Tutorial - 8

1. Show that when X and Y are jointly Gaussian with zero mean & PDF

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\left[\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right]}$$

then, $f_y(y/x)$ is also Gaussian.

$\Rightarrow$

$$f_y(y/x) = \frac{f_{xy}(x, y)}{f_x(x)}$$

$$= \frac{\dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\left[\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma^2}\right)\right]}}{\dfrac{1}{\sqrt{2\pi}\sigma_1} e^{-x^2/2\sigma_1^2}}$$

$$= \frac{1}{\sigma_2\sqrt{2\pi(1-\rho^2)}} \exp\left(-\left[\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right) + \frac{x^2}{2\sigma_1^2}\right]\right)$$

$$= \text{''} \exp\left(-\left[\frac{1}{2(1-\rho^2)}\cdot\left(\frac{y^2}{\sigma_2^2} - \frac{2\rho xy}{\sigma_1\sigma_2}\right) + \frac{x^2}{2(1-\rho^2)\sigma_1^2} - \frac{x^2}{2\sigma_1^2}\right]\right)$$

$$= \text{''} \exp\left(-\left[\text{''} + \frac{\cancel{x^2} - \cancel{x^2} + \rho^2 x^2}{2(1-\rho^2)\sigma_1^2}\right]\right)$$

$$= \text{''} \exp\left(-\left[\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_2^2} - \frac{2\rho xy}{\sigma_1 \sigma_2} + \frac{\rho^2 x^2}{\sigma_1^2}\right)\right]\right)$$

$$= \text{''} \exp\left(-\left[\frac{1}{2\sigma_2^2(1-\rho^2)}\left(y^2 - 2\frac{\sigma_2}{\sigma_1}\rho xy + \frac{\rho^2 x^2 \sigma_2^2}{\sigma_1^2}\right)\right]\right)$$

$$= \text{''} \exp\left(-\left[\text{''} \left(y - \frac{\rho\sigma_2}{\sigma_1}x\right)^2\right]\right)$$

$$f_Y(y/x) = \frac{1}{\sigma_2\sqrt{2\pi(1-\rho^2)}} \exp\left[-\frac{\left(y - \frac{\rho\sigma_2}{\sigma_1}x\right)^2}{2\sigma_2^2(1-\rho^2)}\right]$$

Gaussian with mean $\frac{\rho\sigma_2}{\sigma_1}x$ &

variance $\sigma_2^2(1-\rho^2)$.

2. Liniear transformation of jointly Gaussian random variable is also Gaussian.

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \\ a_{1N} & a_{2N} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \qquad (Y = AX)$$

where, $X_1, X_2, \ldots X_N$ are jointly Gaussian.

$$f_{X_1, X_2 \ldots X_N}(x_1, x_2 \ldots x_N) = \frac{1}{(2\pi)^{N/2} |C_X|^{1/2}} e^{-\frac{1}{2}\left[(x - m_x)^T C_X^{-1} (x - m_x)\right]}$$

Proof: Let $Z = aX + bY$

$W = cX + dY$

$$J\left(\frac{z, w}{x, y}\right) = \left| \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \right| = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = |A|$$

$$f_{Y_1, Y_2 \ldots Y_N}(y_1, y_2 \ldots y_N) = \frac{f_{X_1, X_2 \ldots X_N}(x_1, x_2, \ldots x_N)}{\det(A)}$$

$$C_Y = A C_X A^T$$

$$\therefore |C_Y| = |A| |C_X| |A^T| = |A|^2 |C_X|$$

$$\therefore \; |A|^2 = \frac{|C_y|}{|C_x|}$$

$$\therefore \; |A|^2 = \frac{|C_y|^{1/2}}{|C_x|^{1/2}}$$

$$f_{Y_1, Y_2, \ldots Y_N}(y_1, y_2 \ldots y_N) = \frac{\dfrac{1}{(2\pi)^{N/2}|C_x|^{1/2}} e^{-\frac{1}{2}[(x-m_x)^T C_x^{-1} (x-m_x)]}}{|A|}$$

$$= \frac{1}{2\pi^{N/2} |C_y|^{1/2}} \cdot \exp\left[\frac{-1}{2} (A^{-1}y - m_x)^T C_x^{-1} (A^{-1}y - m_x)\right]$$

$$= \text{''} \exp\left[\frac{-1}{2} \times [A^{-1}(y - Am_x)]^T C_x^{-1} A^{-1}(y - Am_x)\right]$$

$$= \text{''} \exp\left[\frac{-1}{2} \left[(y - Am_x)^T . (A^{-1})^T C_x^{-1} A^{-1} (y - Am_x)\right]\right]$$

$$= \text{''} \exp\left[\frac{-1}{2} (y - m_y)^T . (AC_xA^T)^{-1} (y - m_y)\right]$$

$$= \text{''} \exp\left[\frac{-1}{2} (y - m_y)^T C_y^{-1} (y - m_y)\right]$$

$$= \frac{1}{2\pi^{N/2} |C_y|^{1/2}} e^{\left[-\frac{1}{2} (y-m_y)^T C_y^{-1} (y-m_y)\right]}$$

Which gives joint PDF for Gaussian Y.

43. Show that the $E(X/Y)$ (best MMSE of X given y) correspond to the liniear estimate of X given y is possible if X and Y are jointly Gaussian

$\Rightarrow$ Que-① $\Rightarrow$ $x = x - m_x$

$y = y - m_y$

$$f_x(x/y) = \frac{1}{\sigma_1\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{\left(x - \rho\frac{\sigma_1}{\sigma_2}y\right)^2}{2\sigma_1^2(1-\rho^2)}\right)$$

$$= \frac{1}{\sigma_1\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{\left((x-m_x) - \frac{\rho\sigma_1(y-m_y)}{\sigma_2}\right)^2}{2\sigma_1^2(1-\rho^2)}\right)$$

$$= \frac{1}{\sigma_1\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{\left(x - \left(\frac{\rho\sigma_1}{\sigma_2}(y-m_y)+m_x\right)\right)^2}{2\sigma_1^2(1-\rho^2)}\right)$$

$$E(x/y) = \frac{\rho\sigma_1}{\sigma_2}(y-m_y) + m_x$$