Groups and linear algebra (SC220) Autumn 2018
In Sem -I Time: 1hr 30 min

Name: ____________________

Student I.D.: ____________________

Section 1. True/False (2 pts. each)

Print "T" if the statement is true, otherwise print "F". In either case give a justification or a counter example.

F If G and H are cyclic groups then $G \times H$ is also cyclic.

Counter Example : $G = \mathbb{Z}_2$, $H = \mathbb{Z}_2$ are cyclic

but $G \times H = \mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic.

F D_6 (Group of Symmetries of a hexagon) is isomorphic to A_4 (Group of even permutations of 4 letters)

D_6 has an element r of order 6

$A_4 = \{e, (123), (132), (124), (142), (234), (243), (134), (143), (12)(34), (13)(24), (14)(23)\}$ has no elements of order 6.

T Z_{11}^* is a cyclic group.

$\mathbb{Z}_{11}^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ has order 10

$\mathbb{Z}_{11}^* = \langle 2 \rangle$ since $\langle 2 \rangle = \{2, 4, 8, 5, 10, 9, 7, 3, 6, 1\} = \mathbb{Z}_{11}^*$

F The remainder when 3^{47} is divided by 23 is 9

By Fermat's little thm. $3^{23-1} \equiv 1 \bmod 23$.

$\therefore 3^{22} \equiv 1 \bmod 23 \Rightarrow (3^{22})^2 \equiv 1^2 \bmod 23 \Rightarrow 3^{44} \equiv 1 \bmod 23$

$\therefore 3^{44+3} = 3^3 \bmod 23 \equiv 27 \bmod 23 \equiv \boxed{4 \bmod 23}$

T In S_4 let $\sigma = (123)(34)$ then σ^{2018} is $(13)(24)$

$\sigma = (123)(34) = (3412) = (1234)$

$|\sigma| = 4 \Rightarrow \sigma^4 = e \Rightarrow (\sigma^4)^{504} = \sigma^{2016} = e$

$\therefore \sigma^{2018} = \sigma^{2016+2} = \sigma^{2016} \cdot \sigma^2 = \sigma^2 = (1234)(1234)$

$= (13)(24)$

T In D_n the subgroup generated by r is a Normal subgroup.

$|\langle r \rangle| = n$ Hence the number of cosets of $\langle r \rangle$ in D_n are $\frac{|D_n|}{|\langle r \rangle|} = 2$. Hence left and right cosets are same and $\langle r \rangle$ is a Normal subgroup of D_n

F Every abelian group is cyclic

$\mathbb{Z}_2 \times \mathbb{Z}_2$ is abelian but not cyclic.

T The matrices of type $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, b, d \in \mathbb{R}, ad \neq 0$ form a subgroup of $GL_2(\mathbb{R})$

By Subgroup criterea $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} x & y \\ 0 & w \end{pmatrix} \in GL_2(\mathbb{R})$ s.t. $ad \neq 0$, $xw \neq 0$ then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} x & y \\ 0 & w \end{pmatrix}^{-1} = \frac{1}{xw}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} w & -y \\ 0 & x \end{pmatrix} = \begin{pmatrix} \frac{a}{x} & \frac{bx-ay}{xw} \\ 0 & \frac{d}{w} \end{pmatrix}$$

and $\frac{ad}{xw} \neq 0$ since $ad \neq 0$ & $xw \neq 0$.

F The group $(\mathbb{Q}, +)$ is isomorphic to $(\mathbb{Q}^+, \times)$

Suppose $(\mathbb{Q}, +) \equiv (\mathbb{Q}^+, \times)$ and let $\Phi : (\mathbb{Q}, +) \to (\mathbb{Q}^+, \times)$ be an isomorphism, then if $\Phi(a) = 2$ then $\Phi\left(\frac{a}{2} + \frac{a}{2}\right) = \Phi\left(\frac{a}{2}\right) \cdot \Phi\left(\frac{a}{2}\right) = \left(\Phi\left(\frac{a}{2}\right)\right)^2 = 2 \Rightarrow \Phi\left(\frac{a}{2}\right) = \sqrt{2}$. Contradiction since $\sqrt{2}$ is not rational.

