Lecture-14

Recap:

Normal distribution

$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\ \sigma}$$

$$\left.\begin{array}{l} E[X] = \mu \\ Var(X) = \sigma^2 \end{array}\right\} \text{H.W.}$$

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if $\mu = 0$ and $\sigma = 1$,
its called standard normal

$$\Phi(a) = \int_{-\infty}^{a} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

$X : N(\mu, \sigma^2)$

What is the distribution of $\cancel{\frac{X-\mu}{\sigma^2}}$ $\frac{X-\mu}{\sigma}$?

is standard normal $N(0,1)$

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if X is normally distributed then so is $aX+b$.

$$\frac{X}{\sigma} - \frac{\mu}{\sigma}$$

$$a = \frac{1}{\sigma}, \quad b = \frac{-\mu}{\sigma}$$

Mean for $\frac{X-\mu}{\sigma}$ is 0

$$a\mu + b = \frac{1}{\sigma}\cdot\mu - \frac{\mu}{\sigma} = 0$$

Var for $\frac{X-\mu}{\sigma}$ is $a^2\sigma^2 = \frac{1}{\sigma^2}\cdot\sigma^2 = 1$

e.g. -

Average for insem I is 10.

$\sigma = 2, \sigma^2 = 4$

if X is your marks,

then $P(8 < X < 12)$

$$\int_8^{12} \frac{e^{-\frac{(x-10)^2}{2 \cdot 4}}}{\sqrt{2\pi} \cdot 2}\, dx$$

Convert this into std. normal.

$$X \longrightarrow \frac{X - \mu}{\sigma}$$

$$P(8 < X < 12)$$

$$\downarrow$$

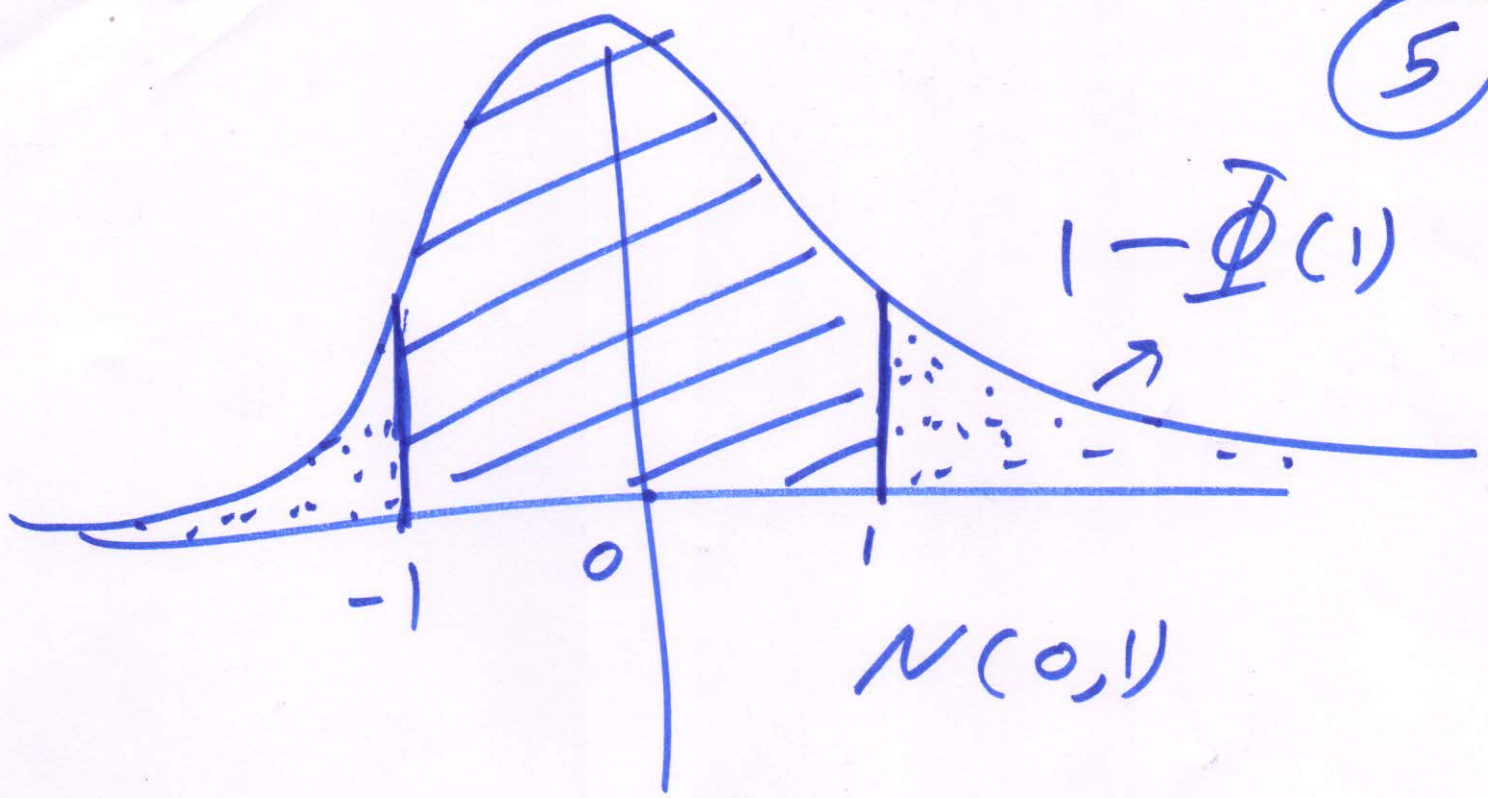
$$P\left(\frac{8-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{12-\mu}{\sigma}\right)$$

