LINEAR ALGEBRA

BASICS OF MATRIX

Matrix Multiplication: Let $A_{m\times n}$, $B_{p\times q}$

1. $(AB)_{m\times q}$ exist $\Leftrightarrow n = p$ 2. $(BA)_{p\times n}$ exist $\Leftrightarrow m = q$ 3. $AB \neq BA$ (Not Cumulative) 4. $A(BC) = (AB)C$ (Associative)	5. $AB = 0 \nRightarrow A = 0 \; or \; B = 0.$ 6. $(AB)_{m\times q}$ involves No. of multiplications = mnp And Number of Additions = $mp(n-1)$ Where, $n = p$.

Trace of Matrix: Let $A_{n\times n}$ Matrix,

Properties: 1. $Tr(A \pm B) = Tr(A) \pm Tr(B)$ 2. $Tr(A) = Tr(A^T)$ 3. $Tr(AB) = Tr(BA)$ The Principle Diagonal = $(a_{11}\; a_{22}\; a_{33} \;...\; a_{nn})$	$A_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ $\boldsymbol{Tr(A) = Sum\ of\ principle\ diagonal\ elements.}$

Diagonal Matrix: If $A_{n\times n}$ is diagonal Matrix, $a_{ij} = 0, \forall i \neq j$.

Diagonal Matrix Denoted By $D = diag(a_{11}\; a_{22}\; a_{33} \;...\; a_{nn})$ **Note:** $D^K = diag(a_{11}^K\; a_{22}^K\; a_{33}^K \;...\; a_{nn}^K)$

Upper Triangular Matrix: In $A_{n\times n}$, $a_{ij} = 0, for\; i > j$	**Lower Triangular Matrix:** In $A_{n\times n}$, $a_{ij} = 0, for\; i < j$

Inverse of Matrix:

1. If $\|A\| = 0$, A is called **Singular Matrix**. And If $\|A\| \neq 0$, A is called **Non-Singular Matrix.** 2. $A^{-1}\; exists \;\Leftrightarrow \|A\| \neq 0$.	3. If $AB = BA = I$, then B is called the inverse of A. $AA^{-1} = A^{-1}A = I$ 4. $A^{-1} = \frac{adj\,(A)}{\|A\|}, Where\; adj\,(A) = (Cofactor\; Matrix)^T$
Properties: 1. $(AB)^T = B^T A^T$ 2. $(AB)^{-1} = B^{-1}A^{-1}$ 3. $(A^T)^T = A$ 4. $(A^{-1})^{-1} = A$	5. $(A^{-1})^T = (A^T)^{-1}$ 6. $\|A^{-1}\| = 1/\|A\|$ 7. $\left\|adj\left(adj\left(adj\; K\; times\; (A)\right)\right)\right\| = \|A\|^{(n-1)^K}$ 8. $adj\left(adj\left(adj\; K\; times\; (A)\right)\right) = \|A\|^{n-2}A$

Determinant of Matrix ($\boldsymbol{|A|}$): Sum of Product of any Row/ Column elements and corresponding cofactors.

Properties: 1. $\|A\| = \|A^T\|$ 2. If matrix has Zero Row/ Column, $\|A\| = 0$. 3. If two Row/ Column of matrix are equal/ proportional, $\|A\| = 0$. 4. If two Row / Column of the matrix interchanged, $\|A\| = (-1)\|A\|$. 5. $\|A\|$ = Product of diagonal elements of Upper/ Lower Triangular Matrix only. 6. $\|A^K\| = \|A\|^K$	7. If all elements of Row / Column are scalar multiple of K then, $\|A\| = K\|A\|$ 8. If all elements of Matrix are multiplied by scalar multiple (K) then, $\|A\| = K^n\|A\|, where\; n = \; dimension$ 9. If every element of a Row/ Column is multiplied by a scalar and added to another Row/ Column, then the determinant remains same. 10. $\|AB\| = \|A\|\|B\|$

RANK OF MATRIX

RANK OF MATRIX: Let A be any zero matrix of order $m \times n$,

$$Rank\;(r) = e(A) = The\; order\; of\; largest\; Non\; Zero\; minor$$

MINOR: Determinant of Sub-matrix (Square matrix).

SUB-MATRIX: matrix Obtained by deleting rows and columns.

