6. VECTOR CALCUSUS

DOT/SCALAR PRODUCT: $\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3 = \|a\|\|b\|\cos\theta$	**CORSS/ VECTOR PRODUCT:** $\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \|a\|\|b\|\sin\theta\ \hat{n}$
DIVERGENCE OF FUNCTION: $\nabla\cdot\bar{F}$	**NIBAL OR DEL** $(\nabla) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
CURL OF FUNCTION: $\nabla\times\bar{F}$	

GRADIENT OF A SCALAR FUNCTION: $\nabla * \bar{F} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

1. Gradient of a scalar function gives vector function.
2. Gradient of a scalar function is rate of change of function with respect to “x”, “y”, “z”.
3. From a scalar field, we can obtain vector filed by gradient.

APPLICATION OF GRADIENT OF A SCALAR FUNCTION:

1.	The unit normal vector to the surface $\emptyset(x,y,z)$ at point p is given by,	$\hat{n} = \left.\frac{\nabla\emptyset}{\|\nabla\emptyset\|}\right\|_{at\ p}$
2.	The directional derivative of surface $\emptyset(x,y,z)$ at point p in the direction $\bar{a}$ of is given by,	$\nabla\emptyset\|_{@p}\cdot\frac{\bar{a}}{\|\bar{a}\|}$
3.	The maximum value of the directional derivative of surface $\emptyset(x,y,z)$ at point p is given by,	$\nabla\emptyset\|_{@p}$

SUMMARY OF DIVERGENCE OF FUNCTION:

1.	The divergence of the vector field given by,	$\nabla\cdot\bar{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$
2.	From a vector field, we can obtain scalar filed by gradient.	
3.	The divergence of the gradient is Laplacian.	$\nabla\cdot(\nabla\bar{a}) = \nabla^2\bar{a} = \frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_2}{\partial y^2} + \frac{\partial^2 a_3}{\partial z^2}$
4.	The divergence measures outflow minus in flow.	
5.	$\bar{a}$ is referred to solenoidal or divergence free, if $\nabla\cdot\bar{a} = 0$	

SUMMARY OF CURL:

1.	$\bar{a}$ is Curl free/ irrotational vector, if $\nabla\times\bar{a} = \bar{0}$	
2.	Gradient field are irrotational.	$\nabla\times(\nabla\bar{a}) = \bar{0}$
3.	Divergence of curl of a vector function is zero.	$\nabla\cdot(\nabla\times\bar{a}) = \bar{0}$

INTRODUCTION TO VECTOR INTEGRATION:

Line Integration: $\int_a^b$ f(x)dx	Line/Contour Integration: $\int_c \vec{F}\cdot d\vec{r}$
Double Integration: $\iint_{Region} f(x,y)\,dxdy$	Closed Contour Integration: $\oint_C \vec{F}\cdot d\vec{r}$
Double Integration: $\iint_{Region} f(x,y)\,dxdy$	Surface Integration: $\iint_S \vec{F}\cdot\hat{n}dS$
Triple Integration: $\iiint f(x,y,z)dxdydz$	Closed Surface Integration: $\oiint_S \vec{F}\cdot\hat{n}dS$

Area of a triangle formed by the tips of vectors $\vec{a},\vec{b},\vec{c} = \frac{1}{2}\left|(\vec{a}-\vec{b})\times(\vec{a}-\vec{c})\right|$

Vector triple product $\vec{a}\times(\vec{b}\times\vec{c})$ of three vectors $\vec{a},\vec{b},\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

The parametric representation of the curve C is given by,

$\vec{r}(t) = (x(t), y(t), z(t)), a \le t \le b, \|\vec{r}(t)\| = 1$ $d\vec{r}(t) = \vec{r}'(t)\,dt = [dx(t), dy(t), dz(t)]$ $\vec{F} = F_1 i + F_2 j + F_3 k$	$\int_c \vec{F}\cdot d\vec{r} = \int_c \vec{F}\cdot\vec{r}'\,dt$

1) In the aerodynamics and fluid mechanics $\oint_C \vec{F}\cdot d\vec{r}$ is called circulation of $\vec{F}$ around C where $\vec{F} = Fluid\ velocity$.

2) Work done by Force: $\vec{F}$ is a force acting on a particle which moves a point $P_1$to P_2, along the line integral $\int_c \vec{F}\cdot d\vec{r}$ gives the total amount of work done by $\vec{F}$.

The value of the integral of a vector point function $\int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$ depends on the path C joining P_1 and P_2(unless the vector function is irrotational.)

