Primal dual relationship

P_1 Primal		P_2 Dual
Maximisation	—	Minimisation
minimisation	—	Maximisation
Number of variables (n)	—	Number of constraints (n)
Number of constraints (m)	—	Number of variables (m)
The "b" value	—	objective function coefficients (C)
Objective function coefficients (c)	—	The "b" value
constraint coefficients (A)	—	constraint coefficients (A^T)

Relationship between types of variables of the primal and the types of constraints of the dual and vice versa.

Primal (maximisation)		Dual (minimisation)
"$\leq$" constraint	——	"$\geq$" variable
"$\geq$" constraint	——	$\leq$ variable
$=$ constraint	——	unrestricted variable
$\geq$ variable	——	$\geq$ constraint
$\leq$ variable	——	$\leq$ constraint
unrestricted variable	——	$=$ constraint.

Duality theories

we are given the primal problem as,

(Primal)

$$\max Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall\, i = 1, 2, \dots, m$$

$$x_j \geq 0 \quad \forall\, j = 1, 2, \dots, n.$$

It corresponding dual is.

(Dual)

$$\min w = \sum_{i=1}^{m} b_i y_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad \forall\, j = 1, 2, \dots, n$$

$$y_i \geq 0, \quad \forall\, i = 1, 2, \dots, m.$$

Theorem: The dual of the dual is primal.

$$\boxed{\text{Primal} \rightarrow \underline{\underline{\text{Dual}}} \rightarrow \begin{array}{c}\text{Dual of Dual} \\ \text{|||} \\ \text{Primal}\end{array}}$$

verify on the examples that we have seen
H.W.

Proof consider the dual in standard form.

$$(\text{Dual}'') \left\{ \begin{array}{l} \text{Max } w' = \min(-w) = \sum_{i=1}^{m} (-b_i)\, y_i \\ \text{s.t. } \sum_{i=1}^{m} (-a_{ij})\, y_i \le \underline{\underline{-c_j}} \quad \forall\, j = 1, 2, \dots n \\ \qquad y_i \geq 0 \quad \forall\, i = 1, 2, \dots m. \end{array} \right.$$

$$(\text{Dual of Dual}') \left\{ \begin{array}{l} \min z' = \sum_{j=1}^{n} (-c_j)\, x_j' \\ \text{s.t. } \sum_{j=1}^{n} (-a_{ij})\, x_j' \geq -b_i \quad \forall\, i = 1, 2, \dots m \\ \qquad x_j' \geq \quad \forall\, j = 1, 2, \dots, n. \end{array} \right.$$

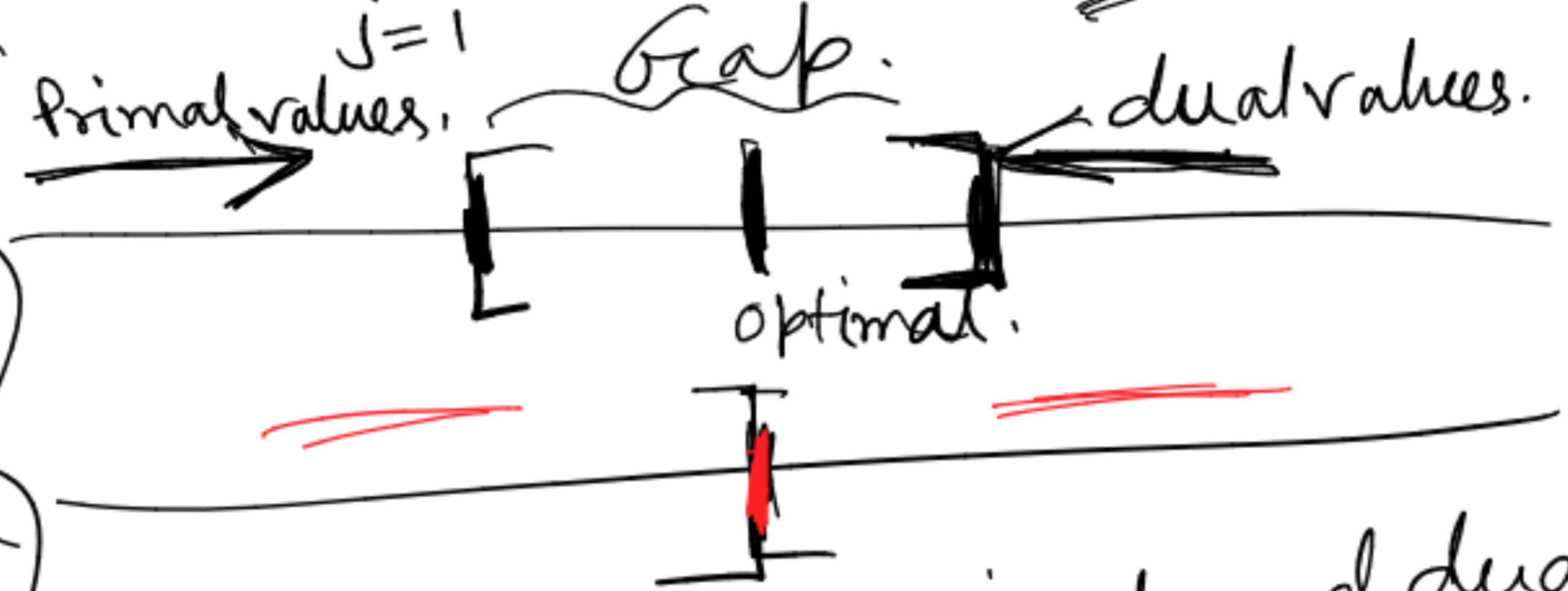
$$
\begin{cases}
\max Z = \min(Z') = \sum_{j=1}^{n} c_j' x_j' \\
\text{s.t.} \quad \sum_{j=1}^{n} a_{ij}' x_j' \le b_i \quad \forall i = 1, 2, \dots, m \\
\qquad x_j' \ge 0 \quad \forall j = 1, 2, \dots m.
\end{cases}
$$

