3. INTEGRATION

First fundamental theorem of integral calculus:

If f(x) is continuous [a,b] then the function F(x) = $\int_a^b$ f(t)dt is,

1) Continuous [a,b]
2) differentiable (a,b)
3) $\frac{dF(x)}{dx} = f(x)$

Second fundamental theorem of integral calculus (Area Under Curve):

If f(x) is continuous [a,b] and F(x) is antiderivative of f(x) then, $\int_a^b \mathrm{f(x)dx} = F(b) - F(a)$

Mean Value theorem of integral:

If f(x) is continuous [a,b] then there exist a real number C ϵ (a,b) such that $F(C) = \frac{\int_a^b \mathrm{f(x)dx}}{b-a}$

Properties of definite integrals:

$\int_a^b \mathrm{f(x)dx} = \int_a^b \mathrm{f(t)dt}$	$\int_a^b \mathrm{f(x)dx} = -\int_b^a \mathrm{f(x)dx}$
$\int_a^b \mathrm{f(x)dx} = \int_a^c \mathrm{f(x)dx} + \int_c^b \mathrm{f(x)dx},\ where, c \in \mathrm{(a,b)}$	
$\int_0^a \mathrm{f(x)dx} = \int_0^a \mathrm{f(a-x)dx}$	$\int_a^b \mathrm{f(x)dx} = \int_a^b \mathrm{f(a+b-x)dx}$
$\int_{-a}^a \mathrm{f(x)dx} = 0$ $where\ f(x) = -f(x), odd\ function$	$\int_{-a}^a \mathrm{f(x)dx} = 2\int_0^a \mathrm{f(x)dx},$ $where\ f(x) = f(-x), even\ function$

Improper integrals:

First Kind (Limits infinite): $\int_a^b$ f(x)dx is said to be Improper integral if a = ∞ **or** b **= -**∞ **or** both.

Second Kind (Function infinite): $\int_a^b$ f(x)dx is said to be Improper integral if a and b are finite but f(x) is infinite for some x ϵ (a, b).

Comparison Test:

1) If 0 ≤ f(x) ≤ g(x) for all x ϵ [a, b] and $\int_a^b$ g(x)dx converges than $\int_a^b$ f(x)dx also converges.
2) If 0 ≤ f(x) ≤ g(x) for all x ϵ [a, b] and $\int_a^b$ g(x)dx diverges than $\int_a^b$ f(x)dx also diverges.

Improper Integrals for First Kind Test: If f(x) and g(x) are two positive functions such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = K\ (Finite)$ then $\int_a^\infty$ f(x)dx and $\int_a^\infty$ g(x)dx both converges or diverges together.

From the definition of limits,

There exists some N (x > N) such that $K - 1 < \frac{f(x)}{g(x)} < K + 1$ ==> f(x) < (K+1) g(x)

$\int_a^\infty$ **f(x)dx =** $\int_a^N$ **f(x)dx +** $\int_N^\infty$ **f(x)dx, Where** $\int_a^N$ **f(x)dx is finite and** $\int_N^\infty$ **f(x)dx** $< (K+1)$ $\int_N^\infty$ **g(x)dx = Finite.**

Improper Integrals for Second Kind Test: If f(x) and g(x) are two positive functions and,

1) f(x) → ∞ as x → a such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l\ (Finite)$
2) f(x) → ∞ as x → b such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l\ (Finite)$

Then, $\int_a^b$ f(x)dx and $\int_a^b$ g(x)dx both converges or diverges together.

