```
(***************************)
(* Kempe Burmester Problem *)
(***************************)

mat = {{1, 2, 3}, {4, 5, 6}, {7, 8, 9}}
{{1, 2, 3}, {4, 5, 6}, {7, 8, 9}}

f1 = x1; f2 = y1;

f3 = x2 - x1 - Cos[p1] * Lap; f4 = y2 - y1 - Sin[p1] * Lap;

f4
-y1 + y2 - Lap Sin[p1]

jacob = D[{f1, f2, f3, f4}, {{x1, y1, p1, x2, y2, p2}}]
{{1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0},
 {-1, 0, Lap Sin[p1], 1, 0, 0}, {0, -1, -Lap Cos[p1], 0, 1, 0}}

{{1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0},
 {-1, 0, Lap Sin[p1], 1, 0, 0}, {0, -1, -Lap Cos[p1], 0, 1, 0}}
{{1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0},
 {-1, 0, Lap Sin[p1], 1, 0, 0}, {0, -1, -Lap Cos[p1], 0, 1, 0}}

f5 = x3 - x1 - Cos[p1] * Lab; f6 = y2 - y1 - Sin[p1] * Lab;

f7 = x3 - x4 + Cos[p3] * Lbr - Cos[p4] * Ltr;
f8 = y3 - y4 + Sin[p3] * Lbr - Sin[p4] * Ltr;

f9 = x3 - x7 + Cos[p3] * Lbc - Cos[p7] * Ldc;
f10 = y3 - y7 + Sinh[p3] * Lbc - Sin[p7] * Ldc;

f11 = x5 - x7 + Cos[p5] * Ltq - Cos[p7] * Lqd;
f12 = y5 - y7 + Sin[p5] * Ltq - Sin[p7] * Lqd;

f13 = x7 - Lad; f14 = y7;

f15 = x6 - Las; f16 = y6;

f17 = x4 - x5; f18 = y4 - y5;
f19 = x5 - x6 - Cos[p6] * Lts; f20 = y5 - y6 - Sin[p6] * Lts;

f21 = x4 - x2 - Cos[p2] * Lpt; f22 = y4 - y2 - Sin[p2] * Lpt;
```

```
jacob = D[{f1, f2, f3, f4, f5, f6, f7, f8, f9, f10, f11, f12, f13,
   f14, f15, f16, f17, f18, f19, f20, f21, f22}, {{x1, y1, p1, x2, y2,
    p2, x3, y3, p3, x4, y4, p4, x5, y5, p5, x6, y6, p6, x7, y7, p7}}]
```

```
{{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {-1, 0, Lap Sin[p1], 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, -1, -Lap Cos[p1], 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {-1, 0, Lab Sin[p1], 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, -1, -Lab Cos[p1], 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 1, 0, -Lbr Sin[p3], -1, 0, Ltr Sin[p4], 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 1, Lbr Cos[p3], 0, -1, -Ltr Cos[p4], 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 1, 0, -Lbc Sin[p3], 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, Ldc Sin[p7]},
 {0, 0, 0, 0, 0, 0, 0, 1, Lbc Cosh[p3], 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -Ldc Cos[p7]},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -Ltq Sin[p5], 0, 0, 0, -1, 0, Lqd Sin[p7]},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, Ltq Cos[p5], 0, 0, 0, 0, -1, -Lqd Cos[p7]},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1, 0, Lts Sin[p6], 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1, -Lts Cos[p6], 0, 0, 0},
 {0, 0, 0, -1, 0, Lpt Sin[p2], 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, -1, -Lpt Cos[p2], 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

jacob // MatrixForm

1	0	0	0	0	0	0	0	0	0	0	0	0	(
0	1	0	0	0	0	0	0	0	0	0	0	0	(
-1	0	Lap Sin[p1]	1	0	0	0	0	0	0	0	0	0	(
0	-1	-Lap Cos[p1]	0	1	0	0	0	0	0	0	0	0	(
-1	0	Lab Sin[p1]	0	0	0	1	0	0	0	0	0	0	(
0	-1	-Lab Cos[p1]	0	1	0	0	0	0	0	0	0	0	(
0	0	0	0	0	0	1	0	-Lbr Sin[p3]	-1	0	Ltr Sin[p4]	0	(
0	0	0	0	0	0	0	1	Lbr Cos[p3]	0	-1	-Ltr Cos[p4]	0	(
0	0	0	0	0	0	1	0	-Lbc Sin[p3]	0	0	0	0	(
0	0	0	0	0	0	0	1	Lbc Cosh[p3]	0	0	0	0	(
0	0	0	0	0	0	0	0	0	0	0	0	1	(
0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	(
0	0	0	0	0	0	0	0	0	0	0	0	0	(
0	0	0	0	0	0	0	0	0	0	0	0	0	(
0	0	0	0	0	0	0	0	0	0	0	0	0	(
0	0	0	0	0	0	0	0	0	1	0	0	-1	(
0	0	0	0	0	0	0	0	0	0	1	0	0	-
0	0	0	0	0	0	0	0	0	0	0	0	1	(
0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	-1	0	Lpt Sin[p2]	0	0	0	1	0	0	0	(
0	0	0	0	-1	-Lpt Cos[p2]	0	0	0	0	1	0	0	(

