Assignment 1: CS 215

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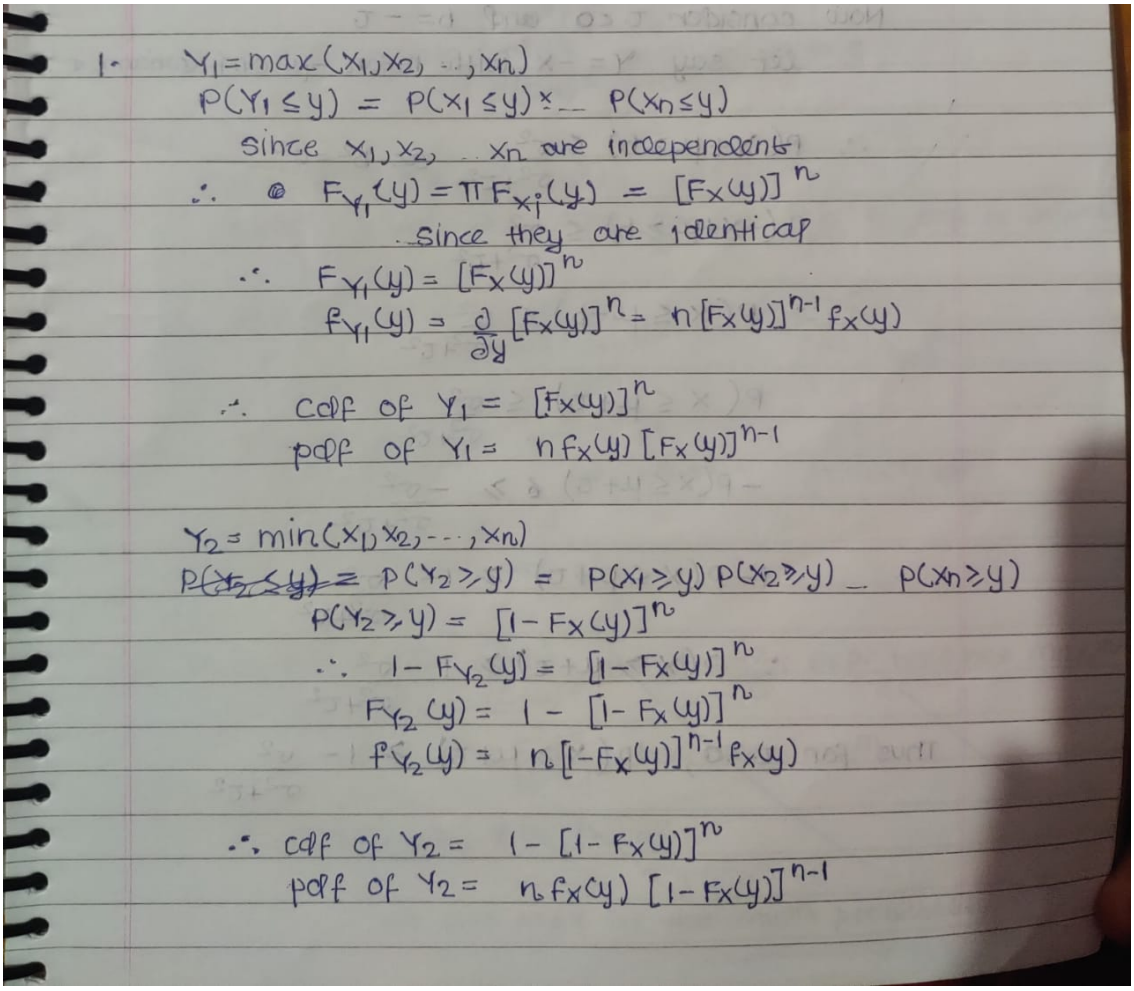
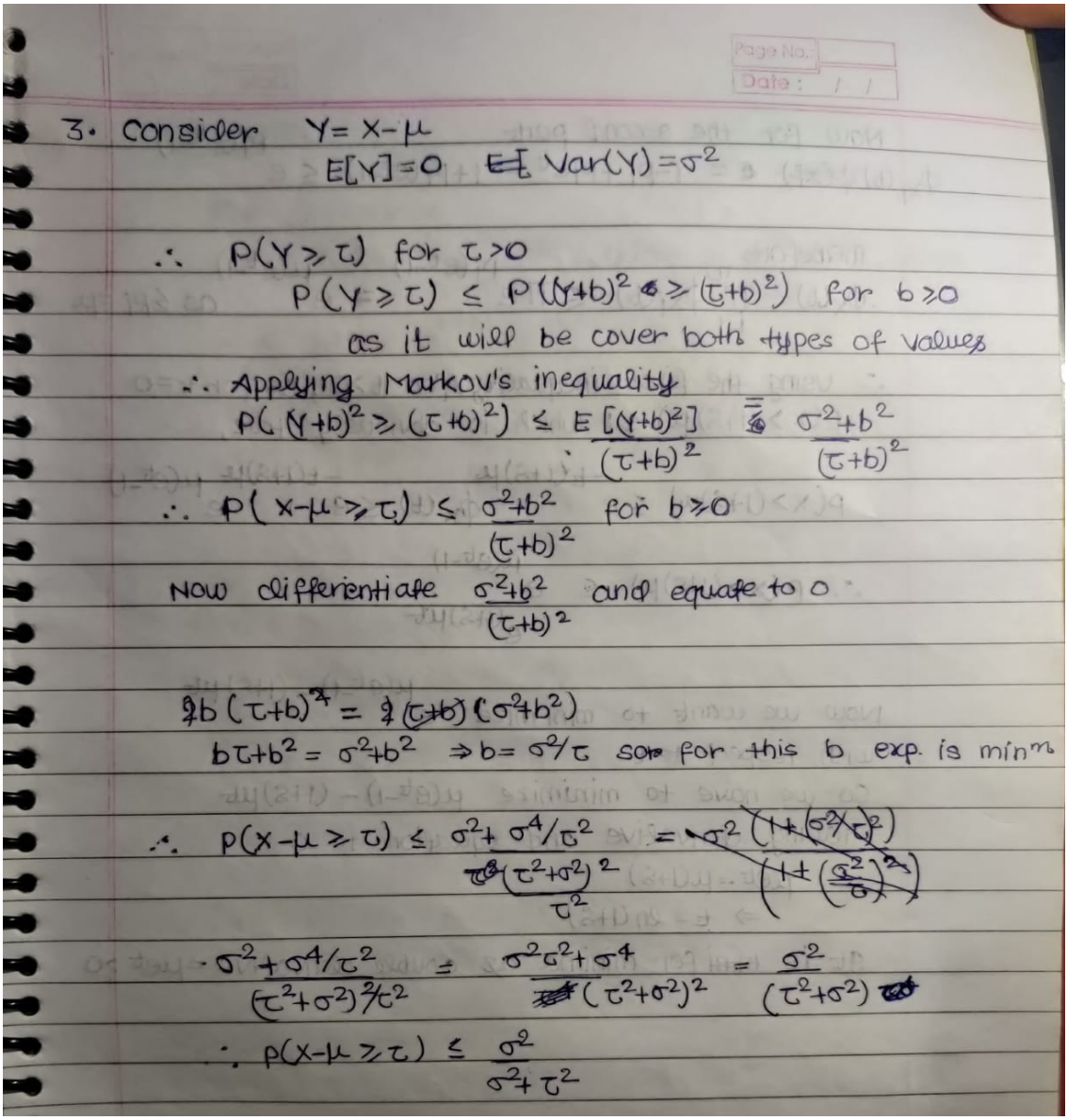
Question 1



