5. Linear Programming Problems

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5.1 Types of solutions, standard and canonical of LPP, Basic and feasible solutions, slack variables, surplus variables, simplex method.

5.2 Artificial variables, Big-M method

5.3 Duality, dual of LPP and dual simplex method.

Problem 1: Determine basic solution to following problem. Also find basic feasible solution & optimal solⁿ. Investigate the basic feasible solution are degenerate or non-degenerate

objective fⁿ $\longrightarrow$ Max $Z = x_1 - 2x_2 + 4x_3$: subject to

constraints $\longrightarrow$ $\begin{cases} x_1 + 2x_2 + 3x_3 = 7 \\ 3x_1 + 4x_2 + 6x_3 = 15 \end{cases}$

$\longrightarrow$ No. of unknowns, $m = 3$, constraints, $n = 2$

$\therefore$ Basic solⁿ is obtained by $m - n = 1$ variable as zero.

& No. of solutions are, ${}^{m}C_{n} = {}^{3}C_{2} = 3$

No. of solⁿ	Non-basic variable	Basic variable	eqⁿ basic solⁿ	Z
1.	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 7$ $4x_2 + 6x_3 = 15$ $\Rightarrow$ No solution	
2.	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 7$ $3x_1 + 6x_3 = 15$ $\Rightarrow x_1 = 1, x_3 = 2$	9
3.	$x_3 = 0$	x_2, x_1	$x_1 + 2x_2 = 7$ $3x_1 + 4x_2 = 15$ $\Rightarrow x_1 = 1, x_2 = 3$	-5

Here, $x_2 = 0$ & $x_3 = 0$, both are feasible solutions

As Z is max at $x_2 = 0 \Rightarrow x_2 = 0$ is optimal solⁿ

Both solⁿˢ are non-degenerate, as $x_1 \neq 0, x_3 \neq 0$ &

$x_1 \neq 0, x_2 \neq 0$

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HW 2) Max $Z = 2x_1 - 2x_2 + 4x_3 - 5x_4$ subject to
$x_1 + 4x_2 - 2x_3 + 8x_4 \le 2$
$-x_1 + 2x_2 + 3x_3 + 4x_4 \le 1$; $x_1, x_2, x_3, x_4 \ge 0$

Simplex Method :

Problem 1: Max $Z = x_1 + 4x_2$ subject to
$2x_1 + x_2 \le 3$
$3x_1 + 5x_2 \le 9$
$x_1 + 3x_2 \le 5$ $\qquad x_1, x_2 \ge 0$

→ Standard form :

$$\text{Max } Z - x_1 - 4x_2 + \underbrace{0S_1 + 0S_2 + 0S_3}_{\text{slack variables}} = 0 \quad \text{sub to}$$
$$2x_1 + x_2 + S_1 + 0S_2 + 0S_3 = 3$$
$$3x_1 + 5x_2 + 0S_1 + S_2 + 0S_3 = 9$$
$$x_1 + 3x_2 + 0S_1 + 0S_2 + S_3 = 5,$$
$$x_1, x_2, S_1, S_2, S_3 \ge 0$$

Iteration ①

most (-ve)

	Basic variable	x_1	x_2	S_1	S_2	S_3	RHS	Ratio
R_1	Z	-1	-4	0	0	0	0	0
R_2 key column	S_1	2	1	1	0	0	3	$3/1 = 3$
R_3 key row	S_2	3	5	0	1	0	9	$9/5 = 1.8$
R_4	S_3	1	3	0	0	1	5	$5/3 = 1.67$

key element

coeff of

Smallest +ve ratio

Iteration②

coeff of

	x_1	x_2	S_1	S_2	S_3	RHS	Ratio
Z	1/3	0	0	0	4/3	20/3	
S_1	5/3	0	1	0	-1/3	4/3	
S_2	4/3	0	0	1	-5/3	2/3	
$\boxed{x_2}$	1/3	1	0	0	1/3	5/3	

$R_3 - 5R_4$, $R_2 - R_4$, $R_1 + 4R_4$

As there is no -ve term in R_1, iterations stop.

