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MAE 263B FINAL EXAM
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PART 1 : KINEMATIC ANALYSIS

(1) Forward Kinematics.

The solution can be found in the file 'Forward Kinematics Analysis.mlx'. It has the required expressions in both symbolic form and numerical form.

(2) Inverse Kinematics

The inverse kinematics was solved with the help of the code 'InverseKinematicsPuma.mlx', to get the expressions of Transformation matrix on each side.

The solution has been presented below.

$\underline{\theta_1}$

$$P_y c_1 - P_x s_1 = d_2 + d_3$$

$$\theta_1 = \text{atan2}\left(\pm\sqrt{P_x^2 + P_y^2 - (d_2+d_3)^2}\right) + \text{atan2}(-P_x, P_y)$$

$\underline{\theta_2 \ \& \ \theta_3}$

$$(P_x c_1 + P_y s_1) c_2 - P_z s_2 = a_2 + a_3 c_3 + d_4 s_3$$

$$-P_z c_2 - (P_x c_1 + P_y s_1) s_2 = a_3 s_3 - d_4 c_3$$

$$(P_x c_1 + P_y s_1)^2 + P_z^2 = a_2^2 + a_3^2 + d_4^2 + 2 a_2 a_3 c_3 + 2 a_2 d_4 s_3$$

$$\Rightarrow a_3 c_3 + d_4 s_3 = \frac{(P_x c_1 + P_y s_1)^2 + P_z^2 - a_2^2 - a_3^2 - d_4^2}{2 a_2}$$

$$= K_3$$

$$\theta_3 = \operatorname{atan2}\left(\pm\sqrt{a_3^2 + d_4^2 - k_3^2},\ k_3\right) + \operatorname{atan2}(d_4, a_3)$$

$$(P_x c_1 + P_y s_1)\, c_2 - P_z s_2 = a_2 + a_3 c_3 + d_4 s_3$$

$$-P_z c_2 - (P_x c_1 + P_y s_1)\, s_2 = a_3 s_3 - d_4 c_3$$

$$\overset{A}{[P_x c_1 + P_y s_1]}\, c_2 + \overset{C}{[-P_z]}\, s_2 = \overset{D}{a_2 + a_3 c_3 + d_4 s_3}$$

$$\underset{E}{[-P_z]}\, c_2 + \underset{F}{[-P_x c_1 - P_y s_1]}\, s_2 = \underset{G}{a_3 s_3 - d_4 c_3}$$

$$-(P_x c_1 + P_y s_1)^2 - P_z^2 < 0$$

$$\Rightarrow \theta_2 = \operatorname{atan2}\Big(-P_z(a_2 + a_3 c_3 + d_4 s_3) - (P_x c_1 + P_y s_1)(a_3 s_3 - d_4 c_3),\ -P_z(a_3 s_3 - d_4 c_3) + (P_x c_1 + P_y s_1)(a_2 + a_3 c_3 + d_4 s_3)\Big)$$

$$\operatorname{atan2}(DE - AG,\ CG - DF)$$

$\underline{\theta_4}$

$$R_{23}\,\sigma_1 - R_{13}\,\sigma_2 + R_{33} S_{23}\, S_4 = 0$$

$$\begin{aligned}
\sigma_1 &= c_1 c_4 - c_2 c_3 s_1 s_4 + s_1 s_2 s_3 s_4 \\
&= c_1 c_4 - s_1 s_4 \left(c_2 c_3 - s_2 s_3 \right) \\
&= c_1 c_4 - s_1 s_4 c_{23}
\end{aligned}$$

$$\sigma_2 = s_1 c_4 - c_1 s_4 c_{23}$$

$$\Rightarrow \left(R_{23} c_1 - R_{13} s_1\right) c_4 - \left(R_{23} s_1 c_{23} - R_{13} c_1 c_{23}\right) s_4 + R_{33} S_{23}\, s_4 = 0$$

