2D TRANSFORMATION

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2D Transformation

Transformation means changing some graphics into something else by applying rules. We can have various types of transformations such as translation, scaling up or down, rotation, shearing, etc. When a transformation takes place on a 2D plane, it is called 2D transformation.

Transformations play an important role in computer graphics to reposition the graphics on the screen and change their size or orientation.

2D Transformation

Given a 2D object, transformation is to change the object’s

Position (translation)

Size (scaling)

Orientation (rotation)

Shapes (shear)

Apply a sequence of matrix multiplication to the object vertices

2D Transformation

Rigid Body Transformations - transformations that do not change the object.

- Translate

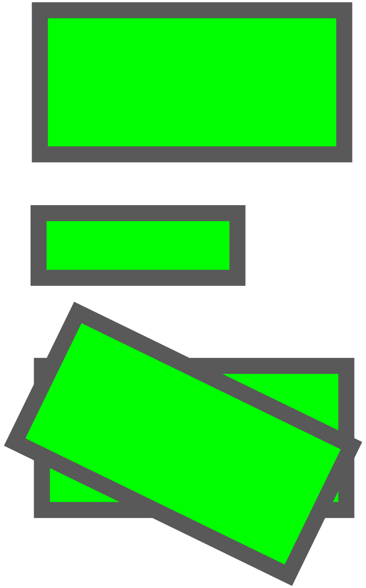
If you translate a rectangle, it is still a rectangle

- Scale

If you scale a rectangle, it is still a rectangle

- Rotate

If you rotate a rectangle, it is still a rectangle



We have always represented vertices as (x,y)

An alternate method is:

$$(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}$$

Example:

$$(2.1, 4.8) = \begin{bmatrix} 2.1 \\ 4.8 \end{bmatrix}$$

Point representation

- We can use a column vector (a 2x1 matrix) to represent a 2D point $\begin{vmatrix} x \\ y \end{vmatrix}$
- A general form of *linear* transformation can be written as:

$$x' = ax + by + c \qquad \text{OR} \qquad \begin{vmatrix} X' \\ Y' \\ 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

$$y' = dx + ey + f$$

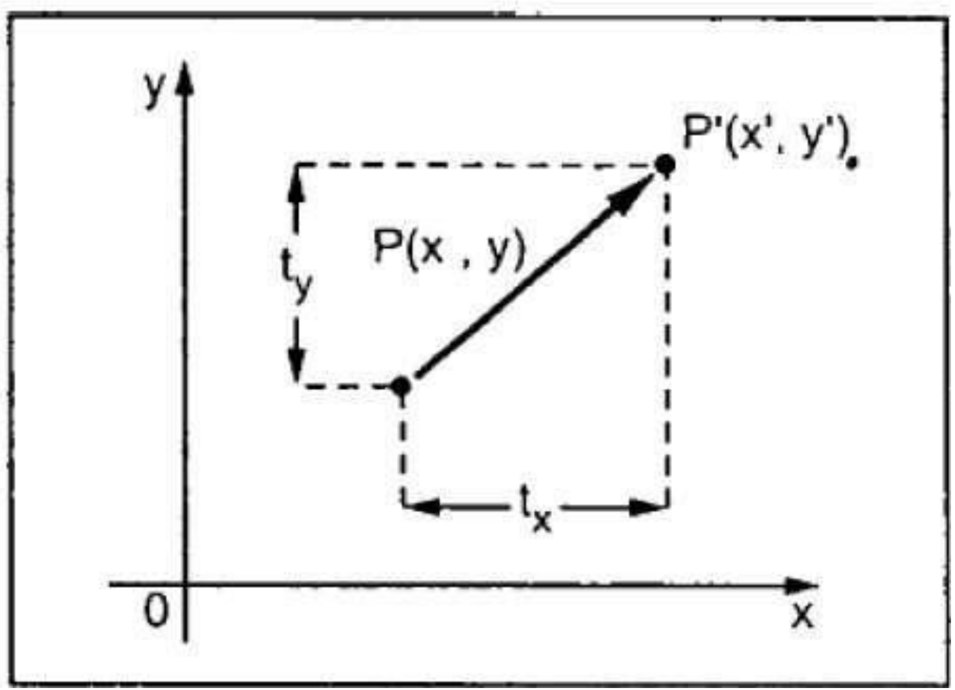
Translation

A translation moves an object to a different position on the screen. You can translate a point in 2D by adding translation coordinate (tx, ty) to the original coordinate (X, Y) to get the new coordinate (X’, Y’).

