Dhirubhai Ambani Institute of Information and Communication Technology (DA-IICT)
Mid-semester Examination
CT314 (Statistical Communication Theory)
Date of Examination: March 23, 2012
Duration: 2 Hours
Maximum Marks: 20

Instructions:
1. Attempt all questions.
2. Use of scientific non programmable calculator is permitted.
3. Figures in brackets indicate full marks.
4. All the acronyms carry their usual meaning.

Q1: Let X and Y be two random variables with Y=aX+b, where a, b are constants. Find the correlation coefficient between X and Y. (2)

Q2: Let X be uniformly distributed in the interval (-1, 1) and $Y = X^3$. Find the linear MMSE of Y in terms of X. Also find best MMSE estimate. Write the final expression for MMSE in each case. Which estimate would be better and why? (4)

Q3: Consider a vector of random variables $\underline{X} = [X_1, X_2]^T$. These random variables have unit variance and are uncorrelated. Now the transformed vector $\underline{Y} = [Y_1, Y_2]^T$ is obtained as $\underline{Y} = A\underline{X}$, where A is the transformation matrix. Find the matrix A so that $\underline{Y}$ has the covariance matrix $C_{\underline{Y}} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$. Hint: First show that $C_{\underline{Y}} = AA^T$ which can then be expressed as $U\Sigma U^{-1}$ (8)

Q4: Consider the experiment of tossing a fair coin. The random process $X(t)$ is defined by $X(t) = \sin \pi t$, when head occurs and $X(t) = 2t$, when tail show up. Sketch the sample functions and write the pdfs at t=0 sec and t=1 sec. Here $-\infty < t < \infty$ (2)

Q5: $X(t) = \cos(w_0 t + \Theta)$, $-\infty < t < \infty$, w_0 is constant and Θ is a uniformly distributed random variable in the interval $(-\pi, \pi)$. Is this a WSS process? Verify by checking the conditions for WSS. Also find autocovariance function for the process. (4)

"BEST WISHES"

MidSem Exam 2011-2012

CT314, Solutions.

ANS 1. Given X & $Y = aX + b$.

$$\rho_{xy} = \frac{\mathrm{cov}(X, Y)}{\sigma_x \sigma_y}$$

$$\mathrm{cov}(X, Y) = E[(X - \mu_x)(Y - \mu_y)]$$

$$Y - \mu_y = (aX + b) - E(aX + b)$$
$$= (aX + b) - a\mu_x - b$$
$$= (aX - a\mu_x) = a(X - \mu_x)$$

$$\therefore \mathrm{cov}(X, Y) = E[a(X - \mu_x)^2]$$

Now $\sigma_x^2 = E[(X - \mu_x)^2]$

& $\sigma_y^2 = E[(Y - \mu_y)^2] = E[a^2(X - \mu_x)^2]$
$$= a^2 E[(X - \mu_x)^2]$$

$$\therefore \sigma_x \sigma_y = a\, E[(X - \mu_x)^2]$$

$$\therefore \rho_{xy} = \frac{aE(X - \mu_x)^2}{a\, E(X - \mu_x)^2} = 1.$$

ANS 2. MMSE estimate of Y in terms of X is

$$\hat{Y} = E[Y/X] = E[Y] + \frac{\mathrm{cov}(X, Y)}{\mathrm{Var}(X)}(x - E(X))$$

From given data, $E[X] = 0$, $E[Y] = E[X^3] = 0$

$$\mathrm{cov}(X, Y) = E[XY] - E[X]E[Y] = E[XY] = E[X^4] = \int_{-1}^{1} x^4 dx$$
$$= \left(\frac{x^5}{5}\right)_{-1}^{1} = \frac{1}{5}(1 + 1) = \frac{2}{5}$$

and $\mathrm{Var}(X) = E[(X-\mu_x)^2] = E[X^2] = \int_{-1}^{1} x^2 dx = \left(\frac{x^3}{3}\right)_{-1}^{1}$

$= \frac{1}{3}(1+1) = \frac{2}{3}$.

$\therefore \hat{y} = 0 + \frac{2}{5 \cdot 2} \cdot 3 \,(x - 0)$

$E[Y/x] = \hat{y} = \frac{3}{5}x$ → This is Linear MMSE estimate of y in terms of x.

