4 Topics → Discrete Mathematics → part of Maths

1. Math Logic
2. Set Theory & Algebra
3. Graph Theory
4. Combinatorics.

→ Study of discrete numbers.
→ If we can tell next with previous
if not : real no. Maths

1) MATHEMATICAL LOGIC

→ Study of reasons.

- provides rules & techniques to write program.

(i) Propositional logic
(ii) Predicative Logic

→ declarative sentence / statement which is either true/false but not both.

eg: Delhi is cap of India ✓
eg: $10 + 11 = 21$ ✓
eg: $10 + 11 = 101$ ✗
eg: What is the time? (not statement)

eg: I am a liar → taking both values → both T/F. (not propositions)

Scanned with CamScanner

If we are in binary no.s,

both (2) & (3) are true.

→ b4 propositions, we need to know the context.

(Eg1) Tomorrow, it may be false.

Take general Meaning at the present time.

* To connect propositional Statements.

* Connectives [¬, ∧, ∨, ⟶, ⟷]

(i) Negation (¬)

p ⟶ Statement

¬p ⟶ It is not the case that p.

p	¬p
F	T
T	F

→ Truth Table

eg: Today is Monday = F.

¬p : It is not the case that today is Monday.

¬p : Today is not Monday. = T.

(ii) Conjunction (∧) → "p and q"

Let p, q be statement. The s. "p & q" is called the conjunction of p & q.

It is defined by:

T → 1
F → 0

p	q	p∧q
F	F	F
F	T	F
T	F	F
T	T	T

eg: Rama likes Sita : p → atomic Statement

Rama likes Laxman : q

p∧q : Rama likes Sita & Lakhan. → molecular stat

eg: Ram & Laxhan are brothers. → atomic Statement

- dont denote p∧q.
- Its a single Statement
- "and" is not a given conjunct.
- cant be split as p and q.

(iii) Disjunction (∨)

The Statement "p or q" is called the disjunct of p and q. It is defined by:

p	q	p V q
F	F	F
F	T	T
T	F	T
T	T	T

is T, if atleast one is true.

eg1: p: Ramesh has a bike.
q: Ramesh has a car.

p V q: Ramesh has a bike or a car.

eg2: Mysore is 135Km or 145 Km away from Bangalore.

d = 140 Km (Both are false, but are generally true)

- Here, or → approximately (not disjunction).

∴ Atomic Statement → stick to meaning.

(iv) Conditional → if-else in C.

The Statement "If p then q" is called the conditional form p to q. It is defined by:

p	q	p → q
F	F	T
F	T	T
T	F	F
T	T	T

p → q → p implies q.

If student studs I can't claim.

If p happens, Q should happen.

eg1: p → q: If Ramu studies well, then he will pass the exam.

p: Ramu studies well.
q: Ramu will pass the exam.

Note: Other rep of p → q are:

Rs
(1) p implies q.
(2) p only if q.
(3) p is sufficient for q.
(4) q if p (if p then q)
(5) q is necessary for p.

(v) BiConditional

The statement "p if and only if q" is called the biconditional Statement. It is defined by:

p	q	p ⟷ q
F	F	T
F	T	F
T	F	F
T	T	T

p ⟷ q → EX-NOR (equality)
if T = = F → F.

eg1: $p \leftrightarrow q$: Krishna is married if and only if he has wife.

p: Krishna is married.
q: Krishna has a wife.

eg2: You can take the flight if & only if, you have a ticket. → General Meaning - dont find loopholes.

Other forms of $p \longleftrightarrow q$ are :

(i) p iff q
(ii) If p then q and conversely
(iii) p is necessary & sufficient for q.

* Negation → unary operator ($\neg$)
$\wedge, \vee, \rightarrow, \leftrightarrow$ = binary operator

* Propositional Function

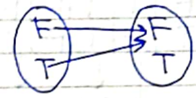
- Is a function whose Variables are Propositions.

eg: $f(x,y,z) = x + y^2 + z3$ → functn of 3 Var. (real)

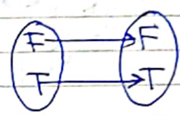
$f(p,q) = (p \wedge q) \rightarrow p$ → propositional functns.

- We have 4 propositional functns of single Variable (p)

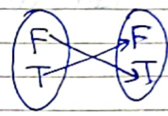
1. $f(p) = F$



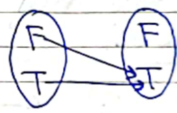
2. $f(p) = p$.



3. $f(p) = \neg p$ (Negation here).



4. $f(p) = T$.



p	0	1	2	3
F	F	F	T	T
T	F	T	F	T

→ 4 Functns.

* We have 16 Propositional Functions of two Variables

p	q	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
F	F	F	F	F	F	F	F	F	F	T	T	T	T	T	T	T
F	T	F	F	F	T	T	T	T	T	F	F	F	F	T	T	T
T	F	F	F	T	F	F	F	T	T	F	F	T	T	F	F	T
T	T	F	T	F	F	F	T	F	T	F	T	F	T	F	T	F

AND (1), EXOR (6), OR (7), $p \downarrow q$ (8), $\leftrightarrow$ (9), $\rightarrow$ (13), $p \uparrow q$ (14)

$1 = p \wedge q$
$7 = p \vee q$
$9 = p \leftrightarrow q$
$13 \Rightarrow p \rightarrow q$

} same under here.

→ In English, we have only these meanings (building blocks).

---

* Exclusive OR ($\nabla / \oplus$) ← Other Notations.

$\boxed{p \nabla q \equiv p \oplus q}$ → 6.

eg: I will go by bus <u>or</u> train. (PL)

→ I can't go by both.

Here, or is connective but not disjunction (but ex-or).

$=$	$\neq$	: equal
$\in$	$\notin$	: belong
$\subset$	$\not\subset$	: subset.

* NAND ($\uparrow$)

$p \uparrow q$.

$\boxed{p \uparrow q \equiv \neg(p \wedge q) \equiv 14}$

* NOR : ($\downarrow$)

$p \downarrow q$.

$\boxed{p \downarrow q \equiv \neg(p \vee q) \equiv 8}$

AND : $\wedge$
product : $\times$
(same meaning).

OR : $\vee$
sum : $+$

(3) The no. of propositional functions of 'n variables is'

$2V \longrightarrow 4$ Rows : . Total func. : $2^4 = 16$
$3V \longrightarrow 8$ Rows : Total funct^n : 256

$\therefore \boxed{\text{Total funct}^n\text{s} = 2^{(2^n)}}$

(4) Well Formed Formula : (WFF)

= Propositional Function.

eg1: $p \rightarrow (q \vee r)$ is a WFF. (functn of 3 var)

eg2: $p \wedge q \wedge r$ is a WFF.

eg3: $p \rightarrow q \rightarrow r$ is not a WFF.

(p, q, r) $f(p, q, r)$.

$FTF \xrightarrow{(1)} F$, $FTF \xrightarrow{(2)} T$

[∵ Not a functn ∵ 2 images].

1) $(p \rightarrow q) \rightarrow r = F$. (order)

$(F \rightarrow T) \rightarrow F \equiv F$

need not check here. ∵ start with false.

2) $p \rightarrow (q \rightarrow r)$

$F \rightarrow (T \rightarrow F)$

T.

But, $(p \rightarrow q) \rightarrow r$: WFF.

* Types of Functions

(1) Tautology:

A WFF which is always true.

eg: $p \vee \neg p \equiv T$.

(2) Contradiction (absurdity)

A WFF which is always false.

eg: $p \wedge \neg p \equiv F$.

(3) Contingency

A WFF which is neither tautology or nor contradiction.

eg. $f(p) = p$.

eg: $p \vee q$.

(4) Satisfiable:

A WFF which has atleast one truth value true

∴ $\boxed{\text{Satisfiable} = \text{not a contradiction}}$

$\boxed{\text{Unsatisfiable} = \text{contradiction}}$

Scanned with CamScanner

(5)

(i) The converse of $p \rightarrow q$ is $q \rightarrow p$.

(ii) The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

(iii) The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

eg: If work is not finished on time, then I am in trouble. (S p)

Converse: If I am in trouble, then work is not finished on time

Inverse: If the work is finished on time, then I am not in trouble.

Contrapositive: If I am not in trouble, then the work is finished on time.

p & Contrapositive is equal in meaning

$\therefore \boxed{S \equiv \neg q \rightarrow \neg p}$

logically equivalent.

* Logical Equivalences:

Two WFF's P & Q are said to be logically equivalent iff $P \leftrightarrow Q$ is a tautology.

$P \leftrightarrow Q$ or $P \equiv Q$ i.e. P & Q have same truth values.

eg: $p \equiv \neg(\neg p)$

p	$\neg p$	$\neg(\neg p)$	$p \leftrightarrow \neg(\neg p)$
F	T	F	T
T	F	T	T
(P)		(Q)	

If same TVS, LE.

eg: $\boxed{p \rightarrow q \equiv \neg p \vee q}$

p	q	$p \rightarrow q$ (P)	$\neg p$	$\neg p \vee q$ (Q)	$P \leftrightarrow Q$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

$\Rightarrow$ 2 Molecular Stat are LE, if they have same TV.

$\therefore$ finite combinations

F → adde...

* 7b: is write in PA ...
* 8a: ∧ → intersection
  ∨ → union.

(p absorbs q).

$p \cdot (p+q)$
$p + pq$ (P)
$p \wedge (p \vee q)$

* Laws of Equivalences

SNo.	Name	Equivalence	
1.	Negation Laws	$\sim F \equiv T$, $\sim T \equiv F$, $\sim(\sim p) \equiv p$, $p \wedge \sim p \equiv F$, $p \vee \sim p \equiv T$ (5 Laws)	
2.	Domination Laws	$p \wedge F \equiv F$	$p \vee T \equiv T$
3.	Identity Laws	$p \wedge T \equiv p$	$p \vee F \equiv F$
*4.	Idempotent Laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
5.	Commutative Laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
6.	Associative Laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
7.	Distributive Laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
8.	Absorption Laws	$p \wedge (p \vee q) \equiv p$	$p \vee (p \wedge q) \equiv p$
9.	De-Morgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$

SNo.	Name	Equivalence
10.	Law of implication	$p \to q \equiv \neg p \vee q$
11.	Law of contrapositive	$p \to q \equiv \neg q \to \neg p$
12.	Law of Biconditional	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

* Logical Implications

A WFF P is said to be logically implies to a WFF Q, iff $P \to Q$ is a tautology.

We write it as $P \Rightarrow Q$

(P logically implies Q)
or P implies Q is a tautology.

eg: $p \wedge (p \to q) \Rightarrow q$

Scanned with CamScanner

if p is true, $p \vee q$ ...

⇔ $\neg q \rightarrow \neg p$,
$\neg q$.

p
$p \rightarrow q$
$\therefore q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

tautology

* Rules of Inference

S.No.	Name	Rule of Inference	Tautological form
1	Addition	p / $\therefore p \vee q$	$p \rightarrow (p \vee q)$
2	Simplification	$p \wedge q$ / $\therefore p$	$(p \wedge q) \rightarrow p$
3	Rules of detachment (Modus Ponens)	$p \rightarrow q$ / p / $\therefore q$	$[(p \rightarrow q) \wedge p] \rightarrow q$
4	Modus Tollens	$p \rightarrow q$ / $\neg q$ / $\therefore \neg p$	$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$

S.No.	Name	Rule of inference	Tautological form
5	Disjunctive syllogism	$p \vee q$ / $\neg p$ / $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow q$
6	Hypothetical Syllogism (Rule of transitive)	$p \rightarrow q$ / $q \rightarrow r$ / $p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
7	Conjunction	p / q / $\therefore p \wedge q$	$(p \wedge q) \rightarrow (p \wedge q)$
8	Constructive dilemma	$(p \rightarrow q) \wedge (r \rightarrow s)$ / $p \vee r$ / $\therefore q \vee s$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$
9	Destructive dilemma	$(p \rightarrow q) \wedge (r \rightarrow s)$ / $\sim q \vee \sim s$ / $\therefore \sim p \vee \sim r$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$

∵ Table too lengthy in Q/ ∴ Table Truth

* Fallacies:

- seems to be true, but its not true.

1. Fallacy of affirming the consequent:

- fallacy of affirming the converse.

$p \rightarrow q$ → consequent

antecedent

$$\frac{p \rightarrow q \qquad q}{\therefore p} \quad \times$$

2. Fallacy of denying the antecedent:

(Fallacy of assuming the opposite)

$$\frac{p \rightarrow q \qquad \neg p}{\therefore \neg q} \quad \times$$

3. Fallacy on non-sequitur

$$\frac{p}{\therefore q}$$ } no relation

If p comes by car, then he should be rich X.

* Problems

Q. 2001 (Gate)

$F_1 : p \rightarrow \neg p$.

$F_2 : (p \rightarrow \neg p) \vee (\neg p \rightarrow p)$.

· Valid → tautology

p	$\neg p$	F_1: $p \rightarrow \neg p$	x: $\neg p \rightarrow p$	F_2: $F_1 \vee x$
F	T	T	F	T
T	F	F	T	T

F_1 is satisfiable, F_2 is Valid (a) [Best ans].

Q. CS-1997 (Gate)

$p \rightarrow q \equiv \neg p \vee q$.

(a) $(p \vee q) \rightarrow p$.

$\neg(p \vee q) \vee p$.

$= (\neg p \wedge \neg q) \vee p$

$\Rightarrow (\neg p \vee p) \wedge (\neg q \vee p)$

$\Rightarrow T \wedge (\neg q \vee p)$.

$\Rightarrow (\neg q \vee p)$. X

(b) $p \vee (q \to p)$

$p \vee (\neg q \vee p)$

$\Rightarrow p \vee \neg q$ X

(c) $p \vee (p \to q)$

$\Rightarrow p \vee (\neg p \vee q)$

$\Rightarrow T \vee q \Rightarrow T$ ✓

(d) X

Q. GATE CS 2000

a, b, c, d → Propositions.

$a \leftrightarrow (b \vee \neg b)$ and $b \leftrightarrow c$ hold. → tautology

$(a \wedge b) \to ((a \wedge c) \vee d)$ is always:

Soln: $a \equiv T$, $b \equiv c$.

$(T \wedge b)$

$c \to ((T \wedge c) \vee d))$

$c \to (c \vee d)$ ∴ Addition rule.

∴ (a) True ✓

Q. GATE IT-2004

$P: [(\neg p \vee q) \wedge (r \to s) \wedge (p \vee r)] \to (\neg s \to q)$.

$\boxed{\text{coll}} \to$ Q/soln ✓

Q. GATE CS 2006

$P_1: ((A \wedge B) \to C) \equiv ((A \to C) \wedge (B \to C))$

$P_2: ((A \vee B) \to C) \equiv ((A \to C) \vee (B \to C))$

Use rules of equivalence.

P1: $\Rightarrow (A \wedge B) \to C$.

$\Rightarrow \neg(A \wedge B) \vee C = (\neg A \vee \neg B) \vee C$ looks as distrib.

$\Rightarrow (\neg A \vee C) \vee (\neg B \vee C)$ (idemp).

$\Rightarrow (A \to C) \vee (B \to C)$.

$P_1 \to$ not true.

(d) Both P_1 and P_2 are not tautologies.

Scanned with CamScanner

Q. P_2: $(A \vee B) \to C$.

$\neg(A \vee B) \vee C$

$\Rightarrow (\neg A \wedge \neg B) \vee C$

$\Rightarrow (\neg A \vee C) \wedge (\neg B \vee C)$. (distributive)

$\Rightarrow A \to C \wedge B \to C$.

Q. Gate CS 2014 (Paper 2)

Not a tautology.

(A) $((a \to b) \wedge (b \to c)) \to (a \to c)$ ✓

(B) $(a \leftrightarrow c) \to (\neg b \to (a \wedge c))$ ✗

(C) $(a \wedge b \wedge c) \to (c \wedge a)$ ✓

(D) $a \to (b \to a)$. ✓

(A) ⟶ Rule of transitivity.

(C) $\dfrac{p \wedge q}{\therefore p}$.

(D) $a \to (\neg b \vee a)$ ⟵ must be.

$\Rightarrow$ ~~$\neg a$~~ if a is true.

Q. Gate CS-2009

Binary operation □ is defined

P	Q	P □ Q
T	T	T
T	F	T
F	T	F
F	F	T

which is equivalent to $P \vee Q$.

$P \square Q \equiv Q \to P$.

(a) $\sim Q \square \sim P \equiv \sim P \to \sim Q = P \vee \neg Q$.

(b) $P \square \neg Q \equiv \neg Q \to P = Q \vee P$ ✓

Q. Gate CS-2018

$P \oplus Q = \overline{P \odot Q} = \bar{P} \odot Q$.

p	q	p ⊙ q	p ⊕ q
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

Scanned with CamScanner

$\boxed{p \oplus q = p\bar{q} + \bar{p}q}$

$\boxed{p \odot q = pq + \bar{p}\bar{q}}$

∴ (A) $\overline{P \oplus Q} = P \odot q$ ✓

(B) $P \odot Q = \overline{P \oplus Q} = \bar{P} \oplus Q$ ✓

(C) $\bar{P} \oplus \bar{Q} = P \oplus Q = P\bar{Q} + \bar{P}Q$. ✓

(D) X.

Q. GATE CS 2019

(B) $x \oplus y = x + y$

(A) $x \oplus y = \overline{(xy + \bar{x}\bar{y})}$

$\Rightarrow x \oplus y = \overline{x \odot y}$ ✓

(B) $x \oplus y = x + y$ ✓

x	y	$= 0$
0	0	✓
0	1	✓
1	0	✓

(C) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ ✓

∵ (Associative)

∴ (D) ⟶ Answer.

(D) $(x+y) \oplus z = x \oplus (y+z)$

$x = 0$

$y \oplus z$ X $0 \oplus (y+z)$ X

Q. Gate CS - 2015

$\boxed{p \leftrightarrow q} \equiv (p \to q) \land q \to p$.

(A) $(\neg p \lor q) \land (p \lor \neg q)$ ✓

(B) $(\neg p \lor q) \land (q \to p)$ ✓

(C) $(\neg p \land q) \lor (p \land \neg q)$ X

(d) $(\neg p \land \neg q) \lor (p \land q)$ ✓

Suppose, (p_T, q_T), then $p \leftrightarrow q \equiv T$.

(C) $(F \land T) \lor (T \land F)$

$F \lor F = F$ ✓

(d) $(F \land F) \lor (T \land T)$.

$F \lor T = T$.

✓

$(F \lor T) \land (T \lor F)$.

