Probability-Stochastic

Probability is a tool to understand randomness

Random Experiment is a experiment whose outcome is unknown

Assumption: all possible outcomes of an random experiment is known

the elements of the sample space need to be exclusive, exhaustive and equally likely

Betrand Paradox(Showing inadequacy of what we learnt in 12,10):

two circles one inside other , r1=2, r2=1;, prob. that a randomly drawn chord intersects inner circle?

answer: when midpoint of that chord is inside inner circle; sample space of points in inner circle is infinite, hence; infinite sample space

p(A) = area of inner circle/ area of outer circle = 1/4

also , if we take a point on outer circle and draw a chord from there then; $-\pi/2 < \theta < \pi/2$, for outer circle

for inner circle, theta of chord will vary only ; $-\pi/6 < \theta < \pi/6$

then $P(A) = \pi/6 \div (\pi/2) = 1/3$, which is different from our previous result.

Kolmogorov axioms of probability(Born from pascal)

$\sigma - algebra$: is a collection of subsets of a sample space(Ω) called (U) if the following properties:

if A is an event of U then A compliment is also event of U

if A ⊂ U then A is an event

Every subset of sample space Ω is not an event

Axioms of Prob.(Ω, U given)

Prob. is a function: P such that it takes a U σ algebra and maps it between 0 and 1

P:U→ [0, 1]

which follows following rules:

1)P(ϕ or null set) = 0, P(Ω) =1

2)Let A1, A2..An is all belongs to U then $P(union of A1, A2..An)$ ≤ $\sum_{i=1}^{n}(prob. of A1, A2, ...An)$

3) if A1, A2....An are mutually disjoint then $P(union of A1, A2..An)$ = $\sum_{i=1}^{n}(Prob. A1, A2..An)$

(P, Ω, U) **form probability space**

Chevelier de mere's problem(gave birth to prob. theory modern)

which has higher probability?

1. getting a 6 in 4 throws of a single die
2. a double 6 in 24 throws in a pair of die

$\Omega 1$ = {1,2,3,4,5,6}, $\Omega 2$ = { (1,1), (1,2)....36 elements}

$\sigma - algebra$ will be subset of these

$A_i = (no. 6 doesn't appear in ith roll of die)$

$P(A_i) = 5/6$, since every throw is an independent event then;-

$P(\hat{A}) = 1 - (5/6)^4 = 51.8\%$; Probability of getting a 6 in 4 throws

Bi ={ a double 6 doesn't show up in ith roll of two die) = 35/36

$P(\hat{B}) = 1 - (35/36)^{24} = 49.1\%$

Buffon's needle Problem

5 parallel lines on a floor 2 inch apart each, a needle thrown, what is the probability that needle will intersect

or touch one of the lines?

1st step : Construct Probability space

https://www.youtube.com/watch?v=sJVivjuMfWA

Stochastic Differential Equation

{X(t) ; t≥0} { S(t), 0≤t≤T }, S1 and S2 are different random variables

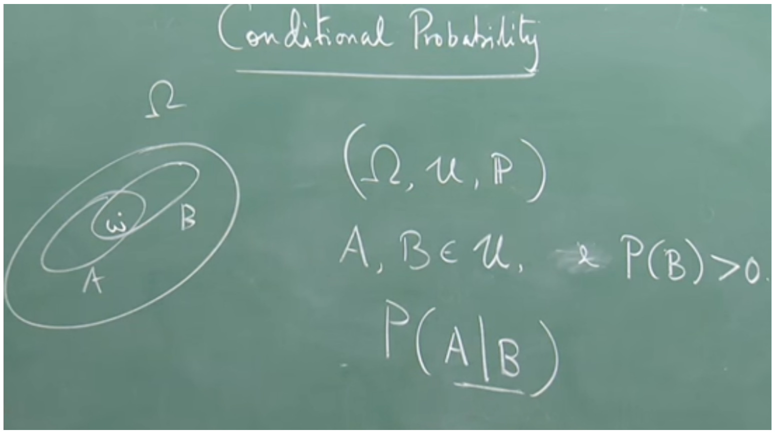
S(t), price of one share at time t

Assuming the same scenario and market conditions where market will go? { X(t)(w) }, fixed w changing t

$E[X] = \int_{\omega}(X dp)$. : Expectation
$V[X] = \int_{\omega}(X - E[X])^2 dp$: Variance

$F_X(x) = P(X \leq x) = P(X^{-1}(-\infty, x)) = \int_{-\infty}^{x} f_X(x)dx$: f(x) is density function, F(X) is cumulative density function
$=> E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$

Conditional Probability



Our sample space becomes B when we are talking about conditional probability of A given B happened

$$P(A/B) = P(AB)/P(B)$$

if A and B are independent that Means there is no overlap and P(A) doesn't depend on B

$$=> P(A/B) = P(A)$$

$$=> P(AB) = P(A) * P(B)$$

Law Of Large Numbers and central limit Theorem

Suppose you want to calculate height of male members of a population, not possible to go every home, so you do small number of random experiment to calculate height of male members

