4. EXTERNAL SURFACES (FINS)

WHY WE NEED FINS: From the Newton's Law of cooling, for increasing heat Transfer,

$T_\infty \downarrow$: Practically not Possible	$h \uparrow$: Practically not Possible	$A_s \uparrow$: Practically Possible

FINS: It's extra solid material attached to the base to increase heat transfer by increasing the surface area.

$L =$ Length of Fins	$W =$ Width of Fins
$t =$ Thickness of Fins	$T_0 =$ Base or Source Temperature of Fins
$\frac{h\delta}{K} < 0.2(Treat\ like\ a\ 1D\ Heat\ Flow)$	$\delta =$ Thickness $A_{fin} = A_{Lateral} + A_{tip} \approx A_{Lateral} = 2(W+t)L = PL$

Heat Transfer in fins: $Q_1(\rightarrow) = Q_2(\uparrow) + Q_3(\rightarrow)$

IMPORTANT POINT W.R.T. FINS: 1. Fins should have higher thermal conductivity 2. Fins should be Strong and Anti-corrosive in nature 3. Fins should have Low Weight (Due to cantilever Struct.) 4. Fins should be environment friendly. 5. Fin cost Should be low or moderate.	**USE-CASES:** 1. Fin is used when h is less (Free Convection with gases) 2. Aluminium Material generally used.

GENERALISED DIFFERENTIAL EQUATION FOR FINS:

ASSUMPTIONS:

• 1D Heat Flow. • Steady State. • No internal heat generation • Material homogenous and isotropic • Thermal conductivity is constant. • Base or Source temperature is constant. $T_0 = C$	• Surrounding fluid temperature is constant. $T_\infty = C$ • Heat transfer coefficient value is constant. • Radiation heat loss neglected. • Prefect contact between fin and base material. • Fin having constant cross-sectional area.

$\frac{d\theta}{dx} = \frac{dT}{dx}$	$\frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2}$	$\theta = T - T_\infty =$ Excess Temperature $\theta_0 = T_0 - T_\infty$	$\frac{hP}{KA_c} = m^2$
$Q_{in} = Q_x = -KA_c\frac{\partial T}{\partial x}$	$Q_{x+dx} = Q_x + \frac{\partial Q_x}{\partial x}dx$	$Q_{Conv.} = hA_s(T - T_\infty) = h\,Pdx(T - T_\infty)$	
From the energy Balance, $Q_x - [Q_{x+dx} + Q_{Conv}] = 0$	$\frac{d^2T}{dx^2} - m^2(T - T_\infty) = 0$	$\frac{d^2\theta}{dx^2} - m^2\theta = 0$	$\boldsymbol{\theta = C_1e^{-mx} + C_2e^{mx}}$ $\boldsymbol{= A\cosh mx + B\sinh mx}$
$\boldsymbol{mL} =$ **Constant**	$(mL)^2 = \frac{hPL}{KA_c/L} = \frac{hPA_s}{KA_c/L} = \frac{R_{Cond.}}{R_{Conv.}} = Hybrid\ Biot\ Number$		

NOTE: At the Base Surface: $Q_{in} = Q_{out} = -KA_c\,dT/dx|_{x=0} = -KA_c\,d\theta/dx|_{x=0} = \boldsymbol{Q_{Fin}}$

Circular Fin	Square Fin	Rectangle Fin	Equilateral Triangle Fin
$D = 4\,A_c/P$	$a = 4\,A_c/P$	$t = 2\,A_c/P\ (\because t \ll W)$	$a = 4\sqrt{3}\,A_c/P$
$m = \sqrt{4\,h/KD}$	$m = \sqrt{4\,h/Ka}$	$m = \sqrt{2\,h/Kt}$	$m = \sqrt{4\sqrt{3}\,h/Ka}$

VERY LONG OR INFINITE LENGTH OF FIN:

$BC\ x = 0, \theta = \theta_0$ $x = \infty, \theta = 0$, $T_L = T_\infty$	$C_2 = 0(From\ 2nd\ BC)$ $C_1 = \theta_0(From\ 1st\ BC)$	$\frac{\theta}{\theta_0} = e^{-mx} = \frac{T - T_\infty}{T_0 - T_\infty} = Exponential$

At the Base Surface: $Q_{in} = Q_{out} = -KA_cm\theta_0 = \sqrt{hPKA_c}\theta_0 = Q_{Fin}$

CASE-I: Same temperature at different length for different fins with same base temperature.