MMSE in this case is $E[(X^3 - 3/5 X)^2]$. ①

You may wish to check this in the Lab. Generate 1000 samples of X drawn from uniform distribution (-1,1). Get $Y = X^3$, actual (true). Get $\hat{Y} = \frac{3}{5}X$ linear est. Find mean of the error which is minimum only if we choose coeff of X as 3/5. Any other coefficient leads to higher MMSE.

Now for best MMSE estimate can be obtained by considering nonlinear estimate

i.e. $\hat{Y} = E[Y/X]$ from this

$E[Y/X=x] = E[X^3/X=x] = x^3$ which is the best estimate of Y given X.

and the MMSE in this case is $E[(Y - g(X))^2]$

$= E[(X^3 - X^3)^2] = 0$

This gives minimum error in terms of MMSE.

Q3. X_1, X_2 have unit variance and they are uncorrelated. $\therefore C_{\underline{X}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = identity matrix (Covariance matrix of $\underline{X}$ is identity matrix)

Now $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$C_{\underline{Y}} = E[(\underline{Y} - m_{\underline{Y}})(\underline{Y} - m_{\underline{Y}})^T]$

$= E[(A\underline{X} - A m_{\underline{X}})(A\underline{X} - A m_{\underline{X}})^T]$

$= A C_{\underline{X}} A^T$ where $C_{\underline{X}}$ = identity matrix

$\therefore C_{\underline{Y}} = AA^T$ = covariance matrix of $\underline{Y}$

It is required that $\underline{C_y}$ i.e. covariance matrix of $\underline{Y}$ has the form
$$\underline{C_y} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

For this we want to find A.

To get A we proceed as follows. Since $\underline{C_y}$ is symmetric it can be diagonalized

Hence we can write

$$\underset{2\times 2}{\underline{C_y}} = \underset{2\times 2}{U}\,\underset{2\times 2}{\Sigma}\,\underset{2\times 2}{U^{-1}} = U \Sigma U^T$$

U - ~~eigen vectors~~ columns are eigen vectors of $\underline{C_y}$ i.e. covariance matrix

Σ - Eigen values are the diagonal elements.

$$= U \Sigma^{1/2} \Sigma^{1/2} U^T$$
$$= U \Sigma^{1/2} (\Sigma^{1/2})^T U^T$$
$$= U \Sigma^{1/2} (U \Sigma^{1/2})^T$$
$$= A \quad A^T$$

$\therefore A = U \Sigma^{1/2}$ where U is the eigen vector matrix and Σ is the diagonal matrix with diagonal entries as the eigen values

So A can be obtained by finding the eigen vector eigen value decomposition of $\underline{C_y}$.

$$C_{\underline{Y}} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$(C_{\underline{Y}} - \lambda I)\underline{p} = \underline{0}$. take determinant of the matrix $(C_{\underline{Y}} - \lambda I)$

$$\left| \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} (1-\lambda) & 0.5 \\ 0.5 & (1-\lambda) \end{vmatrix}$$

i.e. $(1-\lambda)^2 - 0.25 = 0$.

$1 + \lambda^2 - 2\lambda - 0.25 = 0$ or $\lambda^2 - 2\lambda + 0.75 = 0$.

So $\lambda = \dfrac{+2 \pm \sqrt{4 - 4.1.0.75}}{2.1}$

$= \dfrac{+2 \pm \sqrt{4-3}}{2} = \dfrac{+2 \pm 1}{2}$

$\therefore \lambda_1 = 3/2, \quad \lambda_2 = 1/2$.

with $\lambda_1 = 3/2$

$$\begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$p_1 + 0.5p_2 = 1.5p_1$ & $0.5p_1 + p_2 = 1.5p_2$.

