CS 357 / MATH 357
Fall 2013

Numerical Methods – Machine Problem 2
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Due: 11/22/2013

Note: MP is due **5pm** on the due date. Please submit your MP through the dropbox in the Siebel Center basement. Make sure to include your name and **netid** in your homework.

Problem Description

You work for the ACME model airplane company. The company is designing a new model sailplane with a wingspan of 2 meters. The new sailplane has a complicated wing design and the product development department can not figure out an analytical solution for the lift distribution along the wing. They have asked you (the company numerical methods expert and web designer) to solve the problem for them. They would like to know the lift force distribution from the root of the wing near the fuselage out to the tip of the wing. They also need to integrate the lift distribution to find out the total lift. They have given you a description of the geometry of the wing and the equations governing the lift distribution.

The wingspan (distance from one tip to the other) is 2.0 meters. The chord (distance from the leading edge of the wing to the trailing edge of the wing) at the center of the wing is 23.5 cm. The chord at the tip is 16.5 cm. There is no sweep. The wing is straight and tapered.

The lift on the wing is given by the following integral.

$$L = \int_{-1}^{1} \Gamma(x)dx \tag{1}$$

Where Γ is the circulation on the wing and can be equated to the increment in lift. The value of Γ changes along the wing from zero at the tips to some non-zero value at the root or center of the wing. The distribution of circulation is symmetrical about the center of the wing so we need only consider half of the wing. The value of the circulation at a particular point $\bar{x}$ measured from the center or root of the entire wing is given by the following equation.

$$\Gamma(\bar{x}) = \int_{\bar{x}}^{1} \gamma(x)dx \tag{2}$$

Examination of equation 2 reveals that Γ depends on the distribution of another function γ. Without going into too much detail the aerodynamics people in product development have told you that γ is the circulation that is shed behind the wing. You stop them there and decide to approximate γ by a piecewise constant function. Noting that the wing is symmetrical (left and right wing identical) you decide to solve for Γ on the right half of the wing. So at the points x_i starting at the root of the wing $x = 0$, $\Gamma(x_i)$ satisfies the following relation:

$$\Gamma(x_i) = \sum_{j=i}^{m-1} \gamma(x_j) \tag{3}$$

In other words Γ satisfies:

$$\begin{aligned}
\Gamma_0 &= \gamma_0 + \gamma_1 + \cdots + \gamma_{m-1} \\
\Gamma_1 &= \gamma_1 + \gamma_2 + \cdots + \gamma_{m-1} \\
\Gamma_2 &= \cdots
\end{aligned}$$

It remains to relate the value of γ_i to the wing geometry and flight conditions. The following equation describes that relationship.

$$\int_{\bar{x}}^{1} \gamma(x)dx = \frac{\bar{c}c_{\ell_\alpha}}{2}(\alpha - \int_{0}^{1} g(x)\gamma(x)dx) \tag{4}$$

Where $\bar{c}$ is the chord length at $x = \bar{x}$, c_{ℓ_α} is the lift curve slope and is equal to 2π, and α is the angle the wing makes with the relative wind which for this problem is .05 radians. The function $g(x)$ relates the wing geometry to the distribution of γ and will be defined below. This equation can be transformed to:

$$\int_{0}^{1} \left[\frac{\bar{c}c_{\ell_\alpha}}{2} g(x) + k\right] \gamma(x)dx = \frac{\bar{c}c_{\ell_\alpha}}{2}\alpha \tag{5}$$

And here $k = 1$ when $x \geq \bar{x}$ and $k = 0$ when $x < \bar{x}$. This equation holds at any $x = \bar{x}$ along the wing.

Problem 1 [40pt]

In order to solve this problem you must use the above equation to determine the $\gamma(x)$, integrate $\gamma(x)$ to determine the value of Γ along the span and finally integrate Γ to find the lift. We begin by considering the first equation above. If Γ were known the lift could be calculated. We can use Gaussian quadrature to evaluate the integral for lift. The interval is even -1 to 1. In order to do this we need the Gauss points and the value of Γ at those points (and the weights of course). We choose to use the Gauss points corresponding to N = 20 for the whole span. There are 21 points total from tip to tip with one point at the center of the wing. There are 11 points on the half span of the wing from x = 0 to x = 1.0.

We now apply equation 5 at each of the gauss points $\bar{x}$ on the half span from $0 \leq x \leq 1$ to obtain a linear system of equations with $\gamma(\bar{x})$ as the unknowns. We approximate the integral in equation 5 by assuming the integrand (f) is a piecewise constant function of x. The value of f is constant on an interval centered at $\bar{x}$ and extending half way to the Gauss point on either side of $\bar{x}$. The coeffiients of the 11x11 matrix corresponding to the linear system of equations is given by the expression below.

$$a_{i,j} = \frac{\bar{C}(xg_i)C_{\ell_\alpha}}{4\pi}\left[\frac{xv_j}{xv_j^2 - xg_i^2}\right] + K \tag{6}$$

Again where $K = 1$ if $j \geq i$ and $K = 0$ if $j < i$. The quantities xg are the locations of the Gauss points. The quantities xv are points locate half way between the gauss points and satisfy the following relation.

$$\begin{aligned} xv_i &= \frac{xg_{i+1} + xg_i}{2};\ i = 0, 1, \cdots, n-2 \\ &= 1.0;\ i = n-1 \end{aligned}$$

In our case $n = 11$ is the number of points on the half span from $0 \leq x \leq 1$. The elements of the right hand side vector is given by the following expression.

$$b_i = \bar{C}(xg_i)C_{\ell\alpha}\alpha/2.0 \tag{7}$$

(a) Write a python program to solve for the unknown vector γ_i at the 11 Gauss points on the half span of the wing by forming a linear system of equations with the elements given above.

(b) Plot the value of γ at the Gauss points for the half span.

(c) Calculate and plot the value of $\Gamma(\bar{x})$ at the Gauss points for the half span by considering the contributions from the individual γ_i. Notice that the value of $\Gamma(0)$, the first Gauss point is just the sum of all the γ_i. Subsequent values of $\Gamma(xg_i)$ are calculated by subtracting γ_i from $\Gamma(xg_{i-1})$.

(d) Calculate the total lift on the wing by using Gaussian quadrature. Utilize the value of Γ at the Gause points and the weights for the n = 20 quadrature rule (use a table to determine the weights and Gauss points).

(e) (extra credit 10 pts) Repeat the calculation for a constant chord wing of chord 0.2. Compare the total lift and Γ distribution along the half span.