Gamma Function: $\Gamma\ (n) = \int_0^\infty e^{-x}x^{n-1}dx = (n-1)!\ \ , for\ n > 0$

$\Gamma\ (1/2) = \sqrt{\pi},\ \ \Gamma\ (-1/2) = -2\sqrt{\pi}$

Beta Function: $\beta\ (m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

$\beta\ (m,n) = \beta\ (n,m)$ And $\beta\ (m,n) = \frac{\Gamma\ (m)+\Gamma\ (n)}{\Gamma\ (m+n)}$ And $\beta\ (m,n) = 2\int_0^{\frac{\pi}{2}} \sin\theta^{2m-1}\cos\theta^{2n-1}\,d\theta$

ILATE: Inverse => Log => Arithmetic => Trigonometric => Exponential

The length of the arc y = f(x) between x = a and x = b is given by $\boldsymbol{l} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

The volume of the solid generated by revolving the area bounded by the curve y = f(x), x-axis and the x = a and x = b about y-axis is $\boldsymbol{V} = \int_a^b \pi y^2 dx$

The region R enclosed by curves $y_1 = f(x)$ and $y_2 = g(x)$ is rotated about the x-axis. the volume of the resulting solid $\boldsymbol{V} = \int_a^b \pi({y_1}^2 - {y_2}^2)dx$

Leibnitz Formula: $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f\big(v(x)\big)\, u'(x) - f\big(u(x)\big)\, v'(x)$

Integral as sum of the limit:

If f(x) is continuous in the interval [a, b] then the definite integral of f(x) with limits a and b is defined by the equation

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a+rh), where\ h = \frac{b-a}{n}$$

To express a given series as definite integral:

1) Write the general term i.e. i.e. $\frac{1}{n} f\left(\frac{r}{n}\right)$
2) Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx
3) integrate the resulting expression taking the **lower limit** = $\lim_{n\to\infty}\frac{r}{n}$ where, r is as in the first term And **Upper limit** = $\lim_{n\to\infty}\frac{r}{n}$ where, r is as in the last term

Double Integrals:

$$\iint_{Region} f(x,y)\, dxdy = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k,y_k)\Delta A_k = \int_{x=a}^{b}\int_{y=g(x)}^{h(x)} f(x,y)\, dydx = \int_{y=c}^{d}\int_{y=p(y)}^{q(y)} f(x,y)\, dxdy$$

Application of Double Integrals:

$$\iint_{Region} dxdy = Area\ of\ region$$

Volume using double integrals:

The volume V beneath the surface z = f(x, y) > 0 and above the region R in the xy-plane is $V = \iint_{Region} z\, dxdy$

Triple Integrals:

$$\iint_{3D\ Region} f(x,y,z)\, dxdydz = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k,y_k,z_k)\Delta V_k = \int_{z=z1}^{z2}\int_{y=y1(z)}^{y2(z)}\int_{x=x1(y,z)}^{x2(y,z)} f(x,y,z)\, dxdydz$$

$$= \int_{x=x1}^{x2}\int_{z=z1(x)}^{z2(x)}\int_{y=y1(z,x)}^{y2(z,x)} f(x,y,z)\, dydzdx = \int_{y=y1}^{y2}\int_{x=x1(y)}^{x2(y)}\int_{z=z1(x,y)}^{z2(x,y)} f(x,y,z)\, dzdxdy$$

Note: $f(x,y,z)$ is continuous over the closed region bounded by surfaces in 3D space

Application of Triple Integrals:

$$\iiint_{Region} dxdydz = Volume\ of\ 3D\ Closed\ bounded\ region\ T$$

Change of Variable:

$$\int_{x=a}^{b} f(x)\, dx\, ; x = g(u), dx = g'(u)\, du \implies \int_{x=a}^{b} f(x)\, dx = \int_{u=c}^{d} f\big(g(u)\big)g'(u)du$$

$$\iint f(x,y)dxdy;\ x = g(u,v), y = h(u,v) \implies \iint f(x,y)dxdy = \iint f\big(g(u,v),h(u,v)\big)\,|\mathrm{J}|\, dudv$$

$$\iiint f(x,y,z)dxdydz = \iiint f(g(u,v,w),h(u,v,w),i(u,v,w))\,|\mathrm{J}|\, dudvdw$$

Where J = Jacobian = $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ (For 2D) = $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$ (For 3D)

For Polar coordinate, J = R And x = R cos θ, y = R sin θ

For Cylindrical coordinate, J = R And x = R cos θ, y = R sin θ, z = z

For Spherical coordinate, J = R^2 sin θ, And x = R sin θ cos φ, y = R sin θ sin φ, z = R cos θ