Differential Equations

Contain Derivatives

Consider the equation of a straight line $\boxed{y = mx + c}$, with m and c being fixed parameters.

Taking the first derivative we get, $\boxed{\frac{dy}{dx} = m}$ and the second derivative gives us $\boxed{\frac{d^2y}{dx^2} = 0}$.

i/. Successive derivatives reduce the number of fixed parameters. This implies greater generalisation and more universal relevance.

ii/. Derivatives capture changes, and are relevant for evolving systems.

These are the two advantages of working with differential equations.

Changes in an <u>independent variable</u>, t, ("<u>time</u>", but it can be anything else).

We use a differential equation to express changes of a <u>variable, x</u>, <u>in time, t</u>.

dependent variable, $x \longrightarrow$ Population, Capital, height, position, etc.

$\frac{dx}{dt} \longrightarrow$ Rate at which x changes with t.

Since $x \equiv x(t)$, i.e. x depends on <u>ONLY</u> <u>one variable</u>, we get a <u>full derivative</u> <u>(or ordinary derivative)</u> in t. This requires an <u>ordinary differential equation</u>.

<u>Orders of a differential equation</u>:

A.) <u>First-order</u>: Highest derivative is $\frac{dx}{dt}$.

B.) <u>Second-order</u>: Higest derivative is $\frac{d^2x}{dt^2}$.

Examples:

A.) First-order ordinary differential equation

$$\boxed{\frac{dx}{dt} = x}$$ Eg. Compound interest.

B.) Second-order ordinary differential equation.

$$\boxed{\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0}$$ Eg. Damped oscillator.

Order of the ~~derivatives~~ differential equation = The number of initial (or boundary) conditions required in an integral solution.

If there are more than one independent variables, as in $\boxed{\psi(x,t)}$, then we have a partial differential equation, such as The Diffusion (or Heat) Equation:

$$\boxed{\frac{\partial \psi}{\partial t} = \nu \frac{\partial^2 \psi}{\partial x^2}}$$ which requires one initial condition (first order in t) and two boundary conditions (second order in space).

The Wave Equation: $\boxed{\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2}}$

Requires two initial conditions and two boundary conditions, because it is second order in t and also second order in x.

Second-order Differential Equations

Consider Newton's Second Law

$\boxed{F = kma} \Rightarrow \boxed{F = m\frac{d^2x}{dt^2}}$ $(k = 1)$

Now we write $\boxed{\frac{d^2x}{dt^2} = \frac{F(x,t)}{m}}$,

in which we substitute,

$\boxed{\frac{dx}{dt} = v} \Rightarrow \boxed{x \equiv x(t)}$

and $\boxed{\frac{dv}{dt} = \frac{F(x,t)}{m}} \Rightarrow \boxed{v \equiv v(t)}$

At ~~at~~ a given time, $t = t_0$, two initial conditions are required, $x(t_0)$ (an initial position) and $v(t_0)$ (an initial velocity). The former specifies the state and the latter the rate at which the state is changing.

Rate $\propto$ State : $\boxed{\frac{dx}{dt} \propto x}$

First-order system

We consider a system $\boxed{\frac{dx}{dt} = \pm ax}$ in which $a > 0$.

Rescaling : $\frac{dx}{d(at)} = \pm x$

~~Now~~ Now we rescale $\boxed{T = at}$, and get $\boxed{\frac{dx}{dT} = \pm x}$

Separation of Variables : $\int \frac{dx}{x} = \pm \int dT$

$\Rightarrow \ln x = \ln A \pm \ln e^{T}$

$\Rightarrow \boxed{x = A e^{\pm T}}$

A linear First-Order Autonomous Ordinary Differential Equation :

$\boxed{\frac{dx}{dt} = f(x) = a \pm bx}$ $a, b > 0$ (An autonomous form)

$\boxed{\frac{dx}{dt} = f(x,t) \text{ is in a } \underline{\text{NON-AUTONOMOUS}} \text{ form.}}$

<u>Transformation of variables:</u>

Write $\boxed{y = a \pm bx}$. $\Rightarrow$ $\frac{dy}{dt} = \pm b \frac{dx}{dt}$

But ~~dx~~ $\frac{dx}{dt} = a \pm bx = y$.

Hence, $\frac{dy}{dt} = \pm by$, which we rescale to get $\frac{dy}{d(bt)} = \pm y$ $\boxed{T = bt}$.

and, therefore, $\boxed{\frac{dy}{dT} = \pm y}$. This equation is in the <u>rate $\propto$ state form</u>.

