Population Dynamics

Use a differential equation, i.e, by a Continuum description (differentiable), $x(t)$.

Rate of per capita growth of the population is

$$\boxed{\frac{\Delta x}{x\,\Delta t} = r(x,t)}$$

$r \rightarrow$ difference between growth rate and death rate.

By assuming a continuously differentiable function, $x(t)$. $\boxed{\frac{1}{x}\frac{dx}{dt} = r(x,t)}$.

Initially (for simplicity), assume that $r = a$ (constant). Hence $\boxed{\frac{dx}{dt} = ax}$. ($a > 0$).

$\Rightarrow \int \frac{dx}{x} = \int a\,dt \Rightarrow \boxed{\ln x = at + \ln A}$.

when $\boxed{t = t_0,\ x = x_0} \Rightarrow \boxed{\ln A = \ln x_0 - at_0}$.

$\Rightarrow \boxed{x = x_0\, e^{a(t-t_0)}}$ Malthusian Law of Population Growth.

THOMAS ROBERT MALTHUS: An Essay on the Principle of Population.

This law shows an exponential growth.

Between 1700 - 1961, world population <u>Doubled every 35 years</u>, approximately.
In 1961 A.D., $x_0 = 3.06 \times 10^9$ and $a = 2\% = 0.02$.

i.) <u>a was measured from</u> $\boxed{\frac{\Delta x}{x} \cdot \frac{1}{\Delta t} = a}$ which is the percentage increase rate (<u>$t \to$ in years</u>)

ii.) For a population size to double, $\boxed{x = 2x_0}$.

Hence, $$T = t - t_0 = \frac{1}{a} \ln\left(\frac{x}{x_0}\right) = \frac{\ln 2}{a}.$$

$$\Rightarrow T = \frac{1}{0.02} \ln 2 = 50 \ln 2 \simeq 35 \text{ years}.$$

Growth at this rate cannot be sustained in the long run. The <u>Malthusian law</u> fails obviously, when long term growth is considered.

<u>The Logistic Model</u>: (PIERRE FRANÇOIS <u>VERHULST</u>).

$$\boxed{\frac{\Delta x}{x \Delta t} = r(x) = a - bx}$$

i) $a, b > 0$
ii) $r(x)$ becomes small for large x.

$$\Rightarrow \boxed{\frac{dx}{dt} = x(a - bx) = ax\left(1 - \frac{x}{a/b}\right)}$$ The Logistic <u>Equation</u>

Define $K = a/b \rightarrow$ The <u>carrying capacity</u> and set $\boxed{x = \frac{K}{1 + c^{-1} e^{-at}}}$. For $t \to \infty$, $x \to K$ <u>(The upper limit)</u>.

Practical Examples of Population Dynamics

I) The World Population : $\boxed{\frac{1}{x}\frac{dx}{dt} = r = a - bx}$

(A) Here $r = r(x) = \underline{0.02}$ per annum in 1961.
(B) $a = \underline{0.029}$ (ecological estimates). (C) $x = \underline{3.06 \times 10^9}$

Hence $r = a - bx \Rightarrow 0.02 = 0.029 - b(3.06 \times 10^9)$

$\Rightarrow \boxed{b = \frac{0.009}{3.06 \times 10^9} \simeq 3 \times 10^{-12}}$. Numerically b is much smaller than a.

Carrying capacity of the world population, $(K = a/b)$, is $\boxed{K \simeq \frac{a}{b} = \frac{0.029}{3 \times 10^{-12}} \simeq 10^{10} \text{ (10 billion)}}$ Estimate of 1961

II.) Population of the U.S.A. : $\boxed{x = \frac{K}{1 + c^{-1}e^{-at}}}$

Write $\boxed{c^{-1} = e^{at_0}} \Rightarrow \boxed{x = \frac{K}{1 + e^{-a(t - t_0)}}}$ Three unknown parameters, a, K, t_0.

Therefore, Census data were taken for 3 years, 1790, 1850 and 1910 by Pearl and Reed (1920).

$\boxed{a \simeq 0.03, \; b \simeq 1.6 \times 10^{-10}}$ Carrying capacity $K = a/b \simeq 200$ million.

But the present U.S. population is more than 300 million.

How? Pearl and Reed estimated in 1920. But after World War II, the vital coefficients changed; a increased and b decreased. (Belgium showed similar changes). France, however, gave a good match with predictions.

Policy Implications: $\boxed{\frac{1}{x}\frac{dx}{dt} = r(x) = a\left(1-\frac{x}{k}\right)}$

Percentage growth rate $\boxed{r = a\left(1-\frac{x}{k}\right) = a\left(\frac{k-x}{k}\right)}$

i.) When $x \ll k$, $r \simeq a$, ii.) When $x \to k$, $r \to 0$, i.e. $\frac{k-x}{k}$, the fractional space for growth, is reduced. Members within the population come in their way. To maintain a high value of r, either Ⓐ reduce x or Ⓑ increase k (by reducing the value of b).

How? War instincts: Lebensraum, ethnic cleansing, external invasion, increasing national wealth by war and colonisation, preventing immigration.

India is a fertile land, and hence can sustain large populations (in the Ganga Valley)

Criticisms (and Scope for improvement):

i.) Technology, environment and sociological factors are changing rapidly, affecting a and b very rapidly as well. So they need re-calibration more frequently.

ii) Model by subdividing groups according to age and gender.

iii.) Large populations live in congested conditions and suffer outbreaks of epidemics. Population sizes can fluctuate, not according to the logistic law

The Laws of Social Dynamics

Elliott W. Montroll

1/. "First Law": In the absence of any social, economic or ~~ec~~ ecological force, $\boxed{\frac{1}{x}\frac{dx}{dt} = \text{constant}}$ $x \equiv x(t)$ is the population size.

2/. "Second Law": The constancy of $\boxed{\frac{1}{x}\frac{dx}{dt}}$ is violated when a force (social, economic or ecological) is applied. "Force" causes "replacements". Constancy of $\boxed{\frac{1}{x}\frac{dx}{dt}}$ is the Malthusian law. The simplest form of the replacing "force" is the linear function: $\boxed{a-bx}$.

$\Rightarrow \frac{1}{x}\frac{dx}{dt} = a - bx$ (No longer a constant).

