Social Network Analysis

NETWORK MEASURES

SLIDE CREDITS: TEACHING MATERIAL ON SOCIAL NETWORK ANALYSIS BY TANMOY CHAKRABORTY, WILEY, 2021

Where's the similarity?

https://hbr.org/2012/09/marketing-gangnam-style

Official Release

Jul 15, 2012 Nov 16, 2011

Popularity

One billion views in 6 months | 30 million views within 2 months

Total YouTube Views

Over 3.9 billion views by 2021 | Over 235 million views by 2020

https://www.businesstoday.in/magazine/case-study/kolaveri-di-success-case-study/story/22957.html

VIRAL MARKETING

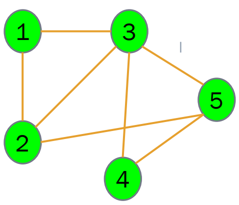
Online Social Media: Some Interesting Questions

- What is the dynamics when one's post receive high visibility on online social media?
- How to publicise one's post in online social media?
- How to find the social media celebrities in such a vast online world?
- How to identify the prolific users in a specific domain in social media?
- What are the role of prolific users when a post becomes viral in social network?
- How to determine if two social media users are similar in terms of online activities?
- How do we know if similar users are connected in a network?
- What are the relevant quantities and how to measure these quantities?

Network Measures: Classification

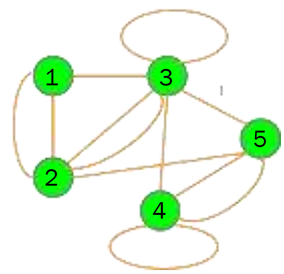
- Microscopic
  - Degree
  - Local clustering coefficient
  - Node centrality
- Mesoscopic
  - Connected components
  - Giant components
  - Group centralities
- Macroscopic
  - Degree Distribution
  - Path and Diameter
  - Edge density
  - Global clustering coefficient
  - Reciprocity and Assortativity

Degree of a Node



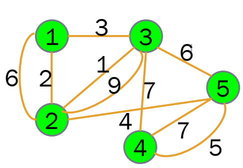
G_1

- For an undirected, unweighted network, degree of a node ***v*** is defined as the number of nodes in the network to which there is an edge from the node ***v***.
- In other words, for an undirected, unweighted network, degree of a node ***v*** is the number of edges of the network that are incident on the node ***v***.
- Putting differently, for an undirected, unweighted network, degree of a node ***v*** is the number of neighbours of the node ***v***.
- In graph G_1, degrees of the nodes 1 through 5 are 2, 3, 4, 2, 3.
- In graph G_2, degrees of the nodes 1 through 5 are 3, 5, 7, 5, 4.
- Note: A self-loop is counted twice in evaluating degree of a node.



G_2

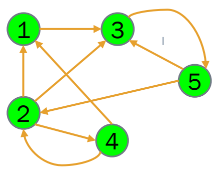
Weighted Degree of a Node



G_3

- For an undirected, weighted network, the weighted degree of a node is defined as the sum of weights of the edges incidents on that node
- For the weighted undirected graph G_3, the weighted degrees of the nodes are as follows:
  - Weighted degree of node 1 is 11
  - Weighted degree of node 2 is 22
  - Weighted degree of node 3 is 26
  - Weighted degree of node 4 is 16
  - Weighted degree of node 5 is 22

Indegree and Outdegree of a Node



G_4

- ❑In a directed network, the indegree of a node is defined as the number of incoming edges to the node
- ❑In a directed network, the outdegree of a node is defined as the number of outgoing edges from the node
- ❑For the directed graph G_4, the indegrees and outdegrees of the nodes are as follows:
  - Indegrees of the nodes 1 through 5 are 2, 2, 3, 1, 1
  - Outdegrees of the nodes 1 through 5 are 1, 3, 1, 2, 2

Sum of the Degrees…

- ❑For an unweighted, undirected network, the sum of the degrees of the nodes in a graph is twice the number of edges in the graph
- ❑Proof
  - ✓When we add an edge **e** to graph, it joins a pair of vertices v_i and v_j of the graph.
  - ✓Prior to the addition of the edge **e** to graph, let the degrees of the nodes v_i and v_j be d_i and d_j .
  - ✓After addition of the edge **e** to graph, the revised degrees of the nodes v_i and v_j be $d_i + 1$ and $d_j + 1$. The degrees of the other nodes remain unaffected.
  - ✓Then, on addition of an edge **e**, the sum of degrees of the nodes in G is incremented by 2 from its previous value. The fact is true for the addition of any edge to the graph.
  - ✓If we add $|E|$ number of edges to the graph one-by-one, the sum of the degrees is enhanced by $2 \times |E|$.
  - ✓If a graph has no edges, all the nodes have degree zero, and so, the sum of the degrees is zero.
  - ✓Thus, a graph with $|E|$ edges has its sum of the degrees of the nodes as $2 \times |E|$.

