# How to solve a m×n game

i) Dominance property is used to reduce the size of the pay-off matrix.

ii) Any m×n game could be reduced to m×2 or 2×n game and then graphical method could be employed to solve the game.

iii) Even m×n game can be reduced to 2×2 game and could be solved either by pure strategy r by mixed strategy.

**In summary**

To solve a m×n game

i) First check dominange
ii) check saddle point
iii) If 2×n or m×2 use graphical method
iv) If m×n where m, n > 2
The what to do??

# Solving nxn square game

**Assumption:**

i) nxn pay-off matrix is obtained after checking dominance property

ii) No saddle point exists

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 1 & -1 & -1 \\ A_2 & -1 & -1 & 3 \\ A_3 & -1 & 2 & -1 \end{array}$$

i) No dominance effect

ii) No saddle point.

Let the strategies for A is $(x_1, x_2, x_3)$ where $0 \le x_i \le 1$ and $\sum_{i=1}^{3} x_i = 1$

the strategies for B is $(y_1, y_2, y_3)$ where $0 \le y_j \le 1$ and $\sum_{j=1}^{3} y_j = 1$

iii). value of the game exists and let it be $v$.

Solving $n \times n$ square matrix using algebraic method.

$$
\begin{array}{cc|ccc|}
 & & y_1 & y_2 & y_3 \\
 & & B_1 & B_2 & B_3 \\
\hline
x_1 & A_1 & 1 & -1 & -1 \\
x_2 & A_2 & -1 & -1 & 3 \\
x_3 & A_3 & -1 & 2 & -1 \\
\hline
\end{array}
$$

$$
\left\{
\begin{array}{ll}
x_1 - x_2 - x_3 = v & \text{—— (1)} \\
-x_1 - x_2 + 2x_3 = v & \text{—— (2)} \\
-x_1 + 3x_2 - x_3 = v & \text{—— (3)} \\
x_1 + x_2 + x_3 = 1 & \text{—— (4)}
\end{array}
\right.
$$

Subtract (2) from (1), we have

$$x_1 - x_2 - x_3 - (-x_1 - x_2 + 2x_3) = v - v$$

$$\Rightarrow 2x_1 - 3x_3 = 0 \Rightarrow x_3 = \frac{2}{3}x_1 \quad \text{—— (5)}$$

Subtract (3) from (1), we have,

$$(x_1 - x_2 - x_3) - (-x_1 + 3x_2 - x_3) = v - v$$

$$\Rightarrow 2x_1 - 4x_2 = 0 \Rightarrow x_2 = \frac{1}{2}x_1 \quad \text{—— (6)}$$

from (4) we have,

$$x_1 + \frac{1}{2}x_1 + \frac{2}{3}x_1 = 1$$

$$\Rightarrow \frac{6+3+4}{6} x_1 = 1$$

$$\Rightarrow x_1 = \frac{6}{13} \quad \text{---} (7)$$

Now from (6) and (5), we have,

$$x_2 = \frac{1}{2} \cdot \frac{6}{13} = \frac{3}{13}$$

$$\text{and } x_3 = \frac{2}{3} \cdot \frac{6}{13} = \frac{4}{13}$$

So the optimal strategy for player A is,

$$\left(\frac{6}{13}, \frac{3}{13}, \frac{4}{13}\right)$$

and the value of the game is,

$$v = x_1 - x_2 - x_3 = \frac{6-3-4}{13} = -\frac{1}{13}$$

Again we have,

$$y_1 - y_2 - y_3 = 0 \quad \text{(10)}$$
$$-y_1 - y_2 + 3y_3 = 0 \quad \text{(11)}$$
$$-y_1 + 2y_2 - y_3 = 0 \quad \text{(12)}$$
$$y_1 + y_2 + y_3 = 1 \quad \text{(13)}$$