Properties: 1. $e(A_{m\times n}) \leq \min\{m, n\}$ 2. If $e(A_{n\times n}) = n$, $\|A\| \neq 0$ else $e(A_{m\times n}) < n$. 3. $e(A) = e(A^T)$ 4. $e(A+B) = e(A) + e(B)$	5. $e(AB) = \min\{e(A), e(B)\}$ 6. If $e(A) = 0$, $A = null\; matrix$ else If $e(A) \neq 0$. 7. If $e(A) = n \;\Rightarrow e\big(adj(A)\big) = n$ $= n-1 \;\Rightarrow e\big(adj(A)\big) = 1$ $\leq n-2 \;\Rightarrow e\big(adj(A)\big) = 0.$

ROW ECHELON FORM OF MATRIX: Let $A_{m\times n}$ Matrix is said to be in Row Echelon Form if,

1. Zero Rows (if any) should be below the non-zero rows.
2. Zero before first non-zero number in row should be less than zeros before first non-zero number in next row.

$$e(A) = No.of\; Non-Zero\; Rows\; in\; Echelon\; Form$$

Note-I: To get Row Echelon Form perform Row Operations only.

Note-II: Rank of the Matrix is not affected by elementary row operations.

SYSTEM OF LINER EQUATIONS

$$\begin{aligned} a_{11}x_1 + \quad a_{12}x_2 + \quad \cdots \quad + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \quad a_{22}x_2 + \quad \cdots \quad + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots \quad \vdots \\ a_m x_1 + \quad a_m x_2 + \quad \cdots \quad + a_{mn}x_n &= b_n \end{aligned} \quad Where, m = \ No. Equations\ and\ n = No. of\ variables$$

Matrix Form: $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Leftrightarrow AX = B\ where, \begin{matrix} A = Coefficient\ Matrix\ and \\ X = Unknown/\ Variable\ Vector \\ B = Right\ hand\ side\ Vector \end{matrix}$$

Solution of System of Equations: The values of vector X satisfies $AX = B$.
Consistent System: The System has at least one solution.
In-Consistent System: The System has no solution.
Homogeneous System of Equation: $AX = O$.
Non-Homogeneous System of Equation: $AX = B$.

METHOD TO SOLVE SYSTEM OF EQUATIONS (m = n = 3)	
Matrix Inversion Method	Cramer's Method
$X = A^{-1}B$	$x_1 = \frac{\Delta_1}{\Delta}, x_2 = \frac{\Delta_2}{\Delta}, x_3 = \frac{\Delta_3}{\Delta}$

SOLUTIONS:

1. **Unique Solution:** $\Delta \neq 0\ And\ x_1 = \frac{\Delta_1}{\Delta}, x_2 = \frac{\Delta_2}{\Delta}, x_3 = \frac{\Delta_3}{\Delta}$
2. **Infinitely many solutions:** $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$
3. **No Solution:** $\Delta = 0\ and\ atleast\ one\ of\ \Delta_1, \Delta_2, \Delta_3\ is\ non\ zero.$

DISADVANTAGES:

1. Applicable only for m = n.
2. Inverse Method Fails When $|A| = 0$.
3. In Cramer's Method We need to calculate *(n+1)* determinants of order *n*. hence, No suitable for *n>4*.

GAUSSIAN ELIMINATION METHOD: Let $AX = B$ be the given system of linear equation.
AUGMENTED MATRIX: $[A|B]$ = A and B together.

$AX = B$		
Consistent System of Equation: $e(A\|B) = e(A)$		Inconsistent System of Equation: $e(A\|B) \neq e(A)$
$e(A\|B) = e(A) = r = n$	$e(A\|B) = e(A) = r < n$	**No Solutions**
Unique Solution	**Infinitely many solutions**	

HOMOGENEOUS SYSTEM OF EQUATIONS: $AX = O$

Trivial/ Zero Solution: $X = O$	**Non-Trivial/ Non-Zero Solution:** $X \neq O$

Note: Every Homogeneous system is always consistent. But Non-Trivial solutions may or may not exists, if exists infinitely many solutions exists.

$AX = O$		
$m = n$		$m \neq n$
$e(A\|B) = e(A) = r = n$ And $\|A\| \neq 0$	$e(A\|B) = e(A) = r < n$ And $\|A\| = 0$	1. $m < n$ Always possesses infinitely many Non-Trivial solutions. 2. $m > n$ If $e(A\|B) = e(A) = r = n$, Only Trivial Solution If $e(A\|B) = e(A) = r < n$, Infinitely many Non-Trivial solutions.
Only Trivial Solution	Infinitely many Non-Trivial solutions	

NULL SPACE: Set of all Solutions of $AX = O$.
NULLITY: Dimensions of null Space. $Nullity = No. of\ linerarly\ independent\ solutions = n - r$
Where r= rank of matrix, and n = number of variables.

$$Rank + Nullity = No. of\ Variables$$

Linearly Dependent: $Y \propto X$	**Linearly Independent:** Y is not proportional to X

EIGEN VALUES AND EIGEN VECTORS

CHARACTERISTIC EQUATIONS: let $A_{n\times n}$ Matrix, then $|A-\lambda I|=0$ is called characteristic equations of A.
EIGEN VALUES: Roots of characteristic equations is called Eigen Values.