Question 2

Question 3



3. Consider $Y = X - \mu$

$E[Y] = 0$ $\quad$ $Var(Y) = \sigma^2$

$\therefore$ $P(Y \geq \tau)$ for $\tau > 0$

$P(Y \geq \tau) \leq P((Y+b)^2 \geq (\tau+b)^2)$ for $b \geq 0$

as it will be cover both types of values

$\therefore$ Applying Markov's inequality

$P((Y+b)^2 \geq (\tau+b)^2) \leq \frac{E[(Y+b)^2]}{(\tau+b)^2} = \frac{\sigma^2+b^2}{(\tau+b)^2}$

$\therefore$ $P(X-\mu \geq \tau) \leq \frac{\sigma^2+b^2}{(\tau+b)^2}$ for $b \geq 0$

Now differentiate $\frac{\sigma^2+b^2}{(\tau+b)^2}$ and equate to 0

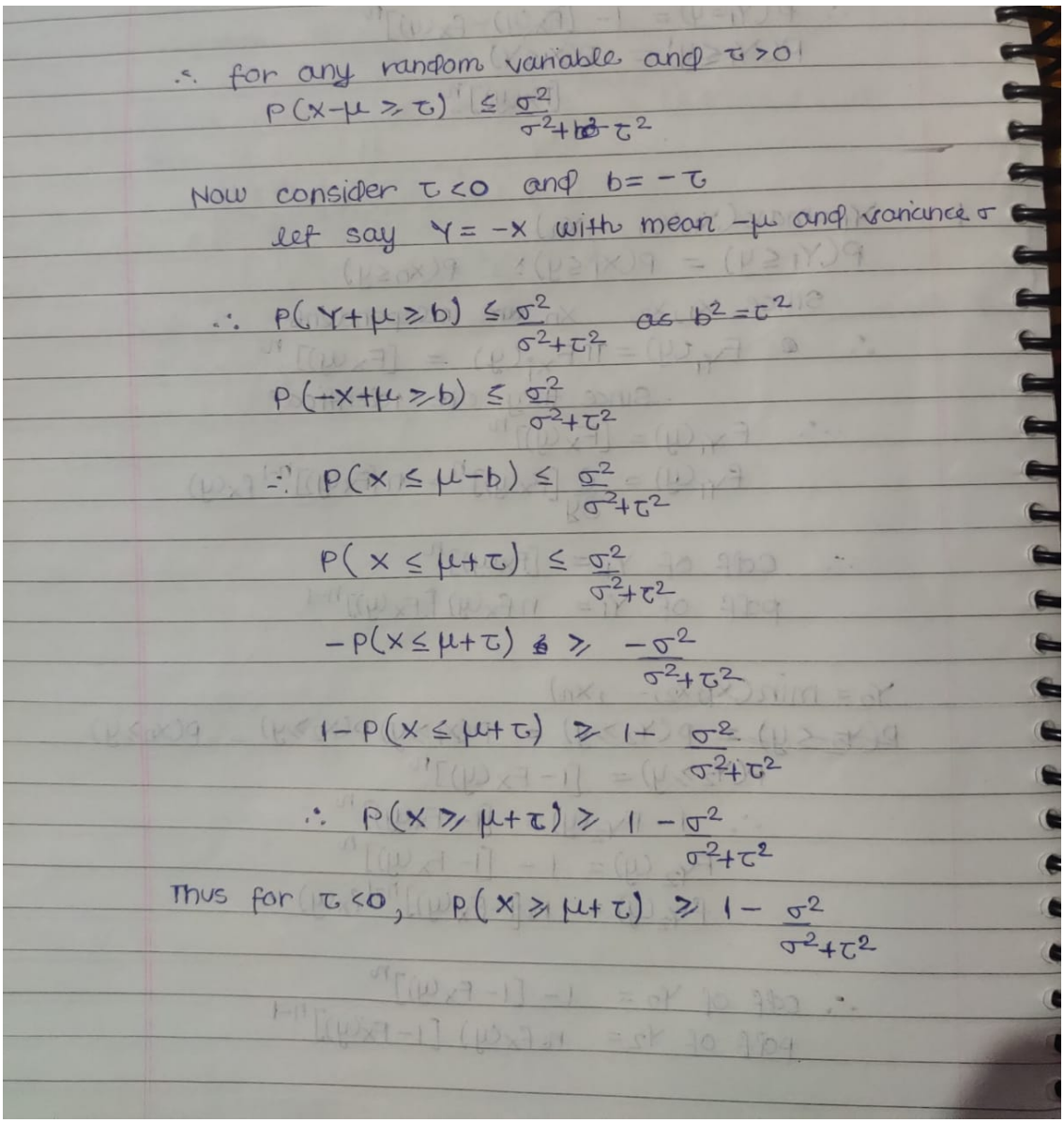
$2b(\tau+b)^2 = 2(\tau+b)(\sigma^2+b^2)$

$b\tau + b^2 = \sigma^2 + b^2 \Rightarrow b = \sigma^2/\tau$ so for this b exp. is min^n

$\therefore$ $P(X-\mu \geq \tau) \leq \frac{\sigma^2 + \sigma^4/\tau^2}{(\tau^2+\sigma^2)^2/\tau^2} = \sigma^2\left(\frac{1+(\sigma^2/\tau^2)}{(1+(\frac{\sigma^2}{\tau}))^2}\right)$

$\frac{\sigma^2+\sigma^4/\tau^2}{(\tau^2+\sigma^2)^2/\tau^2} = \frac{\sigma^2\tau^2+\sigma^4}{(\tau^2+\sigma^2)^2} = \frac{\sigma^2}{(\tau^2+\sigma^2)}$

$\therefore$ $P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$



$\therefore$ for any random variable and $\tau > 0$

$$P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

Now consider $\tau < 0$ and $b = -\tau$

let say $Y = -X$ with mean $-\mu$ and variance σ

$\therefore$ $P(Y+\mu \geq b) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$ as $b^2 = \tau^2$

$$P(-X+\mu \geq b) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$\therefore P(X \leq \mu - b) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$P(X \leq \mu + \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$$

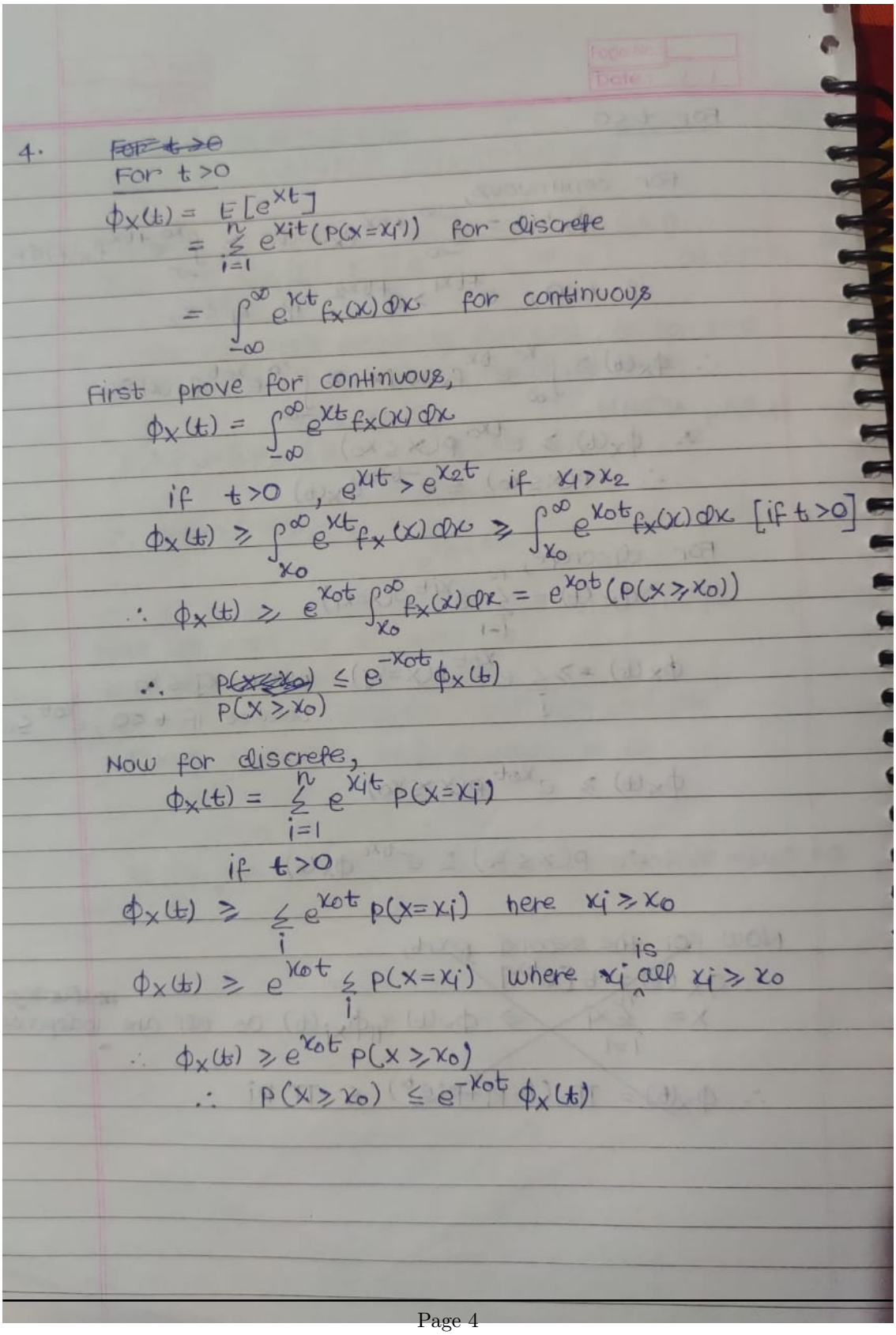
$$-P(X \leq \mu+\tau) \geq -\frac{\sigma^2}{\sigma^2+\tau^2}$$

