7. FOURIER SERIES

PERIODIC FUNCTION: A function is called a periodic function, if $f(x)$ is defined for all real x except possibly at some points, and if there is some positive number p, called a period of $f(x)$, such that,

$$f(x+p) = f(x), \forall x$$

The smallest positive period is often called the fundamental period.

FOURIER SERIES:

Fourier series is representation of a non-sinusoidal periodic function sum of sinusoids.
Consider a periodic function $f(x)$ with periodicity $2l$, the trigonometric Fourier series (TFS) of $f(x)$ is given by,

$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)\right]$	Where, a_n, b_n = Trigonometric Fourier series coefficient a_0 = Average value of $f(x)$ = Term independent of "x" = Constant term
$a_0 = \frac{1}{2l}\int_{<2l>} f(x)dx$	
$a_n = \frac{1}{l}\int_{<2l>} f(x)\cos\left(\frac{n\pi}{l}x\right)dx$	$b_n = \frac{1}{l}\int_{<2l>} f(x)\sin\left(\frac{n\pi}{l}x\right)dx$

EVEN FUNCTION:	**ODD FUNCTION:**
A function is said to be an even function if $f(x) = f(-x)$	A function is said to be an even function if $f(x) = -f(-x)$
$h(x) = h(-x)$, Rotation about Y-axis	$g(x) = -g(x)$, Rotation about X-axis

SYMMETRY	CONDITION	a_0	a_n	b_n	PROPERTY
EVEN	$f(x) = f(-x)$	YES	YES	0	Cosine Term only
ODD	$f(x) = -f(-x)$	0	0	YES	Sine Term only

HALF RANGE SERIES:

$$f(x) = neither\ even\ not\ odd\ function$$
$$f_e(x) = even\ function\ obtained\ from\ f(x)\ by\ rotation$$
$$f_o(x) = odd\ function\ obtained\ from\ f(x)\ by\ rotation$$

- Half Range Cosine series of $f(x)$ in range *(0, l)* is same as Even Fourier series of $f_e(x)$.
- Half Range Sine series of $f(x)$ in range *(0, l)* is same as Odd Fourier series of $f_o(x)$.

Note: Recall Integration by parts rule.

EXISTANCE OF FOURIER SERIES:

Functions that have Fourier series representation are those periodic functions which satisfy three Dirichlet conditions.

1) $f(x)$ is absolutely integrable over one period. ($f(x) = \tan x, 0 \le x \le \frac{\pi}{2}, violates$)
2) $f(x)$ has an infinite number of maxima and minima over one period. ($f(x) = \sin\left(\frac{2\pi}{x}\right), 0 \le x \le 1, violates$)
3) $f(x)$ had an infinite number of finite discontinuities over one period.
   ($f(x) = 0\ for\ irrational\ number\ and\ 1\ for\ rational\ number\ over\ (-\pi,\pi), violates$)

The periodic function equals to their Fourier series representation, except at some values of "x" where $f(x)$ has finite discontinuity. At these values of "x", the Fourier series converges to the average value of the function values on either side of the discontinuity.