1. A dipole $\vec{\mathrm{p}}$ is at a distance r from a point charge q and oriented so that $\vec{\mathrm{p}}$ makes an angle θ with the vector $\vec{\mathrm{r}}$ from q to $\vec{\mathrm{p}}$.
   (a) What is the force on $\vec{\mathrm{p}}$?
   (b) What is the force on q ?

   soln
   In both the parts it is easier if we take the dipole along $\hat{z}$.

   (a) Due to q at the origin the force on the dipole $\vec{p}$ is

   $$\vec{F} = (\vec{p}\cdot)\vec{\nabla})\vec{E} = p\frac{\partial \vec{E}}{\partial z}$$

   where $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$

   $$\begin{aligned} \therefore \quad \vec{F}_p &= \frac{qp}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left(\frac{\vec{r}}{r^3}\right) \\ &= \frac{qp}{4\pi\epsilon_0}\left[\vec{r}\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\frac{\partial \vec{r}}{\partial z}\right] \\ &= \frac{qp}{4\pi\epsilon_0}\left[-\frac{3z\vec{r}}{r^5} + \frac{\hat{z}}{r^3}\right] \\ &= \frac{q}{4\pi\epsilon_0}\left[-\frac{(3\vec{p}\cdot\vec{r})\vec{r}}{r^5} + \frac{\vec{p}}{r^3}\right] \\ &= -\frac{q}{4\pi\epsilon_0 r^3}\left[(3\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right] \end{aligned}$$

   (b) For this part we place the dipole at the origin.

   The electric field at q due to $\vec{p}$ is

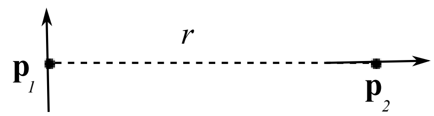
   $$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r}')\hat{r}' - \vec{p}\right]$$

   Now $\vec{r}' = -\vec{r}$ (used in part (a)).
   $\therefore \quad \hat{r}' = -\hat{r}$.
   $\therefore \quad$ force on q is

   $$\vec{F}_q = a\vec{E} = \frac{q}{4\pi\epsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

   We see that the forces are equal and opposite.

2. $\vec{\mathrm{p}}_1$ and $\vec{\mathrm{p}}_2$ are perfect dipoles a distance r apart. $\vec{\mathrm{p}}_2$ is along $\vec{\mathrm{r}}$ while $\vec{\mathrm{p}}_1$ is orthogonal to $\vec{\mathrm{r}}$. Calculate the torque on the dipoles. Are they equal and opposite?



soln

To calculate torque on $\vec{p}_2$ we consider $\hat{z}$ along $\vec{p}_1$. So at $\vec{p}_2$ the electric field is

$$\begin{aligned} \vec{E}_{p1} &= \frac{p_1}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) \\ &= \frac{p_1}{4\pi\epsilon_0 r^3}\hat{\theta} \quad \text{since} \quad \theta = \frac{\pi}{2} \\ \therefore \quad \vec{\tau}_{p2} &= \vec{p}_2 \times \vec{E}_{p1} = \frac{p_1 p_2}{4\pi\epsilon_0 r^3}(-\hat{n}) \end{aligned}$$

where $\hat{n}$ is a normal to the paper outward.

To calculate the torque on $\vec{p}_1$ due to $\vec{p}_2$ we consider the origin at $\vec{p}_2$ with $\hat{z}$ along $\hat{p}_2$.

$$\begin{aligned} \vec{E}_{p2} \text{ at } \vec{p}_1 &= \frac{p_2}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) \\ \frac{p_2}{4\pi\epsilon_0 r^3}(-2\hat{r}) \\ \therefore \quad \vec{\tau}_{p1} &= \vec{p}_1 \times \vec{E}_{p2} = \frac{2p_1 p_2}{4\pi\epsilon_0 r^3}(-\hat{n}) \end{aligned}$$

Note that the torques are not equal and opposite. Did you expect them to be so?