I.I.d random variables(independently and identically distributed)

$$X1, X2, X3 ... X_n$$

$$i.i.d => F_{X1}(x) = F_{X2}(x) = ...F_{Xn}(x)$$

let X1, X2, ...Xn be a sequence of i.i.d random variables defined on the same probability space. Let $E[X_i] = m$

Then $P(lim_{n->\infty} \frac{X1+X2+X3+..Xn}{n} = m) = 1$

Bernouli Random Variable(binomial distribution)

X be a random variable which takes values 0 or 1

X1, X2, X3...Xn.   i.i.d and bernouli random variable

$P(x_i = 1) = p$

$p(x_i = 0) = q$

$p + q = 1$

Mean = np

variance = npq

Laplace De - Moivre theorem

X1, X2, X3...be i.i.d bernouli random variables

$S_n = X1 + X2 + X3....X_n$

then for any $-\infty < a, b < \infty$

$limit_{n->\infty} P(a <= \frac{S_n - np}{\sqrt{npq}} <= b) = Normal Prob. Density function = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$

above is normalized random variable which as we increase, **n will start behaving as standard normal**

$limit_{n->\infty} P(a <= \frac{S_n - n\mu}{\sigma\sqrt{n}} <= b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$, generalized

Cheybyshev Inequality, Borel-Cantelli lemmas

Cheybyshev Lemma

Let X be a random variable then for λ>0, $1 < p < \infty$

$$=> P(|X| >= \lambda) <= \frac{1}{\lambda^p} E(|X|^p)$$

Borel-Cantelli lemmas

$$\sum_{n=1}^{\infty} P(A_n) < \infty => P(A_n, i.o) = 0$$

if the left side happens⇒Null set

$$A_{n,i.o} \subset U_{m=n}^{\infty} A_m$$

Convergence in Probability

$X_n \to X$ in probability if

$$limit_{n->\infty} P(|X_n - X| > \epsilon) = 0$$

Conditional Expectation-I

(Ω, F, P) be a probability space

F is a power set of Ω, $F = 2^{\Omega}$

$$A \epsilon F$$

$$E[X|A] = \sum_{\omega \epsilon \Omega} X(w) P(w/A)$$

$$=> E[X|A] = \frac{1}{P(A)} \sum_{\omega \epsilon \Omega} X(w) P(w)$$

for every w in A we can define as

$$E[X|A] = \frac{1}{P(A)} \int_A X dP$$

Martingales

Filtration: Subset of σ algebra is revealed as we get new information in random experiment eg: we throw a fair coin there are 8 possibilities, if 1st coin is told to be head ..we can know which subset of outcomes we are in....

$$\Omega = HHT, HHH, HTH, HTT, TTH, THH, THT, TTT$$

$$A_H = HHT, HHH, HTT, HTH$$

$$A_T = TTH, THH, THT, TTT$$

when we know 1st is head out sigma algebra is revealed, and it's compliment is revealed

$F_1 = [A_H, A_T, \phi, \Omega], F_1 \subset F$ is itself a sigma -algebra

if 2 tosses are known:

$$A_{HH} = HHH, HHT$$

$$A_{HT} = HTH, HTT$$

$$A_{TH} = THH, THT$$

$$A_{TT} = TTH, TTT$$

$$F_2 = [A_H, A_T, \phi, \Omega, A_{HH}, A_{HT}, A_{TH}, A_{TT}, UNIONOFA_{HH}WITHOTHERS]$$

$F_1 \subset F_2 \subset F_3 = F$, definition of filtration

for $0 < t < T$, suppose there exist a sigma-algebra $F_t \subset F$ and whenever $S <= t$, F(s) will be contained in F(t), then F(t) is called **filtration associated with probability space**

$X(t) \rightarrow$ stochastic process, adapted to filtration $F(t)$ if $X(t)$ is $F(t)$ - measurable

$[F_n]_0^N$ is given filtration and $[X_n]_0^N$ is stochastic process adapted to filtration

X_n will be called a **discreet martingale**, if:

$E[X_{n+1}/F_n] = X_n$, expectation at (n+1)th time is same as nth time

$E[X_{n+2}/F_{n+1}] = X_{n+1}$

$E[E[X_{n+2}/F_{n+1}]/F_n] = X_n$

$E[X_{n+2}/F_n] = X_n$, by **tower law** as **F_n is larger sigma algebra than** F_{n+1}

continuous martingale:

$E[X(t)/F(s)] = X(s), s <= t$

$E[X_n] = \int_\Omega X_n dP = \int_\Omega E[X_{n+1}/F_n] dP = \int_\Omega X_{n+1} dP = E[X_{n+1}]$