$$1 - P(X \leq \mu+\tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$$

$$\therefore P(X \geq \mu+\tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$$

Thus for $\tau < 0$, $P(X \geq \mu+\tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$

Question 4



4. ~~For $t \to 0$~~
For $t > 0$

$\phi_X(t) = E[e^{Xt}]$

$= \sum_{i=1}^{n} e^{x_i t}(P(X=x_i))$ for discrete

$= \int_{-\infty}^{\infty} e^{xt} f_X(x)\,dx$ for continuous

First prove for continuous,

$\phi_X(t) = \int_{-\infty}^{\infty} e^{xt} f_X(x)\,dx$

if $t>0$, $e^{x_1 t} > e^{x_2 t}$ if $x_1 > x_2$

$\phi_X(t) \geq \int_{x_0}^{\infty} e^{xt} f_X(x)\,dx \geq \int_{x_0}^{\infty} e^{x_0 t} f_X(x)\,dx$ [if $t>0$]

$\therefore \phi_X(t) \geq e^{x_0 t} \int_{x_0}^{\infty} f_X(x)\,dx = e^{x_0 t}(P(X \geq x_0))$

$\therefore \frac{P(X \leq x_0)}{P(X \geq x_0)} \leq e^{-x_0 t} \phi_X(t)$

Now for discrete,

$\phi_X(t) = \sum_{i=1}^{n} e^{x_i t} P(X=x_i)$

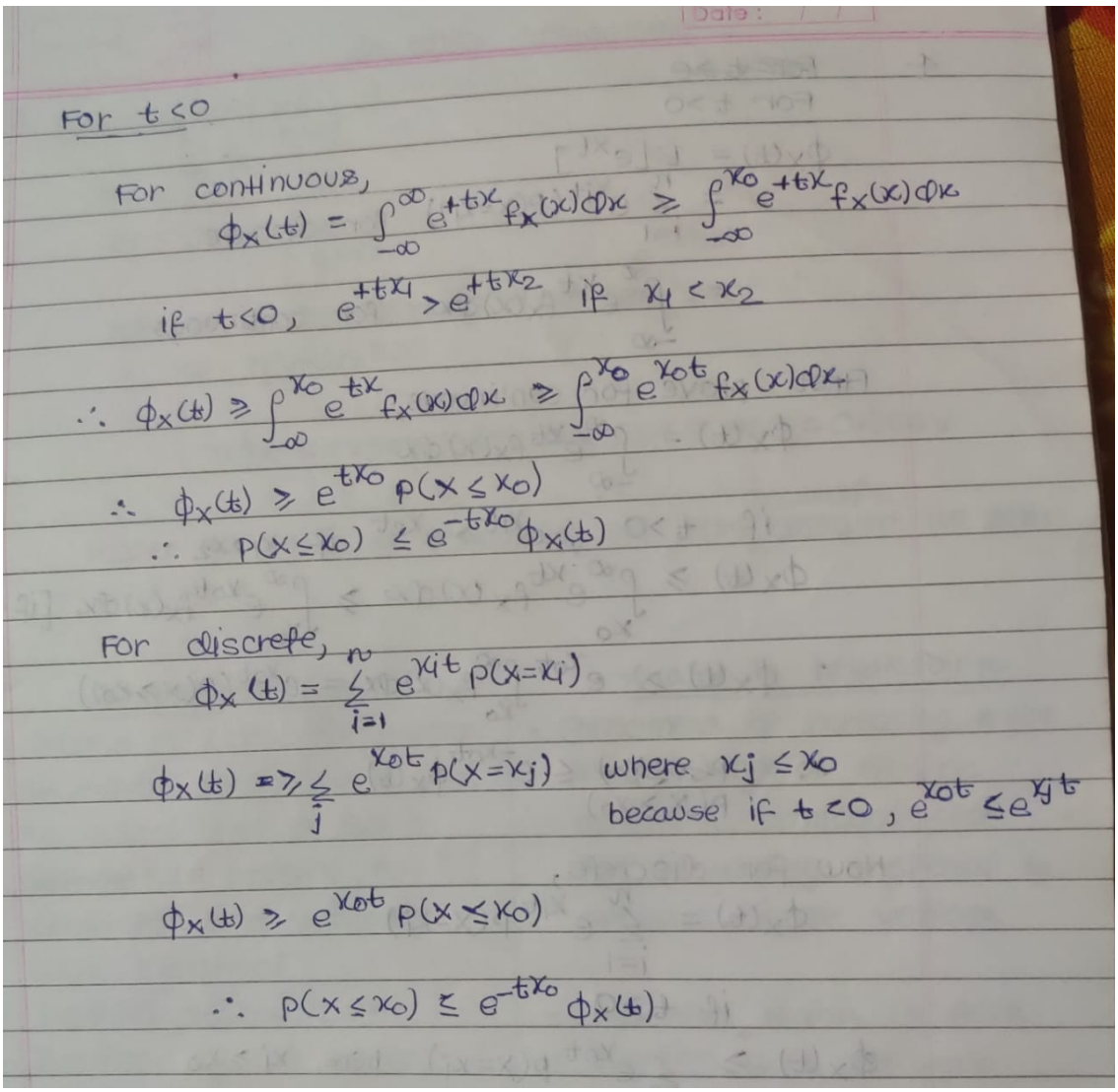
if $t>0$

$\phi_X(t) \geq \sum_{i} e^{x_0 t} P(X=x_i)$ here $x_i \geq x_0$

$\phi_X(t) \geq e^{x_0 t} \sum_{i} P(X=x_i)$ where x_i is all $x_i \geq x_0$

$\therefore \phi_X(t) \geq e^{x_0 t} P(X \geq x_0)$

$\therefore P(X \geq x_0) \leq e^{-x_0 t} \phi_X(t)$



For $t \leq 0$

For continuous,

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{+tx} f_X(x)\,dx \geq \int_{-\infty}^{x_0} e^{+tx} f_X(x)\,dx$$

if $t<0$, $e^{+tx_1} > e^{+tx_2}$ if $x_1 < x_2$

$$\therefore \phi_X(t) \geq \int_{-\infty}^{x_0} e^{tx} f_X(x)\,dx \geq \int_{-\infty}^{x_0} e^{x_0 t} f_X(x)\,dx$$