$\therefore Z_{max} = \frac{20}{3}$, $x_2 = \frac{5}{3}$, $x_1 = 0$

Problem 2) Max $Z = 6x_1 - 2x_2 + 3x_3$ subject to

$$2x_1 - x_2 + 2x_3 \le 2;$$
$$x_1 + 4x_3 \le 4 \quad ; x_1, x_2, x_3 \ge 0$$

Sol^n 1] Standard form :

$$Z - 6x_1 + 2x_2 - 3x_3 + 0S_1 + 0S_2 = 0 \quad \text{sub to}$$
$$2x_1 - x_2 + 2x_3 + S_1 + 0S_2 = 2$$
$$x_1 + 0x_2 + 4x_3 + 0S_1 + S_2 = 4$$
$$x_1, x_2, x_3, S_1, S_2 \ge 0$$

2] Simplex table is as follows:

	Basic Variable	coeff of x_1	x_2	x_3	S_1	S_2	R.H.S	Ratio
Iteration 1	Z	-6	2	-3	0	0	0	0
	S_1	2^*	-1	2	1	0	2	1
	S_2	1	0	4	0	1	4	4
Iteration 2	Z	0	-1	3	3	0	6	
	x_1	1	$-1/2$	1	$1/2$	0	1	-2
	S_2	0	$1/2^*$	3	$-1/2$	1	3	6
Iteration 3	Z	0	0	9	2	2	12	
	x_1	1	0	$8/2$	0	1	4	
	x_2	0	1	6	-1	2	6	

$\therefore x_1 = 4, x_2 = 6, x_3 = 0$ and $Z_{max} = 12$

Problem 3 Maximize $Z = 3x_1 + 5x_2$ subject to

$3x_1 + 2x_2 \le 18; \quad x_1 \le 4, \; x_2 \le 6 \quad ; x_1, x_2 \ge 0$

Problem 4 Maximize $Z = 3x_1 + 2x_2 + 5x_3$ subject to

$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_3 \le 460$$
$$x_1 + 4x_2 \le 420$$
$$x_1, x_2, x_3 \ge 0$$

Sol^n: $Z_{max} = 1350, \; x_2 = 100, \; x_3 = 230, \; x_1 = 0$

Problem 3] Minimize $Z = -4x_1 - x_2 - 3x_3 - 5x_4$ subject to

$$4x_1 - 6x_2 - 5x_3 - 4x_4 \geq -20$$
$$-3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20 \; ; \; x_1, x_2, x_3, x_4 \geq 0.$$

solution: Max $Z^* = -Z = 4x_1 + x_2 + 3x_3 + 5x_4$ subject to

$$-4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 20$$
$$-3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20; \; x_1, x_2, x_3, x_4 \geq 0.$$

1] Standard form :

$$Z^* - 4x_1 - x_2 - 3x_3 - 5x_4 + 0S_1 + 0S_2 + 0S_3 = 0$$

subject to $-4x_1 + 6x_2 + 5x_3 + 4x_4 + S_1 + 0S_2 + 0S_3 = 20$

$$-3x_1 - 2x_2 + 4x_3 + x_4 + 0S_1 + S_2 + 0S_3 = 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 + 0S_1 + 0S_2 + S_3 = 20$$

where $x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

2] Simplex table :

	Basic variable	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS	Ratio
Iteration ①	z	-4	-1	-3	-5	0	0	0	0	0
	s_1	-4	6	5	4*	1	0	0	20	5
	s_2	-3	-2	4	1	0	1	0	10	10
	s_3	-8	-3	3	2	0	0	1	20	10
Iteration ②	z	-9	13/2	13/4	0	5/4	0	0	25	
	x_4	-1	3/2	5/4	1	1/4	0	0	5	-5
	s_2	-2	-7/2	11/4	0	-1/4	1	0	5	-5/2
	s_3	-6	-6	1/2	0	-1/2	0	1	10	-5/3

As no ratio is positive, the problem has no solution.

Note: Problem should be always of maximization type & if constraint $\leq$ then add slack variable and constraint $\geq$ then subtract surplus variable and add artificial variable

Big-M method :

Problem 1: Using Penalty method Max $Z = 3x_1 - x_2$

subject to $2x_1 + x_2 \geq 2$;

$x_1 + 3x_2 \leq 3$; $x_2 \leq 4$; $x_1, x_2 \geq 0$

Sol^n: Standard form: Max $Z = 3x_1 - x_2$ subject to

$2x_1 + x_2 - S_1 + 0S_2 + 0S_3 + A_1 = 2$ —— ②

$x_1 + 3x_2 + 0S_1 + S_2 + 0S_3 + 0A_1 = 3$ —— ③

$0x_1 + x_2 + 0S_1 + 0S_2 + S_3 + 0A_1 = 4$ —— ④

$x_1, x_2, x_3, S_1, S_2, S_3, A_1 \geq 0$

max $Z = 3x_1 - x_2 - 0S_1 - 0S_2 - 0S_3 - MA_1 = 0$ —— ①

To eliminate MA_1 from ①, multiply ② by M & add to ①

Max $Z = (3+2M)x_1 + (-1+M)x_2 - MS_1 - 0S_2 - 0S_3 - 0A_1 - 2M$

$\therefore Z - (3+2M)x_1 - (-1+M)x_2 + MS_1 + 0S_2 + 0S_3 + 0A_1 = -2M$

2] Big M table :

