NUMERICAL METHODS

POLYNOMIAL: A function is of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ $(a_0 \neq 0)$ and $a_0, a_1, a_2, \ldots a_n$ are the constants is called n^{th} degree polynomial. Where n is Positive integer.
POLYNOMIAL EQUATION: If $f(x)$ is polynomial, then $f(x) = 0$ is called polynomial equation.
n^{th} Degree polynomial equation has 'n' roots.
ALGEBRAIC FUNCTION: A function obtained by applying finite number of algebraic operations on polynomials is called Algebraic function. All Polynomial functions are algebraic functions.
ALGEBRAIC EQUATION: If $f(x)$ is Algebraic function, then then $f(x) = 0$ is called Algebraic Equation.
Every polynomial equation is algebraic equation but converse need not be true.
TRANSCENDENTAL EQUATION: An equation other than Algebraic equation is called transcendental equation.
Transcendental equation may have no root or finite number of roots or infinite roots.
DESCARTES' RULE OF SIGN: If $f(x)$ is polynomial, then

- No. of Positive real roots of $f(x)$ $\leq$ No. of changes of signs in $f(x)$
- No. of Negative real roots of $f(x)$ $\leq$ No. of changes of signs in $f(-x)$

No. of Roots		
Real Roots		**Complex Roots**
+ ve	**- ve**	

REAL ROOTS OF AN EQUATION: If $'\alpha'$ is a real root of an equation $f(x) = 0$, then the curve intersects real axis at $x = \alpha$.
INTERMEDIATE VALUE THEOREM: If $f(x)$ is continuous on $[a, b]$ and if $f(a), f(b)$ have opposite signs, then $\exists$ at least one real root lies between a & b.

ERRORS OF APPROXIMATION:

ABSOLUTE ERROR: $\varepsilon = |Exact\ value - approximate\ value|$
RELATIVE ERROR: $\varepsilon = |Exact\ value - approximate\ value| / |Exact\ value|$
PERCENTAGE ERROR: $Relative\ Error\ \times 100$

$\varepsilon_{n+1} \leq C\varepsilon_n^p$ $C = Asymptotic\ Error\ Constant$	$\varepsilon_i = Error\ in\ i^{th}\ stage\ of\ series$ $p =\ Order\ of\ Convergence$

Here, If p is large, convergence is fast. And If p is small, convergence is slow.
If $p = 1$, the convergence is called Liner convergence.
If $p = 2$, the convergence is called Quadratic convergence.
If $p = 3$, the convergence is called Cubic convergence.
TRUCATED ERROR: The error obtained by truncations of the infinite sum to approximate it to finite sum is called traction error.

BISECTION METHOD:

If $f(x) = 0$ has a real root in (a, b), **1st Approximation:** $\boldsymbol{x_1 = (a + b)/2}$ If $f(x_1) = 0$, x_1 is root, Else CASE-I: If $f(a), f(x_1)$ have opposite signs root lies between a and x_1. CASE-II: If $f(x_1), f(b)$ have opposite signs root lies between x_1 and b. **2nd Approximation:** $\boldsymbol{x_2 = (a + x_1)/2\ Or\ (x_1 + b)/2}$	**Advantages and Disadvantages:** • Convergence is guaranteed. • Method never fails. • Convergence is very slow. • Order of Convergence $p = 1$ • Nth Stage Interval length = $(b - a)/2^N$ • **Permissible error** $\boldsymbol{\varepsilon \geq (b - a)/2^N}$

REGULA-FALSI METHOD (METHOD OF FALSE POSITION):

If $f(x) = 0$ has a real root in (a, b), **1st Approximation**: $\boldsymbol{x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}}$ If $f(x_1) = 0$, x_1 is root, Else CASE-I: If $f(a), f(x_1)$ have opposite signs root lies between a and x_1. CASE-II: If $f(x_1), f(b)$ have opposite signs root lies between x_1 and b.	**2nd Approximation:** $\boldsymbol{x_2 = \dfrac{af(x_1) - x_1f(a)}{f(x_1) - f(a)}}$ $\boldsymbol{Or\ \dfrac{x_1f(b) - bf(x_1)}{f(b) - f(x_1)}}$

Advantages and Disadvantages: Convergence is very slow And Order of Convergence $p = 1$.

NEWTON'S RAPHSON METHOD (METHOD OF TANGENT):

If $f(x) = 0$ has a real root in (a, b), (n+1)th iteration $x_{n+1} = x_n - {f(x_n)}/{f'(x_n)}$ with initial guess x_0.

Advantages and Disadvantages:	NOTE:
• The method converges fast. Eg. if choose x_0 nearest to the roots, convergence is fast otherwise it's slow or sometimes it diverges also. • Order of Convergence $p = 2$. • It improves the results of previous methods.	If $f'(x_n) = 0$, in the neighbour hood of x_n then this method fails.

