Assignment 1: CS 215

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Contents

Question 1

1. $Y_1 = \max(X_1, X_2, \dots, X_n)$

$P(Y_1 \le y) = P(X_1 \le y) \times \dots P(X_n \le y)$

since $X_1, X_2, \dots X_n$ are independent

$\therefore F_{Y_1}(y) = \prod F_{X_i}(y) = [F_X(y)]^n$

since they are identical

$\therefore F_{Y_1}(y) = [F_X(y)]^n$

$f_{Y_1}(y) = \frac{\partial}{\partial y}[F_X(y)]^n = n[F_X(y)]^{n-1} f_X(y)$

$\therefore$ cdf of $Y_1 = [F_X(y)]^n$

pdf of $Y_1 = n f_X(y) [F_X(y)]^{n-1}$

$Y_2 = \min(X_1, X_2, \dots, X_n)$

$\cancel{P(Y_2 \le y)} = P(Y_2 \ge y) = P(X_1 \ge y) P(X_2 \ge y) \dots P(X_n \ge y)$

$P(Y_2 \ge y) = [1 - F_X(y)]^n$

$\therefore 1 - F_{Y_2}(y) = [1 - F_X(y)]^n$

$F_{Y_2}(y) = 1 - [1 - F_X(y)]^n$

$f_{Y_2}(y) = n[1 - F_X(y)]^{n-1} f_X(y)$

$\therefore$ cdf of $Y_2 = 1 - [1 - F_X(y)]^n$

pdf of $Y_2 = n f_X(y) [1 - F_X(y)]^{n-1}$

Question 2

Q.2. $\left\{ X \sim \sum_{i=1}^{K} p_i \mathcal{N}(\mu_i, \sigma_i^2) \right\}$ 1

$$\Rightarrow E(X) = \int_{-\infty}^{\infty} x \, P(X=x) \, dx$$

$$= \int_{-\infty}^{\infty} dx \, x \sum_{i=1}^{K} P(X=x \mid X \sim \mathcal{N}(\mu_i, \sigma_i^2)) \, P(X \sim \mathcal{N}(\mu_i, \sigma_i^2))$$

$$= \sum_{i=1}^{K} p_i \int_{-\infty}^{\infty} dx \; x \; P(X=x \mid X \sim \mathcal{N}(\mu_i, \sigma_i^2))$$

$$= \sum_{i=1}^{K} p_i \, E(X \mid X \sim \mathcal{N}(\mu_i, \sigma_i^2)) = \boxed{\sum_{i=1}^{K} p_i \mu_i}$$

We know that $Var(X) = E(X^2) - (E(X))^2$ and following similar calculation done for the above part, we get :

$$\Rightarrow Var(X) = \sum_{i=1}^{K} p_i \left[E(X^2 \mid X \sim \mathcal{N}(\mu_i, \sigma_i^2)) - (E(X \mid X \sim \mathcal{N}(\mu_i, \sigma_i^2)))^2 \right]$$

$$= \sum_{i=1}^{K} p_i \, Var(X \mid X \sim \mathcal{N}(\mu_i, \sigma_i^2))$$

$$= \boxed{\sum_{i=1}^{K} p_i \sigma_i^2}$$

$\Rightarrow$ Similarly, for the MGF :

$$MGF(X) = E(e^{tX}) = \sum_{i=1}^{K} p_i \int_{-\infty}^{\infty} dx\, e^{tx} P(X=x \mid X \sim \mathcal{N}(\mu_i, \sigma_i^2))$$

$$= \sum_{i=1}^{K} p_i \, E(e^{tX} \mid X \sim \mathcal{N}(\mu_i, \sigma_i^2))$$

$$= \boxed{\sum_{i=1}^{K} p_i \, e^{\left(\mu_i t + \frac{\sigma_i^2 t^2}{2}\right)}}$$

$\left\{ Z = \sum_{i=1}^{K} p_i X_i \right\}$ (2)

$$\Rightarrow E(Z) = E\left(\sum_{i=1}^{K} p_i X_i\right) = \sum_{i=1}^{K} p_i \, E(X_i)$$

$$= \boxed{\sum_{i=1}^{K} p_i \mu_i}$$

$$\Rightarrow Var(Z) = Var\left(\sum_{i=1}^{K} p_i X_i\right) = \sum_{i=1}^{K} p_i^2 \, Var(X_i)$$

(as they are independent)

$$= \boxed{\sum_{i=1}^{K} p_i^2 \sigma_i^2}$$

$\Rightarrow$ $MGF(Z) = E(e^{tZ}) = E\left(e^{t\sum_{i=1}^{k} p_i x_i}\right)$

(as all X_i are independent)

$$= \prod_{i=1}^{k} E(e^{tp_i x_i}) = \prod_{i=1}^{k} \phi_{X_i}(tp_i) \quad \left[\text{where } \phi_{X_i} \text{ is MGF of } \mathcal{N}(\mu_i, \sigma_i^2)\right]$$

$$= \prod_{i=1}^{K} e^{\left(\mu_i p_i t + \frac{\sigma_i^2 p_i^2 t^2}{2}\right)}$$

$$= e^{\sum_{i=1}^{k}\left(\mu_i p_i t + \frac{\sigma_i^2 p_i^2 t^2}{2}\right)}$$

$$= \boxed{e^{t\left(\sum_{i=1}^{k} \mu_i p_i\right) + \frac{t^2}{2}\left(\sum_{i=1}^{k} \sigma_i^2 p_i^2\right)}}$$

$\Rightarrow$ We observe that $MGF(Z)$ is same as that of a gaussian, i.e. $MGF\left(\mathcal{N}\left(\sum_{i=1}^{k}\mu_i p_i, \sum_{i=1}^{k}\sigma_i^2 p_i^2\right)\right)$

So, due to uniqueness of MGF, we can say that $Z \sim \mathcal{N}\left(\sum_{i=1}^{k} \mu_i p_i, \sum_{i=1}^{k} \sigma_i^2 p_i^2\right)$

