1. Two infinitely long wires running parallel to the x axis carry uniform charge densities $+\lambda$ and $-\lambda$.

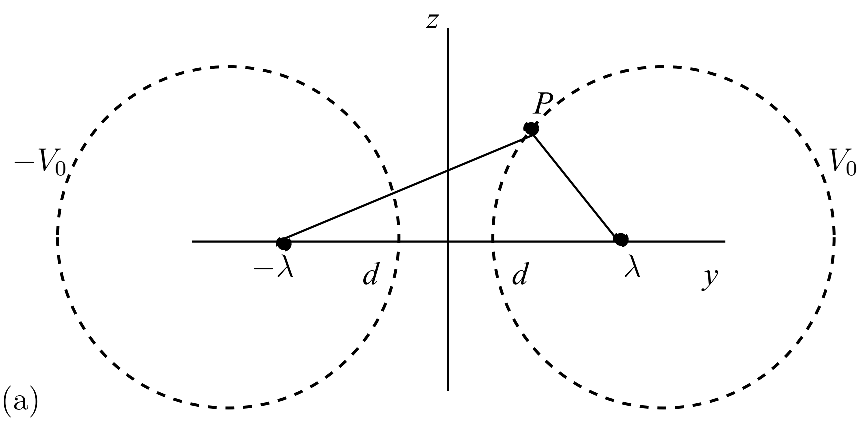
   (a) Find the potential at any point using the origin as the reference.

   (b) Show that the equipotential surfaces are circular cylinders. Locate the axis and radius of the cylinder corresponding to a given potential V_0.

soln

The two line charges $+\lambda$ and $-\lambda$, parallel to the x axis cuts the y axis at $y = -d$ and $y = d$ respectively. Consider a point $P(x, y, z)$. The distance of P from the line charges are s_1 and s_2 given by

$$s_1 = \sqrt{(y+d)^2 + z^2} \qquad \text{and} \qquad s_2 = \sqrt{(y-d)^2 + z^2}$$



(a)

The potential at point P is

$$V = \frac{-\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_1}{k_1}\right) + \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_2}{k_2}\right) = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\left(\frac{s_2}{s_1}\right) + \ln\left(\frac{k_1}{k_2}\right)\right]$$

If we want $V = 0$ at the origin where $s_1 = s_2 = d$ then $k_1 = k_2$. So

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\sqrt{(y-d)^2 + z^2}}{\sqrt{(y+d)^2 + z^2}}\right) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{(y-d)^2 + z^2}{(y+d)^2 + z^2}\right)$$

(b) Consider a surface of constant potential V_0. Then from the above eqn. we have

$$(y-d)^2 + z^2 = \left[(y+d)^2 + z^2\right] K \qquad \text{where} \qquad K = \exp\left[\frac{4\pi\epsilon_0 V_0}{\lambda}\right]$$

$$\begin{aligned} \therefore\ \ y^2(1-K) + z^2(1-K) - 2yd(1+K) + d^2(1-K) &= 0 \\ \therefore\ \ y^2 + z^2 - 2yd\left(\frac{1+K}{1-K}\right) + d^2 &= 0 \end{aligned}$$

This is the equation of a circle $(y - y_0)^2 + z^2 = R^2$ with center at $(y_0, 0) = (d\left(\frac{1+K}{1-K}\right), 0)$ and radius $R = \frac{2d\sqrt{K}}{1-K}$.

2. A conducting sphere of radius R has an amount of charge Q over it. This sphere is placed in an otherwise uniform electric field $\vec{E}_0$. Find the potential in the region outside the sphere.

soln:
The given electrostatic configuration has an azimuthal symmetry if we consider the uniform electric field along $\hat{z}$. The general variable separated form of the potential is given as

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Far away from the sphere the influence of the sphere gets negligible. So the electric field is $E_0\hat{z}$. This corresponds to an electric potential function $-E_0 z = -E_0 r\cos\theta$. Now as $r \to \infty$, $V(r,\theta) \to \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$. So we have