$\Rightarrow \boxed{\frac{dx}{dt} = x(a-bx)} \rightarrow$ The Logistic Equation

3/. "Third Law": Evolution is the natural response to a replacement. The "force" brings about change. (Eg. Genetic mutation brings about extinction and replacement of species).

Problem of Sharks and Salmon

$\boxed{\dfrac{dx}{dt} = ax - bx^2 - c}$, $a, b, c > 0$

$\Rightarrow \boxed{\dfrac{dx}{dt} = ax - bx^2 + (-c)}$. Now we already know that a system like $\boxed{\dfrac{dx}{dt} = ax - bx^2 + c}$ can be transformed to a form $\boxed{\dfrac{dy}{dt} = \alpha^2 - by^2}$, in which $\boxed{y = x - \dfrac{a}{2b}}$ and $\boxed{\alpha^2 = \dfrac{a^2}{4b} + c}$. We, thus replace all "c" with "$-c$", i.e. $\boxed{c \longrightarrow -c}$, and rescale further ~~to~~ by $\boxed{X = \dfrac{y}{\alpha/\sqrt{b}}}$ and $\boxed{T = \alpha\sqrt{b}\,t}$ to get $\boxed{\dfrac{dX}{dT} = 1 - X^2}$, whose integral solution is $\boxed{X = \dfrac{A - e^{-2T}}{A + e^{-2T}}}$

A is an integra-tion constant

This is then written as,

$$\boxed{x = \frac{a}{2b} + \frac{\alpha}{\sqrt{b}}\left[\frac{A - e^{-2\alpha\sqrt{b}t}}{A + e^{-2\alpha\sqrt{b}t}}\right]}$$

When $t \to \infty$,

$$x \to \frac{a}{2b} + \frac{1}{\sqrt{b}} \cdot \sqrt{\frac{a^2}{4b} - c} \Rightarrow \boxed{x \longrightarrow \frac{a}{2b}\left(1 + \sqrt{1 - \frac{4bc}{a^2}}\right)}$$

This limiting value of the population does not depend on ~~of~~ the value of A.

Population of New York City : A Critical Case

$$\boxed{\frac{dx}{dt} = ax - bx^2 - c} \qquad \boxed{a, b, c > 0}.$$

$a \rightarrow$ growth parameter, $b, c \rightarrow$ decline parameters.

$$\boxed{a = \frac{1}{25} = 4\times10^{-2}, \; b = \frac{1}{25\times10^6} = 4\times10^{-8}, \; c = 10^4}$$

$$\therefore \frac{4bc}{a^2} = \frac{4\times4\times10^{-8}\times10^4}{4\times4\times10^{-4}} = 1 \Rightarrow a^2 = 4bc \Rightarrow \boxed{\frac{a^2}{4b} - c = 0}$$

$$\Rightarrow \boxed{\alpha^2 = 0} \Rightarrow \boxed{\frac{dy}{dt} = -by^2} \Rightarrow \int y^{-2}dy = -b\int dt$$

$$\Rightarrow \boxed{-y^{-1} = -bt + \text{Constant}}. \Rightarrow \boxed{\frac{1}{y} = bt + A}$$ A is the integration constant.

$$\Rightarrow \boxed{y = \frac{1}{bt+A}} \Rightarrow \boxed{x = \frac{a}{2b} + \frac{1}{bt+A}}.$$

When $t \rightarrow \infty$, $y \rightarrow 0$, and $x \rightarrow \frac{a}{2b}$.
This is the limiting value of the population, $\Rightarrow x \rightarrow \frac{4\times10^{-2}}{2\times4\times10^{-8}}$, $x \rightarrow$ 0.5 million.

This convergence is slow as in a power-law. This happens in all critical phenomena, such as phase transitions. Power laws are also seen in gas laws, Zipf's law (GEORGE KINGSLEY ZIPF) and Pareto's law in income and wealth distributions (VILFREDO PARETO). They are SCALE-FREE.

Fall of a Parachute

Also $\boxed{Re = \frac{\rho l v}{\eta}}$

Reynold's Number: $\boxed{Re = \frac{l v}{\nu}}$

$l \to$ characteristic length, $v \to$ characteristic velocity.

$\boxed{\nu = \eta/\rho} \longrightarrow$ Kinematic viscosity, in which ρ is the density, η is the dynamic viscosity.

$\boxed{\nu_{water} \sim 10^{-2} \text{ S.I. units}}$ $\boxed{\nu_{air} \sim 10^{-4} \text{ S.I. units}}$

$\boxed{\text{Drag force, } D \propto v^{\gamma}}$, where v is the velocity.

i/. When $\boxed{R_e \sim 10}$ (low v, high ν), $\boxed{\gamma = 1}$.
ii/. When $\boxed{Re \sim 10^3}$ (high v, low ν), $\boxed{\gamma = 2}$.
(This is due to turbulence in air).
iii/. $10 < R < 10^3$, γ is uncertain. $\gamma = ?$

For a falling parachute, $\gamma = 2$. $\boxed{D = kv^2}$

$\Rightarrow \boxed{m\frac{dv}{dt} = mg - kv^2}$, $k > 0$. We now get, $\boxed{\frac{dv}{dt} = g - \frac{k}{m}v^2}$.

$\Rightarrow \frac{1}{g}\frac{dv}{dt} = 1 - \frac{v^2}{(\sqrt{mg/k})^2}$ We rescale, $\boxed{X = \frac{v}{\sqrt{mg/k}}}$.

$\Rightarrow \frac{\sqrt{mg/k}}{g}\frac{dX}{dt} = 1 - X^2$. Now rescale $\boxed{T = \sqrt{kg/m}\, t}$.

$\Rightarrow \boxed{\frac{dX}{dT} = 1 - X^2}$ The initial condition is $t = 0$, $v = 0$.

$\Rightarrow \boxed{T = 0}$ and $\boxed{X = 0}$ are the rescaled initial conditions.

The integral solution is $\boxed{X = \tanh(T)}$.

