Assignment 7

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***Abstract*—This document explains the method of performing QR decomposition on a 2×2 matrix.**

Download all python codes from

https://github.com/shivangi−975/EE5609−Matrix_Theory/tree/master/Assignment7/codes

and latex-tikz codes from

https://github.com/shivangi−975/EE5609−Matrix_Theory/edit/master/Assignment7/Assignment7.tex

1 Problem

Find the QR decomposition on a given 2×2 matrix.

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{1.0.1}$$

2 Solution

The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \tag{2.0.1}$$

Here $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{2.0.2}$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.0.3}$$

Equation (2.0.2) can be written in QR form as:

$$\mathbf{QR} = \begin{pmatrix} \mathbf{q_1} & \mathbf{q_2} \end{pmatrix} \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} \tag{2.0.4}$$

Now,

$$u_1 = ||\mathbf{a}|| = \sqrt{1^2 + 2^2} = \sqrt{5} \tag{2.0.5}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.0.6}$$

$$u_3 = \frac{\mathbf{q_1}^T \mathbf{b}}{||\mathbf{q_1}||^2} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0 \tag{2.0.7}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3 \mathbf{q_1}}{||\mathbf{b} - u_3 \mathbf{q_1}||} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.0.8}$$

$$u_2 = \mathbf{q_2}^T \mathbf{b} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \sqrt{5} \tag{2.0.9}$$

Substituting equation (2.0.5) to (2.0.9) in (2.0.4),to obtain the QR Decomposition of the given matrix as:

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \tag{2.0.10}$$

In equation (2.0.10) $\mathbf{R}$ is orthogonal because the columns and rows are orthogonal to each other. Simplifying (2.0.10) it turns out that if we have orthogonal matrix then $\mathbf{QR}$ decomposition will be equal to $\mathbf{Q}$= $\mathbf{A}$ and$\mathbf{R}$ = $\mathbf{I}$ (R would be an identity matrix). Hence,

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.11}$$