Sum of the Weighted Degrees…

- Sum of the weighted degrees of the nodes in an undirected weighted graph is twice the sum of weights of the edges in the graph
  - Proof: Proved following the same line of approach

Sum of Indegrees and Outdegrees

- In a directed network, the sum of indegrees is same as the sum of outdegrees.
- Proof. Proved following the same line of approach

Number of Odd-degree Nodes…

- Number of odd-degree nodes in an undirected network is always even.
- Proof.
  - ✓If possible, let the number of odd degree nodes of the graph $G(V, E)$ be an odd integer.
  - ✓Then, the sum of the degrees of these odd-degree nodes is an odd integer, say N_{odd}.
  - ✓All the remaining nodes of the graph have even degrees.
  - ✓Clearly, the sum of the degrees of these even-degree nodes is an even integer, say N_{even}.
  - ✓Then, the sum of the degrees of all the nodes of the graph is $N_{odd} + N_{even}$, which is odd integer.
  - ✓However, the sum of the degrees of all the nodes is $2 \times |E|$
  - ✓So, $N_{odd} + N_{even} = 2 \times |E|$, which is a contradiction, as the LHS is odd and RHS is even!
  - ✓Hence the result.

Degree Distribution

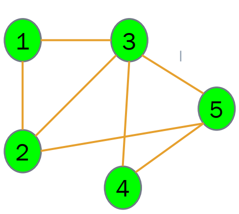
- Degree distribution of a network is the (probability) distribution of the degrees of nodes over the whole network.
- Let a network has $N = |V|$ nodes.
- Let P_k denotes the probability that a randomly chosen node has degree k.
- Then, $P_k = \frac{N_k}{N}$, where N_k refers to the number of nodes of degree k in the network.
- The distribution (k, P_k) represents the degree distribution of the concerned graph,
- The mean degree, denoted $\langle k \rangle$, is given by $\langle k \rangle = \sum_k k.P_k$.

Cumulative Degree Distribution

- Cumulative degree distribution (CDD) is given by the fraction of nodes with degree smaller than k.
- In other words, it is the distribution (k, C_k), where $C_k = \frac{\sum_{k'<k} N_{k'}}{N}$

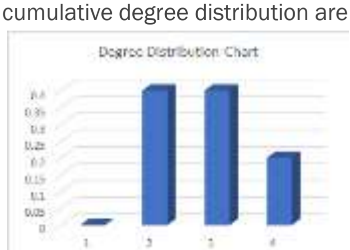
- Complementary cumulative degree distribution (CCDD) is given by the fraction of nodes with degree greater than or equal to k.
- In other words, it is the distribution (k, CC_k), where $CC_k = 1 - C_k$

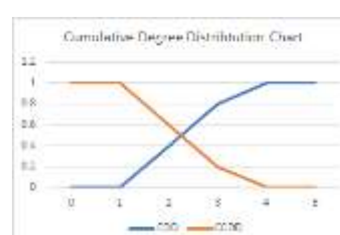
Degree Distribution: Example



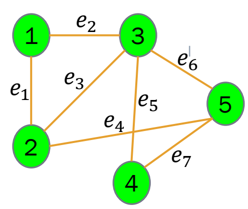
G_1

- For the graph G_1, we have the following:
  $N = 5$, and $N_1 = 0, N_2 = 2, N_3 = 2, N_4 = 1.$
- The above implies, $P_1 = 0, P_2 = 0.4, P_3 = 0.4, P_4 = 0.2,$
  $C_1 = 0, C_2 = 0.4, C_3 = 0.8, C_4 = 1.0,$
  and $CC_1 = 1.0, CC_2 = 0.6, CC_3 = 0.2\ CC_4 = 0.0.$
- Then the degree distribution, Cumulative degree distribution and Complementary cumulative degree distribution are as follows:





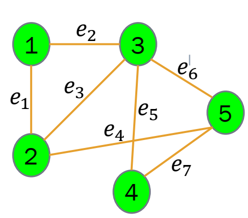
Some Graph Preliminaries…



G_1

- In an undirected network,
  - Two nodes are called adjacent if they are linked by an edge.
  - Two edges are called incident if they share a common end-node.
- In graph G_1, the nodes **1 and 2** are adjacent, **1 and 3** are adjacent, and so on.
- In graph G_1, the edges $\boldsymbol{e_1}$ **and** $\boldsymbol{e_2}$ are incident, $\boldsymbol{e_1}$ **and** $\boldsymbol{e_3}$ are incident, and so on
- A walk in a network is an alternating sequence of nodes and edges, where every consecutive node pair is adjacent, and every consecutive edge pair is incident.
- A walk may pass through a node or an edge more than once. Length of a walk is the number of edges in the sequence.
- In graph G_1, the sequence **{3, e_3, 2, e_4, 5, e_6, 3, e_5, 4, e_7, 5, e_4, 2}** is a walk of length 6.
- For a simple graph, the edges from the above sequence may be omitted.

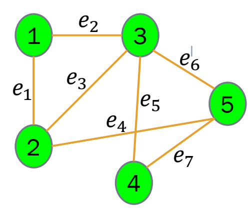
Some Graph Preliminaries…



G_1

- A walk in a network is called
  - a closed walk if the last node in the sequence is same as the first node; else it is called an open walk.
  - a trail if the sequence has no repeated edge.
  - a path if the sequence has neither a repeated edge nor a repeated node. In other words, a path is an open trail having no repeated nodes.
  - a cycle if the sequence has all the edges distinct, and all the nodes, except the first and the last nodes, are also distinct. In other words, a cycle is a closed path with the only repetition of the first and the last nodes in the sequence.
- In graph G_1,
  - the sequence **{2,5, 4, 3, 2, 1, 3, 4, 5, 2}** is a closed walk.
  - the sequence **{5, 4, 3, 2, 1, 3}** is a trail.
  - the sequence **{5, 4, 3, 2, 1}** is a path.
  - the sequence **{5, 4, 3, 2, 5}** is a cycle.