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta$$

This gives $A_1 = -E_0$, $A_0 = A_2 = A_3 = ... = 0$. Let us rewrite the potential using these values of the constants A_l.

$$V(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

The sphere is an equipotential surface. So $V(R,\theta)$ must be independent of θ.

$$\begin{aligned} V(R,\theta) &= -E_0 R\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \\ &= \frac{B_0}{R} + \left(-E_0 R + \frac{B_1}{R^2} \right)\cos\theta + \sum_{l=2}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \end{aligned}$$

If this has to be independent of θ then we must have $B_1 = E_0 R^3$ and $B_2 = B_3 = ... = 0$. Let us rewrite the potential again

$$V(r,\theta) = \frac{B_0}{r} + \left(-E_0 r + \frac{E_0 R^3}{r^2} \right)\cos\theta$$

The total charge on the sphere is Q. Q/ϵ_0 will be equal to the total flux of the electric field over the surface of the sphere. Over the surface of the sphere the electric field is normal, which is the radial direction $\hat{r}$. The radial component of the electric field can be derived from $V(r,\theta)$. This is $-\frac{\partial V}{\partial r}$ given by

$$E_r(r,\theta) = \frac{B_0}{r^2} + \left(E_0 + \frac{2E_0 R^3}{r^3} \right)\cos\theta$$

At $r = R$ we get

$$E_r(R,\theta) = \frac{B_0}{R^2} + 3E_0\cos\theta$$

This is the normal component of the electric field over the surface of the sphere. Integrating this over the surface of the sphere will give the total charge on the sphere. The constant term $\frac{B_0}{R^2}$ when integrated over the sphere will give $\frac{B_0}{R^2}4\pi R^2 = 4\pi B_0$. We integrate the other term

$$\begin{aligned}
\therefore \quad \frac{Q}{\epsilon_0} &= 4\pi B_0 + \int_0^{2\pi}\int_0^{\pi} 3E_0 \cos\theta R^2 \sin\theta d\theta d\phi \\
&= 4\pi B_0 + 6\pi R^2 E_0 \int_0^{\pi} \cos\theta \sin\theta d\theta \\
&= 4\pi B_0
\end{aligned}$$

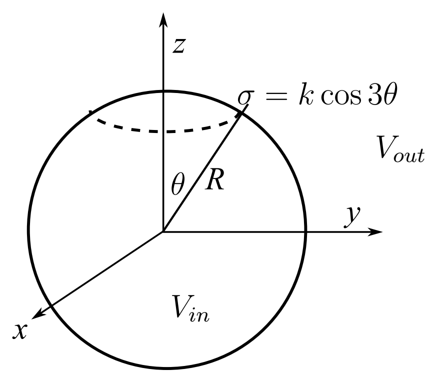
$$\therefore \quad B_0 = \frac{Q}{4\pi\epsilon_0}$$

Now we have fixed all the constants. The potential is given as

$$V(r,\theta) = \frac{Q}{4\pi\epsilon_0 r} + \left(-r + \frac{R^3}{r^2}\right) E_0 \cos\theta$$

3. A sphere of radius R has a surface charge given by the surface charge density $\sigma = k\cos 3\theta$ where k is a constant. Find the potential inside and outside the sphere.

soln



For a problem with azimuthal symmetry potential is

$$\begin{aligned}
V_{out} &= \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) \\
V_{in} &= \sum_{l=0}^{\infty}\left(A_l' r^l + \frac{B_l'}{r^{l+1}}\right) P_l(\cos\theta)
\end{aligned}$$

Since there are no sources at infinity, we will demand $V_out \to 0$ as $r \to \infty$. This implies $A_l = 0$ for $l = 0, 1, 2, ...$ So we have

$$V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

As $r \to 0$,

$$V_{in} \to \sum_{l=0}^{\infty} \frac{B_l'}{r^{l+1}} P_l(\cos\theta) \to \infty$$

So for V_{in} to be finite inside we must have $B_l' = 0$ for $l = 0, 1, 2, ...$

$$\therefore \quad V_i n = \sum_{l=0}^{\infty} A_l' r^l P_l(\cos\theta)$$

At $r = R$, $V_{in}(R) = V_{out}(R)$.