(10) $-$ (11)

$$(y_1 - y_2 - y_3) - (-y_1 - y_2 + 3y_3) = 0$$

$$\Rightarrow 2y_1 - 4y_3 = 0 \Rightarrow y_3 = \frac{1}{2} y_1$$

(10) $-$ (12), we have,

$$(y_1 - y_2 - y_3) - (-y_1 + 2y_2 - y_3) = 0$$

$$\Rightarrow 2y_1 - 3y_2 = 0$$

$$\Rightarrow y_2 = \frac{2}{3} y_1$$

From 13 we have,

$$y_1 + y_2 + y_3 = 1$$

$$\Rightarrow y_1 + \frac{2}{3}y_1 + \frac{1}{2}y_1 = 1$$

$$\Rightarrow \frac{6+4+3}{6} y_1 = 1$$

$$\Rightarrow y_1 = \frac{6}{13}$$

$$y_2 = \frac{2}{3}y_1 = \frac{2}{3} \cdot \frac{6}{13} = \frac{4}{13}$$

$$y_3 = \frac{1}{2}y_1 = \frac{1}{2} \cdot \frac{6}{13} = \frac{3}{13}$$

optimal strategy for player B is $\left(\frac{6}{13}, \frac{4}{13}, \frac{3}{13}\right)$ ✓

value of the game is,

$$v = y_1 - y_2 - y_3 = \frac{6-4-3}{13}$$

$$= -\frac{1}{13}$$ ✓

## Result :-

If we add a fix number to each element of a pay-off matrix then the optimal stategies for players remain same, However the value of the game will be increased by that number.

**Proof** Let $A_{m\times n} = (a_{ij})_{m\times n}$ be the pay-off matrix and $v$ be the value of the game.

also let $X = (x_1, x_2, \ldots, x_m)$ where $0 \le x_i \le 1$ and $\sum_{i=1}^{m} x_i = 1$

and $Y = (y_1, y_2, \ldots, y_n)$ where $0 \le y_j \le 1$ and $\sum_{j=1}^{n} y_j = 1$

be the optimal strategies.

Then,
$$v = \max_X \min_Y E(X,Y)$$
$$= \min_Y \max_X E(X,Y)$$

Let $k$ be the numbe added to each element of the pay-off matrix.

Then the new pay-off matrix is

$$A_k = (a_{ij} + k)_{m \times n}$$

$$E_k(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} + k)\, x_i y_j$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j + k \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j$$

$$= E(X,Y) + k \quad \text{as } \sum x_i = 1,\ \sum y_j = 1$$

Thus,

$$\max_X \min_Y E_k(X,Y) = \max_X \min_Y [E(X,Y) + k]$$

$$= \max_X \min_Y E(X,Y) + k$$

Similarly

$$\min_Y \max_X E_k(X,Y) = \underline{\min_Y \max_X E(X,Y)} + k.$$

$v$

Hence,

$$\min_Y \max_X E_k(X,Y) = \max_X \min_Y E_k(X,Y) = \underline{v + k}$$

# Reduction of a $m \times n$ game to LPP

Let $A'_{m\times n} = (a'_{ij})_{m\times n}$ be a pay-off matrix.

and $X = (x_1, x_2, \ldots, x_m)$, $0 \le x_i \le 1$, and $\sum_{i=1}^{m} x_i = 1$ and $Y = (y_1, y_2, \ldots, y_n)$, $0 \le y_j \le 1$ and $\sum_{j=1}^{n} y_j = 1$ be the strategies for player A and player B respectively.

consider a new game $A = (a'_{ij} + K)$ such that $a_{ij} = a'_{ij} + K$ are positive for all $i$ and $j$

$$
\begin{array}{cc|cccc|}
 & & y_1 & y_1 & \cdots & y_n \\
 & & B_1 & B_2 & \cdots & B_n \\
\hline
x_1 & A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\
x_2 & A_2 & a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & & & \\
x_m & A_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\
\hline
\end{array}
$$