Basic Variable	x_1	x_2	S_1	S_2	S_3	A_1	RHS	Ratio
Z	$-3-2M$	$1-M$	M	0	0	0	$-2M$	
A_1	2*	1	-1	0	0	1	2	1
S_2	1	3	0	1	0	0	3	3
S_3	0	1	0	0	1	0	4	—
Z	0	5/2	-3/2	0	0		3	
x_1	1	1/2	-1/2	0	0		1	-2
S_2	0	5/2	1/2*	1	0		2	4
S_3	0	1	0	0	1		4	—
Z	0	10	0	3	0		9	
x_1	1	3	0	-1	0		3	
S_1	0	5	1	2	0		4	
S_3	0	1	0	0	1		4	

$\therefore Z_{max} = 9$, $x_1 = 3$, $x_2 = 0$.

HW 1] Min $Z = 2x_1 + 3x_2$ subject to

$x_1 + x_2 \geq 5$

$x_1 + 2x_2 \geq 6$

$x_1, x_2 \geq 0$

Solution: Max $Z^* = -Z = -2x_1 - 3x_2$ subject to

$x_1 + x_2 \geq 5$

$x_1 + 2x_2 \geq 6$, $x_1, x_2 \geq 0$

Standard form is :

Max $Z^* = -2x_1 - 3x_2 - 0S_1 - 0S_2 - mA_1 - mA_2$ — ①

subject to $x_1 + x_2 - S_1 + 0S_2 + A_1 + 0A_2 = 5$ — ②

$x_1 + 2x_2 + 0S_1 - S_2 + 0A_1 + A_2 = 6$ — ③

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0.$

To eliminate MA_1 & MA_2 from ①, multiply ② & ③ with M and add with ①.

$\therefore Z^* = -2x_1 - 3x_1 - 0S_1 - 0S_2 - MA_1 - MA_2 + Mx_1 + Mx_2 - MS_1 + 0S_2 + MA_1 + 0A_2 - 5M + Mx_1 + 2Mx_2 + 0S_1 - MS_2 + 0A_1 + MA_2 - 6M$

$\therefore Z^* = (-2 + 2M)x_1 + (-3 + 3M)x_2 - MS_1 - MS_2 + 0A_1 + 0A_2 - 11M$

$\therefore Z^* - (-2 + 2M)x_1 - (-3 + 3M)x_2 + MS_1 + MS_2 - 0A_1 - 0A_2 = -11M$

Iteration		Basic Variable	x_1	x_2	S_1	S_2	A_1	A_2	RHS	Ratio
			coeff of							
Iteration ①	R_1	Z^*	$2-2M$	$3-3M$	M	M	0	0	$-11M$	
	R_2	A_1	1	1	-1	0	1	0	5	5
	R_3	A_2	1	2^*	0	-1	0	1	6	3
Iteration ②	R_1	Z^*	$\frac{1}{2} - \frac{M}{2}$	0	M	$\frac{3}{2} - \frac{M}{2}$	0		$-2M-9$	
	R_2	A_1	$\frac{1}{2}^*$	0	-1	$\frac{1}{2}$	1		2	4
	R_3	x_2	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0		3	6
Iteration ③		Z^*	0	0	1	1			-11	
		x_1	1	0	-2	1			4	
		x_2	0	1	1	-1			1	

$\therefore Z^* = -11$

$\boxed{\therefore Z_{min} = 11 \quad \& \quad x_1 = 4, \; x_2 = 1}$

Duality and dual simplex method:

Find the dual of following problems:

Problem 1) Max $Z = 40x_1 + 35x_2$, subject to

$2x_1 + 3x_2 \le 60$,

$4x_1 + 3x_2 \le 96$, $x_1, x_2 \ge 0$

$\rightarrow$ Dual is given by, Min $W = 60y_1 + 96y_2$

subject to $2y_1 + 4y_2 \ge 40$

$3y_1 + 3y_2 \ge 35$, $y_1 y_2 \ge 0$

Problem 2] Min $Z = 10x_1 + 20x_2$ subject to

$3x_1 + 2x_2 \ge 18$

$x_1 + 3x_2 \ge 8$

$2x_1 - x_2 \le 6$, $x_1, x_2 \ge 0$

Primal is Min $Z = 10x_1 + 20x_2$ subject to

$3x_1 + 2x_2 \ge 18$

$x_1 + 3x_2 \ge 8$

$-2x_1 + x_2 \ge -6$, $x_1, x_2 \ge 0$

Dual is given by, Max $W = 18y_1 + 8y_2 - 6y_3$

subject to $3y_1 + y_2 - 2y_3 \le 10$

$2y_1 + 3y_2 + y_3 \le 20$, $y_1, y_2, y_3 \ge 0$.

