Scalars

- single number
- write scalar in *italics*
- E.g. *Let s ∈ R be the slope of the line*

Vectors

- Array of numbers
- Identifying points in space, with each element giving the coordinate along a different axis.

Matrices

- 2-D Array of numbers
- uppercase, bold-face e.g. **A**

Properties of Matrices and Vectors

S.No	Matrix/Vector Property	Description
1.	**Transpose of Matrix**	$A = [a_{ij}]_{mxn}$, then $A^T = [a_{ji}]_{mxn}$
2.	**Matrix Addition**	$C_{i,j} = A_{i,j} + B_{i,j}$.
3.	**Matrix Multiplication**	$C_{i,j} = \sum_k A_{i,k} B_{k,j}$.
4.	**Transpose of Matrix Product**	$(\boldsymbol{AB})^\top = \boldsymbol{B}^\top \boldsymbol{A}^\top$.
5.	**Identity Matrix**	$\forall \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{I}_n \boldsymbol{x} = \boldsymbol{x}$.
6.	**Inverse Matrix**	$\boldsymbol{A}^{-1} \boldsymbol{A} = \boldsymbol{I}_n$.

S.No	Matrix/Vector Property	Description
7.	**Linear Combination of vectors**	$\boldsymbol{Ax} = \sum_i x_i \boldsymbol{A}_{:,i}$
8.	**Span of set of vectors**	$\sum c_i \boldsymbol{v}^{(i)}$
9.	**Linearly Indipendent set of vectors**	If **no vector is linear combination** of other vectors in the set;
10.	**Square Matrix**	Equal no. of rows and columns
11.	**Left/Right Inverse**	For square matrix, **left inverse = right inverse**
12.	**Norms**	Determines **size of vector**; $\|\|\boldsymbol{x}\|\|_p = \left(\sum_i \|x_i\|^p\right)^{\frac{1}{p}}$ for $p \in \mathbb{R}, p \geq 1.$
13.	**Properties of Norms (function *f*)**	• $f(\boldsymbol{x}) = 0 \Rightarrow \boldsymbol{x} = \boldsymbol{0}$ • $f(\boldsymbol{x} + \boldsymbol{y}) \leq f(\boldsymbol{x}) + f(\boldsymbol{y})$ (the **triangle inequality**)

S.No	Matrix/Vector Property	Description
15.	**L1 Norm**	exactly 0 and elements that are close to 0; $\|\|\boldsymbol{x}\|\|_1 = \sum_i \|x_i\|.$
16.	**Max Norm**	$\|\|\boldsymbol{x}\|\|_\infty = \max_i \|x_i\|.$
17.	**Frobenius Norm**	Measures **size of matrix**. (Analogus to euclidean-norm)

S.No	Matrix/Vector Property	Description
18.	**Dot product of vectors (x, y)**	$\boldsymbol{x}^\top \boldsymbol{y} = \|\|\boldsymbol{x}\|\|_2 \|\|\boldsymbol{y}\|\|_2 \cos\theta$ where θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$.
19.	**Diagnol Matrix**	$D_{i,j} = 0$ all $i \neq j$.
20.	**Identity Matrix**	Matrix whose diagnol enteries are 1
21.	**Symmetric Matrix**	$\boldsymbol{A} = \boldsymbol{A}^\top$.
22.	**Unit Vector**	$\|\|\boldsymbol{x}\|\|_2 = 1$.
23.	**Orthogonal Vectors**	$\mathbf{x}^\top \mathbf{y} = 0$
24.	**Orthonormal Vectors**	**Orthogonal** vectors with **unit norm**; $\boldsymbol{A}^\top \boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{I}$.
25.	**Orthogonal Matrix**	Square matrix with rows and columns **orthonormal respectively**.

Matrix Decomposition

Eigen Decomposition

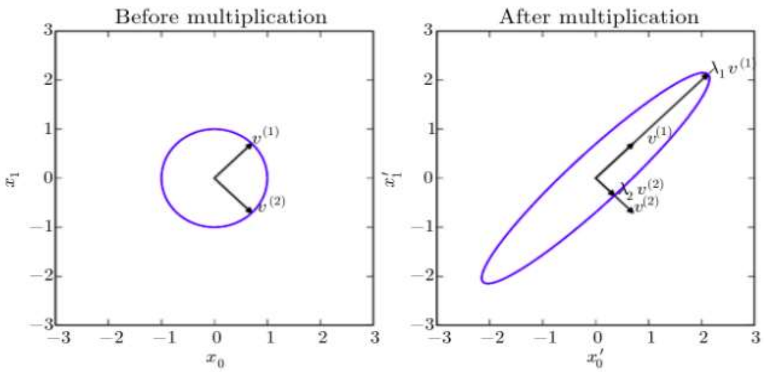
Eigendecomposition is a way of **breaking/decomposing** matrix into smaller matrices (analogus to *prime factorization*).

$$\boldsymbol{A} = \boldsymbol{V}\mathrm{diag}(\boldsymbol{\lambda})\boldsymbol{V}^{-1}.$$

Eigen Vector* is a non-zero vector ***v*, which upon being multiplied by matrix **A**, alters only the scale of ***v***.

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}.$$

Below figure shows before and after multiplying *eigen vector* with *eigen value* :



- Matrix whose all eigen values are -
  - positive is called **positive definite**
  - positive or zero-valued is called **positive semi-definite**
  - negative is called **negative definite**
  - negative or zero-valued is called **negative semi-definite**

Single Value Decomposition

SVD factorizes matrix into singular values and singular vectors. In SVD, matrix **A** can be decomposed as follows -

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^{\top}.$$

Properties -

- **A** - (m, n) matrix
- **U** - (m, m) orthogonal matrix, columns of **U** are called **left-singular vectors**

- **D** - (m, n) diagnol matrix, not necessarily square, elements along diagnol **D** are called **singular values of A**
- **V** - (n, n) orthogonal matrix, columns of **V** are called **right-singular vectors**

Moore-Penrose Pseudoinverse

Usually matrix inversion is not possible for non-square matrices. To solve below linear equation, in case **A is non-square matrix** , we use Moore-Penrose pseudoinverse formula to **find solution to x** -

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$$

$$\boldsymbol{A}^{+} = \lim_{\alpha \searrow 0} (\boldsymbol{A}^{\top}\boldsymbol{A} + \alpha \boldsymbol{I})^{-1}\boldsymbol{A}^{\top}.$$

Here, **U, D, V** are SVD of **A**.

*Pseudo-inverse of **D** is obtained by -*

- take reciprocal of non-zero elements
- take transpose of resultant matrix

Trace Operator

- Gives **sum of diagnol enetries** of matrix.

$$\mathrm{Tr}(\boldsymbol{A}) = \sum_i \boldsymbol{A}_{i,i}.$$

- **Frobenius norm** can be re-written in terms of Trace operator as follows -

$$||A||_F = \sqrt{\mathrm{Tr}(\boldsymbol{A}\boldsymbol{A}^{\top})}.$$

- **Properties of Trace operator** -
  i. $\mathrm{Tr}(\boldsymbol{ABC}) = \mathrm{Tr}(\boldsymbol{CAB}) = \mathrm{Tr}(\boldsymbol{BCA})$
  ii. $\mathrm{Tr}(\boldsymbol{A}) = \mathrm{Tr}(\boldsymbol{A}^{\top}).$