The PDF then becomes:

$$\boxed{p(z) = \frac{1}{\sqrt{2\pi \sum_{i=1}^{k}\sigma_i^2 p_i^2}}\, e^{-\frac{\left(z - \sum_{i=1}^{k}\mu_i p_i\right)^2}{2\left(\sum_{i=1}^{k}\sigma_i^2 p_i^2\right)}}}$$

Question 3

3. Consider $Y = X - \mu$

$E[Y] = 0$ $\quad Var(Y) = \sigma^2$

$\therefore$ $P(Y \geq \tau)$ for $\tau > 0$

$P(Y \geq \tau) \leq P((Y+b)^2 \geq (\tau+b)^2)$ for $b \geq 0$

as it will be cover both types of values

$\therefore$ Applying Markov's inequality

$$P((Y+b)^2 \geq (\tau+b)^2) \leq \frac{E[(Y+b)^2]}{(\tau+b)^2} = \frac{\sigma^2+b^2}{(\tau+b)^2}$$

$$\therefore \; P(X-\mu \geq \tau) \leq \frac{\sigma^2+b^2}{(\tau+b)^2} \quad \text{for } b \geq 0$$

Now differentiate $\frac{\sigma^2+b^2}{(\tau+b)^2}$ and equate to 0

$$\cancel{2}b(\tau+b)^{\cancel{2}} = \cancel{2}\cancel{(\tau+b)}(\sigma^2+b^2)$$

$b\tau + b^2 = \sigma^2 + b^2 \Rightarrow b = \sigma^2/\tau$ so for this b exp. is min^m

$$\therefore \; P(X-\mu \geq \tau) \leq \frac{\sigma^2+\sigma^4/\tau^2}{(\tau^2+\sigma^2)^2/\tau^2} = \cancel{\sigma^2\frac{(1+(\sigma^2/\tau^2))}{(1+(\sigma^2/\tau)^2)}}$$

$$\frac{\sigma^2+\sigma^4/\tau^2}{(\tau^2+\sigma^2)^2/\tau^2} = \frac{\sigma^2\tau^2+\sigma^4}{(\tau^2+\sigma^2)^2} = \frac{\sigma^2}{(\tau^2+\sigma^2)}$$

$$\therefore \; p(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$\therefore$ for any random variable and $\tau > 0$

$$P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\cancel{b^2}\tau^2}$$

Now consider $\tau < 0$ and $b = -\tau$

let say $Y = -X$ with mean $-\mu$ and variance σ

$$\therefore \quad P(Y+\mu \geq b) \leq \frac{\sigma^2}{\sigma^2+\tau^2} \quad \text{as } b^2 = \tau^2$$

$$P(-X+\mu \geq b) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$\therefore \quad P(X \leq \mu-b) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$P(X \leq \mu+\tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$-P(X \leq \mu+\tau) \geq -\frac{\sigma^2}{\sigma^2+\tau^2}$$

$$1-P(X \leq \mu+\tau) \geq 1-\frac{\sigma^2}{\sigma^2+\tau^2}$$

$$\therefore \quad P(X \geq \mu+\tau) \geq 1-\frac{\sigma^2}{\sigma^2+\tau^2}$$

Thus for $\tau < 0$, $P(X \geq \mu+\tau) \geq 1-\dfrac{\sigma^2}{\sigma^2+\tau^2}$

Question 4

4. ~~For $t>0$~~

For $t>0$

$\phi_X(t) = E[e^{Xt}]$

$= \sum_{i=1}^{n} e^{x_i t}(P(X=x_i))$ for discrete

$= \int_{-\infty}^{\infty} e^{xt} f_X(x)\,dx$ for continuous

First prove for continuous,

$\phi_X(t) = \int_{-\infty}^{\infty} e^{xt} f_X(x)\,dx$

if $t>0$, $e^{x_1 t} > e^{x_2 t}$ if $x_1 > x_2$

$\phi_X(t) \geq \int_{x_0}^{\infty} e^{xt} f_X(x)\,dx \geq \int_{x_0}^{\infty} e^{x_0 t} f_X(x)\,dx$ [if $t>0$]

$\therefore \phi_X(t) \geq e^{x_0 t} \int_{x_0}^{\infty} f_X(x)\,dx = e^{x_0 t}(P(X \geq x_0))$

$\therefore$ ~~$P(X \leq x_0)$~~ $P(X \geq x_0) \leq e^{-x_0 t} \phi_X(t)$

Now for discrete,

$\phi_X(t) = \sum_{i=1}^{n} e^{x_i t} P(X=x_i)$

if $t>0$

$\phi_X(t) \geq \sum_i e^{x_0 t} P(X=x_i)$ here $x_i \geq x_0$

$\phi_X(t) \geq e^{x_0 t} \sum_i P(X=x_i)$ where ~~x_i~~ is all $x_i \geq x_0$

$\therefore \phi_X(t) \geq e^{x_0 t} P(X \geq x_0)$

$\therefore P(X \geq x_0) \leq e^{-x_0 t} \phi_X(t)$

For $t<0$

For continuous,

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{+tx} f_X(x)\,dx \geq \int_{-\infty}^{x_0} e^{+tx} f_X(x)\,dx$$

if $t<0$, $e^{+tx_1} > e^{+tx_2}$ if $x_1 < x_2$

$$\therefore \phi_X(t) \geq \int_{-\infty}^{x_0} e^{tx} f_X(x)\,dx \geq \int_{-\infty}^{x_0} e^{x_0 t} f_X(x)\,dx$$

$$\therefore \phi_X(t) \geq e^{tx_0} P(X \leq x_0)$$

$$\therefore P(X \leq x_0) \leq e^{-tx_0} \phi_X(t)$$

For discrete,

$$\phi_X(t) = \sum_{i=1}^{n} e^{x_i t} P(X=x_i)$$

$\phi_X(t) \geq \sum_{j} e^{x_0 t} P(X=x_j)$ where $x_j \leq x_0$

because if $t<0$, $e^{x_0 t} \leq e^{x_j t}$

$$\phi_X(t) \geq e^{x_0 t} P(X \leq x_0)$$

$$\therefore P(X \leq x_0) \leq e^{-tx_0} \phi_X(t)$$

Now for the second part

$\phi_{X_i}(t)$ ~~$\phi(x_i)$~~ $= 1-p_i+p_ie^t = 1+p_i(e^t-1) \leq e^{p_i(e^t-1)}$

