Q.1. (a) The random variable $X = (X_1, X_2)$ has PDF :

$$P(X) = \begin{cases} C & (-1 < X_1 < 1) \text{ and } (-1 < X_2 < 1) \\ 0 & \text{otherwise} \end{cases}$$

as X_1 and X_2 are uniformly distributed and they are independent $(C_1 C_2 = C)$

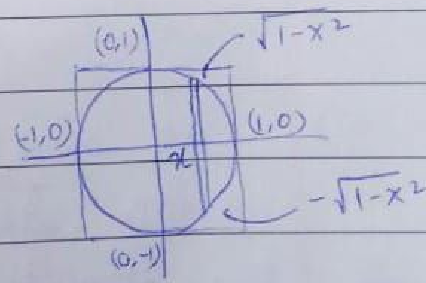
$$1 = \int_{-1}^{1}\int_{-1}^{1} C\,dx\,dy \Rightarrow 1 = 4C \Rightarrow \boxed{C = 1/4}$$

Let S be the region of interest (circle of radius 1). The probability p is

$$P = \iint_S C\,dx\,dy \quad \left(\text{as } S \text{ is inside } -1 < X_1 < 1 \text{ and } -1 < X_2 < 1\right)$$

$$P = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} C\,dx\,dy$$

$$P = \int_{-1}^{1} 2C\sqrt{1-x^2}\,dx$$



[Let $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$]

$$P = \int_{-\pi/2}^{\pi/2} 2C\cos^2\theta\,d\theta$$

$$P = \int_{-\pi/2}^{\pi/2} C(1+\cos 2\theta)\,d\theta = C\pi + \left(\frac{\sin(\pi) - \sin(-\pi)}{2}\right)^{0}$$

So, $\boxed{P = \pi/4}$

(b) We know that the probability that $X = (X_1, X_2)$ lies within circle of radius is p:

$$p = \pi/4 \Rightarrow \boxed{\pi = 4p}$$

Let Z denote a bernoulli random variable with success probability p. Then according to the weak law of large numbers for N trials

$$\frac{\sum_N Z_i}{N} \to p = \pi/4 \quad \text{as } N \to \infty$$

$$\Rightarrow \left(\frac{\text{\# of successes}}{N} \to \frac{\pi}{4} \right) \quad \text{as } (N \to \infty)$$

So, for large enough N

$$\left[\pi \approx 4 \left(\frac{\text{\# of successes}}{N} \right) \right]$$

(c) We may split the 10^9 samples into 100 batches of 10^7 samples and then calculate the collective mean.

Result : $N = 10 \to 3.60$
$N = 10^2 \to 3.08$
$N = 10^3 \to 3.18$
$N = 10^4 \to 3.1448$
$N = 10^5 \to 3.14412$
$N = 10^6 \to 3.14198$
$N = 10^7 \to 3.1422676$
$N = 10^8 \to 3.14159044$
$N = 10^9 \to 3.1415379047$

(d) An easy way of doing this is as follows:

We know that as each z_i $i \in [1, M]$ is a bernoulli random variable, $\sum_{M} z_i$ will be a binomal random variable

We can iterate through all values of M starting from 1 $\Rightarrow$

1) Calculate all integers in the range
$$C = \left[M\left(\frac{\pi - 0.01}{4}\right), M\left(\frac{\pi + 0.01}{4}\right) \right]$$
2) Sum their probabilities $= \sum_{x \in C} \binom{M}{x} \left(\frac{\pi}{4}\right)^{x} \left(1 - \frac{\pi}{4}\right)^{M-x}$

3) Check if sum of probabilities $\geq 95\%$
If it is the case report the value of M.

Practically it makes more sense to multiply M with 10 on each step. Doing this we get the value of $M \sim 10^5$ (of the order of 10^5)