1. $Y_1 = \max(X_1, X_2, \ldots, X_n)$

$P(Y_1 \leq y) = P(X_1 \leq y) \times \ldots P(X_n \leq y)$

since $X_1, X_2, \ldots X_n$ are independent

$\therefore \quad F_{Y_1}(y) = \prod F_{X_i}(y) = [F_X(y)]^n$

since they are identical

$\therefore \quad F_{Y_1}(y) = [F_X(y)]^n$

$f_{Y_1}(y) = \frac{\partial}{\partial y}[F_X(y)]^n = n[F_X(y)]^{n-1} f_X(y)$

$\therefore$ cdf of $Y_1 = [F_X(y)]^n$

pdf of $Y_1 = n f_X(y)[F_X(y)]^{n-1}$

$Y_2 = \min(X_1, X_2, \ldots, X_n)$

$\cancel{P(Y_2 \leq y)} = P(Y_2 \geq y) = P(X_1 \geq y) P(X_2 \geq y) \ldots P(X_n \geq y)$

$P(Y_2 \geq y) = [1 - F_X(y)]^n$

$\therefore \quad 1 - F_{Y_2}(y) = [1 - F_X(y)]^n$

$F_{Y_2}(y) = 1 - [1 - F_X(y)]^n$

$f_{Y_2}(y) = n[1 - F_X(y)]^{n-1} f_X(y)$

$\therefore$ cdf of $Y_2 = 1 - [1 - F_X(y)]^n$

pdf of $Y_2 = n f_X(y)[1 - F_X(y)]^{n-1}$

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3. Consider $Y = X - \mu$

$E[Y] = 0$ & $Var(Y) = \sigma^2$

$\therefore P(Y \geq \tau)$ for $\tau > 0$

$P(Y \geq \tau) \leq P((Y+b)^2 \geq (\tau+b)^2)$ for $b \geq 0$

as it will be cover both types of values

$\therefore$ Applying Markov's inequality

$$P((Y+b)^2 \geq (\tau+b)^2) \leq \frac{E[(Y+b)^2]}{(\tau+b)^2} = \frac{\sigma^2+b^2}{(\tau+b)^2}$$

$$\therefore P(X-\mu \geq \tau) \leq \frac{\sigma^2+b^2}{(\tau+b)^2} \text{ for } b \geq 0$$

Now differentiate $\frac{\sigma^2+b^2}{(\tau+b)^2}$ and equate to 0

$$\cancel{2}b(\tau+b)^{\cancel{2}} = \cancel{2}\cancel{(\tau+b)}(\sigma^2+b^2)$$

$b\tau + b^2 = \sigma^2 + b^2 \Rightarrow b = \sigma^2/\tau$ so for this b exp. is min^m

$$\therefore P(X-\mu \geq \tau) \leq \frac{\sigma^2 + \sigma^4/\tau^2}{\frac{(\tau^2+\sigma^2)^2}{\tau^2}} = \cancel{\frac{\sigma^2(1+(\sigma^2/\tau)^2)}{(1+(\frac{\sigma^2}{\tau})^2)}}$$

$$\frac{\sigma^2+\sigma^4/\tau^2}{(\tau^2+\sigma^2)^2/\tau^2} = \frac{\sigma^2\tau^2+\sigma^4}{(\tau^2+\sigma^2)^2} = \frac{\sigma^2}{(\tau^2+\sigma^2)}$$

$$\therefore P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$\therefore$ for any random variable and $\tau > 0$

$$P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2 + \cancel{b^2}\tau^2}$$

Now consider $\tau < 0$ and $b = -\tau$

let say $Y = -X$ with mean $-\mu$ and variance σ

$$\therefore \; P(Y+\mu \geq b) \leq \frac{\sigma^2}{\sigma^2+\tau^2} \quad \text{as } b^2 = \tau^2$$

$$P(-X+\mu \geq b) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$\therefore \; P(X \leq \mu - b) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$P(X \leq \mu+\tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$-P(X \leq \mu+\tau) \geq \frac{-\sigma^2}{\sigma^2+\tau^2}$$

$$1-P(X \leq \mu+\tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$\therefore \; P(X \geq \mu+\tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$$