Theorem: The weak duality theorem

If $(x_1, x_2, \dots, x_n)$ is a feasible solution for the primal and $(y_1, y_2, \dots, y_m)$ is a feasible solution for the dual, then,

$$\sum_{j=1}^{m} c_j x_j \le \sum_{i=1}^{n} b_i y_i$$

idea



both primal and dual reach the optimal in that case gap $= 0$

Proof

$$\sum_{j=1}^{m} c_j x_j \le \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} y_i \right) x_j$$

since y_i is a feasible solution to dual therefore it satisfies constraints of dual and x_j's ≥ 0.

$$\sum_{i=1}^{n} b_i y_i \ge \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} x_j \right) y_i$$

Theorem: If $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ be a feasible solution to the primal problem and $y^* = (y_1^*, y_2^*, \cdots, y_m^*)$ be a feasible solution to the dual problem such that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

then both x^* and y^* are optimal solutions to primal and dual respectively.

Proof we are given that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* \quad \text{--- (1)}$$

consider any feasible solution say $x' = (x_1', x_2', \cdots, x_n')$ to primal $\checkmark$ from weak duality theorem we have.

$$\sum_{j=1}^{n} c_j x_j' \leq \sum_{i=1}^{n} b_i y_i^* \quad \text{--- (2)}$$

From (1) & (2) we have,

$$\sum_{j=1}^{n} c_j x_j' \leq \sum_{j=1}^{n} c_j x_j^* \quad \text{--- (3)}$$

This implies that x_j^* is an optimal solution.

Theorem:

If primal has an optimal solution
$$x^* = (x_1^*, \ldots, x_n^*)$$
and the dual has also an optimal solution
$$y^* = (y_1^*, \ldots, y_m^*)$$
such that $\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$

If a finite optimal solution exists for the primal then there exists a finite optimal solution exists for the dual and conversely.

Summarize the results

Primal	Dual	conclusion
Feasible	Feasible	Finite optimal for both exists.
No feasible	Feasible	Dual objective function is unbounded.
Feasible	No feasible	Primal objective function is unbounded.
No feasible	No feasible	No solution exists.

Duality and simplex method

Construct the dual of the following problem and solve both primal and its dual using simplex.

$$\begin{cases} \text{Max } Z = 3x_1 + 4x_2 \\ \text{s.t. } 1x_1 + x_2 \le 12 & \text{— } 1 \cdots y_1 \\ 2x_1 + 3x_2 \le 21 & \text{— } 2 \cdots y_2 \\ 1x_1 \le 8 & \text{— } 3 \cdots y_3 \\ x_2 \le 6 & \text{— } 4 \cdots y_4 \end{cases}$$

$$x_1, x_2 \geq 0$$

Solution:

$$\begin{cases} \text{The Dual is.} \\ \min w = 12y_1 + 21y_2 + 8y_3 + 6y_4 \\ \text{s.t. } 1y_1 + 2y_2 + 1y_3 \geq 3 \\ y_1 + 3y_2 + y_4 \geq 4 \\ y_1, y_2, y_3, y_4 \geq 0. \end{cases}$$

$\underline{\underline{Ex^m}}$ $\max\ Z = 2x_1 - 6x_2$

$$\begin{aligned} \text{s.t.}\quad & x_1 - 3x_2 \le 6 \\ & 2x_1 + 4x_2 \ge 8 \\ & x_1 - 3x_2 \ge -6 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Find the dual.

<u>Solution</u> Standard form.

$\max\ Z = 2x_1 - 6x_2$

$$\begin{aligned} \text{s.t.}\quad & x_1 - 3x_2 \le 6 & \cdots\quad y_1 \\ & -2x_1 - 4x_2 \le -8 & \cdots\quad y_2 \\ & -x_1 + 3x_2 \le 6 & \cdots\quad y_3 \\ & x_1, x_2 \ge 0. \end{aligned}$$

<u>Dual</u>

$\min\ w = 6y_1 - 8y_2 + 6y_3$

$$\begin{aligned} \text{s.t.}\quad & y_1 - 2y_2 - y_3 \ge 2 \\ & -3y_1 - 4y_2 + 3y_3 \ge -6 \\ & y_1, y_2, y_3 \ge 0. \end{aligned}$$

$$\min w = 6y_1 - 8y_2 + 6y_3$$

$$\text{s.t.} \quad y_1 - 2y_2 - y_3 \geq 2$$

$$3y_1 + 4y_2 - 3y_3 \leq 6$$

$$y_1, y_2, y_3 \geq 0.$$