SC217: Electromagnetic Theory, **AUTUMN'19 MIDSEM 2**

DA-IICT, B.Tech, Sem III 02:00-04:00 PM 22^{nd}Oct.2019 **25 marks**

1. Let $\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$, where p is a constant, be the electric field in a region expressed in the spherical polar coordiante system. Find the potential difference between the points
$\vec{r}_1 : (r = r_1, \theta = 0, \phi = 0)$ and $\vec{r}_2 : (r = r_2, \theta = \theta_2, \phi = 0)$. **(5)**
soln:
Method 1:

$$\begin{aligned} -\vec{\nabla}\Phi &= \vec{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \\ \therefore\ -\frac{\partial\Phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta} &= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \end{aligned}$$

$$\therefore\ \frac{\partial\Phi}{\partial r} = -\frac{p\cos\theta}{2\pi\epsilon_0 r^3} \quad \text{and} \quad \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

This gives $\Phi = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} + h(\theta)$ and $\Phi = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} + g(r)$.
Comparing the teo expressions of Φ we get

$$\Phi = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} + c$$

where c is a constant.
$\therefore$ the potential difference between $\vec{r}_1$ and $\vec{r}_2$ is

$$\Phi_2 - \Phi_1 = \frac{p\cos\theta_2}{4\pi\epsilon_0 r_2^2} - \frac{p}{4\pi\epsilon_0 r_1^2} = \frac{p}{4\pi\epsilon_0}\left[\frac{\cos\theta_2}{r_2^2} - \frac{1}{r^2}\right]$$

Method 2:
Consider a curve $r = f(\theta)$ on the plane $\phi = 0$ connecting $\vec{r}_1$ to $\vec{r}_2$. Then $f(0) = r_1$ and $f(\theta_2) = r_2$.

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot\vec{dl} = \int_{\vec{r}_1}^{\vec{r}_2} E_r dr + \int_{\vec{r}_1}^{\vec{r}_2} E_\theta r d\theta$$

Along the chosen path $dr = f'(\theta)d\theta$.

$$\begin{aligned}
\therefore \int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{l} &= \int_0^{\theta_2} \frac{2p\cos\theta}{4\pi\epsilon_0 f^3(\theta)} f'(\theta)d\theta + \int_0^{\theta_2} \frac{p\sin\theta}{4\pi\epsilon_0 f^3(\theta)} f(\theta)d\theta \\
&= \frac{p}{4\pi\epsilon_0}\left[2\int_0^{\theta_2} \frac{f'(\theta)\cos\theta}{f^3(\theta)}d\theta + \int_0^{\theta_2}\frac{\sin\theta}{f^2(\theta)}d\theta\right] \\
&= \frac{p}{4\pi\epsilon_0}\int_0^{\theta_2} \frac{2f'(\theta)\cos\theta + f(\theta)\sin\theta}{f^3(\theta)}d\theta \\
&= \frac{p}{4\pi\epsilon_0}\int_0^{\theta_2} d\left[-\frac{\cos\theta}{f^2(\theta)}\right] \\
&= -\frac{p}{4\pi\epsilon_0}\left[\frac{\cos\theta_2}{f^2(\theta_2)} - \frac{1}{f^2(0)}\right] \\
&= -\frac{p}{4\pi\epsilon_0}\left[\frac{\cos\theta_2}{r_2^2} - \frac{1}{r_1^2}\right]
\end{aligned}$$

$$\therefore\ \Phi_2 - \Phi_1 = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{l} = \frac{p}{4\pi\epsilon_0}\left[\frac{\cos\theta_2}{r_2^2} - \frac{1}{r_1^2}\right]$$