For Fin-1: $At\ x = x_1, T = T_1$ $\frac{T_1 - T_\infty}{T_0 - T_\infty} = e^{-m_1x_1}$	For Fin-2: $At\ x = x_2, T = T_1$ $\frac{T_1 - T_\infty}{T_0 - T_\infty} = e^{-m_2x_2}$	$\frac{m_1}{m_2} = \frac{x_2}{x_1}$

For Circular Fins,

$\frac{K_2D_2}{K_1D_1} = \left(\frac{x_2}{x_1}\right)^2$	For $K_1 = K_2$, $\frac{D_2}{D_1} = \left(\frac{x_2}{x_1}\right)^2$	For $D_1 = D_2$, $\frac{K_2}{K_1} = \left(\frac{x_2}{x_1}\right)^2$	For $K_1 = K_2 \,\&\, D_1 = D_2$, $x_1 = x_2$ If $x_1 \neq x_2$, There is impurities

CASE-II: Different temperature at different length in same fin with unknown base temperature.

For Fin-1: $At\ x = x_1, T = T_1$ $\frac{T_1 - T_\infty}{T_0 - T_\infty} = e^{-mx_1}$	For Fin-1: $At\ x = x_2, T = T_2$ $\frac{T_2 - T_\infty}{T_0 - T_\infty} = e^{-mx_2}$	$\frac{T_1 - T_\infty}{T_2 - T_\infty} = e^{m\Delta x} \neq f(time)$

FIN OF FINITE LENGTH:

CASE-I: Insulated tip or adiabatic tip.

$BC\ x = 0, \theta = \theta_0$ $x = L, d\theta/dx = 0', T_L \neq T_\infty$	$A = \theta_0\ (From\ 1st\ BC)$ $B = -A \tanh mL\ (From\ 2nd\ BC)$	$\frac{\theta}{\theta_0} = \frac{\cosh m(L-x)}{\cosh mL} = \frac{T - T_\infty}{T_0 - T_\infty}$
At Tip, $x = L, T = T_L, \frac{T_L - T_\infty}{T_0 - T_\infty} = \frac{1}{\cosh mL}$	$Q_{in} = Q_{out} = KA_c m\theta_0 \tanh mL = \sqrt{hPKA_c}\theta_0 \tanh mL = Q_{Fin}$	

CASE-II: Convective Heat Loss from tip

$BC\ x = 0, \theta = \theta_0$ $x = L, -K\ d\theta/dx = h(T_L - T_\infty)$	$\frac{\theta}{\theta_0} = \frac{\cosh m(L-x) + \frac{h}{mK}\sinh m(L-x)}{\cosh mL + \frac{h}{mK}\sinh mL} = \frac{T - T_\infty}{T_0 - T_\infty}$
$Q_{in} = Q_{out} = KA_c m\theta_0 \left[\frac{\tanh mL + \frac{h}{mK}}{1 + \frac{h}{mK}\tanh mL}\right] = Q_{Fin}$	We can use corrected length approach for the same problem, $\frac{\theta}{\theta_0} = \frac{\cosh m(L_C - x)}{\cosh mL_C} = \frac{T - T_\infty}{T_0 - T_\infty}$ & $Q_{Fin} = KA_c m\theta_0 \tanh mL_C = \sqrt{hPKA_c}\theta_0 \tanh mL_C$

CORRECTED LENGTH (L_C) APPROACH:

$Q_{tip} = Q_{New\ Area} \Rightarrow hA_c\Delta T = hP\Delta L\Delta T$	$\Delta L = A_c/P = Volume/Surface\ Area$

$L_C = L + \Delta L$	For Circular Fin, $L_C = L + D/4$	For Rectangular Fin, $L_C = L + t/2 \approx L$

SIGNIFICATION OF (h/mK): $h/mK = \sqrt{Bi}$

$h/mK < 1$	$h/mK = 1$	$h/mK > 1$
Fin Acts as Heat Conductor	No use of fins	Fin Acts like a heat insulator

HEAT TRANSFER WITHOUT USE OF FIN: $Q_{Without\ Fin} = hA_c\theta_0$

CONDITION FOR INFINITE LENGTH: $\tanh mL = 1\ or\ mL \geq 5$

$Q_{Finite} = KA_c m\theta_0 \tanh mL = \sqrt{hPKA_c}\theta_0 \tanh mL$	$Q_{Infinite} = KA_c m\theta_0 = \sqrt{hPKA_c}\theta_0$
For $\tanh mL = 1\ or\ mL \geq 5$, $Q_{Finite} = Q_{Infinite}$	$\%\ \begin{matrix} Error\ in \\ Heat\ Transfer \end{matrix} = \frac{Q_{Infinite}}{Q_{Finite}} - 1 = \frac{1}{\tanh mL} - 1$
$L \geq 5/m, Mathematically$ (Cost is more)	$L \geq 2.65/m, Practically$ (Cost is Less)

TWO RESERVOIRS AT DIFFERENT TEMPERATURE: Fin of finite length connected between two reservoirs.

$T_1 > T_2 > T_\infty$ $Q_{in} = Q_1 + Q_2$	$BC\ x = 0, \theta = \theta_1$ $x = L, \theta = \theta_2$	$\theta_1 = T_1 - T_\infty$ $\theta_2 = T_2 - T_\infty$	$A = \theta_1$	$B = \frac{\theta_2 - A\cosh mL}{\sinh mL}$
$\theta = \theta_1 \frac{\sinh m(L-x)}{\sinh mL} + \theta_2 \frac{\sinh mx}{\sinh mL}$		$\frac{d\theta}{dx} = \frac{m}{\sinh mL}[\theta_1 \cosh mx + \theta_2 \cosh m(L-x)]$		
$Q_{Fin1} = -KA_c\ d\theta/dx\|_{x=0}$ $Q_{Fin2} = -KA_c\ d\theta/dx\|_{x=0}$		$Q_{Fin} = Q_{Fin1} + Q_{Fin2} = KA_c m(\theta_1 + \theta_2)\left[\frac{\cosh mL - 1}{\sinh mL}\right]$		
$at\ \frac{d\theta}{dx} = 0, \frac{\theta_2}{\theta_1} = \frac{\cosh m(L - \boldsymbol{x_{min}})}{\cosh mL}$		$at\ x_{min}, \theta = \theta_{min} = T_{min} - T_\infty$ • Draw the Diagram & Show L		

At $\theta_1 = \theta_2 = \theta_0$,

$\frac{\theta}{\theta_0} = \frac{\sinh m(L-x)}{\sinh mL} + \frac{\sinh mx}{\sinh mL}$	$Q_{Fin} = 2KA_c m\theta_0 \left[\frac{\cosh mL - 1}{\sinh mL}\right]$	$x_{min} = \frac{L}{2}$	$@x_{min}, \frac{\theta_{min}}{\theta_0} = \frac{1}{\cosh(mL/2)}$

FIN EFFICIENCY (η):

Real Fin	**Ideal Fin**	
Temperature varies w. r. t. x ($R_{Cond.} \neq 0$)	Temperature doesn't vary w. r. t. x ($R_{Cond.} = 0$)	$\eta = \frac{Q_{Fin}}{Q_{Fin\ Max}}$
$Q_{RealFin} = \int_0^L hPdx\ \theta$	$Q_{IdealFin} = hA_s\theta_0, A_s = A_{fin} = PL$ $\&\ \theta_0 = T_0 - T_\infty$	$= \frac{Q_{RealFin}}{Q_{IdealFin}}$