Some Graph Preliminaries…



G_1

- The distance between nodes v_i and v_j in a graph is defined as the length of the shortest path between the nodes v_i and v_j.
- In graph G_1, the distance between 1 and 4 is 2, the same between 1 and 5 is also 2.
- The diameter of a network is defined as the maximum distance between any pair of nodes in the network.
- The diameter of the graph G_1 is 2.
- For a graph G with n nodes, the average path length l_G is defined as the average number of steps along the shortest paths for all possible pairs of nodes in the network.

$$l_G = \frac{\sum_{i \neq j} d_{ij}}{n(n-1)},$$ where d_{ij} is distance between nodes v_i and v_j

Some Graph Preliminaries…

- The density of a graph $G(V, E)$, denoted $\rho(G)$, is defined as the ratio of the number of edges in the graph to the total number of possible edges in the network. Mathematically,

$$\rho(G) = \frac{2 \times |E|}{|V| \times (|V| - 1)}$$

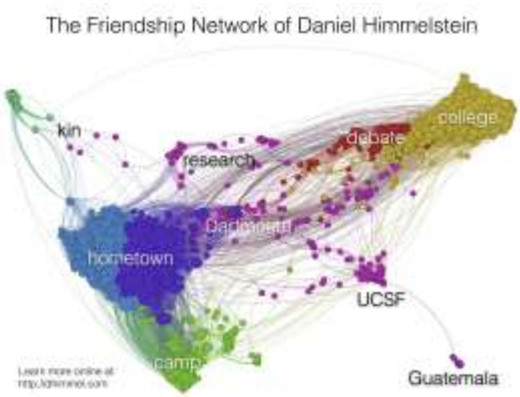
- For the graph G_1, the average path length is:

$$\frac{2 \times (1 + 1 + 2 + 2 + 1 + 2 + 1 + 1 + 1 + 1)}{5 \times 4} = \frac{26}{20} = 1.3$$

- For the graph G_1, the network density is:

$$\frac{2 \times 7}{5 \times 4} = 0.7$$

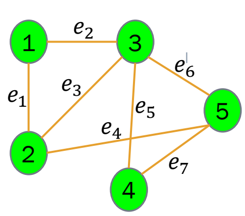
Clusters in Social Networks



A Facebook Friendship Network Example
https://blog.dhimmel.com/friendship-network/

- In social networks, we often find
  - tightly-knit groups here and there
  - less dense ties away from these groups
- Indicative of friendship structures in social media
- Measure used to capture these phenomena
  - Local clustering coefficient
  - Global clustering coefficient

Local Clustering Coefficient



G_1

- In a network $G(V, E)$, the local clustering coefficient of node $v_i \in V$, denoted C_i, is defined as

$$C_i = \frac{Number\ of\ edges\ between\ neighbors\ of\ v_i}{Number\ of\ maximum\ possible\ edges\ between\ neighbors\ of\ v_i}$$

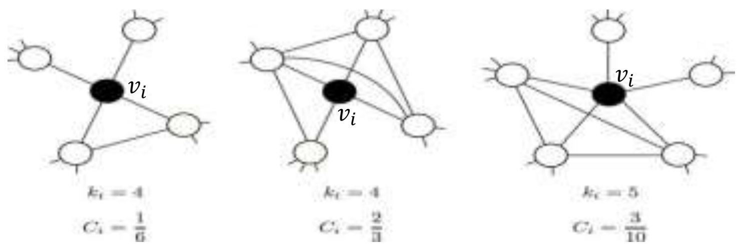
- In graph G_1,
  - the local clustering coefficient of node 2 is $^2/_3$
  - the local clustering coefficient of node 3 is $^3/_6$ i.e. $^1/_2$
  - and so on…

Local Clustering Coefficient

The local clustering coefficient C_i for a vertex v_i in a network $G(V,E)$ is given by the proportion of edges between the vertices within its neighborhood divided by the number of links that could possibly exist between them.

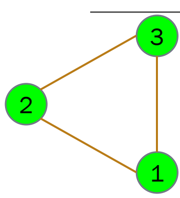
$$C_i = \frac{2 \times |\{e_{jk}|\ v_j, v_k \in N_i, e_{jk} \in E\}|}{k_i.(k_i - 1)}$$

Where N_i is the neighbourhood of the vertex v_i, and $k_i = |N_i|$.

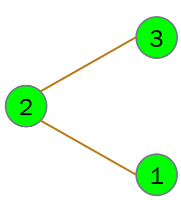


https://www.researchgate.net/publication/236604411_Suicide_Ideation_of_Individuals_in_Online_Social_Networks/figures?lo=1

Global Clustering Coefficient



Closed Triplet

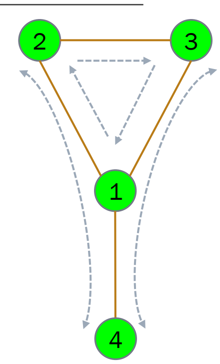


Open Triplet

- The global clustering coefficient C of a network G is defined as

$$C = \frac{Total\ number\ of\ closed\ triplets\ in\ G}{Total\ number\ of\ triplets\ (open\ \&\ closed)\ in\ G}$$

- In the graph G_2 , there is three closed triplet viz., [1,2,3], [2,3,1], and [3,1,2].
- In the graph G_2 , there are five open and closed triplets, viz., (1,2,3), (2,3,1), (3,1,2), (2,1,4), and (3,1,4).
- Thus, the global clustering coefficient of the graph G_2 is $^3/_5$.