$$\sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = \sum_{l=0}^{\infty} A_l' R^l P_l(\cos\theta)$$

Since $P_l(\cos\theta)$ are orthogonal functions and hence linearly independent we have

$$\frac{B_l}{R^{l+1}} = A_l' R^l \quad \Longrightarrow \quad B_l = A_l' R^{2l+1} \tag{1}$$

Over the surface of the sphere we have

$$E_{r \ \ out} - E_{r \ in} = \frac{\sigma}{\epsilon_0} = \frac{k\cos 3\theta}{\epsilon_0}$$

So we calculate the radial component of the electric field inside and outside.

$$\begin{aligned} E_{rout} = -\frac{\partial V_{out}}{\partial r} &= -\frac{\partial}{\partial r}\left[\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)\right] \\ &= \sum_{l=0}^{\infty} (l+1)\frac{B_l}{r^{l+2}} P_l(\cos\theta) \\ E_{rin} &= \sum_{l=0}^{\infty} (-l) A_l' r^{l-1} P_l(\cos\theta) \end{aligned}$$

So at the surface $r = R$ we will have

$$\begin{aligned} (E_{rout} - E_{rin})|_{r=R} &= \sum_{l=0}^{\infty} \left[(l+1)\frac{B_l}{R^{l+2}} + l A_l' R^{l-1}\right] P_l(\cos\theta) \\ &= \frac{k\cos 3\theta}{\epsilon_0} \end{aligned}$$

Substituting for B_l from eq.(1) we get

$$\sum_{l=0}^{\infty} (2l+1) A_l' R^{l-1} P_l(\cos\theta) = \frac{k\cos 3\theta}{\epsilon_0} \tag{2}$$

Multiplying both sides by $P_m(\cos\theta)$ and integrating from $\theta = 0$ to π will give A_l'. But let us do it differently.

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = \frac{8}{5} P_3(\cos\theta) - \frac{3}{5} P_1(\cos\theta)$$

Substituting in eq.(2) we get

$$\sum_{l=0}^{\infty}(2l+1)A'_l R^{l-1}P_l(\cos\theta) = \frac{k}{5\epsilon_0}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right]$$

We see that only the $l=1$ and $l=3$ terms survive on l.h.s.

$$\begin{aligned}
\therefore\quad 3A'_1 = -\frac{3k}{5\epsilon_0} &\qquad\Longrightarrow\qquad A'_1 = -\frac{k}{5\epsilon_0} \\
7A'_3R^2 = \frac{8k}{5\epsilon_0} &\qquad\Longrightarrow\qquad A'_3 = \frac{8k}{35\epsilon_0 R^2}
\end{aligned}$$

All other $A'_l = 0$.
From Eq.(1) we get

$$\begin{aligned}
B_1 &= A'_1R^3 = -\frac{kR^3}{5\epsilon_0} \\
B_3 &= A'_3R^7 = \frac{8kR^5}{35\epsilon_0}
\end{aligned}$$

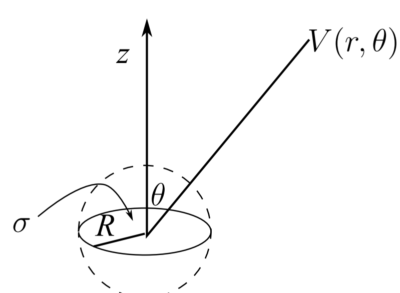
$$\begin{aligned}
\therefore\quad V_{out} &= \frac{kR^3}{5\epsilon_0}\frac{\cos\theta}{r^2} + \frac{8kR^5}{35\epsilon_0}\frac{P_3(\cos\theta)}{r^4} \\
V_{in} &= \frac{k}{5\epsilon_0}r\cos\theta + \frac{8kR^5}{35\epsilon_0 R^2}r^3P_3(\cos\theta)
\end{aligned}$$

Check that the potential is continuous at $r = R$.

