Problem Set 12

(1) N types of stock ... ↔ double or nothing ..

(a) Expected amount for a single stock of 1$ $= \frac{1}{2} \times 2 + \frac{1}{2} \times 0 - 1 = \underline{0}$

expected amount for N stocks $= N \times 0 = \underline{\underline{0}}$.

Variance $= |Var_1| + |Var_2| + \ldots + |Var_N|$.

as they are mutually independent.

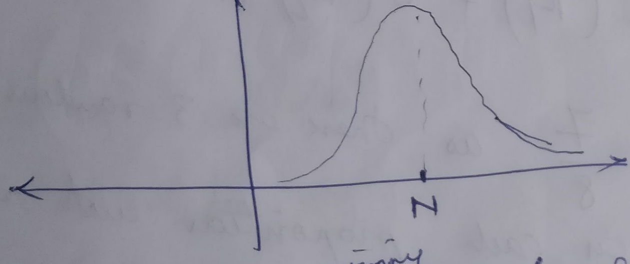
$= N\,|Var_1| = N \times \left[\frac{1}{2}\,[1-0]^2 + \frac{1}{2}\,[-1-0]^2\right]$

$= \underline{\underline{N \times 1}} = \underline{\underline{N}}$.

(b) if all stocks are different:

→ It'll be the same, as they have the same price = $1 & the same behaviour double-or-nothing.

(c) the expected winning amount follows the below distribution.



As she is conservative, its good if they invest or play.

As the expected ~~return~~ winning return is ~~same~~ 0.

Any amount @ the end of the play will vary by std. deviation.

But as she's conservative, it's better not to play @ all.

(d) expected payoff =

$$= \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \dots + \frac{1}{6} \times 6 = 3.5$$

$$\text{Variance} = \frac{1}{6}\left[2.5^2 + 1.5^2 + 0.5^2 + 0.5^2 + 1.5^2 + 2.5^2\right]$$

$$= \frac{2}{6} \times \left[6.25 + 2.25 + 0.25\right]$$

$$= \frac{1}{3} \times 8.75 = \underline{\underline{2.9166}}$$

e) expected payoff

$$= \frac{1}{6}\left[1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3\right]$$

$$= \frac{441}{6} = \frac{147}{2} = \underline{\underline{73.5}}$$

$$\text{Variance} = \frac{1}{6}\left[72.5^2 + 65.5^2 + 46.5^2 + 9.5^2 + 51.5^2 + 142.5^2\right]$$

2 (a) by linearity principle :

$$E(R) = E(R_1) + E(R_2) + \dots$$

$E(R_1) = \frac{7}{8}$ as there are 3 random variables ~~f~~ in each proposition with an OR statement b/w them

$$\therefore E(R) = \frac{7}{8} + \frac{7}{8} + \dots + \frac{7}{8} = \frac{49}{8}.$$

(b) As there are 7 proportions, with

$$E[T_1 + T_2 + \dots + T_7] = \frac{49}{8}$$

means at least ~~true~~ should all

$T_1, T_2 \dots T_7 = 1$ else the average value can't be $6\frac{1}{8}$ or $\frac{49}{8}$.

(3) (a) Busy student: must complete 3 problem set. Each P.S → 1 day → probl $= 2/3$
→ 2 days → prob $= 1/3$.

$$E(B) = 1 \times \frac{2}{3} + 2 \times \frac{1}{3} = \frac{4}{3}$$

for each P.S

$$E(B) = E(B_1) + E(B_2) + E(B_3)$$

$$= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4.$$

(b) Relaxed Student :-

"mean no. of days" $= \frac{1}{p}$

here $p = \frac{1}{6}$ = probability of getting a 1 on a rolled dice.

∴ "mean no. of days" $= \frac{1}{1/6} = 6$.

(c) unlucky student =

$E(d_1 \times d_2) =$ as the events are independent

$$E(d_1 \times d_2) = E(d_1) \cdot E(d_2)$$

$$= \frac{7}{2} \times \frac{7}{2} = \frac{49}{4} = 12.25$$

∴ he needs to wait for 12.25 days before doing laundry.

(d) $E(\text{student}) = \frac{1}{2} E(\text{Busy student})$

$$+ \frac{1}{3} E(\text{relaxed})$$

$$+ \frac{1}{6} E(\text{unlucky})$$

$$= \frac{1}{2} \times 4 + \frac{1}{3} \times 6 + \frac{1}{6} \times \cancel{12} \frac{49}{4}$$

$$= 4 + \frac{\overset{2.04}{\cancel{12.25}}}{\cancel{6}} = 6.04$$

(4). 2 coins, 1 fair coin & 1 biased coin.