G_2

Global Clustering Coefficient

- The global clustering coefficient may also be written as

$$C = \frac{3 \times Total\ number\ of\ triangles\ in\ G}{Total\ number\ of\ triplets\ (open\ \&\ closed)\ in\ G}$$

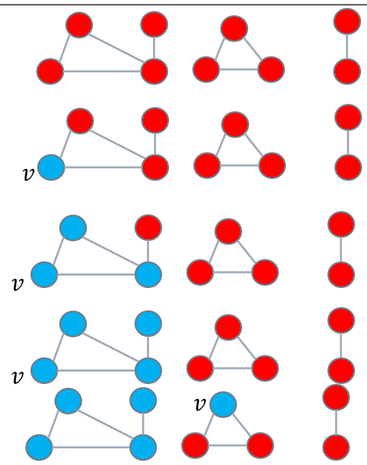
- In other words, if $A = (A_{ij})$ is the adjacency matrix of the graph G, then

$$C = \frac{\sum_{i,j,k}(A_{ij}A_{jk}A_{ki})}{\sum_i k_i(k_i - 1)} \text{ where } k_i = \sum_j A_{ij}$$

Connected Components

- In a typical social network, there are loose links that connects the tightly-knit clusters
- In an undirected network G, two nodes v_i and v_j are said to be connected if there exists a path between v_i and v_j .
- An entire network is said to be connected if any pair of nodes in the network is connected.
- Connected subnetworks of a network, if exist, are called components of the network.
- In real-world networks, there often exist one giant component (consuming major chunk of nodes) and many smaller components.
- In a network, connectedness shows resilience to link breakdowns.

Finding Connected Components



$n = 0$ — The network G with all nodes coloured red

$n = 1$ — Choose a random node v, colour it blue, and set n to 1

$n = 3$ — Apply BFS from node v, and colour with blue all the nodes reached thereof, and increment n each time

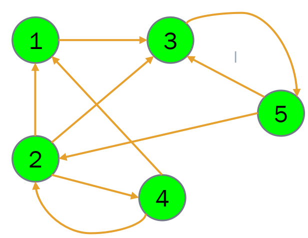
$n = 4$ — No more node can be reached from v using BFS. We get a component in blue.

$n = 4$ — Since $n \neq 9$, we choose a red node as v, repeat the steps above to find other components

Connectedness in Directed Networks

- A directed network G is strongly connected if there exists a (directed) path between every pair of nodes in G.
- If we replace all the directed edges of a directed network G with undirected edges, then the resultant network is called an undirected version of the directed network G.
- A directed network G is said to be weakly connected if its undirected version is connected.

Can you say the below graph G_4 is strongly connected or weakly connected?



G_4

Centrality in a Network

- Influential players often play central roles in a network
- Defining/Identifying influential players always remain hard
  - Some players attract limelight
  - Some others play behind the scene
  - Many others do important linkage
  - and so on…
- To identify influential players, we require
  - to define a notion of influence
  - to device measure that can capture that influence

Degree Centrality

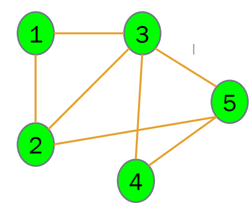
❑Centrality of the simplest kind

❑In a sense, captures the popularity of a player within a network

❑Quantifies the direct influence of a node on its local neighbourhood

❑The degree centrality $C_d(v)$ of a node v in a network $G(V, E)$ is defined as:

$$C_d(v) = \frac{\deg(v)}{\max_{u \in V} \deg(u)}$$



G_1

❑Particularly useful for marketing scenarios, wherein the detected influential user can promote a product/service across her followers

❑Degree centrality of the nodes 1 through 5 in network G_1 are $^2/_4$, $^3/_4$, $^4/_4$, $^2/_4$, and $^3/_4$, respectively; i.e., 0.5, 0.75, 1.0, 0.5, and 0.75, respectively. So, node 3 is most central according to degree centrality measure.

Closeness Centrality

❑A means for detecting nodes that can spread information very efficiently through a graph

❑The measure is useful in

- Examining/restricting the spread of fake news/misinformation in social media
- Examining/restricting the spread of a disease in epidemic modelling
- Controlling/restricting the flow of vital information and resources within an organization (a terrorist network, for example)

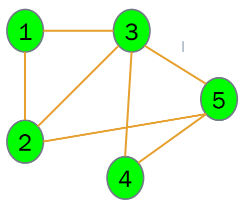
❑The closeness centrality $C(v)$ of a node v in a network $G(V, E)$ is defined as

$$C(v) = \frac{|V| - 1}{\sum_{u \in V \setminus \{v\}} d(u, v)}$$

Where $d(u, v)$ denotes the distance of node u from node v

❑The measure indicates how close a node from the rest of the network

Closeness Centrality



G_1

- In graph G_1, the closeness centrality for the nodes are as follows

$$C(1) = \frac{5-1}{1+1+2+2} = \frac{4}{6} = 0.67$$
$$C(2) = \frac{5-1}{1+1+2+1} = \frac{4}{5} = 0.80$$
$$C(3) = \frac{5-1}{1+1+1+1} = \frac{4}{4} = 1.0$$
$$C(4) = \frac{5-1}{2+2+1+1} = \frac{4}{6} = 0.67$$
$$C(1) = \frac{5-1}{2+1+1+1} = \frac{4}{5} = 0.80$$