Its solution is $\boxed{y = c\, e^{\pm T}}$, as before.

$\Rightarrow \quad a \pm bx = c\, e^{\pm bt}$

$\Rightarrow \quad \mp bx = a - c e^{\pm bt}$

$\Rightarrow \quad \mp x = \frac{a}{b} - \frac{c}{b} e^{\pm bt}$

$$\Rightarrow \quad \boxed{x = \mp\left(\frac{a}{b} - \frac{c}{b} e^{\pm bt}\right)}.$$

The choice of the lower (~~[illegible]~~) sign gives $\boxed{x = \frac{a}{b} - \frac{c}{b} e^{-bt}}$ from $\boxed{\frac{dx}{dt} = a - bx}$

Solving $\boxed{\dfrac{dx}{dt} = a - bx}$ where $a, b > 0$:

Separation of variables:

$$\frac{dx}{a-bx} = dt \Rightarrow \int \frac{d(-bx)}{a-bx} = \int d(-bt)$$

$$\Rightarrow \ln(a-bx) = \ln c - bt = \ln c + \ln e^{-bt}$$

$$\Rightarrow a - bx = c e^{-bt} \Rightarrow bx = a - ce^{-bt}$$

$$\Rightarrow \boxed{x = \frac{a}{b} - \frac{c}{b} e^{-bt}}$$

Since we started with a first-order differential equation in t, we require ONE INITIAL condition, which is

when $t = 0$, $x = 0$ $\Rightarrow 0 = \frac{a}{b} - \frac{c}{b} e^{-b0}$

$\Rightarrow$ $\boxed{c = a}$, by which we get.

$\boxed{x = \frac{a}{b}\left(1 - e^{-bt}\right)}$. We now define a scale for x as $\boxed{x_0 = a/b}$ and a scale for t as $\boxed{\tau = 1/b}$. Using these scales we can write $\boxed{x = x_0 \left(1 - e^{-t/\tau}\right)}$.

<u>Rescaling</u> $\boxed{X = \frac{x}{x_0}}$ and $\boxed{T = \frac{t}{\tau}}$,

we get $\boxed{X = 1 - e^{-T}}$. We can also perform a rescaling on $\boxed{\frac{dx}{dt} = a - bx}$ to obtain $X = 1 - e^{-T}$. This can be done as

$$\frac{1}{b}\frac{dx}{dt} = \frac{a}{b} - x .$$

$$\Rightarrow \quad \frac{dx}{d(bt)} = \frac{a}{b} - x .$$

Since $\boxed{T = bt}$ and $\boxed{x_0 = a/b}$ (<u>x_0 and τ are NATURAL scales</u>)

we write

$$\frac{dx}{dT} = x_0 - x .$$

$$\Rightarrow \quad \frac{d(x/x_0)}{dT} = 1 - (x/x_0) .$$

Since $\boxed{X = \frac{x}{x_0}}$

we finally get $\boxed{\frac{dX}{dT} = 1 - X}$, a rescaled differential equation whose solution is as before, $\boxed{X = 1 - e^{-T}}$. The limiting cases of this solution are when $\boxed{T = 0, \quad X = 0}$ and when $T \longrightarrow \infty$, $X \longrightarrow 1$, which is a convergence to a finite value.

Plotting $\boxed{X = 1 - e^{-T}}$:

i.) We know that when $T = 0$, $X = 0$. Now, when $\boxed{0 < T \ll 1}$, we expand

$$e^{-T} = 1 - T + \frac{T^2}{2!} - \frac{T^3}{3!} + \cdots$$

Successive terms in this series diminish very rapidly since $T \ll 1$. Hence,

$$\boxed{e^{-T} \simeq 1 - T} \quad \text{when } T \ll 1.$$

$$\therefore \quad X = 1 - e^{-T} \simeq 1 - (1 - T)$$

$$\Rightarrow \quad \boxed{X \simeq T} \Rightarrow \frac{x}{x_0} \simeq \frac{t}{\tau}.$$

$$\therefore \quad x \simeq x_0 \frac{t}{\tau} = \frac{a}{b} b t = at$$

Hence, for $t \ll \tau$ (or $T \ll 1$), $\boxed{x \simeq at}$.

ii.) In the opposite limit when $T \to \infty$,

$$\boxed{X = 1 - e^{-\infty} = 1} \Rightarrow X \to 1, \text{ when } T \to \infty.$$

