4. PARTIAL DERIVATIVE

If $u = f(x, y)$ is a function of x and y then,

$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$	$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$

Homogeneous Function:

If degree of each term in the function is same, function is said to be Homogeneous Function.
If $\boldsymbol{f(kx, ky) = K^n f(x, y)}$, f is Homogeneous function with degree **"n".**

1) Euler's Theorem:

If $u = f(x, y)$ is Homogeneous function with degree "n" in x and y,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$	$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

2) Euler's Theorem:

If $u = f(x, y)$ is not Homogeneous function but function of u (let's say F(x)) is Homogeneous function with degree "n" in x and y,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)} = G(u)$	$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = G(u)[G'(u) - 1]$

3) Euler's Theorem:

If $u = f(x, y) + g(x, y) + h(x, y)$, where f, g and h are homogenous functions of degree m, n ,p respectively,

$xu_x + yu_y = mf + ng + ph$	$x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy} = m(m-1)f + n(n-1)g + p(p-1)h$

Total Derivative:

If $u = f(x, y)$ where x and y are functions of t, the total derivative of u with respect to t is given by,

$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$	$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$ $for\ u = f(x, y, z)$
Total Differential: $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$ $for\ u = f(x, y, z)$	

Here, "x", "y", "z" are intermediate variables. "t" is independent variable. "u" is dependent variable.

Chain rule for Partial differentiation:

If $u = f(x, y)$ where $x = g(r, s), y = h(r, s)$,

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$	$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$

Here, "x", "y" are intermediate variables. "r" and "s" are independent variable. "u" is dependent variable.
If $u = f(x, y, z)$ where $x = g(r, s, t), y = h(r, s, t), z = i(r, s, t)$,

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$	$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$	$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$

Here, "x", "y", "z" are intermediate variables. "r", "s", "t" are independent variable. "u" is dependent variable.