3. A sphere of radius R carries a polarization $\vec{\mathrm{P}}(\vec{\mathrm{r}}) = k\vec{\mathrm{r}}$
(a) Calculate the bound charges ρ_b and σ_b and the electric field caused due to them inside and outside the sphere.
(b) Find the electric field using the Gauss' law for the displacement vector $\vec{D}$ given as $\oint_S \vec{D}\cdot\hat{n}da = Q_{f(enc)}$.

soln:
(a)
The bound volume charge density is given as

$$\begin{aligned} \rho_b &= -\vec{\nabla}\cdot\vec{P} = -3k \\ \sigma_b &= \vec{P}\cdot\hat{n} = kR \end{aligned}$$

The given electrostatic configuration has a spherical symmetry. So by Gauss's law inside the sphere $r < R$ we have

$$E_{in}4\pi r^2 = -\frac{3k}{\epsilon_0}\frac{4}{3}\pi r^3$$

This gives $\vec{E}_{in} = -\frac{k\vec{r}}{\epsilon_0}$.
Outside the sphere $r > R$ we have

$$E_{out}4\pi r^2 = \frac{1}{\epsilon_0}\left[-3k\frac{4}{3}\pi R^3 + kR \times 4\pi R^2\right] = 0$$

This gives $\vec{E}_{out} = 0$.

(b)
Since there is no free charges anywhere we have $Q_{f(enc)} = 0$. So using the Gauss' law for the displacement vector we get $\vec{D} = 0$ everywhere.
Since $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ we have $\vec{E} = -\frac{\vec{P}}{\epsilon_0}$.
This directly gives

$$\vec{E}_{in} = -\frac{\vec{P}}{\epsilon_0} = -\frac{k\vec{r}}{\epsilon_0}, \quad \text{and} \quad \vec{E}_{out} = 0$$

4. A point charge q is imbedded at the center of a sphere of linear dielectric material with susceptibility χ_e and radius R. Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

soln
The problem has a spherical symmetry.
Consider a Gaussian sphere of radius r. We have $D4\pi r^2 = q$.
$\therefore \quad D = q/4\pi r^2$.

$$D = \epsilon_0 E + P = \epsilon_0 E + \epsilon_0 \chi_e E = \epsilon_0(1+\chi_e)E$$

$$\therefore \quad E = \frac{D}{\epsilon_0(1+\chi_e)} = \frac{q}{4\pi\epsilon_0(1+\chi_e)r^2}$$

Polarization $\vec{P} = \epsilon_0 \chi_e \vec{E}$.

$$\therefore \quad P = \frac{\chi_e}{1+\chi_e}\frac{q}{4\pi r^2}$$

$$\text{Hence} \quad \vec{\nabla}\cdot\vec{P} = 0 \quad \text{for} \quad r > 0$$

$$\therefore \quad \rho_b = -\vec{\nabla}\cdot\vec{P} = 0$$

On the surface of the sphere

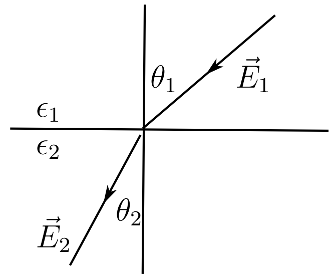
$$\sigma_b = \vec{P}\cdot\hat{r} = \frac{\chi_e}{(1+\chi_e)}\frac{q}{4piR^2}$$

Total bound charge on the surface of the sphere is $\frac{\chi_e}{1+\chi_e}q$.
Since the total bound charge has to be 0, the remaining bound charge is concentrated at the center surrounding the point charge. We can write

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{\chi_e}{1+\chi_e}q\delta^{(3)}(\vec{r})$$

Inside the dielectric the charge q is screened by ρ_b and reduces the electric field

5. At the interface between one linear dielectric and another the electric field lines bend. Show that $\tan\theta_2/\tan\theta_1 = \epsilon_2/\epsilon_1$ assuming there is no free charge at the boundary. Refer to fig.1 below.



