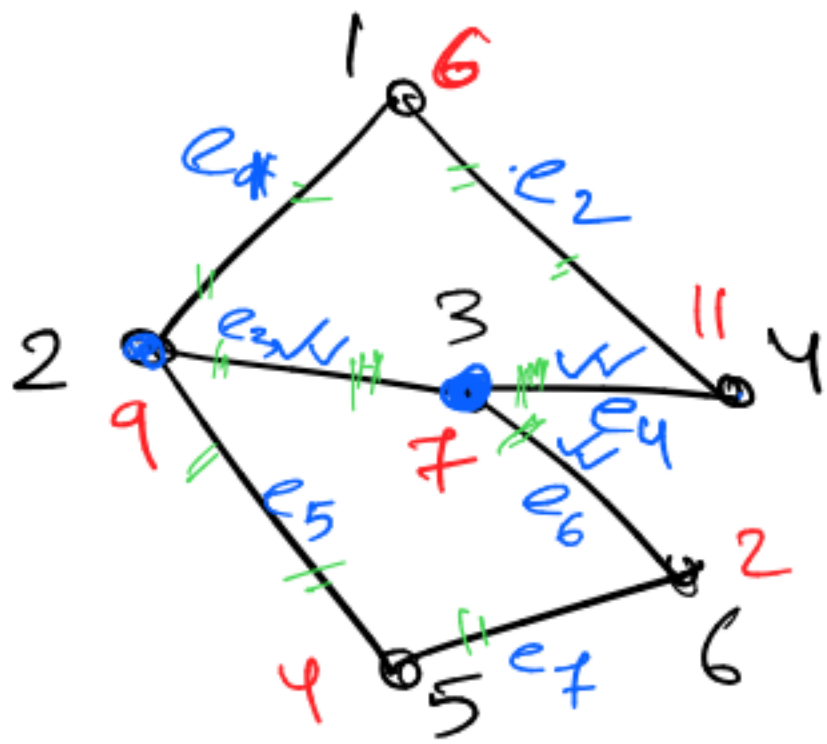
Approximation algorithm

Vertex cover problem (VCP)

We are given a graph $G(V,E)$ and a weight function $w : V \to \mathbb{R}^+$

Find a minimum cost set $V' \subseteq V$ such that at least one end point of each edge belongs to V'.



vertex 3 covers the edges e_3, e_4 and e_6.

$\{2, 3, 4, 5\}$ is a vertex cover.

weight of this vertex cover is

$$9 + 7 + 11 + 4 = 31$$

$\{1, 3, 5\}$ is a vertex cover of cost $6+7+4 = 17$.

Can a vertex cover problem be converted to a LPP. ??

For each vertex $v_i \in V$ take a binary decision variable say x_i where $x_i = 1$ if $v_i \in V'$ otherwise $x_i = 0$.

one needs to find a minimum cost vertex cover.

$$\min \sum_{i=1}^{n} w_i x_i$$

s.t. $\quad x_i + x_j \geq 1 \quad \forall\, e = (v_i, v_j) \in E$

$\quad x_i \in \{0, 1\} \quad \forall\, v_i \in V$

What can be possible values of x_i ??

either the vertex $v_i \in$ the vertex cover $x_i = 1$

or

it can be in the vertex cover $x_i = 0$

LP formulation of the VCP is

$$\begin{cases} \min \sum_{i=1}^{n} w_i x_i \\ \text{s.t. } x_i + x_j \geq 1 \quad \forall e = (v_i, v_j) \in E \\ \qquad x_i \in \{0,1\} \quad \forall v_i \in V \end{cases}$$

Observe that the decision variables are integer valued or binary valued.

We call such linear program as integer linear program. (ILP)

$$\boxed{\text{ILP is NP-complete}}$$

It is unlikely to get a polynomial time algorithm for ILP.

Solving ILP is NP-hard. Therefore we use LP to approximate the optimum solution.

First we relax the ILP.

$x_i \in \{0,1\} \rightarrow$ relax it to $x_i \in [0,1]$

So relaxed linear program is

$$\min \sum_{i=1}^{n} w_i x_i$$

s.t. $x_i + x_j \geq 1 \quad \forall e = (v_i, v_j) \in E$

$\Longrightarrow \quad x_i \in [0,1] \quad \forall v_i \in V$

$x_i \geq 0$

x_i can take value > 1

It is really possible??

NO

$x_i = 5.5$

$5.5 \cdot w_i$

P_R

$$\min \sum_{i=1}^{n} w_i x_i$$

$$\text{s.t.} \quad x_i + x_j \geq 1 \quad \forall e = (v_i, v_j) \in E$$

$$x_i \geq 0 \quad \forall v_i \in V$$

Suppose we solve P_R then How can we compare the solution of P_R with the solution of the VCP.

Here some vertices take fractional value ??.

VCP using linear program

- First formulate the VCP using ILP.
- Relax the ILP to a LP.
- Solve the LP.
- Let x^* be an optimal fractional solution.
- Let $C = \{v_i \mid x_i^* \geq \frac{1}{2}\}$
- Return C as a solution to the VCP.

Deterministic rounding

Need to show

i) C is a vertex cover.

ii) $w(C) \leq 2 \cdot opt_f$

where opt_f is the value of the optimum solution of the relaxed linear program.

Proof i)

C is a vertex cover.

