# Phase Plot of the Gompertz Law

$$\boxed{\dot{x} = \frac{dx}{dt} = f(x) = -ax\ln(bx)} \quad \underline{a, b > 0}$$

When $\boxed{\dot{x} = 0}$, i.) $\ln(bx) = 0 \Rightarrow bx = 1$

($x_c \to$ fixed points) $\Rightarrow \boxed{x = b^{-1}} \quad \therefore \boxed{x_c = b^{-1}}$.

Also ii) when $x \to 0$, $\boxed{\dot{x} = -\frac{a\ln(bx)}{x^{-1}}}$

both $\boxed{\ln(bx) \to -\infty}$ and $\boxed{x^{-1} \to \infty}$.

We apply L'Hospital Rule when $x \to 0$.

$$\Rightarrow \dot{x} = -\frac{a\cancel{b}(1/\cancel{b}x)}{-x^{-2}} = -\frac{ax^{-1}\big|_{x=0}}{-x^{-2}\big|_{x=0}} = a\frac{\infty}{\infty^2} = 0$$

$\Rightarrow \boxed{\dot{x} \to 0 \text{ when } x \to 0}$ by L'Hospital's Rule

Turning point of $f(x)$: $\boxed{f(x) = -ax\ln(bx)}$

$$\Rightarrow f'(x) = -a\left[\ln(bx) + x\cdot\frac{1}{\cancel{b}x}\cdot\cancel{b}\right] = -a\left[1 + \ln(bx)\right]$$

If $f'(x) = 0$, $\Rightarrow \ln(bx) = -1 \Rightarrow bx = 1/e$.

$\Rightarrow x = 1/be$.

$f''(x) = -a\left[\frac{1}{\cancel{b}x}\cdot\cancel{b}\right] = -a/x$ when $x = 1/be$, $f''(x) = -abe$

Since $f''(x) < 0$ at $x = 1/be \Rightarrow$ Maximum.

i.) $x = 0 \to$ Unstable repeller

ii) $x = 1/b \to$ Stable attractor

iii) The peak has an asymmetric position