3. COMPLEX VARIABLES

COMPLEX NUMBER (Z)	$z = x + iy$	$x = Re(z)$	$y = Im(z) =$ Complex Part	$Iota = i = \sqrt{-1}$
	$z = re^{i\theta}$	$r = \sqrt{x^2 + y^2}$	$\theta = \tan^{-1}(y/x)$	$x = r\cos\theta, y = r\sin\theta$
Magnitude of Complex No. $\lvert z\rvert = r$			Argument of complex No. $\arg z = \theta$	
Addition of Two Complex No.: $z = z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$				
Multiplication of Two Complex No.: $z = z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$				
Complex Conjugate of a complex No.: $conj\ z = \bar{z} = x - iy$			$z \cdot \bar{z} = \lvert z\rvert^2$	
Division of Two Complex No.: $z = z_1/z_2 = z_1\bar{z}_2/\lvert z_2\rvert^2$				

PROPERTIES OF MODULUS:

$x = Re(z) \le \lvert z\rvert$	$y = Im(z) \le \lvert z\rvert$
$\lvert z_1 \pm z_2\rvert \le \lvert z_1\rvert + \lvert z_2\rvert$	$\lvert z_1 \pm z_2\rvert \ge \lvert z_1\rvert - \lvert z_2\rvert$
$\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$	$\left\lvert\frac{z_1 + z_2}{z_3 + z_4}\right\rvert \le \frac{\lvert z_1\rvert + \lvert z_2\rvert}{\lvert\lvert z_3\rvert - \lvert z_4\rvert\rvert}$
$\lvert z_1 \cdot z_2 \cdot z_3 \cdot \ldots \cdot z_n\rvert = \lvert z_1\rvert \cdot \lvert z_2\rvert \cdot \lvert z_3\rvert \cdot \ldots \cdot \lvert z_n\rvert$	$\lvert z_1 + z_2\rvert^2 + \lvert z_1 - z_2\rvert^2 = 2(\lvert z_1\rvert^2 + \lvert z_2\rvert^2)$

EULER'S RULE: $e^{i\theta} = \cos\theta + i\sin\theta$

$sin\ \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$	$cos\ \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$	$sinh\ \theta = \frac{e^{\theta} - e^{-\theta}}{2}$	$cosh\ \theta = \frac{e^{\theta} + e^{-\theta}}{2}$

$\sin iz = i\sinh z$	$\cos iz = \cosh z$	$\tan iz = i\tanh z$
$\sinh iz = i\sin z$	$\cosh iz = \cos z$	$\tanh iz = i\tan z$

CUBE ROOTS OF UNITY:	1	$\omega = (-1 + i\sqrt{3})/2$	$\omega^2 = (-1 - i\sqrt{3})/2$

PROPERTIES OF CUBE ROOTS OF UNITY:

One root is real and other two are complex.	Square of one complex root results in the other complex root.		
Complex roots are conjugates of each other.	$1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$		
$\omega^{3n} = 1$	$\omega^{3n+1} = \omega$		$\omega^{3n+2} = \omega^2$
CUBE ROOTS OF -1:	-1	$-\omega$	$-\omega^2$

FUNCTION OF COMPLEX VARIABLE (w): $w = f(z) = u + iv$

ANALYTIC FUNCTIONS: If $f(z)$ is differentiable at a point say $z = z_0$, then we can say that $f(z)$ is an analytic function at $z = z_0$ provided $f'(z)$ exists at all points lying in neighbourhood of $z = z_0$ also.

ANALYTIC FUNCTIONS AT $z = z_0$	ANALYTIC FUNCTIONS IN A REGION "R"
$f(z)$ is analytic at $z = z_0$ if $f'(z)$ exists at $z = z_0$ & in neighbourhood of z_0.	$f(z)$ is analytic at all points in region R then the function is analytic over the region "R".

SINGULAR POINT/ SINGULARITY: If $f(z)$ ceases to be analytic at a point $z = z_1$ in region "R", then z_1 is called a Singular point or Singularity.

ENTIRE FUNCTION: If $f(z)$ is analytic in the region "R" and not having a singular point in region "R" then $f(z)$ in the region "R" then $f(z)$ is said to be entire function.

CAUCHY'S RIEMANN EQUATIONS:

If $f(z) = u + iv$, is analytic function in the region "R", then it must satisfy, CR equations			
RECTANGULAR FORM		**POLAR FORM**	
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \cdots (1)$	$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \cdots (2)$	$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \cdots (3)$	$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \cdots (2)$

$f(z) = u + iv$ $where\ u\ \&\ v = f(r,\theta) = f(x,y)$	$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} (From\ CR\ Equations)$

LAPLACE EQUATION: $\boldsymbol{\nabla^2 f = 0}$

HARMONIC FUNCTION: If $\emptyset$ (2D) is a harmonic function, then it must satisfy Laplace Equation.