- consider any edge $e = (v_i, v_j)$
- x^* is a optimum fractional solution.
- constraint for e is.

$$x_i + x_j \geq 1$$

This implies.

$$x_i^* + x_j^* \geq 1 \text{ for the edge } e.$$

- Then either $x_i^* \geq 1/2$ or $x_j^* \geq 1/2$
- by the step of the algorithm.

either $v_i \in C$

or $v_j \in C$

Then clearly C is a vertex cover.

<u>Proof ii)</u>

$$opt_f = \sum_{i=1}^{n} w_i x_i^*$$

by the step of the algorithm.

$$x_i^* \geq 1/2$$

$$opt_f = \sum_{i=1}^{n} w_i x_i^*$$

$$\geq \frac{1}{2} \sum_{i=1}^{n} w_i$$

$$= \frac{1}{2} \cdot w(C)$$

$$\Rightarrow w(C) \leq \frac{1}{2} \cdot opt_f$$

$*$

Let opt_I is the value of optimum solution to the ILP.

What is the relation between

$$opt_I \quad opt_f \quad ??$$

$$\Downarrow$$

$$\geq$$

Then clearly

$$opt_I \geq opt_f.$$

From (*) we get

$$w(C) \leq 2 \cdot opt_f \leq 2 \cdot opt_I$$

Set cover problem

Input: a universe U of n elements.
$a_1, a_2, \dots, a_n$

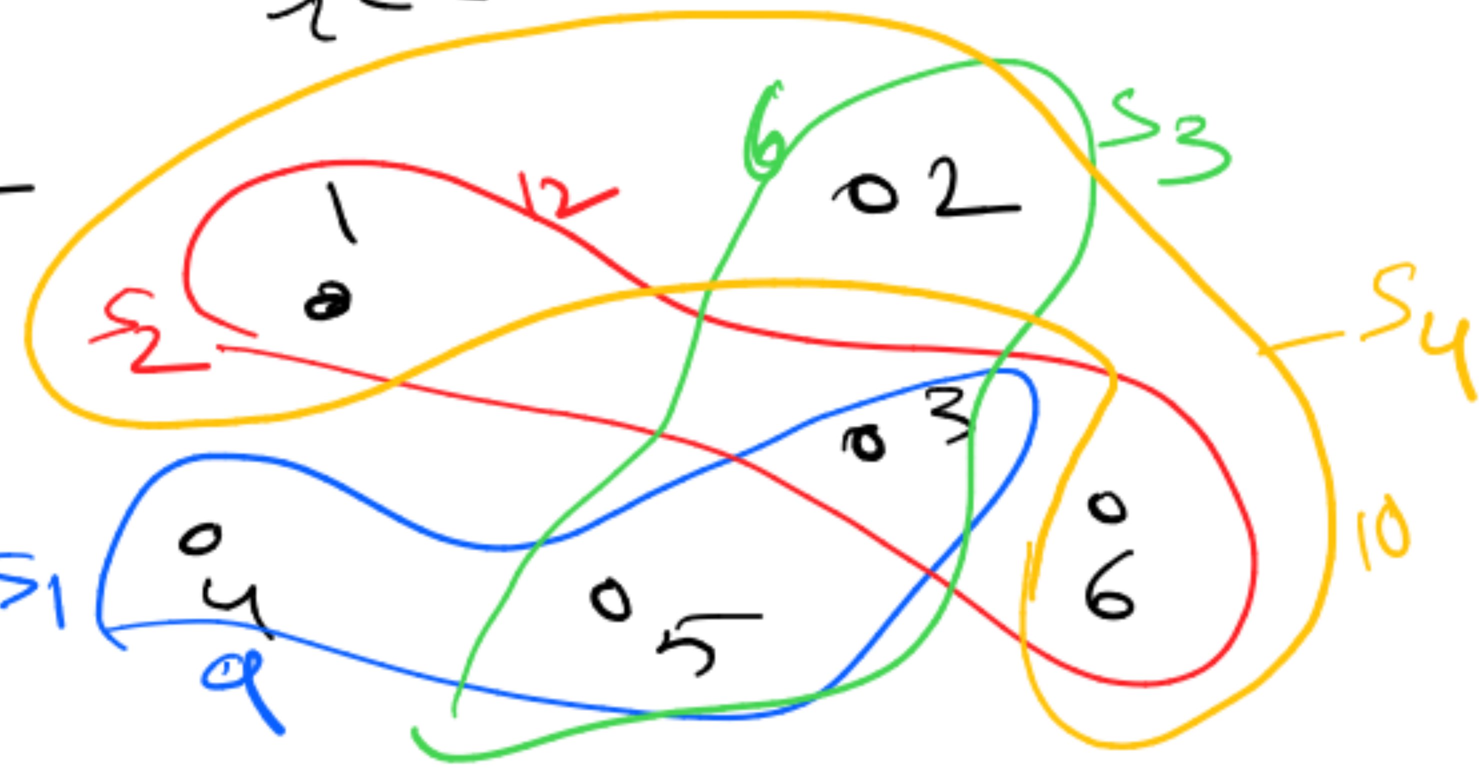
a collection C of m subsets of U
$S_1, S_2, \dots, S_m$

a weight function
$w : C \to \mathbb{R}^+$

Output: Find a minimum cost subcollection $C' \subseteq C$ such that

$$\bigcup_{S_i \in C'} S_i = U$$

Example



S_1 and S_4 covers U

H.W

i) How set cover problem related to VCP. ??

ii) Find ILP formulation of the set cover problem.

iii) Let each set S_i is bounded ie, each S_i contains at most t elements.

$$|S_i| \le t.$$

Hint:
VCP
$\equiv$
SCP
with
$t=2$

Then provide a t. approximation for the set cover problem.