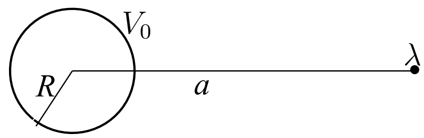
<u>Method 3</u>:
The path from $\vec{r}_1$ to $\vec{r}_2$ is broken into two parts. The first from $(r_1, 0, 0) \to (r_2, 0, 0)$ and then from $(r_2, 0, 0) \to (r_2, \theta_2, 0)$. So we have

$$\begin{aligned}
\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{l} &= \int_{\vec{r}_1}^{(r_2,0,0)} \vec{E}\cdot\hat{r}dr + \int_{r_2,0,0}^{\vec{r}_2} \vec{E}\cdot\hat{\theta}rd\theta \\
&= \int_{r_1}^{r_2} \frac{2p\cos(0)}{4\pi\epsilon_0 r^3}dr + \int_0^{\theta_2}\frac{p\sin\theta}{4\pi\epsilon_0 r_2^3}r_2 d\theta \\
&= -\frac{p}{4\pi\epsilon_0}\left[\frac{1}{r^2}\right]_{r_1}^{r_2} + \frac{p}{4\pi\epsilon_0 r_2^2}\left[-\cos\theta\right]_0^{\theta_2} \\
&= -\frac{p}{4\pi\epsilon_0}\left[\frac{1}{r_2^2} - \frac{1}{r_1^2} + \frac{\cos\theta_2}{r_2^2} - \frac{1}{r_2^2}\right] \\
&= -\frac{p}{4\pi\epsilon_0}\left[\frac{\cos\theta_2}{r_2^2} - \frac{1}{r_1^2}\right]
\end{aligned}$$

$$\therefore\ \Phi_2 - \Phi_1 = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{l} = \frac{p}{4\pi\epsilon_0}\left[\frac{\cos\theta_2}{r_2^2} - \frac{1}{r_1^2}\right]$$

2. An infinite line charge with charge density λ is placed at a distance a from the axis of an infinite conducting cylinder of radius $R < a$, maintained at a potential $V_0 = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{a}{R}\right)$. The line charge is parallel to the axis of the cylinder.

   (a) Find the potential at all points outside the cylinder. **(3)**
   soln:



A combination of parallel line charges λ and $-\lambda$ produces equipotential circular cylinders. Since we want the potential outside the given cylinderical conductor, let us consider an image charge as a line charge $-\lambda$ at a distance b from the axis within the cylinder. The potential due to λ and $-\lambda$ everywhere is given as $V = -\frac{\lambda}{2\pi\epsilon_0}\ln(s_1/s_2)$ where s_1 and s_2 are distances of a point from λ and $-\lambda$ respectively.

We want the potential of the given cylinder as V_0. Cpnsider a point P on the cylinder that lies on the line joining the two line charges. For P, $\;s_1 = a - R,\;\; s_2 = R - b$.

$$\therefore \quad -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{a-R}{R-b} = V_0$$

$$\therefore \quad \frac{a-R}{R-b} = e^{-2\pi\epsilon_0 V_0/\lambda}$$

This gives $b = R^2/a$. We may verify that this will make the potential at all points of the cylinder equal to V_0.

Now we can find the potential at all points outside the cylinder due to λ and $-\lambda$. This is given as $V = -\frac{\lambda}{2\pi\epsilon_0}\ln(s_1/s_2)$. In cartesian coordinates where we take the line charges along the z axis we have $s_1 = \sqrt{(x-a)^2 + y^2}$ and $s_2 = \sqrt{(x-b)^2 + y^2}$.

$$\begin{aligned} V &= -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{\sqrt{(x-a)^2+y^2}}{\sqrt{(x-b)^2+y^2}} \\ &= -\frac{\lambda}{4\pi\epsilon_0}\ln\frac{(x-a)^2+y^2}{(x-b)^2+y^2} \end{aligned}$$

(b) Find the equation of the surface with potential 0. **(2)**

soln:

The surface with $V = 0$ will be given by $s_1 = s_2$, i.e, $(x-a)^2 + b^2 = (x-b)^2 + y^2$. This gives $x = \frac{a+b}{2}$.

Substituting for b we get $x = \frac{a^2+R^2}{2a}$. This is equation of a plane parallel to the yz plane midway between the line charge λ and its image $-\lambda$.

