

1. A conducting sphere of radius R has an amount of charge Q over it. This sphere is placed in an otherwise uniform electric field \vec{E}_0 . The potential of the sphere is found to be V_0 . Find the potential in the region outside the sphere.

soln:

The given electrostatic configuration has an azimuthal symmetry if we consider the uniform electric field along \hat{z} . The general variable separated form of the potential is given as

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

Far away from the sphere the influence of the sphere gets negligible. So the electric field is $E_0 \hat{z}$. This corresponds to an electric potential function $-E_0 z + c = -E_0 r \cos \theta + c$ where c is a constant that can always be added to the potential function.

Now as $r \rightarrow \infty$, $V(r, \theta) \rightarrow \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$. So we have

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) = -E_0 r \cos \theta + c$$

This gives $A_0 = c$, $A_1 = -E_0$ and $A_2 = A_3 = \dots = 0$. Let us rewrite the potential using these values of the constants A_l .

$$\begin{aligned} V(r, \theta) &= -E_0 r \cos \theta + c + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) \\ &= c + \frac{B_0}{r} + \left(-E_0 r + \frac{B_1}{r^2} \right) \cos \theta + \sum_{l=2}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) \end{aligned}$$

The sphere is an equipotential surface. So $V(R, \theta)$ must be independent of θ .

$$\begin{aligned} V(R, \theta) &= -E_0 R \cos \theta + c + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta) \\ &= c + \frac{B_0}{R} + \left(-E_0 R + \frac{B_1}{R^2} \right) \cos \theta + \sum_{l=2}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta) \end{aligned}$$

If this has to be equal to V_0 which is independent of θ then we must have

$$c + \frac{B_0}{R} = V_0, \quad B_1 = E_0 R^3 \text{ and } B_2 = B_3 = \dots = 0$$

Let us rewrite the potential again

$$V(r, \theta) = V_0 - \frac{B_0}{R} + \frac{B_0}{r} + \left(-E_0 r + \frac{E_0 R^3}{r^2} \right) \cos \theta$$

The total charge on the sphere is Q . Q/ϵ_0 will be equal to the total flux of the electric field over the surface of the sphere. Over the surface of the sphere the electric field is normal, which is the radial direction \hat{r} . The radial component of the electric field can be derived from $V(r, \theta)$. This is $-\frac{\partial V}{\partial r}$ given by

$$E_r(r, \theta) = \frac{B_0}{r^2} + \left(E_0 + \frac{2E_0 R^3}{r^3} \right) \cos \theta$$

At $r = R$ we get

$$E_r(R, \theta) = \frac{B_0}{R^2} + 3E_0 \cos \theta$$

This is the normal component of the electric field over the surface of the sphere. Integrating this over the surface of the sphere will give the total charge on the sphere. The constant term $\frac{B_0}{R^2}$ when integrated over the sphere will give $\frac{B_0}{R^2} 4\pi R^2 = 4\pi B_0$. We integrate the other term

$$\begin{aligned} \therefore \frac{Q}{\epsilon_0} &= 4\pi B_0 + \int_0^{2\pi} \int_0^\pi 3E_0 \cos \theta R^2 \sin \theta d\theta d\phi \\ &= 4\pi B_0 + 6\pi R^2 E_0 \int_0^\pi \cos \theta \sin \theta d\theta \\ &= 4\pi B_0 \end{aligned}$$

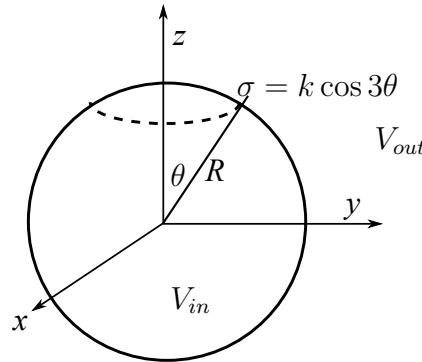
$$\therefore B_0 = \frac{Q}{4\pi\epsilon_0}$$