Thus for $\tau < 0$, $P(X \geq \mu+\tau) \geq 1 - \dfrac{\sigma^2}{\sigma^2+\tau^2}$

4. ~~For $t>0$~~

For $t>0$

$$\phi_X(t) = E[e^{Xt}]$$
$$= \sum_{i=1}^{n} e^{x_i t}(P(X=x_i)) \text{ for discrete}$$
$$= \int_{-\infty}^{\infty} e^{xt} f_X(x)\, dx \text{ for continuous}$$

First prove for continuous,

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{xt} f_X(x)\, dx$$

if $t>0$, $e^{x_1 t} > e^{x_2 t}$ if $x_1 > x_2$

$$\phi_X(t) \geq \int_{x_0}^{\infty} e^{xt} f_X(x)\, dx \geq \int_{x_0}^{\infty} e^{x_0 t} f_X(x)\, dx \; [\text{if } t>0]$$

$$\therefore \phi_X(t) \geq e^{x_0 t} \int_{x_0}^{\infty} f_X(x)\, dx = e^{x_0 t}(P(X \geq x_0))$$

$$\therefore \cancel{P(X \leq x_0)}\; P(X \geq x_0) \leq e^{-x_0 t} \phi_X(t)$$

Now for discrete,

$$\phi_X(t) = \sum_{i=1}^{n} e^{x_i t} P(X=x_i)$$

if $t>0$

$$\phi_X(t) \geq \sum_i e^{x_0 t} P(X=x_i) \text{ here } x_i \geq x_0$$

$$\phi_X(t) \geq e^{x_0 t} \sum_i P(X=x_i) \text{ where } \cancel{x_i} \text{ is all } x_i \geq x_0$$

$$\therefore \phi_X(t) \geq e^{x_0 t} P(X \geq x_0)$$
$$\therefore P(X \geq x_0) \leq e^{-x_0 t} \phi_X(t)$$

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For $t<0$

For continuous,

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{+tx} f_X(x)\,dx \geq \int_{-\infty}^{x_0} e^{+tx} f_X(x)\,dx$$

if $t<0$, $e^{+tx_1} > e^{+tx_2}$ if $x_1 < x_2$

$$\therefore \phi_X(t) \geq \int_{-\infty}^{x_0} e^{tx} f_X(x)\,dx \geq \int_{-\infty}^{x_0} e^{x_0 t} f_X(x)\,dx$$

$$\therefore \phi_X(t) \geq e^{tx_0} P(X \leq x_0)$$

$$\therefore P(X \leq x_0) \leq e^{-tx_0} \phi_X(t)$$

For discrete,

$$\phi_X(t) = \sum_{i=1}^{n} e^{x_i t} P(X=x_i)$$

$\phi_X(t) \geq \sum_{j} e^{x_0 t} P(X=x_j)$ where $x_j \leq x_0$

because if $t<0$, $e^{x_0 t} \leq e^{x_j t}$

$$\phi_X(t) \geq e^{x_0 t} P(X \leq x_0)$$

$$\therefore P(X \leq x_0) \leq e^{-tx_0} \phi_X(t)$$

Now for the second part

$\phi_{x_i}(t)$ ~~$\phi(x_i)$~~ $= 1-p_i+p_ie^t = 1+p_i(e^t-1) \leq e^{p_i(e^t-1)}$

Therefore

$\phi_x(t) = \prod_{i=1}^{n} \phi_{x_i}(t) \leq \prod e^{p_i(e^t-1)} = e^{\mu(e^t-1)}$ as $\sum p_i = \mu$

$\therefore$ Using the first inequality for $t>0$, as for $t=0$

$p(x>(1+\delta)\mu) \leq 1$ which is obviousely true,

$p(x>(1+\delta)\mu) \leq e^{-t(1+\delta)\mu} \phi_x(t) \leq e^{-t(1+\delta)\mu} e^{\mu(e^t-1)}$

$\therefore p(x>(1+\delta)\mu) \leq \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}}$

Now we want to minimize $e^{\mu(e^t-1)-(1+\delta)\mu t}$

with respect to t.