$$P\left(\frac{8-10}{2} < \frac{X-\mu}{\sigma} < \frac{12-10}{2}\right)$$

$$\downarrow$$

$$P(-1 < Z < 1)$$

$$= \underset{0.8413}{\Phi(1)} - \Phi(-1)$$



$\Phi(1) - \Phi(-1)$

$= \Phi(1) - (1 - \Phi(1))$

$= 2 \cdot \Phi(1) - 1$

$= 2 * 0.8413 - 1$

$= 1.6826 - 1$

$= 0.6826$

$$P(\mu < X < \mu + \sigma)$$

$$= P\left(\frac{\mu - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{\mu + \sigma - \mu}{\sigma}\right)$$

$$= P(0 < Z < 1)$$

$$= \Phi(1) - \Phi(0)$$

$$= 0.8413 - 0.5$$

$$= 0.3413 \quad * 322$$

$$P(\mu + \sigma < X < \mu + 2\sigma)$$

$$= P\left(\frac{\mu + \sigma - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{\mu + 2\sigma - \mu}{\sigma}\right)$$

$$= P(1 < Z < 2)$$

eg: Grading on the curve

Grade	marks	no. of students
AA	more than $\mu+2\sigma$	7
AB	$\mu+\sigma$ to $\mu+2\sigma$	44
BB	μ to $\mu+\sigma$	110
BC	$\mu-\sigma$ to μ	110
CC	$\mu-2\sigma$ to $\mu-\sigma$	44
CD	$\mu-3\sigma$ to $\mu-2\sigma$	7
DD	$\mu-4\sigma$ to $\mu-3\sigma$	
DE	$\mu-5\sigma$ to $\mu-4\sigma$	
F	less than $\mu-5\sigma$	

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$\Phi(2) - \Phi(1)$

$= 0.9772 - 0.8413$

$= 0.1359 \quad * 322$

$= 44$

$P(\cancel{x}\ 26 < X)$

$= P(2 < Z)$

$= 1 - 0.9772$

$= 0.0228 \quad * 322$

$= 7$

eg: Binary
Communication Channel.

$0 \rightarrow -2$ volts

$1 \rightarrow 2$ volts.

You receive R.

$R < 0.5$, it is 0

$R > 0.5$, its 1

$$R = \underbrace{X}_{\text{sent}} + \underbrace{N}_{N(0,1)}$$

What is the probability of error? if $P(X=0) = 1/3$
$P(X=1) = 2/3$

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p = probability of error

$$= P(\text{error} \mid 0 \text{ is sent})\, P(\underset{1/3}{0 \text{ is sent}})$$

$$+$$

$$P(\text{error} \mid 1 \text{ is sent})\, P(\underset{2/3}{1 \text{ is sent}})$$