Therefore

$\phi_X(t) = \prod_{i=1}^{n} \phi_{X_i}(t) \leq \prod e^{p_i(e^t-1)} = e^{\mu(e^t-1)}$ as $\sum p_i = \mu$

$\therefore$ Using the first inequality for $t>0$, as for $t=0$
$p(X > (1+\delta)\mu) \leq 1$ which is obviousely true,

$p(X>(1+\delta)\mu) \leq e^{-t(1+\delta)\mu} \phi_X(t) \leq e^{-t(1+\delta)\mu} e^{\mu(e^t-1)}$

$\therefore\ p(X>(1+\delta)\mu) \leq \dfrac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}}$

Now we want to minimize $e^{\mu(e^t-1)-(1+\delta)\mu t}$
with respect to t.

So we have to minimize $\mu(e^t-1)-(1+\delta)\mu t$

Taking derivative and equation to 0

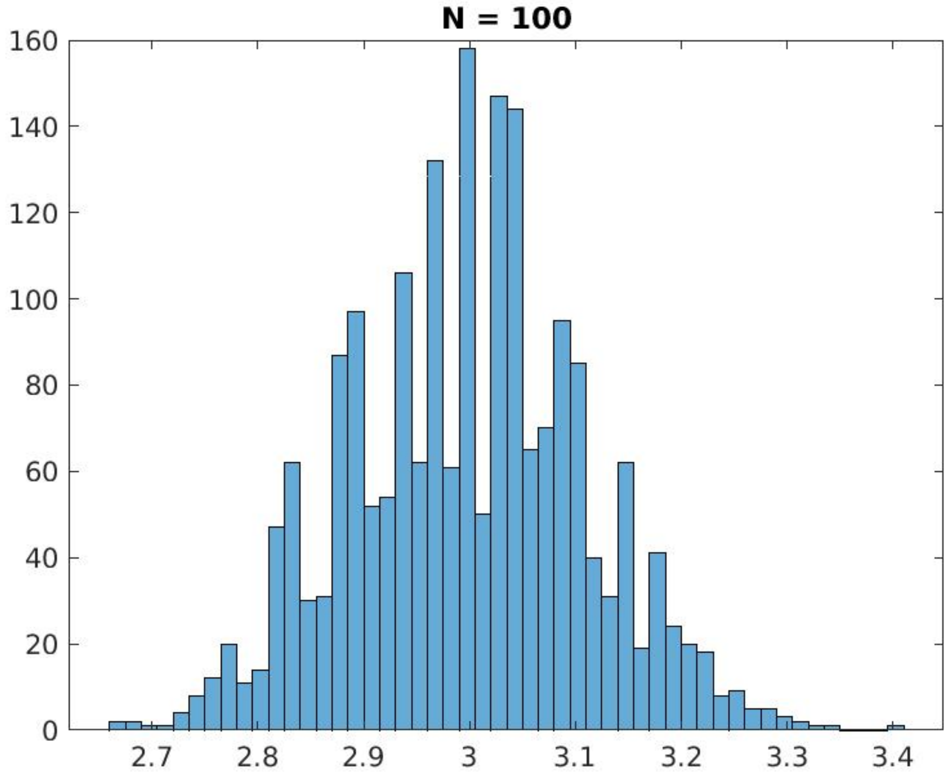
$\mu e^t = \mu(1+\delta)$

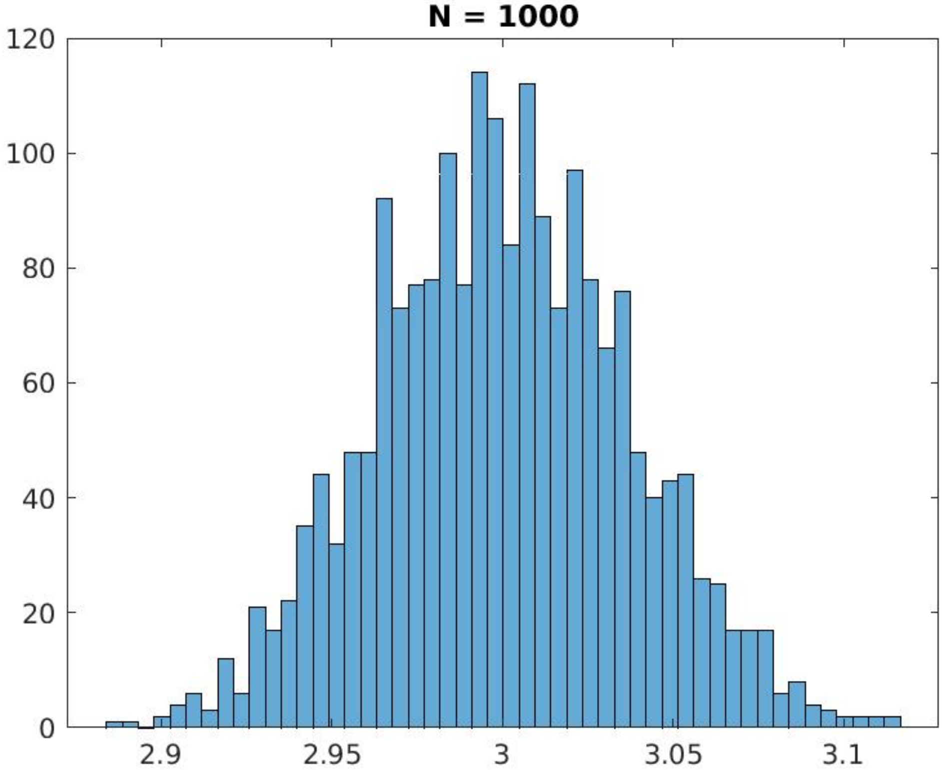
$\Rightarrow t = \ln(1+\delta)$

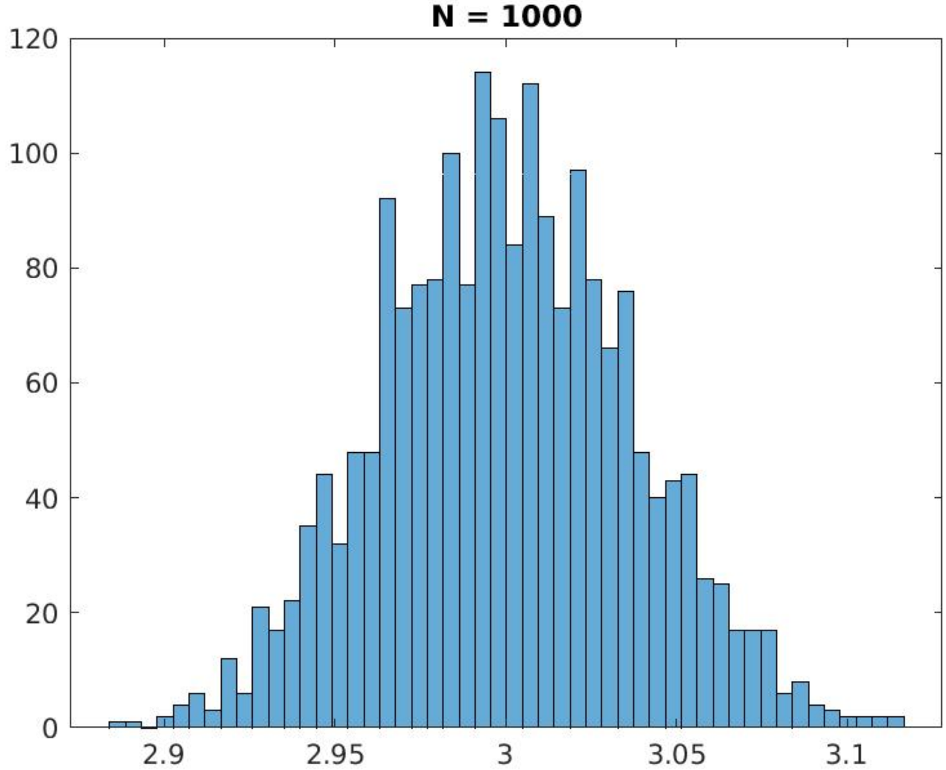
It is ~~mini~~ for minima as double derivative $= \mu e^t > 0$

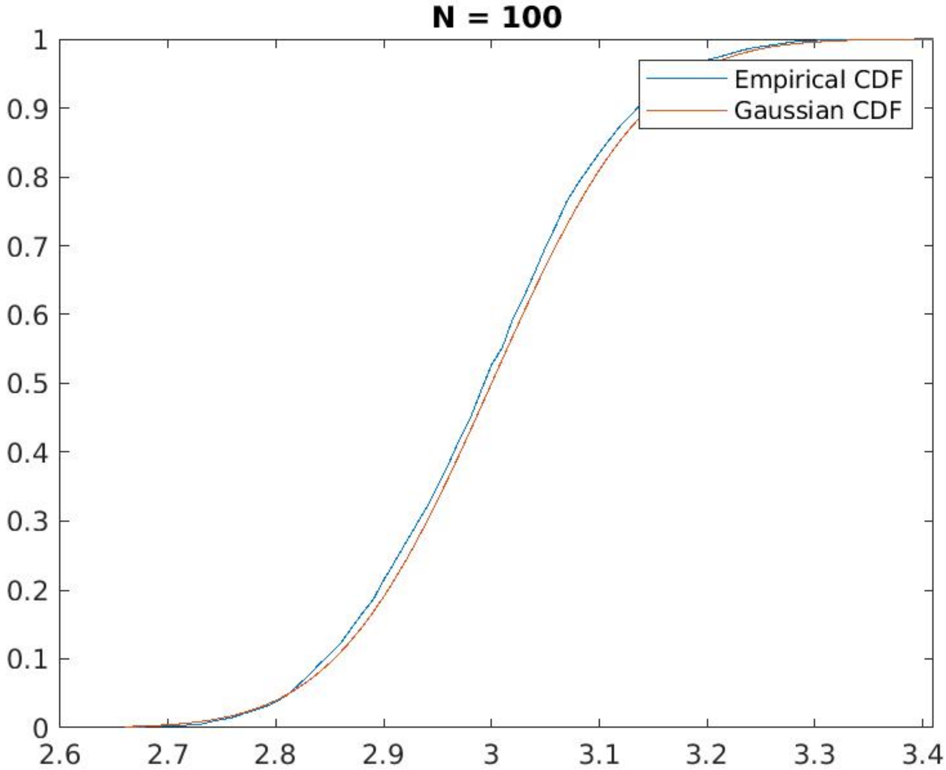
Question 5

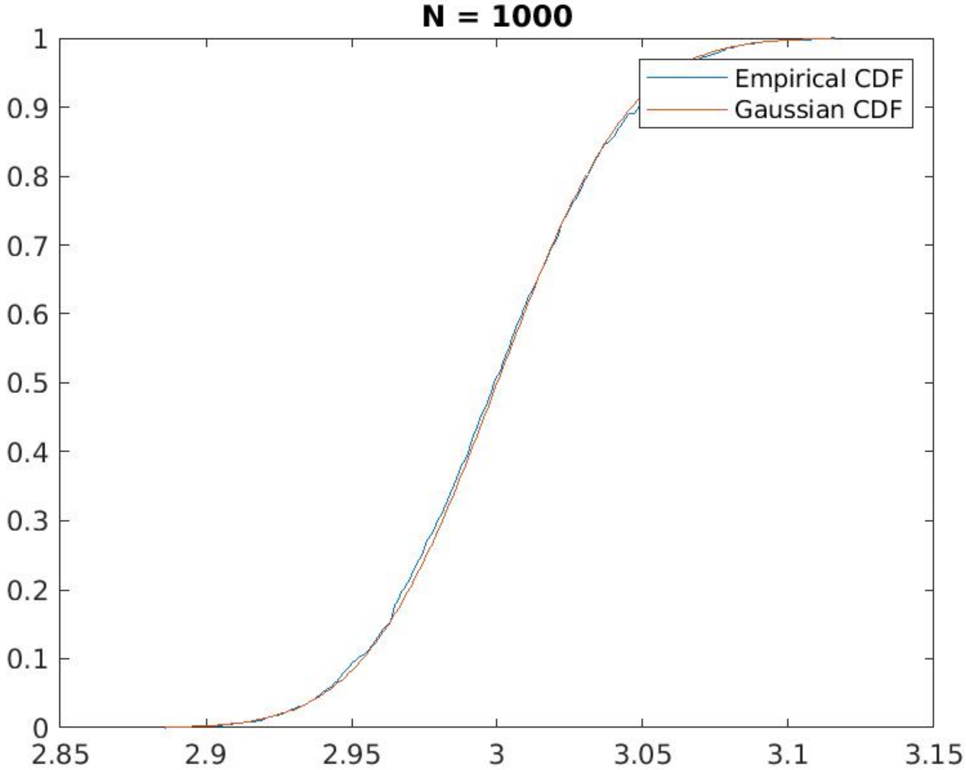
Code for this qsn is in file named 'q5.m'