CAYLEY-HAMILTON THEOREM: Every Square Matrix of order (n>1) satisfies it's own characteristic equation.
ADVANTAGES OF THEOREM: Easily find A^{-1}. And A^K can be express in lower power of A and I.

ALGEBRAIC MULTIPLICITY OF EIGEN VALUE (λ): No. of times eigen value occurred.
EIGEN VECTOR: A Non-Zero vector (X) is said to be eigen vector corresponds to the eigen value (λ) of the matrix (A) if $(A-\lambda I)X=0$.
NOTE:
1. Corresponding to one eigen value infinitely many eigen vector exists.
2. **GEOMETRIC MULTIPLICITY** = No. of linearly independent eigen vectors = $n-r$.
3. $\boldsymbol{GEOMETRIC\ MULTIPLICITY \leq\ ALGEBRAIC\ MULTIPLICITY}$ $(For\ any\ eigen\ value)$**.**

PROPERTY: If λ is eigen value of non-singular matrix (A), **1)** 1/ λ is eigen value of A^{-1} **2)** $|A|/\lambda$ is eigen value of $adj\ (A)$.

DIAGONALIZATION: Given Matrix (A) is said to be diagonalizable if there exists a non-singular matrix P such that, $AP=PD \ \Leftrightarrow A=PDP^{-1}$.
Here $P=$ Matrix with columns are eigen values of A and $D=$ Diagonal matrix with eigen values of A as elements.
PROPERTY:
1. $A^K=PD^KP^{-1}$.
2. A is diagonalizable ⇔ $|P|\neq 0$. Eg. A had n linearly in-depend solutions.

PROPERTY OF EIGEN VALUES AND EIGEN VECTORS:
1. Sum of the eigen values = Trace of the vector
2. Product of the eigen values = determinant of the vector.
3. Zero is one of the eigen value of the matrix (A) ⇔ $|A|=0$ and $e(A)=r<n$.
4. Eigen values of $A\ and\ A^T$ are same.
5. The eigen values of Upper/ Lower Triangular/Diagonal Matrix are just diagonal elements of Matrix only.
6. If $a+bi$ is eigen value of the matrix, $a-bi$ is also eigen value. **[COMPLEX RULE]**
7. If $a+\sqrt{b}$ is eigen value of the matrix, $a-\sqrt{b}$ is also eigen value. **[CONJUGATE RULE]**
8. Eigen vectors corresponding to distinct eigen values are linearly independent.
9. For different values of eigen value $\lambda_1, \lambda_2, \lambda_3, \ldots$ and Corresponding Eigen Vectors $X_1, X_2, X_3, \ldots$ following rules applicable.
    a. 1/ λ is eigen value of A^{-1}.
    b. λ^K is eigen value of A^K.
    c. $(\lambda \pm K)^m$ is eigen value of $(A\pm KI)^m$.
    d. $f(A)=a_0A^K+a_1A^{K-1}+\cdots+a_nI$, then $f(\lambda)$ is eigen value.

MODEL-I: For given matrix, find eigen value.
MODEL-II: For given matrix and eigen value/vector, find eigen vector/vector.
MODEL-III: For given Eigen value And Eigen vectors, find matrix.
MODEL-IV: Find Values using Cayley-Hamilton Theorem.
MODEL-V: Problem Related to properties.

SPECIAL MATRICES

Purely Real Number: $\bar{z}=z$	Purely Imaginary Number: $\bar{z}=-z$

Conjugate of a complex number is Mirror of the point about X axis.

Symmetric Matrix: $A=A^T$	**Hermitian Matrix:** $A=A^\theta$
Skew-Symmetric Matrix: $A=-A^T$	**Skew-Hermitian Matrix:** $A=-A^\theta$
Orthogonal Matrix: $AA^T=A^TA=I$	**Unitary Matrix:** $AA^\theta=A^\theta A=I$

OBSERVATIONS:	1. Diagonal Element of Hermitian Matrix are real number. 2. Diagonal Element of Skew-Symmetric Matrix are Zero. 3. Diagonal Element of Skew- Hermitian Matrix are Zero.