$T \land T = T$

Q CSE-Gate 2016 Set 2

How many are implied by:

$P \land (P \Rightarrow Q)$ is ____

$$\begin{array}{l} P \\ P \to Q \\ \hline \therefore Q \end{array}$$

I ~~false~~ ~~loose~~
II Q ✓
III true ✓
(IV) $P \lor Q$ ✓
(V) $\neg Q \lor P$ ✓

$P \quad \therefore P \lor Q$

$\therefore P \quad \therefore \neg Q \lor P$

Now, if P $\therefore$ T.
$\therefore$ if P is true, true is true.

Q p: I am intelligent
q: I am lazy

Which of the following notation represents "I am intelligent but I am lazy".

(a) $p \land q$
(b) $p \lor q$
(c) $p \land \neg q$
(d) $p \lor \neg q$. ✓

→ I am intelligent & lazy → meaning.

Q p: You listen in the class
q: You will pass the exam

$\to q$

$p \to q$

"You will pass the exam if he listens in the class

q if p

(C) $p \to q$. ✓

(A) $p \land q$.
(B) $p \lor q$.

"You will pass the exam only if he listens in the class".

$q \to p$ (D). ✓

Q Verify the following arguments:

(1) If today is Madan's bday, then today is 2nd Sept,
Today is 2nd Sept,
Hence, today is Madan's bday.

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array} \quad \times \text{ fallacy.}$$

Correct it using bi-conditional.

$p \leftrightarrow q$.

Scanned with CamScanner

(2) If it rains, I will be sick.
I was not sick.
therefore it didn't rain

$p \rightarrow q$
$\neg q$
$\therefore \neg p$ ✓ $\therefore$ (Valid)
[Modulus tollens].

(3) Aish is very beautiful. p
Aish is very rich q
Hence, Aish is very nice. r.

p
q
$\therefore r$ (No-relation)

$\therefore$ In-Valid. ✓

Q CS-2014 (paper-3)

P: Good mobile phones are not cheap
Q: Cheap mobile phones are not good.

$g \rightarrow \neg c$.
$c \rightarrow \neg g$. } contrapositive

$\therefore$ logically equivalent.

$[\neg g \vee \neg c. \equiv \neg c \vee \neg g]$.

(D) L, M, N are true.
✓

Q CS-Gate 2017-Set 2

$(\neg p \wedge r) \wedge ((p \wedge q) \rightarrow \neg r)$

$p \rightarrow q$
p only if q.
(A) $(\neg p \wedge r) \wedge (\neg r \rightarrow (p \wedge q))$.

if only not there, $\rightarrow$ (B)

Q Verify the following logical inference.

All students are good. p
Rama is a student q
Therefore, Rama is good $\therefore r$.

(logic so far, not sufficient to solve.
$\therefore$ (common predicates).
functions.
✓

31

(2) PREDICATIVE LOGIC

An open proposition (predicate) is a func $f: U^n \rightarrow \{T, F\}$

eg1: x is a Prime no : $P(x)$

P(2) ✓ P(5) ✓
P(3) ✓ P(6) X
P(4) X

($\forall x$, we are getting T/F).

NOTE: [open Proposition is not a proposition].
(it gives prop, when you replace the variables with values from universal Set).

eg2: $y < 5$: $L(y)$.

L(7) X

Eg3: $x + y = 6$: $S(x, y)$.

– open Proposition of 2 Var.
S(2,3) X
S(2,4) ✓.

* Quantifiers

1: Universal Quantifier

for all, $\forall$ / for every

→ $\forall x P(x)$: "P(x) is true for all values of x in the universal Set".

2: Existential Quantifier

there Exist, $\exists$

$\exists x P(x)$: "P(x) is true for some value of x in the universal Set".
(atleast one value of x).

Statement	True if:	False if:
$\forall x (P(x))$	P(C) is true for all values of 'C' in Universal Set U	There exist atleast one 'C' in U for which P(C) is F.
$\exists x P(x)$	There exist atleast one 'C' in U for which P(C) is True.	P(C) is false for all values of 'C' in Universal Set U.

eg: U = Set of reals.

(i) $P(x)$: "$x < x + 1$" $\forall x P(x)$ is true,
not true / false $\exists x P(x)$ is true.
∵ open Propos.

(ii) $Q(x)$: "$x < x^2$"
↳ Not T/F.

$\forall x\, Q(x)$ is false. $[0,1]$.
$\exists x\, Q(x)$ is true.

(iii) $R(x)$ "$x^2+1=0$".

$\forall x\, R(x)$ is false.
$\exists x\, R(x)$ is false.

$\therefore \forall x\, P(x) \longrightarrow \exists x\, P(x)$

Q. Write the following statements in symbolic form:

(i) All men are Good → All men are not good.
(ii) No men are Good
(iii) Some men are Good.
(iv) Some men are not good.

is men → Predicate
is good → Predicate

Let: $M(x)$: "x is a man".
$G(x)$: "x is Good".

$\forall \equiv \wedge$
$\exists \equiv \vee$
$\forall (\times)\ \wedge$
$\exists$ (add) $\vee$

(i) $\forall x\,[M(x) \longrightarrow G(x)]$.
(ii) $\forall x\,[M(x) \longrightarrow \neg G(x)]$
(iii) $\exists x\,[M(x) \wedge G(x)]$.
(iv) $\exists x\,[M(x) \wedge \neg G(x)]$ (Rule)

* Equivalences.

1. all true ≡ none false (not at least one false)

$$\boxed{\forall x\, P(x) \equiv\ \sim \exists x \sim P(x)}$$

2. all false ≡ none true (not at least one T)

$$\boxed{\forall x \sim P(x) \equiv\ \sim \exists x\, P(x)}$$

3. not all true ≡ at least one false.

$$\boxed{\sim \forall x\, P(x) \equiv \exists x \sim P(x)}$$

4. not all false ≡ at least one true.

$$\boxed{\sim \forall x \sim P(x) \equiv \exists x\, (P(x))}$$

If snake cross $\forall x \longrightarrow \exists x$ ($\sim$ cancels $\sim$)

$\wedge$ is applied between propositions apart from predicates... (partially visible: "∧ ... apin P.")

5. Distributive of $\forall$ over $\wedge$. (RL)

$\forall x[P(x) \wedge Q(x)] \equiv$

$[\forall x P(x)] \wedge [\forall x Q(x)]$.

$\wedge$ is used b/w propositions.

6. Distributive of $\exists$ over $\vee$.

$\exists x[P(x) \vee Q(x)] \equiv$

$[\exists x P(x)] \vee [\exists x Q(x)]$.

$\left[\begin{array}{l}\forall \text{ over } \vee \\ \exists \text{ over } \wedge\end{array}\right]$ not valid.

* Inferences:

7. $\forall x P(x) \Rightarrow \exists x P(x)$

8. $[\forall x P(x)] \vee [\forall x Q(x)] \Rightarrow$

$\forall x[P(x) \vee Q(x)]$. (Joke comm.)

9. $\exists x[P(x) \wedge Q(x)] \Rightarrow$

$[\exists x P(x)] \wedge [\exists x Q(x)]$

(Can't distribute).

NOTE:

Let $U = \{1, 2\}$ (Universal Set).

Proof of 5:

$\forall x[P(x) \wedge Q(x)]$

$\Rightarrow P(1) \wedge Q(1), P(2) \wedge Q(2)$

$\Rightarrow P(1), Q(1), P(2), Q(2)$.

$\forall x P(x) \wedge \forall x Q(x)$

$L \to R$, $R \to L$ (Same Logic).

Proof of 6: (Negate (5))

$\exists x[\neg P(x) \vee \neg Q(x)] =$

$[\exists x \neg P(x)] \vee [\exists x \neg Q(x)]$.

* Converse of (8) is false:

$\forall x[P(x) \vee Q(x)] \not\Rightarrow [\forall x P(x)] \vee [\forall x Q(x)]$

Produce an eg, where LHS = T, RHS = F.

Let $P(1), Q(2)$ are true.

$P(2), Q(1)$ are False

$T \Rightarrow F \vee F$ X.

Scanned with CamScanner

(i) $P(1) \vee Q(1) \rightarrow$ True.

$P(2) \vee Q(2) \rightarrow$ True.

$\therefore$ LHS = True.

Now, $\forall x P(x) = T$ (if both $P(1)$, $P(2) \rightarrow T$).

(ii) $\therefore$ RHS = False.

* Sentences with Multiple quantifiers

$L(x,y)$: "x loves y".
↳ open P. (T/F).

S.No.	Notation	Statement
1.	$(\forall x)(\forall y) L(x,y)$	Everybody loves everybody
2.	$(\forall x)(\exists y) L(x,y)$	Everybody loves somebody
3.	$(\exists x)(\forall y) L(x,y)$	Somebody loves everybody.
4.	$(\exists x)(\exists y) L(x,y)$	Somebody loves Somebody.
5.	$(\forall y)(\forall x) L(x,y)$	Everybody is loved by everybody
6.	$(\forall y)(\exists x) L(x,y)$	Everybody is loved by someone
7.	$(\exists y)(\forall x) L(x,y)$	Somebody is loved by everybody.
8.	$(\exists y)(\exists x) L(x,y)$	Somebody is loved by Somebody.

(Change the order anywhere)

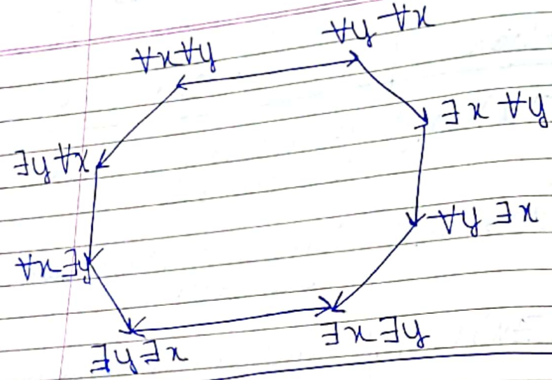
$\forall x \forall y L(y,x)$ Everybody is loved by everybody.

Logical equivalence

$1 \equiv 5$
$4 \equiv 8$.

$3 \rightarrow 6$ $\qquad$ $6 \not\rightarrow 3$

if 6 is true, 3 not true (may/not)



Q Let U = Set of real numbers.

(i) $P(x, y)$: xy is real.

$\forall x \forall y\, P(x, y)$ is true.

(ii) $Q(x, y)$: $x + y = 0$.

$\forall x \exists y\, Q(x, y)$ is true.

$x = 1$, $1 + y = 0$, true for AL 1 y.

$x = 2$, $2 + y = 0$, true for atleast 1 y.

(iii) $\exists y \forall x\, Q(x, y)$ is False

$y = 1$, $x + 1 = 0$; not true, for all values of x.

$y = 2$, $x + 2 = 0$; not true $\forall x$.

give me atleast one y value, for this it is True.

(iii) $R(x, y, z)$: $x + y = z$

$\forall x \forall y \exists z\, R(x, y, z)$ is True.

$\exists z \forall x \forall y\, R(x, y, z)$ is False.

$z = 1$, x give me all values of x & y.

Q Gate CS 2009.

$\neg \forall x\, (P(x))$

$\exists x\, \neg P(x)$ ∴ (I) & (IV)

Q CSE-Gate-2016-Set 2

(A) $(\forall x\, P(x) \Rightarrow \forall x\, q(x)) \Rightarrow$

$(\exists x\, \neg p(x) \vee \forall x\, q(x))$.

$\forall x\, (p(x) \vee q(x)) \Rightarrow$

$\forall x\, p(x) \vee \forall x\, q(x))$.

(D) X.

Scanned with CamScanner

Q CS-2012

"Some real no are rational".

$\exists x[\text{real}(x) \land \text{rational}(x)]$

Q CS-2013

"None of my friends are perfect".

"All of my frnds are not perfect"

$\forall x[F(x) \longrightarrow \neg P(x)]$.

$\forall x[\neg F(x) \lor \neg P(x)]$.

$\neg\exists x[F(x) \land P(x)]$ (d)

Q CS-2007

G(x): x is a Graph.
C(x): x is connected

"There are graphs which are not connected".

$\exists x[G(x) \land \neg C(x)]$ (B)

$\neg G(x)$

$\therefore \neg\forall x[\neg G(x) \lor C(x)]$ (C)

$\neg\forall x[G(x) \Rightarrow C(x)]$.

$\Rightarrow \neg\forall x[\neg G(x) \lor C(x)]$ (A).

Q CS-2009

(B) $\forall x[(G$

(D) $\forall x[(G(x) \lor S(x)) \longrightarrow P(x)]$.

Q Gate CS-2006

(D) $(\forall x)[(T(x) \lor L(x)) \longrightarrow ((H(x) \lor D(x)) \longrightarrow A(x))]$.

Q Gate CS-2004

(b) rule of thumb.

Q Gate CS-2005

(b).

Scanned with CamScanner

Q CS-2010

$\forall x \exists y \exists t (\neg F(x,y,t))$

$\neg(\exists x \forall y \forall t (F(x,y,t))$

(b)

Q Gate-2017-Set1

I $\forall x(\exists y R(x,y))$
can imply
$\exists y(\exists x R(x,y))$ & $\neg \exists x(\forall y(\neg R(x,y)$

(b) I & IV only.

Q CSE-2019

$x=5$; $z|5 \Rightarrow z=\{1,5\}$

$x=6$; $z|6 \Rightarrow z=\{1,2,3,6\}$

P: x is a Prime no.
R: w is a prime no.
Q: For every Prime x, there exists a prime w, such that $w>x$.

(C) S_2 and S_3.

2] SET THEORY

* Set: Well defined collection of objects.

eg: Collection of all students of CS — Set (S)
[we can say precisely $x \in S$]
(no ambiguity)

eg: Collection of clever students of this class.
- (not clearly defined).
- membership:

eg: Collection of all students who got above 80% in X boards → Set.

eg: if we are having membership ambiguity for atleast 1 member; → Not Set.

* Empty/Null Set ϕ or $\{\}$

- All Natural no b/w 0.1 & 0.5 . $\{\}$
- Contains no elements.

Scanned with CamScanner

* Universal Set (U) → $U = \phi$ (can be)

contains all the elements.

- Whatever, we are discussing, it is controlled by Universal Set.

* Subset : ($A \subseteq B$)

Every element of A is in B.

* Equality of Sets

$A = B$ if, $A \subseteq B$ and $B \subseteq A$. ✓

* Proper Subset ($A \subset B$)

every element of A is in B, & $A \neq B$

eg: $A = \{1, 2, 3\}$; $B = \{1, 3, 2, 2\}$. ← Set.

Here, $A = B$

eg: $A = \{1, 2, 3\}$ $B = \{\{1, 2\}, 3\}$.

$\downarrow$ no. of e = 3 $\downarrow$ no. of e = 2. (1 not an ele of B).

Here, $A \neq B$

* Power Set $P(A)$ or 2^A.

Set of all Subsets of A.

eg: $A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

each having 2 poss. (P/A)

$\therefore 2^3 = 8$.

Note:

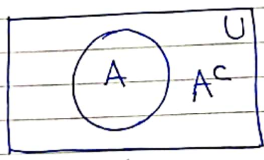
If $|A| = n$, then $|P(A)| = 2^n$.
↳ cardinality

i.e. $\boxed{|2^A| = 2^{|A|}}$

* Operations on Sets

(1) Complement of a Set:

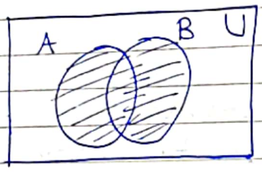
$A^c = \{x \mid x \notin A\}$. ↳ doesn't belong to A.



Scanned with CamScanner

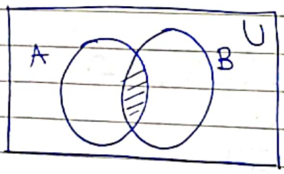
(2) Union

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



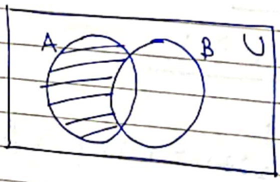
(3) Intersection

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.



(4) Difference of Sets :

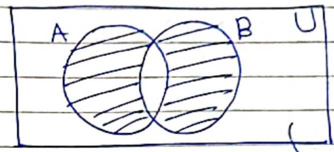
$A - B = \{x \mid x \in A \text{ and } x \notin B\}$.



$\Rightarrow$ $A \setminus B \longrightarrow A - B$

(5) Symmetric Difference of Sets

$A \oplus B = (A - B) \cup (B - A)$.



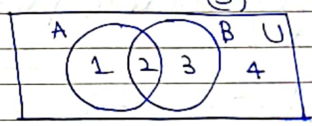
$\longrightarrow$ EX-OR.

$(A \cup B) - (A \cap B)$

* Problems

Q1 CS-1996

$\underset{(1)}{(A-B)} \cup \underset{(3)}{(B-A)} \cup \underset{(2)}{(A \cap B)}$.



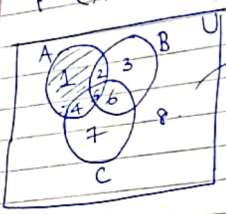
$\because$ 2 Var — 2 Parts.

$\Rightarrow$ $A \cup B$ ✓

Q CS-2005

A, B, C $\longrightarrow$ non-empty Sets.

$X = (A-B) - C$
$Y = (A-C) - (B-C)$



→ 8 Pieces

$X = (A-B) - C$
$[1,4] - [4,5,6,7]$
$\Rightarrow 1$

$Y = (A-C) - (B-C)$
$\Rightarrow [1,2] - [2,3] \Rightarrow 1$

(a) $X = Y$ ✓

Q Which of the following is a power set of empty set ϕ.

$\phi = \{\} \longrightarrow |\,| = 0$

ϕ is a subset of every Set.

$P(\phi) = \{\phi\} \longrightarrow |\,| = 1$

Power Set of ϕ is not empty.

$\boxed{P(P(\phi)) = \{\phi, \{\phi\}\}}$

$P(P(\phi)) = \{\{\}, \{\{\}\}\}$

$\{a, b\}$ where $a = \{\}$, $b = \{\{\}\}$

$P(P(P(\phi))) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$\Rightarrow \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$ ✓

Q Gate CS-2012

$2^{11} = 2048$

Q Gate CS-1995

Set P(S) of Set = $S = \{\{\phi\}, 1, \{2,3\}\}$

$2^3 = 8$

Q CS-2000

$S = \{1\}$

$P(S) = \{\phi, S\}$

$P(P(S)) = \{\phi, \{\phi\}, \{S\}, P(S)\}$

(a) $P(P(S)) = P(S)$ X
(b) $P(S) \cap S = P(S)$ X
(c) $P(S) \cap P(P(S)) = \{\phi\}$

Q. Gate CS-2001

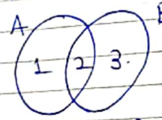
A = {1}, B = {2}.