$$\therefore \phi_X(t) \geq e^{tx_0} p(X \leq x_0)$$

$$\therefore p(X \leq x_0) \leq e^{-tx_0} \phi_X(t)$$

For discrete,

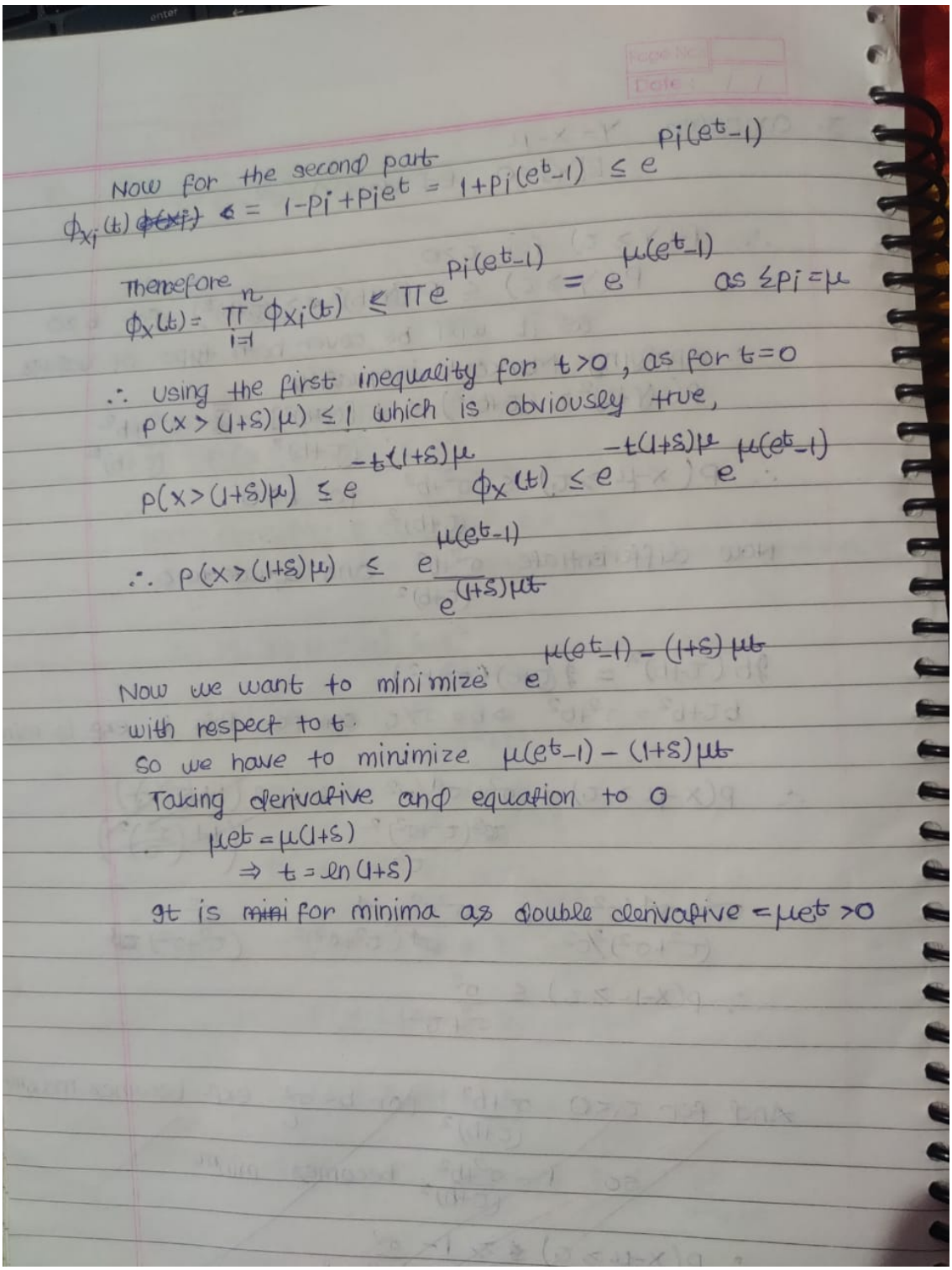
$$\phi_X(t) = \sum_{i=1}^{n} e^{x_i t} p(X = x_i)$$

$$\phi_X(t) \geq \sum_{j} e^{x_0 t} p(X = x_j) \quad \text{where } x_j \leq x_0$$

because if $t<0$, $e^{x_0 t} \leq e^{x_j t}$

$$\phi_X(t) \geq e^{x_0 t} p(X \leq x_0)$$

$$\therefore p(X \leq x_0) \leq e^{-tx_0} \phi_X(t)$$



Now for the second part

$\phi_{X_i}(t) = 1 - p_i + p_i e^t = 1 + p_i(e^t - 1) \leq e^{p_i(e^t-1)}$

Therefore

$\phi_X(t) = \prod_{i=1}^{n} \phi_{X_i}(t) \leq \prod e^{p_i(e^t-1)} = e^{\mu(e^t-1)}$ as $\sum p_i = \mu$

$\therefore$ using the first inequality for $t > 0$, as for $t = 0$ $P(X > (1+\delta)\mu) \leq 1$ which is obviously true,

$P(X > (1+\delta)\mu) \leq e^{-t(1+\delta)\mu} \phi_X(t) \leq e^{-t(1+\delta)\mu} e^{\mu(e^t-1)}$

$\therefore P(X > (1+\delta)\mu) \leq \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}}$

Now we want to minimize $e^{\mu(e^t-1) - (1+\delta)\mu t}$ with respect to t.