$=> E[X_0] = E[X_1] = E[X_2]........$

Brownian Motion-1

how do we model sample zig zag path?Brownian motion is not differentiable so no derivative rather we will deal with integrals

symmetric random walk(Drunkards walk):stochastic process generated by repeated tosses of a fair coin

$\omega = H, T, T, H, H, H, T, T, H, T, H......$

$\bar{\omega} = T, T, H, T, T, T, H, H, T.....$

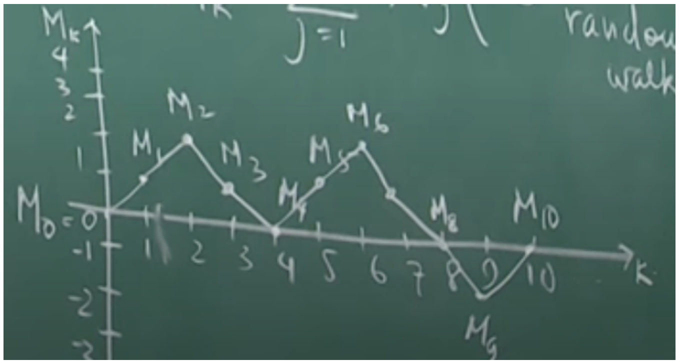
TWO different scenarios

define a random variable $X_j = [1, \omega_j = H; -1, \omega_j = T]$

$E(X_j) = 0 = (1 * 1)/2 + (1 * -1)/2$

$Var(X_j) = 1 = (1 * 1/2) + (1 * 1/2)$

stochastic process $M_k = \sum_{j=1}^{k} X_j$, $\quad M_0 = 0$, has independent increments



Properties:

$$E(M_{k_{i+1}} - M_{k_i}) = 0$$

$$Var(M_{k_{i+1}} - M_{k_i}) = K_{i+1}$$

`symmetric random walk is a discrete` **`martingale`**, so

$$E[M_i/F_k] = E[(M_i - M_k) + M_k|F_k]$$
$$= E[(M_i - M_k)/F_k] + E[M_k|F_k]$$

$M_l - M_k$ is independent of F_k

$$= E[M_i - M_k] + M_k E[1/F_k]$$
$$= 0 + M_k * 1 = M_k$$

Scaled random walk; Nth level approximation of brownian motion

$W^n(t) = \frac{1}{\sqrt{n}} M_{nt}$, where nt is an integer

as $n->\infty, W^n(t)->W(t)$

when t is not an take two nearest points u,s on both sides of 't' such that tu and ts is an integer

and then compute the approximation for nu, ns

Brownian Motion 2

A stochastic process $[W(t)]_{t>=0}, with, w(0) = 0$, w(0) is itself a random variable

a brownian motion if any give point in time, the increments:

$[W(t_1) - W(t_0)], [W(t_2) - W(t_1)]$........$[W(t_n) - W(t_{n-1})]$

are independent random variables, $W(t_{i+1}) - W(t_i)$ follows normal distribution with $N(0, t_{i+1} - t_i)$

Brownian motions can take -ve values, geometric brownian motion solves this(using exponential function)

$F(t)_{t>=0}]$ is a collection of σ algebra such that;

1. $0 <= s <= t, F(s) \subset F(t)$
2. $[W(t)]W(t)]$ must be adapted to filtration
3. $0 <= t <= u$ then $[W(u) - W(t)] is independent of F(t)$

$F(t)$ can also be viewed as smallest sigma algebra generated by $W(t)$

Exponential Martingale(Used in Finance)

$$Z(t) = \exp[\sigma W(t) - \frac{1}{2}\sigma^2 t], \sigma > 0$$

inside square bracket is itself a random variable

$W(t) = W(t) - W(0)$, W(0) belons to normal dist.

can be computed by;

$[a_1 <= W(t) <= b_1] = \frac{1}{\sqrt{2\pi t}} \int_{a_1}^{b_1} \exp^{-x^2/2t} dt$

suppose $W(t_1) = x_1, a_1 <= x_1 <= b_1$

then conditional probability

$P((a_2 <= W(t_2) <= b_2)/(W(t_1) = x_1)) = \frac{1}{\sqrt{2\pi(t_2-t_1)}} \int_{a_2}^{b_2} \exp^{-(x_2-x_1)^2/2(t_2-t_1)} dt$

$P[a_1 <= W(t) <= b_1], [(a_2 <= W(t_2) <= b_2)] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} g(x_1, t_1|0) g(x_2, t_2 - t_1|x_1) dx_2 dx_1$

Transitional density function : $g(x,t|y) = \frac{1}{\sqrt{2\pi t}} \exp^{-(x-y)^2/2t}$

suppose $[W(t_2) - X_1] = Z$, a random variable then $W(t_2)$ follows $N(x_1, (t_2 - t_1))$

Brownian Motion 3

Quadratic Variation of Brownian motion [W(t)]