For very long or infinite length of Fin, $\eta = 1/mL$	For Finite length Fin, Insulated Tip with Adiabatic Tip, $\eta = \tanh mL/mL$ & as $mL \to 0, \eta = 1$	For Finite length Fin, Convection at tip, $\eta = \tanh mL_c/mL_c$

OVERALL EFFICIENCY (η_0): It's used for multiple fins.

$\eta_0 = \frac{Q_{Tot.}}{Q_{Tot.Max}} = 1 - \frac{A_F}{A_T}(1-\eta)$		A_F = Area of all finned surfaces, A_{UF} = Area of un-finned surfaces = $A_T - A_F$,	
$Q_{Fin} = \frac{\theta_0}{1/NKA_c m \tanh mL} = \frac{\Delta T}{R_F}$	$Q_{UF} = \frac{\theta_0}{1/hA_{UF}} = \frac{\Delta T}{R_{UF}}$	$R_{Eq} = \frac{1}{R_{UF}} + \frac{1}{R_F}$	$R_T = R_{Pipe} + R_{Eq}$
$Q_{Fin} = \eta Q_{max} = \eta h A_F \theta_0$	$Q_{Tot.} = Q_{Fin} + Q_{UF} = hA_F\theta_0\left[1 - \frac{A_F}{A_T}(1-\eta)\right]$		At $\eta = 1$, $Q_{Tot.Max} = hA_F\theta_0$

FIN EFFECTIVENESS (ϵ): $\epsilon = Q_{Fin}/Q_{Without\ Fin}$

$\epsilon > 1$	$\epsilon = 1$	$\epsilon < 1$
Fin Acts as Heat Conductor	No use of fins	Fin Acts like a heat insulator

Note: To justify the cost $\epsilon > 2$

For Very long or Infinite length of fin, $\epsilon = \frac{1}{\sqrt{Bi}} = \frac{mK}{h} = \sqrt{\frac{PK}{hA_c}}$ 1. $\propto \sqrt{K}$ 2. $\propto \sqrt{1/h}$ 3. $\propto \sqrt{P/A_c}$ (Thin fins are more effective)	For finite length of fin & Insulated tip or adiabatic tip, $\epsilon = \frac{\tanh mL}{\sqrt{Bi}} = \frac{mK}{h}\tanh mL = \sqrt{\frac{PK}{hA_c}}\tanh mL$ ϵ increases and η decreases with length. So, to get higher both values select the optimum length. Effectiveness is related to Thermal & economics.

$\frac{\epsilon}{\eta} = \frac{Q_{Max}}{Q_{Without\ Fin}} = \frac{A_F}{A_c} = \frac{Surface\ Area}{C/s\ \ Area}$	For Circular Fin, $\epsilon/\eta = 4\,L/D$

Fin Effectiveness can be increased by,

1. Select a fin material high thermal conductivity.
2. Attach a fin in environment having low value of heat transfer coefficient (Free convection with Gases).
3. Select a geometry having high value of ratio of perimeter to cross sectional area (Thin fin preferred).

Note:

1. In actual practice short, thin multiple fins in closed space arrangement required.
2. With increasing length if fin effectiveness increases but efficiency decreases.

OVERALL EFFECTIVENESS (ϵ_0): It's used for multiple fins.

$\epsilon_0 = \frac{Q_{Tot.}}{Q_{No\ Fin}} = \frac{Q_{Tot.}}{Q_{Base}} = \frac{\eta A_F + A_{UF}}{A_{Base}}$	$Q_{Tot.} = \eta h A_F\theta_0 + hA_{UF}\theta_0$ $Q_{Base} = hA_{Base}\theta_0$