Problem 3: Form dual of following primal.

Max $Z = x_1 + 2x_2$ subject to

$2x_1 + 4x_2 \le 160$

$x_1 - x_2 = 30$

$x_1 \ge 10$ $x_1, x_2 \ge 0$

Answer: Max $Z = x_1 + 2x_2$ subject to

$2x_1 + 4x_2 \le 160$

$x_1 - x_2 \le 30$, $x_1 - x_2 \ge 30$

$x_1 \ge 10$, $x_1, x_2 \ge 0$

Max $Z = x_1 + 2x_2$ subject to

$2x_1 + 4x_2 \le 160$

$x_1 - x_2 \le 30$

$-x_1 + x_2 \le -30$

$-x_1 \le -10$. $x_1, x_2 \ge 0$

{ whenever maximise problem has = contraint, use $\le$ first and $\ge$ later. }

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Dual is $\text{Min } W = 160y_1 + 30y_2 - 30y_3 - 10y_4$
subject to $2y_1 + y_2 - y_3 - y_4 \geq 1$
$4y_1 + y_2 + y_3 + 0y_4 \geq 2$, $y_1, y_2, y_3, y_4 \geq 0$

Put $y_2 - y_3 = y'$

$\text{Min } W = 160y_1 + 30y' - 10y_4$ subject to
$2y_1 + y' - y_4 \geq 1$
$4y_1 - y' \geq 2$, $y_1, y_4 \geq 0$, y' is unrestrick

Dual simplex method :

Note : The problem should always be of minimization type & all constraints are ($\leq$).

Problem 1: Solve using dual simplex method.

$\text{Max } Z = -3x_1 - 2x_2$ subject to
$x_1 + x_2 \geq 1$, $x_1 + x_2 \leq 7$
$x_1 + 2x_2 \geq 10$, $x_2 \leq 3$, $x_1, x_2 \geq 0$

Solution ~~Primal is~~ $\text{Min } Z^* = -Z = 3x_1 + 2x_2$
subject to $-x_1 - x_2 \leq -1$, $x_1 + x_2 \leq 7$
$-x_1 - 2x_2 \leq -10$, $x_2 \leq 3$, $x_1, x_2 \geq 0$

Standard form $\text{Min } Z^* = 3x_1 + 2x_2 - 0s_1 - 0s_2 - 0s_3 - 0s_4$
$\therefore Z^* - 3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 = 0$ sub to
$-x_1 - x_2 + s_1 + 0s_2 + 0s_3 + 0s_4 = -1$
$x_1 + x_2 + 0s_1 + s_2 + 0s_3 + 0s_4 = 7$
$-x_1 - 2x_2 + 0s_1 + 0s_2 + s_3 + 0s_4 = -10$
$0x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 + s_4 = 3$
$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$.

Iteration ①.

	Basic variable	coeff of x_1	x_2	s_1	s_2	s_3	s_4	RHS	
R_1	Z^*	-3	-2	0	0	0	0	0	
R_2	s_1	-1	-1	1	0	0	0	-1	
R_3	s_2	1	1	0	1	0	0	7	
R_4	s_3	-1	-2	0	0	1	0	-10	most -ve
R_5	s_4	0	1	0	0	0	1	3	
	Ratio	3	1	smallest +ve ratio	—	0	—	0	

Iteration ② : key element is -2

Basic var.		x_1	x_2	s_1	s_2	s_3	s_4	RHS
R_1	z^*	-2	0	0	0	-1	0	10
R_2	s_1	$-1/2$	0	1	0	$-1/2$	0	4
R_3	s_2	$1/2$	0	0	1	$1/2$	0	2
R_4	x_2	$1/2$	1	0	0	$-1/2$	0	5
R_5	s_4	$\boxed{-1/2}$	0	0	0	$1/2$	1	$\boxed{-2}$
Ratio		4	–	–	–	-2	0	-5

Iteration ③ : key element is $-1/2$

Basic Var		x_1	x_2	s_1	s_2	s_3	s_4	RHS
R_1	z^*	0	0	0	0	-3	-4	18
R_2	s_1	0	0	1	0	-1	-1	6
R_3	s_2	0	0	0	1	1	1	0
R_4	x_2	0	1	0	0	0	1	3
R_5	x_1	1	0	0	0	-1	-2	4
Ratio								

As there is no $-$ve value in RHS, iterations stop.

Here, $z^* = 18 \Rightarrow z_{min} = -18$

$x_2 = 3$ & $x_1 = 4$, $x_3 = 0$, $x_4 = 0$