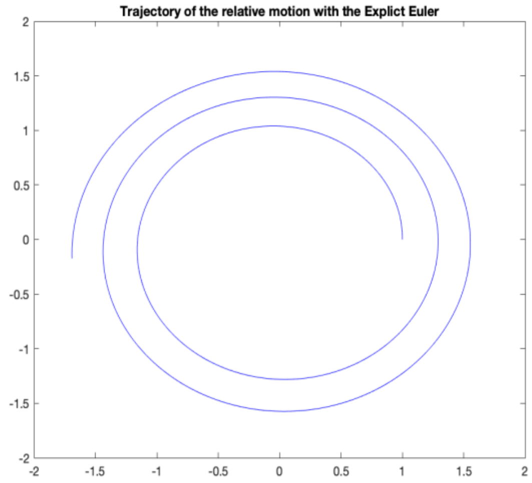
PART 1- Numerical Methods Coursework

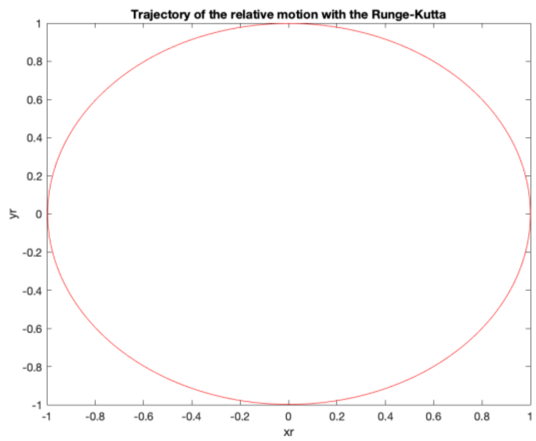
1. Vector v is given by $v = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix}$, where the system of second order ODEs have been resolved into a system of first order ODE's yielding $\dot{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ -G\frac{m_1+m_2}{(x_r{}^2+y_r{}^2)^{\frac{3}{2}}}x_r \\ -G\frac{m_1+m_2}{(x_r{}^2+y_r{}^2)^{\frac{3}{2}}}y_r \end{pmatrix}$, $v(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 25 \end{pmatrix}$

2.

A.



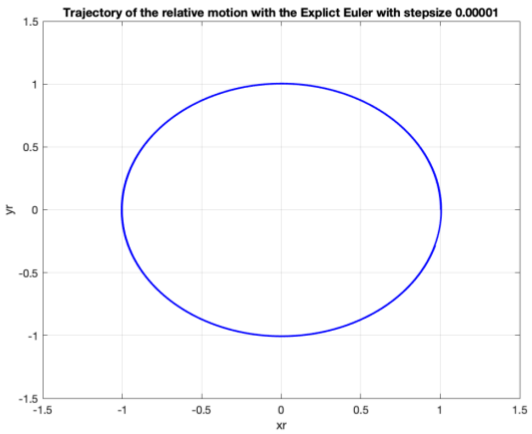
B.

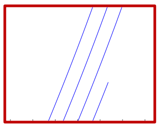


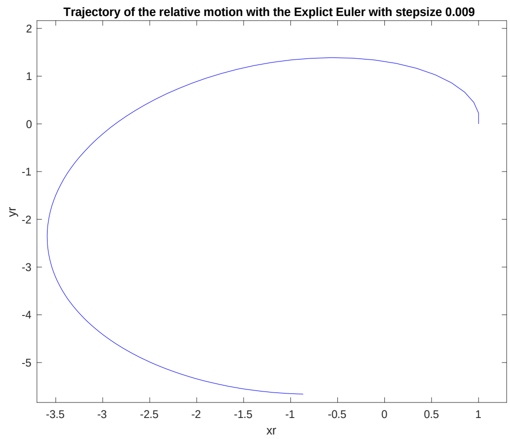
C. The Runge- Kutta methods represents the exact solutions well as they give similar plots of the trajectory of the relative motion, whilst the Explicit Euler tends to grow unbounded. Note that the computational complexity of the 4th order Runge Kutta is greater than that of the Explicit Euler method as the Runge Kutta computes higher order derivatives compared to the Explicit Euler, this leads to there being a greater requirement of a greater number of steps to be used by the Explicit method in order to reach the same level of accuracy. A greater computational complexity is also associated with a greater runtime. Overall, both methods can produce numerical solutions to represent the exact solution, but the Runge-Kutta provides a more reliable representation, as it gives greater accuracy, but comes at a cost of a greater runtime.

