SC217: Electromagnetic Theory **Assignment 4** **Aug. 12-16**

DA-IICT, B.Tech, Sem III Autumn2016

1. Some particles carry electrical charge. Experiments suggest that there are two kinds of charges. Suppose there were three kinds of charges in nature which we call red, blue and green. Charges of the same kind repel while charges of different kind attract. Ofcourse there are particles which neither repel nor attract other particles. You can't see the colors on the particles. You can only observe the repulsion and attraction between the particles. Treating these repulsion and attraction as relations on the set of particles how will you partition the particles and thus discover the existence of the three kinds of charges.

   soln:
   Let us define relations on the set of particles. Let R_A be the relation such that a particle is related to another if they attract each other. It is obvious that R_A is symmetric due to the third law of mechanics. We can't test whether this relation is reflexive, i.e, whether a particle attracts itself. So we will assume that the relation is reflexive. Now we investigate whether R_A is transitive. So consider three particles A, B and C. Suppose A attracts B and B attracts C. Then A and B are of different colors. Say A is red and B is blue. Now B attracts C. So B and C are of different colors. It is possible that C is red. So A and C has the same color. Hence they repel each other. So R_A is not an equivalence relation on the set of particles. Hence we can't partition the particles based on this relation.

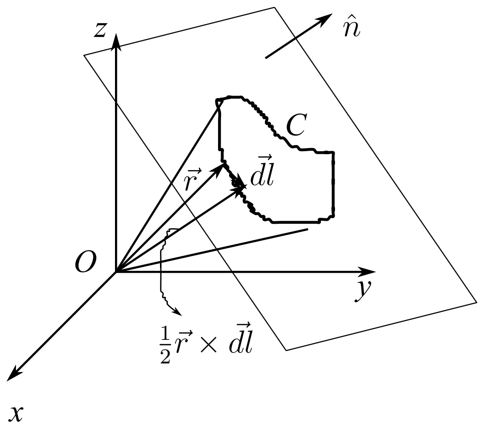
   Now let R_R be the repulsive relation on the set of particle. Again this relation is obviously symmetric and we assume it to be reflexive. Now suppose A repels B and B repels C. Then A and B are of the same color. Also B and C has the same color. So all the particles A, B and C have the same color. Hence R_R is an equivalence relation on the set of particles. The set of particles get partitioned into particles of three colors, red, blue and green.

   There may be particles which neither attract, nor repel other particles. Each of these particles make a class of itself under R_A and R_R. So we have three classes of particles each characterized by their color, red, blue and green, that form equvalence classes under R_R and several classes, each containing a singles colorless particle.

2. Let C be a closed planar curve contained by a plane $\vec{r} \cdot \hat{n} = p$. Here $\hat{n}$ is the unit normal vector to the plane and p is the distance of the plane from the origin. If a is the area of the region enclosed by the curve C then show that

$$a\hat{n} = \frac{1}{2} \oint_C \vec{r} \times \vec{dl}$$

   soln



In order to show two vectors equal we will show the projections of the vectors on either side on an arbitrary unit vector $\hat{u}$ to be same. Consider

$$\begin{aligned}\hat{u} \cdot \left[\frac{1}{2}\oint \vec{r} \times \vec{dl}\right] &= \frac{1}{2}\oint \hat{u} \cdot (\vec{r} \times \vec{dl}) \\ &= \frac{1}{2}\oint (\hat{u} \times \vec{r}) \cdot \vec{dl} \\ &= \frac{1}{2}\int_S [\vec{\nabla} \times (\hat{u} \times \vec{r})] \cdot \hat{n} da \qquad \text{Stokes' Theorem}\end{aligned}$$

We had evaluated earlier that $\vec{\nabla} \times (\vec{\omega} \times \vec{r}) = 2\vec{\omega}$.

$$\begin{aligned}\therefore\ \hat{u} \cdot \left[\frac{1}{2}\oint \vec{r} \times \vec{dl}\right] &= \frac{1}{2}\int_S 2\hat{u} \cdot \hat{n} da \\ &= \hat{u} \cdot \hat{n} \int_S da \\ &= \hat{u} \cdot \hat{n} a\end{aligned}$$

So the projection of an arbitrary vector $\hat{u}$ is same on both the vectors. Hence

$$a\hat{n} = \oint_C \vec{r} \times \vec{dl}$$

3. Consider a general curvilinear coordinate system u, v, w. The cartesian co-ordinates x, y, z are given as $x(u, v, w), y(u, v, w), z(u, v, w)$.