P(1,2) = P(1) ∩ P(2).

⇒ { X

∴ A∩B = {2}

A = {1,2}



$P(A \cap B) = \{\phi, \{2\}\}$ ✓

$P(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$ ✓

$P(B) = \{\phi, \{2\}, \{3\}, \{2,3\}\}$ ✓

S_1: ✓, S_2: X.

$P(A \cap B) = P(A) \cap P(B)$ ✓

$P(A \cup B) = P(A) \cup P(B)$ X.

Q. ∵ $A \cup B = \{1,2,3\}$

$P(A \cup B) = \{\{1\}, \{2\}, \{3\},$
$\{1,2\}, \{2,3\}, \{1,3\},$
$\phi, \{1,2,3\}\}$.

3/8/19 Saturday

Lecture 2

* Relations

- Cartesian Product

$A \times B = \{(a,b) \mid a \in A, b \in B\}$

Given 2 Sets, create new Set

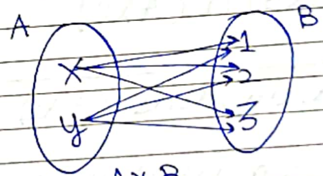
unordered.

eg: $A = \{x, y\}$, $B = \{1,2,3\}$

ordered

$A \times B = \{(x,1), (x,2), (x,3), (y,1), (y,2), (y,3)\}$.

6 ordered Pairs



A × B

∵ $x \notin A \times B$

$x, 1 \notin A \times B$

$(x,1) \in A \times B$ ✓

$\underbrace{(1,x)}_{\text{∵ ordered Pair.}} \notin A \times B$

$\phi \notin A \times B$; but $\phi \subseteq A \times B$.
$\because$ (ϕ subset of every set)

NOTE:

(1) $|A \times B| = |A| \cdot |B|$

(2) $A \times B \neq B \times A$

(3) $A \times B = B \times A$ iff $A = B$ or $A = \phi$ or $B = \phi$

(4) $(A \times B) \times C \neq A \times (B \times C)$ [associative]

$\because ((a,b),c) \neq (a,(b,c))$.

$\because (a,b)$ has to be a.

(5) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(6) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

* Relations → Binary relation (Gate)

A binary relation is a subset of $A \times B$.

eg: $A = \{x, y\}$; $B = \{1, 2, 3\}$.

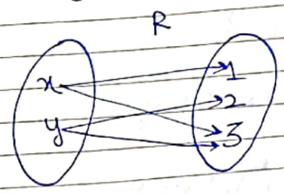
y related to 3

$R = \{(x,1), (x,3), (y,2), (y,3)\}$

$S = \{(x,2), (x,3), (y,2)\}$

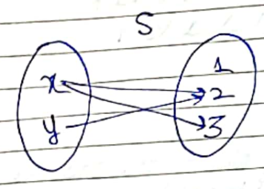
among 6, select 4 in R.

R



(Subset of Cartesian Product).

S



Note:

Total no. of relations from A to B is $|P(A \times B)|$

→ How many subsets of $A \times B$ (total no. of subsets).

$\boxed{2^{|A| \cdot |B|}}$

Empty relation: ϕ.

Universal relation : $A \times B$

Diagonal relation: $\Delta = \{(a,a) \mid a \in A\} \subseteq A \times A$

A → n elements , diag rel → n ele

$A = \{1, 2, 3\}$ → $(1,1)$; $(2,2)$; $(3,3)$

* Types of binary relation

Let R be relation on set A
i.e. $R \subseteq A \times A$.
→ Denote $(x,y) \in R$ as xRy.

- (x,y) belong to R.
- x Related to Y.

1. Reflexive

$\forall x, xRx$ (every elem related to itself).

2. Irreflexive → if (x,x) there / not.

$\forall x, x \not R x$ (x,x) doesnot belong here.

3. Symmetric

$\forall x,y, xRy \Rightarrow yRx$.

(All Males) if x is brother of y, y is brother of x.

4. Asymmetric

$\forall x,y ; xRy \Rightarrow y \not R x$

5. AntiSymmetric

$\forall x,y ; xRy \ \& \ yRx \Rightarrow x=y$.

⇒ if my set = N.
∴ $x \le y$ & $y \le x \Rightarrow x = y$.

6. Transitive

$\forall x,y,z ; xRy \wedge yRz \Rightarrow xRz$.

↳ even if $|A| = 2$, we can explain it by repeating

* Compatible relation

A relation which is reflexive & symmetric.

* Equivalence relation

A relation which is reflexive, symmetric & transitive.

* Partial order relation

A relation which is reflexive, antisymmetric and transitive.

Scanned with CamScanner

* Examples

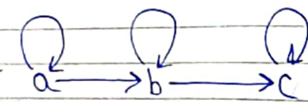
relation from A to A (represent using directed Graph)

Let $A=\{a,b,c\}$

$A \times A = 9$ element
$\therefore 2^9$ relations

1. $R=\{(a,a),(b,b),(c,c),(a,b),(b,c)\}$

if all loops reflexive

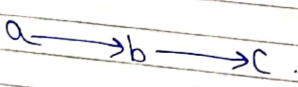


: reln on the set $A \times A$. (direct Graph)

Reflexive : every elem related to itself.

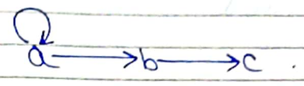
$\therefore$ Reflexive ✓ $\quad$ $\because$ all related to itself
irreflexive X $\quad$: $\forall x, x \not R x$
(every element should be related to itself)
✓ $\quad$ $\because aRa$ is present.

2. $R=\{(a,b),(b,c)\}$



- reflexive : X is (a,a) & (b,b) & (c,c) present
- irreflexive : ✓

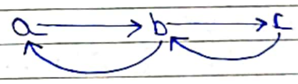
3. $R=\{(a,a),(a,b),(b,c)\}$



reflexive : X $\quad$ $\because$ (b,b) not there & (c,c) not
irreflexive : X $\quad$ (check for a) [fails] (a,a) is there.

if every loop → reflexive
no loop → irreflexive
some loop → nothing ✓

4. $R=\{(a,b),(b,c),(b,a),(c,b)\}$



Symmetry
Asymmetry
Antisymm.

- Symmetric : $\forall x,y \; xRy \longrightarrow yRx$ ✓

(a,a) false $\therefore$ T.
(a,b) ✓ $\quad$ (b,a) ✓ $\quad$ $\therefore$ T.
(a,c) X $\quad$ $\therefore$ T
(b,a) ✓ $\quad$ (a,b) ✓ $\quad$ $\therefore$ T.
(b,b) X $\quad$ $\therefore$ T.
(b,c) ✓ $\quad$ (c,b) ✓ $\quad$ $\therefore$ T.
(c,b) ✓ $\quad$ (b,c) ✓ $\quad$ $\therefore$ T.

Scanned with CamScanner

- Symmetric: check every pair & check if opp present. ✓
- Asymmetric: X (a,b) & (b,a) both exists.
- Antisymmetric: X

(Check 9 entries)

(a,b) & (b,a) both present but a=b (F)

5. R={(a,a),(a,b),(b,c)}

$a \circlearrowleft \longrightarrow b \longrightarrow c$

- Symmetric: X
  (a,a) ✓
  (a,b) ✓ (b,a) X ∴ F.
- Asymmetric: X
  (a,a) ✓ (a,a) ✓ ∴ T.
  ∵ (a,a & a,a both there).
- Antisymmetric ✓
  for (a,a) ✓
  (a,b) ✓ (b,a) X ⇒ F → T/F ∴ T.
  (a,c) X ∴ T.

(b,b) X ∴ T
(b,c) ✓ (c,b) X ∴ F → ∴ T.
(c,c) X ∴ T
(c,b) X. ∴ T
(remaining 8, left side fails) ✓

6. R={(a,b),(b,c)}

$a \longrightarrow b \longrightarrow c$

- Symmetric: (a,b) ✓ (b,a) X.
  X
- Asymmetric: (a,b) ✓, b(a) (not there) ∴ ✓
  ✓
  (b,c) → ✓.
- Antisymmetric (a,b) ✓ (b,a) X.
  ∴ F → ∴ T.
  ✓

7. R={(a,a),(c,c)}

$a \circlearrowleft \quad b \quad c \circlearrowleft$

Symmetric: (b,a): left not there
(b,b): left not there.
✓

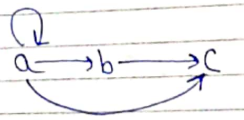
Asymmetric : X

(a,a) ✓ (a,a) ✓ ∴ X
(c,c) ✓ (c,c) ✓ ∴ X

Antisymmetric ✓

(a,a) ✓ (a,a) ✓ ∴ ✓
(c,c) ✓ (c,c) ✓ ∴ ✓

8. $R=\{(a,a),(a,b),(b,c),(a,c)\}$.



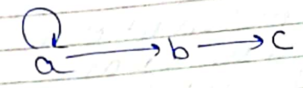
Transitive : $\forall x,y,z,\ xRy,\ yRz \rightarrow xRz$

$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$
3 3 3 ⇒ 27 possibilities.

(a,a) ✓
(a,a) (a,b) ∴ (a,b) ✓
(a,b) (b,a) ∴ T.
✓ X
(a,b), (b,b) ∴ T.
(a,b), (b,c) ∴ T.

∴ Transitive ✓

9. $R=\{(a,a),(a,b),(b,c)\}$ → Anti-Symmetric



Transitive : X.

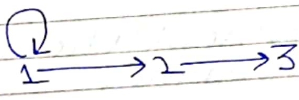
(a,b), (b,c) → (a,c) ∴ False.
✓ T T F X

* Relation b/w these binary relⁿ.

eg Let $R=\{(1,1),(1,2),(2,3)\}$ be relation on set $A=\{1,2,3\}$

Find the reflexive (symmetric, transitive) closure of R.

Solⁿ:



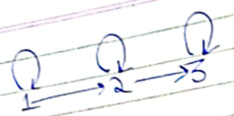
(i) The reflexive closure of R is $R \cup \{(a,a) \mid a \in A\}$

$\Rightarrow \{(1,1),(1,2),(2,3)\} \cup \{(1,1),(2,2),(3,3)\}$.

$\Rightarrow \{(1,1),(2,2),(3,3),(1,2),(2,3)\}$

⇒ 5 ✓

Scanned with CamScanner



(if set is reflexive, Ref C = itself)

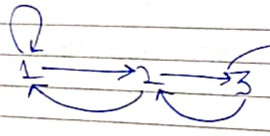
→ [Smallest reflexive relation which contains the given relation is termed as reflexive closure]. (no further extra)

(ii) Symmetric closure of R is $R \cup R^{-1}$

$\{(1,1),(1,2),(2,3)\} \cup$

$\{(1,1),(2,1),(3,2)\}$.

$\Rightarrow \{(1,1),(1,2),(2,1),(2,3),(3,2)\}$ → symmetric closure.



→ [Smallest symmetric relⁿ which contains R → symmetric closure]

(iii) $R^2 = R \cdot R$

$\{(1,1),(1,2),(2,3)\} \cdot \{(1,1),(1,2),(2,3)\}$

$\Rightarrow \{(1,1),(1,2),(1,3)\}$.

$\boxed{\{(a,b)\} \cdot \{(b,c)\} = \{(a,c)\}}$ → dot Product.

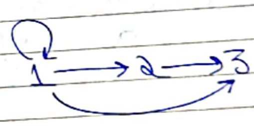
$R^3 = R \cdot R^2 = \{(1,1),(1,2),(2,3)\}$.
$\{(1,1),(1,2),(1,3)\}$.

$\Rightarrow \{(1,1),(1,2),(1,3)\} \Rightarrow R^2$.

$\boxed{\text{The transitive closure of R is } R \cup R^2 \cup R^3 \cup \dots}$

$\{(1,1),(1,2),(2,3)\} \cup \{(1,1),(1,2),(1,3)\}$
$\cup \dots$

$\Rightarrow \{(1,1),(1,2),(1,3),(2,3)\}$ → Transitive Closure.



$a \to b \to c \to d \to e$.
$a \to c$.

• $R \cup R^2 \cup R^3$ → ∵ 3 elements.

* Formulae :

Let $S = \{1, 2, 3 \ldots\ldots n\}$

(1) The no. of relations on S is $2^{n \cdot n} \Rightarrow 2^{(n^2)}$

[No. of Subsets of $S \times S$] = $|P(S \times S)|$

(2) The no. of reflexive relations on S is

$\forall x,\ xRx$

$(1,1), (2,2), (3,3) \ldots\ldots (n,n),$ — p p p p

$(1,2), (2,1), (1,3), (3,1) \ldots\ldots$ — p/a p/a p/a p/a

$2^{(n^2 - n)} \Rightarrow 2^{n(n-1)}$

(3) The no. of irreflexive relations on S is

a a a a

$(1,1), (2,2), (3,3) \ldots\ldots (n,n), (1,2),$ — p/a

$(2,1), (1,3), (3,1) \ldots\ldots$ — p/a p/a p/a

$2^{(n^2 - n)} \Rightarrow 2^{n(n-1)}$

(4) The no. of Symmetric relations on S is :

p/a p/a p/a p/a

$(1,1), (2,2), (3,3) \ldots\ldots (n,n), |(1,2), (2,1)|$ — a/p, a/p

$|(1,3), (3,1)| \ldots\ldots$ — p/a, p/a

if (1,1) taken, reverse pair not, F → T

$$\boxed{2^n * 2^{\left[\frac{n^2 - n}{2}\right]}} \Rightarrow 2^{\frac{n(n+1)}{2}}$$

(5) No. of Asymmetric relations on S is :

a a a a

$(1,1), (2,2), (3,3) \ldots\ldots (n,n), |(1,2), (2,1)|$ — p a / a p / a a

$|(1,3), (3,1)| \ldots\ldots$ — p a / a p / a a

$$\boxed{3^{\left[\frac{n^2 - n}{2}\right]}} \Rightarrow 3^{\frac{n(n-1)}{2}}$$

(6) No. of antisymmetric relations on S is :

p/a p/a p/a p/a

$(1,1), (2,2), (3,3) \ldots\ldots (n,n), |(1,2), (2,1)|$ — p a / a p / a a

$|(1,3), (3,1)| \ldots\ldots$ — p a / a p / a a

(Then ... transitive)

$$\boxed{2^n * 3^{\left[\frac{n^2-n}{2}\right]}}$$

NOTE:

(1) Asymmetry $\Rightarrow$ Irreflexive.
(should not have loop).

(2) Asymmetry $\Rightarrow$ Anti-Symmetry.
opp edge not there

xRy, yRx.
T F
$\therefore$ always T

(3) Irreflexive & transitive $\Rightarrow$ Anti-symmetry
No loops

xRy, yRx
but xRx X.

(4) If R, S are equivalence relations then:
(i) $R \cap S$ is an equivalence relation.
(ii) $R \cup S$ need not be an equivalence relation.

PROBLEMS

Q. CS-2010

No. of reflexive relⁿ on Set of 5 element.

(1,1), (2,2), (3,3) .... (n,n), (1,2), (2,1) ...
(1,3) .... (3,1) ...

$\Rightarrow 2^{(n^2-n)} = 2^{5(5-1)} \Rightarrow 2^{20}$ (C)

Q. The no. of compatible relation on a set containing 'n' elements.
reflexive & Symmetric

(1,1), (2,2) .... (n,n), | (1,2), (2,1) |
| (1,3), (3,1) |

$$\boxed{2^{\frac{n^2-n}{2}} = 2^{\frac{n(n-1)}{2}}}$$

Q CS-2007

S (n elements)
no. of OP in largest & smallest equiv relations on S are:

$S = \{1, 2, 3 \cdots n\}$.

Largest equivalence reln = $S \times S$ → n^2
Smallest eq reln =
$\{(1,1), (2,2), (3,3) \cdots (n,n)\}$ → n. (b) ✓

Q CS-2009

$R = \{(x,y), (x,z), (z,x), (z,y)\}$

Symmetric X
anti-symmetric X. (d)

b/w distinct nodes, opp edges shouldn't exist.

Q CS-2002:

$S = \phi$ (binary relation Set $A = \{1, 2, 3\}$.

$\phi \subseteq A \times A$ ✓

if $(1,1), (2,2), (3,3)$ is in ϕ. ϕ = reflexive

$(1,1) \notin \phi$ ∴ not reflexive.

$(a,b) \in \phi \Rightarrow (b,a) \in \phi$. ∴ Left fails.

∴ symmetric ✓

(d) transitive & symmetric.
→ Left fails too.

$\boxed{\phi \text{ is everything except reflexive}}$
→ not equivalence reln.

[empty relation on empty set is everything (both reflexive & irreflexive)].

Q CS-2015 (Paper-2)

aRb $(1,1)$ not there

check $(1,1)$ 1 & 1 not distinct

- Reflexive X.
- Symmetric ✓

$(a,b) \in R \Rightarrow (b,a) \in R$.

if, $(3,6) \in R \Rightarrow (6,3) \in R$ ✓

- Transitive. X

$(2,6) \in R$, $(6,9) \in R$,

but $(2,9) \notin R$.

(d) ✓

Q CS-2016 (Set 2)

$(a,b)\,R\,(c,d)$ if $a \le c$ or $b \le d$

- reflexive: check $(a,b)\,R\,(a,b)$.

✓ if $a \le a$ or $b \le b$

- $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f)$

$\Rightarrow (a,b)\,R\,(e,f)$.

$\overset{F}{(2,3)\,R\,(3,1)}$ and $\overset{T}{(3,1)\,R\,(1,2)} \Rightarrow$

$\underbrace{}_{T}$ $\underbrace{}_{T}$

$(2,3) \not{R} (1,2)$

if and, true. ∴ Not transitive

but reflexive (B)

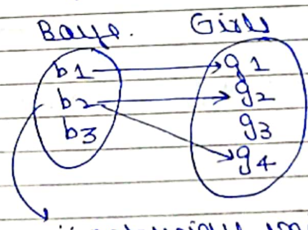
* Functions

A relation f from X to Y is called a function if for every $x \in X$, there is a unique $y \in Y$ such that $(x,y) \in f$.

The range of f is $f(X) = \{ f(x) \mid x \in X \}$.

eg:

Boys to their gfs is not a function.



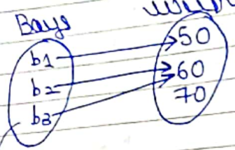
among 12 arrows, select 3 arrows.