Now we have fixed all the constants. The potential is given as

$$V(r, \theta) = V_0 - \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{R} - \frac{1}{r} \right) + \left(-r + \frac{R^3}{r^2} \right) E_0 \cos \theta$$

2. A sphere of radius R has a surface charge given by the surface charge density $\sigma = k \cos 3\theta$ where k is a constant. Find the potential inside and outside the sphere.

soln



For a problem with azimuthal symmetry potential is

$$\begin{aligned} V_{out} &= \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta) \\ V_{in} &= \sum_{l=0}^{\infty} \left(A'_l r^l + \frac{B'_l}{r^{l+1}} \right) P_l(\cos \theta) \end{aligned}$$

Since there are no sources at infinity, we will demand $V_{out} \rightarrow 0$ as $r \rightarrow \infty$. This implies $A_l = 0$ for $l = 0, 1, 2, \dots$. So we have

$$V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

As $r \rightarrow 0$,

$$V_{in} \rightarrow \sum_{l=0}^{\infty} \frac{B'_l}{r^{l+1}} P_l(\cos \theta) \rightarrow \infty$$

So for V_{in} to be finite inside we must have $B'_l = 0$ for $l = 0, 1, 2, \dots$.

$$\therefore V_{in} = \sum_{l=0}^{\infty} A'_l r^l P_l(\cos \theta)$$

At $r = R$, $V_{in}(R) = V_{out}(R)$.

$$\sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta) = \sum_{l=0}^{\infty} A'_l R^l P_l(\cos \theta)$$

Since $P_l(\cos \theta)$ are orthogonal functions and hence linearly independent we have

$$\frac{B_l}{R^{l+1}} = A'_l R^l \implies B_l = A'_l R^{2l+1} \quad (1)$$

Over the surface of the sphere we have

$$E_{r \text{ out}} - E_{r \text{ in}} = \frac{\sigma}{\epsilon_0} = \frac{k \cos 3\theta}{\epsilon_0}$$

So we calculate the radial component of the electric field inside and outside.

$$\begin{aligned} E_{rout} = -\frac{\partial V_{out}}{\partial r} &= -\frac{\partial}{\partial r} \left[\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) \right] \\ &= \sum_{l=0}^{\infty} (l+1) \frac{B_l}{r^{l+2}} P_l(\cos \theta) \\ E_{rin} &= \sum_{l=0}^{\infty} (-l) A'_l r^{l-1} P_l(\cos \theta) \end{aligned}$$

So at the surface $r = R$ we will have

$$\begin{aligned} (E_{rout} - E_{rin})|_{r=R} &= \sum_{l=0}^{\infty} \left[(l+1) \frac{B_l}{R^{l+2}} + l A'_l R^{l-1} \right] P_l(\cos \theta) \\ &= \frac{k \cos 3\theta}{\epsilon_0} \end{aligned}$$

Substituting for B_l from eq.(1) we get

$$\sum_{l=0}^{\infty} (2l+1) A'_l R^{l-1} P_l(\cos \theta) = \frac{k \cos 3\theta}{\epsilon_0} \quad (2)$$

Multiplying both sides by $P_m(\cos \theta)$ and integrating from $\theta = 0$ to π will give A'_l . But let us do it differently.

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta = \frac{8}{5} P_3(\cos \theta) - \frac{3}{5} P_1(\cos \theta)$$

Substituting in eq.(2) we get

$$\sum_{l=0}^{\infty} (2l+1) A'_l R^{l-1} P_l(\cos \theta) = \frac{k}{5\epsilon_0} [8P_3(\cos \theta) - 3P_1(\cos \theta)]$$

We see that only the $l = 1$ and $l = 3$ terms survive on l.h.s.

$$\begin{aligned} \therefore 3A'_1 &= -\frac{3k}{5\epsilon_0} \implies A'_1 = -\frac{k}{5\epsilon_0} \\ 7A'_3 R^2 &= \frac{8k}{5\epsilon_0} \implies A'_3 = \frac{8k}{35\epsilon_0 R^2} \end{aligned}$$

All other $A'_l = 0$.