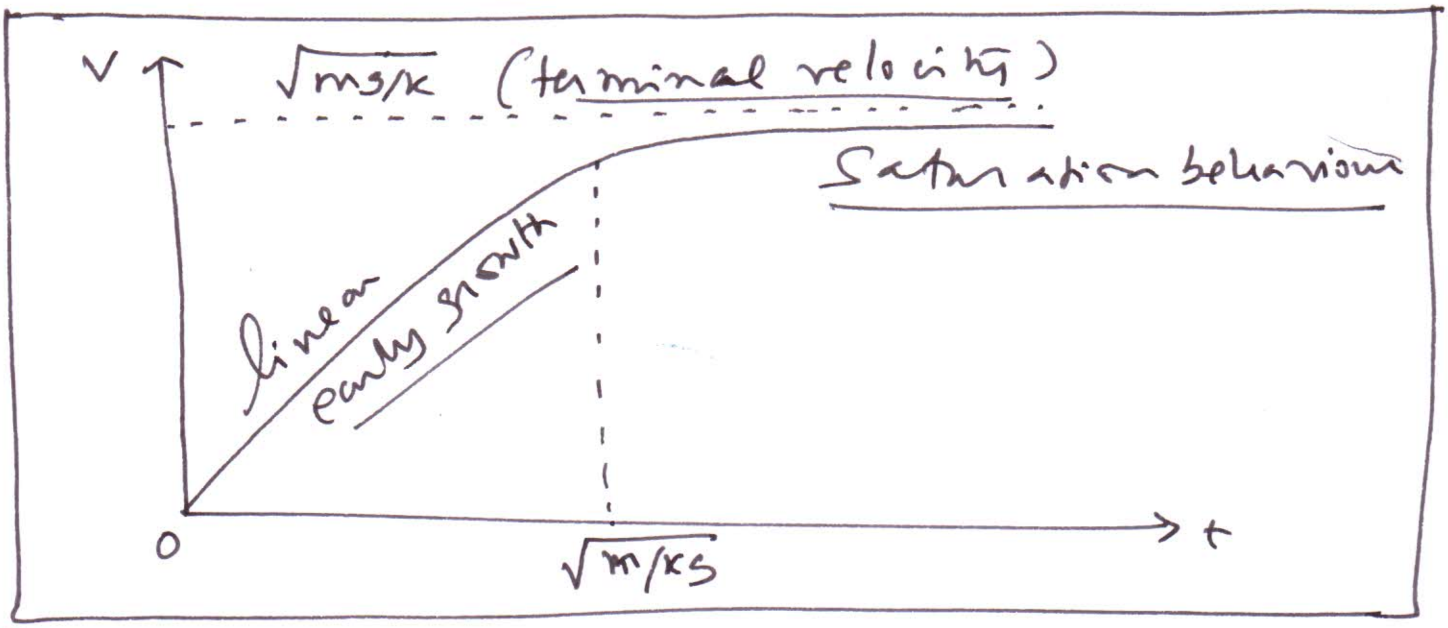
$$\Rightarrow \quad \frac{V}{\sqrt{mg/k}} = \tanh\left(\sqrt{kg/m}\, t\right)$$

$$\Rightarrow \quad \boxed{V = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}\, t\right)}$$

When $t \to \infty$, $V \to \sqrt{\frac{mg}{k}}$.

When $t \to 0$, $\boxed{\tanh\left(\sqrt{\frac{kg}{m}}\, t\right) \simeq \sqrt{\frac{kg}{m}}\, t}$

$$\Rightarrow \quad \boxed{V \simeq \sqrt{\frac{\cancel{m}}{\cancel{k}}}\sqrt{g} \cdot \sqrt{\frac{\cancel{k}}{\cancel{m}}}\sqrt{g}\, t \simeq gt} \quad \text{(linear)}$$



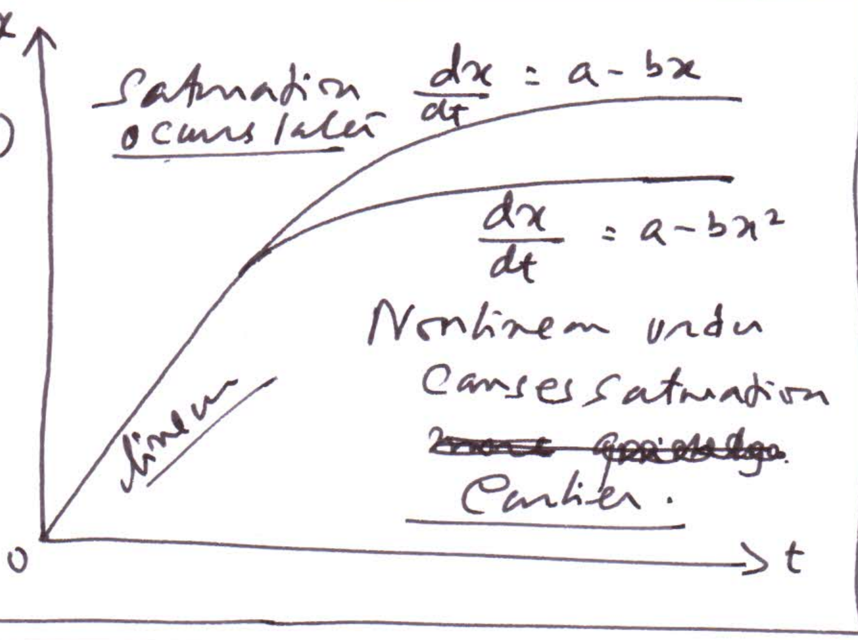
Comparison of $\boxed{\frac{dx}{dt} = a - bx}$ (linear)

and $\boxed{\frac{dx}{dt} = a - bx^2}$

(Nonlinear).

For $t \to 0$,

$\boxed{x \simeq at}$ for both (linear).



Item Response Theory

Singh, Pathak & Pandey

$$\boxed{P(\theta) = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$$ Item Response Function

$\theta \rightarrow$ Ability, $P(\theta) \rightarrow$ Performance Index.
$c \rightarrow$ Probability that a candidate with low ability will respond correctly to an item.
$\omega \rightarrow$ Item discrimination parameter.
$b \rightarrow$ Item difficulty parameter.