RESULTS:	1. Let $A_{n\times n}$ Matrix, $A_{n\times n}$ can be expressed as sum of symmetric and Skew symmetric matrices. 2. Let $A_{n\times n}$ Matrix, $A_{n\times n}$ can be expressed as sum of Hermitian and Skew Hermitian matrices. 3. If A is an orthogonal Matrix, the $|A| = \pm 1$. 4. If A is an orthogonal Matrix, the $A = A^{-1}$.

CONJUGATE MATRIX ($\bar{A}$): $\bar{A} = replace\ each\ element\ by\ corrosponding\ conjugates.$
CONJUGATE TRANSPOSE MATRIX (A^{θ}): $A^{\theta} = (\bar{A})^T = \overline{A^T}$

PROPERTY: 1. $\left(A^{\theta}\right)^{\theta} = A$ 2. $(AB)^{\theta} = B^{\theta}A^{\theta}$	1. $\bar{\bar{A}} = A$ 2. $\overline{A+B} = \bar{A} + \bar{B}$ 3. $\overline{(KA)} = K\bar{A}$ 4. $\overline{AB} = \bar{B}\bar{A}$

POSITIVE INTEGER POWER OF $A_{n\times n}$: A^K
IDEMPOTENT MATRIX: $A^2 = A$.
NILPOTENT MATRIX: $A^m = O$ where $m =$ Index of Nilpotent matrix (least positive integer).
INVOLUTORY MATRIX: $A^2 = I$. Where $A =$ Square Matrix
PERIODIC MATRIX: $A^{K+1} = A$. Where $K =$ Periodicity (least positive integer)

MATRIX	EIGEN VALUE
Hermitian Matrix	Always Real
Skew-Hermitian Matrix	Either Zero or Pure Imaginary
Orthogonal/ Unitary Matrix	$|A| = 1$ And can be Real or Complex conjugate
Idempotent Matrix	0 or 1
Nilpotent Matrix	0
Involutory Matrix	± 1

RESULTS: Let A be an orthogonal matrix. If λ is eigen value, $1/\lambda$ is also eigen value.

Orthogonal Vectors (90°): Two vectors are said to be orthogonal if $XY^T = O\ or\ Y^TX = O$.
Set of Orthogonal Vectors: $S = \{X_1, X_2, \dots, X_n\}$, where $X_iX_j^T = O$. Every pair is orthogonal.
Norm of Vector: $\|X\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$
Normalized Vector: $X/\|X\|$
Set of Orthonormal Vectors: $S = \{X_1, X_2, \dots, X_n\}$, where $X_iX_j^T = X_i^TX_j = O$ and $\|X_i\| = 1$.
Or If $X_iX_j^T = 0, i \neq j\ else\ 1$.
Result: If A is an orthogonal matrix, then it's rows/ columns are orthonormal.

LU DECOMPOSITION

OBJECTIVE: 1) Solving system of linear equations, 2) Finding A^{-1}, 3) $A = LU$
In $A_{n\times n} = LU$, Each matrix contains $n(n+1)/2$ unknowns/ Variables. hence total (L+U) has $n(n+1)$ unknowns. By comparing we get n^2 equations.
DOOLITTLE'S METHOD: $l_{ij} = 1, where\ i = j\ in\ the\ matrix\ L$
CROUT'S METHOD: $u_{ij} = 1, where\ i = j\ in\ the\ matrix\ U$
CHOLESKY METHOD: If a is symmetric matrix then, $A = LL^T$

Note:	1. LU Decomposition fails if any of the diagonal elements of L or U is zero. 2. LU Decomposition Exists if the matrix is positive definite.

POSITIVE DEFINITE: $A_{n\times n}$ is said to be positive definite if all loading minors of a are positive.
Gauss Elimination method operation involves = $\boldsymbol{2n^3/3}$
LU Decomposition method operation involves = $\boldsymbol{n^3/3}$
Method LU: First find forward substitute (UX=Z and LZ=B) (U) after that backward substitute(L).
RESULT: If a is non singular matrix, then $A_{n\times n} = LU$ can be obtained by row addition operation.
Where, L = Lower triangular matrix with diagonal element 1 And U = Upper triangular matrix.

MINIMAL POLYNOMIAL:	1. If $f(A) = 0$, then we say that the polynomial is annihilates the matrix A. 2. **Monic Polynomial:** The Coefficient of highest power of x is unity. **Minimal polynomial:** 1) Lowest Degree monic polynomial 2) annihilates the matrix A.