Assignment 5

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***Abstract*—This document explains the the concept of finding two straight lines from given second degree equation**

Download all python codes from

https://github.com/shivangi−975/EE5609−Matrix_Theory/tree/master/Assignment5/Codes

Download latex-tikz codes from

https://github.com/shivangi−975/EE5609−Matrix_Theory/blob/master/Assignment5/Assignment5.tex

1 Problem

Find the value of k such that $x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0$ represent pairs of straight lines.

2 Theory

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{2.0.5}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{2.0.6}$$

Equating their product with (2.0.2), we get

$$(\mathbf{n}_1^T\mathbf{x} - c_1)(\mathbf{n}_2^T\mathbf{x} - c_2)$$
$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.7}$$

(2.0.7) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.8}$$

3 Solution

Given,

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \tag{3.0.1}$$

Equating (3.0.1) to (2.0.2), we get

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{u}^T = \begin{pmatrix} \frac{-5}{2} & \frac{-7}{2} \end{pmatrix} \tag{3.0.3}$$

Substituting $\mathbf{V}$ and $\mathbf{u}^T$ in (2.0.8), we obtain

$$\begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & k \end{vmatrix} = 0 \tag{3.0.4}$$

$$\implies \left(k - \left(\frac{49}{4}\right)\right) - \frac{5}{3}\left(\frac{5}{3}k - \frac{35}{4}\right)$$
$$-\frac{5}{2}\left(\frac{-35}{6} + \frac{5}{2}\right) = 0 \tag{3.0.5}$$

$$\implies \frac{64}{k}36 - \frac{128}{12} = 0 \tag{3.0.6}$$

$$\implies \boxed{k = 6} \tag{3.0.7}$$

Substituting (3.0.7) in (3.0.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \tag{3.0.8}$$

Hence value of k=6 represents pair of straight lines.

4 Graphical Illustration

Substituting value of k =6 in equation (3.0.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & 6 \end{vmatrix} \tag{4.0.1}$$

Simplyfying the above determinant , we get

$$\delta = 0 \tag{4.0.2}$$

Since equation (2.0.8) is satisfied, we could say that the given equation (3.0.8) represents two straight lines

$$\det(V) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \tag{4.0.3}$$

Since $\det(V) < 0$ lines would intersect each other pair of straight lines in vector form is :

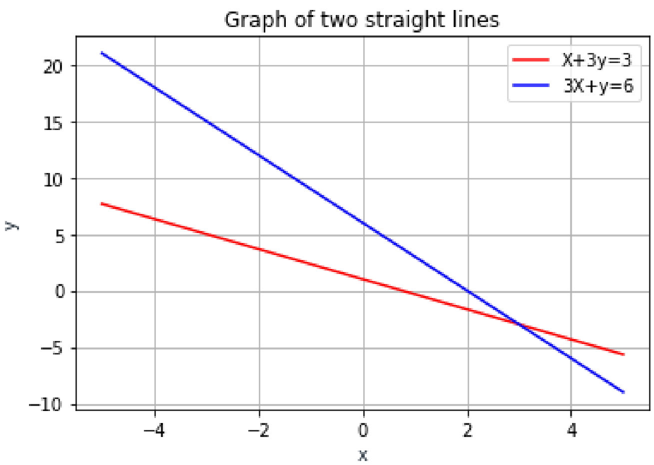

Fig. 1: Pair of straight lines

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{4.0.4}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{4.0.5}$$

Equating their product with (2.0.7)

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{-5}{2} & \frac{-7}{2} \end{pmatrix} \mathbf{x} + 6 \tag{4.0.6}$$

$$\mathbf{n_1} * \mathbf{n_2} = \{1, \frac{10}{3}, 1\} \tag{4.0.7}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{4.0.8}$$

$$c_1c_2 = 6 \tag{4.0.9}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{4.0.10}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{4.0.11}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{4.0.12}$$

Substituting in above equations (4.0.10) we get,

$$m^2 + \frac{10}{3}m + 1 = 0 \tag{4.0.13}$$

$$\implies m_i = \frac{\frac{-10}{3} \pm \sqrt{-(\frac{-16}{9})}}{1} \tag{4.0.14}$$

Solving equation (4.0.14) we have ,

$$m_1 = \frac{-1}{3} \tag{4.0.15}$$

$$m_2 = -3 \tag{4.0.16}$$

$$\mathbf{n_1} = k_1\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{4.0.17}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{4.0.18}$$

Substituting equations (4.0.17), (4.0.18) in equation (4.0.7) we get

$$k_1k_2 = 1 \tag{4.0.19}$$

Possible combination of (k_1, k_2) is (1,1) Lets assume $k_1 = 1$, $k_2 = 1$, we get

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{4.0.20}$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{4.0.21}$$

we have:

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.0.22}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a teoplitz matrix and multiplying with $\mathbf{n_2}$ From equation (4.0.20) and (4.0.21)

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{4.0.23}$$

$$\implies \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.0.24}$$

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{4.0.25}$$

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{4.0.26}$$

Substituting (4.0.20) and (4.0.21) in (4.0.26), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow 3 \times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \tag{4.0.27}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \tag{4.0.28}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \tag{4.0.29}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 9 \times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \tag{4.0.30}$$

From above we get

$$c_1 = 1 \tag{4.0.31}$$

$$c_2 = 6 \tag{4.0.32}$$

Hence pair of straight lines from (4.0.4), (4.0.5) in vector form

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \tag{4.0.33}$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{4.0.34}$$