Define $\boxed{\phi = P(\theta) - c}$ $\Rightarrow$ $\boxed{\phi = (1-c)\left[1+e^{-\frac{\theta-b}{\omega}}\right]^{-1}}$

$$\Rightarrow \frac{d\phi}{d\theta} = (1-c) \cdot \times -1 \left[1+e^{-\frac{\theta-b}{\omega}}\right]^{-2} \times e^{-\left(\frac{\theta-b}{\omega}\right)} \times -\frac{1}{\omega}$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{(1-c)}{\omega} \cdot \frac{\phi^2}{(1-c)^2} \cdot e^{-\left(\frac{\theta-b}{\omega}\right)}.$$

Now $\boxed{\left[1+e^{-\frac{\theta-b}{\omega}}\right]^{-1} = \frac{\phi}{1-c}}$ $\Rightarrow$ $\boxed{e^{-\frac{\theta-b}{\omega}} = \frac{1-c}{\phi} - 1}$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{(1-c)}{\omega} \cdot \frac{\phi^2}{(1-c)^2} \cdot \left[-1 + \frac{1-c}{\phi}\right].$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{\cancel{(1-c)}}{\omega} \cdot \frac{\phi}{\cancel{1-c}} \left[1 - \frac{\phi}{1-c}\right].$$

$$\Rightarrow \boxed{\frac{d\phi}{d\theta} = \frac{\phi}{\omega}\left[1 - \frac{\phi}{1-c}\right]}$$ $\rightarrow$ The logistic equation.

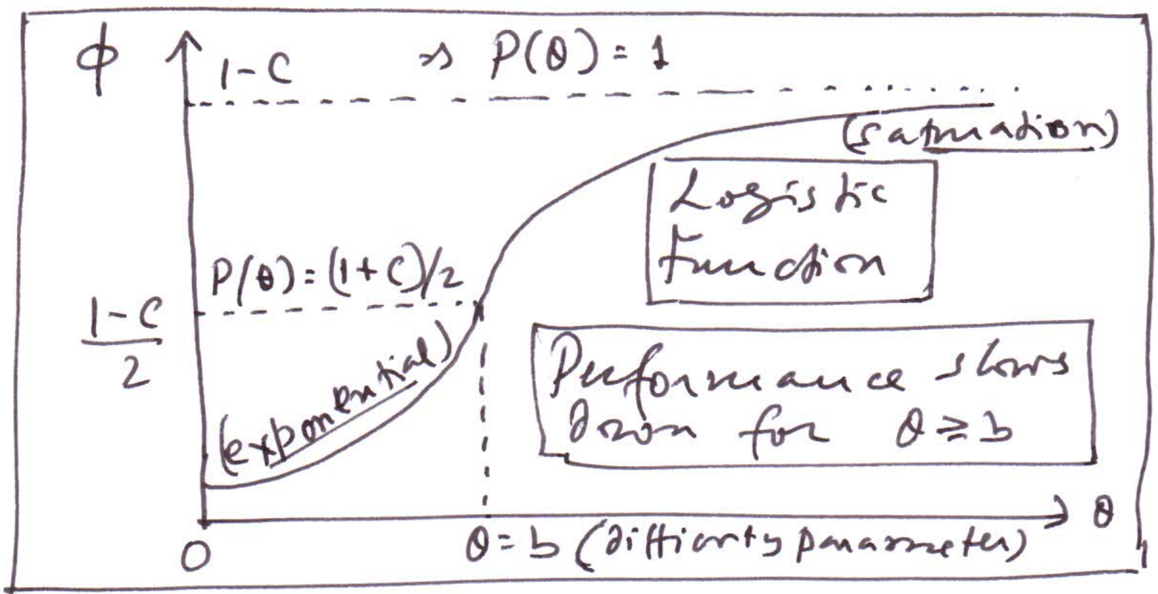
Compare with $\boxed{\frac{dx}{dt} = ax\left(1 - \frac{x}{K}\right)}$. $\Rightarrow$ The limiting value of ϕ is $1-c$ (like carrying capacity)

By definition

$\Rightarrow P(\theta) - c = \phi \rightarrow 1 - c$ when $\theta \rightarrow \infty$.

$\because P(\theta) - c = 1 - c \Rightarrow \boxed{P(\theta) \rightarrow 1}$ when $\theta \rightarrow \infty$.

(Absolutely perfect performance when ability is infinite).



When $\boxed{\phi = \frac{1-c}{2}}$,

$\Rightarrow P - c = \frac{1-c}{2}$

$\Rightarrow \boxed{P = \frac{1+c}{2}}$

Performance starts to decelerate after this value.

Using the $P(\theta)$ function, we write $\frac{1+c}{2} = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}$

$\Rightarrow \boxed{\frac{\cancel{1-c}}{2} = \frac{\cancel{1-c}}{1+e^{-(\theta-b)/\omega}}} \Rightarrow e^{-(\theta-b)/\omega} = 1$

$\Rightarrow \boxed{\theta = b}$ when $\boxed{P = \frac{1+c}{2}}$.

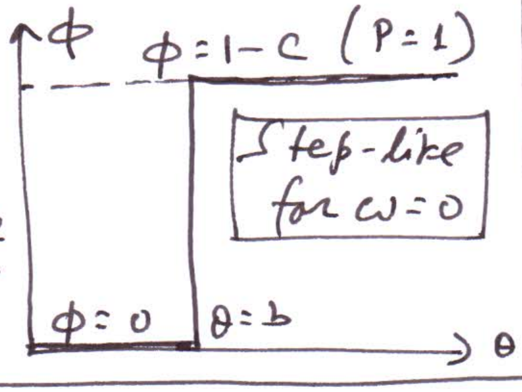
$\Rightarrow$ Beyond $\theta = b$ (item difficulty parameter), performance increases at a decreasing rate (i.e. slows down)

Item Discrimination parameter, ω: When ω is small the logistic function is step-like and steep.

Let $\boxed{\omega = 0}$, in $\boxed{P = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$

i/. when $\theta > b$, $e^{-(\theta-b)/\omega} = e^{-\infty} = 0$, and $P = 1$, $\phi = 1-c$

ii/. when $\theta < b$, $e^{-(\theta-b)/\omega} = e^{\infty} \rightarrow \infty \Rightarrow P = c$, and $\phi = 0$



The Spread of Technological Innovations

Agricultural Innovation :

> i) Total number of farmers in a farming Community is N.
> ii) $x(t) \to$ Number of farmers who have adopted an innovation.
> iii) $N-x(t) \to$ Number of farmers who have not adopted the innovation.

$\boxed{\Delta x \propto \Delta t}$, $\boxed{\Delta x \propto x}$ and $\boxed{\Delta x \propto (N-x)}$

$\therefore \ \Delta x \propto x(N-x)\Delta t \quad \Rightarrow \boxed{\frac{\Delta x}{\Delta t} \equiv \frac{dx}{dt} = C\,x(N-x)}$.

