Matrix theory Assignment 15

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***Abstract*—This document contains the concept of linear functionals.**

Download all python codes from

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https://github.com/shivangi-975/EE5609-
    Matrix_Theory/tree/master/Assignment15/
    Codes
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Download latex-tikz codes from

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https://github.com/shivangi-975/EE5609-
    Matrix_Theory/blob/master/Assignment15/
    Assignment_15.tex
```

1 Problem

In R^3, let $\alpha_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $\alpha_2 = \begin{pmatrix}0\\1\\-2\end{pmatrix}$ and $\alpha_3 = \begin{pmatrix}-1\\-1\\0\end{pmatrix}$. Describe a linear functional f on R^3 such that $f(\alpha_1) = f(\alpha_2) = 0$ but $f(\alpha_3) \neq 0$.If α=$\begin{pmatrix}2\\3\\-1\end{pmatrix}$ such that $f(\alpha) \neq 0$

2 Solution

Let us consider $\alpha = \begin{pmatrix}a\\b\\c\end{pmatrix}$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha \tag{2.0.1}$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.2}$$

Hence:

$$A = \begin{pmatrix}1 & 0 & -1\\0 & 1 & -1\\1 & -2 & 0\end{pmatrix} \tag{2.0.3}$$

$$\begin{pmatrix}1 & 0 & -1\\0 & 1 & -1\\1 & -2 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.4}$$

$x = A^{-1}\alpha$ will give solution of equation.

$$\begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\1 & -2 & 0 & 0 & 0 & 1\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \tag{2.0.5}$$

$$\begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\0 & -2 & 1 & -1 & 0 & 1\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + 2R_2} \tag{2.0.6}$$

$$\begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\0 & 0 & -1 & -1 & 2 & 1\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3/(-1)} \tag{2.0.7}$$

$$\begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\0 & 0 & 1 & 1 & -2 & -1\end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_3} \tag{2.0.8}$$

$$\begin{pmatrix}1 & 0 & 0 & 2 & -2 & -1\\0 & 1 & -1 & 0 & 1 & 0\\0 & 0 & 1 & 1 & -2 & -1\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_3} \tag{2.0.9}$$

$$\begin{pmatrix}1 & 0 & 0 & 2 & -2 & -1\\0 & 1 & 0 & 1 & -1 & -1\\0 & 0 & 1 & 1 & -2 & -1\end{pmatrix} \tag{2.0.10}$$

Hence,

$$A^{-1} = \begin{pmatrix}2 & -2 & -1\\1 & -1 & -1\\1 & -2 & -1\end{pmatrix} \tag{2.0.11}$$

Simplifying,

$$\mathbf{x} = A^{-1}\alpha \tag{2.0.12}$$

$$\implies \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = A^{-1}\begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.13}$$

$$\implies \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2 & -2 & -1\\1 & -1 & -1\\1 & -2 & -1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.14}$$

$$x_1 = \begin{pmatrix}2\\-2\\-1\end{pmatrix}^T \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.15}$$

Taking $\mathbf{u_1}=\begin{pmatrix}2 & -2 & -1\end{pmatrix}$ and $\mathbf{y}=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$$x_1 = \mathbf{u_1}^T\mathbf{y} \tag{2.0.16}$$

$$x_2 = \begin{pmatrix}1\\-1\\-1\end{pmatrix}^T\begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.17}$$

Taking $\mathbf{u_2}=\begin{pmatrix}1 & -1 & -1\end{pmatrix}$ and $\mathbf{y}=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$$x_2 = \mathbf{u_2}^T\mathbf{y} \tag{2.0.18}$$

$$x_3 = \begin{pmatrix}1\\-2\\-1\end{pmatrix}^T\begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.19}$$

Taking $\mathbf{u_3}=\begin{pmatrix}1 & -2 & -1\end{pmatrix}$ and $\mathbf{y}=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$$x_3 = \mathbf{u_3}^T\mathbf{y} \tag{2.0.20}$$

Given, f is a linear functional on R^3,

$$\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \tag{2.0.21}$$

$$\implies f(\alpha) = \begin{pmatrix}\mathbf{u_1}^T\mathbf{y} & \mathbf{u_2}^T\mathbf{y} & \mathbf{u_3}^T\mathbf{y}\end{pmatrix}\begin{pmatrix}f(\alpha_1)\\f(\alpha_2)\\f(\alpha_3)\end{pmatrix} \tag{2.0.22}$$

$$\implies f(\alpha) = \mathbf{x}^T\begin{pmatrix}f(\alpha_1)\\f(\alpha_2)\\f(\alpha_3)\end{pmatrix} \tag{2.0.23}$$

Given, $f(\alpha_1) = f(\alpha_2) = 0$ and $f(\alpha_3) \neq 0$.

$$f(\alpha) = \mathbf{x}^T\begin{pmatrix}0\\0\\f(\alpha_3)\end{pmatrix} \tag{2.0.24}$$

$$\implies f(\alpha) = \left(\mathbf{u_3}^T\mathbf{y}\right)f(\alpha_3) \tag{2.0.25}$$

Now for $\alpha=\begin{pmatrix}2, 3, -1\end{pmatrix}$ to prove $f(\alpha) \neq 0$,

$$\implies f(\alpha) = \left(\mathbf{u_3}^T\mathbf{y}\right)f(\alpha_3) \tag{2.0.26}$$

$$\implies -3f(\alpha_3) \neq 0 \tag{2.0.27}$$

Hence, proved $f(\alpha) \neq 0$