   (a) Determine h_u, h_v and h_w.
   soln:
   The displacement $\vec{dl}_u$ when only u change while v and w don't is given by

$\vec{dl}_u = (\hat{i}\frac{\partial x}{\partial u} + \hat{j}\frac{\partial y}{\partial u} + \hat{k}\frac{\partial z}{\partial u})du.$
$dl_u = h_u du$
$\therefore\ h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}.$
Similarly $h_v = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2}$ and $h_w = \sqrt{\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2}.$

(b) Find $\hat{u}, \hat{v}, \hat{w}$.
soln:

$$\begin{aligned}\hat{u} &= \frac{\vec{dl}_u}{dl_u} \\ &= \frac{1}{h_u}\left(\hat{i}\frac{\partial x}{\partial u} + \hat{j}\frac{\partial y}{\partial u} + \hat{k}\frac{\partial z}{\partial u}\right)\end{aligned}$$

Similarly

$$\begin{aligned}\hat{v} &= \frac{1}{h_v}\left(\hat{i}\frac{\partial x}{\partial v} + \hat{j}\frac{\partial y}{\partial v} + \hat{k}\frac{\partial z}{\partial v}\right) \\ \hat{w} &= \frac{1}{h_w}\left(\hat{i}\frac{\partial x}{\partial w} + \hat{j}\frac{\partial y}{\partial w} + \hat{k}\frac{\partial z}{\partial w}\right)\end{aligned}$$

(c) Find an expression for $\vec{\nabla} \cdot \vec{A}$ in the u, v, w coordinate system.
soln:
To calculate the expression for divergence in the (u, v, w) system, we consider a infinitesimal box of dimension dl_u, dl_v, dl_w. We calculate the flux across the surface $dl_v dl_w$. This will be given as $\vec{A} \cdot \hat{u}(dl_v dl_w) = h_v h_w A_u dv dw$.
There are two opposite faces on the box along $\hat{u}$ and $-\hat{u}$.
The flux over these faces gets subtracted and the net flux through the box over these surfaces is

$$\frac{\partial}{\partial u}\left(h_v h_w A_u dv dw\right) du = \frac{\partial}{\partial u}\left(h_v h_w A_u\right) du dv dw$$

Considering similar flux along $\hat{v}$ and $\hat{w}$ we get the total flux over the surface of this infinitesimal box as

$$\frac{\partial}{\partial u}\left(h_v h_w A_u\right) du dv dw + \frac{\partial}{\partial v}\left(h_u h_w A_v\right) du dv dw + \frac{\partial}{\partial w}\left(h_u h_v A_w\right) du dv dw$$

By divergence theorem this flux is equal to $(\vec{\nabla}\cdot\vec{A}) dl_u dl_v dl_w = (\vec{\nabla}\cdot\vec{A}) h_u h_v h_w du dv dw$.
Equating the two we get

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}\left(h_v h_w A_u\right) + \frac{\partial}{\partial v}\left(h_u h_w A_v\right) + \frac{\partial}{\partial w}\left(h_u h_v A_w\right)\right)$$

4. In the spherical polar system:

(a) Evaluate $\frac{\partial \hat{r}}{\partial \theta}, \quad \frac{\partial \hat{\theta}}{\partial \theta}, \quad \frac{\partial \hat{\phi}}{\partial \theta}, \quad \frac{\partial \hat{r}}{\partial \phi}, \quad \frac{\partial \hat{\theta}}{\partial \phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi}$

soln

In the spherical polar system

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

In this system $h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$.

$$\begin{aligned}
\hat{r} &= \frac{1}{h_r}\left(\frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}\hat{j} + \frac{\partial z}{\partial r}\hat{k}\right) dr \\
&= \left(\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}\right)
\end{aligned}$$

