Dhirubhai Ambani Institute of Information and Communication Technology (DA-IICT)
Mid Semester Examination
CT314 (Statistical Communication Theory)
Date of Examination: March 18, 2011
Duration: 2 Hours;
Maximum Marks; 20

Instructions:

1. Attempt all questions.
2. Use of scientific non programmable calculator is permitted.
3. Figures in brackets indicate full marks.
4. All the acronyms carry their usual meaning.
5. Unless specified assume real random process

Q1; Consider the experiment of throwing a fair die. The sample space consists of six sample points $s_1, s_2, s_3, s_4, s_5, s_6$. Let the sample functions be given by $x_i(t) = t + (i-1)$, for $s = s_i$, $i = 1,2,....6$. (a) Write the expression for first order probability density function at t=2, (a) Find the autocorrelation value $R_X(2,4)$ (b) Is the process WSS? Give proper reasoning for your answer. (4 Marks)

Q2: Show that power spectral density (PSD) of a WSS process is non negative. Hint; Consider a WSS random process X(t) that has negative PSD over a small frequency band and check the output average power when X(t) is applied as input to a LTI system. (2 marks)

Q3: Check if the following functions can be valid autocorrelation functions:

(a) $R_X(\tau) = \exp(-|\tau|), \quad -\infty < \tau < \infty$

(b) $R_X(\tau) = \cos(\tau), \quad -\frac{\pi}{2} \le \tau \le \frac{\pi}{2}$ and 0 otherwise.

(5 Marks)

Q4: Consider the modulated random process $Y(t) = X(t)\cos(w_c t + \Theta)$ where $X(t)$ is WSS process with known $R_x(\tau)$ and $) S_x(f$. Θ is uniformly distributed random variable in the range $(0,-2\pi)$. $X(t)$ and Θ are independent. Find the autocorrelation function and PSD of $Y(t)$. (5 marks)

Q5: Show that for jointly wide sense stationary processes $X(t)$ and $Y(t)$, $R_{XY}(\tau) \le [R_X(0)R_Y(0)]^{\frac{1}{2}}$ (4 marks)

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Q.1

We have six sample f's each with probability of occurance of 1/6.

$x_1(t) = t$

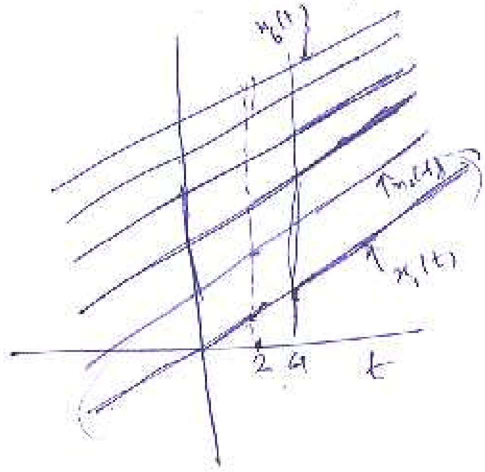
$x_2(t) = t+1$

$x_3(t) = t+2$

$x_4(t) = t+3$

$x_5(t) = t+4$

$x_6(t) = t+5$



$p[x_i(t)] = \frac{1}{6}$

(a) At $t = 2$, we have r.v. $X(2)$, taking values 2, 3, 4, 5, 6, 7 each with probability of 1/6

So the pdf is

$$p_{X(2)}(x) = \frac{1}{6}\left[\delta(x-2) + \delta(x-3) + \delta(x-4) + \delta(x-5) + \delta(x-6) + \delta(x-7)\right]$$

(b). To find $R_X(2,4)$, we have to consider random variables at $t=2$ and $t=4$

At $t = 2$, we have values 2, 3, 4, 5, 6, 7

" $t = 4$, " " " 4, 5, 6, 7, 8, 9

$$\text{So } R_X(2,4) = E[\underline{X(2)}\,\underline{X(4)}] = E[X(t_1)X(t_2)]$$

$$= \frac{8+15+24+35+48+63}{6} = \frac{193}{6}$$

The process is not WSS, as the mean and autocorrelation are dependent on time.

For example $E[X(2)] = \frac{2+3+4+5+6+7}{6}$

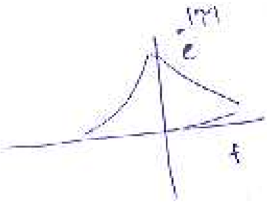
$E[X(4)] = \frac{4+5+6+7+8+9}{6}$, $E[X(2)] \neq [E X(4)]$

No need to check for autocorrelation.

Q3

(a) $R_x(\tau) \longleftrightarrow S_x(f)$

$$S_x(f) = \int_{-\infty}^{\infty} e^{-|\tau|} e^{-j\omega\tau}\, d\tau$$

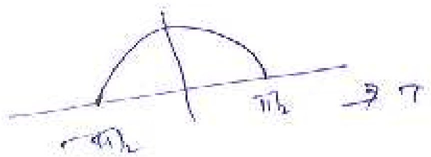


$$= \int_{-\infty}^{0} e^{\tau} e^{-j\omega\tau}\, d\tau + \int_{0}^{\infty} e^{-\tau} e^{-j\omega\tau}\, d\tau$$

$$= \frac{2}{2+\omega^2} = \frac{2}{2+(2\pi f)^2}$$ ie $S_x(f) > 0$ hence this is a valid autocorrelation fn.