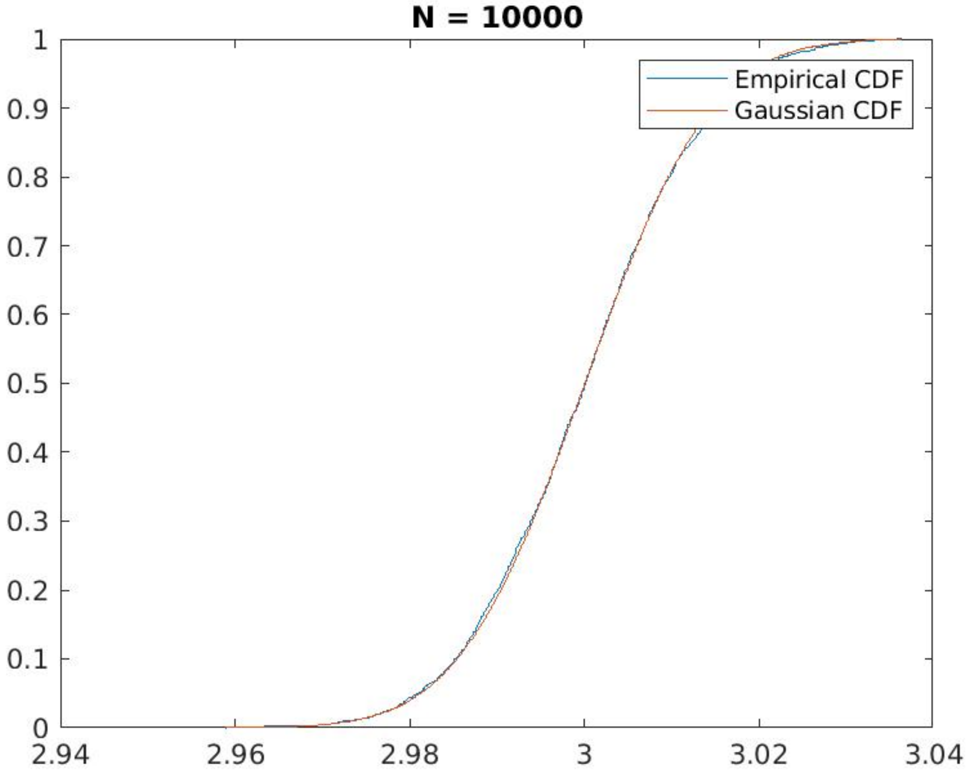


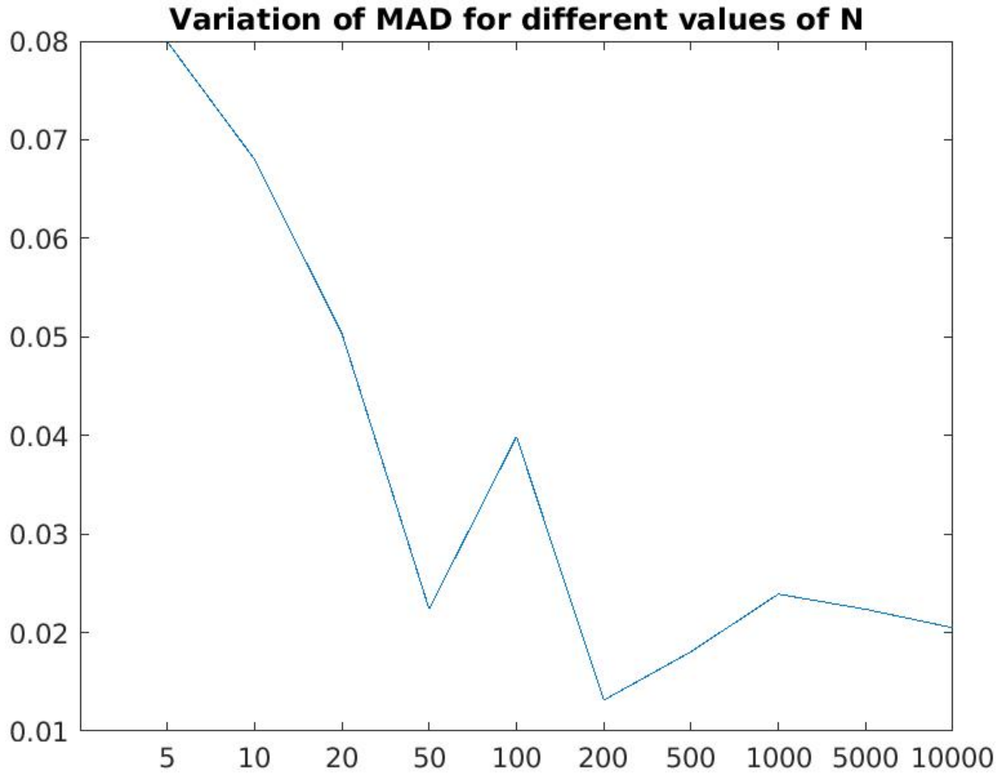

N = 1000
120
100
80
60
40
20
0
2.9
2.95
3
3.05
3.1


N = 1000
120
100
80
60
40
20
0
2.9
2.95
3
3.05
3.1


N = 100
Empirical CDF
Gaussian CDF
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2.6
2.7
2.8
2.9
3
3.1
3.2
3.3
3.4