F Let α and β be any two permutations in S_n then $\alpha^4\beta^{-2}\alpha$ is an odd permutation if α is an even permutation.

$\operatorname{sgn}(\alpha^4 \beta^{-2} \alpha) = \operatorname{sgn}(\alpha^4) \cdot \operatorname{sgn}(\beta^{-2}) \cdot \operatorname{sgn}(\alpha)$

But for any permutation α, β α^4 and $\beta^{-2} = (\beta^2)^{-1}$ is even $\Rightarrow \operatorname{sgn}(\alpha^4\beta^{-2}\alpha)$
$= \operatorname{sgn}(\alpha)$

Hence if $\operatorname{sgn}(\alpha^4\beta^{-2}\alpha)$ is even/odd $\Leftrightarrow$ $\operatorname{sgn}(\alpha)$ is even/ood.

Section 2. Short Answer (10 pts each)

Answer all problems in as thorough detail as possible.

1. Prove that the subgroup of S_4 generated by (12) and ~~(12)(34)~~ (13)(24) is isomorphic to D_4.

Proof:

We have $(12)(13)(24) = (1324)$

Consider the mapping $\Phi : D_4 \to \langle (12), (13)(24) \rangle$

2 that maps $r \xrightarrow{\Phi} (1324) \in \langle (12), (13)(24) \rangle$

$s \xrightarrow{\Phi} (13)(24) \in \langle (12), (13)(24) \rangle$

4 Now $\underbrace{\langle (12), (13)(24) \rangle}_{H} = \underbrace{\langle (13)(24), (1324) \rangle}_{K}$

Since $(12)(13)(24) = (1324)$ so $K \subseteq H$

Also $(13)(24)(1324) = (12)$ so $H \subseteq K$

Now in $\langle (13)(24), (1324) \rangle$ we have

$(1324)^4 = e$, $[(13)(24)]^2 = e$ and

4 $(1324)(13)(24) = (13)(24)(1324)^{-1} = (12)$

and therefore

$\langle (13)(24), (1324) \rangle = \langle (1324), (13)(24) \mid (1324)^4 = ((13)(24))^2 = e$ & $(1324)(13)(24) = (13)(24)(1324)^{-1} \} \rangle$

which is isomorphic to

$D_4 = \langle r, s \mid r^4 = s^2 = e, rs = sr^{-1} \} \rangle$

The mapping Φ takes generators to generators the remaining elements are determined by the isomorphism.

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* – Alternatively show group table and appropriate mapping

2. Let G be a group and $|G| = pq$ where p and q are primes. Show that any proper subgroup of G is cyclic.

Proof:

1. By Lagrange's theorem the order of a subgroup divides the order of a group. Since $|G| = pq$ the proper subgroups of $|G|$ are of order p or order q (since p and q are primes)

(5)

2. Also every group of prime order is cyclic. (Elements other than identity are generator). This is also a consequence of Lagranges thm.

(5)

Hence every proper subgroup of G where $|G| = pq$ is cyclic.

3. Let H and K be Normal subgroups of a group G such that $H \cap K = e$. Show that every element of H commutes with every element of K.

Proof: Since $H \trianglelefteq G$ and $K \trianglelefteq G$

$ghg^{-1} \in H$ $\forall$ $g \in G$ and $h \in H$ and

$gkg^{-1} \in K$ $\forall$ $g \in G$ and $k \in K$.

(6) Consider $h, \in H$ and $k \in K$ ~~respectively~~

then $\underbrace{hkh^{-1}}_{k'}k^{-1} = k'k^{-1}$ for some $k' \in K$ (Since K is a subgroup)

$\therefore$ $hkh^{-1}k^{-1} \in K$

Also $h\underbrace{kh^{-1}k^{-1}}_{h'} = hh'$ for some $h' \in H$ (since H is a subgroup)

Hence $hkh^{-1}k^{-1} \in H \cap K$

(4) But Since $H \cap K = e$

$hkh^{-1}k^{-1} = e \Rightarrow hk = kh$

Since h and k were chosen arbitrarily we get $hk = kh$ $\forall$ $h \in H$ an $k \in K$

$\therefore$ Every element of H commutes with every element of K.