∵ not unique image for b2 ∴ X

∵ b3 doesn't have gf ∴ X.

functn: relation is well up

eg: Boys to their weights is a function



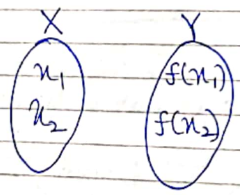
$f(b1) = 50$
$f(b2) = f(b3) = 60$

every boy should take 1 arrow exactly.

$\boxed{\text{Range} = \{50, 60\}}$

*Note:

If $f: X \to Y$ is a function then for any x_1, x_2 in X,



• if $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ ∵ else, relation.

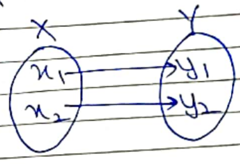
[attach f]
(function)

* Types of function

(1) One-one function / injective:

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (detector f) [one one]

— diff element should have diff image.



(or) $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

(2) Onto function (Surjective)

$\boxed{\text{range} = \text{co-domain}}$

— every element in Y shud have pre-image.

(3) Bijective

— Both one-one & onto.

(4) Many to one

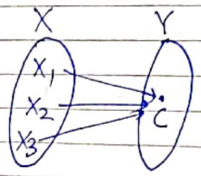
- which is not one-one.

(5) Into : not onto.

(6) Constant function

$f: X \longrightarrow Y$

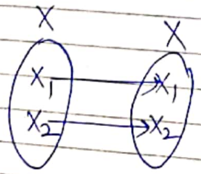
$f(x) = c, \ \forall x \in X$



(7) Identity function

$I: X \longrightarrow X$

$I(x) = x, \ \forall x \in X$



(8) Inverse of a function:

If $f: A \longrightarrow B$ is a bijective function then

$f^{-1}: B \longrightarrow A$ is also a bijective defined as:

$f^{-1} = \{(b,a) \mid (a,b) \in f\}$

for inverse, functn needs to be one one & onto.

(9) Composition of funct

If $f: A \longrightarrow B$ and $g: B \longrightarrow C$ then,

$g \circ f: A \longrightarrow C$ is defined as

$$\boxed{g \circ f(x) = g[f(x)]}$$

eg:

$f(x) = x^2, \ g(x) = x+1$.

- $g \circ f(x) = g[f(x)]$.

  $\Rightarrow g[x^2] = \underline{x^2+1}$

- $f \circ g(x) = f[g(x)]$.

  $\Rightarrow f(x+1) = (x+1)^2 = \underline{\underline{x^2+2x+1}}$

Total relations = ...

how many rel are functions?

Note:

(1) $f\circ g \neq g\circ f$ $\quad f: A\to A,\ g: A\to A$.

(2) $(f\circ g)\circ h = f\circ(g\circ h)$

$h: A\to B,\ g: B\to C,\ f: C\to D$.

(3) f, g are one-one $\Rightarrow g\circ f$ is one-one

$f: A\to B,\ g: B\to C$.

(4) f, g are onto $\Rightarrow g\circ f$ is onto

(5) f, g are bijections $\Rightarrow g\circ f$ is bijection.

and $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$.

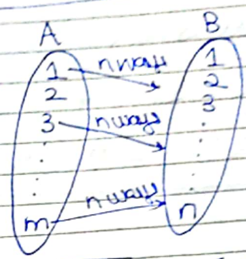
(6) $g\circ f$ is one-one $\Rightarrow f$ is one-one (no guess of g).

(7) $g\circ f$ is onto $\Longrightarrow g$ is onto

* Formula

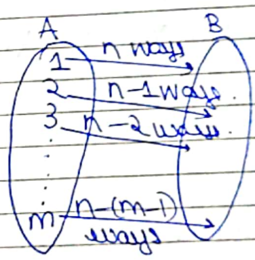
Let $|A| = m,\ |B| = n$.

(1) The no. of functions from A to B is



$\Rightarrow \boxed{n^m}$

(2) The no. of 1-1 functions from A to B is.



$\therefore\ n(n-1)(n-2)\cdots(n-(m-1))$

$\Rightarrow \boxed{\dfrac{n!}{(n-m)!} = {}^nP_m}\ ;\ m\le n$.

(3) No. of onto functions from A to B is :

(from princ of inclusion & exclusion)

$$\boxed{{}^nC_0\, n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - {}^nC_3 (n-3)^m \ldots}$$

(till 0).

(4) The no. of bijective functⁿ, from A to B is $n!$

$[n \times n-1 \times n-2 \times \cdots\cdots 1] = n!$

(5) The no. of many to one functⁿ, from A to B is.

$$\boxed{n^m - {}^nP_m}$$

(6) The no. of into functions is :

$${}^nC_1 (n-1)^m - {}^nC_2 (n-2)^m + {}^nC_3 (n-3)^m \ldots$$

$\because {}^nC_0\, n^m = n^m$

(7) The no. of constant functions from A to B is n.

(8) The no. of identity functⁿ, from A to A is 1.

* Problems

Q CS-2006

$X, Y, Z \longrightarrow x, y, z$ (size).

$W = |X \times Y| = xy$

E = all subsets of $W = 2^{xy}$.

$Z \longrightarrow E$.

$(z) \longrightarrow 2^{xy}$.

No. of functⁿ, from Z to E is e^z

$\therefore (2^{xy})^z = 2^{xyz}$. (D)

Q CS-2014 (Paper 1)

$f : \{0,1\}^4 \longrightarrow \{0,1\}$. — S denote set of fun

2^{16}

$\log_2 \log_2 N$ is ______

$\rightarrow 2\times2\times2\times2 = 16$ elements or 2^{16} relations

N: No. of functions from S to set {0,1}

$\{0,1\}^4 = \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\}$

$\Rightarrow \{(0,0,0,0), (0,0,0,1) \ldots (1,1,1,1)\}$

$\Rightarrow 2^4 = 16$ op

$S = 2^{2^4}$ $\boxed{N = 2^{|S|}}$

$N = 2^{2^{16}}$ $\log_2 N = \log_2 2^{2^{16}}$

$\Rightarrow 2^{16}$

$\log_2(2^{16}) \Rightarrow 16$

Q. CS-2015 (Paper 2)

$X = \{\underbrace{1,2,3,4}_{m}\}$; $Y = \{\underbrace{a,b,c}_{n}\}$

${}^nC_0 n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m$

$\Rightarrow {}^3C_0 3^4 - {}^3C_1 2^4 + {}^3C_2 1^4$

$\cancel{- {}^3C_3 0^4}$

$\boxed{\infty \rightarrow \text{not defined} \times}$

$\because$ can be expressed in known words

$\Rightarrow 3^4 - 3 \cdot 2^4 + 3 \cdot 1$

$\Rightarrow 81 - 3 \cdot 16 + 3$

$\Rightarrow \underline{\underline{36}}$

* GROUPS

1. Natural no: $N = \{1,2,3,4 \ldots\}$
2. Whole no: $W = \{0,1,2,3,4 \ldots\}$

Infinite is a symbol which is not N, but greater than every natural no.

. Largest Natural no $\longrightarrow$ dne.
($\infty \rightarrow$ not natural)

3. Integers: $Z = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$
4. Rational no: $Q = \left\{ \frac{p}{q} / p,q \in Z \ \& \ q \neq 0 \right\}$
5. Real no.s: All points on a straight line.

Scanned with CamScanner

* Binary operation

A function $b: A \times A \longrightarrow A$ is called a binary operation on the Set A.

$A \in N$; $b = +$ (add)

eg: (2,3) is an element of $N \times N$

5 is an element of N.

∴ '+' is a funct^n from $N \times N$ to N.

eg1: '+', 'x' are binary operation on the set of positive integers N.

But '−' is not a binary oper^n on N

∵ $2 - 3 = -1 \notin N$.

eg2: '+', 'x', '−', are binary operations on the set of Integers Z.

eg3: 'Union', 'Intersection' are binary operation on the Set P(A).

Given a subsets of A, its union is also subset of A.

* Algebraic System

A set together with one or more binary operations is called an algebraic System.

eg: (Z, +), (P(A), ∪, ∩)

binary oper^n [oper^n to do cal on Set Z].

Set, BO; oper^n on element of set P(A).

* Notation

Let * be a Binary oper^n on Set A. (i.e * is closed on A).

i.e, $*: A \times A \longrightarrow A$ is a function.

We denote *(2,3) with '2*3'.

* is −, A : Natural

(* is not closed) ; (* is not BO)

* General Properties

Let * be a binary operation on the Set A.

(1) Associative $\therefore$ [$*$ is associative on A].

$\forall a,b,c \in A$

$(a*b)*c = a*(b*c)$

(2) Identity

There exist $e \in A$ such that:

$\forall a \in A,\ a*e = a = e*a$.

eg: $e=0,\ * = +$

$a+0 = a = 0+a$ ✓.

[0 → additive identity
1 → multiplic identity].

(3) Inverse

For each $a \in A$, there exists $b \in A$, such that:

$a*b = e = b*a$.

eg: $* \to +$, (addit inverse)

$a+(-a) = 0 = (-a)+a$.

$a \cdot (1/a) = 1 = (1/a) * a$

[$-a$: additive inverse
$1/a$: mult inverse].

(4) Commutative

$\forall a,b \in A,\ a*b = b*a$.

funct^n $(A \times A) \to A$.

$*$ Definitions

(1) Semi Group: An algebraic System $(A,*)$ is said to be a semi Group if $*$ is associative on A.

(2) Monoid: A Semi Group $(S,*)$ is said to be a monoid if there is an identity element in S with respect to $*$.

(3) Group: A monoid $(M,*)$ is said to be a group if each element in M has inverse with respect to $*$.

(4) Commutative Group/Abelian Group: A Group $(G,*)$ is said to be commutative group if $*$ is commutative on G.

Examples

1. $(N,+) \longrightarrow$ Associative,
(0 not Present – Identity).

$\therefore$ Semi Group, but not monoid

2. $(W,+)$

- associative ✓
- Identity ✓
- Inverse $(-a)$ ✗

$\therefore$ Monoid but not Group.

3. $(Z,+)$

- associative ✓
- Identity ✓ 0.
- Inverse ✓ $(-a)$
- Commutative ✓ $\therefore$ abelein Grp

4. $\underbrace{(N,\times),(W,\times),(Z,\times)}_{\text{Monoids}}$

($1/a$ not present - Inverse).

Set of integers wrt. Mul is Monoid not Group.

5. $(Q,+),(R,+)$ are abelian Groups.

6. $(Q,\times),(R,\times)$ are Monoids. not Groups.

$2/3 - R$, $3/2 - R$.
$\sqrt{2} - R$, $1/\sqrt{2} - R$.
every no. shud have inverse (except 0).

(if we delete 0).

7. $(Q-\{0\},\times)$, $(R-\{0\},\times)$ are abelian Groups.

8. The set of all $n \times n$ non-singular matrices is a Group but not abelian with respect to multiplication.

taken all non-singular (3×3) matrices.

$\because AB \neq BA \therefore$ Commutativity fail ✓

$\Rightarrow$ finite no. of elements

9. The set $G=\{1,-1,i,-i\}$ wrt. multiplic is an abelian Group.
order = 4.

- Take 2 no (same/diff) $\to$ closed.
- associative ✓
- Inverse ✓ $1/i = -i$.
- Commu

$\boxed{\text{identity} = 1}$

* Order of a Group

$O(G)$ = No. of distinct elements of G.

Note:

Scanned with CamScanner

(1) Every finite group of order less than 6 must be abelian.

- BO (closure)
- associative
- identity
- inverse

AB = BA (always happens) when Order <= 6

- Smallest Non-Abelein Grp is 6 elements

(2) Every finite group of prime order must be abelian

O(G) = 19 ⟶ abelian.

(3) In a Group, the inverse of identity ite element is itself.

in X $2 \to 1/2$
$3 \to 1/3$
$1 \to 1$
(ident). (inverse)

in +. $0 \to -0$.
(ident) (inverse)

[identity element ... SG, Group is always SG)
∴ Improper SG

* SubGroups

Let (G, *) be a group & H be a non-empty subset of G.

If (H, *) is a group, then H is called a SubGroup of G.

eg: G(X) is a Group where.

G = {1, −1, i, −i}.

- H_1 = {1, −1} is a Subgroup of G. [Proper]
  - closure
  - identity : 1
  - inverse : 1/a
  - ∴ Group

  4/2

- H_2 = {i, −i} is not a Subgroup of G. [Subset]
  ∵ not closure (i * −i) = 1
  [not binary opern]

- H_3 = {1} is a SubGroup of G. [Improper].
  inv = 1, identity = 1.

  4/1

H_4 = {−1} −1 X.
∵ (−1)(−1) = 1 (not closed)

(−1) x e = −1   e = 1 ⟶ not present.

eg: Let $(Z,+)$ is a Group where

$Z = \{ \ldots\ldots, -2, -1, 0, 1, 2 \ldots \}$

- $E = \{ \ldots -4, -2, 0, 2, 4 \ldots \}$ is a Subgroup of Z.
- Set of all odd no. = not SG ∵ O(H) → Semi Group.
- Set of all n Multiples = SG.

*** Lagrange's Theorem**

If H is any subgroup of a finite group G, then $O(H)$ divides $O(G)$. But the converse is not true.

like $O(G) / O(H_2)$ ✓.
But, $H_2 \rightarrow$ not SG.

PROBLEMS

CS-2013

Q. $x \oplus y = x^2 + y^2$. on Set of Integers.

$(2,3) \longrightarrow O/P = 13$ ∴ binary oper^n

- Commutative

$x \oplus y = x^2 + y^2 = y^2 + x^2 = y \oplus x$ ✓

- Associative

$\{x \oplus y\} \oplus z = \{x^2 + y^2\} \oplus z$
$\Rightarrow (x^2 + y^2)^2 + z^2$.

$\{x \oplus \{y \oplus z\} = x \oplus \{y^2 + z^2\}$
$\Rightarrow x^2 + (y^2 + z^2)^2$. ✗.

∵ LHS ≠ RHS. (a)

Q. The algebraic System $(G, *)$, where G is the Set of all real numbers and $*$ is a binary operation defined by

$a * b = \frac{2ab}{3}$. Then the identity element in G is ______

$(5,7) \xrightarrow{o/p} \frac{2\times5\times7}{3}$

Identity:

$a * e = a = e * a \Rightarrow$

$a * e = \frac{2ae}{3} = a$

$\boxed{e = 3/2}$ (d)

Q Gate-2006

Set {1, 2, 3, 5, 7, 8, 9} under mul. modulo 10 is not a group.

- $(x \cdot y) \% 10 \Rightarrow$

$2 \times 5 = 10 \% 10 \Rightarrow 0$ (not closed)

- e = 1 (identity element) $(9\times1)\%10 = 9$

$(8\times1)\%10 = 8$

$2 \times_{10} b = 1$; $b \Rightarrow$

$2\times1 = 2$

$2\times2 = 4$ $(2\times b)\%10 = 1$.

$2\times5 = 0$ . . .

- $3 \times_{10} b = 1 \Rightarrow b = 7$

$3\times7 = 1$.

3 has an inverse $\longrightarrow$ 7

$(2\times e)\%15 = 2$

Q Gate-2005

Set = {1, 2, 4, 7, 8, 11, 13, 14}.

e = 1 $(2\times1)\%15 = 2$.

- $(4\times b)\%15 = 1$. $b = 4$
- $(7\times b)\%15 = 1$. $b = 13$

$a * b = e = b * a$

(c) 4 and 13

Q CS-2014 (Paper-3)

G be a Group with 15 element.
L be a subgroup of G.
$L \neq G$.
Size of L is atleast 4. } Proper SG.

Size of L is $\rightarrow$ [4, 14].

Lagrange's Theorem

$|L| = \cancel{1}, \cancel{3}, 5, \cancel{15}$ (factors of G)

5 $\rightarrow$ Ans.

$\therefore$ 5.

Q CS-2018

$|H| = 1, 2, 4, \ldots\ldots 42, 84$

42 Ans
[Proper]

Q $G \rightarrow$ arbitrary Group.

equivalence?

R_1: $\forall a, b \in G$, aR_1b if & only if $\exists g \in G$ such that $a = g^{-1}bg$.

reflexive: ✓ aR_1a

$a = g^{-1}ag$.
$\therefore R_1$ is reflexive.

$g \in G$ $\therefore g$ can be identity elem.

Symmetric: ✓

$a = g^{-1}bg$. aR_1b.

$\Rightarrow$

$gag^{-1} = g \cdot g^{-1}bgg^{-1}$

$\Rightarrow$ ~~a~~ $ebe \Rightarrow b$

$\Rightarrow b = (g^{-1})^{-1} a (g^{-1})$.

$\therefore$ Symmetric.

transitive: ✓

$a = g^{-1}bg$, $b = h^{-1}ch \Longrightarrow$

$a = g^{-1}h^{-1}chg$.

$a = (hg)^{-1}c(hg)$.

aRc

R_2: $\forall a, b \in G$, aR_2b if & only if $a = b^{-1}$

(every element can't be inverse of itself).

$\forall a$, $a = a^{-1}$ is wrong.

$\therefore R_2$ X (not reflexive).

Scanned with CamScanner

* Posets & Hasse Diagrams

⇒ A Set P is said to be a partially ordered set (Poset) by the relation $\leq$

If it is:

(i) reflexive : $\forall a,\ a \leq a$

(ii) Antisymmetric :
$\forall a, b,\ a \leq b$ and $b \leq a \Rightarrow a = b$

(iii) Transitive :
$\forall a, b, c,\ a \leq b, b \leq c \Rightarrow a \leq c$.

eg:

(1) $[N, \leq]$ is a Poset.
subset of $N \times N$

- every N is $\leq$ itself.

(2) $[P(A), \subseteq]$ is a Poset.

(3) $[N, |]$ is a Poset.

$a \leq a$ ✓

$a \leq b$ & $b \leq a$: $a = b$ ✓

$a \leq b, b \leq c \Rightarrow a \leq c$ ✓.

(4) $[Z, |]$ is not a poset.

reflexive ✓.

$2 | -2$ ✓ ; $-2 | 2$ ✓ ∴ but $2 \neq -2$.

* 2 elements a and b in a poset $(P, \leq)$ are said to be comparable if either $a \leq b$ or $b \leq a$, otherwise they are incomparable.

If every pair of elements in a poset $(P, \leq)$ are comparable, then we say that $(P, \leq)$ is totally order or linearly ordered or a chain.

Note: if $a \leq b$ & $a \neq b$, we denote it as $a < b$.

a rel to B, & a/b are distinct.

* Let $(P, \leq)$ be a poset and let $a, b \in P$. We say 'b' covers 'a' iff $a \leq b$ (covers)

and there is no element x in P such that $a < x$ and $x < b$.