$p(H) = 1/2$ $\qquad$ $P_H = 3/4$

★ taken from solutions by Dr-Nagpal.

setting a threshold $= \frac{5}{8}$ s.t.

proportion of heads $< T \Rightarrow$ fair coin was picked

" heads $> T \Rightarrow$ biased coin was picked

for 95% confidence. Fair coin

$$\Rightarrow \Pr\left(\frac{H_n}{n} > \frac{5}{8}\right) = \Pr\left(\frac{H_n}{n} - \frac{1}{2} > \frac{5}{8} - \frac{1}{2}\right)$$

$$= \Pr\left(H_n - \frac{n}{2} > \frac{n}{8}\right)$$

$$= \Pr\left(H_n - E(H_n) > \frac{n}{8}\right)$$

$$\leq \cancel{\frac{E(H_n)}{\quad}} \frac{Var(H_n)}{(n/8)^2}$$

$$\leq \frac{16}{n}$$

(iii) for biased coin

$$P\left(\frac{X_n}{n} < \frac{5}{8}\right) = P_r\left(\frac{3}{4} - \frac{X_n}{n} > \frac{3}{4} - \frac{5}{8}\right)$$

$$= P_r\left(\frac{3n}{4} - X_n > \frac{n}{8}\right)$$

$$\leq \frac{\operatorname{Var} X_n}{(n/8)^2} = \frac{\frac{3n}{16}}{n^2/64} = \frac{12}{n}.$$

for 95% confidence $\underline{n = 320}$

(6) $p(X_i = 1) =$ all balls are landed on the other boxes

$$= \left(1 - \frac{1}{n}\right)^n.$$

They are not independent as,

$$p(X_1 = 1 \wedge X_2 = 1 \wedge X_3 = 1 \ldots) = 0$$
$$\neq p(X_1 = 1)\, p(X_2 = 1) \ldots p(X_n = 1)$$

(b) expected no. of empty boxes

$$E(\text{empty boxes}) = E(EB_1) + E(EB_2) + \ldots + E(EB_n).$$
$$= \left(1 - \frac{1}{n}\right)^n + \left(1 - \frac{1}{n}\right)^n + \ldots + \left(1 - \frac{1}{n}\right)^n$$

$$= n\left(1 - \frac{1}{n}\right)^n.$$

$$= n\left[1 - n \times \frac{1}{n} + {}^nC_2 \times \left(\frac{1}{n}\right)^2 - {}^nC_3 \times \left(\frac{1}{n}\right)^3 + \ldots\right]$$

for large n

$$= n\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots\right] \leq \underline{n \times e^{-1}}$$

constant c is at least e^{-1}.

(c). $P_r\left(\text{at least } k \text{ balls in } 1^{st} \text{ box}\right) \leq \binom{n}{k}\left(\frac{1}{n}\right)^k$

for 1^{st} ball to be in box 1 $= \frac{1}{n}$.

out of n throws any k throws can be seleced $= \binom{n}{k}$.

$$P\left(k \text{ balls in } 1^{st} \text{ box}\right) = \frac{\binom{n}{k}}{2^n}\left(\frac{1}{n}\right)^k$$

$$P\left((k+1) \text{ balls in } 1^{st} \text{ box}\right) = \frac{\binom{n}{k+1}}{2^n}\left(\frac{1}{n}\right)^{k+1}$$

$$\therefore P\left(\text{at least } k \text{ balls in box } 1\right) = \frac{\binom{n}{k}\left(\frac{1}{n}\right)^k}{2^n} + \frac{\binom{n}{k+1}\left(\frac{1}{n}\right)^{k+1}}{2^n} + \dots$$

$$\leq \binom{n}{k}\left(\frac{1}{n}\right)^k$$

(c). $P[R \geq k] \leq n \cdot P_r\left[\text{at least } k \text{ balls fall in box } 1\right]$

$$\leq n \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$\leq \frac{n}{k!}$$

(d). $\lim_{n\to\infty} P(R \geq n^t) = 0 \Rightarrow \lim_{n\to\infty} \frac{n}{(n^t)!} = 0$.

Apply stirling formula for $(n^t)! = \frac{e^{(n^t \ln n^t)}}{e^{\ln n^t}}$