

Assignment 6

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Abstract—This document explains the concept of affine transformation of equations when the origin is moved to the point

Download all latex-tikz codes from

https://github.com/shivangi-975/EE5609-Matrix_Theory/blob/master/Assignment6/Assignment6.tex

1 PROBLEM

To what point must origin be shifted so that

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} + (10 \quad -19) \mathbf{x} + 23 = 0 \quad (1.0.1)$$

is transformed to

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} = 1 \quad (1.0.2)$$

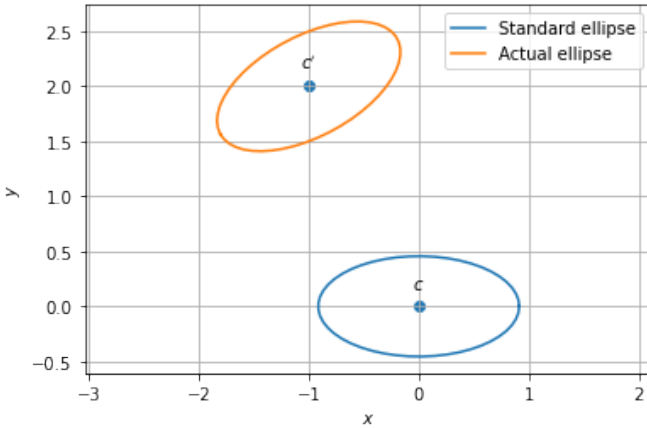


Fig. 1: Ellipse when origin is shifted

2 SOLUTION

Given,

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} + (10 \quad -19) \mathbf{x} + 23 = 0 \quad (2.0.1)$$

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.2)$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ \frac{-19}{2} \end{pmatrix} \quad (2.0.4)$$

$$f = 23 \quad (2.0.5)$$

Origin which is moved to the point is given by \mathbf{c}
The above equation (2.0.2) can be modified as

$$(\mathbf{x} + \mathbf{c})^T \mathbf{V} (\mathbf{x} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{x} + \mathbf{c}) + 23 = 0 \quad (2.0.6)$$

From equation (2.0.6) consider,

$$\Rightarrow (\mathbf{x} + \mathbf{c})^T \mathbf{V} (\mathbf{x} + \mathbf{c}) \quad (2.0.7)$$

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{x}^T \mathbf{V} \mathbf{c} + \mathbf{c}^T \mathbf{V} \mathbf{c} \quad (2.0.8)$$

we know that

$$\mathbf{x}^T \mathbf{V} \mathbf{c} = \mathbf{c}^T \mathbf{V} \mathbf{x} \quad (2.0.9)$$

Substituting equation (2.0.9) in equation (2.0.8)

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} \quad (2.0.10)$$

Equation (2.0.6) is modified as

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{c} + 23 = 0 \quad (2.0.11)$$

Equating (2.0.6) and (1.0.2):

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{c} + 23 = \mathbf{x}^T \mathbf{V} \mathbf{x} - 1 \quad (2.0.12)$$

From above equation (2.0.12) we have,

$$2\mathbf{c}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} = 0 \quad (2.0.13)$$

and

$$2\mathbf{u}^T \mathbf{x} + \mathbf{v}^T \mathbf{c} = -22 \quad (2.0.14)$$

From (2.0.13)

$$\mathbf{c}^T \mathbf{V} \mathbf{x} = -\mathbf{u}^T \mathbf{x} \quad (2.0.15)$$

$$\mathbf{c}^T \mathbf{V} = -\mathbf{u}^T \quad (2.0.16)$$

$$\mathbf{c}^T = -\mathbf{V}^{-1} \mathbf{u}^T \quad (2.0.17)$$

Adjoining \mathbf{V} with identity matrix to compute inverse:

$$\begin{pmatrix} 2 & \frac{-3}{2} & 1 & 0 \\ \frac{-3}{2} & 4 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{1}{2} R_1} \begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ \frac{-3}{2} & 4 & 0 & 1 \end{pmatrix} \quad (2.0.18)$$

$$\begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ \frac{-3}{2} & 4 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + \frac{3}{2} R_1} \begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & \frac{23}{8} & \frac{3}{4} & 1 \end{pmatrix} \quad (2.0.19)$$

$$\begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & \frac{23}{8} & \frac{3}{4} & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{8}{23} R_2} \begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{6}{23} & \frac{8}{23} \end{pmatrix} \quad (2.0.20)$$

$$\begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{6}{23} & \frac{8}{23} \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + \frac{3}{4} R_2} \begin{pmatrix} 1 & 0 & \frac{16}{23} & \frac{6}{23} \\ 0 & 1 & \frac{6}{23} & \frac{8}{23} \end{pmatrix} \quad (2.0.21)$$

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{16}{23} & \frac{6}{23} \\ \frac{6}{23} & \frac{8}{23} \end{pmatrix} \quad (2.0.22)$$