N = 1000
Empirical CDF
Gaussian CDF
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2.85
2.9
2.95
3
3.05
3.1
3.15


N = 10000
Empirical CDF
Gaussian CDF
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2.94
2.96
2.98
3
3.02
3.04


Variation of MAD for different values of N
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
5
10
20
50
100
200
500
1000
5000
10000

Question 6

Code of this question is in the file 'q6.m'

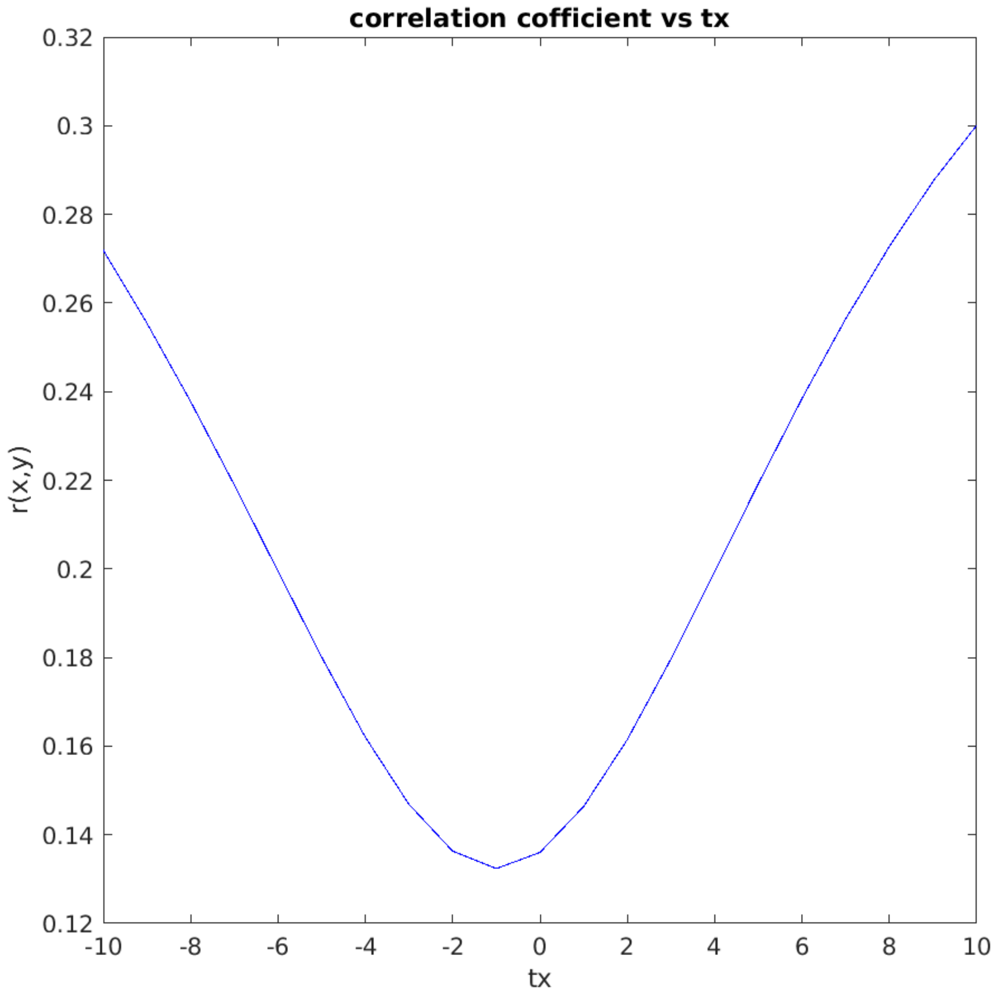


Figure 1: This plot correspond to T1.jpg and T2.jpg

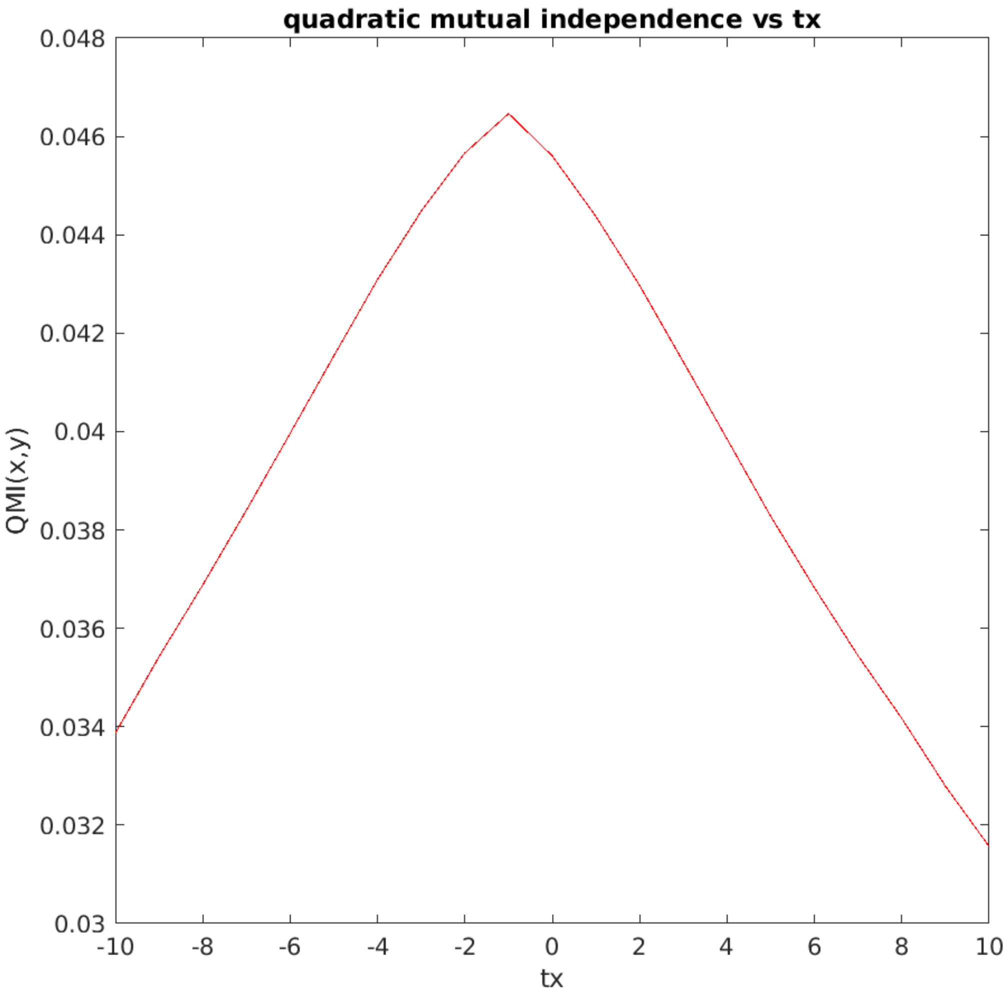


Figure 2: This plot correspond to T1.jpg and T2.jpg

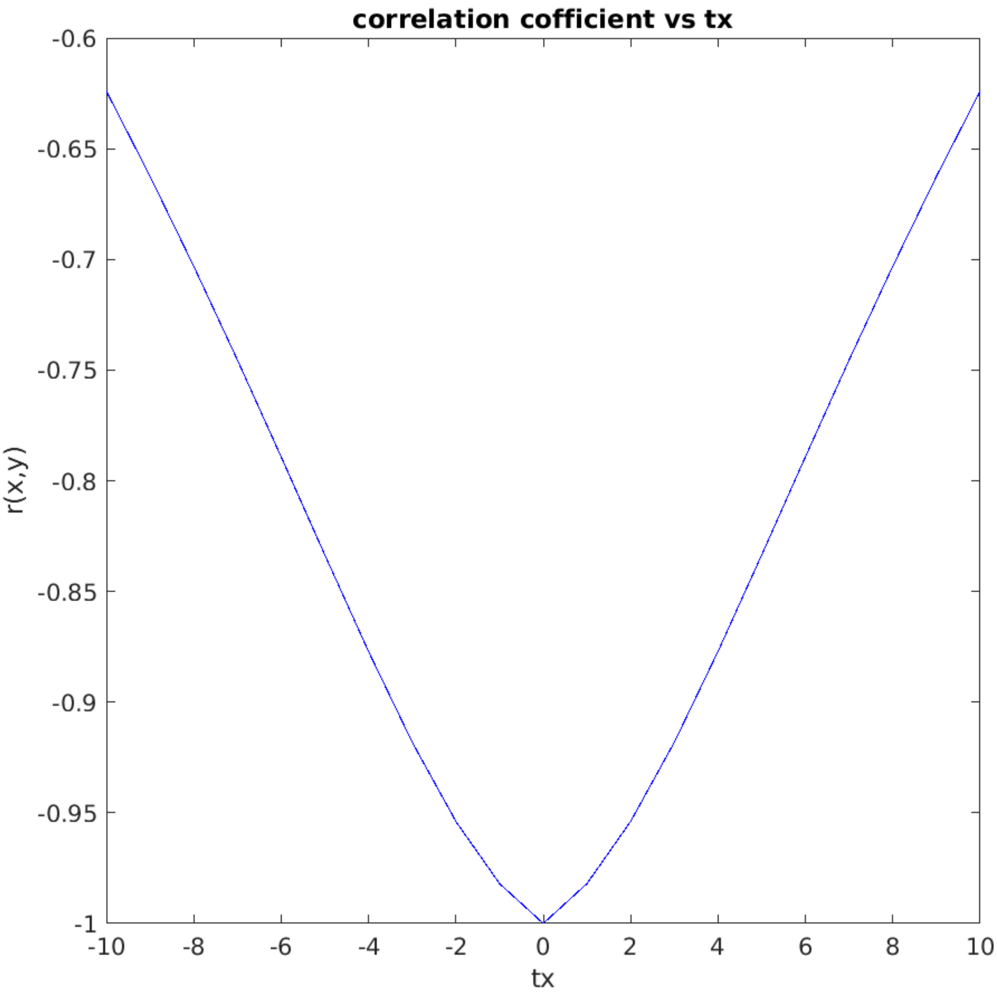


Figure 3: This plot correspond to T1.jpg and negative of T1.jpg

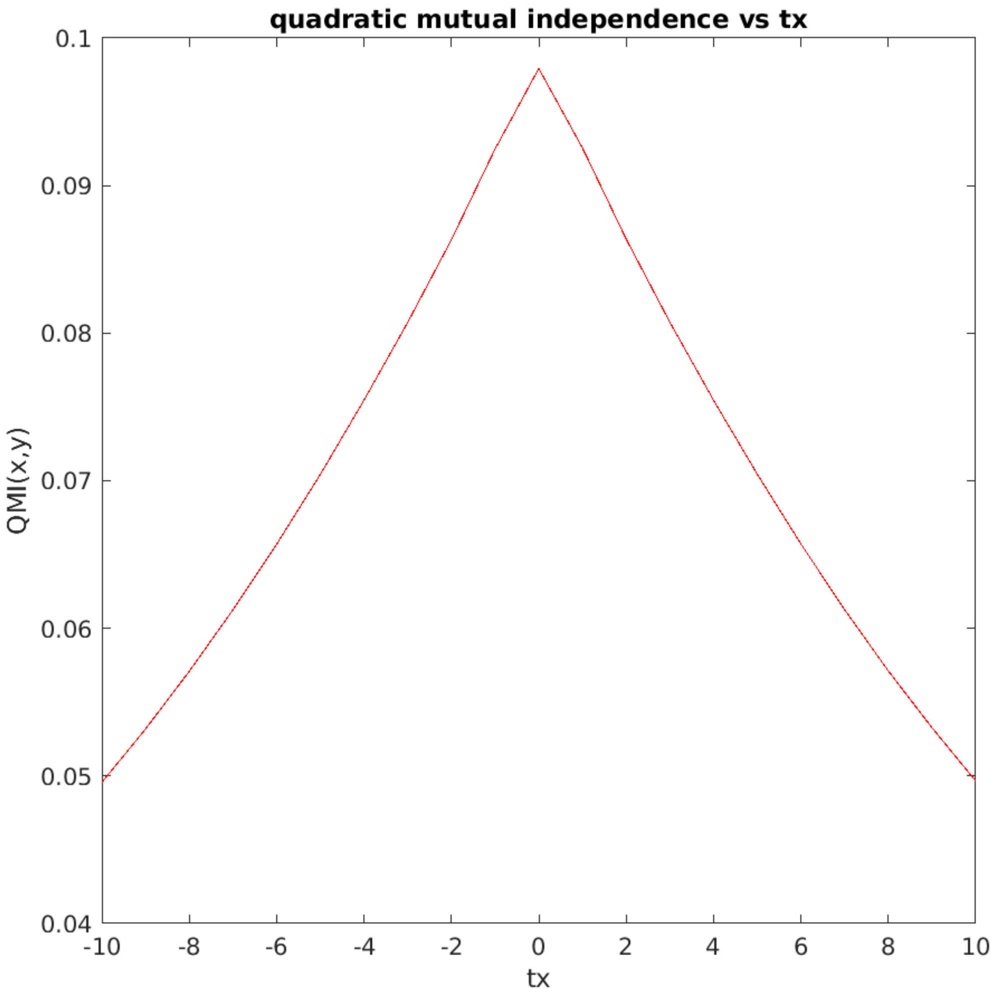


Figure 4: This plot correspond to T1.jpg and negative of T1.jpg

By observation we can see that correlation coefficient between the two images will be always postive, and it will minimum when the shift is equal to -1(that is one unit along negative x axis).By similar type of arguments we can say that QMI will attains it's maximum when the shift is equal to 1.We can say that correlation coefficient increases and QMI decreases when images are moved out of alignment as the point of minimum correlation coefficient and maximum QMI is somewhat close to no shift.

