Structure of an LPP

The mathematical structure of a general LPP is as follows.

$$\begin{cases} \text{optimise } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq = \geq) \ b_1 \\ \qquad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\leq = \geq) \ b_2 \\ \qquad \vdots \\ \qquad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\leq = \geq) \ b_m \\ \qquad x_1, x_2, \ldots, x_n \geq 0 \end{cases}$$

Here,

$x_1, x_2, \ldots, x_n \leftarrow$ are n decision variables

$c_1, c_2, \ldots, c_n \leftarrow$ are n cost coefficient

$b_1, b_2, \ldots, b_m \leftarrow$ are m constant or requirement parameter

$a_{ij} \leftarrow$ constraint coefficient / activity parameter $(i = 1, 2, \ldots, m \ , \ j = 1, 2, \ldots m)$

- The objective function Z is a linear function of the decision variables.
- The constraints are also linear functions of the decision variables.
- The decision variables are non-negative.

Note :- F.

Note:

For some real situations where the decision variables are unrestricted in sign.

$$x \geq 0 \quad x \leq 0 \quad x = 0$$

These variables are somehow converted to non-negative variables.

A more compact representation

$$\text{optimise } Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t. } \sum_{j=1}^{n} a_{ij} x_j \ (\leq = \geq) \ b_i \quad \forall\ i = 1, 2, \ldots, m$$

$$x_j \geq 0 \quad \forall\ j = 1, 2, \ldots, n$$

Feasible solution

A set of values to the decision variables $x_1, x_2, \dots, x_n$ that satisfies the set of constraints and the non-negativity restrictions is called a feasible solution.

Optimal solution:

A feasible solution which in addition optimises the objective function is called the optimal solution.

The value of the optimal solution is after replacing the variable by its value in the objective function and simplifying the expression.

Example minimise $Z = 4x_1 + x_2$

$$\text{s.t. } 3x_1 + x_2 \geq 10 \quad \text{— (1)}$$
$$x_1 + x_2 \geq 5 \quad \text{— (2)}$$
$$x_1 \geq 3 \quad \text{— (3)}$$
$$x_1, x_2 \geq 0$$

$x_1 = 5, x_2 = 4$ is a feasible solution.
Value :- $Z = 24$

$x_1 = 3, x_2 = 2$ it is also a feasible solution and this is the optimal solution $Z = 12 + 2 = 14$

Solving an LPP using graphical method

- If an LPP involves only 2 decision variables then the problem can be solved graphically
- Two major steps to solves a problem graphically.

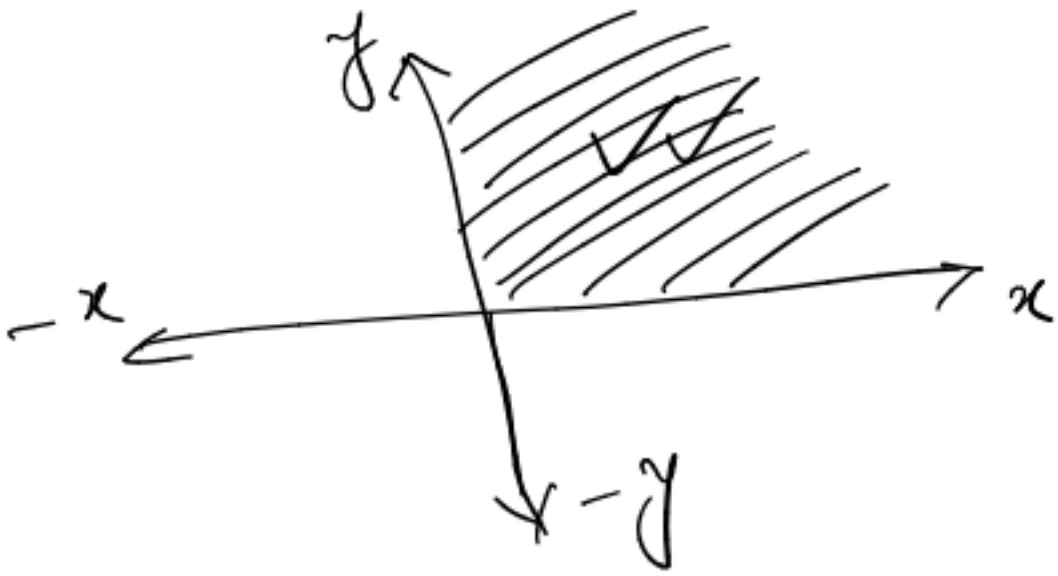
Step 1:- Finding a feasible solution region. (Set of all feasible solutions) we call this as a feasible region.

