③ $P(\theta) \propto \left(\frac{\theta_m}{\theta}\right)^{\alpha} \quad (\theta > \theta_m)$

$\Rightarrow P(\theta) = K\left(\frac{\theta_m}{\theta}\right)^{\alpha}$ (where K is some constant),

$(\theta > \theta_m)$

$$\Rightarrow P(\theta) = \begin{cases} K\left(\frac{\theta_m}{\theta}\right)^{\alpha} & \theta > \theta_m \\ 0 & \text{otherwise.} \end{cases}$$

$$P_X(x) = \begin{cases} \frac{1}{\theta} & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

a) θ_{ML} :-

P_X needs to be maximised.

also $\theta \neq 0$ otherwise the sample set will be a singleton set.

To minimise θ and also includes all the data points.

$\theta_{mlm} = \max_i x_i = \theta^{ML}$

θ^{MAP} :-

$$p(x_i|\theta) = \begin{cases} \left(\frac{1}{\theta}\right) & \text{if } x_i \in [0,\theta] \\ 0 & \text{otherwise} \end{cases}$$

n = no. of data points.

let's consider $\forall i$ $x_i \in [0,\theta]$
$\theta_0 = \max(x_i, \theta_m)$.

$$p(\{x_i\}_{i=1}^{n} | \theta) = \left(\frac{1}{\theta}\right)^n.$$

$$p(\{x_i\}_{i=1}^{n}) = \int_{\theta_0}^{\infty} \left(\frac{1}{\theta}\right)^n \times k\left(\frac{\theta_m}{\theta}\right)^{\alpha} d\theta.$$

$$= k\,\theta_m^{\alpha} \int_{\theta_0}^{\infty} \frac{1}{\theta^{n+\alpha}}.$$

$$= \frac{k\,\theta_m^{\alpha}}{(-n-\alpha+1)\,\theta^{n+\alpha-1}} \Bigg|_{\theta_0}^{\infty}$$

$$= \frac{-K \cdot \theta_m^{\alpha}}{(1-n-\alpha)\,\theta_0^{n+\alpha-1}} = \frac{K\,\theta_m^{\alpha}}{(n+\alpha-1)\,\theta_0^{n+\alpha-1}}$$

$$\Rightarrow P(\{x_i\}_{i=1}^{n}) = \frac{K \cdot \theta_m^{\alpha}}{(n+\alpha-1)\,\theta_0^{n+\alpha-1}}$$

$$P(\theta|\{x_i\}) = \frac{P(\{x_i\}_{i=1}^{n}|\theta) \cdot P(\theta)}{P(x)} \quad (\theta > \theta_m)$$

$$= \frac{\left(\frac{1}{\theta}\right)^{n} K \left(\frac{\theta_m}{\theta}\right)^{\alpha} (n+\alpha-1)\,\theta_0^{n+\alpha-1}}{K \cdot \theta_m^{\alpha}}$$

$$P(\theta \mid \{x_i\}_{i=1}^{n}) = \frac{(n+\alpha-1)\left(\theta_m^{\alpha+n-1}\right)}{\theta^{n} \cdot \theta^{\alpha}}$$

$$= \frac{(n+\alpha-1)\left(\theta_m^{\alpha+n-1}\right)}{\theta^{n+\alpha}}$$

$\therefore$ $P(\theta \mid \{x_i\}_{i=1}^{n})$ is max at lowest possible value of θ.

$$\theta \geq \max_i \{x_i\}$$

$$\theta \geq \theta_m.$$

$$\theta = \max\left(\max_i \{x_i\}, \theta_m\right).$$

(b) $\theta^{map} = \max(\max_i \{x_i\}, \theta_m)$.

if θ_m is ~~reasonably~~ small than the true value of θ then as sample size ~~the~~ tends to infinity, we will find an i, such ~~that~~ $x_i > \theta_m$.

$\hat{\theta}^{map} = \max(\max_i \{x_i\}, \theta_m)$

$= \max_i \{x_i\}$.

So, if $\theta_m < \theta_{true}$ then as sample size $\to \infty$

$\theta^{map} = \theta^m$.

but if $\theta_m > \theta_{true}$, the θ^{map} will always be equal to θ^m and will never tend to true value of θ. whereas θ^{ml} will tend to true value of θ.

So this makes the MAP estimator not desirable as the estimator is dependent on θ_m.

c). $\hat{\theta}_{\text{posteriormean}} = E[\theta]$

$$= \int_{-\infty}^{\infty} \theta \, p(\theta ; \{x_i\}_{i=1}^{n}) \, d\theta$$

$$= \int_{\theta_0}^{\infty} \theta \cdot \frac{(n+\alpha-1)\,(\theta_0^{\,n+\alpha-1})}{\theta^{n+\alpha}} \, d\theta$$

$$= \int_{\theta_0}^{\infty} \frac{(n+\alpha-1)\,(\theta_0^{\,n+\alpha-1})}{\theta^{n+\alpha-1}} \, d\theta$$

$$= (n+\alpha-1)\,\theta_0^{(n+\alpha-1)} \int_{\theta_0}^{\infty} \theta^{-n-\alpha+1} \, d\theta$$

$$= (n+\alpha-1)\,\theta_0^{n+\alpha-1} \left. \frac{\theta^{-n-\alpha+2}}{-n-\alpha+2} \right|_{\theta_0}^{\infty}$$

$$= (n+\alpha-1)\,\theta_0^{n+\alpha-1} \, \frac{\theta_0^{-n-\alpha+2}}{n+\alpha-2}$$

$$= \theta_0 \left(\frac{n+\alpha-1}{n+\alpha-2} \right)$$

$$\hat{\theta}^{\text{Posterior Mean}} = \max\left(\theta_m, \{x_i\}_{i=1}^{n}\right) \frac{n+\alpha-1}{n+\alpha-2}$$

d). When $n \to \infty$,

$$\hat{\theta}^{\text{Posterior Mean}} \longrightarrow \max\left(\theta_m, \{x_i\}_{i=1}^{n}\right)$$

$\therefore$ As $n \to \infty$ $\hat{\theta}^{\text{Posterior Mean}} \longrightarrow \hat{\theta}^{MAP}$.

and $n \to \infty$ $\hat{\theta}^{MAP} \longrightarrow \hat{\theta}^{ML}$. (if $\theta_m < \theta_{true}$)

$\therefore$ if $\theta_m < \theta_{true}$
then $\hat{\theta}^{\text{Posterior Mean}} \longrightarrow \hat{\theta}^{ML}$ (if $\theta_m < \theta_{true}$)

So again, if $\theta_m < \theta_{true}$ then $\hat{\theta}^{\text{posterior Mean}}$ tends to $\hat{\theta}^{ML}$, but if $\theta_m > \theta_{true}$, the $\hat{\theta}^{\text{Posterior Mean}}$ will stuck at θ_m. This is not desirable feature of this estimator.