[0,T] take n large , step length - T/n, steps are not of equal size and $T_n = max$

$f = \sum_0^{n-1} [W((J+1)T/n) - W(jT/n)]^2$

$limit_{n->\infty} f = T$

First Variation of the function = $FV_T(f) = lim_{n->\infty} \sum_0^{n-1} |F(t_{J+1}) - F(t_j)|$
$= lim_{n->\infty} \sum_0^{n-1} |F'(x_j^*)(t_{j+1} - t_j)|, x_j^* \subset (t_{j+1}, t_j)$
$= \int_0^1 |F'(t)| dt$

it's basically counting the ups and downs

2nd Variation of the function $= [f,f](T) = lim_{n->\infty} \sum_0^{n-1} |F(t_{j+1}) - F(t_j)|^2$
$= lim_{n->\infty} \sum_0^{n-1} |f'(t_j^*)|^2 |(t_{j+1} - t_j)|^2, t_j^* \subset (t_j, t_{j+1})$

$$<= ||T_n|| lim_{n->\infty} \sum_0^{n-1} |f^{'}(t_j^*)|^2 |(t_{j+1} - t_j)|$$
$$= ||T_n|| lim_{n->\infty} \int_0^T |f^{'}(t_j^*)|^2 dt = 0$$

$[f, f](T) = 0$, sum of squares of ups and downs is zero

<u>$Theorem, [W, W](T) = T$**, almost always surely**</u>

Ito Integral-1

$\int_0^T \Delta(t) dW(t), \Delta(t)$ is a process adapted to filtration $F(t)$ associated with $W(t)$

$\Delta(t),$ is a simple process, on each interval, $[t_j, t_{j+1})$-(step function) is constant

$\Delta(t)$;no. of stocks i am holding at time t

$I(t)$;gain by trading at time t

where, W(t) represents price at time t

$I(t) = W(t)\Delta(t_0) - W(t_0)\Delta(t_0), 0 <= t <= t_1$

$I(t) = W(t_1)\Delta(t_0) + \Delta(t_1)W(t) - W(t_1)\Delta(t_1), t_1 <= t <= t_2,$

sold and bought again with earned money at t_1

for $t_k <= t <= t_{k+1}$

$I(t) = \sum_0^{k+1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]$

$I(t) = \int_0^t \Delta(t) dW(t)$, $I(t)$,<u>**it self is a stochastic process**</u>(ito integral of stochastic process), every step we are generating stochastic process

<u>**Properties**</u>

1. ito integral is a martingale

2. ito isometry: $E(I^2(t)) = E\int_0^t \Delta^2(u)du$, this integral is not a process, it's a general integral
3. ito integral itself has quadratic variation, $[I,I](t) = \int_0^t \Delta^2(u)du$, it is a stochastic process

$dW(t)dW(t) = t$ or $[W,W](t) = t$, is a shorthand

Ito integral 2

$[\Delta_n(t)]$

as $n \to \infty$, $limit_{n\to\infty}\Delta_n(t) \approx \Delta(t)$

meaning, $limit_{n\to\infty}E(\int_0^T |\Delta_n(t) - \Delta(t)|^2 dt)$

Expectation of distance between $\Delta_n(t), \Delta(t)$ is 0

$\int_0^t \Delta(u)dW(u) = lim_{n->\infty}\int_0^t \Delta_n(u)dW(u)$

Properties

1. $I(t)$ is continuous for any t and given scenario ω
2. $I(t)$ is $F(t)$ measurable
3. $I(t)$ is a martingale
4. $\int_0^t(\Delta(u) + F(u))dW(u) = \int_0^t \Delta(u)dW(u) + \int_0^t F(u)dW(u)$
5. Ito Isometry
6. ito integral is a stochastic process
7. $[I,I](t) = \int_0^t \Delta^2(u)du$, this is itself a stochastic process

Computing Ito integral:

$\int_0^t W(t)dW(t)$, 0→T is divided into n intervals not equal size , T/n

$\Delta_n(t)$ ={ $W(0) = 0, 0 \leq t \leq t/n$;

$W(T/n), T/n \leq t \leq 2T/n$;

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.

