

Acc. to Theorem 3

$$\frac{1}{D+a}X = e^{-ax}\int X e^{ax}\,dx$$

$$I_2 = e^{-ix}\int \sec x\, e^{ix}\,dx$$

$$= e^{-ix}\int \sec x\,(\cos x + i\sin x)\,dx \qquad \sec x \to \frac{1}{\cos x}$$

$$= e^{-ix}\int (1 + i\tan x)\,dx$$

$$\boxed{I_2 = e^{-ix}\left[x - i\log|\cos x|\right]}$$

So,

$$PI = \frac{e^{-x}}{2i}\left[e^{ix}\left(x + i\log|\cos x|\right) - e^{-ix}\left(x - i\log|\cos x|\right)\right]$$

$$PI = \frac{e^{-x}}{2i}\left[x\left(e^{ix} - e^{-ix}\right) + i\log|\cos x|\left(e^{ix} + e^{-ix}\right)\right]$$

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$\therefore$ solution is :-

$$y = CF + PI$$

$$y = e^{-x}(c_1\cos x + c_2\sin x) + \frac{e^{-x}}{2i}\left[x(e^{ix}-e^{-ix}) + i\log|\cos x|(e^{ix}+e^{-ix})\right]$$

$$= e^{-x}\left[(c_1\cos x + c_2\sin x) + \frac{1}{2i}\left\{x(e^{ix}-e^{-ix}) + i\log|\cos x|(e^{ix}+e^{-ix})\right\}\right]$$

* Cauchy's Homogeneous Linear Equation :-

Any equation of the form -

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \dots\dots a_ny = X$$

(where a_i's are constants and X is a function of x) is known as Cauchy's ~~linear~~ Homogeneous Linear eq^n

for eg - $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = X$

$\Rightarrow$ We can solve this by reducing Cauchy linear differential eq^n into linear differential eq^n with constant cofficients.

To reduce to the eq^n :-

Step 1 Put $x = e^z \Rightarrow z = \log x \rightarrow \frac{dz}{dx} = \frac{1}{x}$

Step 2 $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{x}$

$$\boxed{x\frac{dy}{dx} = \frac{dy}{dz} = Dy} \quad \text{where } D = \frac{d}{dz}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dz^2}\cdot\overset{\frac{1}{x}}{\frac{dz}{dx}} + \left(\frac{-1}{x^2}\right)\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

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$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D^2y - Dy$$

$$\boxed{x^2\frac{d^2y}{dx^2} = D(D-1)\,y}$$

$$\boxed{x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)\,y}$$

$$\left(D = \frac{d}{dz}\right)$$

Step 3 find CF

Step 4 find PI

Step 5 Its solution is :-

$$y = CF + PI$$

Ques- Solve the differential eq^n -

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x+\frac{1}{x}\right) \quad \text{—— } (1)$$

Sol- The given eq^n is of the form Cauchy homogeneous linear equation.

So, Put $x = e^z \Rightarrow z = \log x$

$$x\frac{dy}{dx} = Dy,\quad x^2\frac{d^2y}{dx^2} = D(D-1)y,\quad x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

where $D = \frac{d}{dz}$

Put in (1)

$$D(D-1)(D-2)y + 2D(D-1)y + 2y = 10\left(x+\frac{1}{x}\right)$$

$$(D^3 - \cancel{2D^2} - D^2 + \cancel{2D})y + (\cancel{2D^2} - \cancel{2D})y + 2y = 10\left(e^z + \frac{1}{e^z}\right)$$

$$(D^3 - D^2 + 2)\,y = 10\left(e^z + \frac{1}{e^z}\right)$$

AE $\Rightarrow$ $D^3 - D^2 + 2 = 0$

$$(D+1)(D^2 - 2D + 2) = 0$$

$$D = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$$D = -1,\ 1+i,\ 1-i$$

$$Cf = C_1 e^{-z} + e^{+z}(C_2 \cos z + C_3 \sin z)$$

$$cf = \frac{C_1}{x} + x\left(C_2 \cos(\log x) + C_3 \sin(\log x)\right)$$

$$PI = \frac{1}{f(D)} X = \frac{1}{D^3 - D^2 + 2} 10(e^z + e^{-z})$$

$$= 10\left[\underbrace{\frac{1}{D^3 - D^2 + 2} e^{z}}_{I_1} + \underbrace{\frac{1}{D^3 - D^2 + 2} e^{-z}}_{I_2}\right]$$