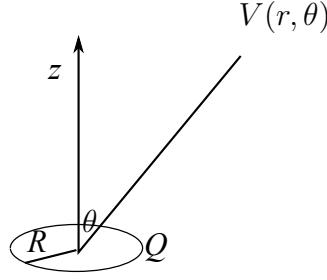
From Eq.(1) we get

$$\begin{aligned} B_1 &= A'_1 R^3 = -\frac{kR^3}{5\epsilon_0} \\ B_3 &= A'_3 R^7 = \frac{8kR^5}{35\epsilon_0} \end{aligned}$$

$$\begin{aligned} \therefore V_{out} &= -\frac{kR^3 \cos \theta}{5\epsilon_0 r^2} + \frac{8kR^5}{35\epsilon_0} \frac{P_3(\cos \theta)}{r^4} \\ V_{in} &= -\frac{k}{5\epsilon_0} r \cos \theta + \frac{8k}{35\epsilon_0 R^2} r^3 P_3(\cos \theta) \end{aligned}$$

Check that the potential is continuous at $r = R$.

3. A ring of radius R has a charge Q uniformly spread along it. The ring is placed on the x - y plane with the z -axis coinciding with its axis. Find the potential $V(r, \theta)$ in the region surrounding the ring.



soln:

It is easy to calculate the potential due to the ring at a distance r from the center of the ring along the z axis. This is given by

$$V(r, 0) = \frac{Q}{4\pi\epsilon_0\sqrt{R^2 + r^2}}$$

First consider $r > R$. We rewrite $V(r, 0)$ as

$$V(r, 0) = \frac{Q}{4\pi\epsilon_0 r} \left(1 + \frac{R^2}{r^2}\right)^{-\frac{1}{2}} = \frac{Q}{4\pi\epsilon_0 r} (1 + x)^{-\frac{1}{2}}$$

where $x = \frac{R^2}{r^2}$. We do a taylor expansion of $V(r, 0)$ around $x = 0$.

$$\begin{aligned} V(r, 0) &= \frac{Q}{4\pi\epsilon_0 r} \left(1 - \frac{1}{2}x + \frac{1}{2} \frac{3}{2} \frac{x^2}{2!} - \frac{1}{2} \frac{3}{2} \frac{5}{2} \frac{x^3}{3!} + \dots\right) \\ &= \frac{Q}{4\pi\epsilon_0} \left(1 - \frac{1}{2} \frac{R^2}{r^3} + \frac{3}{8} \frac{R^4}{r^5} - \frac{10}{16} \frac{R^6}{r^7} + \dots\right) \end{aligned}$$

Using the fact that $P_l(0) = 1 \forall l$ and from the general form of the potential in an azimuthal symmetric electrostatic configuration

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos \theta)$$

we can write down the potential at all points (r, θ) as

$$V(r, \theta) = \frac{Q}{4\pi\epsilon_0} \left(1 - \frac{1}{2} \frac{R^2}{r^3} P_2(\cos \theta) + \frac{3}{8} \frac{R^4}{r^5} P_4(\cos \theta) - \frac{10}{16} \frac{R^6}{r^7} P_6(\cos \theta) + \dots\right)$$

For $r < R$, $V(r, 0)$ can be written as

$$V(r, 0) = \frac{Q}{4\pi\epsilon_0 R} \left(1 + \frac{r^2}{R^2}\right)^{-\frac{1}{2}} = \frac{Q}{4\pi\epsilon_0 R} (1 + x)^{-\frac{1}{2}}$$

where $x = \frac{r^2}{R^2}$.