PROPERTIES OF ANALYTIC FUNCTION:

1. If $f(z)$ & $g(z)$ both analytic functions, then $f(z) + g(z), f(z) - g(z), f(z) \cdot g(z), f(z)/g(z)\,, where\ g(z) \neq 0$ are also analytic functions.
2. If $f(z) = u + iv$ be analytic function, then u & v both are harmonic functions.
3. If $f(z)$ is analytic, then all the successive derivative of $f(z)$ will be analytic.
4. If $f(z) = u + iv$ be analytic function, then the curves $u(x, y) = C_1$ & $v(x, y) = C_2$ are orthogonal trajectories.
5. If $f(z)$ is analytic function, then $f(z)$ is continuous as well as differentiable.
6. If $f(z) = u + iv$ be analytic function, then $Re[u(x, y)]$ is called **Conjugate harmonic function.**
7. $Complex\ Potential\ Function = f(z) = (u = Potential\ function) + i(v = Stream\ function)$

MULTIPLE POINT: Points where curve intersect it self are called multiple points.

<table>
<tr><td>Types of Curve</td><td>With or Without Multipoint</td><td>Open</td><td>Closed</td><td>Neither Open nor Closed</td></tr>
<tr><td colspan="3">Open/ Simple Curve: Starting point doesn't meet ending point.</td><td colspan="2">Closed Curve: Starting point meets ending point.</td></tr>
</table>

SMOOTH CURVE:
If $z'(t)$ exists and $z'(t) \neq 0, \forall\, t \in [a, b]\ Where\ z(t) = x(t) + iy(t)$ & $a \leq t \leq b$ then, $z\ (t)$ is smooth curve.
PIECE WISE SMOOTH CURVE: If $z(t)$ is a smooth curve in $[a, b]$ but there are few points in $[a, b]$ where $z'(t)$ doesn't exists then $z\ (t)$ is called Piece Wise Smooth Curve.

CONTOUR: All the curves which are piece wise smooth or smooth curve they are called as contour.

SIMPLY CONNECTED REGIONS	MULTIPLY CONNECTED REGIONS
If a closed curve "c" in given region "R" which encloses only the points of region "R". During shrinking, if "c" consist only the point of region "R" is called simply closed region. E.g. Paper Without Holes	If a region inside closed curve "c" is not part of given region "R". During shrinking, if "c" consist the point of region "R" and other empty points is called Multiply Connected region. E.g. Paper with Holes

COMPLEX INTEGRATION:

CAUCHY'S INTEGRATION THEOREM: If $f(z)$ is analytic function & If $f'(z)$ exists in region "R" and on the boundary of region "R", then	$\oint_c f(z)\, dz = 0$

Gourset Theorem: If $f(z)$ is not continuous in region "R" and on the boundary of region "R", above result is valid.
Cauchy's Gourset Theorem: If $f(z)$ is analytic function in region "R" and on the boundary, above result is valid.

CAUCHY'S INTEGRAL FORMULA: If $g(z)$ is analytic function in the region "R" & on the curve "c" except the point $z = z_0$, then by Cauchy's integral formula	$\oint_c g(z)\, dz = \oint_c \frac{f(z)}{(z - z_0)} dz = 2\pi i\, f(z_0)$

CAUCHY'S INTEGRAL FORMULA FOR REPEATED SINGULARITY: If $g(z)$ is analytic function in the region "R" & over the curve "c" except the point $z = z_0$(Singular Point) with index "$n + 1$" (repeated times),

$$\oint_c g(z)\, dz = \frac{2\pi i}{n!} \left[\frac{d^n f(z)}{dz^n}\right]_{z=z_0}, where\ g(z) = \frac{f(z)}{(z - z_0)^{n+1}}$$

<table>
<tr><td colspan="2">CALCULATION OF RESIDUES</td></tr>
<tr><td>Residue of $f(z)$ at a simple pole (Pole which is not repeated) $z = z_0$
$$R = \lim_{z \to a} (z - a)\, f(z)$$</td><td>Residue of $f(z)$ at a repeated pole with index "$n + 1$" at $z = z_0$
$$R = \frac{1}{n!} \lim_{z \to a} \frac{d^n}{dz^n} (z - a)^{n+1}\, f(z)$$</td></tr>
</table>

CAUCHY'S RESIDUES THEOREM	$\oint_c f(z)\, dz = 2\pi i \sum (All\ the\ Residues)$

<table>
<tr><td colspan="2">LAURENT SERIES</td><td colspan="2">$$f(z) = \sum_{n=0}^{\infty} a_n\, (z - z_0)^n + \sum_{n=1}^{\infty} b_n\, (z - z_0)^{-n}$$</td></tr>
<tr><td colspan="2">Negative Power: Principle Part</td><td colspan="2">Positive Power: Analytic Part</td></tr>
<tr><td colspan="4">TYPES OF SINGULARITIES BASED ON LAURENT SERIES</td></tr>
<tr><td>Essential: Principle Part has infinite terms</td><td colspan="2">Removable: Principle Part is absent.</td><td>Pole of order "m": Principle Part has infinite terms</td></tr>
</table>