Lecture - 11

P 1

Recap:

Hypergeometric random variable

Cumulative distribution function

↑ Discrete random variables

Continuous random variables

e.g. life time of a mobile in hours $= X$

$$f(x) = \begin{cases} 0 & , \quad x \leq 100 \\ \dfrac{100}{x^2} & , \quad x > 100 \end{cases}$$

What is the probability that exactly 2 out of 5 such mobiles will need to be replaced within first 150 hours of use?

$P(x < 150)$ ② ②

$$= \int_{100}^{150} \frac{100}{x^2}\, dx = \frac{1}{3}$$

$p = \frac{1}{3}, n = 5, i = 2$

Binomial.

$$\binom{5}{2} p^2 (1-p)^3$$

Cumulative distribution.

$$F(a) = P(X \leq a) = P(X < a)$$

$$\quad\quad\quad \|$$

$$\int_{-\infty}^{a} f(x)\, dx$$

e.g. $X \to \cancel{f(X)}\ g(X)$

$f(x)$

Given the density function for ~~#~~ X, what is the density function for $g(X)$?

e.g. X: continuous random variable.

$f(x)$

$$Y = 2X$$

Step 1: Compute F_Y

Step 2: differentiate F_Y to get f_Y

$$F_Y(a) = P(Y \le a)$$
$$= P(2X \le a)$$
$$= P(X \le a/2)$$
$$= F_X(a/2)$$

$$f_Y(a) = \frac{d(a/2)}{da} \cdot f_X(a/2)$$

$$\boxed{f_Y(a) = \frac{1}{2} f_X(a/2)}$$

$\Sigma x_i p_i$

Expectation, Variance of a Continuous random variable

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\text{Variance} = E[X^2] - (E[X])^2$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2$$

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e.g.: $f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$E[X]$, $Var(X)$

$$E[X] = \int_0^1 x \cdot 2x \cdot dx = \frac{2}{3}$$

Feb 29, Saturday 10am to 1pm | Insem 2, lab building.

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Var (X)

$$= \int_0^1 x^2 \cdot 2x \cdot dx - \frac{4}{9}$$

$= 1/18$

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e.g. $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$

Compute $E[e^X]$

Let $Y = e^X$

Compute $E[Y]$.

1. Compute F_Y
2. differentiate it to get f_Y
3. Compute $\int y \, f_Y \, dy = E[Y]$

$$F_Y(a) = P(Y \leq a)$$

$$= P(e^X \leq a)$$

$$= P(X \leq \log a) = \log a$$

Compute $F_X(a)$

$$F_X(a) = P(X \leq a)$$

$$= \int_{-\infty}^{a} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{a} f(x)\,dx$$

$$F_X(a) = a$$

$$f_Y(a) = \frac{d}{da} \log a = \frac{1}{a}$$

$f_Y(a) = \frac{1}{a};$

$$E[Y] = \int_{e^0}^{e^1} y \cdot \frac{1}{y} \cdot dy$$

$$= e - 1$$

Theorem:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

$$= \int_0^1 e^x \cdot dx = e - 1$$

Now, we prove this theorem.

Assume that $g(x)$ is non-negative.

<u>Lemma</u>: Y is a non-negative random ~~x~~ variable.

