Solve the game

	B_1	B_2	B_3	B_4
A_1	6	5	2	3
A_2	1	2	6	3

$$E_1(x) = 6x + 1\cdot(1-x)$$
$$= 6x + 1 - x$$
$$= 5x + 1$$

$$E_2(x) = 5x + 2(1-x)$$
$$= 5x + 2 - 2x$$
$$= 3x + 2$$

$$E_3(x) = 2x + 6(1-x)$$
$$= 2x + 6 - 6x$$
$$= 6 - 4x$$

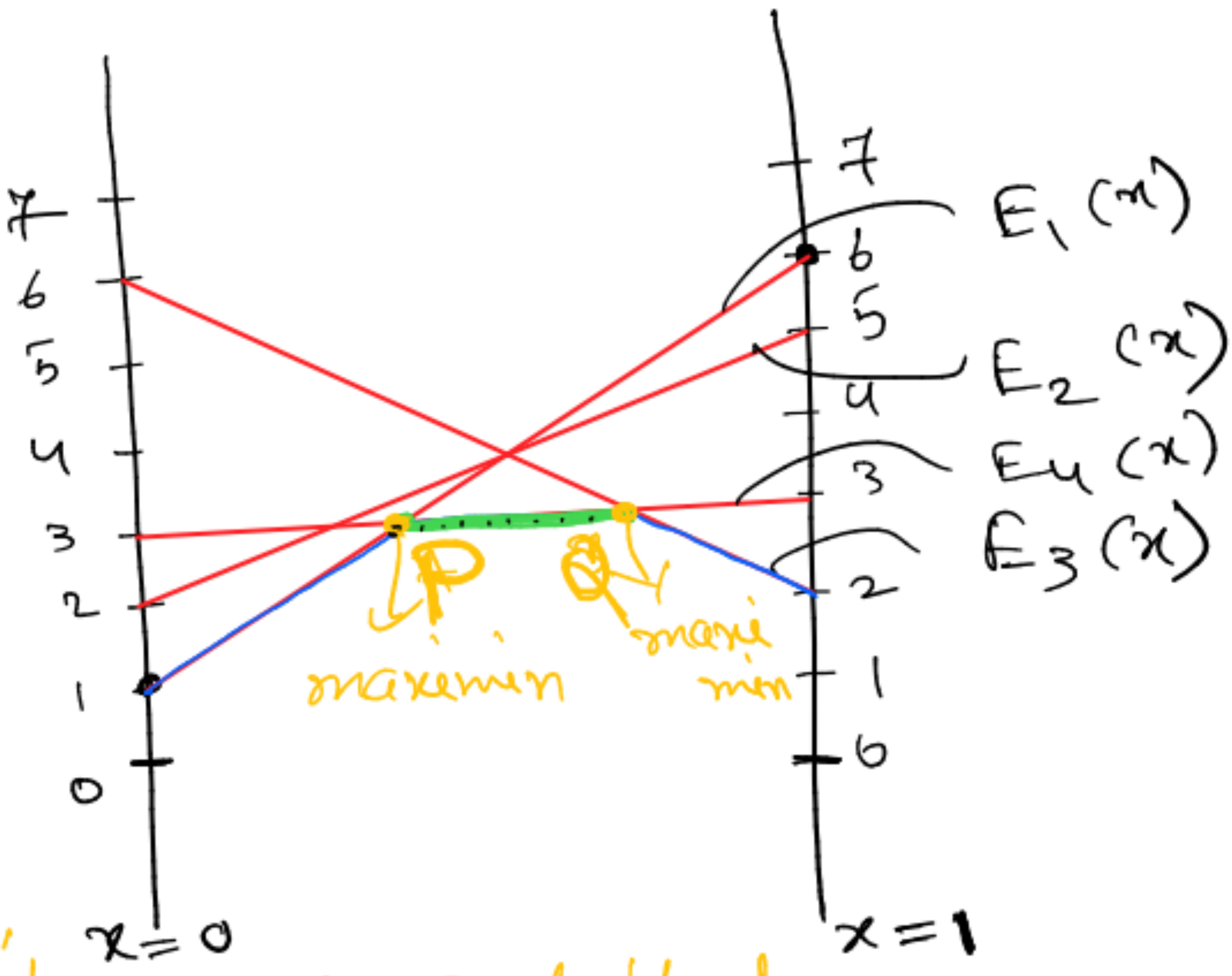
$$E_4(x) = 3x + 3(1-x)$$
$$= 3x + 3 - 3x$$
$$= 3$$