- Clearly, node 3 is most central according to closeness centrality measure

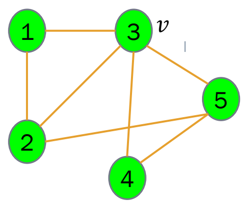
Betweenness Centrality

- A measure to compute how central a node is in between paths of the network
- A measure to compute how many (shortest) paths of the network pass through the node
- Useful in identifying
  - the articulation points, i.e., the points in a network which, if removed, may disconnect the network
  - The super spreaders in analyzing disease spreading in epidemiology
  - the suspected spies in security networks
- The betweenness centrality $C_B(v)$ of a node v in a network $G(V, E)$ is defined as

$$C_B(v) = \sum_{x,y \in V \setminus \{v\}} \frac{\sigma_{xy}(v)}{\sigma_{xy}}$$

where σ_{xy} denotes the number of shortest paths between nodes x and y in the network, $\sigma_{xy}(v)$ denotes the same passing though v. If $x = y$, then $\sigma_{xy} = 1$.

Betweenness Centrality



G_1

- To find the betweenness centrality of node $v = 3$ in graph G_1
- The following matrix is of the form $\sigma_{xy}(v)|\sigma_{xy}$

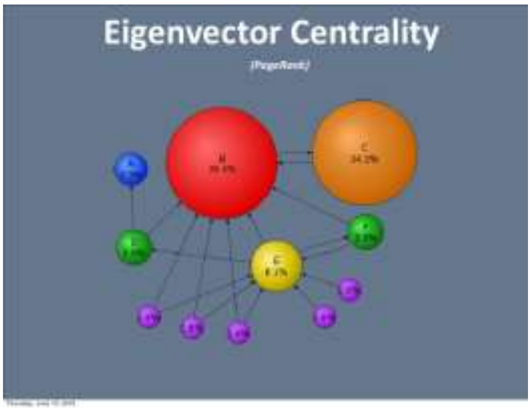
$\sigma_{xy}(v)\|\sigma_{xy}$	1	2	3	4	5
1	0|1	0|1	--	**1|1**	**1|2**
2	0|1	0|1	--	**1|2**	0|1
3	--	--	--	--	--
4	**1|1**	**1|2**	--	0|1	0|1
5	**1|2**	0|1	--	0|1	0|1

- Thus the betweenness centrality of node 3 $= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} = 4$

Betweenness Centrality: Variants

- The edge betweenness centrality refers to the fraction of all pairs of shortest paths of the network that pass through a given edge.
- Computation is more-or-less similar to that of betweenness centrality
- The flow betweenness centrality the fraction of all paths (not necessarily the shortest paths) of the network that passes through a given edge.
- Clearly, flow betweenness centrality measure is computationally expensive than betweenness or edge betweenness centrality measures.

Eigenvector Centrality



https://www.slideshare.net/mdeiters/you-might-also-like-implementing-user-recommendations-in-rails/63-Eigenvector_Centrality_PageRankThursday_June_10

- ❑Measures a node's importance by taking into consideration the preference of its neighbors
- ❑Uses a recursive approach
- ❑A node has a higher eigenvector centrality, if it is directly connected to other nodes having high eigenvector centrality
- ❑Generally applied on directed networks

Eigenvector Centrality

❑The eigen vector centrality x_v of a node v in a network $G(V, E)$ is given by

$$x_v = \frac{1}{\lambda_1} \sum_{t \in N(v)} x_t = \frac{1}{\lambda_1} \sum_{t \in V} (a_{vt} \times x_t)$$

where λ_1 is the largest eigen value of the matrix $A = (a_{ij})$, the adjacency matrix of the network G

❑The largest eigen value $\lambda_1$is obtained by solving the equation

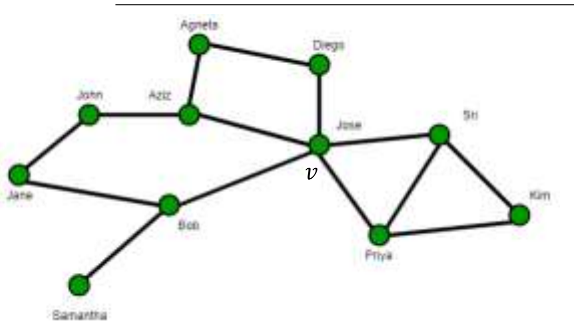
$$A.X = \lambda_1.X$$

❑X above is a column vector, whose v^{th} entry is x_v , the eigen vector centrality of the node v

Katz Centrality

- An extension of eigenvector centrality
- Can be used to compute centrality in directed networks such as citation networks and the World Wide Web
- Mostly suitable in the analysis of directed acyclic graphs
- Computes the relative influence of a node in a network by considering all immediate neighbors and all further nodes connected to the node
- Connections with distant neighbors are, however, penalized by an attenuation factor

Katz Centrality: Attenuation Factor



https://www.geeksforgeeks.org/katz-centrality-centrality-measure/

- Let us consider the influence of *Jose* in the network, and also let the attenuation factor be α, $0 < \alpha < 1$
- Immediate neighbours of Jose are ***Diego, Aziz, Bob, Priya,*** and ***Sri.*** Influence of these neighbours on Jose would be attenuated at a factor of α
- Second order neighbours of Jose are ***Agneta, John, Samantha,*** and ***Kim.*** Influence of these neighbours on Jose would be attenuated at a factor of α^2
- The (only) third order neighbour of Jose is ***Jane.*** Influence of these neighbours on Jose would be attenuated at a factor of α^3