From the figure, you can write that −

X’ = X + tx

Y’ = Y + ty



The pair (t_x, t_y) is called the translation vector or shift vector. The above equations can also be represented using the column vectors.

$$P = \frac{[X]}{[Y]} \qquad p' = \frac{[X']}{[Y']} \qquad T = \frac{[t_x]}{[t_y]}$$

We can write it as −

$$\mathbf{P' = P + T}$$

3x3 2D Translation Matrix

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} tx \\ ty \end{vmatrix}$$

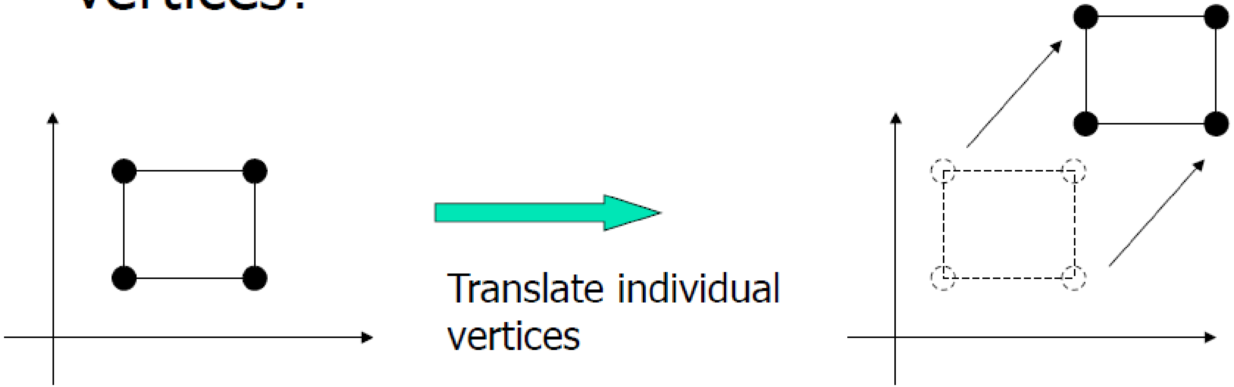
$\Downarrow$ Use 3 x 1 vector

$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

- Note that now it becomes a matrix-vector multiplication

Translation

- How to translate an object with multiple vertices?



Translation Example

- Given:

$$P = (x, y)$$
$$T = (t_x, t_y)$$

$$P = (-3.7, -4.1)$$
$$T = (7.1, 8.2)$$

- We want:

$$x' = x + t_x$$
$$y' = y + t_y$$

$$x' = -3.7 + 7.1$$
$$y' = -4.1 + 8.2$$

- Matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$
$$P' = P + T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3.7 \\ -4.1 \end{bmatrix} + \begin{bmatrix} 7.1 \\ 8.2 \end{bmatrix}$$
$$x' = 3.4$$
$$y' = 4.1$$

Translation Examples

- P=(2,4), T=(-1,14), P’=(?,?)
- P=(8.6,-1), T=(0.4,-0.2), P’=(?,?)
- P=(0,0), T=(1,0), P’=(?,?)

Rotation

In rotation, we rotate the object at particular angle θ (theta) from its origin. From the following figure, we can see that the point P(X, Y) is located at angle φ from the horizontal X coordinate with distance r from the origin.

Let us suppose you want to rotate it at the angle θ. After rotating it to a new location, you will get a new point P' (X', Y').

Using standard trigonometric the original coordinate of point P(X, Y) can be represented as −