So we have to minimize $\mu(e^t - 1) - (1+\delta)\mu t$

Taking derivative and equation to 0

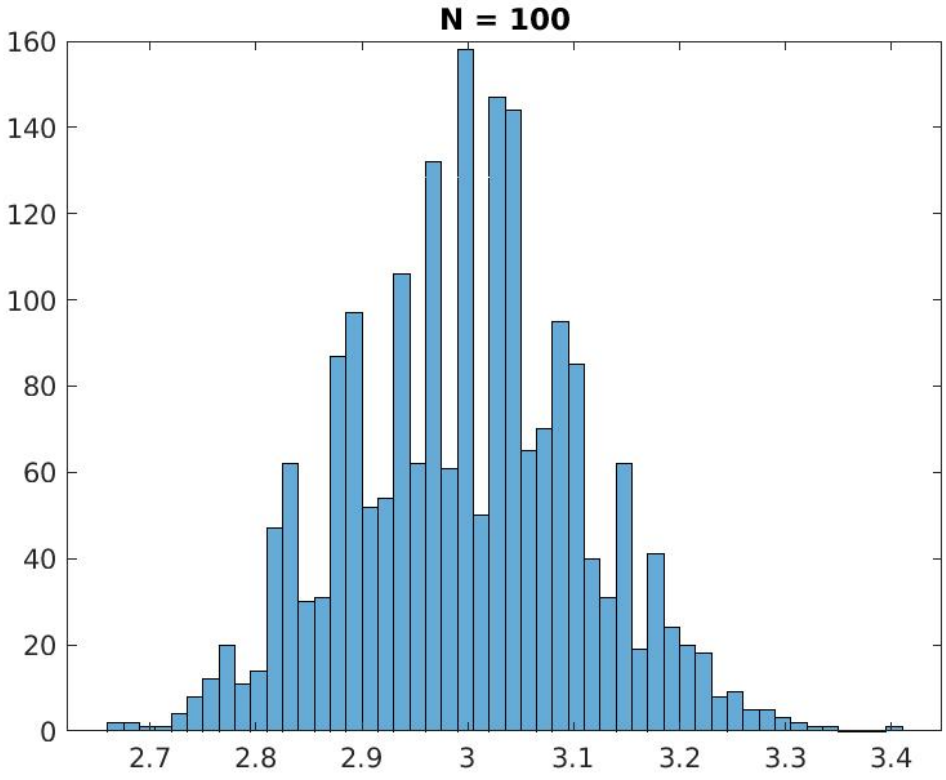
$\mu e^t = \mu(1+\delta)$

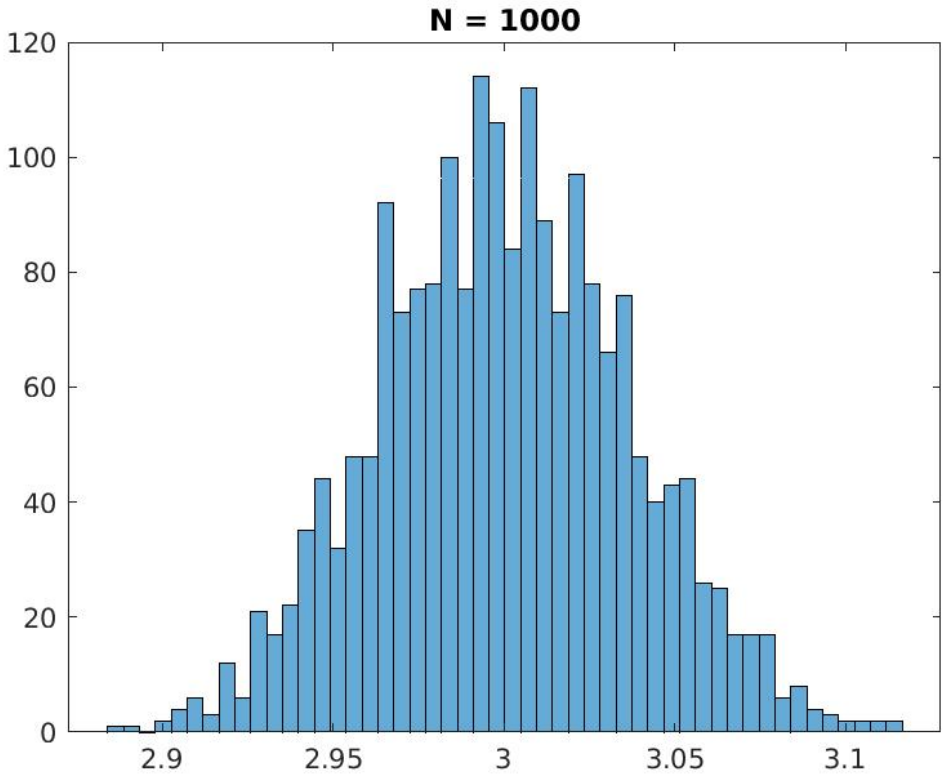
$\Rightarrow t = \ln(1+\delta)$

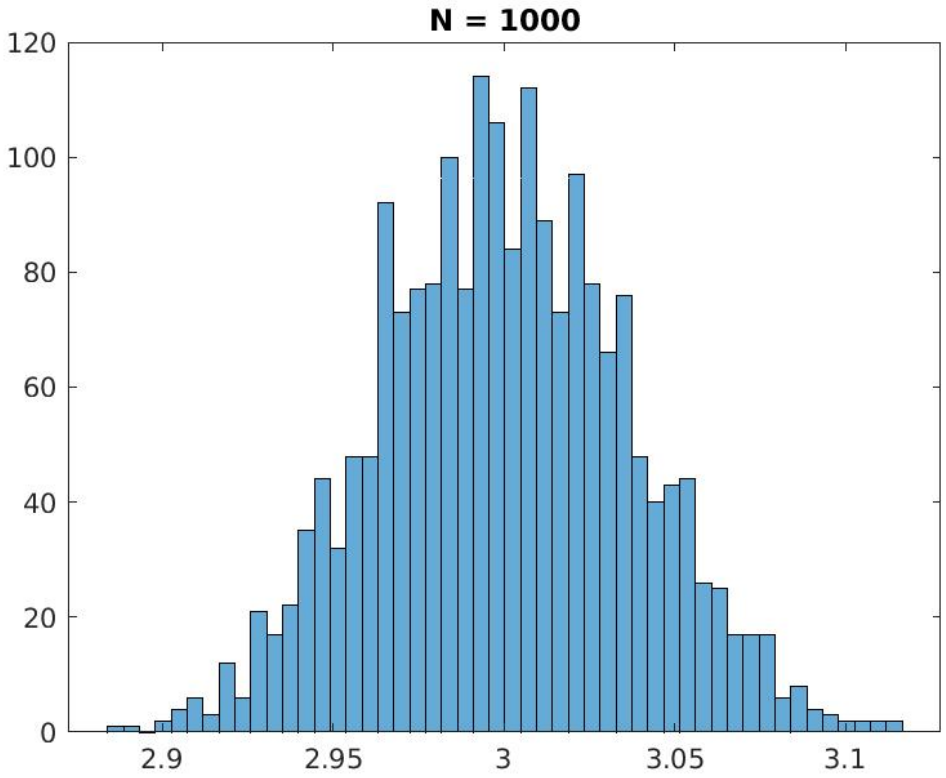
It is for minima as double derivative $= \mu e^t > 0$