$$\Rightarrow \left(R_{23} c_1 - R_{13} s_1\right) c_4 = \left(R_{23} s_1 c_{23} - R_{13} c_1 c_{23} - R_{33} S_{23}\right) s_4$$

$$t_4 = \frac{R_{23} c_1 - R_{13} s_1}{R_{23} s_1 c_{23} - R_{13} c_1 c_{23} - R_{33} S_{23}}$$

$$\theta_4 = \operatorname{atan2}\left(R_{23} c_1 - R_{13} s_1 ,\; R_{23} s_1 c_{23} - R_{13} c_1 c_{23} - R_{33} S_{23} \right)$$

θ_5

$$R_{31}\,\sigma_5 - R_{21}\,\sigma_1 - R_{11}\,\sigma_3 = 0$$

$$\sigma_1 = c_1 s_4 s_5 + c_2 s_1 s_3 c_5 + c_3 s_1 s_2 c_5$$
$$+ c_2 c_3 c_4 s_1 s_5 - c_4 s_1 s_2 s_3 s_5$$

$$= c_1 s_4 s_5 + s_1 s_{23} c_5$$
$$+ s_1 c_{23} c_4 s_5$$

$$\sigma_1 = (c_1 s_4 + s_1 c_4 c_{23})\, s_5 + (s_1 s_{23})\, c_5$$

$$\sigma_3 = c_1 c_2 c_5 s_3 - s_1 s_4 s_5 + c_1 c_3 c_5 s_2$$
$$+ c_1 c_2 c_3 c_4 s_5 - c_1 c_4 s_2 s_3 s_5$$

$$= c_1 s_{23} c_5 - s_1 s_4 s_5 + c_1 c_4 c_{23} s_5$$
$$\sigma_3 = (-s_1 s_4 + c_1 c_4 c_{23})\, s_5 + (c_1 s_{23})\, c_5$$

$$\sigma_5 = s_2 s_3 c_5 - c_2 c_3 c_5 + c_2 c_4 s_3 s_5$$
$$+ c_3 c_4 s_2 s_5$$
$$= (-c_{23})\, c_5 + (s_{23} c_4)\, s_5$$

$$R_{31}\,\sigma_5 - R_{21}\,\sigma_1 - R_{11}\,\sigma_3 = 0$$

$$c_5\left[-R_{31}\,c_{23} - R_{21}\,s_1 s_{23} - R_{11}\,c_1 s_{23}\right]$$
$$+\, s_5\left[R_{31}\,s_{23} c_4 - R_{21}\left(c_1 s_4 + s_1 c_4 c_{23}\right) - R_{11}\left(-s_1 s_4 + c_1 c_4 c_{23}\right)\right] = 0$$

$$s_5\left[R_{31}\,s_{23} c_4 - R_{21}\left(c_1 s_4 + s_1 c_4 c_{23}\right) - R_{11}\left(-s_1 s_4 + c_1 c_4 c_{23}\right)\right]$$
$$= c_5\left[R_{31}\,c_{23} + R_{21}\,s_1 s_{23} + R_{11}\,c_1 s_{23}\right]$$

$$\theta_5 = \operatorname{atan2}\Big(\left[R_{31}\,c_{23} + R_{21}\,s_1 s_{23} + R_{11}\,c_1 s_{23}\right],$$
$$\left[R_{31}\,s_{23} c_4 - R_{21}\left(c_1 s_4 + s_1 c_4 c_{23}\right) - R_{11}\left(-s_1 s_4 + c_1 c_4 c_{23}\right)\right]\Big)$$