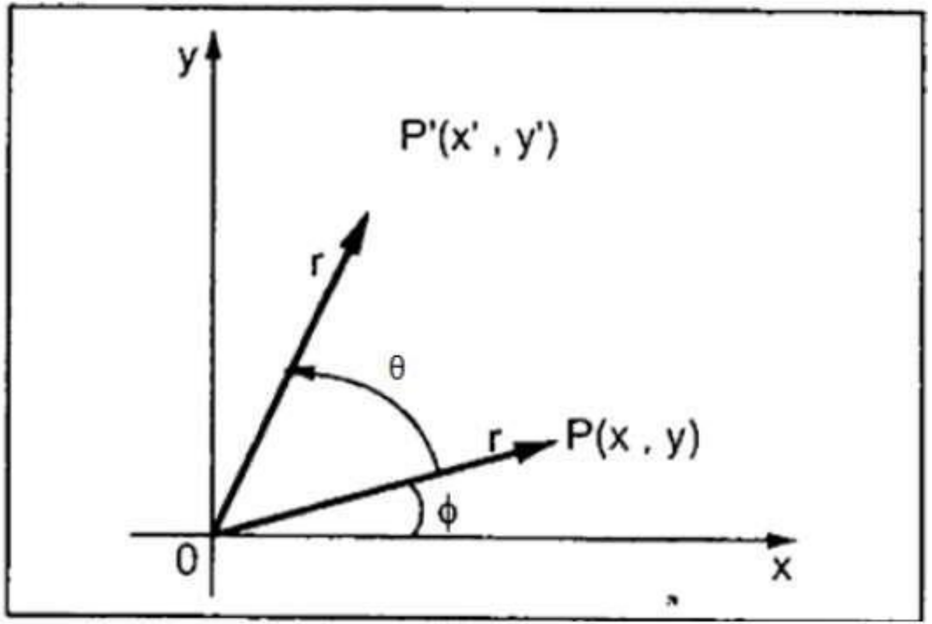
$$X = r\cos\phi \ldots\ldots(1)$$

$$Y = r\sin\phi \ldots\ldots(2)$$

Same way we can represent the point P′ (X′, Y′) as −

$$x' = r\,cos\ (\phi + \theta) = r\,cos\,\phi\,cos\,\theta\ -\ r\,sin\,\phi\,sin\,\theta \ldots\ldots.(3)$$

$$y' = r\,sin\ (\phi + \theta) = r\,cos\,\phi\,sin\,\theta + r\,sin\,\phi\,cos\,\theta \ldots\ldots.(4)$$



Substituting equation (1) & (2) in (3) & (4) respectively, we will get

$$x' = x\cos\theta - \ y\sin\theta$$

$$y' = x\sin\theta + \ y\cos\theta$$

Representing the above equation in matrix form,

$$[X'Y'] = [XY]\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} OR$$

P’ = P . R

Where R is the rotation matrix

$$R = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

The rotation angle can be positive and negative.

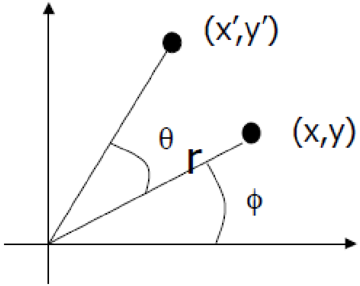
For positive rotation angle, we can use the above rotation matrix. However, for negative angle rotation, the matrix will change as shown below –

$$R = \begin{bmatrix} cos(-\theta) & sin(-\theta) \\ -sin(-\theta) & cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} (\because cos(-\theta) = cos\theta \; and \; sin(-\theta) = -sin\theta)$$

3x3 2D Rotation Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

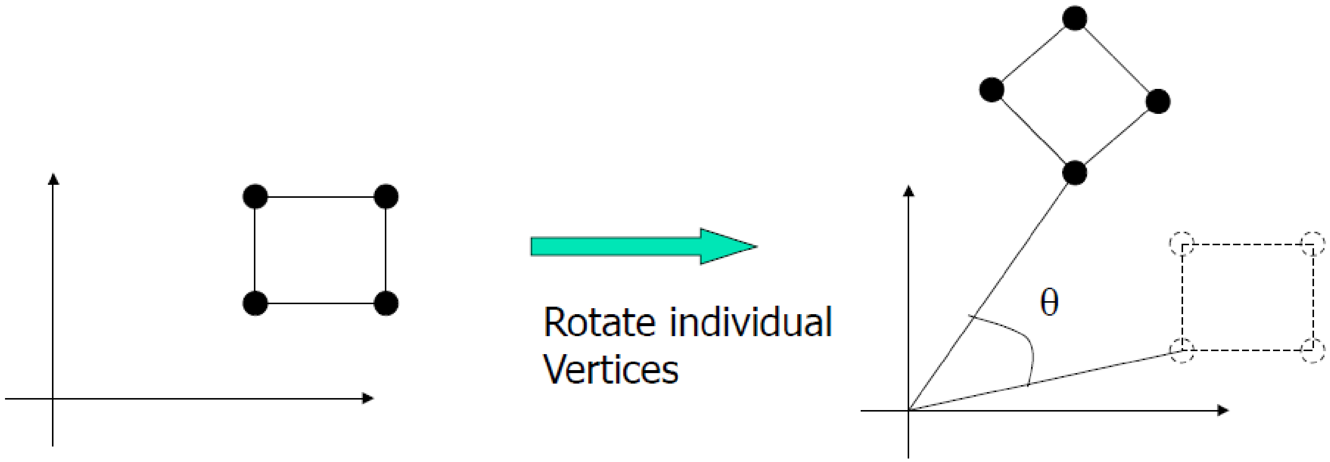


$$\Downarrow$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation

- How to rotate an object with multiple vertices?



