Second-Order Systems of Differential Equations

i.) First-Order Autonomous System : $\boxed{\frac{dx}{dt} = f(x)}$.

ii.) Coupled Second-Order Autonomous System:

$$\boxed{\frac{dx}{dt} = f(x,y)} \text{ and } \boxed{\frac{dy}{dt} = g(x,y)}$$

(2-dimensional System) → Dynamical

An Economic Analogy (IBM): Coupled growth of revenue and human resources, given by

$$\boxed{\frac{dR}{dt} = \varphi(R,H)} \text{ and } \boxed{\frac{dH}{dt} = \eta(R,H)}.$$

R → Revenue
H → Human resources.

iii.) Coupled Third-Order Autonomous System:

$$\boxed{\frac{dx}{dt} = f(x,y,z)}, \quad \boxed{\frac{dy}{dt} = g(x,y,z)}, \quad \boxed{\frac{dz}{dt} = h(x,y,z)}.$$

iv.) Coupled N-Order Autonomous Dynamical System:

$$\boxed{\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, x_2, x_3, \ldots, x_N) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, x_3, \ldots, x_N) \\
\frac{dx_3}{dt} &= f_3(x_1, x_2, x_3, \ldots, x_N) \\
&\vdots \\
\frac{dx_N}{dt} &= f_N(x_1, x_2, x_3, \ldots, x_N)
\end{aligned}}$$

N-dimensional System of Coupled N first-order differential equations. All are autonomous.

General Second-Order Autonomous Differential Equation:

$$\boxed{A\left(x,\frac{dx}{dt}\right)\frac{d^2x}{dt^2} + B\left(x,\frac{dx}{dt}\right)\frac{dx}{dt} + C\left(x,\frac{dx}{dt}\right)x = 0}\,.$$

This can then be recast as (dividing by A),

$$\boxed{\frac{d^2x}{dt^2} + F\left(x,\frac{dx}{dt}\right)\frac{dx}{dt} + G\left(x,\frac{dx}{dt}\right)x = 0}\,.$$

Writing, $\dfrac{dx}{dt} = y = 0.x + 1.y$

we get $\dfrac{dy}{dt} = -F(x,y)\,y - G(x,y)\,y$ $\left(\because \text{ since } y = \dfrac{dx}{dt}\right)$

Similarly for a third-order differential equation,

$$\boxed{\frac{d^3x}{dt^3} + F\left(x,\frac{dx}{dt},\frac{d^2x}{dt^2}\right)\frac{d^2x}{dt^2} + G\left(x,\frac{dx}{dt},\frac{d^2x}{dt^2}\right)\frac{dx}{dt} + H\left(x,\frac{dx}{dt},\frac{d^2x}{dt^2}\right)x = 0}$$

we write $\boxed{\dot{x} = y}$, $\boxed{\ddot{x} = \dot{y} = z}$ and $\boxed{\dddot{x} = \dot{z}}$,

in all of which the "dot" implies a time derivative, $\left(\dfrac{d}{dt}\right)$.

Hence we get a coupled set of three equations.

$$\boxed{\begin{aligned}
\frac{dx}{dt} &= \dot{x} = 0.x + 1.y + 0.z \\
\frac{dy}{dt} &= \dot{y} = 0.x + 0.y + 1.z \\
\frac{dz}{dt} &= \dot{z} = -F(x,y,z)\,z - G(x,y,z)\,y - H(x,y,z)\,x
\end{aligned}}\,.$$

An N-order coupled system can always be crafted out of an N-order autonomous differential equation.

Coupled Linear Autonomous Second-Order System:

$$\boxed{\frac{dx}{dt} = Ax + By + C} \quad \text{and} \quad \boxed{\frac{dy}{dt} = Dx + Ey + F}.$$

The most general linear form. Anything else is nonlinear, such as x^3, $\cos y$, xy, e^x, $\ln y$, etc.

Consider a simple system $\boxed{\frac{dx}{dt} = Ax + By}$ and $\boxed{\frac{dy}{dt} = Cx + Dy}$. (without any free constant).

$$\Rightarrow \frac{d^2x}{dt^2} = A\frac{dx}{dt} + B\frac{dy}{dt} = A\frac{dx}{dt} + B\left(Cx + Dy\right)$$

Substituting $\boxed{By = \frac{dx}{dt} - Ax}$, we get,

$$\frac{d^2x}{dt^2} = A\frac{dx}{dt} + BCx + D\left(\frac{dx}{dt} - Ax\right).$$

$$\Rightarrow \frac{d^2x}{dt^2} = (A+D)\frac{dx}{dt} - (AD - BC)x.$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} - (A+D)\frac{dx}{dt} + (AD - BC)x = 0}$$

Also $\boxed{\frac{d^2y}{dt^2} - (A+D)\frac{dy}{dt} + (AD - BC)y = 0}$

Writing in a matrix form

$$\boxed{\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}} \Rightarrow \boxed{\frac{d\vec{X}}{dt} = \tilde{A}\vec{X}}$$

$\vec{X} = \{x_i\}$

We see that

$\boxed{A + D = T}$, the trace of matrix $\tilde{A}$, and $\boxed{AD - BC = \Delta}$, the determinant of matrix $\tilde{A}$.