We denote it as : b | a — b covers a.

eg1 Let $A = \{a, b, c\}$, then:

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

- $[P(A), \subseteq]$ is a poset. (not TO)

– subset of itself ∴

- $\{a\}$ & $\{a,b\}$ are comparable ∵ $a \in \{a,b\}$
- A & $\{a,c\}$ are comparable $\{a,c\} \subseteq A$.
- $\{a,b\}$ & $\{a,c\}$ are not comparable.

[If every pair are comp → TO].

- A covers $\{a,b\}$.

A

|

$\{a,b\}$.

∵ $(a,b) \subseteq A$.

- A covers $\{a,c\}$.
- A covers $\{b,c\}$

* A doesnot cover $\{a\}$.

∵ $\underbrace{\{a\} \subseteq \{a,b\}}_{\text{small umbrella}} \subseteq A$.

- $\{a,b\}$ covers $\{a\}$

$\{a,b\}$

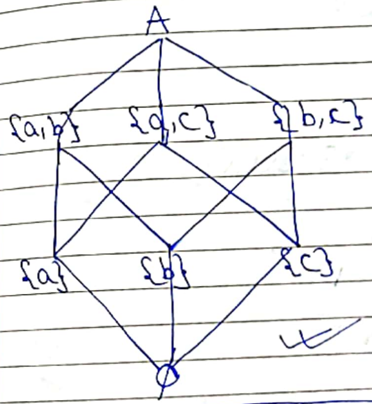
|

a

- $\{a,b\}$ doesnot cover ϕ.

$\phi \subseteq \{a\} \subseteq \{a,b\}$.

* Hasse Diagram



these 3 not comparable.

∴ order.

Subset of Poset → Poset

eg: D_n : Set of positive divisors of 'n'.

| : The relation "divides".

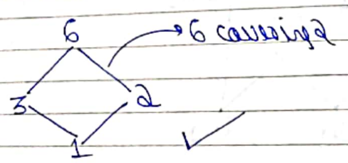
[D_n, |] is a poset ✓

(i) $D_6 = \{1,2,3,6\}$ Poset D of 6.

[D_6, |] is a poset.
but it is not totally ordered since 2,3 are not comparable.

$2/3 \neq 3/2$ ✗

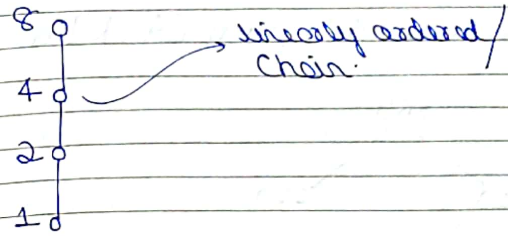
Hasse Diagram



(ii) $D_8 = \{1,2,4,8\}$

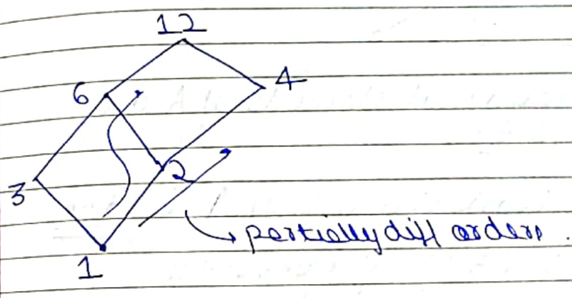
[D_8, |] is totally ordered

→ every pair are comparable. ✓



(iii) $D_{12} = \{1,2,3,4,6,12\}$

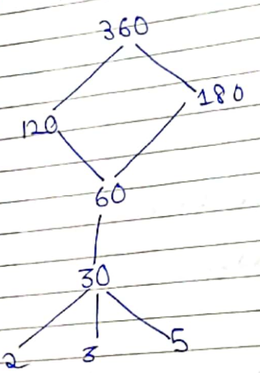
Not, totally ordered.



Note: [D_n, |] is totally ordered iff $n = p^m$ for some prime p

Scanned with CamScanner

* The hasse diagram [A; R] is



(ii) The minimal elements of A are
2,3,5.

(iii) The least element of A is none.
∵ (2,3,5) not comparable.
[not 2].

(iii) The glb of {60, 120, 180} is : 60
Greatest Lower Bound

2 is related to all ∴ LB.
3, 5, 30
60 is also related to all: LB.
120 is related to 120.
not related to 60. (antisymmetry)

∴ GLB = 60 ✓

(4) The lub of {60, 120, 180}
least upper bound.

360 : ✓ (all 3 related to 360).
180 : X ∵ 120 not rel to 180.

⇒ 360 (only 1 upper Bound).

(5) lub & glb of set {3,5} respectively are:

Ub = {30, 60, 120, 180 ...}

∵ 30 related to all other UB.

∴ LUB = 30 ✓
∴ GLB ⟶ does not exist. (∵ LB don't there).

- For any subset of poset, glb & lub may/may not exist.

Scanned with CamScanner

* LATTICES

- A meet Semi-lattice is a poset $(L;\le)$ in which each pair of elements a and b of L has a glb.

Meet or Product of a and b is :
$a \wedge b$ or $a * b = glb\{a,b\}$.
take a intersection : minimizing.

- A Join Semi-Lattice is a poset $(L;\le)$ in which each pair of elements a and b of L has a lub.

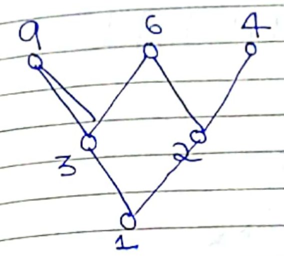
Join or Sum of a and b is :
$a \vee b$ or $a \oplus b = lub\{a,b\}$.
think as union : so up ↑.

- A lattice is a poset $(L;\le)$ in which each pair of elements a and b of L has a glb and lub.

eg1: $[\{1,2,3,4,6,9\};|]$ is a poset.

Subset of Poset → poset.

∴ $[N,|]$ is poset.



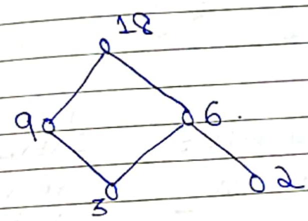
$\{9,6\} \to 3$
$\{3,2\} \to 1$
$\{9,4\} \to 1$.

meet. every pair we have GLB. Meet Semi Lattice

But not Join Semi-lattice.

$\{6,4\} \to$ dne.

eg2: $[\{2,3,6,9,18\};|]$

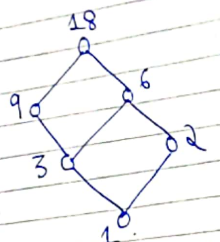


$\{9,6\} \to 18$.
$\{3,2\} \to 6$.
$\{6,3\} \to 18$.

$\{2,3\} \to$ No GLB
But every have LUB.

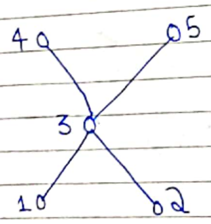
∴ Join Semi Lattice, But not Meet.

eg3: [{1,2,3,6,9,18}; |]



∴ Both Join & Meet → Lattice.

eg4: let the Hasse Diagram:



{1,2} → No GLB.
{4,5} → No ~~GLB~~ LUB.

neither meet nor Join.

Subset of AXA.
∴ Among 25, select a relation which is ref, anti, trans.

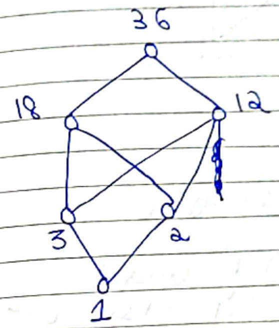
A = {1,2,3,4,5}.

R = {(1,1),(2,2),(3,3),(4,4),(5,5),
(1,3),(1,4),(1,5),
(2,3),(2,4),(2,5) ∵ Transit.
(3,4),(3,5)}

[Reverse edge not taken].

→ Partial order relation.

eg5: [{1,2,3,12,18,36; |] is a Poset.



• {18,12} → lower bounds [3,2,1]

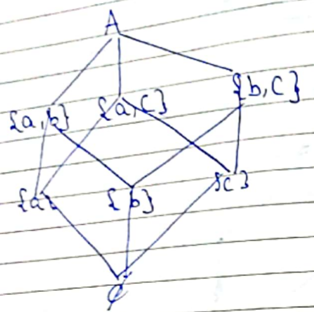
glb: dne ∵ {3,2} not comparable.

• {3,2} → upper bounds [18,12,36]

lub = dne

if you are claiming 1 is least, it should divide others.

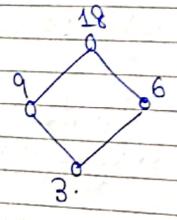
eg6: $[(P(A); \subseteq]$ is a lattice.



Let $x, y \in P(A)$, then

$x \wedge y = x \cap y$ (meet)
$x \vee y = x \cup y$ (join)

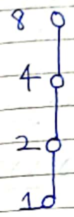
eg7: $[P; |]$ is a lattice where P is the set of positive integers.



meet
for every pair, we can have GCD } unique
" , we can have LCM. }
join.

Let $x, y \in P$ then
$x \wedge y = \gcd(x, y)$
$x \vee y = LCM(x, y)$.

eg8: Every totally ordered set is a lattice.



Let $x, y \in T$, then

$x \wedge y = \min\{x, y\}$
$x \vee y = \max(x, y)$

3. GRAPH THEORY

1. A Graph G is a pair of Sets (V, E).
   V → non empty Set
   E → a

   Coll

2. If the elements of E are ordered Pairs, it is called a directed Graph.

3. The elements of V are called vertices

4. The elements of E are called edges.

5. The no. of vertices in G is called the order of G.

6. No. of edges in G is called the size of G

7. An edge drawn from vertex to itself : loop

8. If more than 1 edge is identified b/w same pair of vertices, edges are called parallel edges. Such a Graph is called multiGraph.

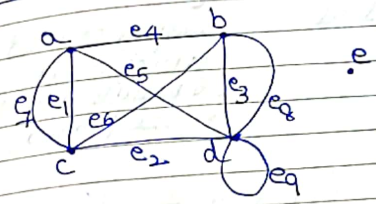
9. A Graph without loops & parallel edges is called a simple Graph.

10. The no. of edges incident at a vertex is called the degree of the vertex. A loop is counted twice at it.

11. The sequence of degrees of vertices of G is called the degree Sequence.

12. A Graph in which vertex is of same degree 'K' is called K-regular.

Examples

1. 

→ e_7 identified with (a,c) / (c,a)
e_5 identified with (a,d) / (d,a) } ∴ Graph.

- undirected Graph
- multiGraph

- Order = 5, Size = 9.
- 'e' is an isolated Vertex. (No edges identified with e)
- e_9 is a loop.
- e_1, e_7 are parallel edges. Also e_3, e_8 are parallel edges.

default → ...

[If ||ₗ edges → multiGraph].

• Vertex: a b c d e → isolated

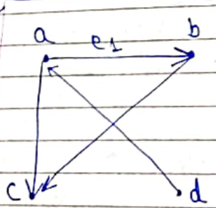
Degree: 4 4 4 6 0

No. of edges incident at a

loop is counted twice at it [6 possible ways to escape from d]

* degree sequence = (0, 4, 4, 4, 6)

increasing order ✓

Q. 

[This is a directed Graph]

Order = 4, Size = 4

- It's a Simple Graph.

(e_1 is identified with ordered pair (a, b))

[No loops, No || edges].

→ No. of Arrows coming in a

Vertex: a b c d.

Indegree: 1 1 2 0.

Outdegree: 2 1 0 1.

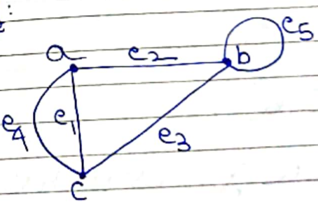
→ No. of arrows coming out from a.

* Theorem:

The sum of degrees of vertices of a graph G is twice the number of edges of G

1st theorem of Graph theory / hand-Shaking theorem

i.e. $\boxed{\sum_{v \in V(G)} \deg(v) = 2|E|}$.

eg: 

→ undirected multi Graph.

$\deg(a) + \deg(b) + \deg(c) = 3 + 4 + 3$

$\Rightarrow \underline{\underline{10}}$.

$2|E| = 2 * 5 \Rightarrow \underline{\underline{10}}$.

Scanned with CamScanner

(Valid for any undirected Graph).

→ 1 edge is contributing to 2 vertices/2 degrees.

NOTE:

(1) In any Graph, the no. of odd degree vertices is even.

a(3) → odd
b(4) → even
c(3) → odd.

• No. of odd degree vertices = 2.

(2) In any directed Graph,

[Sum of indegrees = Sum of out-degrees = No. of edges.]

* Problem

Q (1,3,3,3,4,4,5,6).

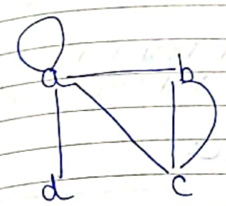
5 odd degree Vertices X. (should be even)

No, since the no. of odd degree vertices is 5, which is odd.

Q (1,3,3,5)?
d c b a

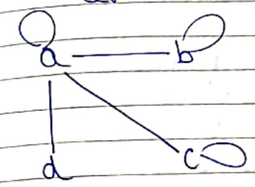
∴ Yes.

• 4 odd degree vertices.



• ~~1~~ ~~3~~ ~~3~~ 5
• 0 2 2
• 0 0 0

Or.

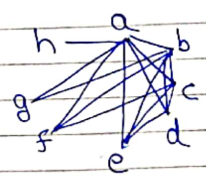


∴ Not unique.

Q CS-2010 Simple Graph.

I 7,6,5,4,4,3,2,1 n(O)=4.
II 6,6,6,6,3,3,2,2 n(O)=2
III 7,6,6,4,4,3,2,2 n(O)=2
IV 8,7,7,6,4,2,1,1 n(O)=2.

I. 7 6 5 4 4 3 2 1
a b c d e f g h ∃ 8 Vertices.



a b c d e f g h
• ~~7~~ 6 5 4 4 3 2 1
• ~~5~~ 4 3 3 2 1 0
~~3~~ ~~2~~ ~~2~~ ~~1~~ 0 0
~~1~~ 1 0 0 0
0 0 0 0 0.

Page No 117

a b c d e f g h
IV. 6, 6, 6, 6, 3, 3, 2, 2 (till 6)
5, 5, 5, 2, 2, 1, 2
4, 4, 1, 1, 1, 1 ← reduce 1 here not req.
3, 0, 0, 0, 1.

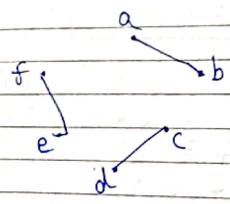
↳ we need loop and (without it imp).

∴ (d) II and IV.

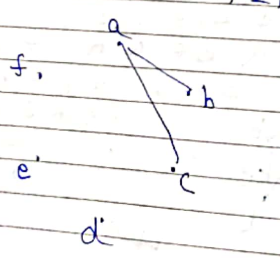
Q CS-2014 (Paper 1)

Simple undirected Graph.

a b c d e f
(A) ~~1~~, 1, 1, 1, 1, 1
0 1, 1, 1, 1
1, 1



a b c d e f
(b) (2, 2, 2, 2, 2, 2)



a b c d e f
2, 2, 2, 2, 2, 2
1, 1, ~~2~~, 2, 2 de, df
1, 1, 0 1, 1

bc & ef.

a b c d e f
(c) (3, 3, 3, 1, 0, 0)
2, 2, 0, 0, 0

cant draw Parallel edges.

Q CS-2016 (Set 2)

23 different compounds.
S of U of 9 compounds, each of which reacts with exactly 3 compounds of U.

∴ 23 Vertices. $|V| = 23$.
9 Vertices with degree 3.

$\sum deg(V) = 2|E|$.

$$9 \times 3 + \sum_{i=10}^{23} deg(v_i) = 2|E| = \text{Even}.$$

must be odd.
[Each compound is ren 14]

∴ atleast one of 14 vertices must have odd degree. (B).

* Find the size of 4-regular graph of order 5.

$|V| = 5$
degree $(V) = 4$.

Sum of degree $= 20 = 2|E|$

$\boxed{|E| = 10}$

* Find the number of edges in a graph containing 3 vertices of degree 4, 2 vertices of degree 3, and 2 vertices of degree 2.

$12 + 6 + 4 \Rightarrow 22$ $\therefore$ 11 edges.

$$\boxed{\sum_{\forall v \in V(G)} deg(V) = 2|E|}$$

Q IT-2004

27 edges.

6 vertices of degree 2
3 vertices of degree 4
remaining vertices of degree 3.

$12 + 12 + 3x = 54$
$3x = 30$
$x = 10$ (a)

$\Rightarrow$ $\therefore$ 19 total vertices. (d)

Q CS-2017 (Paper 2)

$|E| = 25$.

degree $(V) \geq 3$; $\forall V$.

$\sum$ degree$(V) \geq \sum 3$

$2|E| \geq 3|V| = 3n$.

$2 * 25 \geq 3n$.
$50 \geq 3n$.
$\boxed{n = 16}$

4/8/19
Sunday

Lecture 3

Q CS-2005

$|E| \le 3|V| - 6$

Min-degree of G is defined as $\underset{v \in V}{\text{Min}}\{\text{degree}(v)\}$

∴ Min-degree of G can't be:

Soln.

Let $K = \underset{v \in V}{\text{Min}}\{\text{degree}(v)\}$

(degree of any vertex should be K or above).

$\therefore \deg(v) \ge K, \forall v$

$\sum \deg(v) \ge K|V|$. (Summn both sides)

$\Rightarrow 2|E| \ge K|V|$ ✓

$\Rightarrow \cancel{2[3|V|-6] \ge K|V|}$

$\Rightarrow \cancel{6|V| - 12 \ge K|V|}$ $\qquad 2|E| \le 6|V| - 12$

$\Rightarrow \cancel{|V|(6-K) \ge 12}$.

$\therefore K|V| \le 2|E| \le 2(3|V|-6)$

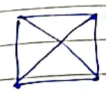
$\therefore (K-6)|V| + 12 \le 0$. (ineqn)

K shouldn't be 6. (d) ✓

Q CS-2002

Max number of edges in a n node undirected graph without self loop

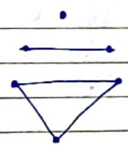
(Parallel edges not allowed)



n = 1 ∴ 0 edges. e = 0.

n = 2 e = 1

n = 3 e = 3



∴ (b) $\frac{n(n-1)}{2}$

Total no. of ways to select 2 vertices from n. (edge)

$\therefore {}^nC_2$. ✓

Q CS-2001

How many undirected graphs (not necessarily connected) can be constructed from n vertices.