Scaling

To change the size of an object, scaling transformation is used. In the scaling process, you either expand or compress the dimensions of the object. Scaling can be achieved by multiplying the original coordinates of the object with the scaling factor to get the desired result.

Let us assume that the original coordinates are (X, Y), the scaling factors are (SX, SY), and the produced coordinates are (X’, Y’). This can be mathematically represented as shown below −

$X' = X \cdot SX$ and $Y' = Y \cdot SY$

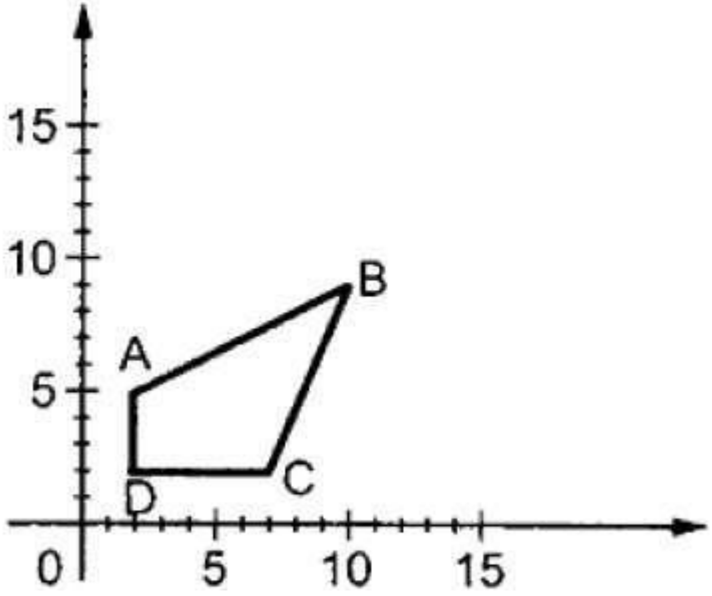
The scaling factor SX, SY scales the object in X and Y direction respectively. The above equations can also be represented as below −

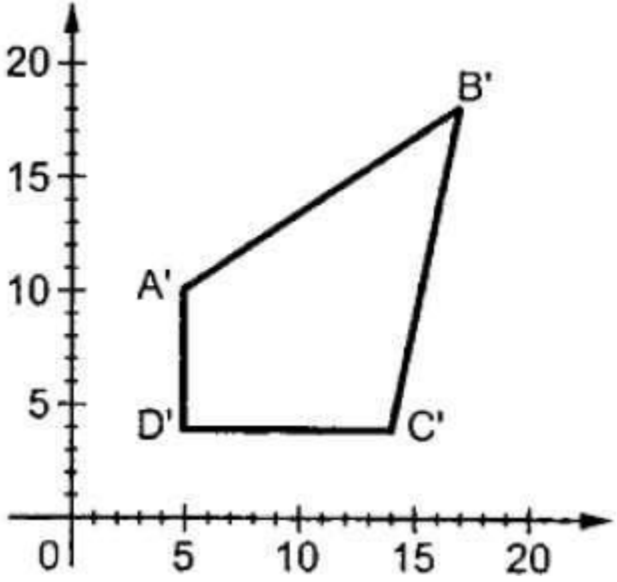
$P' = P \cdot S$

Where S is the scaling factor.

The scaling process is shown in the following figure.

If we provide values less than 1 to the scaling factor S, then we can reduce the size of the object. If we provide values greater than 1, then we can increase the size of the object.



