# DIFFERENTIAL EQUATIONS

**DIFFERENTIAL EQUATIONS:**

The equation contains independent variable ($x$), dependent variable ($y$) and it's derivative ($y'$).

| ORDINARY DIFFERENTIAL EQUATION (ODE) | PARTIAL DIFFERENTIAL EQUATION (PDE) |
|---|---|
| ODEs are formed, When $y = f(only\ x)$ | PDEs are formed, When $y = f(x, z, \ldots)$ |

| ORDER OF EQs. | DEGREE OF EQs. |
|---|---|
| Highest order derivative term present in the equation.<br>E.g. 1st order, 2nd order, 3rd order. | Power of Highest order derivative term in the equation.<br>E.g. $(y')^2 + x = 0$ is 2nd Degree Equation. |

If there is fraction in power, eliminate fraction and make whole number.

| ORDINARY DIFFERENTIAL EQUATIONS (ODEs) | |
|---|---|
| **FIRST ORDER DIFFERENTIAL EQUATIONS** | **HIGHER ORDER DIFFERENTIAL EQUATIONS** |
| 1st order & 1st degree (Linear) D.E.,<br>1. Variable Separable Form,<br>2. Homogenous D.E.<br>3. Linear D.E.<br>4. Exact D.E. | Higher order & 1st degree (Linear) D.E.,<br>1. With Constant coefficients<br>2. With Variable coefficients |

| SOLUTION OF A DIFFERENTIAL EQ.: | TYPES OF SOLUTIONS | |
|---|---|---|
| | **General Solution** | **Particular Solution** |
| • Relation Between dependent & independent variables<br>• Should not contain derivative/ differential term<br>• Solution must satisfy given D.E. | Contains Arbitrary Constant | Don't Contains Arbitrary Constant. E.g. BCs are given |

| VARIABLE SEPARABLE FORM | EQUATIONS REDUCIBLE TO VARIABLE SEPARABLE FORMS | |
|---|---|---|
| $f_1(x)dx = f_2(y)dy$ | **FORM – I** | **FORM – II** |
| • Separate the variable,<br>• Integrate Both Side,<br>• Add appropriate Constant, | $\frac{dy}{dx} = f(ax + by + c)$,<br>$\boldsymbol{Let, ax + by + c = v}$ | $\frac{dy}{dx} = \frac{f(a_1x + b_1y + c_1)}{f_2(a_2x + b_2y + c_2)}$<br>Limiting Condition: $a_1/a_2 = b_1/b_2$ |

**HOMOGENOUS FUNCTION:**

$$y = f(x, y, z, \ldots) \Leftrightarrow y = K^n f(x, y, z, \ldots)\ is\ homogenous\ function\ with\ degree\ n$$

| HOMOGENOUS DIFFERENTIAL EQUATIONS | |
|---|---|
| $\boldsymbol{TYPE - A}$: $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)} = \frac{\emptyset_1(y/x)}{\emptyset_2(y/x)}, \boldsymbol{Let\ v = \frac{y}{x}}$ | $\boldsymbol{TYPE - B}$: $\frac{dx}{dy} = \frac{f_1(x,y)}{f_2(x,y)} = \frac{\emptyset_1(x/y)}{\emptyset_2(x/y)}, \boldsymbol{Let\ v = \frac{x}{y}}$ |
| Where, $f_1(x, y)$ & $f_2(x, y)$ both are homogenous equations with same degree. | |

