Assignment 6

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AI20MTECH14012

***Abstract*—This document explains the concept of affline transformation of equations when the origin is moved to the point**

Download all latex-tikz codes from

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https://github.com/shivangi−975/EE5609−
    Matrix_Theory/blob/master/Assignment6/
    Assignment6.tex
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1 Problem

To what point must origin be shifted so that

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 & -19 \end{pmatrix} \mathbf{x} + 23 = 0 \tag{1.0.1}$$

is transformed to

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} = 1 \tag{1.0.2}$$

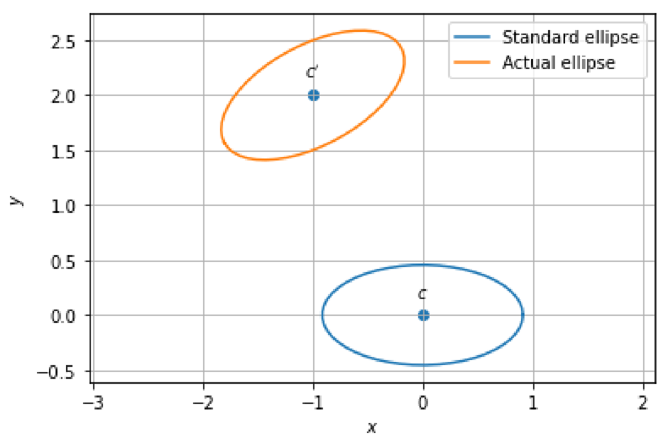

Fig. 1: Ellipse when origin is shifted

2 Solution

Given,

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 & -19 \end{pmatrix} \mathbf{x} + 23 = 0 \tag{2.0.1}$$

The general second degree equation can be expressed as follows,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.0.2}$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ \frac{-19}{2} \end{pmatrix} \tag{2.0.4}$$

$$f = 23 \tag{2.0.5}$$

Origin which is moved to the point is given by c
The above equation (2.0.2) can be modified as

$$(\mathbf{x}+\mathbf{c})^T\mathbf{V}(\mathbf{x}+\mathbf{c}) + 2\mathbf{u}^T(\mathbf{x}+\mathbf{c}) + 23 = 0 \tag{2.0.6}$$

From equation (2.0.6) consider,

$$\implies (\mathbf{x}+\mathbf{c})^T\mathbf{V}(\mathbf{x}+\mathbf{c}) \tag{2.0.7}$$

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{x}^T\mathbf{V}\mathbf{c} + \mathbf{c}^T\mathbf{V}\mathbf{c} \tag{2.0.8}$$

we know that

$$\mathbf{x}^T\mathbf{V}\mathbf{c} = \mathbf{c}^T\mathbf{V}\mathbf{x} \tag{2.0.9}$$

Substituting equation (2.0.9) in equation (2.0.8)

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{c} \tag{2.0.10}$$

Equation (2.0.6) is modified as

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{x}2\mathbf{u}^T\mathbf{c} + 23 = 0 \tag{2.0.11}$$

Equating (2.0.6) and (1.0.2):

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{c}^T\mathbf{V}\mathbf{x}+\mathbf{c}^T\mathbf{V}\mathbf{c}+2\mathbf{u}^T\mathbf{x}2\mathbf{u}^T\mathbf{c}+23 = \mathbf{x}^T\mathbf{V}\mathbf{x}-1 \tag{2.0.12}$$

From above equation (2.0.12) we have,

$$2\mathbf{c}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} = 0 \tag{2.0.13}$$

and

$$2\mathbf{u}^T\mathbf{x} + \mathbf{v}^T\mathbf{c} = -22 \tag{2.0.14}$$

From (2.0.13)

$$\mathbf{c}^T\mathbf{V}\mathbf{x} = -\mathbf{u}^T\mathbf{x} \tag{2.0.15}$$
$$\mathbf{c}^T\mathbf{V} = -\mathbf{u}^T \tag{2.0.16}$$
$$\mathbf{c}^T = -\mathbf{V}^{-1}\mathbf{u}^T \tag{2.0.17}$$

Adjoining V with identity matrix to compute inverse:

$$\begin{pmatrix} 2 & \frac{-3}{2} & 1 & 0 \\ \frac{-3}{2} & 4 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ \frac{-3}{2} & 4 & 0 & 1 \end{pmatrix} \tag{2.0.18}$$

$$\begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ \frac{-3}{2} & 4 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + \frac{3}{2}R_1} \begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & \frac{23}{8} & \frac{3}{4} & 1 \end{pmatrix} \tag{2.0.19}$$

$$\begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & \frac{23}{8} & \frac{3}{4} & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{8}{23}R_2} \begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{6}{23} & \frac{8}{23} \end{pmatrix} \tag{2.0.20}$$

$$\begin{pmatrix} 1 & \frac{-3}{4} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{6}{23} & \frac{8}{23} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & \frac{16}{23} & \frac{6}{23} \\ 0 & 1 & \frac{6}{23} & \frac{8}{23} \end{pmatrix} \tag{2.0.21}$$

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{16}{23} & \frac{6}{23} \\ \frac{6}{23} & \frac{8}{23} \end{pmatrix} \tag{2.0.22}$$