$\underline{\theta_6}$

$$s_6 = R_{21}\,\sigma_7 - R_{11}\,\sigma_8 + R_{31}\,s_{23}\,s_4$$
$$c_6 = R_{22}\,\sigma_7 - R_{12}\,\sigma_8 + R_{32}\,s_{23}\,s_4$$

$$\sigma_7 = c_1 c_4 - c_2 c_3 s_1 s_4 + s_1 s_2 s_3 s_4$$
$$= c_1 c_4 - s_1 s_4 c_{23}$$

$$\sigma_8 = s_1 c_4 + c_1 c_2 c_3 s_4 - c_1 s_2 s_3 s_4$$
$$= s_1 c_4 + c_1 s_4 c_{23}$$

$$s_6 = R_{21} c_1 c_4 - R_{21} s_1 s_4 c_{23} - R_{11} s_1 c_4 - R_{11} c_1 s_4 c_{23} + R_{31} s_{23} s_4$$

$$c_6 = R_{22} c_1 c_4 - R_{22} s_1 s_4 c_{23} - R_{12} s_1 c_4 - R_{12} c_1 s_4 c_{23} + R_{32} s_{23} s_4$$

$$\theta_6 = \text{atan2}(s_6, c_6)$$

Now all the expressions derived were written in a function 'IKPuma.m' which outputs all possible solutions. The function was tested in the code 'InverseKinematicsTesting.m' with different angles and it was successful.

Later these expressions have been used in 'IKPuma2.m' and 'IKPuma3.m' which outputs elbow-up position and nearest solution for θ_1. Hence we get just one solution from these functions.

(3) Jacobian Matrix

The Jacobian Matrix has been found using "Explicit derivation of the Jacobian" in the code 'JacobianPuma.mlx'.

(4) Singularities

The determinant of the Jacobian has been found in the code 'JacobianPuma.mlx'.

when we equate the determinant to 0, we get two cases.

- Case 1:

$$\sin(\theta_5) = 0$$

Hence we have singularities at
$\theta_5 = 0, \pi, -\pi$.

- Case 2:

$$a_3^2 s_2 + d_4^2 s_2 - (a_3^2 + d_4^2)\sin(\theta_2 + 2\theta_3) + a_2 d_4 \cos(\theta_2 - \theta_3) + 2 a_3 d_4 \cos(\theta_2 + 2\theta_3) + a_2 a_3 \sin(\theta_2 - \theta_3) + a_2 d_4 c_{23} - a_2 a_3 s_{23} = 0$$

$$\Rightarrow (a_3^2 + d_4^2) s_2 + 2 a_2 d_4 c_2 c_3 + 2 a_2 a_3 s_2 s_3 - (a_3^2 + d_4^2)\sin(\theta_2 + 2\theta_3) + 2 a_3 d_4 \cos(\theta_2 + 2\theta_3) = 0$$

$$\Rightarrow (a_3^2 + d_4^2) s_2 + 2 a_2 (a_3 s_2 s_3 + d_4 c_2 c_3) - (a_3^2 + d_4^2)\, s(\theta_2 + 2\theta_3) + 2 a_3 d_4\, c(\theta_2 + 2\theta_3) = 0$$

PART 2 : DYNAMICS

(6) Dynamics Equation

The dynamics in parametric form has been solved in the code 'DynamicsPuma.m(x'. we get the solution in the form.

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

we want to represent in matrix form like this.

$$\tau = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} V_1 + f_1 \\ V_2 + f_2 \\ V_3 + f_3 \end{bmatrix}$$

$$+ \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$$

The solution for each term can be found in the same code 'DynamicsPuma.mlx'.

