Dhirubhai Ambani Institute of Information and Communication Technology (DA-IICT)
Mid-semester Examination
CT314 (Statistical Communication Theory)
Date of Examination: March 24, 2017
Duration: 2 Hours
Maximum Marks: 25

Instructions:

1. Attempt all questions.
2. Use of scientific non programmable calculator is permitted.
3. Figures in brackets indicate full marks.
4. All the acronyms carry their usual meaning.

25

Q1: Let X and Y be two random variables with Y=cX+d, where c, d are constants. Find the correlation coefficient between X and Y. (2)

Q2: Consider a vector of random variables $\underline{X} = [X_1, X_2]^T$. These random variables have unit variance and are uncorrelated. Now the transformed vector $\underline{Y} = [Y_1, Y_2]^T$ is obtained as $\underline{Y} = A\underline{X}$, where A is the transformation matrix. Find the matrix A so that $\underline{Y}$ has the covariance matrix $C_{\underline{Y}} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ (8)

Q3: Let X_1, X_2, and X_3 be the zero mean random variables having same variance. We wish to predict X_3 as $aX_1 + bX_2$, a and b are constants. (a) Find the MMSE estimate of X_3. Now assuming that covariance does not depend on the specific index of random variables, but rather on the distance between them [meaning $COV(X_1, X_2) = COV(X_2, X_3)$], express a and b in terms of correlation coefficients. (7)

Q4: Consider jointly Gaussian random variables X_1 and X_2 with mean vector $m_{\underline{X}}$ and covariance matrix $C_{\underline{X}}$. Now define $\underline{Y} = A\underline{X}$ to get Y_1 and Y_2, where A is a invertible matrix. (a) Show that $\underline{Y}$ is jointly Gaussian (4) (b) Write the mean vector and covariance (2) matrix for the vector $\underline{Y}$ (c) Now choose A to make $\underline{Y}$ as (1) statistically independent (d) Reason out why A has to be invertible. (8) (1)

"BEST WISHES"

Second Insem CT314 March 24, 2017

Answers.

1. $Y = cX + d$.

$\mathrm{cov}(X, Y) = E[(X - m_X)(Y - m_Y)]$

$Y = cX + d$ So $m_Y = c E(X) + d = c m_X + d$

$Y - m_Y = cX + \cancel{d} - c m_X - \cancel{d} = c(X - m_X)$

So $\mathrm{cov}(X, Y) = E[c(X - m_X)^2] = c\sigma_X^2$

$\sigma_Y^2 = E(Y - m_Y)^2 = c^2 \sigma_X^2$

So $\rho_{XY} = \dfrac{c\sigma_X^2}{\sigma_X c\sigma_X} = 1$

Q2. $C_{\underline{Y}} = \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix}$ Eigen vectors $= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

normalized – because columns are orthonormal

$\Sigma^{1/2} = \begin{bmatrix} (0.5)^{1/2} & 0 \\ 0 & (1.5)^{1/2} \end{bmatrix}$

$\therefore U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$A = U\Sigma^{1/2} = \begin{bmatrix} \left(\frac{.5}{2}\right)^{1/2} & \left(\frac{1.5}{2}\right)^{1/2} \\ -\left(\frac{.5}{2}\right)^{1/2} & \left(\frac{1.5}{2}\right)^{1/2} \end{bmatrix}$

Verify $C_{\underline{Y}} = AA^T$?

Q3. $\hat{X}_3 = a^* X_1 + b^* X_2$ $\underset{a, b}{\text{Minimize}}\ E(X_3 - \hat{X}_3)^2$

Diff. wrt a and b and equate $= 0$.

we set

$$\begin{bmatrix} E(X_1^2) & E(X_1X_2) \\ E(X_1X_2) & E(X_2^2) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} E(X_1X_3) \\ E(X_2X_3) \end{bmatrix}$$

So solving these two equations

$$a^* = \frac{\sigma_{X_2}^2 \, Cov(X_1,X_3) - Cov(X_1,X_2)\, Cov(X_2,X_3)}{\sigma_{X_1}^2 \sigma_{X_2}^2 - (Cov(X_1,X_2))^2}$$

$$b^* = \frac{\sigma_{X_1}^2 \, Cov(X_2,X_3) - Cov(X_1X_2)\, Cov(X_1X_3)}{\sigma_{X_1}^2 \sigma_{X_2}^2 - (Cov(X_1,X_2))^2}$$

Now Given: $Cov(X_1X_2) = Cov(X_2X_3)$

$\rho_1\sigma^2 = \rho_1\sigma^2$

$Cov(X_2X_3) = \rho_2\sigma^2$

$\cancel{\rho_{X_2X_3} = \rho_1\sigma^2}$

So $\cancel{\rho_{X_1X_2} = \frac{Cov(X_1,X_2)}{\sigma_{X_1}\sigma_{X_2}} = \rho_1}$