$$I_1 = \frac{1}{D^3 - D^2 + 2} e^z$$

Put $D = 1$

$$= \frac{1}{1^3 - 1^2 + 2} e^z = \frac{1}{2} e^z$$

$$I_2 = \frac{1}{D^3 - D^2 + 2} e^z$$

Put $D = -1$

$$= \frac{1}{-1 - 1 + 2} = \frac{1}{0} \quad \text{(case of faliur)}$$

$$\Rightarrow z \cdot \frac{1}{3D^2 - 2D} e^{-z}$$

Put $D = -1$

$$z \cdot \frac{1}{3+2} e^{-z} = \frac{1}{5} z e^{-z}$$

So, $PI = 10\left[\frac{1}{2} e^z + \frac{1}{5} z e^{-z}\right]$

$$PI = 5e^z + 2ze^{-z}$$

$$= 5x + \frac{2\log x}{x}$$

$\therefore$ solution is:-

$$y = \frac{1}{x} C_1 + x\left(C_2 \cos(\log x) + C_3 \sin(\log x)\right) + 5x + \frac{1}{x} 2\log x$$

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Solve the differential eqⁿ.

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x \quad \text{--- (1)}$$

Sol. The given eqⁿ is Cauchy homogeneous linear eqⁿ.

Put $x = e^z \Rightarrow z = \log x$

So, $x\dfrac{dy}{dx} = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

Put in (1)

$$D(D-1)y - Dy - 3y = ze^{2z}$$

$$(D^2-2D-3)y = ze^{2z}$$

$$AE \Rightarrow D^2-2D-3=0$$

$$D = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2\pm 4}{2} = 1 \pm 2$$

$$D = -1, 3$$

$$Cf = C_1e^{-z} + C_2e^{3z} = \frac{C_1}{x} + C_2x^3$$

~~So $y_1 = e^{-x}$, $y_2 = e^{3x}$~~

~~$y_1' = -e^{-x}$, $y_2' = 3e^{3x}$~~

$$\cancel{W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix}}$$

$$\cancel{= 3e^{3x}e^{-x} + e^{3x}e^{-x}}$$

$$\cancel{= 4e^{3x}e^{-x} = 4e^{2x}}$$

$$\cancel{u = -\int \frac{y_2 X}{W} = \int e^{3x} \cdot x^2\log x}$$

$$PI = \frac{1}{f(D)}e^{2z}\cdot z$$

$$= \frac{1}{D^2-2D-3}e^{2z}\cdot z$$

Put $D = D+2$

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$$= e^{2z}\left[\frac{1}{(D+2)^2-2(D+2)-3}\right]z$$

$$= e^{2z}\left[\frac{1}{D^2+2D-3}\right]z$$

$$= \frac{e^{2z}}{-3}\left[\frac{1}{1-\left[\frac{D^2}{3}+\frac{2D}{3}\right]}\right]z$$

$$= \frac{e^{2z}}{-3}\left[1-\left(\frac{D^2}{3}+\frac{2D}{3}\right)\right]^{-1}z$$

$$= \frac{e^{2z}}{-3}\left[1-\left(\frac{D^2+2D}{3}\right)\right]^{-1}z$$

$$= \frac{e^{2z}}{-3}\left[1+\frac{D^2+2D}{3}+\left(\frac{D^2+2D}{3}\right)^2+ - - - - - -\right]z$$

$$= \frac{e^{2z}}{-3}\left[1+\frac{2D}{3}\right]z$$

$$= \frac{e^{2z}}{-3}\left(z+\frac{2}{3}(1)\right)$$

$$= -\frac{e^{2z}}{3}\left(z+\frac{2}{3}\right) = -\frac{e^{2z}}{9}(3z+2)$$

$$PI = -\frac{x^2}{9}(3\log x+2)$$

$\therefore$ solution is :-

$$y = Cf + PI$$

$$y = \frac{C_1}{x} + C_2x^3 - \frac{x^2}{9}(3\log x+2)$$

Ques- Solve - $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$ — (1)

Sol- The given eq^n is Cauchy homogeneous linear eq^n.

So, put $x = e^z \Rightarrow z = \log x$

$\therefore x^2\frac{d^2y}{dx^2} = D(D-1)y$, $x\frac{dy}{dx} = Dy$

where $D = \frac{d}{dz}$

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Put in ①, we get -

$$D(D-1)y + Dy + y = z\sin z$$

$$(D^2 - \cancel{D} + \cancel{D} + 1)y = z\sin z$$

$$(D^2+1)y = z\sin z$$

AE $\Rightarrow$ $D^2+1=0$

$D^2 = -1$

$D = \pm i$

$$CF = e^{0z}(C_1\cos z + C_2\sin z)$$

$$= C_1\cos(\log x) + C_2\sin(\log x)$$

$$PI = \frac{1}{D^2+1} z\sin z$$

$$= \left[z - \frac{2D}{D^2+1}\right] \cdot \frac{1}{D^2+1}\sin z$$

Put $D^2 = -(1)^2 = -1$

$$= \left[z - \frac{2D}{D^2+1}\right] \frac{1}{0}\sin z$$

Case of failure

$$= \left[z - \frac{2D}{D^2+1}\right] z\frac{1}{2D}\sin z$$

$$= \left[z - \frac{2D}{D^2+1}\right] \frac{z}{2}\cos z \qquad \left(\because \frac{1}{D}\sin z = \int \sin z\right)$$

$$= \left[\frac{z^2}{2}\cos z - \frac{D}{D^2+1} z\cos z\right]$$

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* Legendre's Linear Differential Equation:-