Similarly

$$\begin{aligned}
\hat{\theta} &= \left(\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}\right) \\
\hat{\phi} &= -\sin\phi\hat{i} + \cos\phi\hat{j}
\end{aligned}$$

So we have

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}, \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}, \quad \frac{\partial \hat{\phi}}{\partial \theta} = 0$$

$$\frac{\partial \hat{r}}{\partial \phi} = \sin\theta\hat{\phi}, \quad \frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta\hat{\phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\sin\theta\hat{r} - \cos\theta\hat{\theta}$$

(b) Using the above partial derivatives evaluate $\vec{\nabla}\cdot\hat{r}, \quad \vec{\nabla}\cdot\hat{\theta}$ and $\vec{\nabla}\cdot\hat{\phi}$ where $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

soln

$$\begin{aligned}
\vec{\nabla}\cdot\hat{r} &= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\hat{r} \\
&= \frac{\hat{\theta}}{r}\cdot\hat{\theta} + \frac{\hat{\phi}}{r\sin\theta}\cdot\sin\theta\hat{\phi} = \frac{2}{r} \\
\vec{\nabla}\cdot\hat{\theta} &= \frac{\hat{\theta}}{r}\cdot(-\hat{r}) + \frac{\hat{\phi}}{r\sin\theta}\cdot\cos\theta\hat{\phi} = \frac{1}{r\tan\theta} \\
\vec{\nabla}\cdot\hat{\phi} &= \frac{\hat{\phi}}{r\sin\theta}\cdot(-\sin\theta\hat{r} - \cos\theta\hat{\theta}) = 0
\end{aligned}$$

5. Cylindrical system of co-ordinate is specified by three variables (s, ϕ, z) given by

$$x = s\cos\phi; \qquad y = s\sin\phi; \qquad z = z$$

Find the unit vectors $\hat{s}, \quad \hat{\phi}, \quad \hat{z}$ in this co-ordinate system. Find $h_s, \quad h_\phi$ and h_z and write down the expression for $\vec{\nabla}F$ for a scalar function F in this system.

soln

$$\begin{aligned}
\vec{dl}_s &= (\cos\phi\hat{i} + \sin\phi\hat{j})ds \implies \quad h_s = 1 \\
\vec{dl}_\phi &= (-s\sin\phi\hat{i} + s\cos\phi\hat{j})d\phi \implies \quad h_\phi = s \\
\vec{dl}_z &= \hat{k}dz \quad \implies \quad h_z = 1
\end{aligned}$$

$$\therefore \quad \hat{s} = \cos\phi\hat{i} + \sin\phi\hat{j}, \hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}, \hat{z} = \hat{z}$$

$$\begin{aligned}
\vec{\nabla}F &= \frac{1}{h_s}\frac{\partial F}{\partial s}\hat{s} + \frac{1}{h_\phi}\frac{\partial F}{\partial \phi}\hat{\phi} + \frac{1}{h_z}\frac{\partial F}{\partial z}\hat{z} \\
&= \hat{s}\frac{\partial F}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial F}{\partial \phi} + \hat{z}\frac{\partial F}{\partial z}
\end{aligned}$$

6. If $\vec{A} = s\hat{z}$ find $\vec{\nabla} \times \vec{A}$.

   soln

$$\vec{\nabla} \times \vec{\mathrm{A}} = \left[\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\mathrm{s}} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right]\hat{\mathrm{z}}$$

   Here the only component we have is $A_z = s$.

$$\therefore \quad \vec{\nabla} \times \vec{A} = \hat{s}\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \hat{\phi}\frac{\partial A_z}{\partial s} = -\hat{\phi}$$

7. Find the divergence of $\vec{\mathrm{v}} = (r\cos\theta)\hat{\mathrm{r}} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}$. Check the divergence theorem for this function, using the volume as the inverted hemispherical bowl of radius R, resting on the x-y plane and centred at the origin.

   soln

$$\begin{aligned}
\vec{\nabla} \cdot \vec{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \\
&= 5\cos\theta - \sin\phi
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \int_V \vec{\nabla} \cdot \vec{v} dV &= \int_0^R \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (5\cos\theta - \sin\phi) r^2 \sin\theta dr d\theta d\phi \\
&= 2\pi\frac{R^3}{3}\int_0^{\frac{\pi}{2}} 5\cos\theta\sin\theta d\theta \\
&= 5\pi R^3/3
\end{aligned}$$

$$\oint_S \vec{v} \cdot \hat{n} da == \int_{hemisphere} \vec{v} \cdot \hat{n} da + \int_{basecircle} \vec{v} \cdot \hat{n} da$$