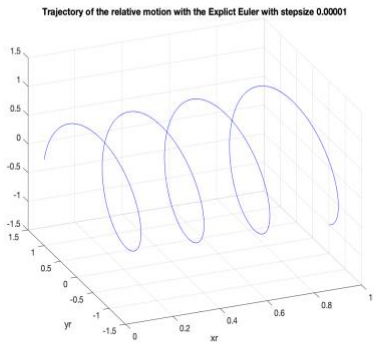
3.

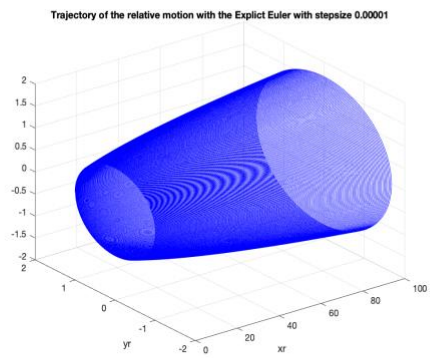
A. Step size has a large impact on the numerical stability of the Explicit Euler; when step size is too large there would be numerical instability and the solution diverges from the expected trajectory. Increasing the step size any greater than 0.009 yields an unstable solution. In the case that the step size is decreased from 0.001, the solution has greater accuracy, and the solution converges. Increasing T leads to divergence from the expected trajectory.



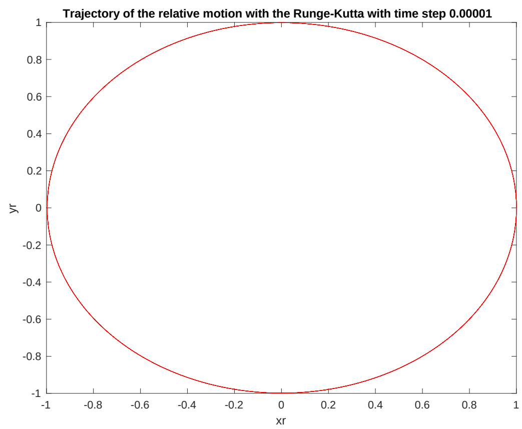


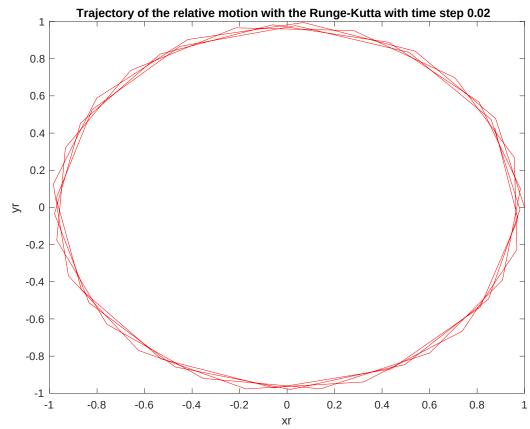


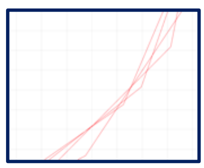




B. The effect of increasing the timestep to 0.02 leads to the solution becoming unstable when the time step is too large; the effect of decreasing the timestep to 0.0001 is that it increases the accuracy of the solution. The range of values that the Runge Kutta method provides a valid solution is greater than that of the Explicit Euler. Increasing the time period to 160s from 1s leads to the solution to diverge as the solution becomes unstable.





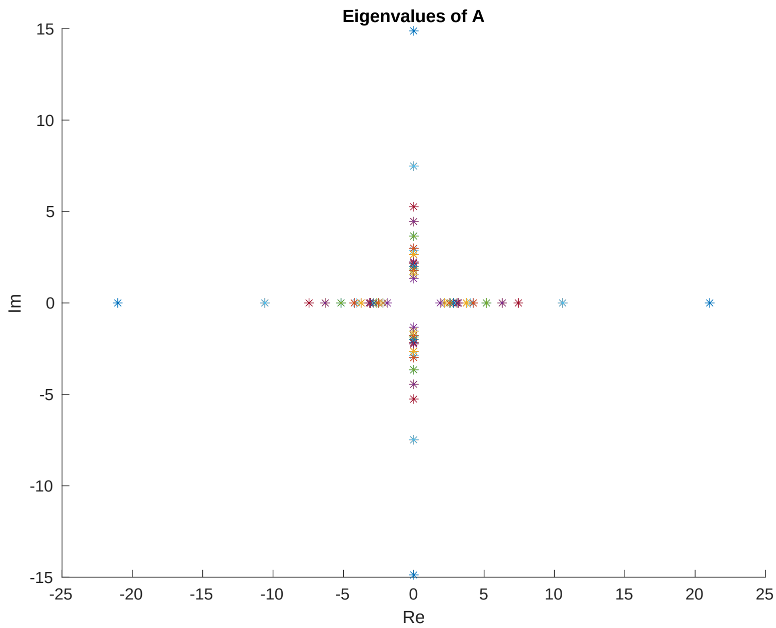


4.