Any eq^n of the form-

$$(ax+b)^n \frac{d^ny}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ---- a_n y = X$$

(where a_i's are constants & X is a function of x) is known as Legendre's Linear Differential eq^n.

⇒ Working rule-

Such eq^n can be solved by reducing the given eq^n to linear differential eq^n with constant coefficients.

Step 1 Put $ax+b = e^z$

$$z = \log(ax+b)$$

$$\frac{dz}{dx} = \frac{a}{ax+b}$$

Step 2 $\boxed{(ax+b)\frac{dy}{dx} = aDy}$ where $D = \frac{d}{dz}$ $\left(\because \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}\right)$

$$\boxed{(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y}$$

and so on...

Step 3 find Cf

Step 4 find PI

Step 5 Its solution is:-

$$y = Cf + PI$$

Ques- Solve the differential eq^n-

$$(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1 \quad \text{—— (1)}$$

Sol- The given eq^n is Legendre's linear differential eq^n.

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Put $(3x+2) = e^z \Rightarrow x = \frac{e^z-2}{3}$

$z = \log(3x+2)$

so that,

$(3x+2)\frac{dy}{dx} = 3Dy$

$(3x+2)^2\frac{d^2y}{dx^2}$ $9(D-1)Dy$

where $D = \frac{d}{dz}$

Put in ①

$$9D(D-1)y + 9Dy - 36y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

$$(9D^2 - 9D + 9D - 36)y = \frac{1}{3}(e^{2z}+4-4e^z) + \frac{4e^z-8}{3} + 1$$

$$(9D^2-36)y = \frac{e^{2z}+4-\cancel{4e^z}+\cancel{4e^z}-8+3}{3}$$

$$(9D^2-36)y = \frac{e^{2z}-1}{3}$$

$$(D^2-4)y = \frac{e^{2z}-1}{27}$$

AE $\Rightarrow D^2-4=0$

$D^2 = 4$

$D = \pm 2$

$Cf = C_1e^{2z} + C_2e^{-2z}$

$= C_1(3x+2)^2 + C_2(3x+2)^{-2}$

$PI = \frac{1}{D^2-4}\left(\frac{e^{2z}-1}{27}\right)$

$= \frac{1}{27}\left[\frac{1}{D^2-4}e^{2z} - \frac{1}{D^2-4}e^{0z}\right]$

Put $D=2$ Put $D=0$

$= \frac{1}{27}\left[\frac{1}{0}e^{2z} - \frac{1}{4}\right]$

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$$= \frac{1}{27}\left[z\frac{1}{2D}e^{2z}+\frac{1}{4}\right]$$

Put $D=2$

$$= \frac{1}{27}\left[\frac{ze^{2z}}{4}+\frac{1}{4}\right]$$

$$= \frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right]$$

$$PI = \frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right]$$

$\therefore$ solution is :-

$y = CF + PI$

$$y = C_1(3x+2)^2 + C_2(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right]$$

Ques- $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\cdot\frac{dy}{dx} + y = \sin 2(\log(1+x))$ —— (1)

Sol- Given eq^n is Legendre's linear diff. eq^n.

Put $(x+1) = e^z \Rightarrow z = \log(x+1)$

so that,

$$(x+1)^2\frac{d^2y}{dx^2} = (1)^2 D(D-1)y \;\&\; (x+1)\frac{dy}{dx} = 1\cdot Dy$$

where $D = \frac{d}{dz}$

Put in (1)

$$D(D-1)y + Dy + y = \sin 2z$$

$$(D^2+1)y = \sin 2z$$

AE $\Rightarrow D^2+1=0$

$D^2 = -1$

$D = \pm i$

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$$CF = e^{0z}(C_1\cos z + C_2\sin z)$$
$$= C_1\cos(\log(x+1)) + C_2\sin(\log(x+1))$$

$$PI = \frac{1}{D^2+1}\sin 2z$$

Put $D^2 = -(2)^2 = -4$

$$PI = \frac{1}{-4+1}\sin 2z = -\frac{1}{3}\sin 2z$$

$$PI = -\frac{1}{3}\sin 2(\log(x+1))$$

$\therefore$ solution is :-

$$y = CF + PI$$
$$y = C_1\cos(\log(1+x)) + C_2\sin(\log(1+x)) - \frac{1}{3}\sin 2(\log(1+x))$$