4. The potential on the axis of a uniformly charged disk of radius R is given as

$$V(r,0) = \frac{\sigma}{2\epsilon_0}(\sqrt{r^2+R^2} - r)$$

Use this together with the fact that $P_l(1) = 1$, to evaluate the first three terms in the expansion of the potential function at points off the axis assuming $r > R$.



soln:
We do a taylor expansion of $V(r,0)$ around $\frac{R}{r}=0$.

$$
\begin{aligned}
V(r,0) &= \frac{\sigma r}{2\epsilon_0}\left(\sqrt{1+\left(\frac{R}{r}\right)^2}-1\right) \\
&= \frac{\sigma r}{2\epsilon_0}\left[\frac{1}{2}\left(\frac{R}{r}\right)^2-\frac{1}{8}\left(\frac{R}{r}\right)^4+\frac{3}{48}\left(\frac{R}{r}\right)^6-....\right] \\
&= \frac{\sigma}{2\epsilon_0}\left[\frac{1}{2}\frac{R^2}{r}-\frac{1}{8}\frac{R^4}{r^3}+\frac{3}{48}\frac{R^6}{r^5}-....\right]
\end{aligned}
$$

Using the fact that $P_l(0)=1\ \forall\ l$ and from the general form of the potential in an azimuthal symmetric electrostatic configuration

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

we can write down the potential at all points (r,θ) as

$$V(r,\theta)=\frac{\sigma}{2\epsilon_0}\left[\frac{1}{2}\frac{R^2}{r}-\frac{1}{8}\frac{R^4}{r^3}P_2(\cos\theta)+\frac{3}{48}\frac{R^6}{r^5}P_4(\cos\theta)-....\right]$$

5. Solve Laplace's equation by separation of variables in cylindrical co-ordinates, assuming there is no dependence on z.
soln
In cylindrical coordinates, the Laplace's Equation is written as

$$\nabla^2 V=\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Since there is no dependence on z we have

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2}=0$$

Let us consider $V(s,\phi)=f(s)g(\phi)$. Then

$$g\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right)+\frac{f}{s^2}\frac{\partial^2 g}{\partial\phi^2}=0$$

Dividing throughout by V and multiplying by s^2 gives

$$\frac{s}{f}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right)+\frac{1}{g}\frac{\partial^2 g}{\partial\phi^2}=0$$

Each term above is a constant. The angular function $g(\phi)$ is expected to be periodic. Hence we consider the constant related to it as negative.
Let $\frac{1}{g}\frac{\partial^2 g}{\partial\phi^2}=-k^2$ and $\frac{s}{f}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right)=k^2$.

$$\therefore\quad g(\phi)=A\sin(k\phi)+B\cos(k\phi) \tag{3}$$

Since $g(\phi) = g(\phi + 2\pi)$ for any physically meaningful solution, k must be an integer. The differential Eqn in s is

$$s\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) = k^2 f$$

$f = s^l$ satisfies the equation. Substituting , we get
$l^2 s^l = k^2 f$, i.e $l^2 f = k^2 f \quad \Longrightarrow \quad l = \pm k$.
$\therefore \quad f = Ds^k + Es^{-k}$.
We note that this is valid when $k \neq 0$. If $k = 0$ then

$$\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) = 0 \quad \Longrightarrow \quad s\frac{\partial f}{\partial s} = c_1$$

$\therefore \quad f(s) = c_1 \ln s + c_2$.
Since k has to be an integer, l is also an integer. Hence the general solution in cylindrical coordinates is

$$V(s,\phi) = (c_1 \ln s + c_2)B_0 + \sum_{k=1}^{\infty}\left(D_k s^k + E_k s^{-k}\right)(A_k \sin k\phi + B_k \cos k\phi)$$

$c_1, c_2, B_0, A_k, B_k, D_k$ and E_k are constants to be determined from the boundary conditions. It is useful to write this potential as

$$V(s,\phi) = (c_1 \ln s + c_2) + \sum_{k=1}^{\infty} s^k (A_k \sin k\phi + B_k \cos k\phi) + s^{-k}(C_k \sin k\phi + D_k \cos k\phi)$$