A. Upon performing the linearisation in the form of $v = Av + \bar{b}t + \bar{c}$, the equation yields :

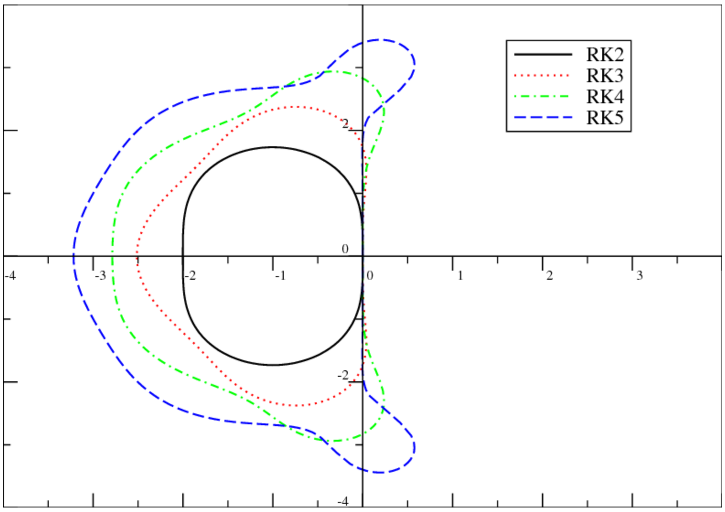
$$\dot{v} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-G(m_1+m_2)({y^*}_r^{\,2} - 2{x^*}_r^{\,2})}{({x^*}_r^{\,2} + {y^*}_r^{\,2})^{\frac{5}{2}}} & \dfrac{3G(m_1+m_2)({x^*}_r{y^*}_r)}{({x^*}_r^{\,2} + {y^*}_r^{\,2})^{\frac{5}{2}}} & 0 & 0 \\ \dfrac{3G(m_1+m_2)({x^*}_r{y^*}_r)}{({x^*}_r^{\,2} + {y^*}_r^{\,2})^{\frac{5}{2}}} & \dfrac{-G(m_1+m_2)({x^*}_r^{\,2} - 2{y^*}_r^{\,2})}{({x^*}_r^{\,2} + {y^*}_r^{\,2})^{\frac{5}{2}}} & 0 & 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ \dfrac{-3G(m_1+m_2)({x^*}_r)}{({x^*}_r^{\,2} + {y^*}_r^{\,2})^{\frac{3}{2}}} \\ \dfrac{-3G(m_1+m_2)({y^*}_r)}{({x^*}_r^{\,2} + {y^*}_r^{\,2})^{\frac{3}{2}}} \end{pmatrix}$$

B. The pattern is symmetric on the real and imaginary axes, arranged in a diamond pattern. The purely real and imaginary eigenvalues are equal in magnitude but opposite in sign. Most points are clustered close to the origin, and the rate of change of the system increases as points tend further from the origin, the points further away would have greater influence on the system.



C. Observations made in question 3, both provide information about the local stability of the system; this is determined by examining the trajectories of the solution over various time periods. These seem to remain stable given that the timestep and time periods is appropriately small, however variation from this displays that methods are unstable. This is a similar observation that confirms the instability of the system. We can deduce that the Explicit Euler is unstable regardless of the change in timestep. The Runge Kutta however seems to be unstable initially remains stable given that the time period is not significantly large; However, as you increase the time period to more realistic values, there is evident divergence as it grows unstable. The information about the eigenvalues and the plot of the eigenvalues of A at each grid point of on the complex plane reveals a pattern that suggests the system will be unstable on a local scale, and the relative motion of the bodies will remain relatively constant.

In this case, the eigenvalues of the matrix A, which have a negative and positive real part, indicate that the linearisation will be unstable as a result of the positive component of the eigenvalue. This coincides with fact that the Runge Kutta is unstable in the positive region, this has been deduced from the figure below that depicts that eigenvalues that lie in the negative real and imaginary region will provide a stable solution whereas the positive real and imaginary components will provide an unstable solution.



Note that the linearisation will only give information about local stability, therefore it does not take into consideration global information and thus large gravitational phenomena are not accounted for.