$$E[Y] = \int_0^\infty P(Y > y)\, dy$$

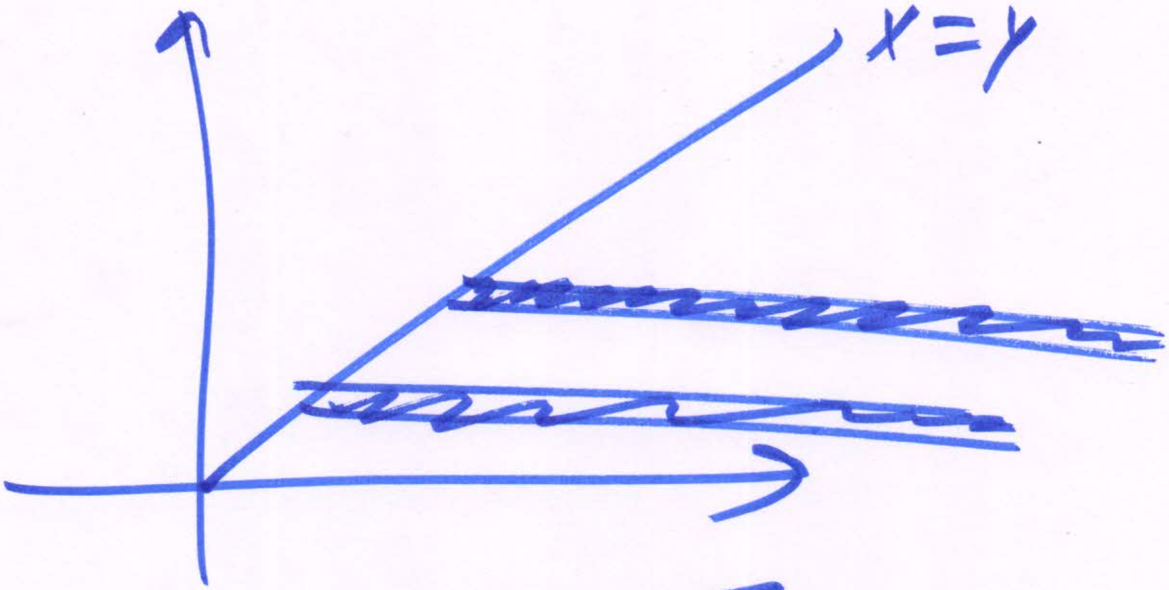
$$\uparrow$$

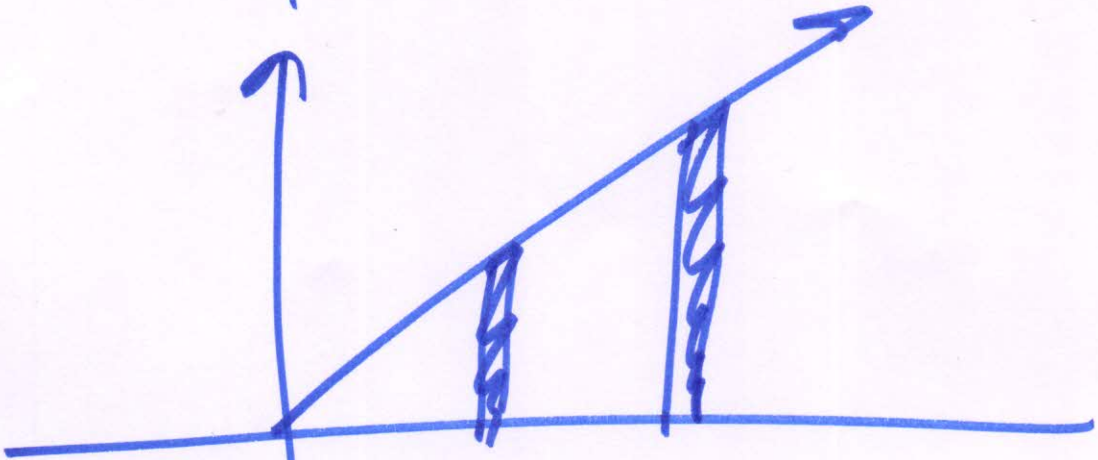
$$P(Y > y) = \int_y^\infty f_Y(x)\, dx$$

$E[Y] =$

$$\int_0^\infty \int_y^\infty f_Y(x)\, dx\, dy$$

over y over x





$$\int_0^\infty \underbrace{\left[\int_0^x dy \right]}_{= x} f_Y(x)\, dx = E[Y]$$