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$W((n-1)T/n); (n-1)/n \leq t \leq T$

}

$limit_{n \to \infty} E(\int_0^T |\Delta_n(t) - W(t)|^2 dt) = 0$,

what is expected value of $W(t)$!!, as n will be very large the distance between two points on a stock chart will be very small

$\int_0^t W(t) dW(t)$=$lim_{n \to \infty} \int_0^t \Delta_n(t) dW(t)$

$= lim_{n \to \infty} \sum_{j=0}^{n-1} W(jT/n)[W((j+1)T/n) - W(jT/n)]$

Let, $W_j = W(jT/n)$

$W_{j+1} = W((j+1)T/n)$

$= limit_{n->\infty} \sum_0^{n-1} W_j (W_{j+1} - W_j)$

Let's start with this;

$1/2 \sum_0^{n-1} (W_{j+1} - Wj)^2 = 1/2 \sum_1^n W_k^2 - \sum_0^{n-1} W_j W_{j+1} + 1/2 \sum_0^{n-1} W_j^2$

$\Rightarrow (1/2) W_n^2 + (1/2) \sum_0^{n-1} W_k^2 - \sum_0^{n-1} W_j W_{j+1} + 1/2 \sum_0^{n-1} W_j^2$

$\Rightarrow (1/2) W_n^2 + \sum_0^{n-1} W_j^2 - \sum_0^{n-1} W_j W_{j+1}$

$\Rightarrow (1/2) W_n^2 + \sum_0^{n-1} W_j (W_{j+1} - W_j)$

$\Rightarrow$ " $+$(Our integral that we wanted)

$\Rightarrow \sum_0^{n-1} W_j (W_{j+1} - W_j) = (1/2) W_n^2 + 1/2 \sum_0^{n-1} (W_{j+1} - Wj)^2$

as $n \to \infty \Rightarrow \int_0^t W(t)dW(t) = (1/2)W^2(T) - (1/2)[W,W](T)$

$= (1/2)W^2(T) - (1/2)(T)$

Ito Calculus-1

$df(W(t))/dt = f^{'}W(t).W^{'}(t)$

$df(W(t)) = f^{'}W(t).W^{'}(t)dt$

$\Rightarrow df(W(t)) = f^{'}W(t).dW(t)$

$df(W(t)) = f^{'}W(t).dW(t) + (1/2)f^{''}(W(t))dt$ **, Stochastic Differential Equation(SDE)**

$[W,W](t) = t \Rightarrow dW(t)dW(t) = t$

$dtdt = 0$

$dW(t)dt = 0$

SDE is shorthand(shortcut) of the following:

$f(W(t)) - f(W(0)) = \int_0^t f^{'}W(t).dW(t) + (1/2)\int_0^t f^{''}(W(t))dt$

$\Rightarrow f(W(t)) = f(W(0)) + \int_0^t f^{'}W(t).dW(t) + (1/2)\int_0^t f^{''}(W(t))dt$

Ito's Formula(ito doebui formula):

$f(t,x)$ is a. given function

$f_x(t,x), f_t(t,x), f_{xx}(t,x)$ exist finitely and is continuous

then for $T \geq 0$

$f(T,W(T)) = f(0,W(0)) + \int_0^T f_t(t,W(t))dt + \int_0^T f_x(t,W(t))dt + (1/2)\int_0^T f_{xx}(t,W(t))dt$

$f(W(T)) - f(W(0)) = \sum_0^{n-1}[f(W(t_{j+1})) - f(W(t_j))]$

$= \sum_0^{n-1}[f'(W(t_j))(W(t_{j+1}) - W(t_j))] + (1/2)\sum_0^{n-1} f''(W(t_j))[W(t_{j+1}) - W(t_j)]^2$

$= \sum_0^{n-1} W(t_j)[W(t_{j+1}) - W(t_j)] + (1/2)\sum_0^{n-1}[W(t_{j+1}) - W(t_j)]^2$

as $||\pi|| \to 0$,

$$f(W(T)) - f(W(0)) = \int_0^T W(t)dW(t) + (1/2)[W,W](T)$$
$$= \int_0^T W(t)dW(t) + (1/2)T$$
$$= \int_0^T f' W(t)dW(t) + (1/2)\int_0^T f''(W(t))dt$$

`We have proved SDE for` $f(x) = (1/2)x^2$

Ito Calculus-2

I**to Process:** any stochastic process is ito process if it can be divided into 3 parts;

1. 0th value of the process
2. an Ito integral
3. an ordinary integral

$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du$

$\Delta(t)$ and $\Theta(t)$ are processes adapted to filtration $F(t)$ associated with brownian motion

Shorthand form(SDE form):

$dX(t) = \Delta(u)dW(u) + \Theta(u)du$

$[X,X](t) = \int_0^t \Delta^2(u)du$

using shorthand;$dX(t)dX(t) = \Delta^2(t)dW(t)dW(t) + \Theta^2(t)dtdt + 2\Delta(t)\Theta(t)dW(t)$