Over the hemisphere $\hat{n} = \hat{r}$ and $da = R^2 \sin\theta d\theta d\phi$

$$\therefore \quad \int_{hem} \vec{v} \cdot \hat{n} da = \int_{hem} (R\cos\theta) R^2 \sin\theta d\theta d\phi = \pi R^3$$

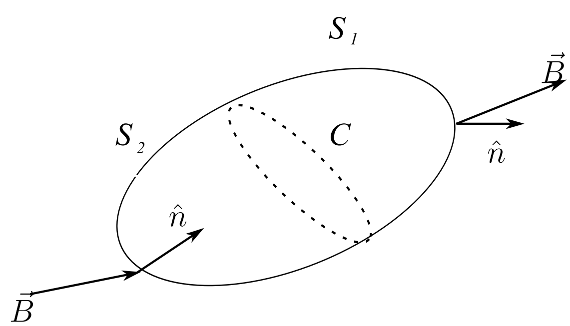
Over the circle $\hat{n} = \hat{\theta}$ and $da = r\sin\theta dr d\phi r dr d\phi$ since $\theta = \pi/2$.

$$\therefore \quad \int_{basecircle} \vec{v} \cdot \hat{n} da = \int_0^R \int_0^{2\pi} (r \sin\frac{\pi}{2}) r dr d\phi = \frac{2\pi R^3}{3}$$

$$\therefore \quad \oint_S \vec{v} \cdot \hat{n} da = \pi R^3 + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}$$

8. If $\vec{\nabla} \cdot \vec{\mathrm{B}} = 0$ show that there exists a vector function $\vec{\mathrm{A}}$ such that $\vec{\nabla} \times \vec{\mathrm{A}} = B$

soln



$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\therefore \quad \oint_S \vec{B} \cdot \hat{n} da = \int_V (\vec{\nabla} \cdot \vec{B}) dV = 0$$

$$\therefore \quad \int_{S_1} \vec{B} \cdot \hat{n} da = \int_{S_2} \vec{B} \cdot \hat{n} da$$

On the surface S_1 $\hat{n}$ is outward to the volume V. On the surface S_2 $\hat{n}$ is inward to the volume V as shown in the figure.

The surface integral will be same over any surface bounded by the dotted curve (loop) C. So these integrals are related to the values of certain fields along the curve C. This can be obtained by a line integral along C. There are two possibilities for these to be scalars:

$$\oint_C \phi dl \quad \text{and} \oint_C \vec{A} \cdot \vec{dl}$$

In the first possibility ϕ is a scalar function integrated over C with length element $dl = |\vec{dl}|$. In the second possibility $\vec{A}$ is a vector function integrated over C with vector length element $\vec{dl}$.

We choose the second possibility because if we reverse the direction of the normals then the surface integrals reverses the sign. This should be accompanied by reversing the way we are traversing C thus replacing $\vec{dl}$ by $-\vec{dl}$. We can see that the first integral will have the same result whether we traverse C clockwise or anticlockwise. So that can't be equal to our surface integral.

$$\therefore \quad \int_{S_1} \vec{B} \cdot \hat{n} da = \oint_C \vec{A} \cdot \vec{dl}$$

By Stokes' theorem

$$\therefore \quad \int_{S_1} \vec{B} \cdot \hat{n} da = \int_{S_1} (\vec{\nabla} \times \vec{A}) \cdot \vec{n} da$$

Since this is true for any arbitrary surface bounded by the curve C we conclude that

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

We have proved that $\exists$ a vector field $\vec{A}$ such that $\vec{B} = \vec{\nabla} \times \vec{A}$. This $\vec{A}$ is not unique. For e.g if $\vec{A}' = \vec{A} + \vec{\nabla}\Phi$ then

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times (\vec{\nabla}\Phi) = \vec{\nabla} \times \vec{A} = \vec{B}$$