3. When an amount of charge Q is placed on a piece of conductor it attains a potential V. Find the electrostatic energy stored in the conductor by evaluating the integral

$$\frac{\epsilon_0}{2}\int_{\text{all space}} |E|^2 d\tau$$

where $\vec{E}$ is the electric field created in the region surrounding the conductor due to the charge Q on it. **(5)**

soln:

The electric field within the conductor is 0. So the energy will be given by the integral becomes

$$\mathcal{E} = \frac{\epsilon_0}{2}\int_{\text{outside}} |E|^2 d\tau$$

$$E^2 = \vec{E}\cdot\vec{E} = \vec{\nabla}\Phi\cdot\vec{\nabla}\Phi = \vec{\nabla}\cdot(\Phi\vec{\nabla}\Phi) - \Phi\nabla^2\Phi$$

This gives

$$\mathcal{E} = \frac{\epsilon_0}{2}\int_{\text{outside}} \vec{\nabla}\cdot(\Phi\vec{\nabla}\Phi)d\tau - \frac{\epsilon_0}{2}\int_{\text{outside}} \Phi\nabla^2\Phi d\tau$$

Since $\nabla^2\Phi = \frac{\rho}{\epsilon_0} = 0$ outside the second integral doesn't contribute to the energy. Using divergence theorem the first integral can be converted to a surface integral over the surface bounding the region. This surface is the surface of the conductor S and the surface at infinity. At infinity we expect the fields go to zero if the given conductor is finite. So we have

$$\begin{aligned}
\mathcal{E} &= \frac{\epsilon_0}{2}\int_{\text{outside}} \vec{\nabla}\cdot(\Phi\vec{\nabla}\Phi)d\tau \\
&= \frac{\epsilon_0}{2}\oint_S \Phi\vec{\nabla}\Phi\cdot\hat{n}da
\end{aligned}$$

Here the normal $\hat{n}$ is directed into the closed surface of the conductor since the region we are integrating over is the outside.

$$\begin{aligned}
\therefore\ \mathcal{E} &= \frac{\epsilon_0}{2}V\oint_S(-\vec{E})\cdot\hat{n}da \\
&= \frac{\epsilon_0}{2}V\oint_S \vec{E}\cdot(-\hat{n})da \\
&= \frac{\epsilon_0}{2}V\frac{Q}{\epsilon_0} \\
&= \frac{1}{2}QV
\end{aligned}$$

4. Consider a point charge q placed at a distance a above the origin on the z axis. Write down the potential due to this point charge at a point given by the spherical polar coordinates (r, θ, ϕ) for $r > a$. Since this situation has an azimuthal symmetry the potential due to the charge configuration can be written as

$$\sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

where $P_l(\cos\theta)$ are called the Legendre polynomials in $\cos\theta$. The first few are
$P_0(\cos\theta) = 1, \quad P_1(\cos\theta) = \cos\theta, \quad P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$,
$P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$
Using the expansion, $\frac{1}{\sqrt{1+x^2-2x\cos\theta}} = \sum_{l=0}^{\infty} x^l P_l(\cos\theta)$, for $x < 1$ find the values of A_l and B_l for all l in terms of q and a. **(5)**
You may use the orthogonality relations of the Legendre polynomials
$\int_0^{\pi} P_l(\cos\theta)P_m(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}\delta_{lm}$

soln:
The potential at the point $P(r,\theta,\phi)$ is given as

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{a^2+r^2-2ar\cos\theta}}$$

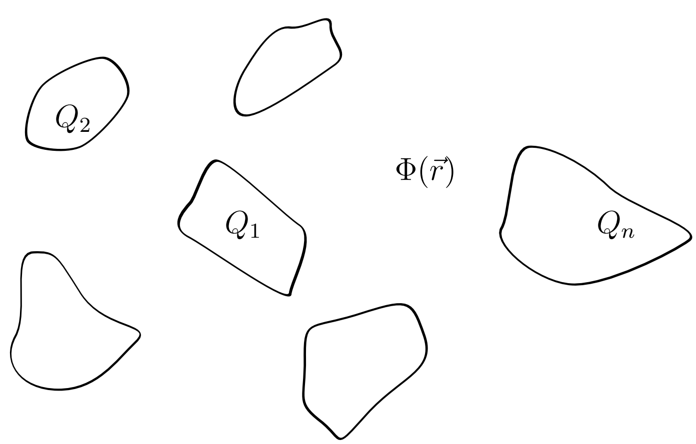
Since $r > a$ we express V as

$$\begin{aligned} V &= \frac{q}{4\pi\epsilon_0 r}\frac{1}{\sqrt{1+(a/r)^2-2(a/r)\cos\theta}} \\ &= \frac{q}{4\pi\epsilon_0 r}\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l P_l(\cos\theta) \\ &= \sum_{l=0}^{\infty}\frac{qa^l}{4\pi\epsilon_0 r^{l+1}}p_l(\cos\theta) \end{aligned}$$

