Matrix formulation of an LPP

The linear programming problem can be written as,

$$\text{optimise } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t. } \sum_{j=1}^{n} a_{ij} x_j \ (\leq = \geq)\ b_i \quad \forall i = 1, 2, \cdots m$$

$$x_j \geq 0 \ \forall j = 1, 2, \cdots, n$$

This can be written as,

$$\left.\begin{array}{l} \text{optimise } Z = CX \\ \text{s.t. } AX \ (\leq = \geq)\ b \\ X \geq 0 \end{array}\right\}$$

where,

$C \Leftarrow$ cost vector, is an n component row vector $(c_1, c_2, \cdots, c_n)$

$X \Leftarrow$ variable vector, is an n component column vector $[x_1, x_2, \cdots, x_n]$

$b \Leftarrow$ constant/requirement vector is an m component column vector $[b_1, b_2, \cdots, b_m]$

$A \leftarrow$ coefficient matrix $[a_{ij}]_{m \times n}$

$$A = \begin{array}{c} \begin{array}{cccc} \downarrow a_1 & \downarrow a_2 & & \downarrow a_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Let a_j denote the j-th column vector of A then we can write

$$Ax \ (\leq = \geq)\ b \text{ as,}$$

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \ (\leq = \geq)\ b}$$

Basic solution:

consider a set of m linear simultaneous equations of n variables $(n > m)$

$$\boxed{Ax = b}$$

Assume that $\operatorname{rank}(A) = \operatorname{rank}(A_b) = m$

($A_b \leftarrow$ Augmented matrix of $Ax=b$)

If any $\boxed{m \times m}$ non-singular square matrix be chosen from A and if all the $n-m$ variables that are not associated with the chosen matrix be set equal to zero, then the solution to the resulting system of equations is called a basic solution.

- The basic solution has not more than m non-zero variable.
- These non-zero variables are called basic variables.
- The variables that are not basic are called non-basic variables.

How to find basic variables

- Find a $m \times m$ non-singular matrix
  i.e., select m linearly independent vectors out of n linearly independent vectors that from A.
- Set variables to zero that are associated with the remaining $(n-m)$ columns.
- Find the solution to the resulting system of equations.

More precisely

$$Ax = b$$

Partition A and x as follows

$$A = [B, R] \quad x = [x_B, x_R]$$

where B is an $m \times m$ non singular matrix

R is an $m \times (n-m)$ matrix.

$$\boxed{x_B = [x_1, x_2, \cdots, x_m]}$$

$$\Rightarrow x_R = [x_{m+1}, x_{m+2}, \cdots, x_n]$$

Then $Ax = b$ can be written as.

$$Ax = Bx_B + Rx_R = b$$

Setting $x_R = 0$

$$Bx_B = b$$

$$\Rightarrow x_B = B^{-1} b$$

$$\underbrace{\begin{bmatrix} 2 & 5 & 4 \\ 1 & 3 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

2×2 non singular matrix

$$\boxed{\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \qquad \boxed{\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}}$$

basic

$x_2 \leftarrow$ non basic

$x_1 \rightarrow$ first column

$x_2 \rightarrow$ second column

$x_3 \rightarrow$ third column.

Set $x_3 \leftarrow 0$ $\quad$ x_3 is a non basic variable.

Now solve:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

x_1 & x_2 are basic variables.

$x_1 = 0 \qquad x_2 = 0$

$$\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

non-basic $\rightarrow x_1$ $\quad x_1 = 0$

basic $\rightarrow x_2, x_3$

Degenerate basic solution:

If the number of basic variables is <u>less than</u> m.
ie, any one of the basic variables be zero
Then the solution is called degenerate basic solution.

Non-degenerate basic solution

If there are exactly m non-zero variables.

$$\boxed{\text{The total number of basic solutions is at most } {}^{n}C_{m}}$$

Example

Find the basic solutions of the following set of equations.

$$\left.\begin{array}{l} 2x_1+3x_2-x_3+4x_4=8 \\ x_1-2x_2+6x_3-7x_4=-3 \\ x_1, x_2, x_3, x_4 \geq 0 \end{array}\right\}$$

Solutions

we write it in matrix form

$$AX=b$$

$$\underbrace{\begin{bmatrix} 2 & 3 & -1 & 4 \\ 1 & -2 & 6 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 8 \\ -3 \end{bmatrix}}_{b}$$

$$x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 = b$$

$$a_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad a_3 = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \quad a_4 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

$$b = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

The maximum number of basic solutions is. $(n=4, m=2)$

$$^{n}C_{m} = {}^{4}C_{2} = 6.$$

$$B_1 = [a_1 \; a_2] = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \qquad \det(B_1) = -7$$

$$B_2 = [a_1 \; a_3] = \begin{bmatrix} 2 & -1 \\ 1 & 6 \end{bmatrix} \qquad \det(B_2) = 13$$

$$B_3 = [a_1 \; a_4] \qquad \det(B_3) = -18$$

$$B_4 = [a_2 \; a_3] \qquad \det(B_4) = 16$$

$$B_5 = [a_2 \; a_4] \qquad \det(B_5) = -13$$

$$B_6 = [a_3 \; a_4] = \begin{bmatrix} -1 & 4 \\ 6 & -7 \end{bmatrix} \qquad \det(B_6) = 7-24 = -17$$

Observation: None of the determinant is zero. Hence every set of two column vectors of A are linearly independent.

For B_1

we set. $x_3 = 0$, $x_4 = 0$

$$x_{B_1} = B_1^{-1} b$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}^{-1} b$$

$$= -\frac{1}{7} \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$$= -\frac{1}{7} \begin{bmatrix} -7 \\ -14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$-2 \times 8 + -3 \cdot -3$
$= -16 + 9$
$= -7$

$-1 \cdot 8 + 2 \cdot -3$
$-8 - 6$
$= -14$

one basic solution to the LPP is
$x_1 = 1$, $x_2 = 2$, $x_3 = 0$, $x_4 = 0$

both basic variables are non-zero hence it is a non-degenerate basic solution.

$\boxed{\text{basic feasible solution}}$

For B_2

we set $x_2 = 0$ and $x_4 = 0$

$$X_{B_2} = B_2^{-1} b$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$$= \frac{1}{13} \begin{bmatrix} 6 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$$= \frac{1}{13} \begin{bmatrix} 45 \\ -14 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{45}{13} \\ -\frac{14}{13} \end{bmatrix}$$

$\boxed{\text{not a basic feasible solution}}$

2nd basic solution is,

$$\underline{x_1 = \frac{45}{13}}, x_2 = 0, \boxed{\underline{x_3 = -\frac{14}{13}}}, x_4 = 0$$

It is also a non-degenerate basic solution.

Basic feasible solution

consider the LPP

optimise $Z = CX$

s.t. $AX \ (\leq = \geq) b$

$X \geq 0$.

Basic feasible solution:

A solution that is basic and satisfies all the constraints and non-negativity restrictions is called a basic feasible solution.

- A feasible solution contains at least $(n-m)$ zero variables.

- A basic feasible solution generates a basic.

Degenerate basic feasible solution

at least one basic variable is zero.

Non-degenerate basic feasible solution

Each basic variable is non-zero