(|| edges not allowed).

$n=1$ $g=1$

$n=2$ $g=2$

$n=3$

g1] $^3C_0$

g2

g3 $^3C_1$

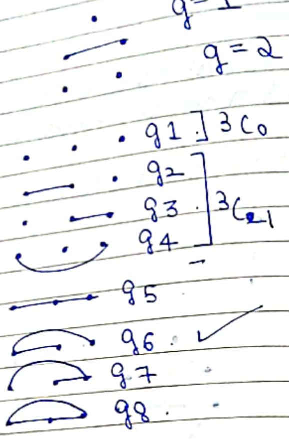
g4

g5

g6 ✓

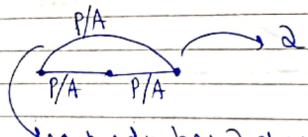
g7

g8



(d) $2^{\frac{n(n-1)}{2}}$

Max no. of edges $= \frac{n(n-1)}{2}$



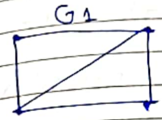
$\rightarrow 2^{\frac{n(n-1)}{2}}$

each edge has 2 choices.

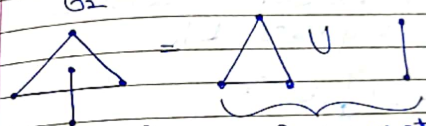
✱ Connectedness

A Graph is said to be connected if there is a path between any given pair of vertices.

eg:

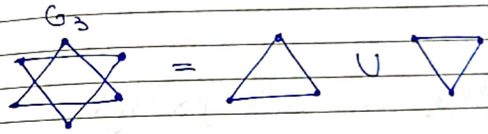


G_1 is connected.



(not conn)

2 connected components.



$G_3 \rightarrow$ not connected.

Q CSE-2018

$v_1 \rightarrow v_2$

1001 Vertices.

edge b/w vertices u & v if & only if, label of u can be obtained by swapping 2 adj nos

$y \rightarrow$ degree

$z \rightarrow$ connected comp.

Scanned with CamScanner

Vertices → (1, 2, 3, 4, 5)

(1, 2, 3, 4, 5) → (2, 1, 3, 4, 5) — (2, 3, 1, 4, 5)

(1, 2, 3, 4, 5) → (1, 3, 2, 4, 5) (2, 3, 4, 1 ...)

(1, 2, 3, 4, 5) → (1, 2, 4, 3, 5) (2, 3, 4, ...)

(1, 2, 3, 4, 5) → (1, 2, 3, 5, 4)

degree = 4

∴ connected Graph

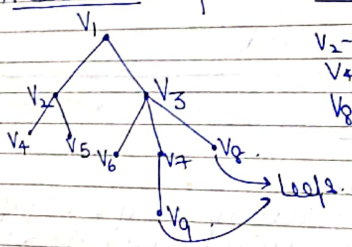
$y = 99$

$z = 1$ (Connected Graph ∴ connected component = 1).

$\boxed{y + 10z = 109}$

* TREE

- A connected Graph with no cycles.



$V_2 - V_3$ X

$V_4 - V_5$ X

$V_8 - V_9$ X

- A tree with n-vertices have (n−1) edges.

$V_1 \to 2$ children

$V_5 \to$ Parent : V_2

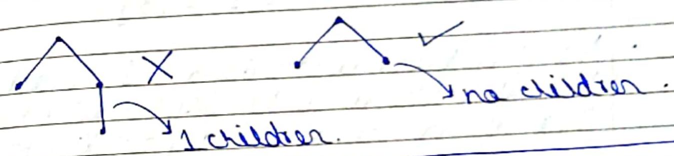
[$V_1 \to$ No Parent]

∴ [for each children, a Parent → edge] (except root).

9 Vertex : 8 Parents → 8 edges.

* Forest :- A Graph with no cycles.
  - Group of trees.
  - need not be connected

* Binary tree : A tree in which each node has 2 or no children.



Q CS-2017

For a tree with 10 Vertices. The Sum of degrees of all the vertices in T is ____.

$n = 10$; $e = 9$

∴ $\sum deg(V) = 2 * 9 = \underline{\underline{18}}$.

Scanned with CamScanner

$E+1$ $E=I+1$

deg 0 = 20
deg 2 = 19

Q CS-2015 (Paper-2)

A binary tree has 20 leaves.
No. of nodes in T having 2 children is ____

eg. ⇒ 5 leaves (Binary)

~~20 + 2x = 2~~

d(leaves) = 1.

degree → {1, 2, 3}

- No. of vertices with deg 2 = 1 (root)
- No. of vertices with deg 1 = 20 (leaf)
- No. of vertices with deg 3 = x (branch)

$\sum deg = 2|E|$

$2 + 3x + 20 = 2(20 + x) = 2|E|$ → $2[(20+x+1)-1]$

(Handshaking)

$E = n - 1$
$\Rightarrow (21 + x) - 1 = 20 + x$

$\Rightarrow 3x + 22 = 40 + 2x$
$x = 218$ ✓

∴ Nodes having 2 children = 19 (inc root)

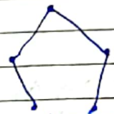
* Special Graphs → no loops / Parallel

1. Null Graphs N_n:

N_5 → 5 connected component.

2. Path Graph P_n → no cycle

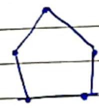
P_5



- All vertices form 1 Path.

3. Cycle Graph C_n

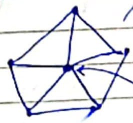
C_5



(All vertices form a cycle).

4. Wheel Graph $W_{1,N}$:

$W_{1,5}$ → W_6

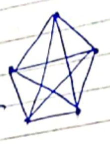


hub of wheel

5. Complete Graph K_n:

$^5C_2$

K_5

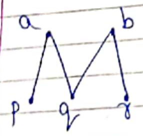


· degree of each vertex = 4

· ∴ 4 regular Graph.

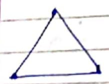
· No of edges in K_n is $\frac{n(n-1)}{2}$

6. BiPartite Graph

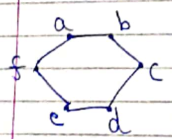


→ G divided into V of 2 sets.

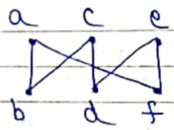
✓



X



⇒

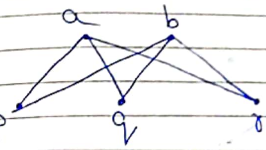


Union of 2 sets of Vertices.

Since, the vertices can be partitioned as:
$M = \{a, c, e\}$ & $N = \{b, d, f\}$

7. Complete BiPartite Graph $K_{m,n}$:

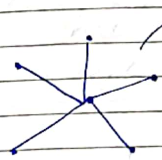
$K_{2,3}$



⇒ m * n edges.

8. Star Graph K_{1N}:

· $K_{1,5}$



→ complete Bi-Partite Graph.

empty Graph: no vertices & no edges

Null Graph: no edges.

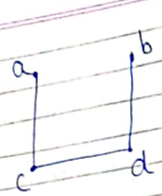
* Complement of a Graph

Complement of a Simple Graph $G = (V, E)$ is the simple Graph $\overline{G} = (V, \overline{E})$ where
$\overline{E} = \{\{V_i, V_j\} \mid \{V_i, V_j\} \notin E\}$

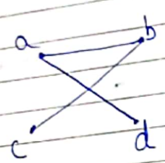
unordered Pair.

1.

G:



$\overline{G}$



Note: $G \cup \overline{G} = K_n$ (complete Graph).

(2) If G has 'n' vertices & 'p' edges then $\overline{G}$ has 'n' vertices and $\frac{n(n-1)}{2} - p$ edges.

* Problems

Q Let G be a Simple Graph, with order 5 & size 7, then size of complement of G is:

V=5 $\boxed{G \cup \overline{G} = K_N}$

e=7.

No. of edges in $K_n = {}^5C_2 \Rightarrow 10$.

$\boxed{e = 10-7 = 3}$; $\boxed{V=5}$

Q If G is a Simple Graph with 15 edges & $\overline{G}$ has 13 edges. How many vertices does it have?

$\frac{n(n-1)}{2} = 28$ → 15+13

$n^2 - n - 56 = 0$

$n^2 + 7n - 8n - 56 = 0$

$\boxed{n = 8}$

Q CS-2015 Paper-2

if isomorphic : simil

$\frac{n(n-1)}{2} - p = p$

$\frac{n(n-1)}{2} = 2p$

$n(n-1) = 4p$

- No. of edges in G is p.
  No. of edges in $\overline{G}$ is ${}^nC_2 - p$.

$\frac{n(n-1)}{\cancel{2}} = 4p$

(D) congruent to 0 mod 4, 1 mod 4

∵ n & n-1 both multiple of 4.

. Euler Path → may / may not be closed.

Q. CS-2014 (Paper 2)

Max no. of edges in a bipartite G on 12 vertices is:

$h_1 + h_2 = 12$

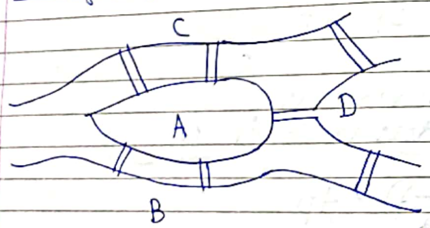
$n_1 \ldots$

$n_2 \ldots$

$Max(n_1 n_2) = \underline{\underline{36}}$.

(1, 11)
(2, 10)
(3, 9) = 27
:
(6, 6) ⇒ 36.

* Konigsberg bridge Problem



* Multi-Graphs & Euler Circuits

(1) An euler ~~Graph in~~ path in a multi Graph is a path that includes each edge of the multiGraph exactly once & intersects each vertex of the multi Graph at least once.

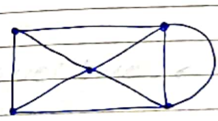
∴ no tail in lord.

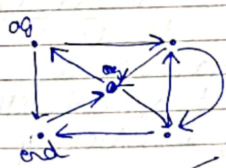
(2) A multiGraph is said to be traversable if it has an Euler Path.

(3) An Euler circuit is an Euler Path whose end points are identical. ↳ need to come to og point.

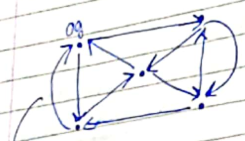
(4) A multiGraph is an Eulerian multi Graph if it has an Euler Circuit.

1. A traversable Multi Graph

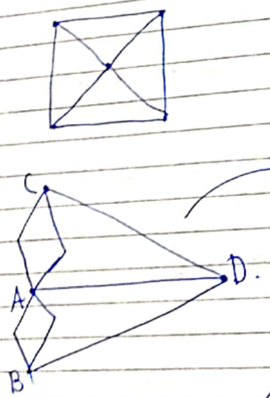




2. An Eulerian Multi graph



3. A multi Graph which is not traversable.



closed tour.

(1) A connected graph G is an Eulerian iff each vertex of G has even degree.
[Eulerian circuit]

(2) A connected graph G is traversable iff G has zero or exactly 2 vertices of odd degree.
open tour.
[Eulerian Path].

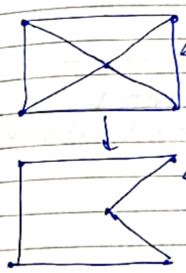
* Hamiltonian Graph

(1) A Path in a graph G is called a Hamiltonian Path if it contains every vertex of G.

(2) A cycle in a Graph G is called a Hamiltonian cycle, if it contains every vertex of G.
(closed

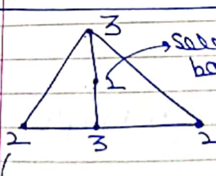
(c) A Graph G is said to be a Hamiltonian graph if it contains a hamiltonian cycle.

eg: Find a Hamiltonian cycle in the Graph.



1. Any Hamiltonian cycle in the graph of order 'n' has 'n' edges.

Q.



· if Eulerian → all degrees must be even. X

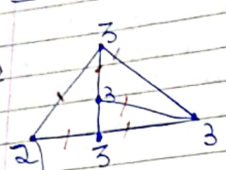
Already edges selected.

$|V| = 5$. (Select 5 edges).

∴ d=2 (Select both)

(D) neither Eulerian nor Hamiltonian.

Q1

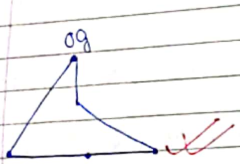


Eulerian X

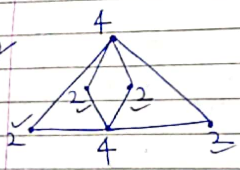
Hamiltonian

|V|=5; Select 5

start 2.



Q2



Eulerian ✓

Hamiltonian X

|V|=6

8 edges must.

Q3 CSE-2019

No. of Hamiltonian Cycles in a complete Graph Kn is

Start

$n \to n-1 \to n-2 \to n-3$.

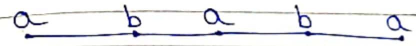
$\therefore \frac{(n-1)!}{2}$; $n \geq 3$.

* Graph Colouring & Chromatic Number

The chromatic no. of graph G is the min no. of colors required to color the vertices of G, such that no two adjacent vertices receive the same colour. It is denoted by $\chi(G)$.

1. The chromatic no. of isolated vertex = 1

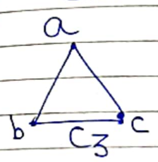
2.



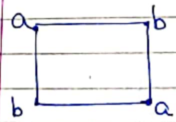
$\chi(P_n) = 2$

Chromatic no. of Path Graph = 2

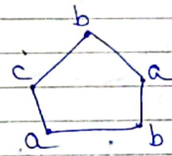
3.



$\chi(C_3) = 3$.



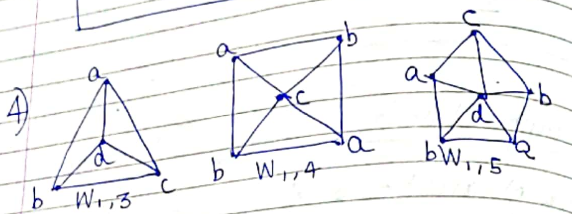
$\chi(C_4) = 2$



$\chi(C_5) = 3$.

$$\boxed{\begin{aligned} \chi(C_n) &= 2, \quad n = \text{even} \\ &= 3, \quad \text{if } n \text{ is odd} \end{aligned}}$$

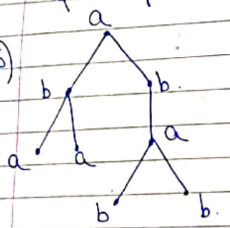
4)



$$\boxed{\begin{aligned} \chi(W_{1,n}) &= 3, \text{ if } n \text{ is even} \\ &= 4, \text{ if } n \text{ is odd} \end{aligned}}$$

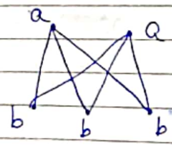
if no. of V = 100 ; n = 99 , ∴ 4.

5)



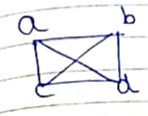
$\boxed{\chi(\text{Tree}) = 2}$

6)



$\boxed{\chi(K_{m,n}) = 2}$

7)

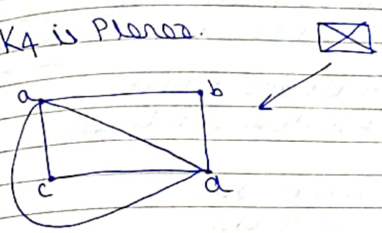


$\boxed{\chi(K_n) = n}$

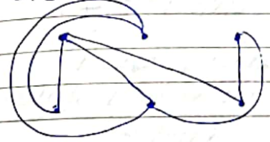
- Each recv diff colour.

* A Graph G is said to be Planar if it can be drawn in a Plane, so that its edges donot cross over.

1. K_4 is Planar.



2. $K_{3,3}$ is not Planar.



Houses

Utilities.

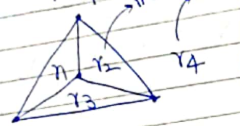
* Four Colour Theorem

$\chi(G) \leq 4$, for any planar Graph G.

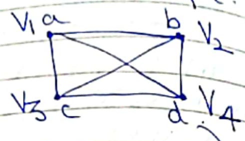
* Colouring Maps, Duality

1.

Map



Dual Graph.



$\chi = 4$.

- Graph without cross over : Map
- Map divides plane into regions.

region → finite area (interior).
→ infinite area (exterior).

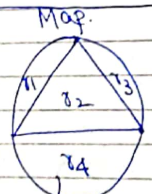
$r_1 r_2$ (share common edge).
$r_1 r_3$ (share common edge)

$\therefore \chi(\text{regions}) = 4$.

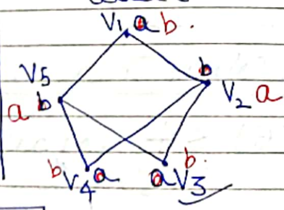
Colouring regions = colouring vertices of dual G.

2.

Map.



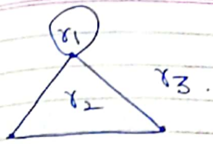
dual G.

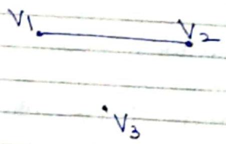


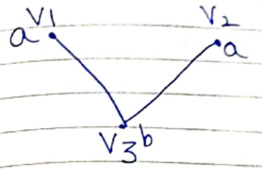
$\boxed{\chi = 2}$

for appearance of this, 2 colours sufficient.

3.





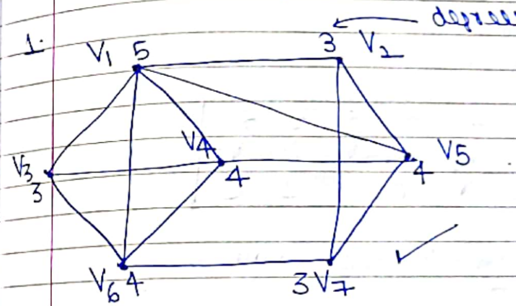


$\boxed{\chi = 2}$

(Min 2 Colours)

Q Find the chromatic no. of the foll Graphs. (Welch-Powell Algo).

1.



decreasing order

Degree	5	4	4	4	3	3	3
Vertex	V_1	V_4	V_5	V_6	V_2	V_3	V_7
Colour	a	b	c	c	b	d	a

$\boxed{\chi \leq 4}$ —— (1)

Col a → over.
Col b → over.

- There is no algo, which gives direct χ.