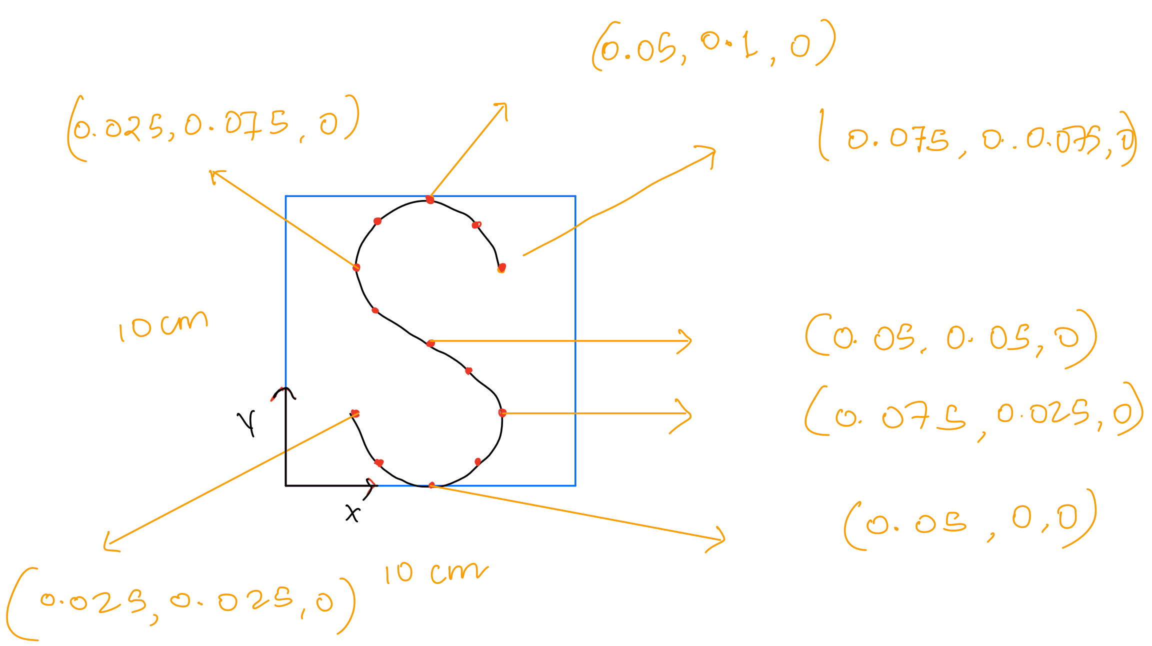
- Another version of analysis has been done by simplifying the dimensions in the previous part. This analysis can be found in the code 'DynamicsPuma2.mlx'.

PART 3 : TRAJECTORY GENERATION

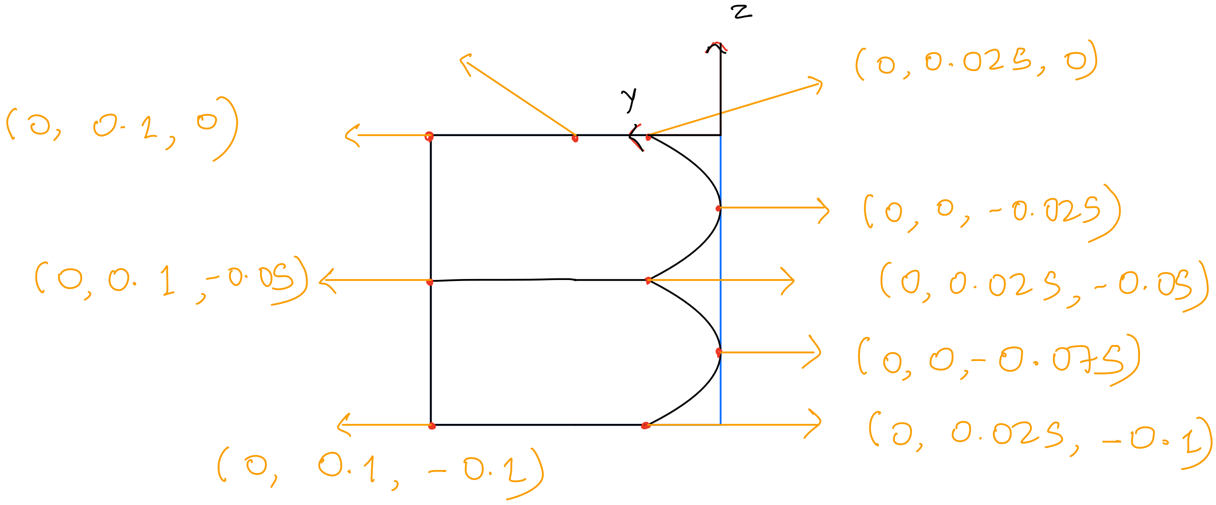
- Trajectory Generation with the help of Joint Space Scheme has been done in the file 'TrajectoryGenerationJointSpace.mlx'. All the solutions are very smooth.
- Trajectory Generation with the help of Task Space Scheme has been done in the file 'TrajectoryGenerationTaskSpace.mlx'. Few jumps in joint angles : like from $-\pi$ to π has been observed though.
- The codes have all the required figures asked in the question. All plots are labelled nicely and are easy to find.
- A sneak peek into the generation of via points for the trajectories have been shown below.