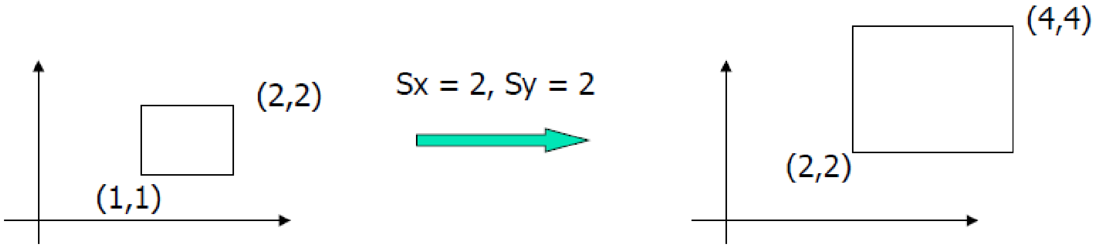


2D Scaling

Scale: Alter the size of an object by a scaling factor (Sx, Sy), i.e.

$$x' = x \,.\, Sx$$
$$y' = y \,.\, Sy$$

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} Sx & 0 \\ 0 & Sy \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}$$



3x3 2D Scaling Matrix

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} Sx & 0 \\ 0 & Sy \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}$$

$$\Downarrow$$

$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

Scale

- Given:

$$P = (x, y)$$
$$S = (s_x, s_y)$$

$$P = (1.4, 2.2)$$
$$S = (3, 3)$$

- We want:

$$x' = s_x x$$
$$y' = s_y y$$

$$x' = 3 * 1.4$$
$$y' = 3 * 2.2$$

- Matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$P' = S \cdot P$$

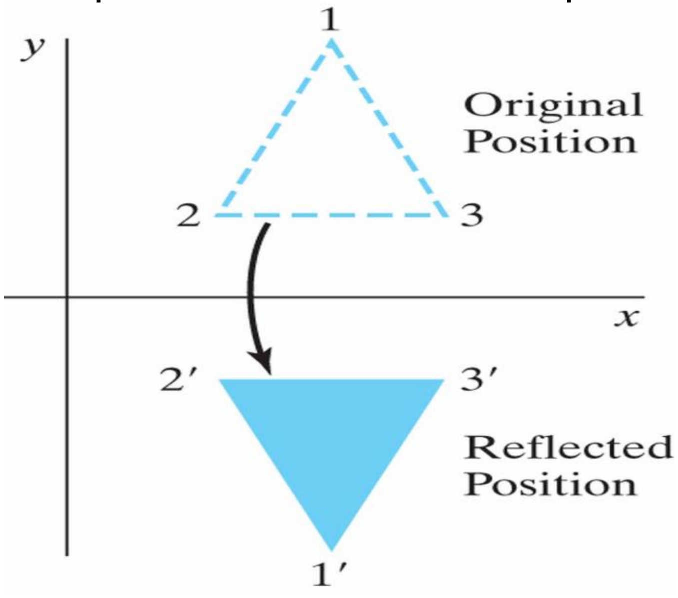
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1.4 \\ 2.2 \end{bmatrix}$$
$$x' = 4.2$$
$$y' = 6.6$$

Reflection

Reflection is the mirror image of original object. In other words, we can say that it is a rotation operation with 180°. In reflection transformation, the size of the object does not change. The following figures show reflections with respect to X and Y axes respectively.

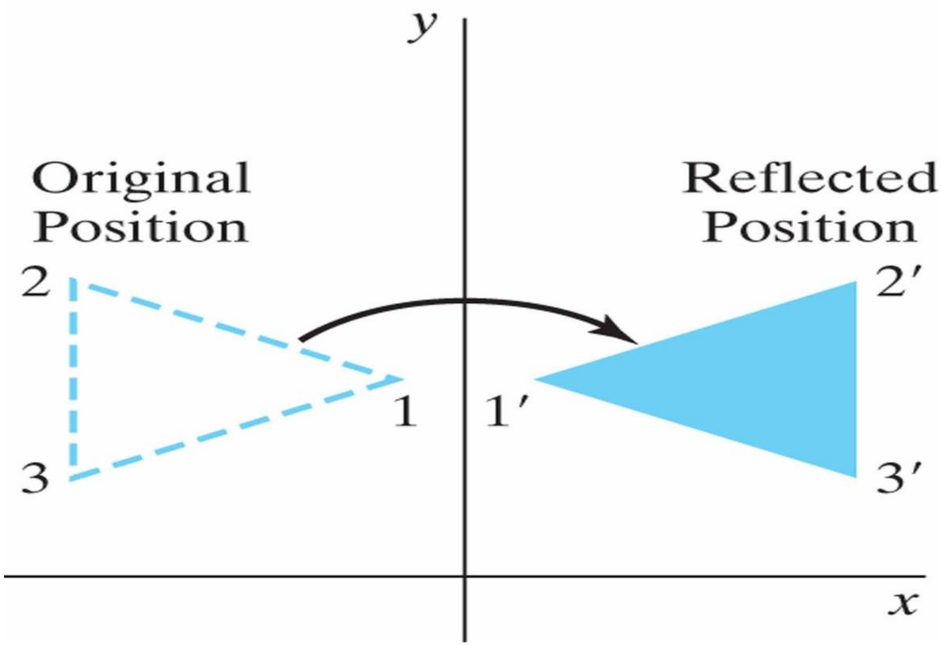
Reflection about the line y=0 (the x axis)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