The initial condition is $\boxed{x(0) = 1}$. $\underline{(C>0)}$

Rescale : $\dfrac{d(x/N)}{dt} = CN\,\dfrac{x}{N}\left(1-\dfrac{x}{N}\right)$. the logistic equation,

define $\boxed{X = x/N}$ and $\boxed{T = CNt}$ to get

$\boxed{\frac{dX}{dT} = X(1-X)}$ whose solution is $\boxed{X = \frac{1}{1+A^{-1}e^{-T}}}$.

Hence $\boxed{x = \frac{N}{1+A^{-1}e^{-CNt}}}$. when $t=0$, $x=1$.
$\Rightarrow 1 = \frac{N}{1+A^{-1}} \Rightarrow \boxed{A^{-1} = N-1}$.

$$\Rightarrow \quad \boxed{x = \frac{N}{1+(N-1)e^{-CNt}} = \frac{Ne^{CNt}}{(N-1)+e^{CNt}}}.$$

> i) A discrepancy arises due to not accounting for information obtained through the mass media.
> ii) The ~~decelera~~ slowing of the growth rate happens later than expected.

$(c'>0)$

Modification: $\boxed{\Delta x = c'(N-x)\Delta t}$ a

(Correction due to ~~xxx~~ impersonal communication.)

The total effect is $\boxed{\Delta x = cx(N-x)\Delta t + c'(N-x)\Delta t}$.

$$\Rightarrow \frac{\Delta x}{\Delta t} = (cx + c')(N-x) \Rightarrow \boxed{\frac{dx}{dt} = N(cx+c')\left(1-\frac{x}{N}\right)}$$

$$\Rightarrow \boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c}\right)\left(1-\frac{x}{N}\right)}, \quad \frac{c'}{c} > 0.$$

Early growth: When $\boxed{x \ll N}$, $\boxed{\frac{dx}{dt} \simeq Nc\left(x + \frac{c'}{c}\right)}$.

i.) Quicker than exponential, if $c'/c > 0$.
ii.) Slower than exponential, if $c'/c < 0$.

Since $c', c > 0$, the non-human intervention boosts early growth of the function, $x(t)$.

$$\Rightarrow \frac{dx}{dt} = Nc\left(x + \frac{c'}{c} - \frac{x^2}{N} - \frac{c'}{c}\frac{x}{N}\right)$$

$$\Rightarrow \frac{dx}{dt} = Ncx + Nc' - cx^2 - c'x$$

$$\Rightarrow \frac{dx}{dt} = -\left[cx^2 - (Nc-c')x - Nc'\right]$$

$$\Rightarrow \frac{dx}{dt} = -\left[(\sqrt{c}x)^2 - 2\,\frac{1}{2}\cdot\frac{\sqrt{c}x}{\sqrt{c}}(Nc-c') + \frac{(Nc-c')^2}{4c} - \frac{(Nc-c')^2}{4c} - Nc'\right]$$

$$\Rightarrow \frac{dx}{dt} = -\left[\sqrt{c}x - \frac{(Nc-c')}{2\sqrt{c}}\right]^2 + \left[Nc' + \frac{(Nc-c')^2}{4c}\right]$$

$$\Rightarrow \frac{dx}{dt} = -c\left[x - \frac{(Nc-c')}{2c}\right]^2 + \left[Nc' + \frac{(Nc-c')^2}{4c}\right]$$

Define $\boxed{y = x - \frac{(Nc - c')}{2c}}$ and $\boxed{\alpha^2 = Nc' + \frac{(Nc-c')^2}{4c}}$

Hence $\frac{dy}{dt} = \frac{dx}{dt} \Rightarrow \boxed{\frac{dy}{dt} = \alpha^2 - cy^2}$

$\Rightarrow \frac{1}{\alpha^2}\frac{dy}{dt} = 1 - \frac{y^2}{\alpha^2/c}$. Rescaling now $\boxed{X = \frac{y}{\alpha/\sqrt{c}}}$ and $\boxed{T = \alpha\sqrt{c}\,t}$,

we get $\boxed{\frac{dX}{dT} = 1 - X^2}$, whose solution is

known to be $\boxed{\left(\frac{1+X}{1-X}\right) = Ae^{2T}}$. The initial condition is

when $\boxed{t=0,\ x=0}$. $\Rightarrow \boxed{y_0 = -\frac{Nc-c'}{2c} = \frac{1}{2}\left(\frac{c'}{c} - N\right)}$

Hence, $\boxed{X_0 = \frac{y_0}{\alpha/\sqrt{c}}}$. Making X the subject of T,

we get, $\boxed{X = \frac{Ae^{2T} - 1}{Ae^{2T}+1}} \Rightarrow \boxed{y = \frac{\alpha}{\sqrt{c}}\left(\frac{Ae^{2\alpha\sqrt{c}t} - 1}{Ae^{2\alpha\sqrt{c}t}+1}\right)}$.

Now $4c\alpha^2 = 4Ncc' + (Nc - c')^2$ (from the definition of α)

$\Rightarrow 4c\alpha^2 = 4Ncc' + N^2c^2 - 2Ncc' + c'^2 = (Nc + c')^2$

$\Rightarrow \boxed{2\alpha\sqrt{c} = (Nc + c')}$ & $\frac{\alpha}{\sqrt{c}} = \frac{2\alpha\sqrt{c}}{2c} = \frac{Nc+c'}{2c}$.

$\Rightarrow \boxed{\frac{\alpha}{\sqrt{c}} = \frac{1}{2}\left(N + \frac{c'}{c}\right)}$. ~~With~~ Further when

$t = 0\ (T=0)$ and $x = 0\ (X = X_0)$, $\boxed{A = \frac{1+X_0}{1-X_0}}$.

Therefore, $A = \frac{1 + y_0/\alpha\sqrt{c}}{1 - y_0/\alpha\sqrt{c}} = \frac{\alpha/\sqrt{c} + y_0}{\alpha/\sqrt{c} - y_0}$

Now: $\boxed{y_0 + \frac{\alpha}{\sqrt{c}} = \frac{1}{2}\frac{c'}{c} - \cancel{\frac{N}{2}} + \cancel{\frac{N}{2}} + \frac{c'}{2c} = \frac{c'}{c}}$.