| LINEAR/ LABNITZ'S DIFFERENTIAL EQUATIONS | | | |
|---|---|---|---|
| Dependent Variable and it's derivative are not multiplied together. | | | |
| $\boldsymbol{TYPE - A}$: $\frac{dy}{dx} + Py = Q$ | | $\boldsymbol{TYPE - B}$: $\frac{dx}{dy} + Px = Q$ | |
| $IF = \int e^{Pdx}$ | $y\ (IF) = \int Q(IF)dx + C$ | $IF = \int e^{Pdy}$ | $x\ (IF) = \int Q(IF)dy + C$ |
| Where, $P, Q$ are function of independent variable. | | | |
| $\boldsymbol{TYPE - C}$: $f'(y)\frac{dy}{dx} + P\,f(y) = Q$ | | $\boldsymbol{TYPE - D}$: $f'(x)\frac{dx}{dy} + P\,f(x) = Q$ | |
| $\boldsymbol{Let\ f(y) = v}$ | | $\boldsymbol{Let\ f(x) = v}$ | |

| EXACT DIFFERENTIAL EQUATIONS | |
|---|---|
| $Mdx + Ndy = 0\ is\ exact\ \Leftrightarrow \left[\frac{\partial M}{\partial y}\right]_x = \left[\frac{\partial N}{\partial x}\right]_y$ | $\int_{y=Constant} Mdx + \int \begin{pmatrix} Terms\ of\ N\ not \\ Containing\ x \end{pmatrix} dy = C$ |

| REDUCING EQUATION TO EXACT FORM | |
|---|---|
| **FORM-I:** If $Mdx + Ndy = 0$ is not exact differential equation but it's homogeneous differential equation, multiply equation with Integrating factor to get Form Exact DE, $IF = \frac{1}{Mx + Ny}$ | **FORM-II:** If $f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = Mdx + Ndy = 0$ is not exact differential equation, multiply equation with Integrating factor to get Form Exact DE, $IF = \frac{1}{Mx - Ny}$ |
| **FORM-III:** If $Mdx + Ndy = 0$ is not exact differential equation, | |
| $\boldsymbol{CASE\ 1}: If\ \frac{\left[\frac{\partial M}{\partial y}\right]_x - \left[\frac{\partial N}{\partial x}\right]_y}{N} = f(x) \Rightarrow IF = e^{\int f(x)dx}$ | $\boldsymbol{CASE\ 2}: If\ \frac{\left[\frac{\partial N}{\partial x}\right]_y - \left[\frac{\partial M}{\partial y}\right]_x}{M} = f(y) \Rightarrow IF = e^{\int f(y)dy}$ |

| INTEGRATING FACTORS BY INSPECTION | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Sr. No.** | **Group of Terms** | **I.F.** | **Exact Differential** | **Sr. No.** | **Group of Terms** | **I.F.** | **Exact Differential** |
| 1 | $xdy - ydx$ | $\frac{1}{x^2}$ | $d\left(\frac{y}{x}\right)$ | 6 | $xdy + ydx$ | 1 | $d(xy)$ |
| 2 | | $\frac{1}{y^2}$ | $d\left(-\frac{x}{y}\right)$ | 7 | | $\frac{1}{xy}$ | $d[\ln(xy)]$ |
| 3 | | $\frac{1}{xy}$ | $d\left[\ln\left(\frac{y}{x}\right)\right]$ | 8 | | $\frac{1}{(xy)^n}$ | $\frac{-1}{n-1} d\left[\ln\left(\frac{1}{(xy)^{n-1}}\right)\right]$ |
| 4 | | $\frac{1}{x^2 + y^2}$ | $d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$ | 9 | | $\frac{1}{x^2 + y^2}$ | $d\left[\frac{1}{2}\ln(x^2 + y^2)\right]$ |
| 5 | | $\frac{1}{x^2 - y^2}$ | $d\left[\frac{1}{2}\ln\left(\frac{x+y}{x-y}\right)\right]$ | 10 | | $\frac{1}{(x^2 + y^2)^n}$ | $\frac{-1}{2(n-1)} d\left[\frac{1}{(x^2 + y^2)^{n-1}}\right]$ |