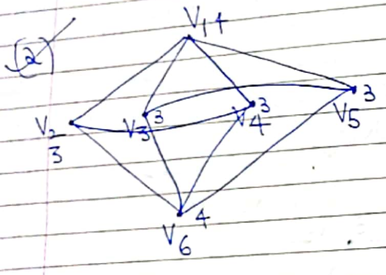
$\chi(K_4) = 4$.

- V_1, V_4, V_6, V_5 forms the SubGraph K_4.

$\therefore \chi \geq 4$ — (2)

$\therefore \boxed{\chi = 4}$ By (1) & (2)

(2)



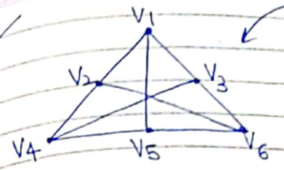
Degree	4	4	3	3	3	3
Vertex	V_1	V_6	V_2	V_3	V_4	V_5
Color	a	a	b	b	c	c

$\boxed{\chi \leq 3}$ ← through algo.

$V_1 V_2 V_4$ forms the subgraph K_3.

$\boxed{\chi = 3}$ $\chi \geq 3$

(3)



Vertex Label do (row wise).

degree	3	3	3	3	3	3
Vertex	V_1	V_2	V_3	V_4	V_5	V_6
colour	a	b	b	a	b	a

$\chi \leq 2$

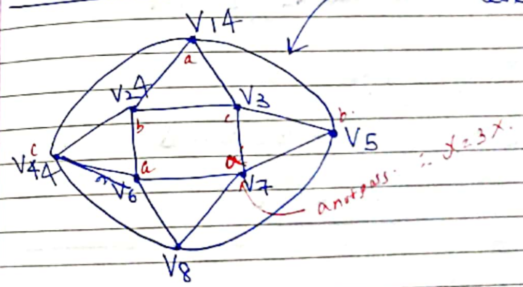
V_1, V_2 is a subgraph K_2

$\chi \geq 2$

$\boxed{\chi = 2}$

(4) CS-2004

numbering row wise



Vertex	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
degree	~~3~~ 4	~~3~~ 4	~~3~~ 4	~~3~~ 4	4	4	4	4
Color	a	b	c	~~d~~ c	b	a	d	~~e~~ e

$\chi \leq 5$

Scanned with CamScanner

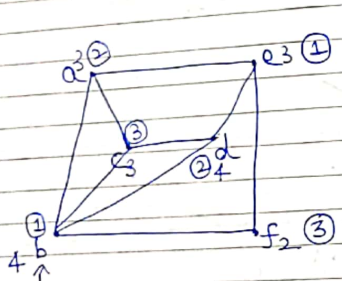
K5 X
K4 X.

≥ 3 · ≤ 5 ∴ planar · ≤ 4.

$\boxed{X=4}$ ∵ Planar Graph.

⇒ 3 is not (just by observation)

Q CSE-2016 Set 2 4



Highest degree

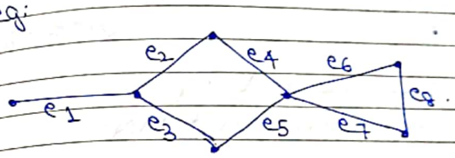
≤ 3 K_3 SG found

∴ $\boxed{X=3}$

* Matching

- A Matching in a Graph G is a subset of edges in which no two edges are adjacent.
- A maximal matching is a matching, to which no edge can be added.
- The no. of edges in the largest maximal matching of G is called the matching number of G. It is denoted by $\alpha'(G)$.

eg:



⇒ 8 edges.

- $\{e_1, \underbrace{e_4, e_5}_{\text{adjacent}}\}$ → subset of edges ∴ Match X (comm. vertex)
- $\{e_1, e_4\}$ is a matching

∵ e_8 can be added, not maximal.

- $\{e_1, e_4, e_8\}$ is a maximal Matching.

$\{e_1, e_5, e_8\}$

$\{e_2, e_5, e_8\}$

$\{e_3, e_4, e_8\}$

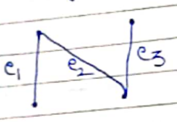
$\{e_1, e_6\}$ is a maximal matching.

- Largest maximal matching : $\{e_2, e_5, e_8\}$.

$\alpha'(G) = 3$

* A Perfect Matching is a matching, in which every vertex is matched.

eg:

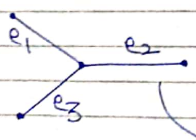


$\alpha'(G) = 2$

$\{e_2\}$ is maximal matching.
$\{e_1, e_3\}$ is maximal matching
→ every vertex is matched
∴ Perfect Matching.

[If the no. of vertices given is odd, ∴ is no perfect matching]

(1) If G has a perfect matching, then $|V(G)|$ is even. Converse is not true.



$\{e_1\}, \{e_2\}, \{e_3\}$
→ maximal mat.

→ no Perfect matching

if odd → definitely not perfect

(2) K_{mn} has a perfect matching iff $m = n$.

(3) The no. of perfect matchings in K_{nn} is $n!$

(4) K_n has a perfect matching iff 'n' is even.

(5) The no. of perfect matching in K_{2n} is
$$\frac{(2n)!}{2^n n!}$$

Q CS-2003 complete graph with 6 vertices.

$2n = 6$.

$$\frac{6!}{2^n 3!} = 15$$

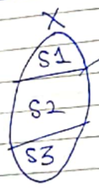
4] COMBINATORICS

- Techniques to reduce the time for counting.

* Basics of Counting

The no. of distinct elements in a Set X is denoted by |X|.

⇒ Sum Rule (Princ. of disjunctive Counting)



→ Parallely counting the no. of student.

X is the union of disjoint non-empty subsets $S_1, S_2, S_3 \cdots S_n$

then,
$|X| = |S_1| + |S_2| + |S_3| + \cdots + |S_n|$

→ ordered Pairs.

Q. How many ways can we get a sum of 4 or 8, when two distinguishable dice are rolled?

Sol) Sum 4 is obtained from
(1,3) (2,2) (3,1)
Sum 8
(2,6), (3,5), (4,4), (5,3), (6,2)

⇒ 2 disjoint Sets : 5+3 = 8.

→ unordered Pairs.

Q. How many ways can we get a sum of 4 or 8, when 2 indistinguishable dice are rolled?

Sum 4 : (1,3), (2,2)
Sum 8 : (2,6), (3,5), (4,4)

No. of ways of getting 4/8 is 2+3 = 5.

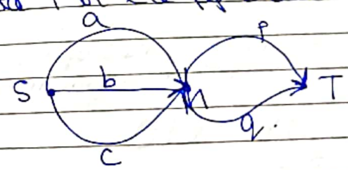
⇒ Product Rule (Princ. of Sequential Counting)

Set $X = S_1 \times S_2 \times S_3 \times \cdots S_n$.

→ ordered (n-tuple)
[Cross Product]

then, $|X| = |S_1| \cdot |S_2| \cdot |S_3| \cdots \cdots |S_n|$.

Q. How many ways can we travel from S to T in the fig shown?



S to M ways $S_1 = \{a, b, c\}$
M to T ways $S_2 = \{p, q\}$
ways from S to T is $X = S_1 \times S_2 =$

$\{(a,p),(a,q),(b,p),(b,q),(c,p),(c,q)\}$

$\therefore$ No. of ways of travelling from S to T is $|X| = |S_1| \cdot |S_2| = 3 * 2 = 6$

X → disjoint.
X → sequentially.
or
if S_1, S_2 element of X : Sum.
S_1, S_2 not done : Product

eg: A computer Password consists of 2 letters of the alphabet followed by 3 or 4 digits. Find:
(i) Total no. of passwords that can be created
(ii) Number of passwords in which no digits repeats.

X → 2 types of Passwords.
S_1: AB123
S_2: AB1234 → this is element of X. [Parallel processing]

(i) $|S_1| = 26 \times 26 \times 10 \times 10 \times 10$
$\Rightarrow 26^2 \times 10^3$

$|S_2| = 26 \times 26 \times 10^4 = 26^2 \times 10^4$

$\therefore$ Total no. of Pass $= 26^2 \times 10^3 \times 11$

$^{26}C_2 \times 2! \times {}^{10}C_4 \times 4!]$

(ii) $S_1 : 26 \times 26 \times 10 \times 9 \times 8$.
$S_2 : 26 \times 26 \times 10 \times 9 \times 8 \times 7$

$= 26^2 \times 10 \times 9 \times 8^2$.

Combinations & Permutations

- A combination of 'n' objects taken 'r' at a time is an unordered selection of 'r' objects from 'n' objects.

- A permutation of 'n' objects taken 'r' at a time is an ordered selection of 'r' objects from 'n' objects. [Arrangement]

eg: List all 2 combinations & 2-permutation of 3 objects.

$n=3, r=2$; Let objects be a, b, c.

I: Without repetitions. $r<n$.

(i) 2-combinations : ab, ac, bc. 3 — bc=cb.

(ii) 2-permutations : ab, ba, ac, ca, bc, cb. 6 — order matters here — possible words of length 2.

II: With repetition $r>n$

(i) 2 combinations : aa, bb, cc, ab, ac, bc.
case 1: $^3C_1$ + case 2: $^3C_2$.

case 1 case 2

$^3C_1 + {}^3C_2 \times 2!$

(vii) 2-permutation:

aa, bb, cc,
ab, ba, ac, ca, bc, cb

r from n
select distinct objects & arrange.

THEOREM (1)

The no. of r-permutations of n-objects without repetitions is

$$\boxed{P(n,r) = {}^nP_r = \frac{n!}{(n-r)!}}$$

VI

1, 2, 3 . . . . . r − 1, r : place

n n−1 n−2 . . . . n−(r−1)
choices

r−1 objects already taken

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$

$\overleftarrow{(n-r)!}$

(1) There are 'n!' permutations of 'n' distinct objects. (r = n)
[Raw Arrangement]

(2) There are '(n−1)!' permutations of 'n' distinct objects in a circle.

(3) If clockwise/anticlockwise represent same permutation, then the no. of circular permutations of 'n' distinct objects is

$\frac{(n-1)!}{2}$. Garland

Let n = 4.

Linear Perm

4 × 3 × 2 × 1 → 4

1/2/3/4   2/3/4   3/4

Circular Perm

1234 = 2341 = 3412 = 4123

1243 = 2431 = 3124 = 4312

$\therefore \frac{4!}{4} \Rightarrow 3!$

Clock / Anti = $3!/2$

THEOREM (2)

Select r from n distinct objects.

The no. of r-combination of 'n' objects without repetitions is:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$n=3$, $r=2$; Let the objects be a, b, c.

from Pg 151:

$\therefore$ $\boxed{\text{no. of comb}^n\text{s} = \frac{\text{no. of permutations}}{r!}}$

$$\boxed{C(n,r) = \frac{n!}{r!(n-r)!}}$$

Note:

(1) $C(n,r) = {}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

(2) ${}^nC_0 = 1 = {}^nC_n$

(3) ${}^nC_r = {}^nC_{n-r}$

$\boxed{\text{V-I}}$

Theorem (3):

No. of r-permutations of 'n' objects with unlimited repetitions is:

$$U(n,r) = n^r.$$

n objects

1	2	3	$\cdots$	$r-1$	r
n	n	n		n	n

$\therefore \underbrace{n \times n \times n \times \cdots}_{r} \Rightarrow \underline{\underline{n^r}}.$

Theorem (4)

$\rightarrow {}^{n+r-1}C_r.$

No. of r-combinations of 'n' objects with unlimited representation is:

$$V(n,r) = C(n-1+r, r) = C(n-1+r, n-1).$$

$\boxed{\text{V-I}}$

Proof:

$V(n,r)$ = No. of r comb. of n objects with repetitions.

5'

2-comb of 3 obj abc with rep
$aa, bb, cc, ab, bc, ac.$

$\Rightarrow$ No. of non-negative integral solutions to $x_1 + x_2 + \cdots + x_n = r$.

$n=3, r=2$

$x_1+x_2+x_3=2$

x_1	x_2	x_3	
2	0	0	aa
0	2	0	bb
0	0	2	cc
1	1	0	ab
1	0	1	ac
0	1	1	bc

Non-negative Integral solns.

⇒ No. of ways of distributing 'r' similar balls into 'n' boxes.

3 boxes, 2 balls. (2 similar balls) if not similar, diff.

```
00|   |
  |00 |
  |   |00
0 |0  |
0 |   |0
  |0  |0
```

$^3C_2$ $^4C_2$

⇒ No. of binary numbers with 'n−1' ones and 'r' zeros.

$n=3 \longrightarrow$ 2 1s, 2 0s.

$\boxed{\begin{matrix}| = 1 \\ 0 = 0\end{matrix}}$

Length of binary no: $n-1+r$.

1	2	3	. . .		$n-1+r$

$C(n-1+r, r)$.

Select r places from $(n-1+r)$ places & put 0.

⇒ $C(n-1+r, n-1)$

Select $n-1$ places from $(n-1+r)$ & put 1.

Problems

Q. How many ways can we distribute 3 similar apples among 5 children?

A1 A2 A3

C_1, C_1, C_2 } same (order don't matter)
C_1, C_2, C_1

∴ unordered Selectn & repetition allowed

No. of ways of selecting 3 children from 5 child; (repetition allowed)

$V(5,3)$

$^7C_3$.   10

$V(5,3) = C(7,3)$

$$\Rightarrow {}^7C_3 \Rightarrow \frac{7\times6\times5}{1\times2\times3} = \underline{\underline{35}}\ (B)$$

- 3 Similar apples, thrown on 5 children. $V(5,3)$

$x_1+x_2+x_3+x_4+x_5 = 3$

Q. How many ways can we distribute 3 different apples among 5 children?

A_1	A_2	A_3	
C_1	C_1	C_2	C_1 is getting 2 A
C_1	C_2	C_1	C_1 is getting 2 A

But diff.

(ordered & repetition).

$U(n,r) = n^r \Rightarrow 5^3$

(5)
A_1 5   5   5. $\Rightarrow \underline{\underline{125}}$

(d) $\underline{\underline{125}}$.

Q. How many ways can we dist. 3 apples among 5 children if each child receives at most one apple?

Similar Apples ✓

A_1	A_2	A_3	
C_1	C_2	C_3	Same
C_1	C_3	C_2	

[Unordered Selection, No repetition]

$$C(n,r) = {}^nC_r \Rightarrow {}^5C_3 = \frac{5!}{3!\,2!}$$

Select 3 children from 5 child. without repetition.

$$\frac{5\times4\times\cancel{3}}{2} \rightarrow \underline{\underline{10}}$$

(A) 10. ✓

Q. 3 diff. apples among 5 children if each child rec. at most 1 apple?

A_1	A_2	A_3	
C_1	C_2	C_3	diff.
C_1	C_3	C_2	

[ordered, No rep]

$$P(n,r) = {}^nP_r = {}^5P_3 = \underline{\underline{60}}\ (C)$$

Scanned with CamScanner

Q In how many ways can a team of 5 be chosen from 10 players so as to include the strongest but exclude the weakest player?

Select 4 from 8. $^8C_4$

out of 5, 1 is selected (strongest)

(unordered, no rep)

Q A shop sells 20 different flavors of ice-cream cones. In how many ways can a customer choose 4 ice cream cones?

Sol^n: order doesn't matter → comb^n. rep^tn

$^{23}C_4$ $V(n,r)$ VI

unordered sel^n of 4 object from 20 object with rep^tn allowed.

$V(20,4)$ $V(4,20)$

F_1	F_2	F_3	…	F_{20}
3		1		
2				2

Q Find the no of non-negative integral solution to $x_1 + x_2 + x_3 + x_4 + x_5 = 10$

$V(5,10) = {}^{14}C_{10}$.   n. r

Q How many diff outcomes are possible by tossing 10 similar coins?

$V(2,10) = C(11,10) = {}^{11}C_1 = 11$.

⇒ 10 similar coins thrown on 2 boxes

(Throw)

10 HS

9 HS, 1T $V(10,2)$

8 HS, 2T $^{11}C_2$

⋮

1 H, 9T

10T.

Q How many diff outcomes are possible by tossing 10 similar dice.

$V(6,10) = {}^{15}C_{10}$, $^{15}C_5$.

Q How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 20$.

→ ∞.

Scanned with CamScanner

$V(5,15) = {}^{19}C_4$

where $x_1 \geq -3$, $x_2 \geq 0$, $x_3 \geq 4$, $x_4 \geq 2$, $x_5 \geq 2$

Soln:

$V(4,12)$

$V(5,15)$

$n=5$
$r=15$

$\boxed{VI}$

Let, $y_1 = x_1 + 3$
$y_2 = x_2$
$y_3 = x_3 - 4$
$y_4 = x_4 - 2$
$y_5 = x_5 - 2$

$(y_1 - 3) + (y_2) + (y_3 + 4) + (y_4 + 2) + (y_5 + 2) = 20$

$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = 15$; $y_i \geq 0$

$\Rightarrow {}^{19}C_{15} = {}^{19}C_4$ ← non-negative integral soln.

$V(4,16) = {}^{19}C_{16}$

Q. Find the no. of ways that can be distributed 16 apples among 4 people. [unordered, repeated] so that each of them gets at least one apple?

$x_1 + x_2 + x_3 + x_4 = 16$, $x_i \geq 1$.

$y_i = x_i - 1$.

$y_1 + y_2 + y_3 + y_4 = 12$.

$V(4,12)$

${}^{15}C_{12}$

${}^{15}C_{12}$

$y_i = x_i - 1$

15

(first give 1 apple to each) then throw 12 Apples $V(4,4,12)$

Q. How many binary seq are there of length 15?

(order matters, rep).

$\Rightarrow 2^{15}$.

2^{15}

Q. How many binary seq are there of length 15, with exactly 10 ones?

→ unordered selection of 10 places among 15, without rep.

1 1 1 1
b_1 b_2 b_3 b_{10} } same.
b_2 b_1 . . .

${}^{15}C_5$. $= {}^{15}C_{10}$

Q. A MCQ test has 25 Qs & 4 choices for each answer.

(i) How many ways can 25 Qs be answered?

1 2 3 . . . . 24 25
a/b/c/d a/b/c/d

$\Rightarrow 4^{25}$

16

(Q) How many ways the 25 Qs be answered so that exactly 3 answers are correct? $\boxed{VI}$

$^{25}C_3 \times 3^{22}$

rem 22 Qs → 3 choices (wrong).

select 3 initially

CS-2003 $\boxed{VI}$

n couples are invited to a party, husband with wife ✓, wife without husband ✓. No. of diff gatherings possible at the party is: ______

Soln

2n total people → 2n ✓, none come, some, only wife.