SECANT METHOD:

If $f(x) = 0$ has a real root in (a, b) and initial guess x_0,

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad (Same\ as\ Regula\ Falsi\ method)$$

Advantages and Disadvantages: Order of Convergence $p = 1.62$ (Super Liner Convergence).

INTERPOLATION AND CURVE FITTING:

LAGRANGE'S INTERPOLATION:

$$f(x) = \frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_1 - x_0)(x_2 - x_0)\dots(x_n - x_0)} f(x_0) +$$

$$\frac{(x - x_0)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_2 - x_1)\dots(x_n - x_1)} f(x_1) + \dots + \frac{(x - x_1)(x - x_2)\dots(x - x_{n-1})}{(x_1 - x_n)(x_2 - x_n)\dots(x_{n-1} - x_n)} f(x_n)$$

NEWTONS DIVIDED DIFFERENCE INTERPOLATION:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
x_0	y_0		
		$f[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$	
x_1	y_1		$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$
		$f[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}$	
x_2	y_2		

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] + \cdots$$

NEWTONS FORMULA OF INTERPOLATION FOR EQUALLY SPACED POINTS:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
x_3	y_3			

Newtons Forward Difference Formula:

$P = \frac{x - x_0}{h}$	$f(x) = y_0 + \frac{P}{1!}\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots$

Newtons Backward Difference Formula:

$P = \frac{x - x_n}{h}$	$f(x) = y_n + \frac{P}{1!}\Delta y_n + \frac{P(P+1)}{2!}\Delta^2 y_n + \frac{P(P+1)(P+2)}{3!}\Delta^3 y_n + \cdots$

Where, h = Length of each equal sized sub-interval.

CURVE FITTING (FITTING OF STRAIGHT LINE):

$\boldsymbol{y = a + bx}$ $\boldsymbol{a}$ **and** $\boldsymbol{b}$ **find by solving 2 Equations.**	$\sum y_i = na + b\sum x_i$
	$\sum x_i y_i = a\sum x_i + b\sum x_i^2$

Where, n = Number of points.

NUMERICAL INTEGRATION:

Using Newton's Forward interpolation **Newton and Cote** Derived Quadrature formula.

NEWTONS TRAPEZOIDAL RULE: n = 1 in Newton's Cote Quadrature formula.

$$\int_a^b f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \ldots + y_{n-1})], where\ h = \frac{b-a}{n}$$

$$Truncation\ Error \le \frac{h^3}{12}\ n\ \left|\operatorname*{Max}_{a\le x\le b} f''(x)\right|$$

SIMPSON'S RULE: n = 2 in Newton's Cote Quadrature formula.

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 \ldots) + 2(y_2 + y_4 + y_6 \ldots)], where\ h = \frac{b-a}{n}$$

$$Truncation\ Error \le \frac{h^5}{180}\ n\ \left|\operatorname*{Max}_{a\le x\le b} f''''(x)\right|$$

NUMERICAL DIFFERENTIATION:

If $\frac{dy}{dx} = f(x,y)$ and initial condition $f(x_0) = y_0$, then

TAYLOR SERIES METHOD:

$$f(x_n) = y_{n-1} + hy'_{n-1} + \frac{h^2}{2!}y''_{n-1} + \frac{h^3}{3!}y'''_{n-1} + \frac{h^4}{4!}y''''_{n-1} + \frac{h^5}{5!}y'''''_{n-1} + \frac{h^6}{6!}y''''''_{n-1} + \cdots, Where\ h = x_n - x_{n-1}$$

RANGE-KUTTA METHOD:

Euler's Method (1st Order R-K Method):		
$f(x_n) = y_{n-1} + hy'_{n-1} = y_{n-1} + h\,f(x_{n-1}, y_{n-1})$		$Truncation\ error =\ Order\ of\ h^2$
Modified Euler's Method (2nd Order R-K Method):		
$y_n = y_{n-1} + \frac{1}{2}(K_1 + K_2)$	$K_1 = hf(x_{n-1}, y_{n-1})$ $K_2 = hf(x_{n-1} + h, y_{n-1} + K_1)$	$Truncation\ error =\ Order\ of\ h^3$
3rd Order R-K Method:		
$y_n = y_{n-1} + \frac{1}{6}(K_1 + 4K_2 + K_3)$	$K_1 = hf(x_{n-1}, y_{n-1})$ $K_2 = hf\left(x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{K_1}{2}\right)$ $K_3 = hf(x_{n-1} + h, y_{n-1} + K_2)$	$Truncation\ error = Order\ of\ h^4$
4rd Order R-K Method:		
$y_n = y_{n-1} + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$	$K_1 = hf(x_{n-1}, y_{n-1})$ $K_2 = hf\left(x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{K_1}{2}\right)$ $K_3 = hf\left(x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{K_2}{2}\right)$ $K_4 = hf(x_{n-1} + h, y_{n-1} + K_3)$	$Truncation\ error =\ Order\ of\ h^5$