- First we aim to choose certain points for a letter. Then we proceeded with straight line interpolation or spline interpolation based on the segment of a letter. And subsequently generated large number of via-point. The way interpolation has been done, the via points can be easily parametrically increased or decreased.

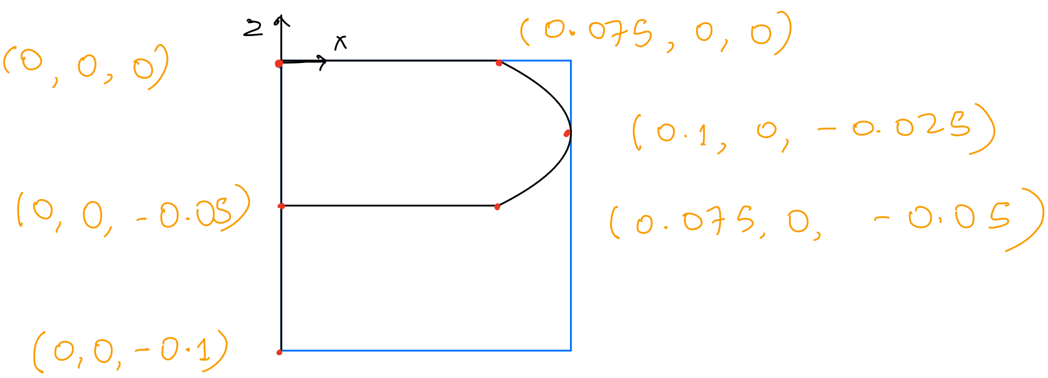
First Letter S

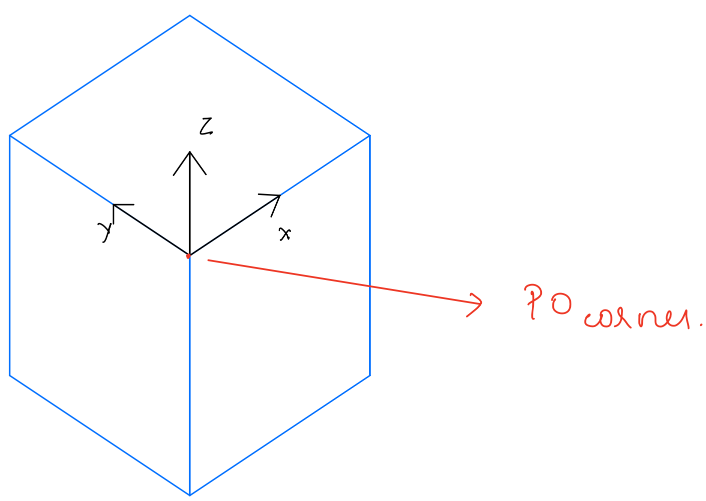


Second letter B



Third letter P





- Next we will show the number of via points used in each phase.

Phases	Via Points
Phase 1	100
Phase 2	200
Phase 3	150
phase 4	350
phase 5	150
phase 6	250
phase 7	150

There are a lot of functions used in this part. To know about them kindly go through the Readme file.