Differentiation and It’s application:

Differentiation:

It is a process of finding a function that outputs the rate of change of one variable with respect to another variable.

Standard formulae of derivatives:

Function $f(x)$	Derivative $f'(x)$
1. $\frac{d}{dx}(k)$, constant	0
2. $\frac{d}{dx}(x)^n$ $(n\epsilon\mathbb{R})$	$n \cdot x^{n-1}$
3. $\frac{d}{dx}(logx)$	$\frac{1}{x}$
4. $\frac{d}{dx}(e^x)$	e^x
5. $\frac{d}{dx}(a^x)$	$a^x \log_e a$
6. $\frac{d}{dx}(\sin x)$	$cosx$
7. $\frac{d}{dx}(cosx)$	$-sinx$
8. $\frac{d}{dx}(tanx)$	sec^2x
9. $\frac{d}{dx}(cosecx)$	$-cosecx\ cotx$

Function $f(x)$	**Derivative $f'(x)$**
10. $\frac{d}{dx}(secx)$	$secx\ tanx$
11. $\frac{d}{dx}(cotx)$	$-cosec^2x$
12. $\frac{d}{dx}(\frac{1}{x})$	$-\frac{1}{x^2}$
13. $\frac{d}{dx}(\sqrt{x})$	$\frac{1}{2\sqrt{x}}$

Note: Formulae 12 and 13 are the particular cases of x^n.

14. $\frac{d}{dx}(\sin^{-1} x)$	$\frac{1}{\sqrt{1-x^2}}, \lvert x\rvert < 1$
15. $\frac{d}{dx}(\cos^{-1} x)$	$\frac{-1}{\sqrt{1-x^2}}, \lvert x\rvert < 1$
16. $\frac{d}{dx}(\tan^{-1} x)$	$\frac{1}{1+x^2}, x \in \mathbb{R}$
17. $\frac{d}{dx}(\cot^{-1} x)$	$\frac{-1}{1+x^2}, x \in \mathbb{R}$
18. $\frac{d}{dx}(\sec^{-1} x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}, x \in \mathbb{R}-(-1,1)$
19. $\frac{d}{dx}(cosec^{-1} x)$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}, x \in \mathbb{R}-(-1,1)$

Working rules of differentiation:

1) Derivative of Sum and Difference:

If u and v are functions of x and $y = u \pm v$, then

$$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

e.g. $\mathbf{y} = \boldsymbol{e^x + x^e + e^e}$

Sol. $\frac{dy}{dx} = \frac{d}{dx}e^x + \frac{d}{dx}x^e + \frac{d}{dx}e^e$

$$= e^x + ex^{e-1} + 0 = e^x + ex^{e-1}$$

2) Derivative of Product:

If u and v are functions of x and $y = uv$, then$\frac{dy}{dx}$

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

e.g. $y = \log x \cdot tanx$

Sol. $y = logx \cdot tanx$

$$\frac{dy}{dx} = logx\ \frac{d}{dx}(tanx) + tanx\ \frac{d}{dx} log(x) \cdots \text{(using product rule)}$$

$$= \log x \cdot sec^2 x + tanx\ \cdot \frac{1}{x}$$

Corollary 1: If $y = uvw$, then

$$\frac{dy}{dx} = uv \cdot \frac{dw}{dx} + vw \cdot \frac{du}{dx} + wu \cdot \frac{dv}{dx}$$

Corollary 2: If $y = ku$, then

$$\frac{dy}{dx} = k \cdot \frac{du}{dx}$$

3) Derivative of a quotient:

If u and v are functions of x and $y = \frac{u}{v}$, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\ , (v \neq 0)$$

e.g. $y = \frac{logx}{x}$

Sol. $y = \frac{logx}{x}$

take $u = logx$ and $v = \frac{1}{x}$

$$\therefore \ \frac{du}{dx} = \frac{1}{x} \text{ and } \frac{dv}{dx} = 1$$

$$\therefore \ \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\therefore \ \frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(logx) - logx \cdot \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x \cdot \frac{1}{x} - logx \cdot 1}{x^2}$$

$$= \frac{1 - logx}{x^2}$$

Differentiation of a composite function:

If y is a function of u and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This rule is known as **Chain rule** and it is useful for finding derivatives of composite function.