So we have to minimize $\mu(e^t-1)-(1+\delta)\mu t$

Taking derivative and equation to 0

$\mu e^t = \mu(1+\delta)$

$\Rightarrow t = \ln(1+\delta)$

It is ~~mini~~ for minima as double derivative $= \mu e^t > 0$

7. Using the same definition as used in slide

$$X = \sum_{i=1}^{k} X_i$$

$\therefore$ ~~$\phi_X(t) = \sum p_i e^{X_i}$~~

$$\phi_X(t_1, t_2, \ldots t_n) = \left(\sum p_i e^{t_i}\right)^n$$

From slides

As $E[X_i X_j] = \frac{\partial}{\partial t_j}\frac{\partial}{\partial t_i}\phi_X(t)\Big|_{t=(0,0,..,0)}$ where $t = (t_1, t_2, \ldots t_k)$

$$= \frac{\partial}{\partial t_j} e^{t_i} n p_i \left(\sum_{i=1}^{k} p_i e^{t_i}\right)^{n-1}\Big|_{t=(0,0,\ldots)}$$

$$= e^{t_i} n p_i \frac{\partial}{\partial t_j}\left(\sum_{i=1}^{k} p_i e^{t_i}\right)^{n-1}\Big|_{t=(0,0,\ldots)}$$

$$= e^{t_i} n p_i (n-1) p_j \left(\sum p_i e^{t_i}\right)^{n-2} \times e^{t_j}\Big|_{t=(0,0,\ldots)}$$

$$= n(n-1) p_i p_j \quad \left(\sum p_i = 1\right)$$

and $E[X_i] = \frac{\partial}{\partial t_i}\phi_X(t)\Big|_{t=(0,0,..0)}$ where $t = (t_1, t_2 \ldots, t_k)$

$$= \frac{\partial}{\partial t_i}\left(\sum p_i e^{t_i}\right)^n\Big|_{t=(0,0,0\ldots)}$$

$$= e^{t_i} p_i n \left(\sum p_i e^{t_i}\right)^{n-1}\Big|_{t=(0,0)}$$

$$= n p_i \quad \text{because } \sum p_i = 1$$

$$E[x_i^2] = \frac{\partial^2}{\partial t_i^2} \phi_x(t) \Big|_{t=(0,0\ldots)} \quad \text{where } t=(t_1, t_2, \ldots t_k)$$

$$= \frac{\partial^2}{\partial t_i^2} (\Sigma p_i e^{t_i})^n$$

$$= \frac{\partial}{\partial t_i} p_i n (\Sigma p_i e^{t_i})^{n-1} \times e^{t_i}$$

$$= p_i n e^{t_i} (\Sigma p_i e^{t_i})^{n-1} + p_i n e^{t_i} (n-1) (\Sigma p_i e^{t_i})^{n-2} \frac{\partial p_i}{\partial t_i}$$

At $\forall t_j = 0$

this becomes

$$p_i n + p_i n (n-1) p_i$$

$$= n p_i [1 + (n-1) p_i]$$

$\therefore$ ~~covariance~~ .

$$\text{cov}(x_i, x_i) = E[x_i^2] - (E[x_i])^2$$

$$= n p_i [1 + (n-1) p_i] - (n p_i)^2$$

$$= n p_i + n p_i^2 (n-1) - n^2 p_i^2$$

$$= n p_i - n p_i^2$$

$$= n p_i (1 - p_i)$$

$$\text{cov}(x_i, x_j) = E[x_i x_j] - E[x_i] E[x_j]$$

$$= n(n-1) p_i p_j - n p_i n p_j$$

$$= -n p_i p_j$$

Therefore matrix is

$$\begin{bmatrix} n p_1 (1-p_1) & -n p_1 p_2 & \cdots & -n p_1 p_k \\ -n p_2 p_1 & n p_2 (1-p_2) & & \vdots \\ -n p_3 p_1 & -n p_3 p_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ & & & n p_k (1-p_k) \end{bmatrix}$$