If the $\vec{F}$ is a conservative field or an irrotational vector (i.e. $\nabla \times \vec{F} = \vec{0}$) in a region "R" then

1. The line integral is $\int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$ independent of path C joining P_1 and P_2 in region "R" and
$$\int_{P_1}^{P_2} \vec{F}\cdot d\vec{r} = \int_{P_1}^{P_2} \nabla\emptyset\cdot d\vec{r} = \int_{P_1}^{P_2} \mathrm{d}\emptyset = \emptyset(P_2) - \emptyset(P_1), where\ \emptyset = scalar\ potential\ function$$
2. $\oint_C \vec{F}\cdot d\vec{r} = 0$ around any closed curve C in region "R".

GREENS THEOREM:

Let "R" be a closed bounded region in the XY-plane whose boundary C. Let $\vec{F} = F_1 i + F_2 j$ be a vector function such that $F_1(x,y)$ and $F_2(x,y)$ are functions that are continuous and have continuous partial derivatives. Then

$$\int_c \boldsymbol{\vec{F}\cdot d\vec{r}} = \oint_C \boldsymbol{F_1 dx} + \oint_C \boldsymbol{F_2 dy} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{dxdy}\ , Where\ The\ direction\ of\ C\ is\ counter\ Clockwise.$$

$$\int_c \boldsymbol{\vec{F}\cdot d\vec{r}} = -\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{dxdy}\ , Where\ The\ direction\ of\ C\ is\ Clockwise.$$

SURFACE INTEGRAL:

Consider a surface "S". Divide the area "S" into "M" elements of area $\Delta S_p$where p = 1, 2, 3, …, M. Choose any point Q_p coordinates are (x_p, y_p, z_p). Let $\vec{F}(x_p, y_p, z_p) = \overrightarrow{F_p}$ and $\overrightarrow{n_p}$ be the positive unit normal vector to ΔS_p and Q_p. Form the sum $\sum_{p=1}^{M} \overrightarrow{F_p}\cdot\vec{n}\,\Delta S_p$. Where, $\overrightarrow{F_p}\cdot\vec{n}$ is normal component of $\overrightarrow{F_p}$ at Q_p. Now take the limit of this sum as M→∞. This limit is called surface integral of the normal component of $\vec{F}$ over S and is denoted by $\iint_S \vec{F}\cdot\vec{n}\,dS$.

$\iint_S \vec{F}\cdot\vec{n}\,dS = \sum_{p=1}^{M} \overrightarrow{F_p}\cdot\vec{n}\,\Delta S_p = \int_S \vec{F}\cdot\vec{n}\,dS$	$\oiint_S \vec{F}\cdot\vec{n}\,dS = closed\ surface\ integral.$

Method of evolution of surface integral:

If R_1 is the projection of S on XY-plane	$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{R_1} (\vec{F}\cdot\vec{n})\frac{dxdy}{\lvert\vec{n}\cdot\vec{k}\rvert}$
If R_2 is the projection of S on YZ-plane	$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{R_2} (\vec{F}\cdot\vec{n})\frac{dxdy}{\lvert\vec{n}\cdot\vec{i}\rvert}$
If R_3 is the projection of S on ZX-plane	$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{R_3} (\vec{F}\cdot\vec{n})\frac{dxdy}{\lvert\vec{n}\cdot\vec{j}\rvert}$

STOKES THEOREM (LINE INTEGRAL TO SURFACE INTEGRAL):

If $\vec{F} = F_1 i + F_2 j + F_3 k$ is a differential vector function defined open surface S bounded by a simple closed curve C,

$$\int_c \boldsymbol{\vec{F}\cdot d\vec{r}} = \oint_C \boldsymbol{F_1 dx} + \oint_C \boldsymbol{F_2 dy} + \oint_C \boldsymbol{F_3 dz} = \iint_S (\boldsymbol{\nabla\times\vec{F}})\cdot\boldsymbol{\vec{n}\,ds},$$

Where, $\vec{n}$ is the outward unit normal vector to the surface S.

GAUSS DIVERGENCE THEOREM (CLOSED SURFACE INTEGRAL TO VOLUME INTEGRAL):

If $\vec{F} = F_1 i + F_2 j + F_3 k$ is a differential vector function defined open surface S enclosing volume V,

$$\iint_S \boldsymbol{\vec{F}\cdot\vec{n}\,dS} = \iiint_V (\boldsymbol{\nabla\times\vec{F}})\,\boldsymbol{dV}$$