Katz Centrality

- The Katz centrality of a node v_i in a network $G(V, E)$, denoted $C_{Katz}(i)$, is defined as

$$C_{Katz}(i) = \sum_{k=1}^{\propto} \sum_{j=1}^{|V|} \alpha^k \times A_{ji}^k$$

where A is the adjacency matrix of G

- Matrix A^k indicates the presence/absence of a path of length k between a node-pair
- The entry A_{ji}^k in A^k matrix indicates the total number of k-hop walks between node j and node i

PageRank

- Devised by Larry Page and Sergey Brin in 1998
- Devised as a part of a research project about a new kind of search engine
- Based upon the concepts of eigenvector centrality and Katz centrality measures
- Used to rate the importance of web pages on the web
- A page's importance is determined by the importance of the web pages linked to the page
- The algorithm is inherently recursive because the page further contributes to the importance of the web pages linked to it

PageRank

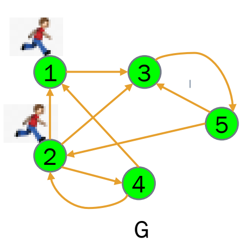
- The PageRank for a network node v_i in a network $G(V,E)$, denoted $\mathrm{PG}(v_i)$, is defined as

$$PG(v_i) = \frac{1-d}{|V|} + d \sum_{\substack{t=1 \\ t \neq i}}^{|V|} \frac{PG(v_t)}{outdeg(v_t)}$$

where d is constant, called the damping factor

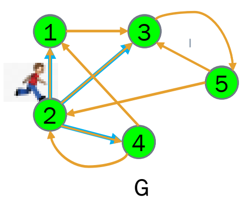
- Though there are many works to determine the optimal value for d, it is usually set as $d = 0.85$

PageRank: The Random Surfer model



- A random surfer surfing through the Internet by
  a. opening a webpage at random, and
  b. moving across webpages by randomly clicking hyperlinks in the page he is in
  c. repeating the steps (a) and (b) at random
- The surfer follows hyperlinks to surf with probability d
- The surfer jumps to pages to surf with probability $(1-d)$
- Since there are $|V|$ number of vertices in the network, the probability of choosing a random webpage is $\frac{1-d}{|V|}$
- Hence, we have the First term of the PageRank equation

PageRank: The Random Surfer model

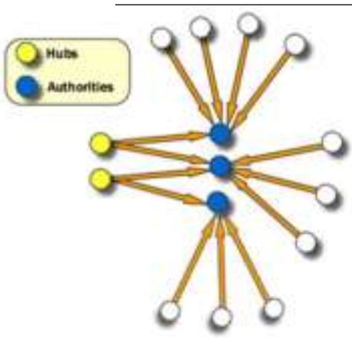


- The surfer is in a page v_t and he decides to follow a hyperlink
- The probability that he decides to follow hyperlink than random jump is d
- At node v_t , he has $outdeg(v_t)$ number of options
- The PageRank contribution of the page v_t is $PG(v_t)$
- The above contribution is divided across the available hyperlinks (outward links)
- However, the surfer could be anywhere in network
- Hence the total possible contribution with this choice $d\sum_{\substack{t=1\\t\neq i}}^{|V|}\frac{PG(v_t)}{outdeg(v_t)}$
- Hence, we have the Second term of the PageRank equation

Hub & Authority

- Nodes having high out-degree are called hubs in a network
- Nodes having high in-degree are called to have authority in a network
- In connection with a citation network
  - Hub nodes are survey papers which cites large number of papers
  - Authoritative nodes are seminal papers that are cited by large number of papers
- PageRank algorithm considers nicely the authoritativeness of a node in a network
- But it does not consider the hubness of a node separately
- However, the later kind of nodes may drag important information regarding the network, too

Hub & Authority



https://slideplayer.com/slide/10495834/

- For node v, its hubness is determined by the cumulative authoritativeness of nodes that v points to.

$$hub(v) = \sum_{u \in out(v)} auth(u)$$

where $out(v)$ denotes the set of nodes pointed by v

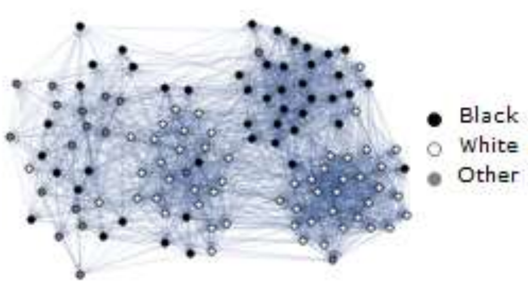
- On the other hand, its authoritativeness is computed by the cumulative hubness of the nodes pointing to v,

$$auth(v) = \sum_{u \in in(v)} hub(u)$$

where $in(v)$ denotes the set of nodes pointing to v

- Kleinberg proposed Hyperlink-Induced Topic Search (HITS) algorithm exploiting these concepts

Assortative Mixing



https://www.wolfram.com/mathematica/new-in-9/social-network-analysis/homophily-and-assortativity-mixing.html

- In friendship kind of social networks,
  - individuals often choose to associate with others having similar characteristics
  - age, nationality, location, race, income, educational level, religion, or language are common characteristics
  - Homophily
- In intimate relationship kind of network,
  - mixing is also disassortative by gender
  - most people prefer to have affair with opposite sex
  - Heterophily
- Assortativity or assortative mixing is a measure to gauge these mixing tendencies