**HIGHER ORDER LINEAR DIFFERENTIAL EQUATION:**
**WITH CONSTANT COEFFICIENTS:**

| | |
|---|---|
| $f(D)\ y = x, Where\ D \equiv d/dx$ | $Complete\ Sol. = \ Complimentary\ Fun. + Particular\ Integral$ |
| $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = X$ | Where, $a_i$ are constant coefficients, $y$ is dependent and $x$ is independent variable. |
| $[a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n]\ y = x = \ f(D)\ y$ | $Let, D \equiv d/dx$ |
| $CS = \ CF \ + \ PI$ | $CS =$ Complete Solution always denote by dependent variable |

| METHOD TO FIND CF | |
|---|---|
| Step 1: Write the given In D-Form. | Step 2: Write Auxiliary (AE) Equation, $AE = f(m) = 0$ |
| Step 3: Solve Auxiliary (AE) Equation & find it's roots. | Step 4: Write CF according to nature of roots. |

| | |
|---|---|
| **CASE-I:** Roots are Real and Distinct | **CASE-II:** Roots are Real & Equal |
| $CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$ | $CF = (C_1 + C_2 x + C_3 x^2) e^{mx}$ |
| **CASE-III:** Roots are Complex | **CASE-IV:** Roots are Complex & Equal |
| For $m = \alpha \pm i\beta$, $CF = (C_1 \cos\beta x + C_2 \sin\beta x)\ e^{\alpha x}$ | For $m_1 = m_2 = \alpha \pm i\beta$, $CF = [(C_1 + C_2 x) \cos\beta x + (C_3 + C_4 x) \sin\beta x]\ e^{\alpha x}$ |

If $x = 0$ in $f(D)\ y = x$, $CS = \ CF$ ($\because PI = 0$).

| METHOD TO FIND PI | |
|---|---|
| 6 BASIC CASES | METHOD OF VARIATION OF PARAMETERS |

| CASE | VALUE OF X | REMARKS |
|---|---|---|
| I | $e^{ax}$ | |
| II | $\sin ax\ OR \cos ax$ | |
| III | $x^m$ | $m$ is positive integer |
| IV | $e^{ax}\ V$ | $V = \sin ax\ OR \cos ax\ OR\ x^m$ |
| V | $x\ V$ | $V = \sin ax\ OR \cos ax$ |
| VI | $x^m\ V$ | $V = \sin ax\ OR \cos ax$ & $m$ is positive integer >1 |

| | |
|---|---|
| $Derivative\ D \equiv \frac{d}{dx}$ | $Integration\ I \equiv \frac{1}{D}$ |

**CASE I:** Replace $'D'$ by $'a'$.

| | |
|---|---|
| $PI = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, Where\ f(a) \neq 0$ | $If\ f(a) = 0, PI = \frac{x}{f'(a)} e^{ax}, Where\ f'(a) \neq 0\ \&\ So\ on \ldots$ |

| | | |
|---|---|---|
| $\sinh x = (e^x - e^{-x})/2$ | $\cosh x = (e^x + e^{-x})/2$ | $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$ |

**CASE II:** Replace $'D^2'$ by $' - a^2'$.

$$PI = \frac{1}{f(D)} \sin ax\ OR \cos ax = \frac{1}{\emptyset(D^2)} \sin ax\ OR \cos ax = \frac{1}{\emptyset(-a^2)} \sin ax\ OR \cos ax\ , Where\ \emptyset(-a^2) \neq 0$$

$$If\ \emptyset(-a^2) = 0, PI = \frac{x}{\emptyset'(D^2)} \sin ax\ OR \cos ax = \frac{x}{\emptyset'(-a^2)} \sin ax\ OR \cos ax\ , Where\ \emptyset'(-a^2) \neq 0\ \&\ So\ on \ldots$$