0 over x

0 over y

$E[g(X)]$

$$= \int_0^\infty \underbrace{P(g(X) > y)}\, dy$$

$$P(g(X) > y) = \int_{x:\, g(x) > y} f(x)\, dx$$

$$= \int_0^\infty \int_{x:\, g(x) > y} f(x)\, dx$$

over y

over x

change the order of integration

$$\int_0^\infty \int_{x:\, g(x)>y} f(x)\, dx\, dy$$

$Y \quad X$

$$\int_{g(x)>0} \left(\int_0^{g(x)} dy \right) f(x)\, dx$$

$X \quad Y$

$$= \int_{g(x)>0} g(x) f(x)\, dx$$

$$= \int_0^\infty g(x) f(x)\, dx$$

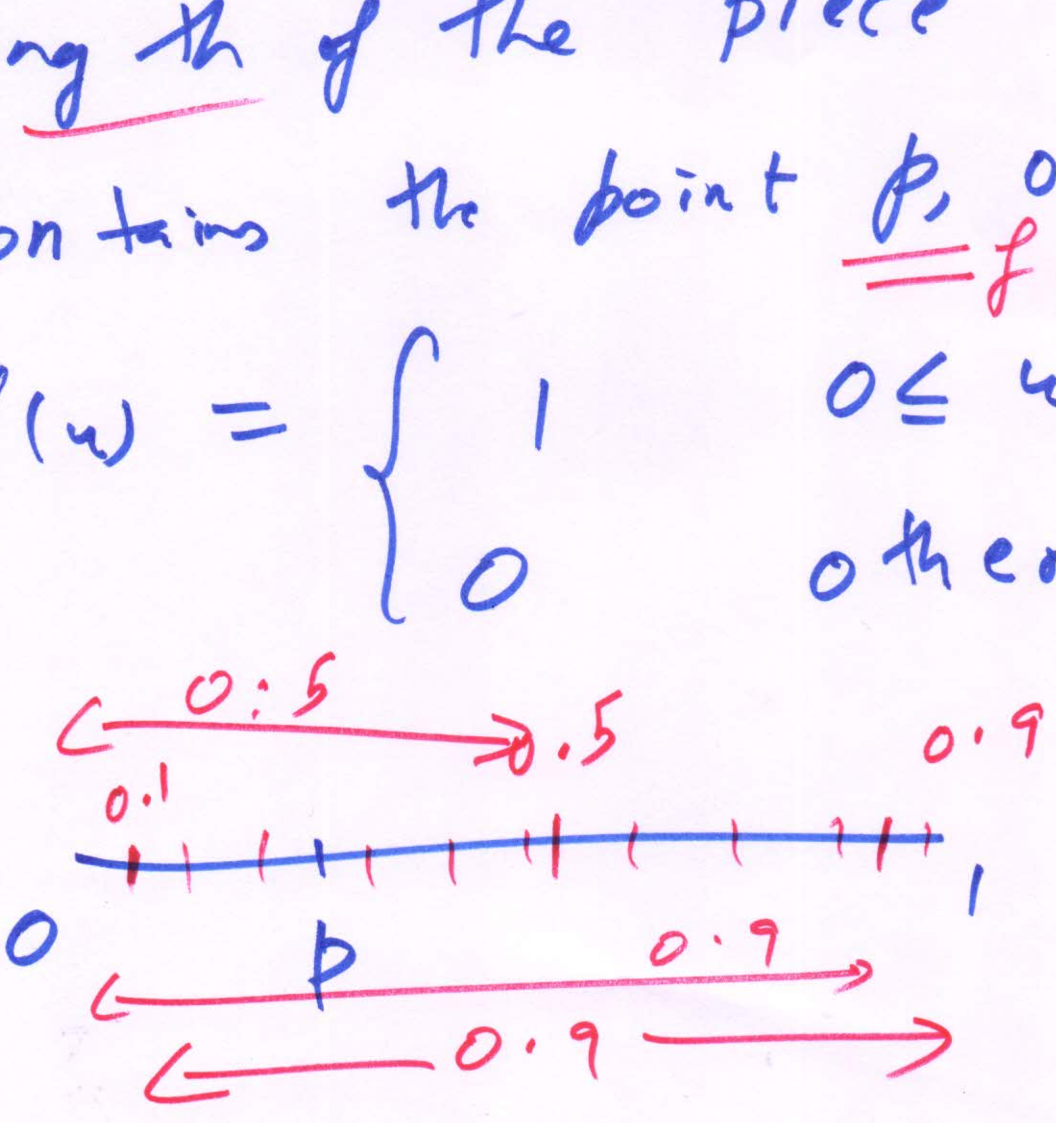
$$= E[g(x)]$$

e.g

A stick of length 1 is split at a point U, which is uniformly distributed over $(0,1)$. Determine the expected length of the piece that contains the point p, $0 \le p \le 1$.

p = fixed

$$f(u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \text{otherwise.} \end{cases}$$



let L be the length (14)
of the piece of the stick,
that contains p