$-0.5p_1 + 0.5p_2 = 0$ & $0.5p_1 - 0.5p_2 = 0$

So we have one eq^n ~~0.5~~ $p_1 - p_2 = 0$ and two unknowns

the sol^n is $p_1 = 1$ & $p_2 = 1$ (~~has~~ any multiple of this is also a solution)

since U ~~has~~ is orthonormal matrix, we normalize this vector to have only direction with magnitude = 1

$$\therefore P_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similary for other eigen vector

$$\begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$p_1 + .5p_2 = 0.5p_1$ & $0.5p_1 + p_2 = .5p_2$

or $.5p_1 + .5p_2 = 0$ and $.5p_1 + .5p_2 = 0$

Again one eqn and two unknows (infinite solutions) is

one solution (~~linearly independent~~) $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

& after normalizing $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

∴ we have two linearly independent eigen vectors

∴ we can write U as $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

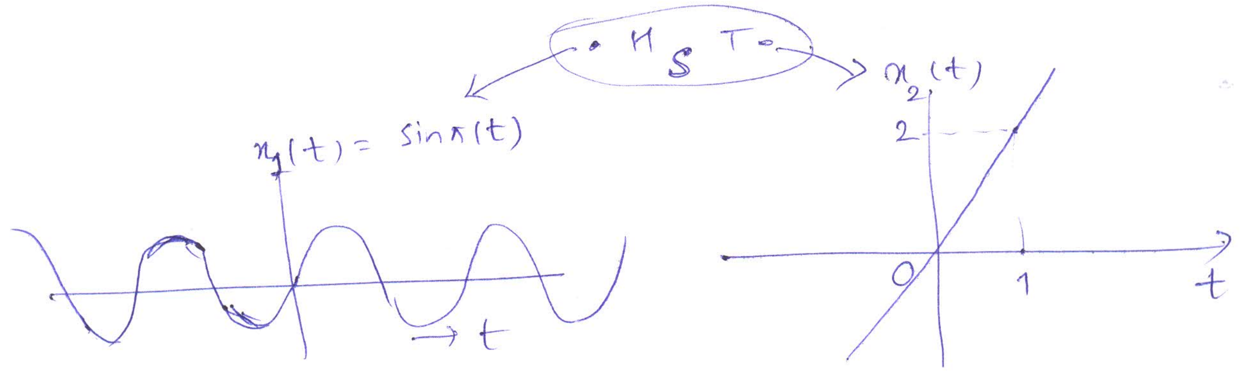
and the corresponding $\Sigma = \begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix}$.

Hence $A = U\Sigma^{1/2}$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{3/2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$$

=

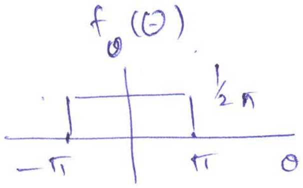
Q 4.



At $t = 0$, we have a r.v. which takes a value 0 (always) So, the pdf is $f_{X(t=0)}(x) = \delta(x)$

At $t = 1$, we have a rv taking values 0 & 2 with equal probability. So $f_{X(t=1)}(x) = \frac{1}{2}\left[\delta(x) + \delta(x-2)\right]$

Q.5.



To verify WSS condition check for mean & auto correlation function of the rp $X(t)$.

$\mu_x(t_1) = \int_{-\infty}^{\infty} x(t_1)\, f_{X(t_1)}(x)\, dx(t_1)$

Since $X(t)$ at $t = t$ is a f$\underline{\underline{n}}$ Θ

we can write $E\left[A\cos(\omega_0 t + \theta)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} A\cos(\omega_0 t + \theta)\, d\theta$

(Make use of $E(g(\theta)) = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta)\, d\theta$

So $\mu_{X(t)} = E\left[A\cos(\omega_0 t + \theta)\right] = 0$.

$R_{XX}(t_1, t_2) = C_{XX}(t_1, t_2) = E\left[A\cos(\omega_0 t_1 + \theta)\, A\cos(\omega_0 t_2 + \theta)\right]$

since mean = 0 at any t

$= \frac{A^2}{2} E\left[\cos\omega_0(t_1 + t_2 + 2\theta) + \cos\omega_0(t_1 - t_2)\right]$

$= \frac{A^2}{2} E\left[\cos\omega_0(t_1 + t_2 + 2\theta)\right] + \frac{A^2}{2} E\left[\cos\omega_0(\underbrace{t_1 - t_2}_{\tau})\right]$

$\Downarrow$

0

$= \frac{A^2}{2}\cos\omega_0\tau$ Since this term is constant

~~Now~~ So the process is WSS as mean is constant & the autocorrelation f$\underline{\underline{n}}$ is a f$\underline{\underline{n}}$ of time difference only.