and $a_{ij} \geq 0$ $\forall i$ and $\forall j$

$\checkmark E_j(X) = a_{1j} x_1 + a_{2j} x_2 + \cdots + a_{mj} x_m$

for $j = 1, 2, \ldots, n$

Let $\min_j E_j(x) = u$

So, A's problem is to find X such that u attains its maximum

As $u = \min_j E_j(x)$ so,

$$E_j(x) \geq u \quad \text{for all } j$$

Thus A's problem is to maximise u subject to

$$E_j(x) \geq u \quad \text{for } j = 1, 2, \ldots, n$$

So, maximise $u \equiv$ minimise $\frac{1}{u}$

Thus the problem could be taken upon as,

(dividing everything by $u$)

minimise $p = \frac{1}{u} = \frac{x_1 + x_2 + \cdots + x_m}{u}$

subject to,

$$a_{11} x_1 + a_{21} x_2 + \cdots + a_{m1} x_m \geq u$$
$$a_{12} x_1 + a_{22} x_2 + \cdots + a_{m2} x_m \geq u$$
$$\vdots$$
$$a_{1n} x_1 + a_{2n} x_2 + \cdots + a_{mn} x_m \geq u$$
$$x_1, x_2, \cdots, x_m \geq 0$$

we take $x_i' = \frac{x_i}{u}$

Then clearly $x_i' \geq 0$.

The problem becomes,

minimize $p = x_1' + x_2' + \cdots + x_m'$

subject to,

$$a_{11} x_1' + a_{21} x_2' + \cdots + a_{m1} x_m' \geq 1$$

$$a_{12} x_1' + a_{22} x_2' + \cdots + a_{m2} x_m' \geq 1$$

$$\vdots$$

$$a_{1n} x_1' + a_{2n} x_2' + \cdots + a_{mn} x_m' \geq 1$$

$$x_i' \geq 0.$$

$$\begin{cases} A^T X' \geq 1 \\ X' \geq 0 \end{cases} \qquad X' = (x_1', x_2', \ldots, x_m')$$

$E_i(Y) = a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n \quad , i = 1, 2, \ldots, m$

Let $\max\limits_{i} E_i(Y) = w$

So B's problem is to find Y such that w attains its minimum

Here, $E_i(Y) \le w \quad , \forall i$

Thus B's problem is to minimise w subject to

$\underline{E_i(Y) \le w}$ for $i = 1, 2, \ldots, m$.

Since all $a_{ij} \geq 0$ so $w \geq 0$

and take

minimise $w \equiv$ maximise $\frac{1}{w}$

Thus the problem could be looked upon as

(dividing everything by $w$)

maximize $q = \frac{1}{w} = \frac{y_1 + y_2 + \cdots + y_n}{w}$

$= y_1' + y_2' + \cdots + y_n'$

(where $y_j' = \frac{y_j}{w}$

$y_j' \geq 0 \ \forall j$)

subject to,

$a_{11} y_1' + a_{12} y_2' + \cdots + a_{1n} y_n' \leq 1$

$a_{21} y_1' + a_{22} y_2' + \cdots + a_{2n} y_n' \leq 1$

$\vdots$

$a_{m1} y_1' + a_{m2} y_2' + \cdots + a_{mn} y_n' \leq 1$

$y_j' \geq 0 \quad \forall j = 1, 2, \ldots, n.$

$$\left[\begin{array}{l} AY' \leq 1 \\ Y' \geq 0 \end{array}\right. \qquad Y' = (y_1', y_2', \ldots, y_n')$$

Finally we have,

$$\left.\begin{aligned} \min\ p &= x_1' + x_2' + \cdots + x_m' \\ \text{s.t.}\quad A^T x' &\geq 1 \\ x' &\geq 0 \end{aligned}\right] \quad P_1$$

and

$$\left.\begin{aligned} \max\ q &= y_1' + y_2' + \cdots + y_n' \\ \text{s.t.}\quad A Y' &\leq 1 \\ Y' &\geq 0 \end{aligned}\right] \quad P_2$$

They are dual to each other
why ?? H.W.

By solving these LPP we have,

$$\min p = \max q = v^*$$

The value of the original game is $\left(\frac{1}{v^*} - K\right)$

## Example

$$A = \begin{bmatrix} -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix}$$

Reduce it to a LPP and solve