Also,

$$\boxed{\frac{\alpha}{\sqrt{c}} - y_0 = \frac{N}{2} + \cancel{\frac{c'}{2c}} - \cancel{\frac{1}{2}\frac{c'}{c}} + \frac{N}{2} = N}.$$

Hence, $\boxed{A = \frac{\alpha/\sqrt{c} + y_0}{\alpha/\sqrt{c} - y_0} = \frac{c'}{cN}}$. Hence we write the final integral solutions as $y = x - \frac{(Nc - c')}{2c} = \frac{\alpha}{\sqrt{c}}\left(\frac{Ae^{2\alpha\sqrt{c}t} - 1}{Ae^{2\alpha\sqrt{c}t} + 1}\right)$

Substituting for A, $2\alpha\sqrt{c}$, and $\alpha/\sqrt{c}$, we get,

$$\boxed{x = \frac{Nc - c'}{2c} + \frac{1}{2}\left(N + \frac{c'}{c}\right) \cdot \frac{(c'/Nc)e^{(Nc+c')t} - 1}{(c'/Nc)e^{(Nc+c')t} + 1}}.$$

$$\Rightarrow x = \frac{Nc - c'}{2c} + \frac{Nc + c'}{2c} \cdot \frac{c'e^{(Nc+c')t} - Nc}{c'e^{(Nc+c')t} + Nc}$$

$$\Rightarrow x = \frac{(Nc - c')\left[c'e^{(Nc+c')t} + Nc\right] + (Nc + c')\left[c'e^{(Nc+c')t} - Nc\right]}{2c\left[c'e^{(Nc+c')t} + Nc\right]}$$

$$\Rightarrow x = \frac{Ncc'e^{(Nc+c')t} - \cancel{c'^2 e^{(Nc+c')t}} + \cancel{(Nc)^2} - Ncc' + Ncc'e^{(Nc+c')t} + \cancel{c'^2 e^{(Nc+c')t}} - Ncc' - \cancel{(Nc)^2}}{2c\left[c'e^{(Nc+c')t} + Nc\right]}$$

$$\Rightarrow x = \frac{\cancel{2}N\cancel{c}c'e^{(Nc+c')t} - 2N\cancel{c}c'}{2\cancel{c}\left[c'e^{(Nc+c')t} + Nc\right]} = \frac{Nc'e^{(Nc+c')t} - Nc'}{Nc + c'e^{(Nc+c')t}}$$

$$\Rightarrow \boxed{x = \frac{Nc'\left[e^{(Nc+c')t} - 1\right]}{cN + c'e^{(Nc+c')t}}}$$ → The integral solution of $\boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c}\right)\left(1 - \frac{x}{N}\right)}$.

The above solution is recast as

$$\boxed{x = \frac{Nc'\left[1 - e^{-(Nc+c')t}\right]}{c' + cNe^{-(Nc+c')t}}}$$ ∴ When $t \to \infty$, $x \to \frac{Nc'}{c'} = N$ | The maximum value.

Industrial Innovations : (Edwin Mansfield)

(Study on coal, iron and steel, brewing and railroads).

i.) Total number of firms in an industry is N.
ii.) $x(t) \rightarrow$ Number of firms that have adopted a technological innovation.

$\boxed{\Delta x \propto \Delta t}$ and $\boxed{\Delta x \propto (N-x)}$ $\Rightarrow \Delta x \propto (N-x)\Delta t$

Jointly, we write $\boxed{\Delta x = \lambda (N-x)\Delta t}$,

in which $\lambda \rightarrow$ proportional factor (not constant)

$\boxed{\lambda = \lambda\left(p, s, \frac{x}{N}\right)}$, in which,

i.) $p \rightarrow$ profitability in investing in an innovation.
ii.) $s \rightarrow$ Investing ability to acquire innovation, as a percentage of the total assets.
iii.) $\frac{x}{N} \rightarrow$ Percentage of firms that have already adopted the innovation.

Edwin Mansfield's Study: (To determine λ).

i.) Carry out a Taylor expansion of λ about some equilibrium values of p, s and x/N, represented with a subscript c $\left(p_c, s_c, \frac{x}{N}\Big|_c\right)$.
ii.) Limit the Taylor expansion only upto the second order, (i.e. orders of p^2, s^2, $\left(\frac{x}{N}\right)^2$.).
iii.) Gather all the coefficients of zeroth, first and second orders.

Accordingly $\boxed{\lambda = f\left(p, s, \frac{x}{N}\right)}$ is Taylor expanded as,

$$\boxed{\begin{aligned}\lambda &= f\left(p_c, s_c, \frac{x}{N}\Big|_c\right) \\ &+ \frac{\partial f}{\partial p}\Big|_c (p-p_c) + \frac{\partial f}{\partial s}\Big|_c (s-s_c) + \frac{\partial f}{\partial (x/N)}\Big|_c \left(\frac{x}{N} - \frac{x}{N}\Big|_c\right) \\ &+ \frac{1}{2!}\frac{\partial^2 f}{\partial p^2}\Big|_c (p-p_c)^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial s^2}\Big|_c (s-s_c)^2 + \frac{\partial^2 f}{\partial^2 (x/N)}\Big|_c \left(\frac{x}{N} - \frac{x}{N}\Big|_c\right)^2 \\ &+ \frac{2}{2!}\frac{\partial^2 f}{\partial p \partial s}\Big|_c (p-p_c)(s-s_c) + \frac{2}{2!}\frac{\partial^2 f}{\partial p\, \partial (x/N)}\Big|_c (p-p_c)\left(\frac{x}{N} - \frac{x}{N}\Big|_c\right) \\ &+ \frac{2}{2!}\frac{\partial^2 f}{\partial s\, \partial (x/N)}\Big|_c (s-s_c)\left(\frac{x}{N} - \frac{x}{N}\Big|_c\right) + \cdots\end{aligned}}$$

In deriving the above expression we have used the mathematical principle $\boxed{\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}}$.

We set on collecting ~~Collecting~~ all the terms of the same order,

$$\boxed{\begin{aligned}\lambda = a_1 + a_2 p + a_3 s + a_4\left(\frac{x}{N}\right) + a_5 p^2 + a_6 s^2 + a_7 ps \\ + a_8 p\left(\frac{x}{N}\right) + a_9 s\left(\frac{x}{N}\right) + a_{10}\left(\frac{x}{N}\right)^2\end{aligned}},$$

in which all a_i are constants, depending on the equilibrium values of p, s and $\frac{x}{N}$, and their derivatives.