$dX(t)dX(t) = \Delta^2(t)dt$

$I(t) = \int_0^t \Delta(u)dW(u)$

$R(t) = \int_0^t \Theta(u)du$, these are continuous in t

$X(t) = X(0) + I(t) + R(t)$

$\sum_0^{n-1}(X(t_{j+1}) - X(t_j))^2 \Rightarrow \sum_0^{n-1}(I(t_{j+1}) - I(t_j))^2 + \sum_0^{n-1}(R(t_{j+1}) - R(t_j))^2 + 2[I(t_{j+1}) - I(t_j))(R(t_{j+1}) - R(t_j)]$

as $||\pi|| \to 0$

$limit_{||\pi||\to 0} \sum_0^{n-1}(I(t_{j+1}) - I(t_j))^2 = \int_0^t \Delta(u)du$

$$\begin{aligned}
|\sum_0^{n-1}(R(t_{j+1}) - R(t_j))^2| &\leq \sum_0^{n-1} |R(t_{j+1}) - R(t_j)||R(t_{j+1}) - R(t_j)| \\
&\leq max|R(t_{j+1}) - R(t_j)| \sum_0^{n-1} |R(t_{j+1}) - R(t_j)| \\
&\leq max|R(t_{j+1}) - R(t_j)| \sum_0^{n-1} \int_{t_j}^{t_{j+1}} |\Theta(u)|du \\
&\leq max|R(t_{j+1}) - R(t_j)| \int_0^t |\Theta(u)|du
\end{aligned}$$

as $||\pi|| \to 0,\ \int_0^t |\Theta(u)|du \to 0$

$\Rightarrow limit_{||\pi||\to 0} \sum_0^{n-1}(R(t_{j+1}) - R(t_j))^2 \to 0$

integrating w.r.t an ito process:

$\int_0^t \Gamma(u)dX(u) = \int_0^t \Gamma(u)[\Delta(u)dW(u) + \Theta(u)du]$

$\Rightarrow \int_0^t \Gamma(u)\Delta(u)dW(u) + \int_0^t \Theta(u)\Gamma(u)du$

Ito Formula for ito process:

$f(T, X(T)) = f(0, X(0)) + \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))dX(t) + (1/2)\int_0^T f_{xx}(t, X(t))dX(t)dX(t)$

We can replace , $dX(t) = \Delta(u)dW(u) + \Theta(u)du$

and $dX(t)dX(t) = \Delta^2(t)dt$

Application to Ito Integral

$dX(t) = \Delta(u)dW(u) + \Theta(u)du$

$\Theta(u) \rightarrow Drift$

$\Delta(u) \rightarrow Volatility$

Geometric Brownian Motion

(Used in finance since stocks are non negative)

$S(t) = S(0)Exp[\sigma W(t) + (\alpha - (1/2)\sigma^2)t]$, α and σ are constants

Defining the ito's process:

$X(t) = \int_0^t \sigma dW(t) + \int_0^t (\alpha - (1/2)\sigma^2)dt, X(0) = 0$

SDE form; $=>$ $dX(t) = \sigma dW(t) + (\alpha - (1/2)\sigma^2)dt$

Quadratic variation; $dX(t)dX(t) = [X, X](t) = \sigma^2 dW(t)dW(t) = \sigma^2 dt$

Let , $S(t) = S(0)exp^{X(t)}$

$$= S(0)Exp[\sigma W(t) + (\alpha - (1/2)\sigma^2)t]$$

What sort of SDE does S(t) follows:

lets take , $f(x) = S(0)e^x, \quad f^{'}(x) = S(0)e^x, \quad f^{''}(x) = S(0)e^x$

applying ito's formula;

$$dS(t) = f(X(t)) = f^{'}(X(t))dX(t) + (1/2)f^{''}(X(t))dX(t)dX(t)$$
$$= S(0)e^{X(t)}dX(t) + (1/2)S(0)e^{X(t)}dX(t)dX(t)$$
$$= S(t)dX(t) + (1/2)S(t)dX(t)dX(t)$$
$$= S(t)(\sigma dW(t) + (\alpha - (1/2)\sigma^2)dt) + (1/2)S(t)\sigma^2 dt$$
$$= \alpha S(t)dt + \sigma S(t)dW(t)$$

$\Rightarrow dS(t) = \sigma S(t)dW(t) + \alpha S(t)dt$, `SDE`

Ito integral:

$I(t) = \int_0^t \Delta(s)dW(s)$, $I(0) = 0$, $E(I(0)) = 0$

$I(t)$ is a martingale

$E(I(t)) = 0$

$var(I(t)) = E[I^2(t)] - E[I(t)]$

$=> var(I(t)) = E[I^2(t)] = \int_0^t \Delta^2(u)du \quad \rightarrow$ `Ito's Isometry Formula`

Application to Ito Integral-2

Instantaneous Interest rate; IR at any point t

Vasicek's Model of Interest rate

$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t), \alpha, \beta, \sigma > 0$, No Drift Term

Closed form solution of above;

$R(t) = e^{-\beta(t)}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta(t)}) + \sigma e^{-\beta(t)} \int_0^t e^{\beta S} dW(s)$

Define $f(t, x) = e^{-\beta(t)}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta(t)}) + \sigma e^{-\beta(t)} x$