By observation we can see that correlation coefficient between the two images will be always negative, and it will minimum when the shift is equal to 0 and it will be equal to -1 as we can clearly see that the images are negative of each other.By similar type of arguments we can say that QMI will attains it's maximum when the shift is equal to 0.We can say that correlation coefficient increases and QMI decreases when images are moved out of alignment as the point of minimum correlation coefficient and maximum QMI is equal to no shift.

Question 7

7. Using the same definition as used in slide

$$X = \sum_{i=1}^{k} X_i$$

$$\therefore \cancel{\phi_X(t) = \sum p_i e^{x_i}}$$

$$\phi_X(t_1, t_2, \ldots t_n) = \left(\sum p_i e^{t_i}\right)^n$$

From slides

As $E[X_i X_j] = \dfrac{\partial}{\partial t_j} \dfrac{\partial}{\partial t_i} \phi_X(t) \Big|_{t=(0,0,..,0)}$ where $t = (t_1, t_2, \ldots t_k)$

$$= \frac{\partial}{\partial t_j} e^{t_i} n p_i \left(\sum_{i=1}^{k} p_i e^{t_i}\right)^{n-1} \Bigg|_{t=(0,0,\ldots)}$$

$$= e^{t_i} n p_i \frac{\partial}{\partial t_j} \left(\sum_{i=1}^{k} p_i e^{t_i}\right)^{n-1} \Bigg|_{t=(0,0,\ldots)}$$

$$= e^{t_i} n p_i (n-1) p_j \left(\sum p_i e^{t_i}\right)^{n-2} \times e^{t_j} \Big|_{t=(0,0,\ldots)}$$

$$= n(n-1) p_i p_j \qquad \left(\sum p_i = 1\right)$$