Comparing with $\sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$ we find

$$A_l = 0, \quad B_l = \frac{qa^l}{4\pi\epsilon_0}$$

5. A chargeless region is bounded by n conducting surfaces as shown in the figure. When a unit charge is placed on conductor i while all other are chargeless, the potential in the region is given by the function $f_i(\vec{r})$. Now if charges $Q_1, Q_2, ..., Q_n$ are placed over the n conductors then justify that the potential in the region is given as $\Phi(\vec{r}) = \sum_{i=1}^{n} Q_i f_i(\vec{r})$ by showing that it satisfies the Laplace's Equation and the appropriate boundary conditions, i.e, this function makes the surfaces of the n conductors equipotential and Gauss's law is satisfied over the surfaces of each conductor.
soln:



(5)

Let us denote the surfaces of the n conductors as $S_1, S_2, ..., S_n$.

$f_i(\vec{r})$ is caused by a unit charge on i and 0 charge on the the other conductors.

$$\therefore\ -\oint_{S_i} \vec{\nabla} f_i(\vec{r})\cdot\hat{n}da = 1 \quad \text{and} \quad -\oint_{S_j} \vec{\nabla} f_i(\vec{r})\cdot\hat{n}da = 0 \ \text{ when } \ j \neq i$$

Each of the functions $f_i(\vec{r})$ are equipotential over all the conducting surfaces.
In the region bounded by the conductors $\vec{\nabla}^2 f_i(\vec{r}) = 0$ since the bounded region is chargeless.

$$\vec{\nabla}^{\Phi}(\vec{r}) = \vec{\nabla}^2 \sum_{i=1}^{n} Q_i f_i(\vec{r}) = \sum_{i=1}]^n Q_i \vec{\nabla}^2 f_i(\vec{r}) = 0$$

$\therefore\ \Phi(\vec{r})$ satisfies the laplace's Equation.
Consider a conductor j. Over this conductor

$$\Phi(\vec{r}) = \sum_{i=1}^{n} Q_i f_i(\vec{r})$$

Since each of the function $f_i(\vec{r})$ is constant over the surface of conductor j, $\Phi(\vec{r})$ is also constant over the surface j.

$$\vec{E} = -del\Phi(\vec{r}) = -\sum_{i=1}^{n} Q_i \vec{\nabla} f_i(\vec{r})$$

Over the surface of the conductor j

$$\oint_{S_j} \vec{E}\cdot\hat{n}da = -sum_{i=1}^{n} Q_i \oint_{S_j} \vec{\nabla} f_i(\vec{r}) = Q_j$$

$\therefore$ the electric field calculated from Φ satisfies the Gauss' law over each conductor.
$\therefore\ \Phi(\vec{r}) = \sum_{i=1}^{n} Q_i f_i(\vec{r})$is the potential in the required region

---

Gradient, divergence and curl

Spherical polar system

$$\vec{\nabla} F = \frac{\partial F}{\partial r}\hat{\mathrm{r}} + \frac{1}{r}\frac{\partial F}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial\phi}\hat{\phi} \qquad \vec{\nabla}\cdot\vec{\mathrm{A}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\vec{\nabla}\times\vec{\mathrm{A}} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathrm{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

Cylindrical System

$$\vec{\nabla} F = \frac{\partial F}{\partial s}\hat{\mathrm{s}} + \frac{1}{s}\frac{\partial F}{\partial\phi}\hat{\phi} + \frac{\partial F}{\partial z}\hat{\mathrm{z}} \qquad \vec{\nabla}\cdot\vec{\mathrm{A}} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\times\vec{\mathrm{A}} = \left[\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\mathrm{s}} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial\phi}\right]\hat{\mathrm{z}}$$