$X(t) = \int_0^t e^{\beta S} dW(s)$, $X(0) = 0$

$dX(t) = e^{\beta t} dW(t); Shorthand$

$f_t(t, x) = -\beta e^{-\beta(t)} + \alpha e^{-\beta(t)} + \sigma \beta e^{-\beta(t)} x$

$f_x(t, x) = \sigma e^{-\beta(t)}$

$f_{xx}(t, x) = 0$

Hence by Ito's Formula we have;

`All these inequalities are in almost surely/everywhere sense`

$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dW(t) + (1/2)f_{xx}(t, X(t))dX(t)dX(t)$

$= [\alpha - \beta f(t, X(t))]dt + \sigma dW(t)$

$f(0, X(0)) = R(0)$

$f(t, X(t)) = R(t)$

$X(t) = \int_0^t e^{\beta S} dW(s), X(0) = 0, E[X(t)] = 0$

$X(t)- > N(0, \frac{1}{2\beta}(e^{2\beta t} - 1)$

$E(R(t)) = e^{\beta t}R(0) + \frac{2}{\beta}(1 - e^{\beta t})$

$Var(R(t)) = \frac{\sigma^2}{2\beta}(1 - e^{2\beta t})$

This can give negative value which do not want, so next model

<u>Cox-Ingersoll-Ross(CIR Model)</u>

$dR(t) = (\alpha - \beta R(t))dt + \alpha\sqrt{R(t)}dW(t), \alpha, \beta, \sigma > 0$

This doesn't have a closed form solution, its difficult to compute the distribution

computing mean of R(t) using ito's calculus:

$f_t(t, x) = e^{\beta(t)}x$

$$\begin{aligned} df(e^{\beta t}R(t)) &= f_t(t, R(t))dt + f_x(t, R(t))dW(t) + (1/2)f_{xx}(t, R(t))dR(t)dR(t) \\ &= (\alpha e^{\beta t})dt + \sigma e^{\beta t}\sqrt{R(t)}dW(t) \end{aligned}$$

integrating both sides:

$e^{\beta t}R(t) = \int_0^t (\alpha e^{\beta t})dt + \int_0^t \sigma e^{\beta t}\sqrt{R(t)}dW(t)$

Ito Integral in Higher Dimension

$W(t) = [W_1(t), W_2(t), ...W_d(t)]$, Higher dimension brownian motion each following a different path

why we need higher dimension? looking at price of 2 stocks

$dW_i(t)dW_i(t) = dt$, Quadratic variation

$dW_i(t)dW_j(t) = 0, i \neq j$, Cross Quadratic

$X(t) = X(0) + \int_0^t \Theta_1(u)du + \int_0^t \Theta_{11}(u)dW_1(u) + \int_0^t \Theta_{12}(u)dW_2(u)$
$Y(t) = Y(0) + \int_0^t \Theta_2(u)du + \int_0^t \Theta_{21}(u)dW_1(u) + \int_0^t \Theta_{22}(u)dW_2(u)$

$[X, X](t) = \int_0^t [\sigma_{11}^2(u) + \sigma_{12}^2(u)]du$

$dX(t)dY(t) = [\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t)]dt$
$[X, Y](t) = \int_0^t [\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t)]dt$

2D Ito Formula

$f(t, x, y)$

assumption.. $[f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}]$ continuous and exists

$f_{xy} = f_{yx}$ symmetry

$f(t, X(t), Y(t)) = f_t(t, X(t), Y(t))dt + f_x(t, X(t), Y(t))dX(t) + f_y(t, X(t), Y(t))dY(t) + (1/2)f_{xx}(t, X(t), Y(t))dX(t)dX(t) + (1/2)f_{yy}(t, X(t), Y(t))dY(t)dY(t) + f_{xy}(t, X(t), Y(t))dX(t)dY(t)$

Levy's Theorem:How to recognize a brownian motion

$stochastic - process[M(t)]_{t \geq 0}$ and $[F(t)]$and $M(t)$is adapted to it

if $[M, M](t) = t$, M is a brownian motion

Two stocks example

$dS_1(t) = \alpha_1 S_1(t)dt + \sigma_1 S_1(t)dW(t)$

$dS_2(t) = \alpha_2 S_2(t)dt + \sigma_2 S_2(t)[\rho dW_1(t) + \sqrt{1 - \rho^2}dW_2(t)], \rho \in [-1, +1]$

$if....W_3(t) = [\rho W_1(t) + \sqrt{1-\rho^2}W_2(t)].....then$

$dW_3(t) = [\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)]$

if replace in the 2nd equation then my stock price will depend on a new brownian motion $W_3(t)$