Two points on $E_1(x)$ are $(0,1)$ and $(1,6)$

Two points on $E_2(x)$ are $(0,2)$ and $(1,5)$

Two points on $E_3(x)$ are $(0,6)$ and $(1,2)$

Two points on $E_4(x)$ are $(0,3)$ and $(1,3)$



For P, we have x_0 such that

$$E_1(x_0) = E_4(x_0)$$

$$\Rightarrow 5x_0 + 1 = 3$$

$$\Rightarrow \quad x_0 = \frac{2}{5}$$

For B,

we have x_0 such that,

$$E_3(x_0) = E_4(x_0)$$

$$\Rightarrow 6 - 4x_0 = 3$$

$$\Rightarrow -4x_0 = -3$$

$$\Rightarrow x_0 = \frac{3}{4}$$

Hence the optimal strategy for player-A is

$$X = \left(\frac{2}{5}, \frac{3}{5}\right) \quad \text{or} \quad X = \left(\frac{3}{4}, \frac{1}{4}\right)$$

The value of the game is.

$$v = 5x_0 + 1$$

$$= 5 \cdot \frac{2}{5} + 1 = 3$$

If we look at the graph the x-value at P is $\frac{2}{5}$ and the x-value at Q is $\frac{3}{4}$.

Player-A has an infinite number of optimal strategies $(x, 1-x)$ where x varies from $\frac{2}{5}$ to $\frac{3}{4}$

We now need to find out an optimal strategy for player - B.

To get the optimal strategy for player - B solve either

for P

	B_1	B_4
A_1	6	3
A_2	1	3

for Q_1

	B_3	B_4
A_1	2	3
A_2	6	3

Example: Solve the following game.

