1. Which of the following can be an electrostatic field?
   (a) $x\hat{\imath}$, **(b)** $y\hat{\imath}$, **(c)** $(1/r)\hat{\theta}$, **(d)** $(1/s)\hat{\phi}$

   soln:
   Electrostatic field is curlless. Only the fields (a) and (c) satisfy this condition. Hence they can be electric fields.

2. A sphere of radius a is maintained at a uniform potential V_0. Find the potential both, inside and outside the sphere.

   soln:
   The problem has spherical symmetry. The potential V is independent of θ and ϕ.

$$\therefore \quad \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)$$

Since there is no charge in the region $r > a$ and $r < a$, we solve

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

in both the region. This gives

$$V = -\frac{c}{r} + d$$

where c and d are constants. The solution will have the same form, both inside and outside the sphere but the constants are different and have to be determined from the boundary conditions. So we have

$$V_{in} = -\frac{c_1}{r} + d_1 \qquad \text{and} \qquad V_{out} = -\frac{c_2}{r} + d_2$$

Inside the sphere $\vec{\nabla}^2 V_{in} = 4\pi c_1 \delta^3(\vec{r})$. This corresponds to a point charge of magnitude $-4\pi\epsilon_0 c_1$ placed at the center of the sphere. Since there is no charge inside the sphere we conclude that $c_1 = 0$.
As $r \to \infty, V_{out} \to d_2$. If we demand the potential far away from the sphere to be 0 then $d_2 = 0$. So now we have

$$V_{out} = -\frac{c_2}{r} \quad \text{and} \quad V_{in} = d_1$$

At $r = a \quad V_{in} = V_{out} = V_0 \quad \Longrightarrow \quad c_2 = -aV_0$ and $d_1 = V_0$.

$$\therefore \quad V_{in} = V_0 \quad \text{and} \quad V_{out} = \frac{a}{r}V_0$$

3. A infinite plane has a uniform charge density σ over it. Find the electric field on either side of the plane by solving the Laplace's equation and using the appropriate boundary condition. Consider the potential of the plane to be 0.

soln:

Let the plane be along the yz plane which is defined by the equation $x = 0$. So the space is partitioned into two regions. Let region 1 be the region $x < 0$ and region 2 be $x > 0$. We denote the potential in region 1 by $\Phi_1(x, y, z)$ and potential of region 2 by $\Phi_2(x, y, z)$. Since both the regions are chargeless Φ_1 and Φ_2 satisfy the Laplace's equation $\nabla^2\Phi_1 = 0$ and $\nabla^2\Phi_2 = 0$.

Due to symmetry the potentials cannot change along y and z. So we have

$$\frac{\partial^2\Phi_1}{\partial x^2} = 0 \qquad \text{and} \qquad \frac{\partial^2\Phi_2}{\partial x^2} = 0$$

The general solution of the above equations are

$$\Phi_1 = c_1 x + d_1 \qquad \text{and} \qquad \Phi_2 = c_2 x + d_2$$

The two potentials must match at the boundary $x = 0$. Since the potential on this plane is taken to be 0 we get $d_1 = d_2 = 0$.

The electric field in the two regions is

$$\vec{E}_1 = -\vec{\nabla}\Phi_1 = -c_1\hat{i} \quad \text{and} \quad \vec{E}_2 = -\vec{\nabla}\Phi_2 = -c_2\hat{i}$$

Applying the boundary condition on the electric fields at the interface of the two regions we get

$$\begin{aligned} E_2 - E_1 &= \frac{\sigma}{\epsilon_0} \\ \therefore \quad -c_2 + c_1 &= \frac{\sigma}{\epsilon_0} \\ \therefore \quad c_1 &= \frac{\sigma}{\epsilon_0} + c_2 \end{aligned}$$

$$\therefore \quad \Phi_1 = (c_2 + \frac{\sigma}{\epsilon_0})x \qquad \text{and} \qquad \Phi_2 = c_2 x$$

which gives

$$E_1 = -(c_2 + \frac{\sigma}{\epsilon_0}) \qquad \text{and} \qquad E_2 = -c_2$$

That is all mathematics can give us. We now have to use physical argument. The electric field on either side must be equal and opposite.

So we have $E_2 = -E_1$. This gives $c_2 = -\frac{\sigma}{2\epsilon_0}$. This gives

$$\vec{E}_2 = \frac{\sigma}{2\epsilon_0}\hat{i} \quad \text{and} \quad \vec{E}_1 = -\frac{\sigma}{2\epsilon_0}\hat{i}$$

4. The screened Coulomb potential

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}$$

commonly occurs in a medium. Calculate the corresponding electric field and charge density.

soln

$$\begin{aligned}\Phi &= \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r} \\ \therefore \vec{E} = \vec{\nabla}\Phi &= \frac{q}{4\pi\epsilon_0}e^{-r/\lambda}\left[\frac{1}{r^2}+\frac{1}{r\lambda}\right]\hat{r}\end{aligned}$$

By Gauss' law

$$\begin{aligned}\rho = \epsilon_0\vec{\nabla}\cdot\vec{E} &= \frac{q}{4\pi}\vec{\nabla}\cdot e^{-r/\lambda}\left[\frac{1}{r^2}+\frac{1}{r\lambda}\right]\hat{r} \\ &= \frac{q}{4\pi}e^{-r/\lambda}\vec{\nabla}\cdot\left(\frac{1}{r^2}+\frac{1}{r\lambda}\right)\hat{r}+\left(\frac{1}{r^2}+\frac{1}{r\lambda}\right)\hat{r}\cdot\vec{\nabla}\left(e^{-r/\lambda}\right) \\ &= \frac{q}{4\pi}e^{-r/\lambda}\left(4\pi\delta^3(\vec{r})+\frac{1}{\lambda r^2}\right)+\left(\frac{1}{r^2}+\frac{1}{r\lambda}\right)\left(-\frac{1}{\lambda}\right)e^{-r/\lambda} \\ &= \frac{q}{4\pi}e^{-r/\lambda}\left(4\pi\delta^3(\vec{r})+\frac{1}{\lambda^2 r}\right)\end{aligned}$$

Interpretation: Inside a medium when we place a point charge a cloud of negative charge accumulate around it. This screens the positive point charge. The charge distribution $\frac{q}{4\pi}\frac{e^{-r/\lambda}}{\lambda^2 r}$ gives the charge density of this negative charge. Evidently the density is very high as $r \to 0$. The total screening cloud integrated from $r = 0$ to $r = \infty$ is $-q$. The constant λ gives the distance scale upto which a reasonable amount of charge q can be felt. For vacuum $\lambda \to \infty$.

5. Given a region of space in which the electric field is everywhere directed parallel to the x axis. Prove that the electric field is independent of the y and the z co-ordinates.

soln
Let $\vec{E} = \hat{\imath}E_x$. Since $\vec{\nabla}\times\vec{E} = 0$ we have

$$\hat{\imath}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+\hat{\jmath}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+\hat{k}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)=0$$

$\therefore\ \hat{\jmath}\left(\frac{\partial E_x}{\partial z}\right)+\hat{k}\left(-\frac{\partial E_x}{\partial y}\right)=0.$
$\therefore\ \frac{\partial E_x}{\partial z}=\frac{\partial E_x}{\partial y}=0.$
So $\vec{E}$ is independent of x and y co-ordinates.