Ques- Solve -

$$(2x+3)^2\frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x \quad \text{—— (1)}$$

Sol- The given eq^n is Legendre's linear diff eq^n.

so, let $2x+3 = e^z \Rightarrow x = \frac{e^z-3}{2}$

$$z = \log(2x+3)$$

$$\therefore (2x+3)\frac{dy}{dx} = 2Dy$$

$$(2x+3)^2\frac{d^2y}{dx^2} = (2)^2 D(D-1)y$$

where $D = \frac{d}{dz}$

Put in (1)

$$4D(D-1)y - 4Dy - 12y = \cancel{6}^{3}\left(\frac{e^z-3}{\cancel{2}}\right)$$
$$(4D^2-4D-4D-12)y = 3(e^z-3)$$
$$(4D^2-8D-12)y = 3(e^z-3)$$
$$(D^2-2D-3)y = \frac{3}{4}(e^z-3)$$

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$AE \Rightarrow D^2 - 2D \cancel{+}3 = 0$

$D = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = 1 \pm 2$

$D = -\cancel{1}, 3$

$Cf = C_1 e^{-\cancel{1}z} + C_2 e^{3z}$

$= C_1 (2x+3)^{-\cancel{1}} + C_2 (2x+3)^{3}$

$PI = \frac{1}{D^2-2D-3} \cdot \frac{3}{4}(e^z - 3)$

$= \frac{3}{4}\left[\frac{1}{D^2-2D-3} e^z - 3\left(\frac{1}{D^2-2D-3}\right)e^{0z}\right]$

Put $D=1$ — Put $D=0$

$= \frac{3}{4}\left[\frac{1}{-4} e^z + \cancel{3}\left(\frac{1}{\cancel{+3}}\right)e^{0z}\right]$

$PI = \frac{3}{4}\left[1 - \frac{e^z}{4}\right] = \frac{3}{4}\left[1 - \frac{(2x+3)}{4}\right]$

∴ solution is :-

$y = Cf + PI$

$y = C_1 (2x+3)^{-3} + C_2 (2x+3)^{5} + \frac{3}{16}(\cancel{7} - 2x)$

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Sequences

$N \rightarrow$ Natural number

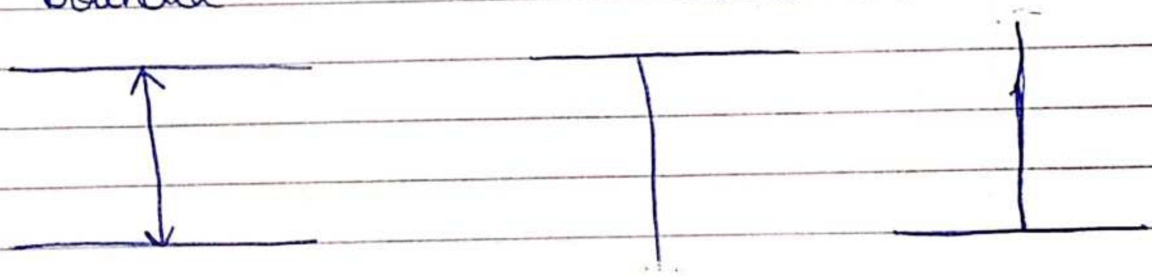
$X \rightarrow$ any set

$f : N \rightarrow X$ Sequence (whose domain is ~~set of~~ N)

Domain   Range

let $\{a_n\}$ and $\{b_n\}$ are two sequences, then $\{a_n\} = \{b_n\}$ only if all the terms of $\{a_n\}$ & $\{b_n\}$ are equal i.e $a_n = b_n$

$\Rightarrow$ Bounded   Unbounded



- A sequence is said to be bounded above if there exist a real number K such that $a_n \leq K \ \forall \ n \in N$ (highest value is K)
  K is called the upper bound of sequence $\{a_n\}$
- A sequence $\{a_n\}$ is said to be bounded below if there exist a real number h such that $h \leq a_n \ \forall \ n \in N$, h is called the lower bound of sequence $\{a_n\}$
- A sequence $\{a_n\}$ is called bounded if there exist two real numbers h & K such that $h \leq a_n \leq K \ \forall \ n \in N$ (bounded above as well as below).

for eg $\rightarrow a_n = \frac{1}{n}$; $\forall n \in N$

$n \geq 1 \Rightarrow \frac{1}{n} \leq 1$

$\frac{1}{n} \geq 0$,

lower bound $\leftarrow 0 < \frac{1}{n} \leq 1 \rightarrow$ upper bound $\therefore \frac{1}{n}$ is bounded.