Edwin Mansfield's study shows $\boxed{a_{10} = 0}$ and

$$\boxed{a_1 + a_2 p + a_3 s + a_5 p^2 + a_6 s^2 + a_7 ps = 0}.$$

The remaining terms $\boxed{\lambda = (a_4 + a_8 p + a_9 s)\frac{x}{N}}$

Define $\boxed{k = a_4 + a_8 p + a_9 s}$ $\Rightarrow$ $\boxed{\lambda = k\frac{x}{N}}$.

$\boxed{K \equiv k(p, s)}$, i.e, k depends on profitability and investing power. k is not to be confused with the carrying capacity in the logistic equation.

$\therefore$ $\boxed{\Delta x = k \frac{x}{N}(N-x)\Delta t}$ $\Rightarrow$ $\frac{\Delta x}{\Delta t} = k\frac{x}{N}(N-x)$

$\Rightarrow$ $\frac{dx}{dt} = k\frac{x}{N}(N-x)$ $\Rightarrow$ $\boxed{\frac{d(x/N)}{dt} = \frac{kx}{N}\left(1-\frac{x}{N}\right)}$

Define $\boxed{X = \frac{x}{N}}$ and $\boxed{T = kt}$, to get

$\boxed{\frac{dX}{dT} = X(1-X)}$, which is the logistic equation.

The solution is $\boxed{X = \frac{1}{1+A^{-1}e^{-T}}}$ $\Rightarrow$ $\boxed{x = \frac{N}{1+A^{-1}e^{-kt}}}$.

Initial condition: when $\boxed{t = t_0,\ x = 1}$.

$\Rightarrow$ $1 = \frac{N}{1+A^{-1}e^{-kt_0}}$ $\Rightarrow$ $\boxed{A^{-1} = (N-1)e^{kt_0}}$

$\Rightarrow$ $\boxed{x = \frac{N}{1+(N-1)e^{-k(t-t_0)}}}$ The integral solution for spread of industrial innovations.

This solution was used to study:

i) The spread of twelve innovations such as the shuttle car, trackless mobile loaders, mining machines, coke ovens, wide strip mills, etc.

ii) Across four major industries like coal, iron and steel, brewing and railroads.

The Dynamics of Free-Living Dividing Cell Growth

$\boxed{x \equiv x(t)} \rightarrow$ Volume of dividing cells at time t.

The growth rate equation is $\boxed{\frac{dx}{dt} = \lambda x}$ $(\lambda > 0)$.

At $\boxed{t = t_0\,,\; x = x_0} \longrightarrow$ Initial condition

$\Rightarrow \boxed{x(t) = x_0 \exp\left[\lambda(t - t_0)\right]}$. Cell doubling happens when $\boxed{x = 2x_0} \Rightarrow$ Doubling time $\boxed{t - t_0 = \frac{\ln 2}{\lambda}}$

Gompertz Law of Tumour Growth

$$\boxed{\frac{dx}{dt} = f(x) = -ax\ln(bx)} \quad a, b > 0.$$

$x(t) \rightarrow$ Number of cells in a tumour.

Scale $\boxed{y = x/b^{-1}}$ and $\boxed{T = at}$.

Rescaling:

$$\Rightarrow \frac{d\left(\frac{x}{b^{-1}}\right)}{d(at)} = -\left(\frac{x}{b^{-1}}\right)\ln\left(\frac{x}{b^{-1}}\right) \Rightarrow \boxed{\frac{dy}{dT} = -y\ln y}$$

Integral Solution: Substitute $\boxed{y = e^X} \Rightarrow \boxed{X = \ln y}$

$$\therefore \frac{dy}{dT} = e^X \frac{dX}{dT} = y\frac{dX}{dT} = -y\ln y = -yX$$

$$\Rightarrow \cancel{y}\frac{dX}{dT} = -\cancel{y}X \quad \Rightarrow \boxed{\frac{dX}{dT} = -X}.$$

$$\Rightarrow \int \frac{dX}{X} = -\int dT \quad \Rightarrow \boxed{X = X_0 e^{-T}}$$

X_0 is the integration constant

$\Rightarrow \ln y = X_0 e^{-T} \Rightarrow \ln(bx) = X_0 e^{-at}$

$\Rightarrow \boxed{x = \frac{1}{b} \exp\left(X_0 e^{-at}\right)}$. Exponential of an exponential.

i.) When $\boxed{t \to \infty \quad x \to b^{-1}}$.

ii.) When $\boxed{t = 0, \quad x = x_0}$ (initial value).

$\therefore \quad x_0 = \frac{1}{b} \exp(X_0) \Rightarrow e^{X_0} = x_0 b$ ~~[illegible]~~

Fixing the unknown $X_0 \Rightarrow \boxed{X_0 = \ln(x_0 b)}$

Since $\boxed{x_0 < b^{-1}}$ (the tumour GROWS),

$\Rightarrow \boxed{\frac{x_0}{b^{-1}} < 1} \Rightarrow \boxed{X_0 = \ln\left(\frac{x_0}{b^{-1}}\right) < 0}$.

Hence, $\boxed{x = \frac{1}{b} \exp\left[\ln\left(\frac{x_0}{b^{-1}}\right) e^{-at}\right]}$,

the Gompertz formula for tumour growth, which satisfies over 1000-fold growth.

We differentiate the $x \equiv x(t)$ equation to get.

$$\frac{dx}{dt} = \frac{1}{b} \exp\left[\ln\left(\frac{x_0}{b^{-1}}\right) e^{-at}\right] . \ln\left(\frac{x_0}{b^{-1}}\right) e^{-at} . (-a)$$

Now $\ln\left(\frac{x_0}{b^{-1}}\right) < 0 \quad \therefore \boxed{-a \ln\left(\frac{x_0}{b^{-1}}\right) = -a X_0 > 0}$

We write $\boxed{\lambda = -a X_0 > 0}$ to get $\boxed{\frac{dx}{dt} = x \lambda e^{-at}}$.

This equation is in the form $\boxed{\frac{dx}{dt} = f(x, t)}$

The non-autonomous form $\frac{dx}{dt} = f(x,t)$.
Can be ~~[illegible]~~ cast in two ways. They are:

i.) $$\boxed{\frac{dx}{dt} = \left(\lambda e^{-at}\right)x = \bar{\lambda}(t)\,x}$$ $\bar{\lambda}$ depends on t.

ii.) OR $$\boxed{\frac{dx}{dt} = \lambda\left(x e^{-at}\right)}$$ λ is a constant.