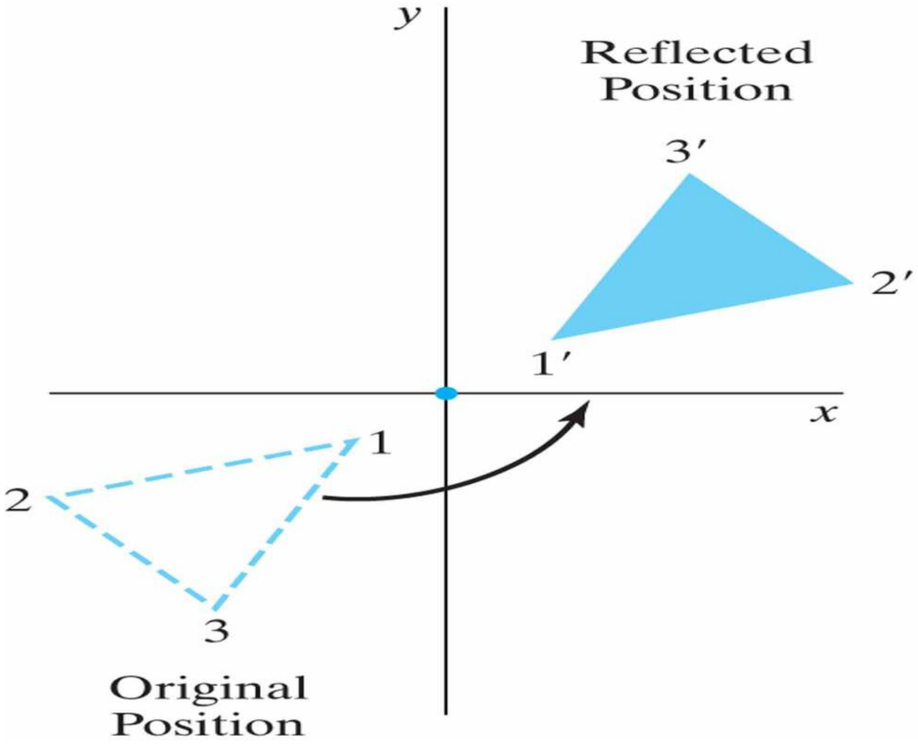
- Reflection about the line x=0 (the y axis)

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Reflection about the Origin

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



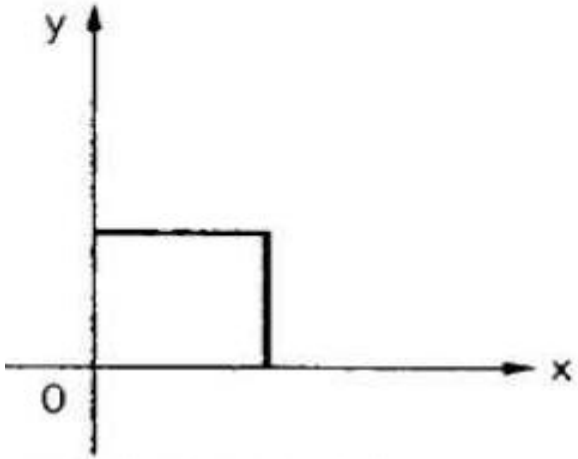
Shear

A transformation that slants the shape of an object is called the shear transformation. There are two shear transformations X-Shear and Y-Shear. One shifts X coordinates values and other shifts Y coordinate values.

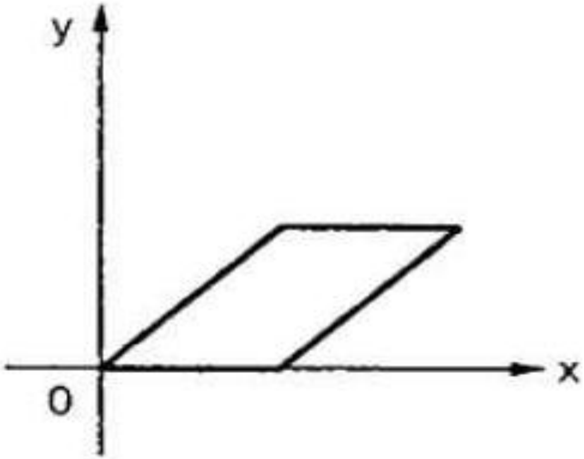
However; in both the cases only one coordinate changes its coordinates and other preserves its values. Shearing is also termed as Skewing.

