Tutorial 10

1. Let $X_1, \ldots, X_n$ be independent geometric random variables, with X_i having parameter p_i for $i = 1, \ldots, n$. If all the p_i are distinct, then, for $k \leq n$,
$$P(S_n = k) = \sum_{i}^{n} p_i q_i^{k-1} \prod_{j \neq i} \frac{p_j}{p_j - p_i}$$

2. If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X given that $X + Y = n$.

3. Consider the multinomial distribution with joint probability mass function
$$P(X_i = n_i, i = 1, \ldots, k) = \frac{n!}{n_1! \ldots n_k!} p_1^{n_1} \ldots p_k^{n_k}$$
Where $n_i \geq 0$ and $\sum_{i=1}^{k} n_i = n$ Such a mass function results when n independent trials are performed, with each trial resulting in outcome i with probability p_i, $\sum_{i=1}^{k} p_i = 1$, The random variables X_i, $i = 1, \ldots, k$, represent, respectively, the number of trials that result in outcome i. Suppose we are given that n_j of the trials resulted in outcome j, for $j = r+1, \ldots, k$, where $\sum_{j=r+1}^{k} n_j = m \leq n$. Then, because each of the other $n - m$ trials must have resulted in one of the trials $1, \ldots, r$, it would seem that the conditional distribution of $X_1, ..., X_r$ is the multinomial distribution on $n - m$ trials with respective trial outcome probabilities
$$P(outcome\, i | outcome\, is\, not\, any\, of\, r + 1, \ldots, k) = \frac{p_i}{F_r}, i = 1, \ldots, r$$
where $F_r = \sum_{i=1}^{r} p_i$ is the probability that a trial results in one of the outcomes $1, \ldots, r$.

4. Suppose that there are N different types of coupons, and each time one obtains a coupon, it is equally likely to be any one of the N types. Find the expected number of coupons one need amass before obtaining a complete set of at least one of each type.

5. Suppose that n elements—call them $1, \ldots, n$—must be stored in a computer in the form of an ordered list. Each unit of time, a request will be made for one of these elements—i being requested, independently of

the past, with probability $P(i), i \geq 1$, $\sum_i P(i) = 1$. Assuming that these probabilities are known, what ordering minimizes the average position in the line of the element requested?

6. Suppose an urn contains $n + m$ balls, of which n are special and m are ordinary. These items are removed one at a time, with each new removal being equally likely to be any of the balls that remain in the urn. The random variable Y, equal to the number of balls that need be withdrawn until a total of r special balls have been removed. Find expectation and varience of Y.