The chain rule can be generalized as follows.

i. If y is a function of u, u is a function of v, v is a function of w and w is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

ii. In general

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dt} \cdots \frac{dz}{dx}$$

e.g. $y = \log \cdot (logx)$

Sol. $y = \log(logx) = logu$

where $u = logx \ \therefore \ \frac{du}{dx} = \frac{1}{x}$

Now, $y = logu \ \therefore \ \frac{dy}{dx} = \frac{1}{u} = \frac{1}{logx}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{logx} \cdot \frac{1}{x}$$

$$\therefore \ \frac{dy}{dx} = \frac{1}{x \, logx}$$

Differentiation of parametric function:

When the variables x and y are expressed in terms of some other variable (called parameter) t or θ, the functions are called parametric functions. e.g. $x = rcos\theta$ and $y = rsin\theta$.

If $x = f(t)$ and $y = g(t)$, where t is a parameter, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}, \text{ where } f'(t) \neq 0.0$$

$e.g. x = at^2, y = 2at,$

Sol. $x = at^2 \ \therefore \ \frac{dx}{dt} = 2at,$

$$y = 2at \ \therefore \ \frac{dy}{dt} = 2a$$

Now $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t} \ (t \neq 0)$

Differentiation of Implicit functions:

Let $\Phi(x, y) = 0$ be the equation defining y implicitly as a function of x i.e. in the equation $\Phi(x, y) = 0$, x and y are related in such a way that y cannot conveniently be expressed directly as a function of x like $y = f(x)$.

e.g. $x^2 + y^2 = xy$

Sol. $x^2 + y^2 = xy$

Differentiating on both sides w.r.t. x,

we get $2x + 2y\frac{dy}{dx} = x\frac{dy}{dx} + y \cdot 1$

$\therefore 2x - y = (x - 2y)\frac{dy}{dx} \ \therefore \ \frac{dy}{dx} = \frac{2x-y}{x-2y}$

Logarithmic differentiation :

When a function $f(x)$ is raised to a power, which itself is a function of x, say $\Phi(x)$ i.e. it is of the form $[f(x)]^{\Phi(x)}$, in that case neither the formula for x^n not that for a^x is applicable. In such a types, we have to take logarithms first then differentiate. This method of differentiation is called logarithmic differentiation.

e.g. $y = x^x$

Sol. $y = x^x$

Now take a log with base e on both sides.

$\therefore \log y = x \log x$

Differentiating on both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + (logx) \cdot 1 = (1 + logx)$$

$$\therefore \frac{dy}{dx} = y\,(1 + logx) = x^x\,(1 + logx)$$

Successive Differentiation:

We have denoted the first derivative of $y = f(x)$ by $f'(x)$ or $\frac{dy}{dx}$. This derivative is called the first derivative of the function f and it is a function of x. This first derivative can again be differentiated with respect to x and the result is called the second derivatives of the function f. It is denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$.

e.g. $y = 4e^{3x} + 2^x$

Sol.$y = 4e^{3x} + 2^x$

$$\therefore \frac{dy}{dx} = 4\,(3\,e^{3x}) + 2^x \log 2 = 12e^{3x} + 2^x\, log2$$

$$\frac{d^2y}{dx^2} = 12\,(3e^{3x}) + log2\,(2^x\, log2)$$

$$= 36\, e^{3x} + 2^x\,(log2)^2$$

Velocity and Acceleration:

Velocity : Velocity is a vector measurement of the rate of motion of an object and the direction in which it is moving.

$$V = \frac{ds}{dt} = f'(t)$$

Acceleration: The rate of change of velocity is called acceleration.

The acceleration a of a particle at time t is given by $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$

e.g. The equation of motion of a particle is $s = t^3 - 3t^2 + 4t + 3$. Find its velocity and Acceleration **.**

Sol. Here $s = t^3 - 3t^2 + 4t + 3$.

Velocity $V = \frac{ds}{dt} = 3t^2 - 6t + 4$

and Acceleration $a = \frac{dV}{dt} = 6t - 6$