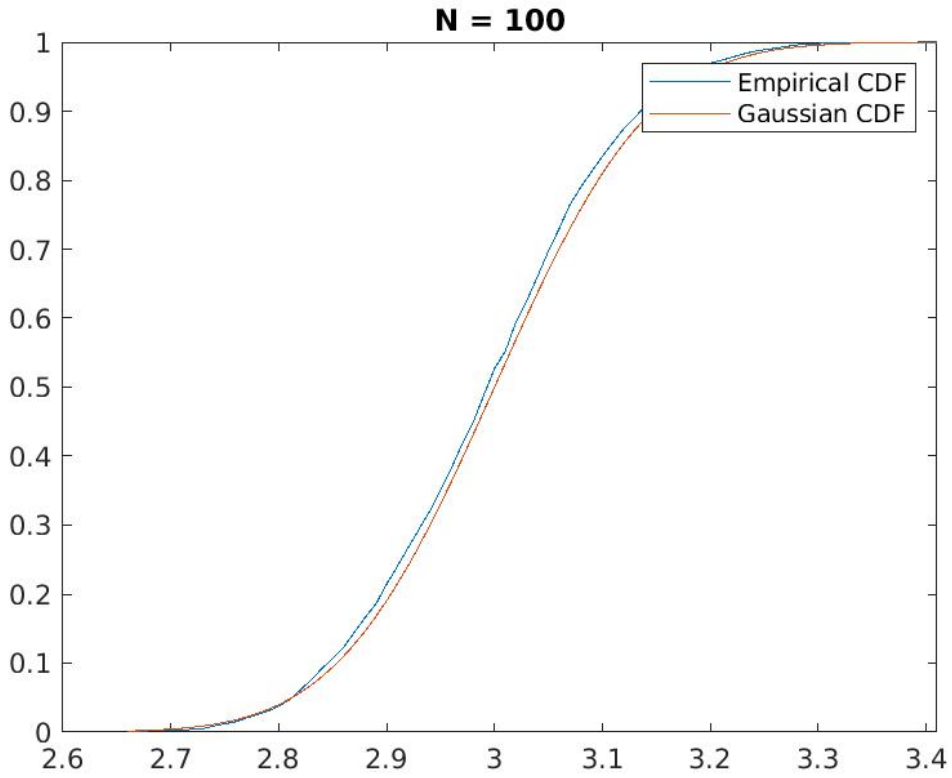
Question 5

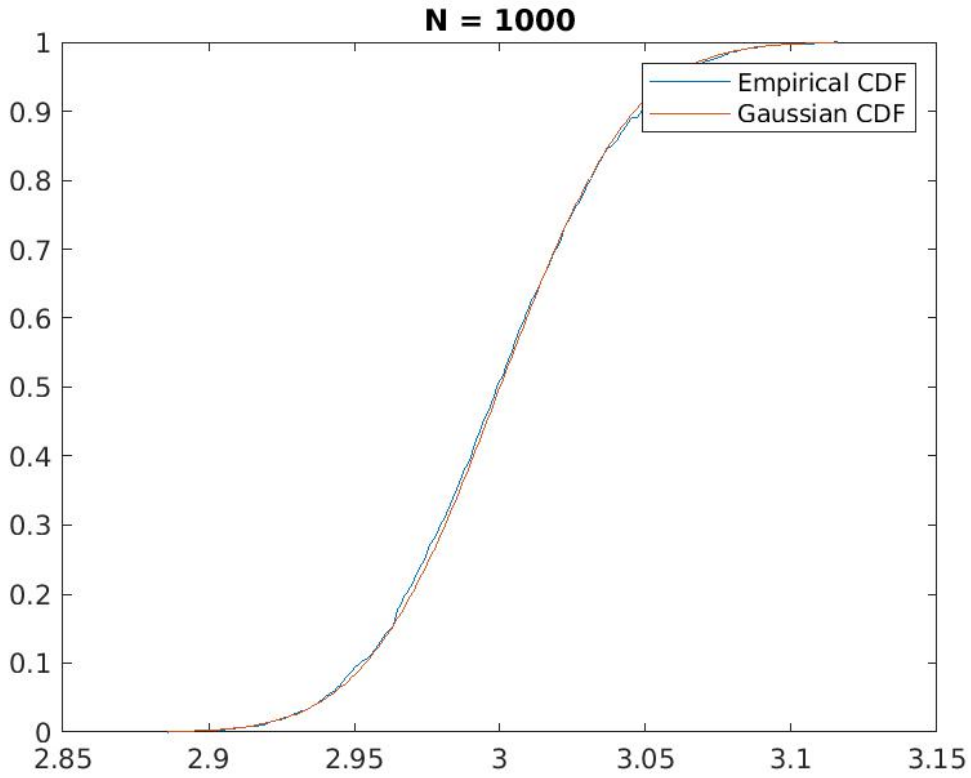
Code for this qsn is in file named 'q5.m'

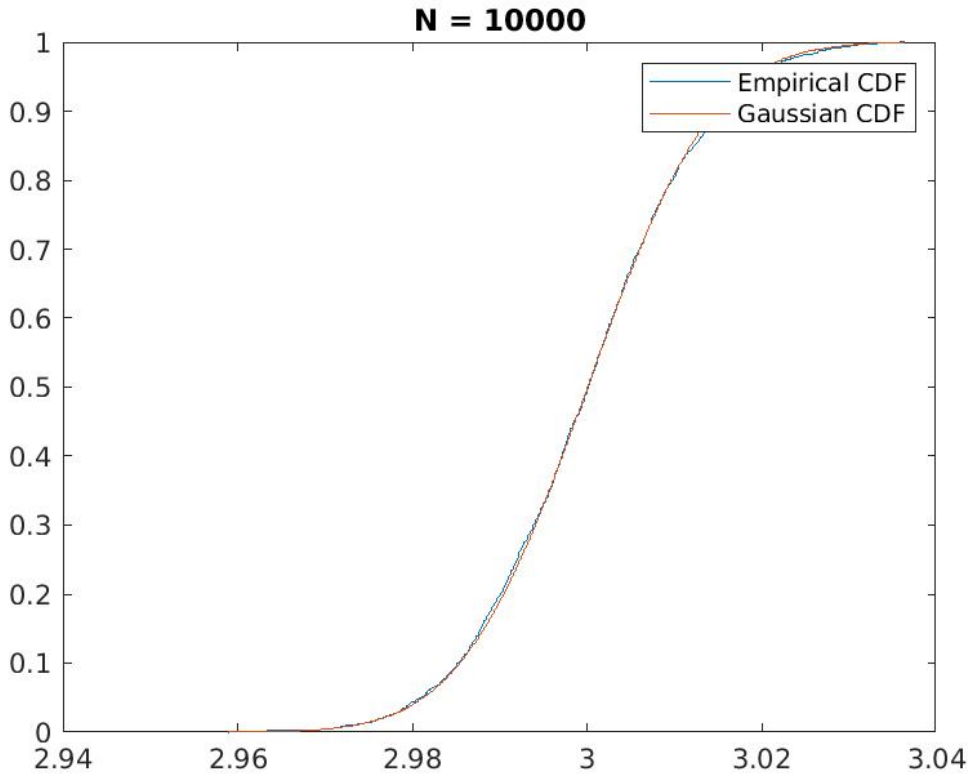


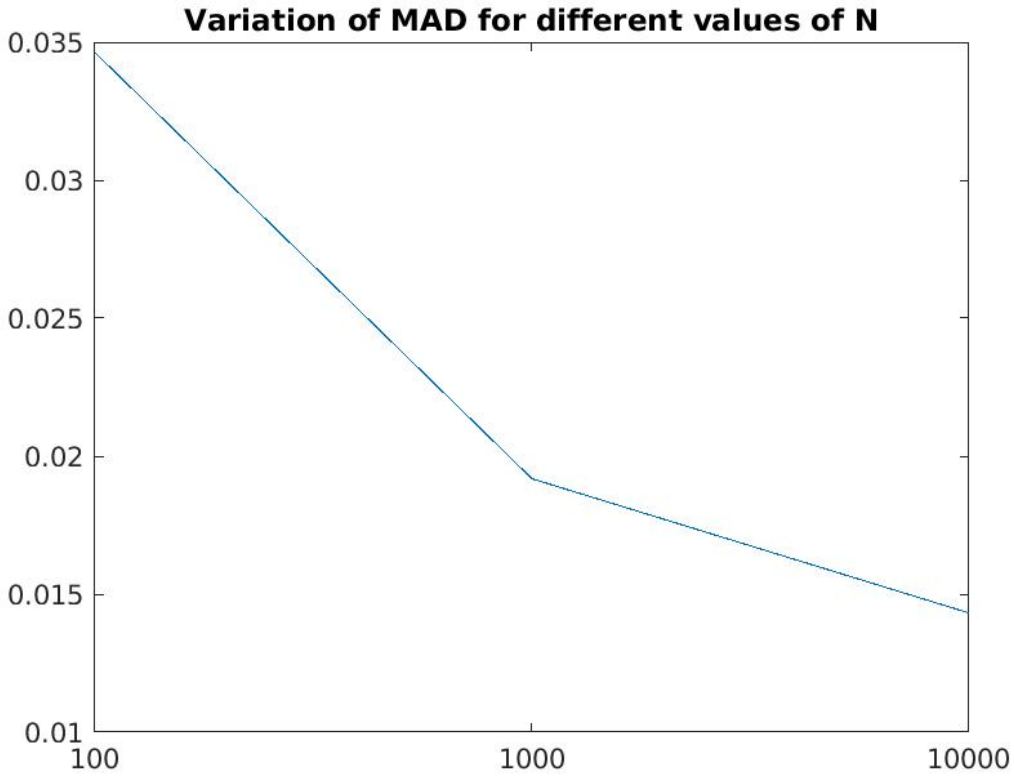

N = 1000
120
100
80
60
40
20
0
2.9
2.95
3
3.05
3.1


N = 1000
120
100
80
60
40
20
0
2.9
2.95
3
3.05
3.1


N = 100
Empirical CDF
Gaussian CDF
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2.6
2.7
2.8
2.9
3
3.1
3.2
3.3
3.4


N = 1000
Empirical CDF
Gaussian CDF
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2.85
2.9
2.95
3
3.05
3.1
3.15


N = 10000
Empirical CDF
Gaussian CDF
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2.94
2.96
2.98
3
3.02
3.04


Variation of MAD for different values of N
0.035
0.03
0.025
0.02
0.015
0.01
100
1000
10000

