Matrix theory Assignment 15

Shivangi Parashar

Abstract—**This document contains the concept of linear functionals.**

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1 PROBLEM

In R^3, let $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ and $\alpha_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$.

Describe a linear functional f on R^3 such that $f(\alpha_1) = f(\alpha_2) = 0$ but $f(\alpha_3) \neq 0$. If $\alpha = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ such that $f(\alpha) \neq 0$

2 SOLUTION

Let us consider $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha \qquad (2.0.1)$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad (2.0.2)$$

Hence:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \qquad (2.0.3)$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad (2.0.4)$$

$x = A^{-1}\alpha$ will give solution of equation.

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftarrow{R_3 \leftarrow R_3 - R_1} \qquad (2.0.5)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{pmatrix} \xleftarrow{R_3 \leftarrow R_3 + 2R_2} \qquad (2.0.6)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{pmatrix} \xleftarrow{R_3 \leftarrow R_3/(-1)} \qquad (2.0.7)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xleftarrow{R_1 \leftarrow R_1 + R_3} \qquad (2.0.8)$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xleftarrow{R_2 \leftarrow R_2 + R_3} \qquad (2.0.9)$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \qquad (2.0.10)$$

Hence,

$$A^{-1} = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \qquad (2.0.11)$$

Simplifying,

$$x_1 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad (2.0.12)$$

$$x_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad (2.0.13)$$

$$x_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad (2.0.14)$$

Given, f is a linear functional on R^3,

$$\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \qquad (2.0.15)$$

$$\implies f(\alpha) = \mathbf{x}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \qquad (2.0.16)$$

Given, $f(\alpha_1) = f(\alpha_2) = 0$ and $f(\alpha_3) \neq 0$.

$$f(\alpha) = \mathbf{x}^T \begin{pmatrix} 0 \\ 0 \\ f(\alpha_3) \end{pmatrix} \qquad (2.0.17)$$

$$\implies f(\alpha) = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} f(\alpha_3) \qquad (2.0.18)$$

$$\implies f\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}^T \begin{pmatrix} a \\ b \\ c \end{pmatrix} f(\alpha_3) \qquad (2.0.19)$$

Hence function is:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} f(\alpha_3) \qquad (2.0.20)$$

Now to prove $f(\alpha) \neq 0$,

$$\implies f\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}^T \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} f(\alpha_3) \qquad (2.0.21)$$

$$\implies -3f(\alpha_3) \neq 0 \qquad (2.0.22)$$

Hence, proved $f(\alpha) \neq 0$