Q.2. $X \sim U(0,1)$ $\quad P(X=x) = p(x) = 1 \quad (0 \leq x \leq 1)$

So, the PDF of Y :

$$P(Y=y) = q(y) = p(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= p(e^{-\lambda y}) \left| \frac{d}{dy} e^{-\lambda y} \right| \qquad \left(\begin{array}{c} y = -\frac{1}{\lambda} \log x \\ x = e^{-\lambda y} \\ y \in [0, \infty) \end{array} \right)$$

$$= \lambda e^{-\lambda y} \quad \text{(for } y \geq 0\text{, 0 otherwise)}$$

The ~~poster~~ prior on λ is :

$$p(\lambda) = \frac{1^{5.5} \lambda^{4.5} e^{-\lambda}}{\Gamma(5.5)} = \frac{\lambda^{4.5} e^{-\lambda}}{\Gamma(5.5)}$$

For ML estimate

$$p = \prod_{i=1}^{N} \lambda e^{-\lambda y_i} = \lambda^{N} e^{-\lambda(\Sigma y_i)}$$

$$\log p = N \log \lambda - \lambda \sum_i y_i$$

$$\frac{1}{p} \frac{dp}{d\lambda} = 0 = \frac{N}{\lambda} - \Sigma y_i \Rightarrow \left(\hat{\lambda}^{ML} = \frac{N}{\Sigma_i y_i} \right)$$

For bayesian estimate, we need to find the mean for

$$P(\theta \mid \{y_i\}_{i=1}^N) = \frac{P(\{y_i\}_{i=1}^N \mid \theta)\, P(\theta)}{P(\{y_i\}_{i=1}^N)}$$

- $\hat{\lambda}^{\text{Posterior mean}} = \int_0^\infty \theta \frac{P(\{y_i\} \mid \theta)\, P(\theta)}{P(\{y_i\})} d\theta$

$$= \int_0^\infty \frac{\theta\, \theta^N e^{-\theta(\Sigma y_i)}\, \theta^{4.5} e^{-\theta}}{\Gamma(5.5)\, P(\{y_i\})} d\theta$$

$$= \frac{\int_0^\infty \frac{\theta^{N+5.5} e^{-\theta(\Sigma y_i + 1)}}{\Gamma(5.5)} d\theta}{\int_0^\infty \frac{\theta^{N+4.5} e^{-\theta(\Sigma y_i + 1)}}{\Gamma(5.5)} d\theta}$$

$$= \frac{\int_0^\infty \theta^{N+5.5} e^{-\theta(\Sigma y_i + 1)} d\theta}{\int_0^\infty \theta^{N+4.5} e^{-\theta(\Sigma y_i + 1)} d\theta}$$

$$= \frac{\left[\theta^{N+5.5} e^{-\theta(\Sigma y_i + 1)} \big/ -(\Sigma y_i + 1)\right]_0^\infty + \frac{N+5.5}{1+\Sigma y_i} \int_0^\infty \theta^{N+4.5} e^{-\theta(\Sigma y_i + 1)} d\theta}{\int_0^\infty \theta^{N+4.5} e^{-\theta(\Sigma y_i + 1)} d\theta}$$

$$= \boxed{\frac{N+5.5}{\Sigma y_i + 1}}$$

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• (i) As expected, the error for both the max. likelihood and bayesian estimate tends to 0 and the variance of λ decreases too.

(ii) I will prefer the bayesian estimate as it gives much lower error than the ML estimate when N is small.

When N is large, $\frac{N+5.5}{\Sigma y_i + 1} \approx N/\Sigma y_i$

So, it doesn't matter which one one chooses.