Question 6

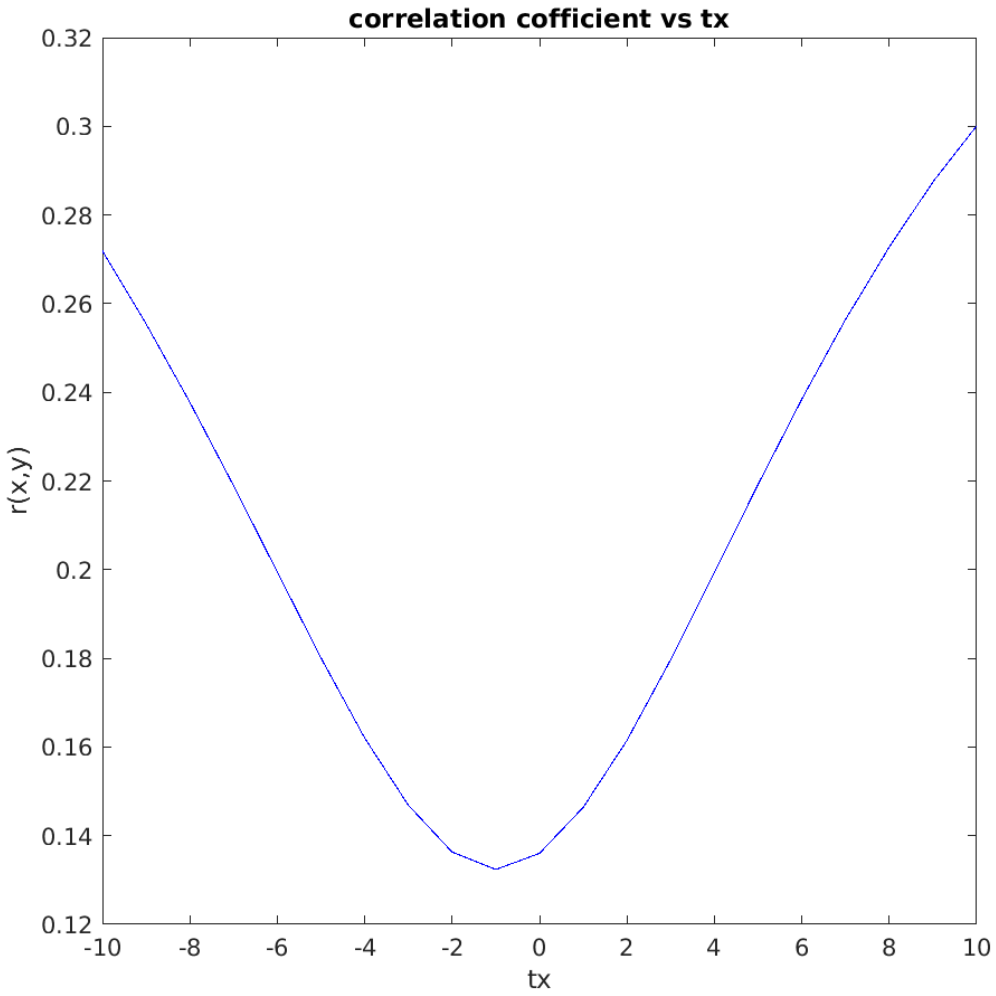


figure This plot correspond to T1.jpg

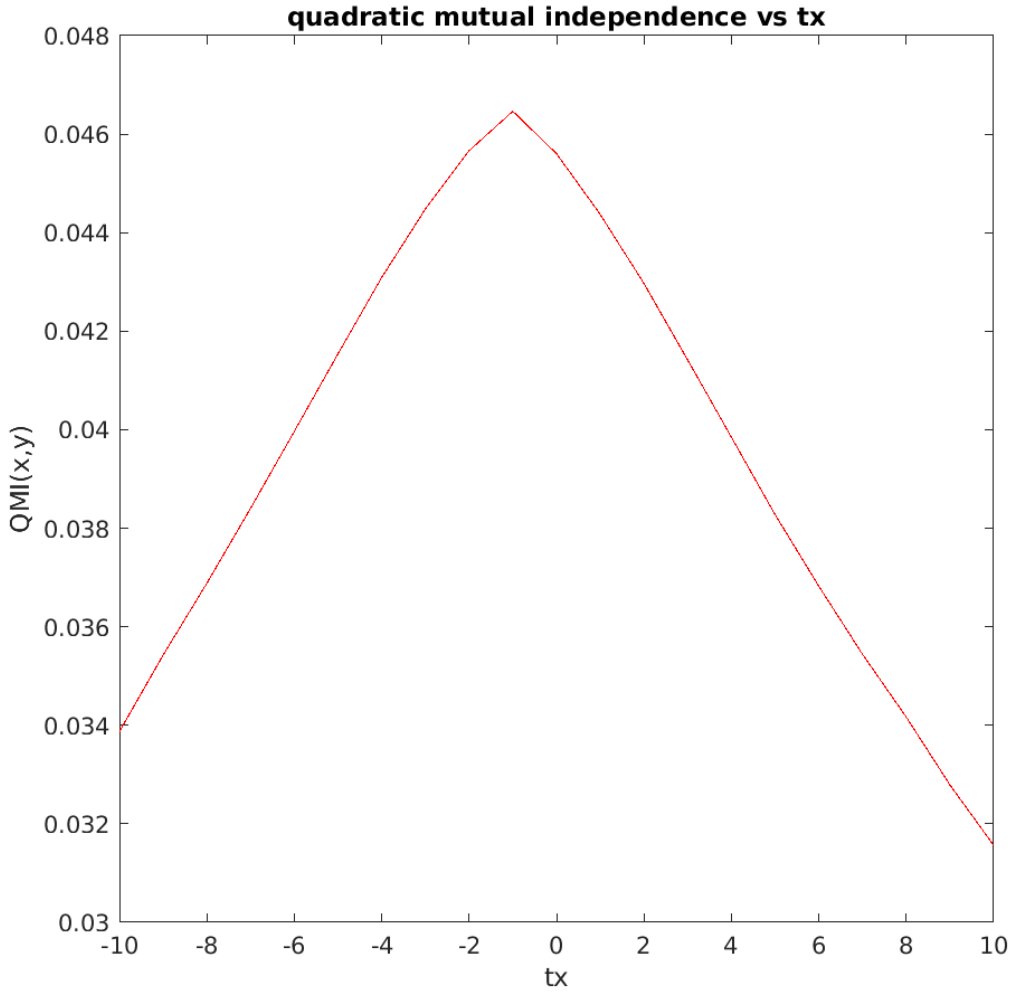


Figure 1: This plot correspond to T2.jpg

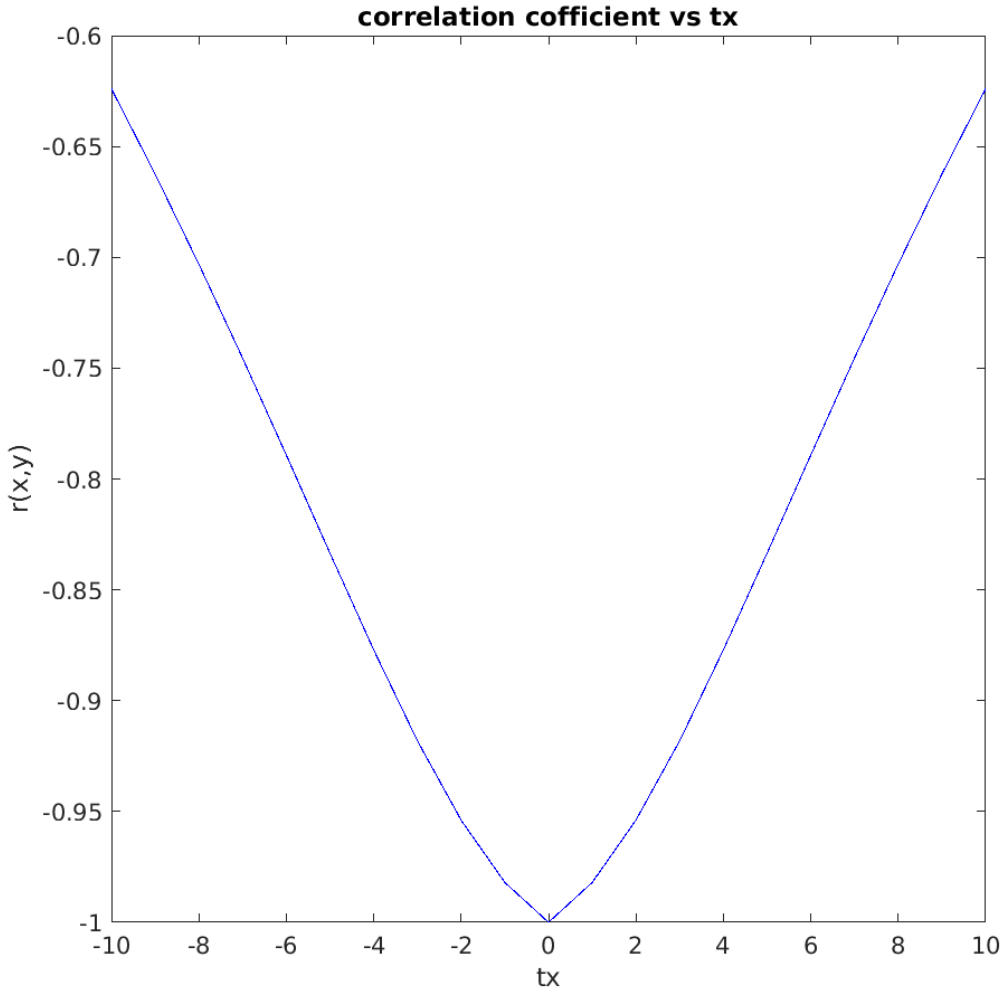
