Matrix Theory (EE5609) Assignment 8

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***Abstract*—This finds the cordinates of foot of perpendicular using Singular Value Decomposition.**

All the codes for the figure in this document can be found at

https://github.com/shivangi−975/EE5609−Matrix_Theory/blob/master/Assignment8/Assignment_8.tex

1 Problem

Find the distance of the point $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ from the plane $\begin{pmatrix} 1 & 2 & -2 \end{pmatrix}\mathbf{x} = 9$

2 Solution

First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{2.0.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 0 \tag{2.0.2}$$

$$\implies a + 2b - 2c = 0 \tag{2.0.3}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.4}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{2.0.5}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.6}$$

Putting values in (2.0.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \tag{2.0.7}$$

In order to solve (2.0.7), perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \tag{2.0.9}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{5}{4} \end{pmatrix} \tag{2.0.10}$$

From (2.0.6) putting (2.0.8) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{2.0.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{2.0.12}$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & 1 \\ \frac{1}{2} & 1 & \frac{5}{2}-\lambda \end{pmatrix} = 0 \tag{2.0.14}$$

$$\implies -4\lambda^3 + 13\lambda^2 - 9\lambda = 0 \tag{2.0.15}$$

Hence eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{9}{4} \tag{2.0.16}$$

$$\lambda_2 = 1 \tag{2.0.17}$$

$$\lambda_3 = 0 \tag{2.0.18}$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{-1}{2} \\ -1 \\ 1 \end{pmatrix} \tag{2.0.19}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{2}{\sqrt{45}} \\ \frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \tag{2.0.20}$$

Hence we obtain $\mathbf{U}$ of (2.0.8) as follows,

$$\begin{pmatrix} \frac{2}{\sqrt{45}} & -\frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{4}{\sqrt{45}} & \frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{5}{\sqrt{45}} & 0 & \frac{2}{3} \end{pmatrix} \tag{2.0.21}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (2.0.8) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.22}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.23}$$

$$\implies \begin{pmatrix} \frac{5}{4} - \lambda & \frac{1}{2} \\ \frac{1}{2} & 2 - \lambda \end{pmatrix} = 0 \tag{2.0.24}$$

$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \tag{2.0.25}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{9}{4} \tag{2.0.26}$$

$$\lambda_5 = 1 \tag{2.0.27}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{2.0.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.0.29}$$

Hence we obtain $\mathbf{V}$ of (2.0.8) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.0.30}$$

From (2.0.8) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} \frac{2}{\sqrt{45}} & -\frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{4}{\sqrt{45}} & \frac{1}{\sqrt{5}} & \frac{-2}{3} \\ \frac{5}{\sqrt{45}} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}^T \tag{2.0.31}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.32}$$

From (2.0.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{3\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} \\ -6 \end{pmatrix} \tag{2.0.33}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{2\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} \end{pmatrix} \tag{2.0.34}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{2.0.35}$$

Verifying the solution of (2.0.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.36}$$

Evaluating the R.H.S in (2.0.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -\frac{1}{2} \end{pmatrix} \tag{2.0.37}$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -\frac{1}{2} \\ -2 \end{pmatrix} \tag{2.0.38}$$

Solving the augmented matrix of (2.0.38) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 2 & -2 \end{pmatrix} \xleftrightarrow{R_1 = \frac{4}{5}R_1} \begin{pmatrix} 1 & \frac{2}{5} & \frac{-2}{5} \\ \frac{1}{2} & 2 & -2 \end{pmatrix} \tag{2.0.39}$$

$$\xleftrightarrow{R_2 = R_2 - \frac{1}{2}R_2} \begin{pmatrix} 1 & -\frac{2}{5} & -\frac{2}{5} \\ 0 & \frac{9}{5} & -\frac{9}{5} \end{pmatrix} \tag{2.0.40}$$

$$\xleftrightarrow{R_2 = \frac{5}{9}R_2} \begin{pmatrix} 1 & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & -1 \end{pmatrix} \tag{2.0.41}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{2}{5}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \tag{2.0.42}$$

From equation (2.0.42), solution is given by,

$$\mathbf{x} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{2.0.43}$$

Comparing results of $\mathbf{x}$ from (2.0.35) and (2.0.43), we can say that the solution is verified.