soln:

$$\vec{D}_1 = \epsilon_0\epsilon_1\vec{E}_1 \qquad \text{and} \qquad \vec{D}_2 = \epsilon_0\epsilon_2\vec{E}_2$$

Since there are no free charges at the interface

$$\begin{aligned} D_1^{\perp} &= D_2^{\perp} \\ \therefore \quad \epsilon_0\epsilon_1 E_1\cos\theta_1 &= \epsilon_0\epsilon_2 E_2\cos\theta_2 \\ \therefore \quad \epsilon_1 E_1\cos\theta_1 &= \epsilon_2 E_2\cos\theta_2 \end{aligned} \tag{1}$$

The parallel component of electric field must be equal.

$$\therefore \quad E_1\sin\theta_1 = E_2\sin\theta_2 \tag{2}$$

From 1 and 2 we have

$$\frac{\tan\theta_1}{\epsilon_1} = \frac{\tan\theta_2}{\epsilon_2} \qquad \Longrightarrow \qquad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_2}{\epsilon_1}$$

6. Suppose the field inside a large piece of dielectric is $\vec{\mathrm{E}}_0$, so that the electric displacement is $\vec{\mathrm{D}}_0 = \epsilon_0\vec{\mathrm{E}}_0 + \vec{\mathrm{P}}$.

   (a) If we have a narrow cylindrical(needle-like) cavity inside the material running parallel to $\vec{\mathrm{P}}$ find the field near the center of the cavity in terms of $\vec{\mathrm{E}}_0$ and $\vec{\mathrm{P}}$. Also find the displacement at the center of the cavity in terms of $\vec{\mathrm{D}}_0$ and $\vec{\mathrm{P}}$.

   (b) Do the same for a thin wafer shaped cavity perpendicular to $\vec{\mathrm{P}}$.

soln:

(a) The tangential component of the electric field along the cylindrical walls of the cavity must be continuous.

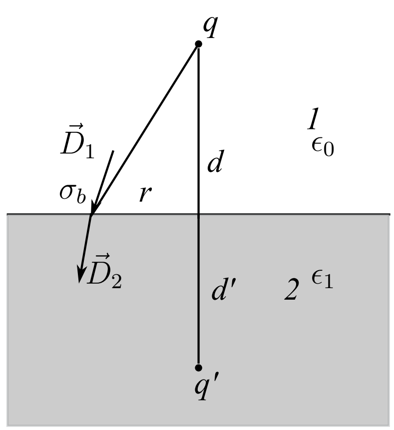
$$\begin{aligned} \therefore \quad \vec{E} &= \vec{E}_0 \\ \vec{D} &= \epsilon_0\vec{E}_0 = \vec{D}_0 - \vec{P} \end{aligned}$$

(b) Here we use the boundary condition on the perpendicular component of $\vec{D}$ since there are no free charges.
Near the center of the cavity

$$\begin{aligned} \vec{D} &= \vec{D}_0 \\ \vec{E} &= \frac{1}{\epsilon_0}\vec{D}_0 = \vec{E}_0 + \frac{1}{\epsilon_0}\vec{P} \end{aligned}$$

7. Suppose the entire region $z < 0$ is filled with uniform linear dielectric material of susceptibility χ_e. Calculate the force on a point charge q situated in air, at the point $(0, 0, d)$.

soln:
We will stretch the method of images here though a number of things are not according to what we have seen earlier in comparison to a point charge placed in front of an infinite grounded conducting plane. For one thing the interface plane $z = 0$ is not grounded. It is not even expected to be equipotential.



The charge q will polarize the dielectric material. Within the homogeneous dielectric material the polarization is
$\vec{P} = \epsilon_0 \chi_e \vec{E} = \frac{\epsilon_0 \chi_e}{\epsilon} \vec{D}$.
$\therefore \quad \vec{\nabla} \cdot \vec{P} = \frac{\epsilon_0 \chi_e}{\epsilon} \vec{\nabla} \cdot \vec{D}$
$\vec{\nabla} \cdot \vec{D}$ is the free charge density ρ_f. There is no free charge inside the dielectric. Hence $\vec{\nabla} \cdot \vec{P} = 0$. This implies the bound charge $\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$.
So there is no charge inside the dielectric material. However since there is a polarization the surface charge density at the interface of the dielectric material and air is $\vec{P} \cdot \hat{n}$ which is not 0. Since $\vec{P}$ is directed into the dielectric and $\hat{n}$ directed out of it, the surface bound charge densidy σ_b is negative. But it is difficult to calculate σ_b.