F

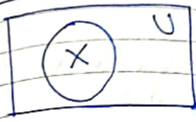
H	W
p	p
a	p
a	a

3 Possib.

3 3 3 ..... n

$\therefore 3^n$

Indirect Counting



$|X| = |U| - |X^C|$

* How many ways are there to pick a man and a woman who are not married from 30 married couples?

Total Poss couples − Married couples

$\Rightarrow (30 \times 30) - 30 = 30 \times (30-1)$

Select a Man in 30. Select a Woman in 29.

$\boxed{^{30}C_1 \times {}^{29}C_1}$

Q In how many ways can 10 people be seated in a row so that a certain pair of them are not next to each other?

Let the people be 1, 2, 3, 4, 5, 6, 7, 8, A, B.

Total ways − no. of ways in which AB are adjacent.

$\Rightarrow 10! - 2 \times 9!$ 　 9!: (AB) → 1 unit, (BA) → 9!

$\Rightarrow 8 \times 9!$

VI

Q. Find the no. of 4 digit telephone no.s having at least one of their digits is repeated.

Total − No. of telep no.s with all 4 distinct digits

$\Rightarrow 10^4 - 10\times9\times8\times7$

$\Rightarrow \underline{4960}$

Q. The no. of squares on a chess board is:

$1\times1 : 64$ $\quad$ $8\times8 : 1$
$2\times2 :$ $\quad$ $7\times7 : 4$

$1+4+9+16+25+\cdots 64$

$\Rightarrow$

$8\times8 : 1$
$7\times7 : 4$
$6\times6 : 9$

$\frac{n(n+1)(2n+1)}{6} = \frac{\cancel{48}\times\cancel{9}^3\times17}{\cancel{36}}$

$\Rightarrow 12 * 17$

$\Rightarrow \underline{204}$.

VI

Q. The no. of rectangles on a chess board is:

Select any 2 Hor lines & then select any 2 vertical lines.

unorder selection, without rep.

$^9C_2 \times {}^9C_2 = \underline{1296}$ (squares are included)

$1296 - 204 = 1092$ (only rectangles)

Q. No. of ways of sitting 7 people on round table? $6!$

Q. No. of ways in which 5 beads of different colours can form a necklace? $4!/2$

$\Rightarrow 12$

* Constrained Repetitions:

Q. How many 8 letter words can be formed using the letters of ENGINEER?

{3E, 2N, G, I, R}

N repeated exactly twice.

$$\frac{8!}{3! \times 2!}$$

even if E interchange remain same

Q. In how many ways, 12 of the 14 people be formed into a 3 teams of 4 each?

12 unordered selⁿ.

$$^{14}C_{12} \times \frac{12!}{4! \times 4! \times 4! \times 3!}$$

No. of permut of $T_1 = 4!$

" " $T_2 = 4!$

" " $T_3 = 4!$

" " $T_1, T_2, T_3 = 3!$

order doesn't matter.

$(T_1)(T_2)(T_3)$

Alter:

$$^{14}C_{12} \times {}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4 \times \frac{1}{3!}$$

select any 4 for 1st team.

* Principle of Inclusion & Exclusion

Q. How many integers b/w 1 & 100 inclusive are divisible by 2 or 3?

Div by 2 : $\left\lfloor \frac{100}{2} \right\rfloor = 50$. $|A_1|$

Div by 3 : $\left\lfloor \frac{100}{3} \right\rfloor = 33$ $|A_2|$

Div by 6 : $\left\lfloor \frac{100}{6} \right\rfloor = 16$ $|A_1 \cap A_2|$

$83 - 16 \Rightarrow$

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 - A_2|$

$\Rightarrow 50 + 33 - 16 = \underline{\underline{67}}$

Q upto 500
3 or 5 or 7.

div by 3, $|A| = \left\lfloor \frac{500}{3} \right\rfloor = 166$

div by 5, $|B| = \left\lfloor \frac{500}{5} \right\rfloor = 100$

div by 7, $|C| = \left\lfloor \frac{500}{7} \right\rfloor = 71$

div by 15, $|A \cap B| = \left\lfloor \frac{500}{15} \right\rfloor = 33$

$|B \cap C| = \left\lfloor \frac{500}{35} \right\rfloor = 14.$

$|A \cap C| = \left\lfloor \frac{500}{21} \right\rfloor = 23$

div by $|A \cap B \cap C| = \left\lfloor \frac{500}{105} \right\rfloor = 4.$

$\Rightarrow |A \cup B \cup C| = 166 + 100 + 71 - 33 - 14 - 23 + 4$

$\Rightarrow \underline{\underline{271}}$

Q CS-2019

$U = \{1, 2 \cdots\cdots n\}$

$A = \{(x, X) \mid x \in X,\ X \subseteq U\}$

ele. set

$X \subseteq U$

$|X| = 2^n.$

$X = \{1\}, \{2\} \cdots\cdots \{n\}.$

$\{1,2\}, \{1,3\} \cdots\cdots \{1, n\}$

$|(1, X)| = 2^{n-1}$

$U = \{1, 2, 3 \cdots\cdots n\}$
p a/p a/p ..... a/p.

$|(2, X)| = 2^{n-1}.$

$|(n, X)| = 2^{n-1}.$

$\therefore n2^{n-1} \longrightarrow \underline{\underline{|A|}} \longrightarrow \mathrm{I})$

(II) $^nC_K =$

$K=1$ $X = \{1\}, x = 1$ $|(x, X)| = \binom{n}{1}$

$K=2$ $X = \{1, 2\}$ $x = 1, 2$ $|(x, X)| = 2\binom{n}{2}$

$|A| = \sum_{K=1}^{n} K \binom{n}{K}$

24

Derangements

A derangement is a permutation of objects that leaves no object in its original position

1. The derangement of {1,2} is {2,1}
2. The derangement of {1,2,3} is {2,3,1}
{3,1,2}
{3,1,2}

Note: The no. of derangements of {1,2,3 .... n} is:

$$\boxed{D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]}$$

include exclude

IT-2004 — Derangements Qs

5 distinct balls distribute in 5 distinct (B1, B2 ... B5) cells $C_1, C_2 \cdots C_5$ such that Ball B_i is not in cell C_i, $\forall i = 1,2,\cdots 5$, & each cell contains exactly 1 ball?

$$D_5 = 5!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$$

$\Rightarrow \frac{5!}{2!} - \frac{5!}{3!} + \frac{5!}{4!} - 1$

$\Rightarrow (5\times4\times3) - (5\times4) - 5$

$\Rightarrow 44$ (a)

Generating Functions

Formulas:

1. $(1-X)^{-1} = \sum_{r=0}^{\infty} X^r = 1 + X + X^2 + X^3 + \cdots$

$\sum_{r=0}^{\infty} (r+1)X^R$ $\quad \sum_{r=0}^{\infty}(r+1)X^r$

2. $(1-X)^{-2} = 1 + 2X + 3X^2 + 4X^3 + \cdots$

3. $$\boxed{(1-X)^{-n} = \sum_{r=0}^{\infty} V(n,r)X^r}$$

4. $V(n,r) = C(n-1+r, r)$

$V(1,r) = C(r,r) = 1$

$V(2,r) = C(1+r, r) = C(1+r, 1) = r+1$

$V(3,r) = C(2+r, r) = C(2+r, 2)$

$\Rightarrow \frac{(r+2)(r+1)}{2}$

${}^{r+2}C_r \quad {}^{r+2}C_2$

${}^{r}C_{r} \quad {}^{r+1}C_r \quad {}^{r+1}C_1 \quad {}^{n+r-1}C_r$

Scanned with CamScanner

* The Power Series

$A(X) = \sum_{r=0}^{\infty} a_r X^r$ is called the

generating function of the sequence $A = \langle a_r \rangle = (a_0, a_1, a_2 \cdots\cdots)$

1. Find the gener. function of the seq. $\{2^r\}_0^{\infty}$

$\sum_{r=0}^{\infty} (2x)^R = \frac{1}{1-2X}$.

$A(X) = \sum_{r=0}^{\infty} 2^r X^r = \frac{1}{1-2X}$. ✓

2. Find the gener. function of the seq. $(1, 2, 3 \cdots\cdots)$

$A(X) = 1 + 2X + 3X^2 + \cdots\cdots$

$\Rightarrow \frac{1}{(1-X)^2}$

$\frac{1}{1-X}$

$A(X) = \sum_{r=0}^{\infty} a_r X \cdot \cancel{R} X^r$.

$1 + 2x + 3X^2 + \cdots$

Q. CSE-2018

$a_n = 2n+3$ ✓

$A(X) = \sum_{r=0}^{\infty} a_r X^r$.

$= \sum_{n=0}^{\infty} (2n+3) X^n$

$\Rightarrow 2 \sum_{n=0}^{\infty} n X^n + 3 \sum_{n=0}^{\infty} X^n$.

$\Rightarrow 2X * \frac{1}{(1-X)^2} + \frac{3 \cdot 1}{1-X}$

$\Rightarrow \frac{2X + 3(1-X)}{(1-X)^2}$ (d)

$\sum_{n=0}^{\infty} nX^n = \sum_{n=1}^{\infty} nX^n$

$= \sum_{n+1=1}^{\infty} (n+1) X^{n+1}$

$\Rightarrow X \sum_{n=0}^{\infty} (n+1) X^n$

Q. Coeff of x^{12} in $(x^3 + x^4 + x^5 + x^6 + \cdots\cdots)^3$ is ______

$\Rightarrow x^9 [1 + x + x^2 + x^3 + \cdots\cdots]^3$

$\Rightarrow x^9 \left[\frac{1}{1-x}\right]^3$ $\boxed{VI}$

$\Rightarrow x^9 [1-x]^{-3} \rightarrow \sum_{r=0}^{\infty} V(3, r) x^r$

$\Rightarrow x^9 \sum_{r=0}^{\infty} V(3,r)\, x^r$.

$\because 12 \to r=3$

$V(3,3) \Rightarrow {}^5C_3$

$\Rightarrow \frac{5\times4\times3}{3\times2} \Rightarrow \underline{\underline{10}}$

CS-2017 (P-2)

$\{a_n\}_{n=0}^{\infty}$ is $\frac{1+z}{(1-z)^3}$

$a_3 - a_0 = ?$

${}^4C_2 = 6$

${}^5C_2 = 10$.

$\sum_{r=0}^{\infty} V(3,r) z^r$

$\Rightarrow (1+z)(1-z)^{-3}$

$\Rightarrow (1+z) \sum_{r=0}^{\infty} V(3,r) z^r$.

$\Rightarrow (1+z) \sum_{r=0}^{\infty} C(2+r,2) z^r$.

$\Rightarrow (1+z) \sum$

$\Rightarrow (1+z)(1+3z+6z^2+10z^3+\cdots)$

$\Rightarrow 1+4z+9z^2+16z^3+\cdots$

15

Recurrence Relations

A recurrence relation for the sequence $a_0, a_1, a_2 \ldots a_n \ldots$ is an equation that relates a_n to one/more of the terms $a_0, a_1, a_2 \ldots a_{n-1}$.

1. Find the recurrence relation for the the sum of first 'n' positive integers.

S_n = The Sum of 1st 'n' positive int.
$\Rightarrow$ Sum of 1st 'n-1' positive integers + n.

$\boxed{S_n \Rightarrow S_{n-1} + n}$ is the recurrence relⁿ.

2. Let P = principle borrowed from bank.
r = interest rate per period.
a_n = amount due after n periods.
Find the recurrence relation?

a_n = Amount due after n periods.

= Amount due after n-1 periods + Int on this amount in nth period.

$\Rightarrow a_{n-1} + r \cdot a_{n-1}$

$\therefore a_n = (1+r)\, a_{n-1}$; $a_0 = P$ is the recurr. relⁿ.

Scanned with CamScanner

3. Find a recurrence relation for the no. of ways can a person climb up a flight of n steps if the person can skip at most 1 step at a time.

7 Steps for success

1 3 5 7
2 4 6 7
2 4 5 6 7
1 3 4 5 7

1 3 4 6 7

Let a_n = no. of ways to climb n steps by skipping at most one step at a time.

$\therefore a_1 = 1,\ a_2 = 2 \rightarrow 0 \to 2,\ 0+1 \to 2$

(Person can start with either 1/2 step no).

Case 1:
- Suppose, start with 1.
- remaining steps in a_{n-1} ways.

Case 2:
- start with step 2
- remaining n-2 steps, he can climb in a_{n-2} ways.

Thus, $\left[a_n = a_{n-1} + a_{n-2}\right]$ recurr. reln.

Note: fibbo seq: $\overset{a_2}{1}, \overset{a_3}{1}, 2, 3, \overset{a_4}{5}, \overset{a_5}{8}, \overset{a_6}{13}, \overset{a_7}{21} \cdots$

$F_n = F_{n-1} + F_{n-2},\ n \geq 2;\ F_0 = F_1 = 1.$

$a_7 = 21$ ways.

Q Gate 2016 Set 1 — n bit string (no conse 1).

1	2	3	4	....	n

Start w. 0 - - - - - - - → a_{n-1} ways.
with no conse 1.

Start w. 1 0 - - - - - - → a_{n-2} ways.

$a_n = a_{n-1} + a_{n-2}$ (B)

$a_n = 2$

Q n bit string, contain 2 consec 1s

1	2	3	4		n

0 → a_{n-1}
1 0 → a_{n-2}
1 1 0/1 0/1 . . . . . 0/1

$\left[2^{n-2} + a_{n-1} + a_{n-2}\right]$ (A)

* Euler phi-function (ϕ)

Let n be a positive integer.

$\boxed{\phi(n) = \text{The no. of positive integers not exceeding } n \text{ \& relatively prime to } n.}$

eg: $\cdot\ \phi(1) = 1$

$\cdot\ \phi(20) = 8$

$\because$ 1, 3, 7, 9, 11, 13, 17, 19 are coprime to 20

$\cdot\ \phi(5) = 4$

$\because$ 1, 2, 3, 4.

$\cdot\ \boxed{\phi(p) = p-1}$; where p is prime.

Note:

If $a = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdots\cdots p_m^{a_m}$.

then $\phi(a) = a \left[1 - \frac{1}{p_1}\right]\left[1 - \frac{1}{p_2}\right] \left[1 - \frac{1}{p_3}\right] \cdots\cdots \left[1 - \frac{1}{p_m}\right]$

eg: $\phi(91) = 7 \times 13 \times \left[1 - \frac{1}{7}\right] \times \left[1 - \frac{1}{13}\right]$

$91 = 7^1 \times 13^1$.

72.

$\cdot\ \phi(90) =$

$\to 2 \times 45 = 2 \times 3^2 \times 5$.

$\Rightarrow 2 \times 3^2 \times 5 \times \left[1 - \frac{1}{2}\right]\left[1 - \frac{1}{3}\right]\left[1 - \frac{1}{5}\right]$

$\Rightarrow$ 24.

Q. IT-2005

Let $n = p^2 q$, p & q are distinct prime no.s

How many no.s 'm' satisfy $1 \le m \le n$ & $\gcd(m, n) = 1$.

$n = p^2 q$.

$\phi(n) \Rightarrow p^2 q \times \left[1 - \frac{1}{p}\right]\left[1 - \frac{1}{q}\right]$.

(d)

Scanned with CamScanner

No. of Problem

No. of positive factors of 18

$18 = 2 \times 3^2$ $\{1,2\}, \{1,3,3^2\}$

Factors of 18 are

$1\times1, 1\times3, 1\times3^2,$
$2\times1, 2\times3, 2\times3^2$

$\Rightarrow \underline{6}$

Note: If $a = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdots p_m^{a_m}$

then no. of factors of a is

$(1+a_1)(1+a_2)\cdots\cdots(1+a_m)$.

CS-2015

No. of divisors of 2100 is ____

$3^1 \times 7^1 \times 2^2 \times 5^2$.

$\Rightarrow 2\times2\times3\times3 = \underline{36}$

$a = b \% m$.

* Modular Arithmetic

Let a & b are integers & m is a positive integer. We say a is congruent to b modulo m, if m divides $a-b$.

i.e. $a \equiv b \pmod m \Leftrightarrow m \mid (a-b)$

a is cong to b mod m

m div $(a-b)$.

$18 \% 5 = 3$

(1) $5 \mid (18-3) \Leftrightarrow 18 \equiv 3 \pmod 5$

(2) $8 \mid (20+4) \Rightarrow 20 \equiv -4 \pmod 8$

(3) $8 \nmid (20-2) \Rightarrow 20 \not\equiv 2 \pmod 8$

8 doesn't divide 18.

8 divides 24

Note: $a \% m = b$.

If $a \equiv b \pmod m$, $c \in \mathbb{Z}$

(i) $a+c \equiv b+c \pmod m$

(ii) $a-c \equiv b-c \pmod m$

(iii) $ac \equiv bc \pmod m$.

(iv)

Note:

$a-c-b+c$.

$ac-bc$

$c(a-b)$.

4

$m \mid (a-b)$

If $a \equiv b \pmod m$ and
$c \equiv d \pmod m$

(i) $a+c \equiv b+d \pmod m$
(ii) $a-c \equiv b-d \pmod m$
(iii) $ac \equiv bd \pmod m$
(iv) $a^x \equiv b^x \pmod m$, $x \in Z^+$

* ~~4~~ Fermat's theorem imp

If p is a prime, & a, p are coprime then:
$a^{p-1} \equiv 1 \pmod p$

* Euler's theorem

If a, n are coprime, then
$a^{\phi(n)} \equiv 1 \pmod n$
if p = prime.

Q What is the remainder when 3^{302} is divided by 5.

$\Rightarrow$ $\boxed{a^{p-1} \equiv 1 \pmod p}$

$\Rightarrow 3^4 \equiv 1 \pmod 5$

$3^{302} = 3^{4(75)} \cdot 3^2 \bmod 5$

$\Rightarrow$ 4

$\Rightarrow (3^4)^{75} \equiv 1^{75} \pmod 5$
$\Rightarrow (3)^{300} \equiv 1 \pmod 5$
$\Rightarrow (3)^{300} \times 3^2 \equiv 3^2 \pmod 5$
$\Rightarrow 3^{302} = 9 \bmod 5 = 4$

$3^{302} \pmod 5 = 4$

Q CS-2019

$3^{51} \bmod 5$ $\quad$ $3^4 \equiv 1 \pmod 5$

$(3^4)^{12} \cdot 3^3 \bmod 5 \Rightarrow 2$

Q $13^{99} \pmod{17}$

4

$13^{16} \equiv 1 \pmod{17}$

$(13^{16})^6 \times 13^3 \bmod 17$

$\Rightarrow (13 \times 13 \times 13) \% 17 \Rightarrow 4$

Scanned with CamScanner