Step 2: Find an optimal solution from the feasible region generated in Step 1.

Step 1:

Here we follow the following steps.

Step 1.1: Since the two decision variables must satisfy the non-negativity restrictions, we only can consider the first quadrant of the xy-plane.



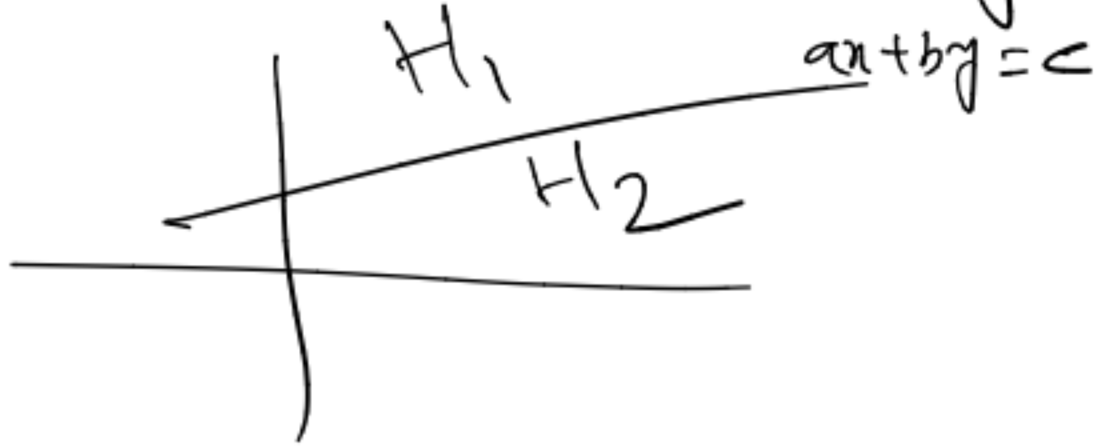
x, and y are the two decision variables.

Step 1.2: Each constraint

$$ax+by\ (\le = \ge)\ c \quad \text{is treated} \quad \text{—} (1)$$

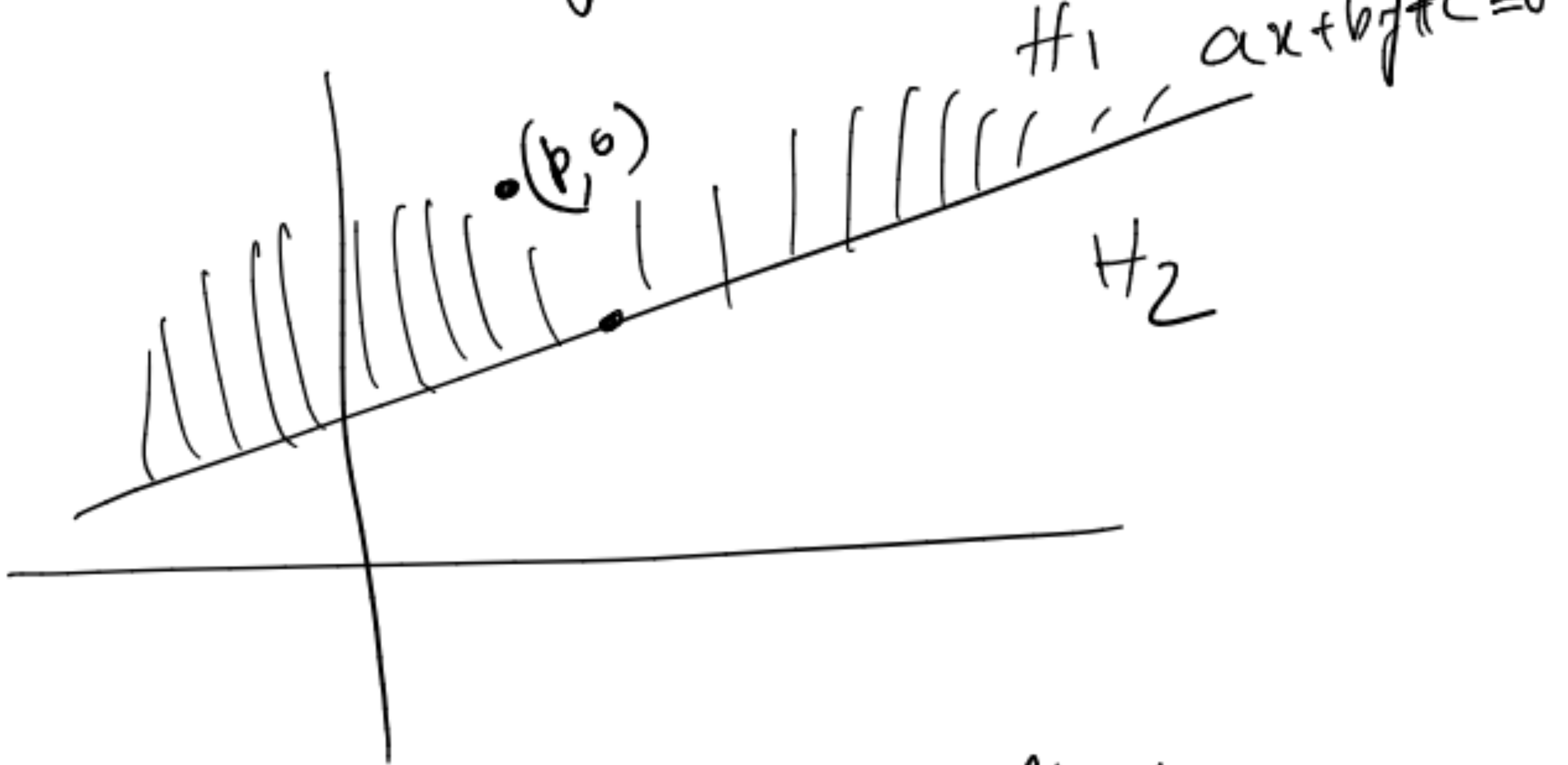
as a equation $ax+by=c$ — (2)