**CASE III:** Use binomial Expansion

$$PI = \frac{1}{f(D)} x^m = \frac{1}{K[1 + \emptyset(D)]} x^m = \frac{[1 + \emptyset(D)]^{-1}}{K} x^m, Use\ binomial\ Expansion$$

| | |
|---|---|
| $(1 + A)^{-1} = 1 - A^1 + A^2 - A^3 + \cdots$ | $(1 - A)^{-1} = 1 + A^1 + A^2 + A^3 + \cdots$ |

| **CASE IV:** | **CASE V:** |
|---|---|
| $PI = \frac{1}{f(D)} e^{ax}\ V = e^{ax} \frac{1}{f(D + a)}\ V, Use\ Case\ II\ \&\ III.$ | $PI = \frac{1}{f(D)} x\ V = x \frac{1}{f(D)}\ V - \frac{f'(D)}{[f(D)]^2}\ V, Use\ Case\ II.$ |

**CASE VI:**

$$PI = \frac{1}{f(D)} x^m\ V = \frac{1}{f(D)} x^m \cos ax = \frac{1}{f(D)} x^m \left[RP \cdot e^{iax}\right] = RP \left[\frac{1}{f(D)} e^{iax} x^m\right], RP = Real, Use\ Case\ IV$$

$$PI = \frac{1}{f(D)} x^m\ V = \frac{1}{f(D)} x^m \sin ax = \frac{1}{f(D)} x^m \left[IP\ e^{iax}\right] = IP \left[\frac{1}{f(D)} e^{iax} x^m\right], IP = Imaginary, Use\ Case\ IV$$

**METHOD OF VARIATION OF PARAMETERS:** Use when none of above useful & Used for 2nd order LDEs.

| | |
|---|---|
| Step 1: Find CF<br>Step 2: Compare with $CF = C_1 y_1 + C_2 y_2$<br>Step 3: Find $y_1$ & $y_2$. Also Find $y_1'$ & $y_2'$.<br>Step 4: Find Common Term $CT = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ | Step 5: Find<br>$u = \int \frac{-y_2 X}{CT} dx\ \&\ v = \int \frac{y_1 X}{CT} dx.$<br>Step 6: Find $PI = u y_1 + v y_2$<br>Step 7: $CS = CF + PI$ |

**WITH VARIABLE (INDEPENDENT VAR) COEFFICIENTS:**

| | |
|---|---|
| Cauchy's Homogenous form | Legendre's Homogenous form |

**CAUCHY'S HOMOGENOUS FORM:**

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = [a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_{n-1} x D + a_n]\ y = X$$

$$\boldsymbol{f(xD)\ y = X, Where\ D \equiv d/dx}$$

| | | | | |
|---|---|---|---|---|
| $\boldsymbol{Let, x = e^z \Rightarrow \ln x = z\ \&\ D_1 \equiv d/dz}$ | $\boldsymbol{xD = D_1}$ | $\boldsymbol{x^2 D^2 = D_1(D_1 - 1)}$ | $\boldsymbol{x^3 D^3 = D_1(D_1 - 1)(D_1 - 2)}$ | $\cdots$ |

**LEGENDRE'S HOMOGENOUS FORM:**

$$a_0 (a + bx)^n \frac{d^n y}{dx^n} + a_1 (a + bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} (a + bx) \frac{dy}{dx} + a_n y$$
$$= [a_0 (a + bx)^n D^n + a_1 (a + bx)^{n-1} D^{n-1} + \cdots + a_{n-1} (a + bx) D + a_n]\ y = X$$

$$\boldsymbol{f[(a + bx)D]\ y = X, Where\ D \equiv d/dx}$$

| | | | |
|---|---|---|---|
| $\boldsymbol{Let, a + bx = e^z \Rightarrow \ln(a + bx) = z\ \&\ D_1 \equiv d/dz}$ | $\boldsymbol{(a + bx)D = bD_1}$ | $\boldsymbol{(a + bx)^2 D^2 = b^2 D_1(D_1 - 1)}$ | $\cdots$ |

Here, z is independent variable.

## FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

| | | | | |
|---|---|---|---|---|
| $p = \frac{\partial f}{\partial x}$ | $q = \frac{\partial f}{\partial y}$ | $r = \frac{\partial^2 f}{\partial x^2}$ | $s = \frac{\partial^2 f}{\partial x\, \partial y}$ | $t = \frac{\partial^2 f}{\partial y^2}$ |
| 1st Order | 1st Order | 2nd Order | 2nd Order | 2nd Order |

**1ST ORDER PDEs:** The equation contains only $p$ & $q$ terms.

| **1ST ORDER PDEs** | |
|---|---|
| **Linear 1ST Order PDEs** | **Non-Linear 1ST Order PDEs** |
| If $p$ & $q$ are with degree one and $p$ & $q$ are not multiplied together are linear 1st Order PDEs. | If $p$ & $q$ are not with degree one or $p$ & $q$ are multiplied together are Non-Linear 1st Order PDEs. |

**LINEAR 1ST ORDER PDEs:**

| | | |
|---|---|---|
| **LAGRANGE'S EQUATION** | $\boldsymbol{Pp + Qq = R}$ | $\boldsymbol{Auxiliary\ Equation: \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$ |

**NON-LINEAR 1ST ORDER PDEs:**

| | | |
|---|---|---|
| **TYPE-I:** $f(p, q\ only) = 0$ | **Sol:** $z = ax + by + c$ | Where, $a$ & $b$ relate with relation $f(a, b) = 0$ |

**CONCEPT:** Non-Linear 1st Order PDEs can only contain 2 constants.

| | | |
|---|---|---|
| **TYPE-II:** $f(z, p, q) = 0, Where\ z = f(x, y)$ & $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$ | $\boldsymbol{Let\ t = x + ay}$ **be a trial solution also,** $\boldsymbol{z = g(t)}$<br>$z$ : Initial Variable; $t$ : Intermediate Var.; $x, y$ : Final Variable<br>$\therefore\ z = f_{composite}(t)$ | |
| $\therefore \frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ $will\ exist\ as\ Total\ Derivative$ | $\frac{\partial z}{\partial x} = \frac{dz}{dt}\frac{\partial t}{\partial x} = \frac{dz}{dt} = p$ | $\frac{\partial z}{\partial y} = \frac{dz}{dt}\frac{\partial t}{\partial y} = a\frac{dz}{dt} = q$ |
| $\therefore f(z, p, q) = f\left[z, \frac{dz}{dt}, a\frac{dz}{dt}\right] = 0, Where\ t = x + ay$ & $z = g(t)$ | | |

| | |
|---|---|
| **TYPE-III:** When Eq. has $p, q, x, y$ such that, $f_1(p, x) = f_2(q, y) = k$ $Where\ p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$ | Find $p, q$ from given conditions. $dz = p\ dx + q\ dy$ $\therefore z = \int p\ dx + \int q\ dy + C$ |

| | |
|---|---|
| **TYPE-IV:** Clairaut's form, $z = px + qy + f(p, q)$ | $\boldsymbol{Sol}: z = ax + by + f(a, b)$ |

| **2ND ORDER PDE (TYPE IDENTIFICATION)** | | | |
|---|---|---|---|
| $A\frac{\partial^2 f}{\partial x^2} + B\frac{\partial^2 f}{\partial x\,\partial y} + C\frac{\partial^2 f}{\partial y^2} + D\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial y} + Ff = G$ | **TYPE OF PDE** | | |
| | **ELLIPTICAL** | **PARABOLIC** | **HYPERBOLIC** |
| $\Delta = B^2 - 4AC$ | $\Delta < 0$ | $\Delta = 0$ | $\Delta > 0$ |

| **WAVE EQUATION** | **HEAT EQUATION** | **LAPLACE EQUATION** |
|---|---|---|
| $\frac{\partial^2 u}{\partial t^2} = C^2\frac{\partial^2 u}{\partial x^2}$ | $\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2}$ | $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ |

| **METHOD OF SEPARATION OF VARIABLE** |
|---|