With these realization we will follow the spirit of method of images. We will call the region $z > 0$ as 1 while $z < 0$ as 2.
In region 1 we will simulate the effect of the surface charge distribution σ_b with a single charge q' placed inside the dielectric at a point $(0, 0, -d')$. Notice that we are not assuming the symmetry we had in the case of the grounded infinite plane. With this the potential everywhere in the region $z > 0$ is

$$V_1 = \frac{q}{4\pi\epsilon_0 \sqrt{r^2 + (z-d)^2}} + \frac{q'}{4\pi\epsilon_0 \sqrt{r^2 + (z+d')^2}} \tag{3}$$

As seen from region 2 the same surface charge will appear to be placed at $(0, 0, d')$. Notice that the real charge is only the surface charge density σ_b. When it is seen from region 1 the effective charge appears to be on the other side, i.e, in region 2. So when we look at σ_b from region 2 it appears at the same distance in region 1. With this understanding we can write down the potential in region 2 as

$$V_2 = \frac{q}{4\pi\epsilon_0\sqrt{r^2+(z-d)^2}} + \frac{q'}{4\pi\epsilon_0\sqrt{r^2+(z-d')^2}} \tag{4}$$

The potentials must match at $z=0$, i.e $V_1(z=0) = V_2(z=0)$. This gives

$$\frac{q}{4\pi\epsilon_0\sqrt{r^2+d^2}} + \frac{q'}{4\pi\epsilon_0\sqrt{r^2+d'^2}} = \frac{q}{4\pi\epsilon_0\sqrt{r^2+d^2}} + \frac{q'}{4\pi\epsilon_0\sqrt{r^2+d'^2}}$$

This is an identity. Hence we can't use it to determine anything. At the interface of regions 1 and 2 we have $D_1^{\perp} = D_2^{\perp}$. This gives

$$-\epsilon_0\frac{\partial V_1}{\partial z} = -\epsilon\frac{\partial V_2}{\partial z}$$

At $z=0$ this gives

$$\begin{aligned}
\epsilon_0\left[-\frac{qd}{(r^2+d^2)^{\frac{3}{2}}} + \frac{q'd'}{(r^2+d'^2)^{\frac{3}{2}}}\right] &= \epsilon\left[-\frac{qd}{(r^2+d^2)^{\frac{3}{2}}} - \frac{q'd'}{(r^2+d'^2)^{\frac{3}{2}}}\right] \\
\therefore\quad \frac{qd}{(r^2+d^2)^{\frac{3}{2}}}(\epsilon-\epsilon_0) &= -\frac{q'd'}{(r^2+d'^2)^{\frac{3}{2}}}(\epsilon+\epsilon_0)
\end{aligned}$$

The above equation is true at all r. We will take two suitable values $r=0$ and $r=d$. We get

$$\frac{q}{d^2}(\epsilon-\epsilon_0) = \frac{q'}{d'^2}(\epsilon+\epsilon_0) \tag{5}$$

$$\frac{q}{2^{3/2}d^2}(\epsilon-\epsilon_0) = \frac{q'}{d^2+d'^2}(\epsilon+\epsilon_0) \tag{6}$$

These two equations can be solved to get d' and q'. This is given by

$$d' = d \quad \text{and} \quad q' = -\frac{q\chi_e}{\chi_e+2}$$

Using these values of q' and d' we can calculate the potential, and electric field in both the regions.
The force on the charge q is equal to the the force between q and q'.

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(2d)^2} = -\frac{1}{4\pi\epsilon_0}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{q^2}{4d^2}$$

The negative sign indicates the force is attractive.