X-Shear

The X-Shear preserves the Y coordinate and changes are made to X coordinates, which causes the vertical lines to tilt right or left as shown in below figure.



(a) Original object



(b) Object after x shear

The transformation matrix for X-Shear can be represented as –

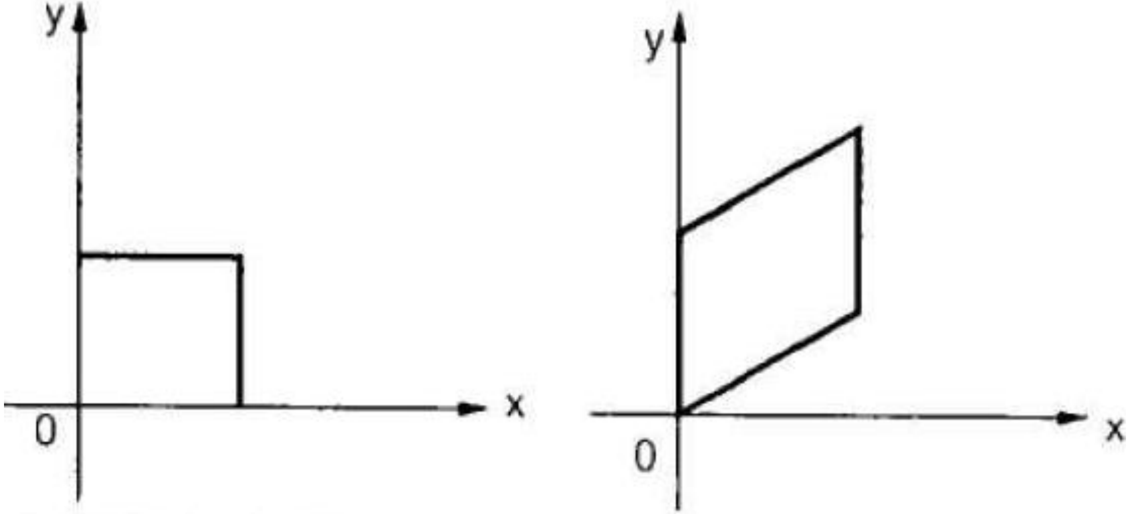
$$X_{sh} = \begin{bmatrix} 1 & shx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X' = X + Sh_x \cdot Y$$

$$Y' = Y$$

Y-Shear

The Y-Shear preserves the X coordinates and changes the Y coordinates which causes the horizontal lines to transform into lines which slopes up or down as shown in the following figure.



(a) Original object (b) Object after y shear

The Y-Shear can be represented in matrix from as –

$$Y_{sh} \begin{bmatrix} 1 & 0 & 0 \\ shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Y' = Y + Sh_y \cdot X$$

$$X' = X$$

Why use 3x3 matrices?

- So that we can perform all transformations using matrix/vector multiplications

- This allows us to *pre-multiply* all the matrices together

- The point (x,y) needs to be represented as (x,y,1) -> this is called Homogeneous coordinates!

Matrix Representations & Homogeneous Coordinates

- Many graphics applications involve sequences of geometric transformations
  - Animations
  - Design and picture construction applications
- We will now consider matrix representations of these operations
  - Sequences of transformations can be efficiently processed using matrices
- To produce a sequence of operations, such as scaling followed by rotation then translation, we could calculate the transformed coordinates one step at a time
- A more efficient approach is to combine transformations, without calculating intermediate coordinate values.
- Multiplicative and translational terms for a 2D geometric transformation can be combined into a single matrix if we expand the representations from 2*2 to 3*3 matrices.

Homogeneous Coordinates

To perform a sequence of transformation such as translation followed by rotation and scaling, we need to follow a sequential process −

- Translate the coordinates,
- Rotate the translated coordinates, and then
- Scale the rotated coordinates to complete the composite transformation.

To shorten this process, we have to use 3×3 transformation matrix instead of 2×2 transformation matrix. To convert a 2×2 matrix to 3×3 matrix, we have to add an extra dummy coordinate W.