From (2.0.17)

$$\mathbf{c}^T = \begin{pmatrix} \frac{-16}{23} & \frac{-6}{23} \\ \frac{-6}{23} & \frac{-8}{23} \end{pmatrix} \begin{pmatrix} 5 \\ \frac{-19}{2} \end{pmatrix} \tag{2.0.23}$$

From above we have :

$$\mathbf{c}^T = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.0.24}$$

Hence,

$$\mathbf{c} = \begin{pmatrix} -1 & 2 \end{pmatrix} \tag{2.0.25}$$

From (1.0.1) and (1.0.2) when the origin is moved to the point c$\begin{pmatrix} -1 & 2 \end{pmatrix}$ $\mathbf{V}$ doesn't change

$$\det(\mathbf{V}) = 5.75 \tag{2.0.26}$$

Since $\det(\mathbf{V}) > 0$ the given equation represents the ellipse. The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \tag{2.0.27}$$

$$\begin{vmatrix} 2-\lambda & \frac{-3}{2} \\ \frac{-3}{2} & 4-\lambda \end{vmatrix} = 0 \tag{2.0.28}$$

$$\implies 4\lambda^2 - 24\lambda + 23 = 0 \tag{2.0.29}$$

The eigenvalues are the roots of equation 2.0.29 is given by

$$\lambda_1 = \frac{6+\sqrt{13}}{2} \tag{2.0.30}$$

$$\lambda_2 = \frac{6-\sqrt{13}}{2} \tag{2.0.31}$$

Hence from above:

$$\mathbf{D} = \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \tag{2.0.32}$$

The eigenvector p is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.33}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.34}$$

For $\lambda_1 = \frac{6+\sqrt{13}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{-\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}+2}{2} \end{pmatrix} \tag{2.0.35}$$

$$\begin{pmatrix} \frac{-\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}+2}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 \div \frac{-\sqrt{13}-2}{2}} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \tag{2.0.36}$$

$$\begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \frac{-3}{2} R_1} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ 0 & 0 \end{pmatrix} \tag{2.0.37}$$

$$\mathbf{p}_1 = \begin{pmatrix} \frac{-\sqrt{13}+2}{2} \\ 1 \end{pmatrix} \tag{2.0.38}$$

For $\lambda_2 = \frac{6-\sqrt{13}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{\sqrt{13}+2}{2} \end{pmatrix} \tag{2.0.39}$$

$$\begin{pmatrix} \frac{\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{\sqrt{13}+2}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 \div \frac{\sqrt{13}-2}{2}} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \tag{2.0.40}$$

$$\begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \frac{-3}{2} R_1} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ 0 & 0 \end{pmatrix} \tag{2.0.41}$$

$$\mathbf{p}_2 = \begin{pmatrix} \frac{\sqrt{13}+2}{2} \\ 1 \end{pmatrix} \tag{2.0.42}$$

Again,for ellipse

$$\mathbf{V} = \mathbf{PDP}^{-1} \tag{2.0.43}$$

Where $\mathbf{D}$ is a diagonal matrix, we get,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.0.44}$$

$$\implies \mathbf{P} = \begin{pmatrix} \frac{-\sqrt{13}+2}{3} & \frac{\sqrt{13}+2}{3} \\ 1 & 1 \end{pmatrix} \tag{2.0.45}$$

$$\mathbf{D} = \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \tag{2.0.46}$$

Standard ellipse can be written in the form:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.0.47}$$

Simplifying we get:

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 5 & \frac{-19}{2} \end{pmatrix} \begin{pmatrix} \frac{16}{23} & \frac{6}{23} \\ \frac{6}{23} & \frac{8}{23} \end{pmatrix} \begin{pmatrix} 5 \\ \frac{-19}{2} \end{pmatrix} = 24 \tag{2.0.48}$$

substituting (2.0.48) in (2.0.47) we have :

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = 1 \tag{2.0.49}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.0.50}$$

$$\mathbf{y} = \begin{pmatrix} \frac{-\sqrt{13}+2}{3} & 1 \\ \frac{\sqrt{13}+2}{3} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{-\sqrt{13}+2}{3} & 1 \\ \frac{\sqrt{13}+2}{3} & 1 \end{pmatrix} \tag{2.0.51}$$

Substituting equation (2.0.32), in equation (2.0.49)

$$\mathbf{y}^T \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \mathbf{y} = 1 \tag{2.0.52}$$

The following figure is the graphical representation of Ellipse when origin is shifted.

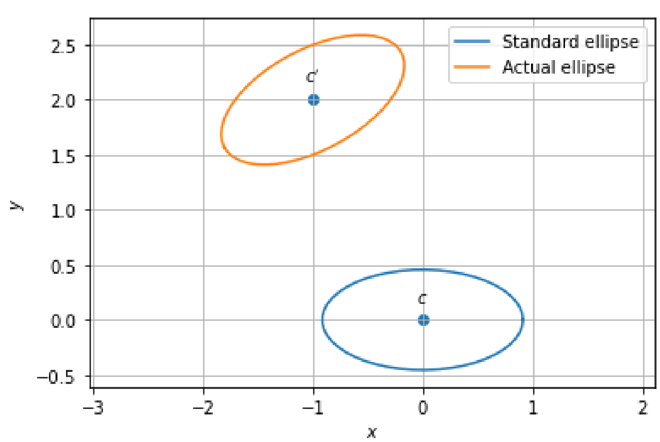

Fig. 1: Ellipse when origin is shifted