$\cancel{\rho = \rho_1\sigma^2}$

$\cancel{Cov(X_1X_2) = \rho_1\sigma_{X_1}\sigma_{X_2} = \rho_1\sigma^2}$

$\cancel{Cov(X_2X_3) = \rho_1\sigma_{X_2X_3} = \rho_1\sigma^2}$

$\cancel{Cov(X_1X_3) = \rho_2\sigma_{X_1}\sigma_{X_3}}$

$$\therefore \cancel{a^*} = \frac{\sigma_{X_2}^2\, \rho_2\, \sigma_{X_1}\sigma_{X_3} - \rho_1\sigma_{X_1}\sigma_{X_2}\, \rho_1\sigma_{X_2}\sigma_{X_3}}{\sigma_{X_1}^2\sigma_{X_2}^2 - \rho_1^2\sigma_{X_1}^2\sigma_{X_2}^2}$$

$$= \frac{\sigma_{X_2}^2\sigma_{X_1}\sigma_{X_3}(\rho_2 - \rho_1^2)}{\sigma_{X_1}^2\sigma_{X_2}^2(1-\rho_1^2)} = \frac{\sigma_{X_3}(\rho_2 - \rho_1^2)}{\sigma_{X_1}(1-\rho_1^2)}$$

verify

$\cancel{b^* = ??}$

$$\text{So } a^* = \frac{\sigma^2\rho_2\sigma^2 - \sigma^4\rho_1^2}{\sigma^4 - \sigma^4\rho_1^2}$$

$$a^* = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2}$$

$$\& \; b^* = \frac{\rho_1(1-\rho_2)}{1-\rho_1^2}.$$

Q4

$$\underline{Y} = A\underline{X} \qquad \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$f_{Y_1Y_2}(y_1 y_2) = \frac{f_{X_1X_2}(x_1, x_2)}{\left| J\left(\frac{y_1, y_2}{x_1, x_2}\right)\right|}$$

$$|J| = \left\|\begin{matrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{matrix}\right\| = \left\|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right\| = \cancel{|A|} = \cancel{|\det(A)|} = \cancel{|\det A|} = \frac{|A|}{|\det(A)|}$$

$$\cancel{f_{X_1X_2}(x_1,x_2) = \frac{e^{-\frac{1}{2}(\underline{x} - \underline{m_x})^T C_x^{-1}(\underline{x} - \underline{m_x})}}{2\ 2\pi}}$$

$$f_{XX_2}(x_1,x_2) = \frac{1}{(2\pi)^{2/2}\, |C_x|^{1/2}}\, e^{-\frac{1}{2}(\underline{x} - \underline{m_x})^T C_{\underline{x}}^{-1} (\underline{x} - \underline{m_x})}$$

$$f_{Y_1Y_2}(y_1,y_2) = \frac{e^{-\frac{1}{2}(A^{-1}\underline{y} - \underline{m_x})^T C_x^{-1} (A^{-1}\underline{y} - \underline{m_x})}}{(2\pi)\, |\det(A)|\, |C_{\underline{x}}|^{1/2}}$$

$A\underline{X} = \underline{Y}$

Now $(A^{-1}\underline{y} - \underline{m_x}) = A^{-1}(\underline{y} - A\underline{m_x})$

$\therefore (A^{-1}y - \underline{m_x})^T = (\underline{y} - A\underline{m_x})^T A^{-1T}$

(b) mean vector of $\underline{Y}$

$\underline{m_y} = A\underline{m_x}$

(a)

$$\therefore f_{Y_1Y_2}(y_1,y_2) = \frac{e^{-\frac{1}{2}(\underline{y} - \underline{m_y})^T A^{-1T} C_{\underline{x}}^{-1} A^{-1} (\underline{y} - \underline{m_y})}}{(2\pi)\, |C_{\underline{y}}|^{1/2}}$$

$(AC_xA^T)^{-1}$

(b) covariance matrix of $\underline{Y}$

$|C_{\underline{y}}| = |A||C_{\underline{x}}||A^T| = |A|^2 |C_{\underline{x}}|$

$|C_y|^{1/2} = |A|\, C_{\underline{x}}^{1/2}$

$|\det \text{ of } C_{\underline{y}}|$

$$= \frac{e^{-\frac{1}{2}(\underline{y} - \underline{m_y})^T C_{\underline{y}}^{-1} (\underline{y} - \underline{m_y})}}{(2\pi)\, |C_{\underline{y}}|^{1/2}}$$

$\Rightarrow$ represents joint pdf of Y_1 and Y_2 which is jointly Gaussian pdf for Y_1 and Y_2 to be independent they have to be uncorrelated for Gaussian case.

So $\underline{Y} = A\underline{x}$ if A is chosen as
using $\underline{A = U^T}$ i.e., eigenvector matrix of $C_{\underline{X}}$ (Covariance matrix of $\underline{X}$)

then Y_1 and Y_2 are uncorrelated, hence independent

(d) if A is not invertible determinant of Jacobian matrix becomes 0, so joint pdf cannot be determined. (of Y_1 and Y_2)