In this way, we can represent the point by 3 numbers instead of 2 numbers, which is called Homogeneous Coordinate system. In this system, we can represent all the transformation equations in matrix multiplication. Any Cartesian point P(X, Y) can be converted to homogeneous coordinates by P’ (Xh, Yh, h).

- Expand each 2D coordinate (x,y) to three element representation (x_h,y_h,h) called **homogeneous coordinates**
- h is the **homogeneous parameter** such that $x = x_h/h, \quad y = y_h/h,$
- A convenient choice is to choose h = 1

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x_h \\ y_h \\ h \end{bmatrix} = \begin{bmatrix} x = \dfrac{x_h}{h} \\ y = \dfrac{y_h}{h} \\ \dfrac{h}{h} \end{bmatrix} Ex. \begin{bmatrix} 4 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, Ex. \begin{bmatrix} 3 \\ 7 \end{bmatrix} \rightarrow \begin{bmatrix} 6 \\ 14 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 = \dfrac{6}{2} \\ 7 = \dfrac{14}{2} \\ \dfrac{2}{2} \end{bmatrix}$$

CG: 2-D Geometric Transormation: Homogeneous Coordinates

#1) Homogeneous Coordinates for Translation

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x+t_x & y+t_y & 1 \end{bmatrix}$$

#2] Homogeneous Coordinates for Rotation

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

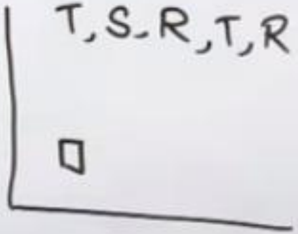
$$= \begin{bmatrix} x\cos\theta - y\sin\theta & x\sin\theta + y\cos\theta & 1 \end{bmatrix}$$

#3) Homogeneous Coordinates for Scaling :-

$$S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x \cdot S_x & y \cdot S_y & 1 \end{bmatrix}$$



CG: 2-D Geometric Transormation: Homogeneous Coordinates – Numerical

Give a 3x3 homegeneous Coordinate Transformation Matrix for Each of the following translations:-

1) Shift The image To The Right 3-Units
2) Shift the image up 2 Units.
3) Move the image down 1/2 unit & Right 1 unit
4) Move the image down $\frac{2}{3}$ unit & left 4 units.

Sol

1) $t_x=3, t_y=0$

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

2) $t_x=0, t_y=2$

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

3) $t_x=1, t_y=-0.5$

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -0.5 & 1 \end{bmatrix}$$

4) $t_x=-4, t_y=-0.66$

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -0.66 & 1 \end{bmatrix}$$

CG: 2-D Geometric Transormation: Homogeneous Coordinates – Numerical

Q) Find The Transformation Matrix That Transform The Square ABCD To Half Its Size With The Centre Still Remaining At The Same Position.

A (1,1), B(3,1), C(3,3), D (1,3) & Centre At (2,2)

Also Find Resultant Coordinates of Square:-

Sol:- 1) Translate The Square So That Its Centre Coincide With The Origin

2) Scale The Square With Respect To Origin

3) Translate Back, The Square To Original Position

$$T_1 \cdot S \cdot T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now

$$\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & 1.5 & 1 \\ 2.5 & 1.5 & 1 \\ 2.5 & 2.5 & 1 \\ 1.5 & 2.5 & 1 \end{bmatrix}$$

CG: 2-D Geometric Transormation: Homogeneous Coordinates – Numerical

Q) Find the Transformation of △ A(1,0), B(0,1), C(1,1) By

1) Rotating 45° about the Origin & translating one Unit in x & y direction.
2) Translating One Unit in x & y direction & then rotating 45° About the Origin.

Sol Rotation matrix $R = \begin{bmatrix} \cos 45° & \sin 45° & 0 \\ -\sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\cos 45 = \sin 45 = \frac{1}{\sqrt{2}}$

Transcation Matrix:- $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

1) $R \cdot T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}}+1 & \frac{1}{\sqrt{2}}+1 & 1 \\ -\frac{1}{\sqrt{2}}+1 & \frac{1}{\sqrt{2}}+1 & 1 \\ 1 & \sqrt{2}+1 & 1 \end{bmatrix}$$

2) $T \cdot R = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 \end{bmatrix}$

$$\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 3/\sqrt{2} & 1 \\ -1/\sqrt{2} & 3/\sqrt{2} & 1 \\ 0 & 2/\sqrt{2} & 1 \end{bmatrix}$$