and draw a line in the xy-plane



Note that each line ② divides the first quadrant into two half-planes H_1 and H_2

Let $(p,0)$ be a point in H_1
If this point $(p,0)$ satisfies the inequation $ax+by\ (\leq = \geq)$ then consider the region H_1 and Shaded the region H_1



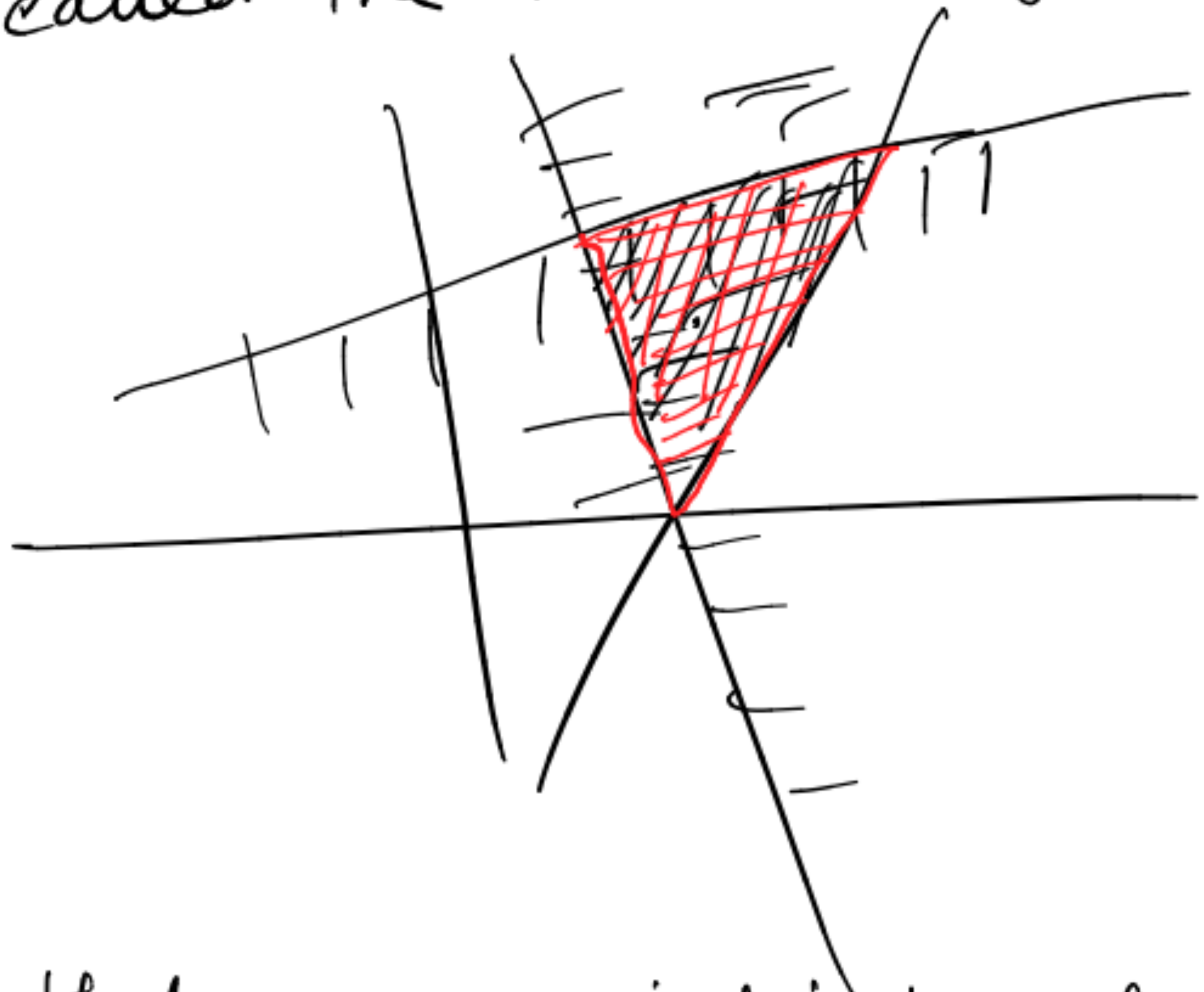
Else if $(p,0)$ does not satisfy the inequality, then consider the region H_2 and shade the region H_2.