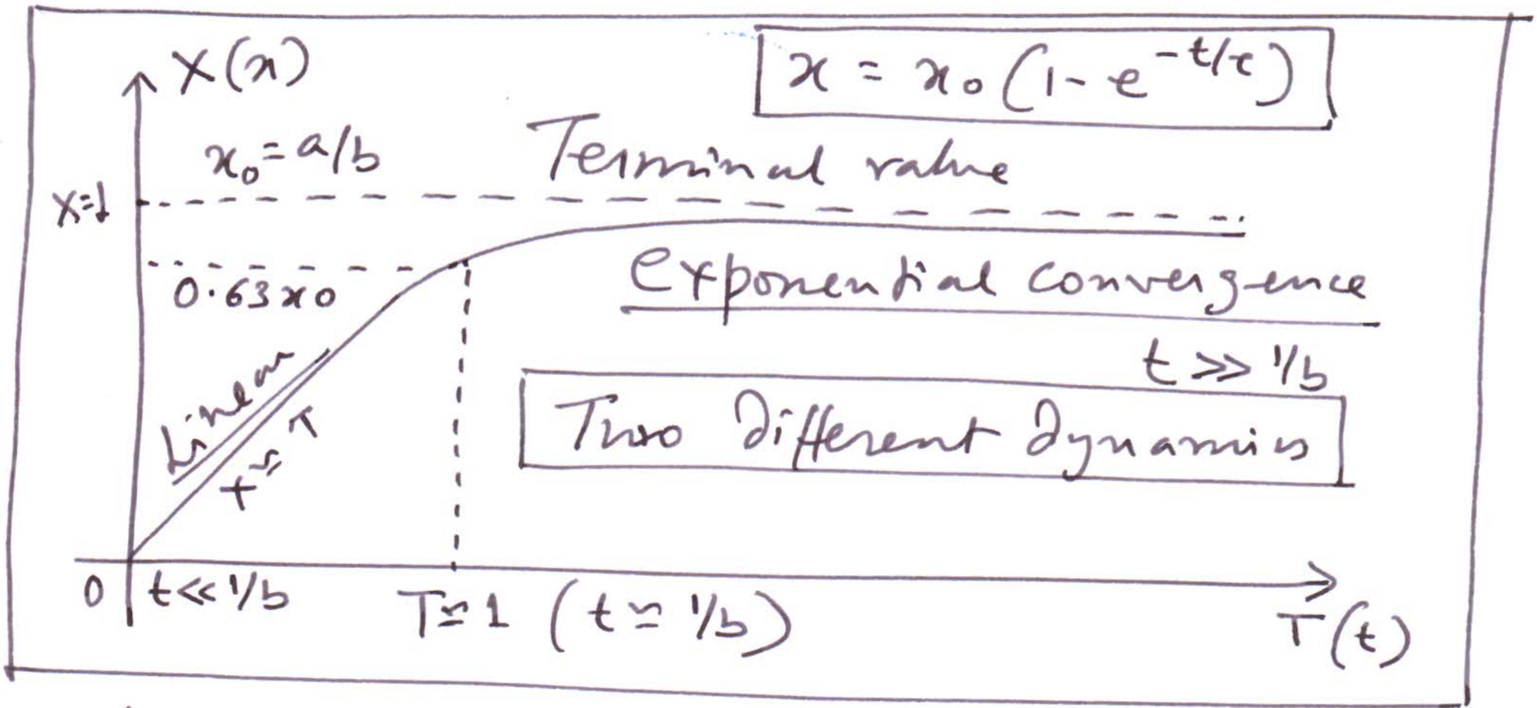
$\Rightarrow$ $x \to x_0 = a/b$, when $t \to \infty$. So for long time scale, x converges towards a limiting value of a/b.

iii) We can obtain the derivative of $X = 1 - e^{-T}$, as $\boxed{\frac{dX}{dT} = e^{-T}}$. If $\frac{dX}{dT} = 0 \Rightarrow T \to \infty$.

The second derivative is $\frac{d^2X}{dT^2} = -e^{-T}$

when $T \to \infty$, $\frac{d^2X}{dT^2} = 0$. Hence this is not a turning point. ~~[illegible]~~.

iv) The transition from the linear behaviour $\boxed{X \simeq T}$ to an exponential convergence of $\boxed{X = 1 - e^{-T}}$ takes place when $\boxed{T \simeq 1}$ or when $t \simeq 1/b$.



When $t = \tau$, $x = x_0(1 - e^{-1}) \Rightarrow \boxed{x \simeq 0{\cdot}63 x_0}$

There are two different dynamics on two different time scales. Eg. Growth of humans or the inflationary Universe.