Assortative Mixing

- The phenomenon of particular interest is the assortative mixing by degree
  - High degree nodes often prefers to connect other high degree nodes
  - Low degree nodes seen to connect other low degree nodes
- Assortative mixing can have impact, for example, on the spread of diseases
  - Many diseases are known to have differing prevalence in different population groups
- Such behaviors are observed in non-social types of networks, too
  - biochemical networks in the cell
  - computer and information networks

Assortative Mixing

- A common practice to find similarity between nodes is to use a correlation coefficient
- The Pearson correlation coefficient is a good choice if we want degree-based assortativity
- For two data (degree) distribution x and y, the Pearson correlation coefficient r_{xy} is given by

$$r_{xy} = \frac{N\sum xy - \sum x \sum y}{\sqrt{(N\sum x^2 - (\sum x)^2)(N\sum y^2 - (\sum y)^2)}}$$

- If $r_{xy} = 1$, then nodes x and y are perfectly assortative (homophily)
- If $r_{xy} = -1$, then nodes x and y are perfectly disassortative (heterophily)
- If $r_{xy} = 0$, then nodes x and y are non-assortative

Transitivity

- A metric to determine the linkage between a pair of nodes
- Very important in social networks, and to a lesser degree in other networks
- In abstract mathematics, if entity x is related to entity y, and also entity y is related to entity z, then the transitivity of the relation ensures that entity x is related to entity z.
- In social networks, a complete transitivity may yield: "Friends of my friends are my friends"
  - Utterly Absurd in real networks!
- In fact, a complete transitivity would imply that each component of a network is a clique!!
- However, partial transitivity is useful: "Friends of my friend are more likely my friend than some randomly chosen member from the population"

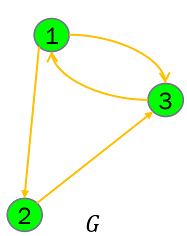
Transitivity

- A complete graph is surely transitive
- A measure of transitivity intends to capture how close a network is to a complete graph
- A network with higher transitivity are likely to form dense clusters
- Two ways to capture this tendency
  - Local clustering coefficient
  - Global clustering coefficient

Reciprocity

- Relevant for directed networks
- A measure of the likelihood of vertices in a directed network to be mutually linked.
- Networks that transport information or material, mutual links facilitate the transportation process
- An important phenomenon for such applications
- Informally, reciprocity refers to: "If you would follow me, most likely I shall follow you back"
- May be considered a simplified version of transitivity

Reciprocity



G

- Reciprocity counts the closed loops of length 2
- The reciprocity R of a network G is defined as

$$C = \frac{Total\ number\ of\ reciprocal\ pairs\ in\ G}{Total\ number\ of\ pairs\ (reciprocal\ \&\ nonreciprocal)\ in\ G}$$

- For graph G, the reciprocity is $\frac{1}{3}$

Reciprocity

- The reciprocity R for a graph $G(V, E)$ having adjacency matrix $A = (a_{ij})$ is given by

$$R = \frac{2}{|E|} \sum_{i<j} (a_{ij} . a_{ji})$$

- On simplification,

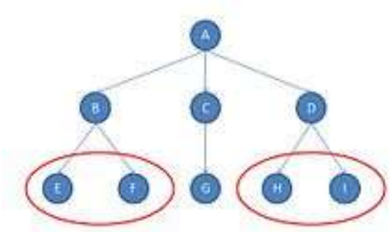
$$R = \frac{2}{|E|} \times \frac{1}{2} Trace(A^2) = \frac{Trace(A^2)}{|E|}$$

- In the above expression, $Trace(\cdot)$ function denotes the sum of the diagonal elements of its argument square matrix

Similarity

- A measure to check whether a pair of nodes in a network are part of the same equivalence class
- An abstract ways of making sense of the patterns of relations among social actors
- Three broad classes of equivalence classes
  - Structural equivalence
  - Automorphic equivalence
  - Regular equivalence
- There is a hierarchy of these three equivalence concepts
  - Any set of structural equivalences are also automorphic and regular equivalences
  - Any set of automorphic equivalences are also regular equivalences
  - Not all regular equivalences are necessarily automorphic or structural
  - Not all automorphic equivalences are necessarily structural

Structural Equivalence



https://en.wikipedia.org/wiki/Similarity_(network_science)#:~:text=Similarity%20in%20network%20analysis%20occurs,automorphic%20equivalence%2C%20and%20regular%20equivalence.

- Two nodes are said to be exactly structurally equivalent if they have the same relationships to all other nodes
- Two actors must be exactly substitutable in order to be structurally equivalent
- In the attached network,
  - nodes E and F are structurally equivalent, since these two nodes have same pattern ties (viz. a single tie) with the node B
  - Also, nodes H and I are structurally equivalent, since these two nodes have same pattern ties (viz. a single tie) with the node D
- Exact structural equivalence is likely to be rare (particularly in large networks)
- the degree of structural equivalence is what interests us the most