$dW_3(t)dW_3(t) = dt$

$[W_3, W_3] = t$, hence W_3 is a brownian motion

$dS_2(t) = \alpha_2 S_2(t)dt + \sigma_2 S_2(t)dW_3(t)$, The correlation between W_1 and W_3 is not zero

we can show that$E[W_1(t)W_3(t)] \neq 0 = \rho t$

Black Scholes Formula-1

At time t_0 , What Should I charge you for an option with strike K

Value of Option $C(T, S(T)) = \max[S_T - K, 0]$

Option Price at $t = 0, = C(0, S(0)) = X(0)$

Price of option **seller portfolio**: $X(T) = C(T, S(T))$

Assumption

1. Market with one stock and one bond:
2. The stock pays no dividends.
3. Market is arbitrage free
4. The stock price follows a geometric Brownian motion with constant driftμ and volatility a: $dS = \mu Sdt + aSdW(t)$.
5. There are no transaction costs or taxes

6. Risk neutral interest rate is constant and known

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

Continuous time interval interest rate: $S_{(t)} = S_{(0)}e^{rT}$

Risk Neutral Pricing:

Black Scholes Formula 2

$\Delta(t) \rightarrow$ no. of stocks held at time t

$X(t) \rightarrow$ Portfolio value at time t (option seller)

$S(t) \rightarrow$ Stock price at time t

$[X(t) - \Delta(t)S(t)] \rightarrow$ Money left after investing in stock, goes to the money market

$r[X(t) - \Delta(t)S(t)], interest...earned..on..the..amount..from..money..market$

$$=> X(t) = \Delta(t)S(t) + r[X(t) - \Delta(t)S(t)]$$

$$=> dX(t) = \Delta(t)dS(t) + r[X(t) - \Delta(t)S(t)]dt$$

We know $\rightarrow dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$

$$= rX(t)dt + S(t)dW(t) + \Delta(t)(\alpha - r)S(t)dt + \sigma\Delta(t)S(t)dW(t)$$

$$f(t,x) = e^{-rt}x$$

Stock price is not martingale, discounted stock price is

$$
\begin{aligned}
d(e^{-rt}S(t)) &= df(t, S(t) \\
&= f_t(t, S(t))dt + f_x(t, S(t))dS(t) + (1/2)f_{xx}(t, S(t))dS(t)dS(t) \\
&= (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t)
\end{aligned}
$$

Discounted Evolution of Portfolio value: Eq. 1

$$
\begin{aligned}
d(e^{-rt}X(t)) &= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) \\
&= \Delta(t)d(e^{-rt}S(t))
\end{aligned}
$$

Now, How my option value $C(t, S(t))$ evolves:

$C(t, x)$

$$dC(t, S(t)) = C_t(t, S(t))dt + C_x(t, S(t))dS(t) + (1/2)C_{xx}(t, S(t))dS(t)dS(t)$$

$dS(t)dS(t) = \sigma^2 S^2(t)dt$

$[S, S](t) = \int_0^t \sigma^2 S^2(t)dt$

$\Rightarrow dC(t, S(t)) = [C_t(t, S(t)) + \alpha S(t)C_x(t, S(t)) +$
$(1/2)\sigma^2 S^2 C_{xx}(t, S(t))]dt + \sigma S(t)C_x(t, S(t))dW(t)$

Discounted Evolution of option value: → Eq. 2

$d(e^{-rt}C(t, S(t)) = df(t, C(t, S(t))$

$= e^{-rt}[-rC(t, S(t)) + C_t(t, S(t) + \alpha S(t)C_x(t, S(t)) +$
$(1/2)\sigma^2 S^2(t)C_{xx}(t, S(t))]dt + e^{-rt}\sigma S(t)C_x(t, S(t))dW(t)$

We need : $d(e^{-rt}X(t)) = d(e^{-rt}C(t, S(t))$

$X(0) = C(0, C(0))$

$X(T) = C(T, S(T))$

Equating the coefficients of $W(t)$ in both equations 1,2:

$\Delta(t) = C_x(t, S(t))$ **Delta of the option, <u>Delta hedging formula(Number of stocks I need to buy at time t)</u>**

$[-rC(t, S(t)) + C_t(t, S(t) + \alpha S(t)C_x(t, S(t)) + (1/2)\sigma^2 S^2(t)C_{xx}(t, S(t))] = \Delta(t)(\alpha - r)e^{-rt}S(t)$

$\Rightarrow [-rC(t, S(t)) + C_t(t, S(t) + \alpha S(t)C_x(t, S(t)) + (1/2)\sigma^2 S^2(t)C_{xx}(t, S(t))] = C_x(t, S(t))(\alpha - r)S(t)$

$\Rightarrow rC(t, S(t)) = C_t(t, S(t)) + rS(t)C_x(t, S(t)) + (1/2)\sigma^2 S^2(t)C_{xx}(t, S(t))$

and $C(T, S(T)) = \max[S_T - K, 0]$

$\Rightarrow rC(t, x) = C_t(t, x) + rxC_x(t, x) + (1/2)\sigma^2 x^2 C_{xx}(t, x)$

under the terminal condition:

$C(T, x) = \max[x - K, 0]$