The taylor expansion of $V(r, 0)$ around $x = 0$.

$$\begin{aligned} V(r, 0) &= \frac{Q}{4\pi\epsilon_0 R} \left(1 - \frac{1}{2}x + \frac{1}{2} \frac{3}{2} \frac{x^2}{2!} - \frac{1}{2} \frac{3}{2} \frac{5}{2} \frac{x^3}{3!} + \dots\right) \\ &= \frac{Q}{4\pi\epsilon_0 R} \left(1 - \frac{1}{2} \frac{r^2}{R^2} + \frac{3}{8} \frac{r^4}{R^4} - \frac{10}{16} \frac{r^6}{R^6} + \dots\right) \end{aligned}$$

Comparing with the general form of an azimuthal symmetric potential we can write

$$V(r, \theta) = \frac{Q}{4\pi\epsilon_0 R} \left(1 - \frac{1}{2} \frac{r^2}{R^2} P_2(\cos \theta) + \frac{3}{8} \frac{r^4}{R^4} P_4(\cos \theta) - \frac{10}{16} \frac{r^6}{R^6} P_6(\cos \theta) + \dots \right)$$

Note that the two potentials for $r < R$ and $r > R$ match at $r = R$.

The potential only contains terms with even l . The given problem has symmetry about the x - y plane, i.e, $\theta = \frac{\pi}{2}$. $P_l(\cos \theta)$ has this property for even l . For odd l the legendre polynomials are antisymmetric about $\theta = \frac{\pi}{2}$. So those terms are not present.

4. Solve Laplace's equation by separation of variables in cylindrical co-ordinates, assuming there is no dependence on z .

soln

In cylindrical coordinates, the Laplace's Equation is written as

$$\nabla^2 V = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Since there is no dependence on z we have

$$\frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Let us consider $V(s, \phi) = f(s)g(\phi)$. Then

$$g \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial f}{\partial s} \right) + \frac{f}{s^2} \frac{\partial^2 g}{\partial \phi^2} = 0$$

Dividing throughout by V and multiplying by s^2 gives

$$\frac{s}{f} \frac{\partial}{\partial s} \left(s \frac{\partial f}{\partial s} \right) + \frac{1}{g} \frac{\partial^2 g}{\partial \phi^2} = 0$$

Each term above is a constant. The angular function $g(\phi)$ is expected to be periodic.

Hence we consider the constant related to it as negative.

Let $\frac{1}{g} \frac{\partial^2 g}{\partial \phi^2} = -k^2$ and $\frac{s}{f} \frac{\partial}{\partial s} \left(s \frac{\partial f}{\partial s} \right) = k^2$.

$$\therefore g(\phi) = A \sin(k\phi) + B \cos(k\phi) \quad (3)$$

Since $g(\phi) = g(\phi + 2\pi)$ for any physically meaningful solution, k must be an integer.

The differential Eqn in s is

$$s \frac{\partial}{\partial s} \left(s \frac{\partial f}{\partial s} \right) = k^2 f$$

$f = s^l$ satisfies the equation. Substituting, we get

$$l^2 s^l = k^2 f, \text{ i.e } l^2 f = k^2 f \implies l = \pm k.$$

$$\therefore f = D s^k + E s^{-k}.$$

We note that this is valid when $k \neq 0$. If $k = 0$ then

$$\frac{\partial}{\partial s} \left(s \frac{\partial f}{\partial s} \right) = 0 \implies s \frac{\partial f}{\partial s} = c_1$$

$$\therefore f(s) = c_1 \ln s + c_2.$$

Since k has to be an integer, l is also an integer. Hence the general solution in cylindrical coordinates is

$$V(s, \phi) = (c_1 \ln s + c_2)B_0 + \sum_{k=1}^{\infty} (D_k s^k + E_k s^{-k}) (A_k \sin k\phi + B_k \cos k\phi)$$

$c_1, c_2, B_0, A_k, B_k, D_k$ and E_k are constants to be determined from the boundary conditions. It is useful to write this potential as

$$V(s, \phi) = (c_1 \ln s + c_2) + \sum_{k=1}^{\infty} s^k (A_k \sin k\phi + B_k \cos k\phi) + s^{-k} (C_k \sin k\phi + D_k \cos k\phi)$$