Measuring Structural Equivalence

- Common Neighbors
  - number of common neighbors shared in the neighborhoods of the nodes a and b

$$\sigma_{CN}(a,b) = |N(a) \cap N(b)|$$

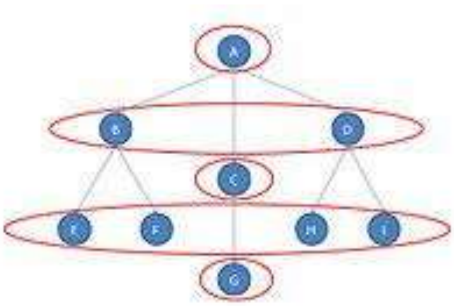
- Jaccard Similarity
  - Normalizes the common neighbors by the combined size of the neighborhoods of the two nodes

$$\sigma_{CN}(a,b) = \frac{|N(a) \cap N(b)|}{|N(a) \cup N(b)|}$$

- Cosine Similarity
  - normalizes the common neighbors by the individual sizes of the neighborhoods

$$\sigma_{CN}(a,b) = \frac{|N(a) \cap N(b)|}{\sqrt{|N(a)||N(b)|}}$$

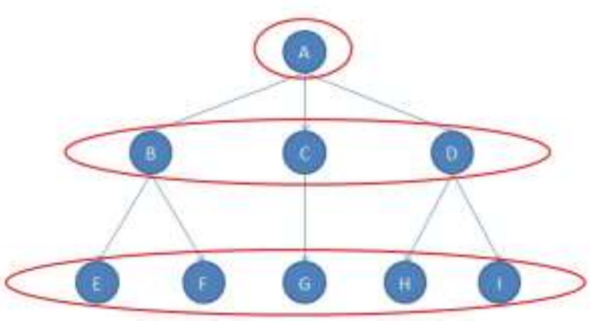
Automorphic Equivalence



https://en.wikipedia.org/wiki/Similarity_(network_science)#:~:text=Similarity%20in%20network%20analysis%20occurs,automorphic%20equivalence%2C%20and%20regular%20equivalence.

- Let us interpret the network as follows: the network describe a franchise group of a restaurant chain
  - A is the general manager at central headquarters
  - B, C, and D are the managers of three different stores.
  - E and F are workers at one store; G is the lone worker at a second store; H and I are workers at the third store
- B and D are equivalent in the following sense
  - B and D report to a boss (same boss here)
  - Each has exactly two workers
- Similarly, E, F, H, and I are also equivalent in the following sense
  - They report to a store manager (different boss here)
  - Nobody report to these persons
- The above approach of equivalence is automorphic equivalence

Regular Equivalence



https://en.wikipedia.org/wiki/Similarity_(network_science)#:~:text=Similarity%20in%20network%20analysis%20occurs,automorphic%20equivalence%2C%20and%20regular%20equivalence.

- Two actors are regularly equivalent if they are equally related to equivalent others
- Two mothers are regularly equivalent, since
  - each has a similar pattern of connections with a husband,
  - with their children,
  - with their in-laws, etc.
- The store managers are regularly equivalent, since
  - each has a similar pattern of connections with their employees at their stores, and
  - with the general manager at the central headquarter

Measuring Regular Equivalence

- The regular equivalence between nodes v_i and v_j in network $G(V, E)$ having adjacency matrix $A = (A_{ij})$ is defined as

$$\sigma_{reg}(v_i, v_j) = \alpha \sum A_{ik} A_{jl} \sigma_{reg}(v_k, v_l)$$

- We may relax the equation as

$$\sigma_{reg}(v_i, v_j) = \alpha \sum_k A_{ik} \sigma_{reg}(v_k, v_j)$$

- Rewrite the above as

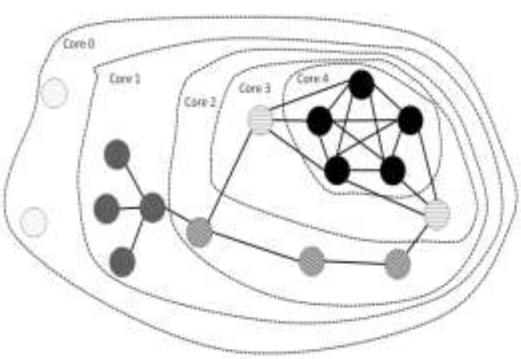
$$\sigma_{reg} = \alpha A \sigma_{reg}$$

- The above imply

$$\sigma_{reg} = (\boldsymbol{I} - \alpha A)^{-1}$$

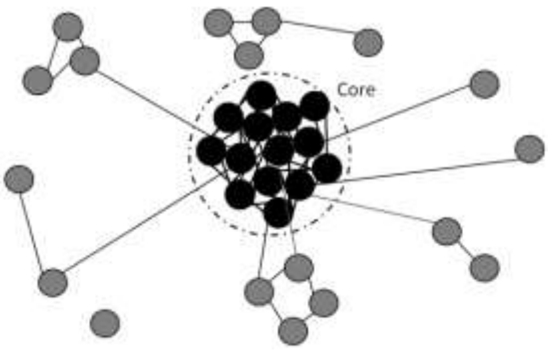
- For convergence of the above, $\alpha < \frac{1}{\lambda_1}$, where λ_1 is the largest eigen value of A

Degeneracy: Core Number



- The coreness or core number of a node is the order of the highest-order core that the node belongs to
- A node has a core number k in network G if
  - It belongs to the k-core subgraph, but
  - does not belong to the $(k + 1)$-core subgraph of G
- In the example network, nodes inside the central-most 4-core subgraph have core number 4
- Similar to centrality, core number is a measure of prestige of a node in a network

Degeneracy: Core-Periphery



- Real-world networks often consists of
  - a dense and connected core, and
  - surrounding the core by disconnected and scrambled periphery

- The structure above is termed as the core-periphery structure of the network