$P(\text{error} \mid 0 \text{ is sent})$

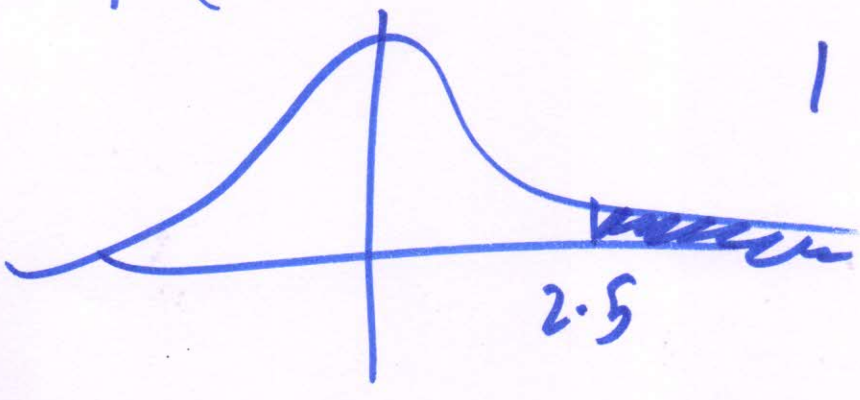
$$R = X + N$$

$$R = -2 + N$$

error if $R > 0.5$

$$-2 + N > 0.5$$

$$P(N > 2.5) = 1 - \Phi(2.5)$$



$$1 - 0.9938$$

$$= 0.0062$$

P(error | 1 is sent)

$X = +2$

error if $R < 0.5$

$2 + N < 0.5$

$P(N < -1.5) =$

$\Phi(-1.5) = 1 - \Phi(1.5)$

$= 1 - 0.9332$

$= 0.0668$

Normal approximation to Binomial

De Moivre-Laplace theorem

n is large

σ is high

e.g.: A coin is tossed 40 times. Coin is fair.

X = no. of heads.

$$P(X=20) = \binom{40}{20} \left(\frac{1}{2}\right)^{20} \left(\frac{1}{2}\right)^{20}$$

$$= \frac{40!}{20!\,20!\,2^{40}} = 0.1254$$

we approximate this as normal distribution.

$$\mu = np = 20$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{10}$$

$$P(X=20) = 0$$

$$\parallel$$

$$P(19.5 < X < 20.5) =$$

Continuity correction (12)

$$P(19.5 < X < 20.5)$$

$$= P\left(\frac{19.5-20}{\sqrt{10}} < Z < \frac{20.5-20}{\sqrt{10}}\right)$$

$$= P(-0.158 < Z < 0.158)$$

$$= 2 \cdot \Phi(0.158) - 1$$

$$= 0.1272 \rightarrow$$ approx. using Normal

$0.1254 \rightarrow$ Binomial distribution

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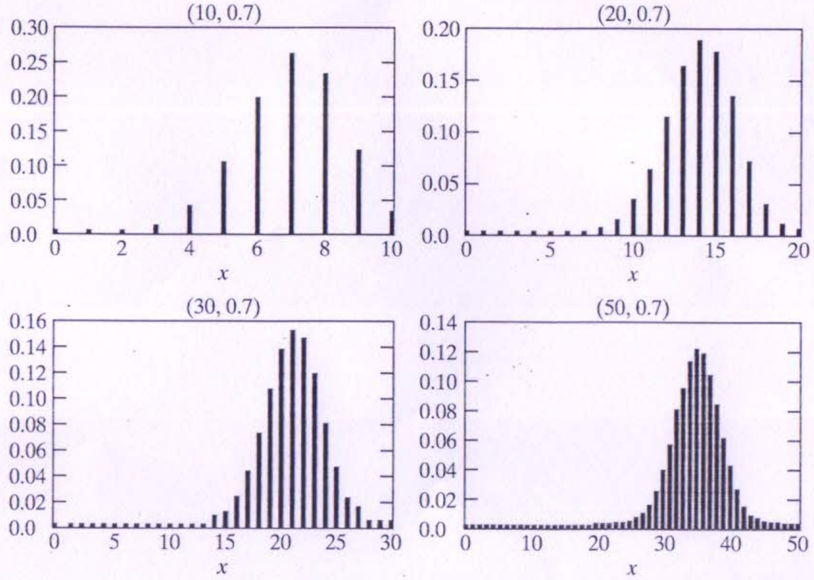
(10, 0.7)
(20, 0.7)
(30, 0.7)
(50, 0.7)
0.30
0.25
0.20
0.15
0.10
0.05
0.0
0.16
0.14
0.12
0.08
0.06
0.04
0.02
x
0
2
4
6
8
10
5
15
20
25
30
40
50

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TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998