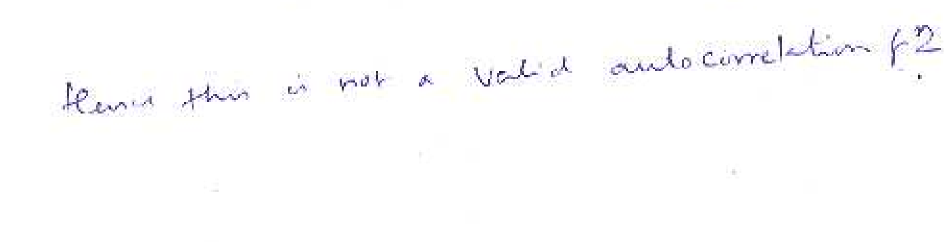
You may also decide on $R_x(t)$ being even.

(b) $R_x(\tau) = \cos\tau \qquad -\pi/2 \le \tau \le \pi/2$



~~That~~ Although $R_x(\tau)$ is even, psd is non negative fn which can be obtained by taking the F.T of $\cos\omega\tau$ ($-\infty < \tau < \infty$) multiplied by rectangle of width π ($\frac{-\pi}{2}$ to $\frac{\pi}{2}$) $1 = \omega = 2\pi f$

This leads to $S_x(f)$



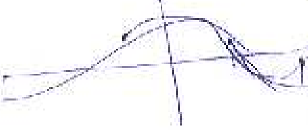
hatched regions show $S_x(f)$ -ve.

Hence this is not a valid autocorrelation fn.

Q.2. To prove $S_x(f) \geq 0$.

we know that $E[Y^2(t)] = \int_{-\infty}^{\infty} S_x(f)|H(f)|^2 df$.

To prove for non negativity, let us consider $S_x(f)$ to be $-ve$ for certain frequency interval



i.e. $|H(f)| = \begin{cases} 1 & f_1 \leq |f| \leq f_1 + \Delta f \\ 0 & \text{else} \end{cases}$

then $E[Y^2(t)] = \int_{-\infty}^{\infty} S_x(f)|H(f)|^2 df$

becomes $-ve$. But expectation of a squared quantity cannot be $-ve$. $\therefore S_y(f)$ should always be $+ve$ or 0

Q4. $Y(t) = X(t)\cos(w_c t + \Theta)$

$R_y(\tau) = E[Y(t)\,Y(t+\tau)]$

$= E\left[X(t)\cos(w_c t + \Theta)\, X(t+\tau)\cos(w_c(t+\tau) + \Theta) \right]$

$= E\left[X(t)\,X(t+\tau) \left\{ \frac{\cos(w_c\tau) + \cos(2w_c t + \tau + 2\Theta)}{2} \right\} \right]$

$X(t)$ and Θ are independent, so

$R_y(\tau) = E[X(t)\,X(t+\tau)] \cdot E\left[\frac{\cos(w_c\tau) + \cos(2w_c t + \tau + 2\Theta)}{2} \right]$

$= R_x(\tau)\,\frac{1}{2}\cos(w_c\tau)$

$\Downarrow$

$S_x(f)$

And the ~~autocorrelation~~ PSD is FT of $R_y(\tau)$

$\therefore S_y(f) = \int_{-\infty}^{\infty} R_y(\tau)\, e^{-j2\pi f\tau}\, d\tau$

$= \frac{1}{2}\left[\frac{S_x(f - f_c) + S_x(f + f_c)}{2} \right]$

$= \frac{1}{4}\left[S_x(f - f_c) + S_x(f + f_c) \right]$

where $R_x(\tau) \leftrightarrow S_x(f)$

$\cos w_c\tau \leftrightarrow \frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c)$

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Consider

$$E\left[\left(X(t) \pm Y(t+\tau)\right)^2\right] \geq 0$$

Since expectation of squared quantity is +ve.

let us take $-$ sign first.

$$E\left[X^2(t) + Y^2(t+\tau) - 2X(t)Y(t+\tau)\right] \geq 0$$

this Gives

$$\left(R_x(0) + R_y(0) - 2R_{xy}(\tau)\right) \geq 0$$

$X(t)$ and $Y(t)$ are jointly WSS.

$$\text{or} \quad \underbrace{R_x(0) + R_y(0) - 2R_x^{\frac{1}{2}}(0)R_y^{\frac{1}{2}}(0) + 2R_x^{\frac{1}{2}}(0)R_y^{\frac{1}{2}}(0)}_{} - 2R_{xy}(\tau) \geq 0 \quad -① $$

$$\left(R_x^{\frac{1}{2}}(0) - R_y^{\frac{1}{2}}(0)\right)^2$$

$$\therefore \left(\sqrt{R_x(0)} - \sqrt{R_y(0)}\right)^2 + 2R_x^{\frac{1}{2}}(0)R_y^{\frac{1}{2}}(0) \geq 2R_{xy}(\tau)$$

$\Downarrow$

This is always ≥ 0.

$$\therefore R_{xy}(\tau) \leq R_x^{\frac{1}{2}}(0)R_y^{\frac{1}{2}}(0).$$

Check ~~for say~~ ① & ② without considering first term.
for say $\sqrt{R_x(0)R_y(0)} = 4$
then $R_{xy}(\tau)$ can take only $-4, -3, \ldots 4$ values
$|R_{xy}(\tau)| \leq \sqrt{R_x(0)R_y(0)}$

If we consider +ve sign, then $\left(\sqrt{R_x(0)} - \sqrt{R_y(0)}\right)^2 + 2\sqrt{R_x(0)}\sqrt{R_y(0)} + 2R_{xy}(\tau) \geq 0 \quad -②$

Combi Looking at ① and ② we conclude that

$$|R_{xy}(\tau)| \leq \sqrt{R_x(0)R_y(0)}$$