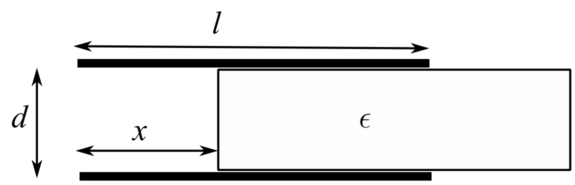
1. A slab of linear dielectric material is partially inserted between the plates of a parallel plate capacitor whose area is l^2 and the distance between the plates is d. Find the force by which the slab is sucked inside the capacitor while it is charged to a potential V.



soln
The width of the capacitor is l and let x be the length over which the capacitor is filled with air while $l-x$ be the length over which it is filled with the dielectric material of permitivity ϵ. The capacitance of this configuration can be shown to be (looked upon as two capacitors connected in parallel):

$$\begin{aligned} C &= \frac{\epsilon_0 l^2}{d}\frac{x}{l} + \frac{\epsilon l^2}{d}\frac{l-x}{d} \\ &= \frac{l}{d}\left[\epsilon l + x(\epsilon_0 - \epsilon)\right] \end{aligned}$$

The work done in charging the capacitor to potential V is

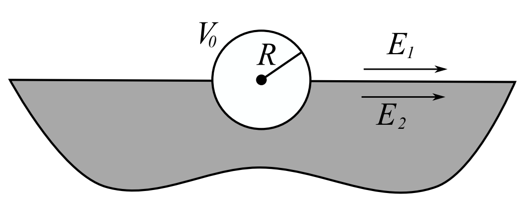
$$W = \frac{1}{2}CV^2 = \frac{lV^2}{2d}\left[\epsilon l + x(\epsilon_0 - \epsilon)\right]$$

The force acting on the dielectric slab is given as

$$F = -\frac{\partial W}{\partial x} = -\frac{V^2}{2}\frac{\partial C}{\partial x} = \frac{V^2}{2}\frac{l}{d}\left(\epsilon - \epsilon_0\right)$$

This is the force with which the dielectric slab is pulled in.

2. A conducting sphere at potential V_0 is half embedded in linear dielectric material of susceptibility χ_e, which occupies the region $z < 0$. Justify that the potential everywhere is exactly the same as it would have been in the absence of the dielectric.



soln
The tangential component of the electric field at the interface of the dielectric and the

air have to be continuous. So $E_1 = E_2 = E$ as shown in the figure. The electric field orginates at the surface of the conducting sphere and is given by $-\vec{\nabla}\Phi$. The boundary condition on Φ is $\Phi = V_0$ on the surface of the sphere. Moreover there is no other charges anywhere. This is because the dielectric is homogeneous. If χ_e is the electric susceptibility of the dielectric material then $\vec{\nabla}\cdot\vec{P} = \frac{\chi_e}{1+\chi_e}\vec{\nabla}\cdot\vec{D} = \frac{\chi_e}{1+\chi_e}\rho_f$. Since there is no free charge anywhere except on the conductor, $\vec{\nabla}\cdot\vec{P} = 0$. So there is no bound charge in the dielectric except at the interface with the sphere. This makes the electric field radial and spheriacally symmetric. This is precisely the solution for the sphere maintained at potential V_0 in free space. So the potential everywhere is the same as in the absence of the dielectric.

Note: The free surface charge on the conducting surface is however not uniformly distributed. In the upper hemisphere $\sigma_1 = \epsilon_0 E$ while in the lower hemisphere $\sigma_2 = D = \epsilon E$. σ_2 will be reduced by the bound surface charges over the interface of the conductor and the dielectric.

3. (a) The electrostatic energy in a uniform dielectric medium is given as $\frac{\epsilon_0\epsilon_r}{2}\int_\tau |E|^2 d\tau$ whereas in free space it is given as $\frac{\epsilon_0}{2}\int_\tau |E|^2 d\tau$. Since we only account for the contribution to the energy by the free charges in the dielectric, we expect this energy to be lower than the actual electrostatic energy when the energy of the bound charges are also included which will be given by the second expression above. But since $\epsilon_r > 1$ we have

$$\frac{\epsilon_0\epsilon_r}{2}\int_\tau |E|^2 d\tau \quad > \quad \frac{\epsilon_0}{2}\int_\tau |E|^2 d\tau$$

How do you explain this discrepancy?

soln

Suppose we have an electric field $\vec{E}(\vec{r})$ in a region of free space caused due to a configuration of charges at the boundary of this region. Now suppose we have some dielectric medium in this region. Then we need more free charges to jack up the same electric field. This is because, just like the conductors, the dielectric material creates its own charges and electric fields that reduces the local electric field. A conductor completely washes it out. The dielectric can only reduce it through polarization. So we need more free charges to work out the same electric field in the presence of the dielectric. Hence the energy required to build this configuration is more. So the reason for the surprising inequality above is that though we have the same electric field everywhere we don't have the same free charge configuration which we used for calculating the electrostatic energy in a dielectric.

(b) Compare the electrostatic energy of two identical parallel plate capacitors with a charge Q on them. One of the capacitors is filled with a dielectric material of relative permitivity ϵ_r, and the other filled with free space.

soln:

Since the capacitors have the same charge Q on them it is convenient to express the energy of the capacitors as $\frac{1}{2}QV$. In the presence of the dielectric the electric field in the region between the plates is small. Hence the potential difference is small. So the capacitor with dielectric has less energy compared to the one without.