From (2.0.17)

$$\mathbf{c}^T = \begin{pmatrix} \frac{-16}{23} & \frac{-6}{23} \\ \frac{-6}{23} & \frac{-8}{23} \end{pmatrix} \begin{pmatrix} 5 \\ \frac{-19}{2} \end{pmatrix} \quad (2.0.23)$$

From above we have :

$$\mathbf{c}^T = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.0.24)$$

Hence,

$$\mathbf{c} = \begin{pmatrix} -1 & 2 \end{pmatrix} \quad (2.0.25)$$

From (1.0.1) and (1.0.2) when the origin is moved to the point $\mathbf{c} \begin{pmatrix} -1 & 2 \end{pmatrix}$ \mathbf{V} doesn't change

$$\det(\mathbf{V}) = 5.75 \quad (2.0.26)$$

Since $\det(\mathbf{V}) > 0$ the given equation represents the ellipse. The characteristic equation of \mathbf{V} is obtained by evaluating the determinant

$$| \mathbf{V} - \lambda \mathbf{I} | = 0 \quad (2.0.27)$$

$$\begin{vmatrix} 2 - \lambda & \frac{-3}{2} \\ \frac{-3}{2} & 4 - \lambda \end{vmatrix} = 0 \quad (2.0.28)$$

$$\Rightarrow 4\lambda^2 - 24\lambda + 23 = 0 \quad (2.0.29)$$

The eigenvalues are the roots of equation 2.0.29 is given by

$$\lambda_1 = \frac{6 + \sqrt{13}}{2} \quad (2.0.30)$$

$$\lambda_2 = \frac{6 - \sqrt{13}}{2} \quad (2.0.31)$$

Hence from above:

$$\mathbf{D} = \begin{pmatrix} \frac{6 + \sqrt{13}}{2} & 0 \\ 0 & \frac{6 - \sqrt{13}}{2} \end{pmatrix} \quad (2.0.32)$$

The eigenvector \mathbf{p} is defined as

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (2.0.33)$$

$$\Rightarrow (\mathbf{V} - \lambda \mathbf{I}) \mathbf{p} = 0 \quad (2.0.34)$$

For $\lambda_1 = \frac{6 + \sqrt{13}}{2}$,

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{-\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}+2}{2} \end{pmatrix} \quad (2.0.35)$$

$$\begin{pmatrix} \frac{-\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}+2}{2} \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 \div \frac{-\sqrt{13}-2}{2}} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \quad (2.0.36)$$

$$\begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{-3}{2} R_1} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ 0 & 0 \end{pmatrix} \quad (2.0.37)$$

$$\mathbf{p}_1 = \begin{pmatrix} \frac{-\sqrt{13}+2}{2} \\ 1 \end{pmatrix} \quad (2.0.38)$$

For $\lambda_2 = \frac{6 - \sqrt{13}}{2}$,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{\sqrt{13}+2}{2} \end{pmatrix} \quad (2.0.39)$$

$$\begin{pmatrix} \frac{\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{\sqrt{13}+2}{2} \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 \div \frac{\sqrt{13}-2}{2}} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \quad (2.0.40)$$

$$\begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - \frac{-3}{2} R_1} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ 0 & 0 \end{pmatrix} \quad (2.0.41)$$

$$\mathbf{p}_2 = \begin{pmatrix} \frac{\sqrt{13}+2}{2} \\ 1 \end{pmatrix} \quad (2.0.42)$$

Again,for ellipse

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad (2.0.43)$$

Where \mathbf{D} is a diagonal matrix, we get,

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.0.44)$$

$$\Rightarrow \mathbf{P} = \begin{pmatrix} \frac{-\sqrt{13}+2}{3} & \frac{\sqrt{13}+2}{3} \\ 1 & 1 \end{pmatrix} \quad (2.0.45)$$

$$\mathbf{D} = \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \quad (2.0.46)$$

Standard ellipse can be written in the form:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (2.0.47)$$

Simplifying we get:

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} = \begin{pmatrix} 5 & -19 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \frac{16}{23} & \frac{6}{23} \\ \frac{23}{6} & \frac{8}{23} \end{pmatrix} \begin{pmatrix} 5 \\ -19 \\ 2 \end{pmatrix} = 24 \quad (2.0.48)$$

substituting (2.0.48) in (2.0.47) we have :

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = 1 \quad (2.0.49)$$

To get \mathbf{y} ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (2.0.50)$$

$$\mathbf{y} = \begin{pmatrix} \frac{-\sqrt{13}+2}{3} & 1 \\ \frac{\sqrt{13}+2}{3} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{-\sqrt{13}+2}{3} & 1 \\ \frac{\sqrt{13}+2}{3} & 1 \end{pmatrix} \quad (2.0.51)$$

Substituting equation (2.0.32), in equation (2.0.49)

$$\mathbf{y}^T \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \mathbf{y} = 1 \quad (2.0.52)$$