```
Dimensions[jacob]
```

```
{22, 21}
```

```
MatrixRank[jacob]
```

```
21
```

```
Lap = 1.0; Lbp = 1.0; Lbr = 0.6614; Lrc = 0.6614;
Clear[Lap, Lbp, Lbr, Lrc]
```

```
Lap = 1.0; Ltq = 1.0; Lab = 2.0; Lbr = 0.6614;
Ltr = 1.25; Lts = 0.6614; Ldc = 2.0; Lqd = 1.0; Lpt = 1.0;
```

```
Dimensions[jacob]
```

```
{22, 21}
```

```
MatrixRank[jacob]
```

```
21
```

```
(*****************************)
(* Here Starts the Method-A+ *)
(*****************************)
```

```
pq1 = jacob[[{1, 2}, All]]; pqm1 = Drop[jacob, {1, 2}, 0];
```

```
r1 = MatrixRank[pq1]
```

```
2
```

```
rm1 = MatrixRank[pqm1]
```

```
20
```

```
(*Used the following bash script to generate formulaes:*)
(*For the 9th constraint, manually wrote the formula*)
(* (j=1;for i in $(seq 1 2 22);do printf'pq'$j'=jacob[[{'$i','$(($i+1))'},All]];';
  ((j++));done;) |oclip *)
```

```
pq1 = jacob[[{1, 2}, All]]; pq2 = jacob[[{3, 4}, All]]; pq3 = jacob[[{5, 6}, All]];
pq4 = jacob[[{7, 8}, All]]; pq5 = jacob[[{9, 10}, All]];
pq6 = jacob[[{11, 12}, All]]; pq7 = jacob[[{13, 14}, All]];
pq8 = jacob[[{15, 16}, All]]; pq9 = jacob[[{17, 18, 19, 20, 21, 22}, All]];
```

```
pq9 // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0.6614\,\text{Sin}[p6] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -0.6614\,\text{Cos}[p6] & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1.\,\text{Sin}[p2] & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1.\,\text{Cos}[p2] & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
(*Again a Bash script for pqm's. And Manual for pqm9*)
(*(j=1;for i in $(seq 1 2 22);do echo'pqm'$j'=Drop[jacob,{'$i','$(($i+1))'},0];';
  ((j++));done;)|oclip*)

pqm1 = Drop[jacob, {1, 2}, 0];
pqm2 = Drop[jacob, {3, 4}, 0];
pqm3 = Drop[jacob, {5, 6}, 0];
pqm4 = Drop[jacob, {7, 8}, 0];
pqm5 = Drop[jacob, {9, 10}, 0];
pqm6 = Drop[jacob, {11, 12}, 0];
pqm7 = Drop[jacob, {13, 14}, 0];
pqm8 = Drop[jacob, {15, 16}, 0];
pqm9 = Drop[jacob, {17, 22}, 0];

(*Now bash script for printing formulaes for ri's and rmi's *)
(* for i in $(seq 9);do echo "r$i"'=MatrixRank[pq'$i'];
 rm'$i'=MatrixRank[pqm'$i'];'; done; *)

r1 = MatrixRank[pq1]; rm1 = MatrixRank[pqm1];
r2 = MatrixRank[pq2]; rm2 = MatrixRank[pqm2]; r3 = MatrixRank[pq3];
rm3 = MatrixRank[pqm3]; r4 = MatrixRank[pq4]; rm4 = MatrixRank[pqm4];
r5 = MatrixRank[pq5]; rm5 = MatrixRank[pqm5]; r6 = MatrixRank[pq6];
rm6 = MatrixRank[pqm6]; r7 = MatrixRank[pq7]; rm7 = MatrixRank[pqm7];
r8 = MatrixRank[pq8]; rm8 = MatrixRank[pqm8];
r9 = MatrixRank[pq9]; rm9 = MatrixRank[pqm9];


rm8 = MatrixRank[pqm8];

pqm8 // MatrixForm
```

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1.\,\text{Sin}[p1] & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1.\,\text{Cos}[p1] & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2.\,\text{Sin}[p1] & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -2.\,\text{Cos}[p1] & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.6614\,\text{Sin}[p3] & -1 & 0 & 1.25\,\text{Sin}[p4] & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.6614\,\text{Cos}[p3] & 0 & -1 & -1.25\,\text{Cos}[p4] & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\text{Lbc}\,\text{Sin}[p3] & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \text{Lbc}\,\text{Cosh}[p3] & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1.\,\text{Sin}[p2] & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1.\,\text{Cos}[p2] & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}$$

```
MatrixRank[pqm8]
```

```
20
```

```
{r1, r2, r3, r4, r5, r6, r7, r8, r9}
```

```
{2, 2, 2, 2, 2, 2, 2, 2, 6}
```

```
{rm1, rm2, rm3, rm4, rm5, rm6, rm7, rm8, rm9}
```

```
{20, 20, 20, 20, 20, 20, 19, 20, 16}
```