and $E[X_i] = \dfrac{\partial}{\partial t_i} \phi_X(t) \Big|_{t=(0,0,..0)}$ where $t = (t_1, t_2 \ldots, t_k)$

$$= \frac{\partial}{\partial t_i} \left(\sum p_i e^{t_i}\right)^n \Bigg|_{t=(0,0,0\ldots)}$$

$$= e^{t_i} p_i n \left(\sum p_i e^{t_i}\right)^{n-1} \Big|_{t=(0,0)}$$

$$= n p_i \quad \text{because } \sum p_i = 1$$

$$E[x_i^2] = \frac{\partial^2}{\partial t_i^2} \phi_x(t) \Big|_{t=(0,0..)} \quad \text{where } t=(t_1, t_2, .. t_k)$$

$$= \frac{\partial^2}{\partial t_i^2} (\Sigma p_i e^{t_i})^n$$

$$= \frac{\partial}{\partial t_i} p_i n (\Sigma p_i e^{t_i})^{n-1} \times e^{t_i}$$

$$= p_i n e^{t_i} (\Sigma p_i e^{t_i})^{n-1} + p_i n e^{t_i} (n-1) (\Sigma p_i e^{t_i})^{n-2} p_i e^{t_i}$$

At $\forall t_j = 0$

this becomes

$$p_i n + p_i n (n-1) p_i$$

$$= n p_i [1 + (n-1) p_i]$$

$\therefore$ ~~Covariance~~ :

$$\text{cov}(x_i, x_i) = E[x_i^2] - (E[x_i])^2$$

$$= n p_i [1 + (n-1) p_i] - (n p_i)^2$$

$$= n p_i + n p_i^2 (n-1) - n^2 p_i^2$$

$$= n p_i - n p_i^2$$

$$= n p_i (1 - p_i)$$

$$\text{Cov}(x_i, x_j) = E[x_i x_j] - E[x_i] E[x_j]$$

$$= n(n-1) p_i p_j - n p_i n p_j$$

$$= -n p_i p_j$$

Therefore matrix is

$$\begin{bmatrix} n p_1 (1-p_1) & -n p_1 p_2 & \cdots & -n p_1 p_k \\ -n p_2 p_1 & n p_2 (1-p_2) & \cdots & \vdots \\ -n p_3 p_1 & -n p_3 p_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ & & & n p_k (1-p_k) \end{bmatrix}$$