	B_1	B_2	B_3	B_4
A_1	-1	3	2	2
A_2	6	2	5	3

$$E_1(x) = -x + 6(1-x)$$
$$= -x + 6 - 6x$$
$$= 6 - 7x$$

$$E_2(x) = x + 2$$

$$E_3(x) = 5 - 3x$$

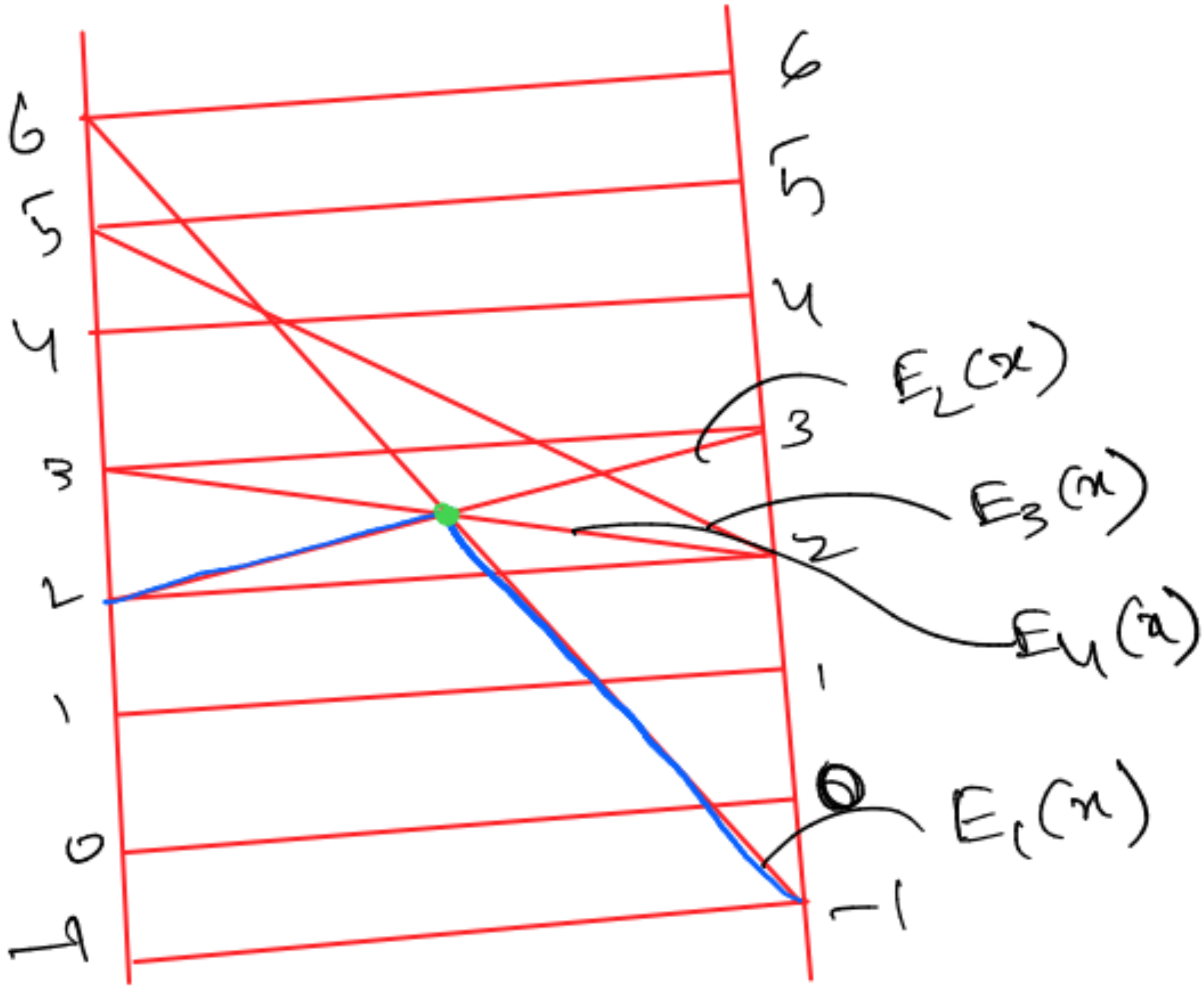
$$E_4(x) = 3 - x.$$

Two points on $E_1(x)$ are $(0,6)$ and $(1,-1)$

" " " $E_2(x)$ are $(0,2)$ and $(1,3)$

" " " $E_3(x)$ are $(0,5)$ and $(1,2)$

" " " $E_4(x)$ are $(0,3)$ and $(1,2)$



We have x_0 such that

$$E_1(x_0) = E_2(x_0) \quad \text{or} \quad E_2(x_0) = E_4(x_0)$$

$$\text{or,} \quad E_1(x_0) = E_4(x_0).$$

$E_1(x_0) = E_{\cancel{4}2}(x_0)$

$\Rightarrow 6 - 7x_0 = x_0 + 2$

$\Rightarrow -8x_0 = -4$

$\Rightarrow x_0 = 1/2$

$E_2(x_0) = E_4(x_0)$

The optimal strategy for player A is
$\left(\frac{1}{2}, \frac{1}{2}\right)$
and the value of the game is,

$$v = 6 - 7x_0$$
$$= 6 - 7 \cdot \frac{1}{2}$$
$$= \frac{12-7}{2} = 5/2 = 2.5$$

Now we need to find an optimal strategy for player - B.

Try all possible submatrices

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & -1 & 3 \\ A_2 & 6 & 2 \end{array} \qquad \begin{array}{c|cc} & B_2 & B_4 \\ \hline A_1 & 3 & 2 \\ A_2 & 2 & 3 \end{array} \qquad \boxed{\begin{array}{c|cc} & B_1 & B_4 \\ \hline A_1 & -1 & 2 \\ A_2 & 6 & 3 \end{array}}$$

value of the game is 3

Dominance

- This is used to remove unnecessary columns and/or rows that have no effect in the optimal solution.

- we get a smaller size instance

General rule for dominance

Let $(a_{ij})_{m \times n}$ be the pay-off matrix.

i) If all the elements of K-th row are less than or equal to the corresponding elements of r-th row, then K-th row is called dominated by the r-th row and hence K-th row can be eleminated from the matrix A.

ii) If all elements of the l-th column are greater than or equal to the corresponding elements of s-th column, then l-th column is dominated by s-th column and hence l-th column can be eleminated from the matrix A.

Example

$$\begin{array}{c|cccc|} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & -3 & 3 & 1 & 20 \\ A_2 & 5 & 5 & 4 & 6 \\ A_3 & -4 & -2 & 0 & 6 \\ \hline \end{array}$$

A_3 is dominated by A_2 since all elements of A_3 are less than or equal to the corresponding elements of A_2.

Therefore the reduced matrix is,

$$\begin{array}{c|cccc|} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & -3 & 3 & 1 & 20 \\ A_2 & 5 & 5 & 4 & 6 \\ \hline \end{array}$$

B_4 is dominated by B_3

Hence, the reduced matrix is.

$$\begin{array}{c|ccc|} & B_1 & B_2 & B_3 \\ \hline A_1 & -3 & 3 & 1 \\ A_2 & 5 & 5 & 4 \\ \hline \end{array}$$

B_2 is dominated by B_3

The reduced matrix is,

$$\begin{array}{c|cc|} & B_1 & B_3 \\ \hline A_1 & -3 & 1 \\ A_2 & 5 & 4 \\ \hline \end{array}$$

A_1 is dominated by A_2

The reduced matrix is.

$$\begin{array}{c|cc|} & B_1 & B_3 \\ \hline A_2 & 5 & 4 \\ \hline \end{array}$$

B_1 is dominated by B_3

The reduced matrix is

$$\begin{array}{c|c|} & B_3 \\ \hline A_2 & 4 \\ \hline \end{array}$$

The optimal strategy for player A is A_2 and for player-B is B_3

The value of the game is 4.

Example

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 1 & 7 & 2 \\ A_2 & 6 & 2 & 7 \\ A_3 & 0 & 1 & 6 \end{array}$$

Use dominance to reduce this matrix.

A_3 is dominated by A_2

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 1 & 7 & 2 \\ A_2 & 6 & 2 & 7 \end{array}$$

B_3 is dominated by B_1

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 1 & 7 \\ A_2 & 6 & 2 \end{array}$$

No further reduction possible.