First form: $\boxed{\frac{dx}{dt} = \bar{\lambda}(t)\,x}$ (Rate $\propto$ State)

The time scale for tumour generation is $\boxed{\bar{t} \sim \frac{1}{\bar{\lambda}}}$ (On comparing with $\boxed{t - t_0 = \frac{\ln 2}{\lambda}}$ in free-living and dividing cells)

$\Rightarrow \boxed{\bar{t} \sim \lambda^{-1} e^{at}}$ $\Rightarrow$ As t increases, longer time is taken for the same amount of growth. The cells mature and divide more slowly.

Second form: $\boxed{\frac{dx}{dt} = \lambda\left(x e^{-at}\right)}$. λ is constant and now rate is proportional to a state, ~~[illegible]~~ $\boxed{x e^{-at}}$.
This effective state, contributing to the growth of the tumour, decreases due to necrosis at the core of the tumour, with lower number of living cells.

First form: Growth process slows down. [SUMMARY]
Second form: Number of cells in the growth is lower.

Bateria versus Toxin: (A non-autonomous system)

$x(t) \rightarrow$	Number of bacteria at time, t.
$T(t) \rightarrow$	Amount of toxin at time, t.

i.) In the absence of toxins bacteria grow, $\boxed{\frac{dx}{dt} = bx}$. $(b>0)$.

ii.) In the presence of toxins, bacteria die out. $\boxed{\frac{dx}{dt} = -axT}$ $(a>0)$.

iii.) Growth rate of toxins, $\boxed{\frac{dT}{dt} = c}$. $(c>0)$.

$\Rightarrow \boxed{T = ct + k}$. Initial condition: When $t=0$, $T=0 \Rightarrow k=0 \Rightarrow \boxed{T = ct}$.

$\therefore \boxed{\frac{dx}{dt} = -axct}$ in the presence of toxins.

Combined Equation: $\boxed{\frac{dx}{dt} = bx - axct}$

$\Rightarrow \boxed{\frac{dx}{dt} = f(x,t) = x(b-act)}$ Non-autonomous Equation

Integral Solution: $\int \frac{dx}{x} = \int (b-act)\,dt$

$\Rightarrow \ln x = \ln x_0 + bt - \frac{act^2}{2}$ | x_0 is a integration constant.

$\Rightarrow \boxed{x = x_0 \exp\left[bt - \frac{act^2}{2}\right]}$. From this solution we see that when $\boxed{t=0,\ x=x_0}$ (initial condition). Further when $t \rightarrow \infty$, the square power dominates and $x \rightarrow 0$ (the limiting condition).

Now we write $bt - \frac{act^2}{2} = \frac{2bt - act^2}{2}$.

This is $-\frac{1}{2}\left[(\sqrt{ac}\,t)^2 - 2\frac{b}{\sqrt{ac}}\sqrt{ac}\,t + \frac{b^2}{ac} - \frac{b^2}{ac}\right]$

which can be written as a square,

$$-\frac{1}{2}\left[\left(\sqrt{ac}\,t - \frac{b}{\sqrt{ac}}\right)^2 - \frac{b^2}{ac}\right] = \frac{b^2}{2ac} - \frac{ac}{2}\left(t - \frac{b}{ac}\right)^2$$

Hence $\boxed{x = x_0\, e^{b^2/2ac} \times \exp\left[-\frac{ac}{2}\left(t - \frac{b}{ac}\right)^2\right]}$

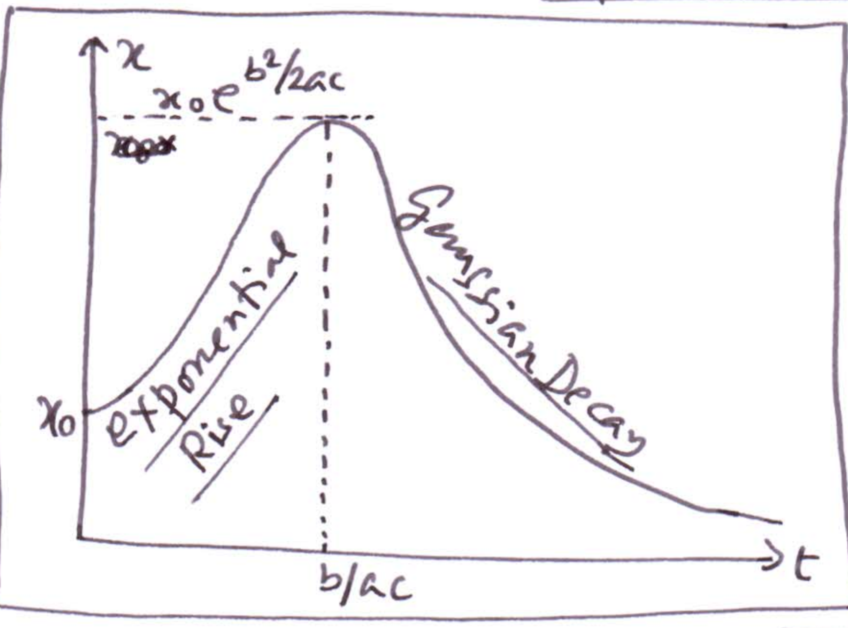
Clearly, i.) When $\boxed{t = 0,\ x = x_0}$, ii.) When $\boxed{t \to \infty,\ x \to 0}$, and iii.) When $\boxed{t = \frac{b}{ac}}$, $\boxed{x = x_0 e^{b^2/2ac} > x_0}$

Looking at $\boxed{\frac{dx}{dt} = x(b - act)}$, we see that when $t = b/ac$, $\frac{dx}{dt} = 0$.

Hence $\boxed{t = b/ac}$ has a turning point for dx/dt.

The Second Derivative: $\frac{d^2x}{dt^2} = \frac{dx}{dt}(b - act) + x(-ac)$

When $t = b/ac$, $\boxed{\frac{d^2x}{dt^2} = -acx < 0}$ This is the condition for a maximum value of $x(t)$.



Rescale: $\boxed{X = x/x_0}$ and $\boxed{T = t/(b/ac)}$. This gives

$$\boxed{X = e^{b^2/2ac} \times \exp\left[-\frac{b^2}{2ac} \times (T-1)^2\right]}.$$

i.) For $T < 1$, early growth is exponential.

ii.) For $T > 1$, the decay is Gaussian.