$=$ can be converted into a $\leq$ and a $\geq$ $\quad ax+by=c \equiv \begin{matrix} ax+by\leq c \\ ax+by\geq c \end{matrix}$

Step 1.3

For each constraint we obtained a shaded region in Step 1.2

The intersection of all these regions is called the feasible region.



Note that: any point in the feasible region satisfies all constraints and the non-negativity restriction and thus the point corresponds a feasible solution.

Example

Maximise $z = 150x + 100y$

s.t. $8x + 5y \leq 60$ — ①

$4x + 5y \leq 40$ — ②

$x, y \geq 0$

Step 1: Finding the feasible region

Step 1.1: we consider only the first quadrant as $x, y \geq 0$

Step 1.2:

constraint 1: $8x + 5y \leq 60$

we consider its corresponding equation

$$8x + 5y = 60$$

Now we draw the straight line of the equation.

To draw the line we consider two points such that they satisfies the equation and plot them on the first quadrant of the xy-plane. The connect them and possible extend them in both directions.

we have the equation

$$8x+5y=60$$

The two points are,

when $x=0$, $y=\frac{60}{5}=12$

when $y=0$, $x=\frac{60}{8}=7.5$

The two points are,

$(0,12)$ and $(7.5,0)$

we perform the similar procedure for all the constraints.

