Assignment 4

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Geometry

***Abstract*—This documnet contains the solution to prove angles of a equilateral triangles are 60 degrees through Linear Algebra .**

Download all python codes from

https://github.com/shivangi−975/EE5609−Matrix_Theory/tree/master/Assignment4/Codes

Download latex-tikz codes from

https://github.com/shivangi−975/EE5609−Matrix_Theory/blob/master/Assignment4/Assignment4.tex

1 Problem

To prove angles of equilateral triangles are 60 °each.

2 Solution

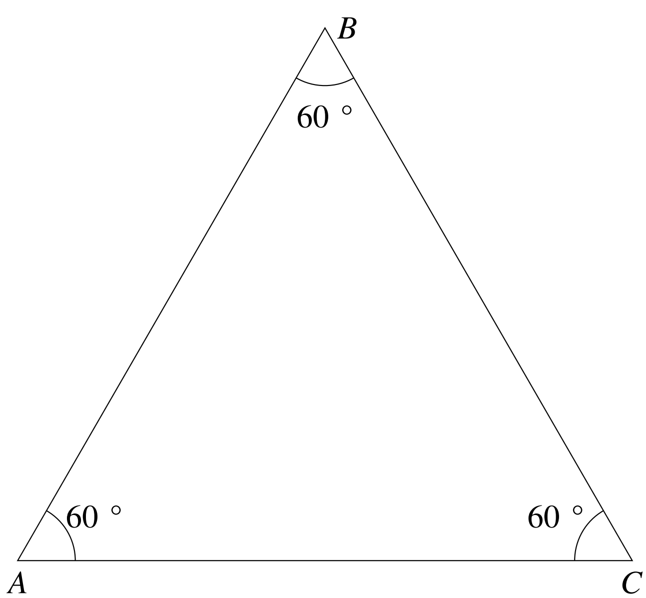

Fig. 1: Equilateral $\triangle ABC$ with A,B and C as vertices

Considering A,B and C as the vertices of triangle:

$$A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

In equilateral triangle all sides are equal.Hence,

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{B} - \mathbf{C}\| = \|\mathbf{A} - \mathbf{C}\| \tag{2.0.1}$$

Putting $\mathbf{B}= 0$ in (2.0.1) we have,

$$\|\mathbf{A}\| = \|\mathbf{C}\| \tag{2.0.2}$$

$$\|\mathbf{A}\| = \|\mathbf{A} - \mathbf{C}\| \tag{2.0.3}$$

Squaring equation (2.0.2)

$$\|\mathbf{A}\|^2 = \|\mathbf{C}\|^2 \tag{2.0.4}$$

Squaring equation (2.0.3)

$$\|\mathbf{A}\|^2 = \|\mathbf{A}\|^2 - 2(\mathbf{A}^T)(\mathbf{C}) + \|\mathbf{C}\|^2$$
$$\implies \|\mathbf{A}\|^2 = 2(\mathbf{A}^T)(\mathbf{C}) \tag{2.0.5}$$

Taking the inner product of sides AB,BC we have:

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = \|\mathbf{A} - \mathbf{B}\| \, \|\mathbf{B} - \mathbf{C}\| \cos ABC \tag{2.0.6}$$

The angle ABC from the above equation is:

$$\cos ABC = \frac{(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \, \|\mathbf{B} - \mathbf{C}\|} \tag{2.0.7}$$

Substituting value in (2.0.7) and putting we have:

$$\cos ABC = \frac{(\mathbf{A})^T(\mathbf{C})}{\|\mathbf{A}\|^2} \tag{2.0.8}$$

From (2.0.5) we have:

$$\cos ABC = \frac{(\mathbf{A})^T(\mathbf{C})}{2(\mathbf{A})^T(\mathbf{C})}$$
$$\implies \cos ABC = 1/2$$
$$\implies \angle ABC = 60^\circ \tag{2.0.9}$$

Taking the inner product of sides AB,AC we have:

$$(\mathbf{B} - \mathbf{A})^T(\mathbf{A} - \mathbf{C}) = \|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{A} - \mathbf{C}\| \cos BAC \tag{2.0.10}$$

The angle BAC from the above equation is:

$$\cos BAC = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{2.0.11}$$

Substituting value in (2.0.11) and putting we have:

$$\cos BAC = \frac{(\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}\|^2} \tag{2.0.12}$$

$$\implies \frac{(\mathbf{A})^T(\mathbf{A})-(\mathbf{A})^T(\mathbf{C})}{\|\mathbf{A}\|^2} \tag{2.0.13}$$

We know $(\mathbf{A})^T(\mathbf{A}) = \|\mathbf{A}\|^2$
From equation (2.0.5) we have: $(\mathbf{A})^T(\mathbf{C}) = \frac{1}{2}\|\mathbf{A}\|^2$
Substituting values in (2.0.12) we have:

$$\cos BAC = \frac{\frac{1}{2}\|\mathbf{A}\|^2}{\|\mathbf{A}\|^2}$$

$$\implies \cos BAC = 1/2$$

$$\implies \angle BAC = 60° \tag{2.0.14}$$

Taking the inner product of sides AC,BC we have:

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{B}) = \|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{B}\|\cos ACB \tag{2.0.15}$$

The angle ACB from the above equation is:

$$\cos ACB = \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{B})}{\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{B}\|} \tag{2.0.16}$$

Substituting value in (2.0.18) and putting we have:

$$\cos ACB = \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{C})}{\|\mathbf{A}\|^2}$$

$$\implies \frac{(\mathbf{C})^T(\mathbf{C})-(\mathbf{A})^T(\mathbf{C})}{\|\mathbf{A}\|^2} \tag{2.0.17}$$

We know $(\mathbf{C})^T(\mathbf{C}) = \|\mathbf{C}\|^2$
From (2.0.4) and (2.0.5) we have:
$(\mathbf{A})^T(\mathbf{C}) = \frac{1}{2}\|\mathbf{A}\|^2 = \frac{1}{2}\|\mathbf{C}\|^2$
Substituting values in (2.0.17) we have:

$$\cos ACB = \frac{\frac{1}{2}\|\mathbf{C}\|^2}{\|\mathbf{C}\|^2}$$

$$\implies \cos ACB = 1/2$$

$$\implies \angle ACB = 60° \tag{2.0.18}$$

Hence from equation (2.0.9),(2.0.14) and(2.0.18)

$$\angle ABC = \angle BAC = \angle ACB = 60° \tag{2.0.19}$$