(c) Compare the electrostatic energy of two identical parallel plate capacitors , both, charged to potential V. One of the capacitors is filled with a dielectric material of relative permitivity ϵ_r, and the other is filled with free space.

soln:
Here it will be easier to express the energy of a capacitor as $\frac{1}{2}CV^2$. The capacitance increases when we insert a dielectric material between the plates. So when the two capacitors are charged up to the same level of potential V, the one with the dielectric material has more energy due to a high capacitance. We need more free charge on the plates to jack up the potential V, hence more energy.

4. For a configuraion of charges and currents confined within a volume $\mathcal{V}$, show that

$$\int_{\mathcal{V}} \vec{\mathrm{J}} d\tau = \frac{d\vec{\mathrm{p}}}{dt}$$

where $\vec{\mathrm{p}}$ is the total dipole moment.

[Hint: Evaluate $\int_{\mathcal{V}} \vec{\nabla} \cdot (x\vec{J}) d\tau$]

soln
The charges and the currents are confined within the volume $\mathcal{V}$. So no current crosses the surface S. Under these conditions we have to show that

$$\int_{\mathcal{V}} \vec{\mathrm{J}} d\tau = \frac{d\vec{\mathrm{p}}}{dt}$$

where $\vec{\mathrm{p}}$ is the total dipole moment of the charge distribution in the volume $\mathcal{V}$. We start with the R.H.S.
Let $\rho(\vec{r}, t)$ be the charge density in the region.

$$\vec{p}(t) = \int_{\mathcal{V}} \vec{r} \rho d\tau$$

$$\therefore \quad \frac{d\vec{p}}{dt} = \int_{\mathcal{V}} \vec{r} \frac{\partial \rho}{\partial t} d\tau = \hat{\imath} \int x \frac{\partial \rho}{\partial t} d\tau + \hat{\jmath} \int y \frac{\partial \rho}{\partial t} d\tau + \hat{k} \int z \frac{\partial \rho}{\partial t} d\tau$$

Let us consider the integral $\int x \frac{\partial \rho}{\partial t} d\tau$. By continuity equation $\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$.

$$\therefore \quad \int x \frac{\partial \rho}{\partial t} d\tau = -\int_{\mathcal{V}} x(\vec{\nabla} \cdot \vec{J}) d\tau$$

$$\vec{\nabla} \cdot (x\vec{J}) = x\vec{\nabla} \cdot \vec{J} + \vec{\nabla} x \cdot \vec{J} = x\vec{\nabla} \cdot \vec{J} + J_x$$

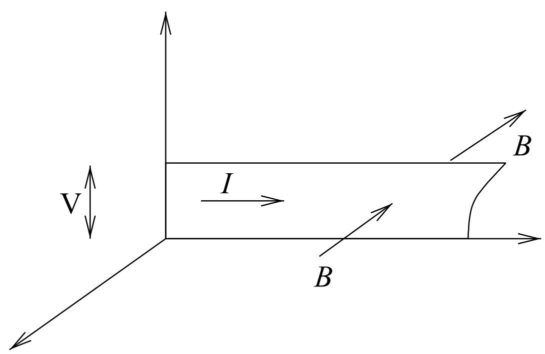
$$\therefore \quad \int_{\mathcal{V}} x \frac{\partial \rho}{\partial t} d\tau = \int_{\mathcal{V}} J_x d\tau - \int_{\mathcal{V}} \vec{\nabla} \cdot (x\vec{J}) d\tau$$
$$= \int_{\mathcal{V}} J_x d\tau - \int_S x\vec{J} \cdot \hat{n} da$$

Since no current crosses the surface S we have $\vec{J} \cdot \hat{n} = 0$.

$$\therefore \quad \int_{\mathcal{V}} x \frac{\partial \rho}{\partial t} d\tau = \int_{\mathcal{V}} J_x d\tau$$

$$\therefore \quad \frac{d\vec{p}}{dt} = \hat{i} \int_{\mathcal{V}} J_x d\tau + \hat{j} \int_{\mathcal{V}} J_y d\tau + \hat{k} \int_{\mathcal{V}} J_z d\tau = \int_{\mathcal{V}} \vec{J} d\tau$$

5. A current I flows through a long rectangular strip of conductor of width a. The surface electron density on the strip is n. If the strip is placed in a magnetic field B perpendicular to its plane, the moving charges constituting the current experiences a force along the width of the strip.
(a) Calculate this force.
(b) As a result of this force charges get accumulated along the two edges of the strip. This produces an electric force which balances the magnetic force and an equilibrium sets in. This phenomenon is called the Hall effect and the electric potential difference between the two edges is called the Hall voltage. Find the Hall voltage in terms of B, I and the electron charge e. This effect is used in the Gaussmeter which measures the magnetic field.



soln:

(a) The given current constitutes of electron moving with velocity v in opposite direction. The force on the electron is upward given by

$$\vec{F}_{mag} = e\vec{v} \times \vec{B} = evB\hat{z}$$

$I = \sigma v a$ where a is the width of the strip and $\sigma = ne$, the charge density.
$\therefore \quad I = neva$
$\therefore \quad F_{mag} = evB = IB/(na)$.
This is the force on the electron in the strip.

(b) Let V be the Hall voltage developed across te edges of the strip. Then the electric field in the strip is $E = V/a$.
The force on an electron due to this field is $F_{elect} = eV/a$.
At equilibrium $F_{elect} = F_{mag}$.

$$\therefore \quad \frac{eV}{a} = \frac{IB}{na} \qquad \Longrightarrow \qquad V = \frac{IB}{ne}$$