constraint 2: $4x+5y\leq 40$

The equation of constraint 1 is

$$4x+5y=40$$

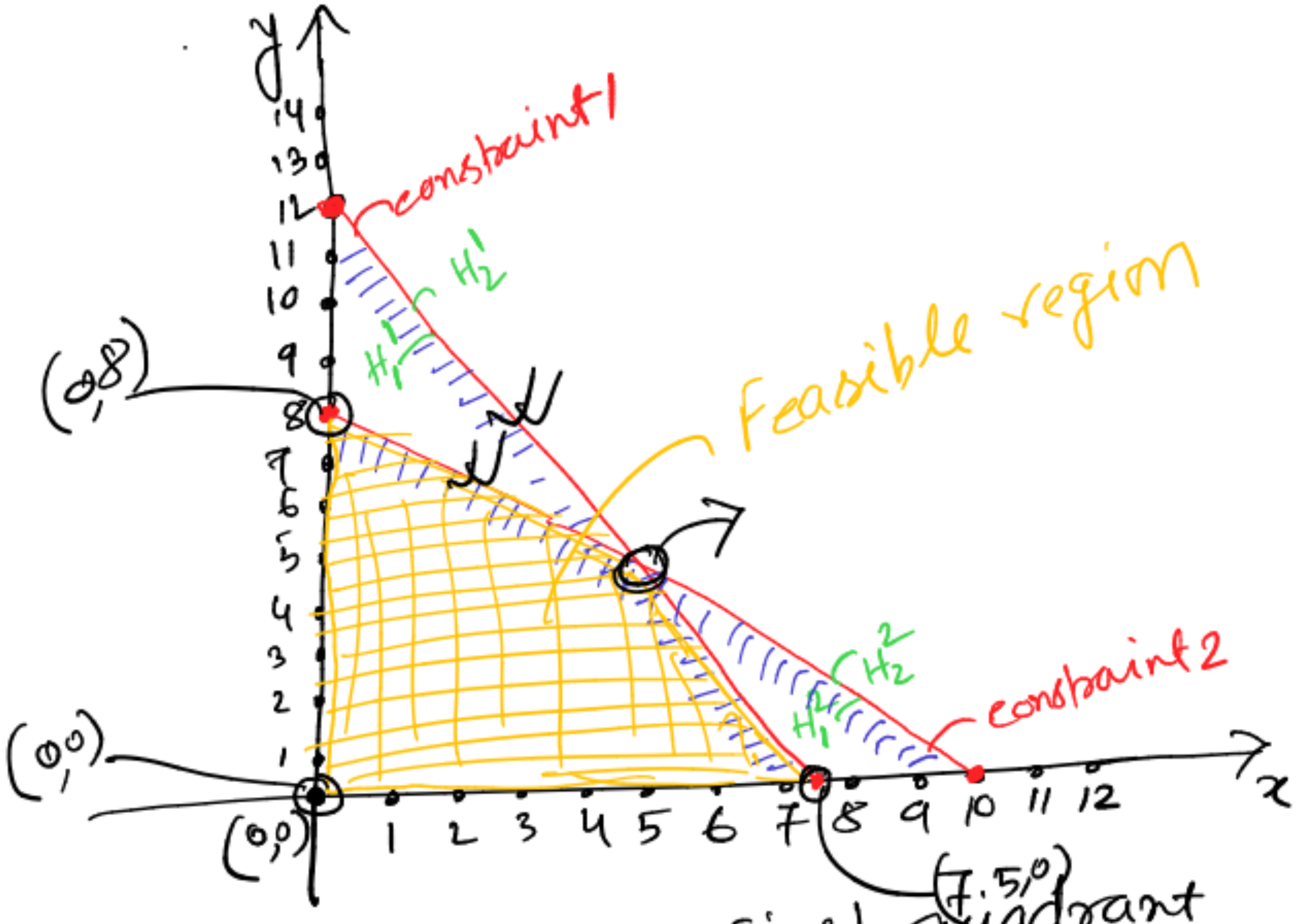
The two points are,

when $x=0$, $y=\frac{40}{5}=8$

when $y=0$, $x=\frac{40}{4}=10$

The two points are,

$(0,8)$ and $(10,0)$



Let line ① divides the first quadrant into two regions H_1^1 and H_2^1

we choose a point say (0,0) in H_1^1

Now, $8x+5y \leq 60$

$\Rightarrow 0+0 \leq 60$

So (0,0) satisfies constraint 1

Therefore we consider the region H_1^1 and shade it.

Similarly (0,0) satisfies constraint 2

Hence H_1^2 is shaded.

Step 1.3:-

The feasible region is the intersection of the two regions H_1^1 and H_1^2.

Step 2: Finding optimal solution

We discuss two methods to find an optimal solution.

Method 1: corner/extreme point method

Method 2: Iso profit/Iso cost method

Method 1 :- corner/extreme point method

Corner/extreme point:
The vertices of a feasible region

Lemma: If there exists an optimum solution to an LPP then the solution attains at one of its corner/extreme points of the feasible region.

Step 2.1 : compute the co-ordinates of each vertex of the feasible region.
These coordinates can be obtained from the graph or by solving the equations of the lines.

Step 2.2 :- A each extreme point compute the value of the objective function.

The extreme point that attains the optimum value of the objective function is the optimum value of the LPP.

The co-ordinates of this extreme point is the optimum solution.

$$\begin{array}{r} 75 \\ 15 \\ \hline 1125 \end{array} \quad 7$$

__Example__

Extreme points	value
$(0,0)$	$150 \cdot 0 + 100 \cdot 0 = 0$
$(7.5, 0)$	$150 \times 7.5 + 0 = 1125$
$(0, 8)$	$0 + 100 \times 8 = 800$
$(5, 4)$	$150 \times 5 + 100 \times 4 = 1150$

$8x + 5y = \cancel{80}\, 60$

$4x + 5y = 40$

$8x + 10y = 80$

$\Rightarrow 8x = \underline{\underline{80 - 10y}}$

$80 - 10y + 5y = 60$

$\Rightarrow -5y = -20$

$\Rightarrow y = 4$

$4x + 5y = 40$

$\Rightarrow 4x = 20 \Rightarrow x = 5$

Since it is a maximization Problem the maximum value attains at $(5, 4)$

So the value of the optimum solution is $Z = 1150$

and the optimum solution is

$$x = 5, \ y = 4$$