
Mathematical Modelling

m_B = Mass of the body

m_w = Mass of the wheel

I_B = Moment of inertia of the body about the axis of wheels

I_w = Moment of inertia of the wheel

L = Height of the centre of mass of the body from the centre of the wheels

r = Radius of the wheels

T = Torque due to both the wheels

ϕ = The angular velocity of the wheel about its axis of rotation.

x = Position of centre of mass of wheel along x-axis

\dot{x} = Linear velocity of centre of mass of the wheel

θ = Angle between z-axis and body of the robot

$\dot{\theta}$ = Angular velocity of the body

E_k = Kinetic Energy

E_p = Potential Energy with the plane through centre of mass of wheels as reference

Assuming the motion of wheels to be pure rolling

$$x = r\phi$$

$$\dot{\phi} = \frac{\dot{x}}{r}$$

Derivation

$$E_k = (m_w + \frac{m_B}{2} + \frac{I_w}{r^2})(\dot{x})^2 + \frac{(m_B L^2 + I_B)}{2}(\dot{\theta})^2 + m_B L \dot{\theta} \dot{x} \cos(\theta)$$

$$E_p = m_B g L \cos(\theta)$$

$$\mathcal{L} = E_k - E_p = (m_w + \frac{m_B}{2} + \frac{I_w}{r^2})(\dot{x})^2 + \frac{(m_B L^2 + I_B)}{2}(\dot{\theta})^2 + m_B L \dot{\theta} \dot{x} \cos(\theta) - m_B g L \cos(\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = m_B L \sin(\theta)(g - \dot{\theta} \dot{x})$$

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}) = (m_B L^2 + I_B)\ddot{\theta} + m_B L \ddot{x} \cos(\theta) - m_B L \dot{x} \dot{\theta} \sin(\theta)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{x}}) = (m_B + 2m_w + 2\frac{I_w}{r^2})\ddot{x} + m_B L \ddot{\theta} \cos(\theta) - m_B L \dot{\theta}^2 \sin(\theta)$$

By Euler-Lagrange Equation

Q = Generalised Force

q = Generalised Co-ordinates

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = Q$$

Equations of Motion

$$\ddot{\theta} = \frac{m_B g L \sin(\theta) - m_B L \ddot{x} \cos(\theta) - T}{(m_B L^2 + I_B)}$$

$$(m_B + 2m_w + 2\frac{I_w}{r^2})\ddot{x} + m_B L \ddot{\theta} \cos(\theta) - m_B L \dot{\theta}^2 \sin(\theta) = \frac{T}{r}$$

For sake of simplicity let's take $k = \frac{m_B^2 L^2}{m_B L^2 + I_B}$

From Equations of Motion

$$\ddot{x} = \frac{1}{m_B + 2m_w + \frac{2I_w}{r^2} - k \cos(\theta)^2} \cdot (-k \cdot g \cdot \cos(\theta) \cdot \sin(\theta) + m_B L^2 \dot{\theta}^2 \sin(\theta) + (\frac{1}{r} + \frac{k \cdot \cos(\theta)}{m_B L}) \cdot T)$$

$$\bar{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

By solving $\dot{\bar{x}} = f(\bar{x}) = 0$ the equilibrium point we get are : $\begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (*Unstable*), $\begin{bmatrix} x \\ 0 \\ \pi \\ 0 \end{bmatrix}$ (*Stable*)

Here x (first element of \bar{x}) is free to take any value.

$$\text{Let's } \bar{x}_o = \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ (*Unstable Equilibrium*)}$$

Linearising $\dot{\bar{x}} = f(\bar{x})$ about \bar{x}_o

$$\frac{\partial \dot{\bar{x}}}{\partial x}(\text{at } \bar{x}_o) = 0$$

$$\frac{\partial \dot{\bar{x}}}{\partial \dot{x}}(\text{at } \bar{x}_o) = \frac{1}{m_B + 2m_w + \frac{2I_w}{r^2} - k} \cdot 0 = 0$$

$$\frac{\partial \ddot{x}}{\partial \theta}(at \bar{x}_o) = \frac{1}{m_B + 2m_w + \frac{2I_w}{r^2} - k} \cdot (-k \cdot g)$$

$$\frac{\partial \ddot{x}}{\partial \dot{\theta}}(at \bar{x}_o) = \frac{1}{m_B + 2m_w + \frac{2I_w}{r^2} - k} \cdot 0 = 0$$

$$\frac{\partial \ddot{x}}{\partial T}(at \bar{x}_o) = \frac{\frac{1}{r} + \frac{k}{m_B L}}{m_B + 2m_w + \frac{2I_w}{r^2} - k}$$

$$\ddot{\theta} = \frac{k \cdot g \cdot \sin(\theta)}{m_B L} - \frac{k \cdot \ddot{x} \cdot \cos(\theta)}{m_B L} - \frac{k \cdot T}{(m_B L)^2}$$

$$\frac{\partial \ddot{\theta}}{\partial x}(at \bar{x}_o) = 0$$

$$\frac{\partial \ddot{\theta}}{\partial \dot{x}}(at \bar{x}_o) = 0$$

$$\frac{\partial \ddot{\theta}}{\partial \theta}(at \bar{x}_o) = \frac{k \cdot g}{m_B L} \cdot \frac{m_B + 2m_w + \frac{2I_w}{r^2}}{m_B + 2m_w + \frac{2I_w}{r^2} - k}$$

$$\frac{\partial \ddot{\theta}}{\partial \dot{\theta}}(at \bar{x}_o) = 0$$

$$\frac{\partial \ddot{\theta}}{\partial T}(at \bar{x}_o) = \frac{-k}{(m_B L)^2} - \frac{k}{m_B L} \cdot \left(\frac{1}{r} + \frac{k}{m_B L} \right) \cdot \frac{1}{m_B + 2m_w + \frac{2I_w}{r^2} - k}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-k \cdot g}{m_B + 2m_w + \frac{2I_w}{r^2} - k} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k \cdot g \cdot (m_B + 2m_w + \frac{2I_w}{r^2})}{m_B L \cdot (m_B + 2m_w + \frac{2I_w}{r^2} - k)} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{\frac{1}{r} + \frac{k}{m_B L}}{m_B + 2m_w + \frac{2I_w}{r^2} - k} \\ 0 \\ \frac{-k}{(m_B L)^2} - \frac{k}{m_B L} \cdot \left(\frac{1}{r} + \frac{k}{m_B L} \right) \cdot \left(\frac{1}{m_B + 2m_w + \frac{2I_w}{r^2} - k} \right) \end{bmatrix}$$

State-Space Equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$