

Assignment 4

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Geometry

Abstract—This document contains the solution to prove angles of an equilateral triangle are 60 degrees through Linear Algebra.

Download all python codes from

https://github.com/shivangi-975/EE5609-Matrix_Theory/tree/master/Assignment4/Codes

Download latex-tikz codes from

https://github.com/shivangi-975/EE5609-Matrix_Theory/blob/master/Assignment4/Assignment4.tex

1 PROBLEM

To prove angles of equilateral triangles are 60° each.

2 SOLUTION

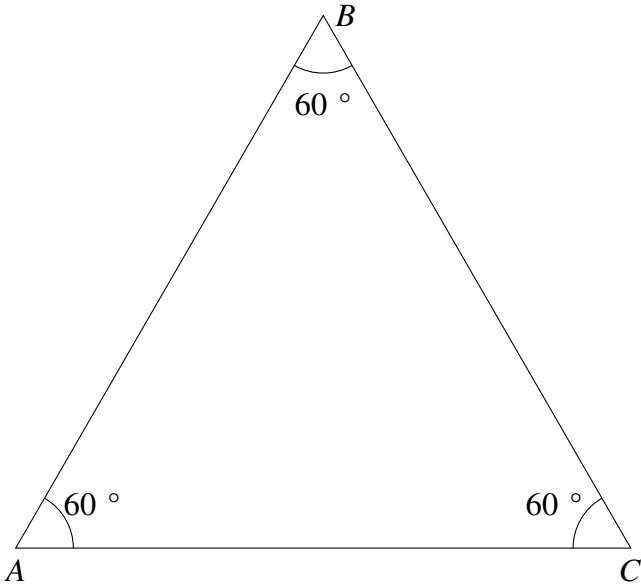


Fig. 1: Equilateral $\triangle ABC$ with A, B and C as vertices

Considering A, B and C as the vertices of triangle:

$$A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

In equilateral triangle all sides are equal. Hence,

$$\|A - B\| = \|B - C\| = \|A - C\| \quad (2.0.1)$$

Putting $B = 0$ in 2.0.1 we have,

$$\|A\| = \|C\| \quad (2.0.2)$$

$$\|A\| = \|A - C\| \quad (2.0.3)$$

Squaring equation 2.0.2

$$\|A\|^2 = \|C\|^2 \quad (2.0.4)$$

Squaring equation 2.0.3

$$\begin{aligned} \|A\|^2 &= \|A\|^2 - 2A^T C + \|C\|^2 \\ \implies \|A\|^2 &= 2A^T C \end{aligned} \quad (2.0.5)$$

Taking the inner product of sides AB, BC we have:

$$(A - B)^T (B - C) = \|A - B\| \|B - C\| \cos ABC \quad (2.0.6)$$

The angle ABC from the above equation is:

$$\cos ABC = \frac{(A - B)^T (B - C)}{\|A - B\| \|B - C\|} \quad (2.0.7)$$

Substituting value in 2.0.7 and putting we have:

$$\cos ABC = \frac{(A)^T (C)}{\|A\|^2} \quad (2.0.8)$$

From 2.0.5 we have:

$$\begin{aligned} \cos ABC &= \frac{(A)^T (C)}{2(A)^T (C)} \\ \implies \cos ABC &= 1/2 \\ \implies \angle ABC &= 60^\circ \end{aligned} \quad (2.0.9)$$

Taking the inner product of sides AB, AC we have:

$$(B - A)^T (A - C) = \|B - A\| \|A - C\| \cos BAC \quad (2.0.10)$$

The angle BAC from the above equation is:

$$\cos BAC = \frac{(\mathbf{B} - \mathbf{A})^T (\mathbf{A} - \mathbf{C})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{A} - \mathbf{C}\|} \quad (2.0.11)$$

Substituting value in 2.0.11 and putting we have:

$$\cos BAC = \frac{(\mathbf{A})^T (\mathbf{A} - \mathbf{C})}{\|\mathbf{A}\|^2} \quad (2.0.12)$$

$$\Rightarrow \frac{(\mathbf{A})^T (\mathbf{A}) - (\mathbf{A})^T (\mathbf{C})}{\|\mathbf{A}\|^2} \quad (2.0.13)$$

We know $(\mathbf{A})^T (\mathbf{A}) = \|\mathbf{A}\|^2$

From equation 2.0.5 we have: $(\mathbf{A})^T (\mathbf{C}) = \frac{1}{2} \|\mathbf{A}\|^2$

Substituting values in 2.0.12 we have:

$$\begin{aligned} \cos BAC &= \frac{\frac{1}{2} \|\mathbf{A}\|^2}{\|\mathbf{A}\|^2} \\ \Rightarrow \cos BAC &= 1/2 \\ \Rightarrow \angle BAC &= 60^\circ \end{aligned} \quad (2.0.14)$$

Taking the inner product of sides AC, BC we have:

$$(\mathbf{C} - \mathbf{A})^T (\mathbf{C} - \mathbf{B}) = \|\mathbf{C} - \mathbf{A}\| \|\mathbf{C} - \mathbf{B}\| \cos ACB \quad (2.0.15)$$

The angle ACB from the above equation is:

$$\cos ACB = \frac{(\mathbf{C} - \mathbf{A})^T (\mathbf{C} - \mathbf{B})}{\|\mathbf{C} - \mathbf{A}\| \|\mathbf{C} - \mathbf{B}\|} \quad (2.0.16)$$

Substituting value in 2.0.18 and putting we have:

$$\begin{aligned} \cos ACB &= \frac{(\mathbf{C} - \mathbf{A})^T (\mathbf{C})}{\|\mathbf{A}\|^2} \\ \Rightarrow \frac{(\mathbf{C})^T (\mathbf{C}) - (\mathbf{A})^T (\mathbf{C})}{\|\mathbf{A}\|^2} \end{aligned} \quad (2.0.17)$$

We know $(\mathbf{C})^T (\mathbf{C}) = \|\mathbf{C}\|^2$

From 2.0.4 and 2.0.5 we have:

$$(\mathbf{A})^T (\mathbf{C}) = \frac{1}{2} \|\mathbf{A}\|^2 = \frac{1}{2} \|\mathbf{C}\|^2$$

Substituting values in 2.0.17 we have:

$$\begin{aligned} \cos ACB &= \frac{\frac{1}{2} \|\mathbf{C}\|^2}{\|\mathbf{C}\|^2} \\ \Rightarrow \cos ACB &= 1/2 \\ \Rightarrow \angle ACB &= 60^\circ \end{aligned} \quad (2.0.18)$$

Hence from equation 2.0.9, 2.0.14 and 2.0.18

$$\angle ABC = \angle BAC = \angle ACB = 60^\circ \quad (2.0.19)$$