Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

Cluster Programme - M.Tech. in Data Science and Engg.

I Semester 2019-20

Course Number	DSECF ZC416
Course Name	Mathematical Foundation for Data Science
Nature of Exam	Closed Book
Weightage for grading	30%
Duration	90 minutes
Date of Exam	22/12/2019 (10:00 a.m - 11:30 a.m)

# Pages	2
# Questions	6

Instructions

1. All questions are compulsory
2. Questions are to be answered in the order in which they appear in this paper and in the page numbers mentioned before each of them.

Pages 2-5

Q1a) Let V be a finite dimensional vector space over $\mathbb{R}$. If S is a set of elements in V such that Span(S) $= V$, are the elements of S necessarily linearly independent? If so, prove it or else disprove using a counter example. (1)

b) Suppose M is an $n \times n$ upper triangular matrix with diagonal entries non-zero, then prove or disprove that the column vectors are linearly independent. How would the scenario change when some of the diagonal entries are zero. (1+1)

c) Construct an example of i) vector spaces V and V' over the same field F such that dim(V over F) $\neq$ dim(V' over F) and ii) vector space V over fields F and G such that dim (V over F) = dim (V over G). (1+1)

Pages 6-9

Q2a) Assume A is a $n \times m$ matrix. Verify whether $T : M_{ln} \to M_{lm}$ with $T(B) = BA$ is linear transformation or not, where M_{rq} denotes the set of all $r \times q$ matrices. (1)

b) Let $T : R^3 \to R^3$ be linear and suppose that $T((1, 0, 1)) = (1, -1, 3)$ and $T((2, 1, 0)) = (0, 2, 1)$. Determine $T((8, 3, 2))$ from the given data. Is this value unique? Justify. (1+1)

c) Verify that Rank-Nullity Theorem for the linear transformation $T : R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (-x_1 + x_2 + x_3, 2x_1 - x_2, x_1 + x_2 + 3x_3)$. (2)

Pages 10-13

Q3a) Consider a real $n \times n$ matrix A. Let $\lambda_1, \lambda_2........\lambda_k.......\lambda_n$ be its eigenvalues. Let $\beta \in \mathbb{R}$ be such that $\beta \neq \lambda_i, \forall i = 1, 2......n$. Also let (λ_k, v) be an eigen pair of A, i.e $Av = \lambda_k v$. Prove that $(A - \beta I)$ is an invertible matrix and then using that show that v is an eigenvector of $(A - \beta I)^{-1}$. Is it possible to find the corresponding eigenvalue? If so, determine that. (1+1)

b) Consider two real $n \times n$ matrices A and B. It is known that $AB = BA$. It is also known that all the eigenvalues of A are distinct. Prove that if x is an eigenvector of A then x is also an eigenvector of B. (2)

c) Let A be a real $n \times n$ matrix. It is also known that for each row, the absolute value of diagonal entry is greater than sum of absolute values of non-diagonal entries i.e., $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \ \forall\ i = 1, 2, \ldots, n$. Prove or disprove that A has only non-zero eigenvalues. (1)

Pages 14-16

Q4a) Applying Gaussian elimination with and without partial pivoting and 4 digit arithmetic with rounding and compare the results with the exact solution $x_1 = 10.00$ and $x_2 = 1.000$ (1.5+1.5)

$$0.003x_1 + 59.14x_2 = 59.17$$
$$5.291x_1 - 6.130x_2 = 46.78$$

b) An LU-factorization of a given square matrix A is of the form $A = LU$ where L is a lower triangular and U is a upper triangular. How many divisions, multiplications and additions are required to obtain L and U, if A is of order $n \times n$ and the factorization exists. (2)

Pages 17-19

Q5a) Let A be an $n \times n$ matrix. Assuming that $Ax = B$ has a unique solution, count the number of divisions, multiplications, additions required to get the solution using Gauss-Jordan method. (3)

b) Test using the spectral method or suitable matrix norms, the convergence of Gauss Jacobi method for the following system (2)

$$2x + y + z = 4$$
$$x + 2y + z = 4$$
$$x + y + 2z = 4$$

Pages 20-22

Q6a) A machine shop has one drill press and five milling machines, which are to be used to produce an assembly consisting of two parts, 1 and 2. The productivity of each machine for the two parts is given below.

	Production time in minutes / piece	
Part	Drill	Mill
1	3	20
2	5	15

It is desired to maintain a balanced loading on all machines such that no machine runs more than 30 minutes per day longer than any other machine (assume that the milling load is split evenly among all the five milling machines).

Divide the work time of each machine to obtain the maximum number of completed assemblies assuming an 8-hour working day. Note that a completed assembly requires equal number of parts 1 and 2. (4)

b) For the following LPP problem, calculate the shadow prices for the resources R_1 and R_2. (1)

Max$\mathbf{Z} = 45x_1 + 30x_2$
Subject to
$3x_1 + 4x_2 \leq 8$ (resource R_1)
$2x_1 + 5x_2 \leq 12$ (resource R_2)
$x_1 \geq 0, x_2 \geq 0$

→ All the best ←