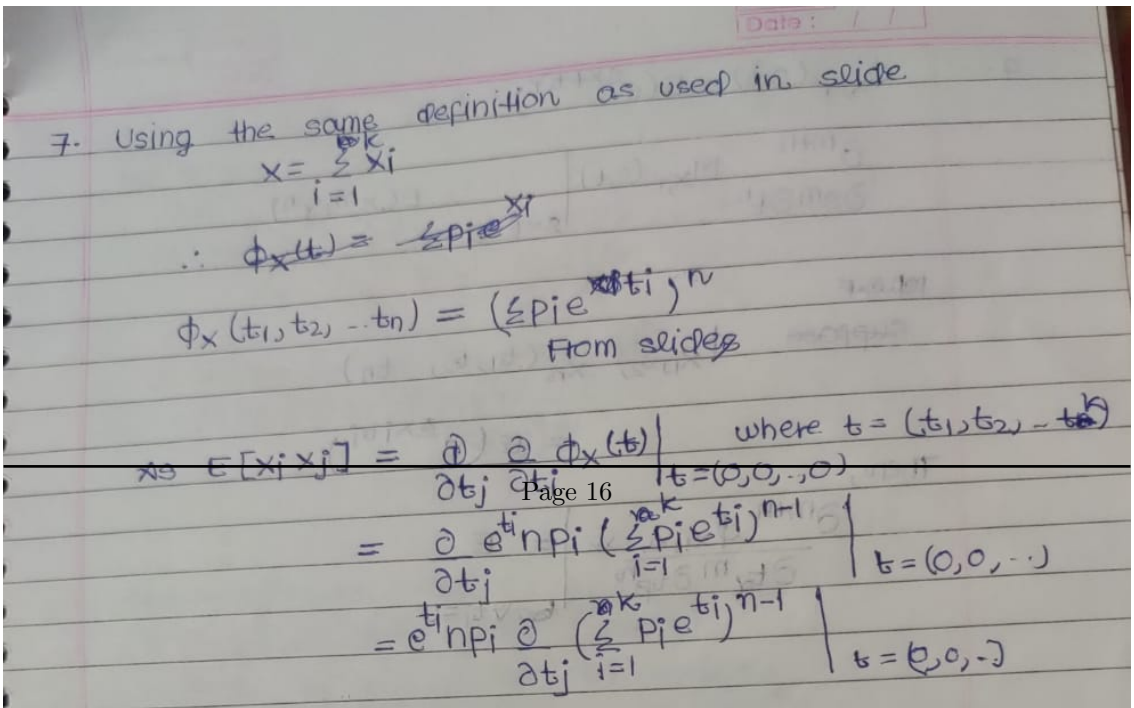


Figure 2: This plot correspond to T1.jpg and negative of T1.jpg

Question 7



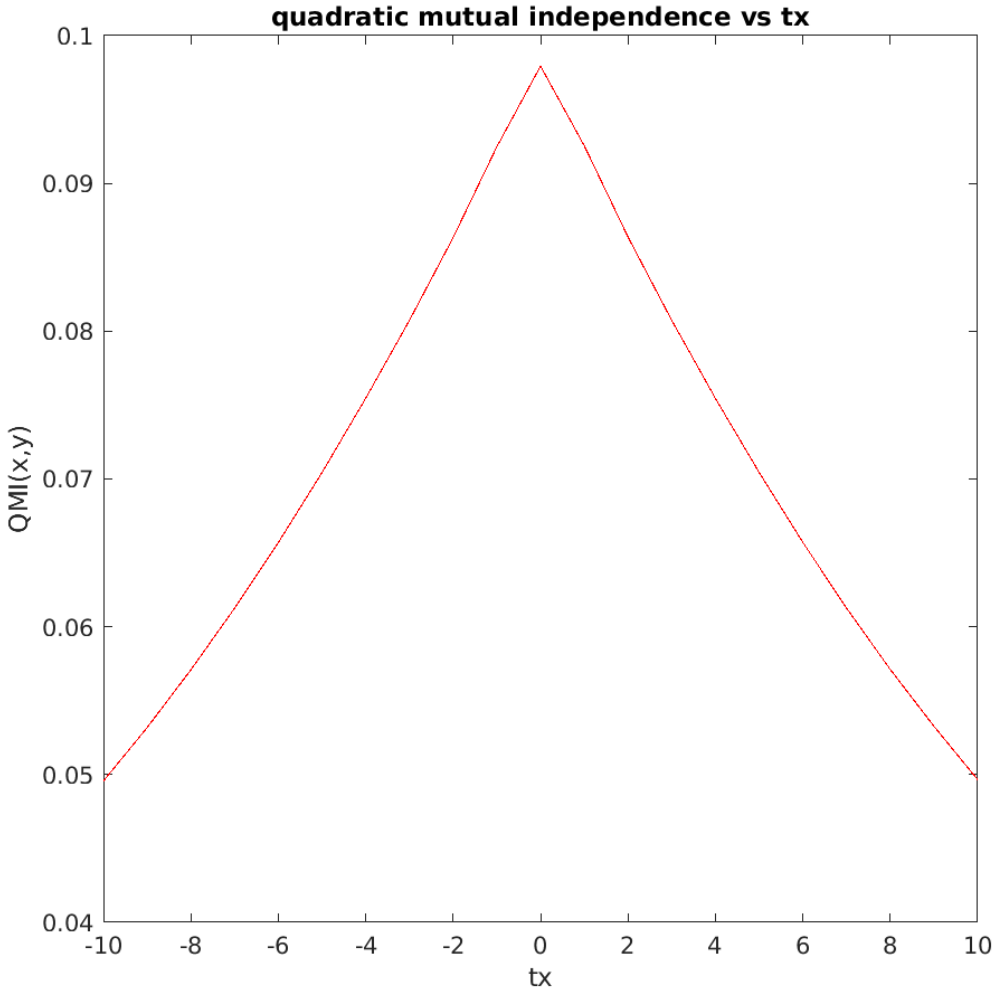


Figure 3: This plot correspond to T1.jpg and negative of T1.jpg