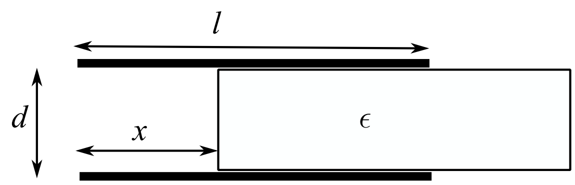
1. A slab of linear dielectric material is partially inserted between the plates of a parallel plate capacitor whose area is l^2 and the distance between the plates is d. Find the force by which the slab is sucked inside the capacitor while it is charged to a potential V.



soln
The width of the capacitor is l and let x be the length over which the capacitor is filled with air while $l - x$ be the length over which it is filled with the dielectric material of permitivity ϵ. The capacitance of this configuration can be shown to be (looked upon as two capacitors connected in parallel):

$$\begin{aligned} C &= \frac{\epsilon_0 l^2}{d}\frac{x}{l} + \frac{\epsilon l^2}{d}\frac{l-x}{d} \\ &= \frac{l}{d}\left[\epsilon l + x(\epsilon_0 - \epsilon)\right] \end{aligned}$$

The work done in charging the capacitor to potential V is

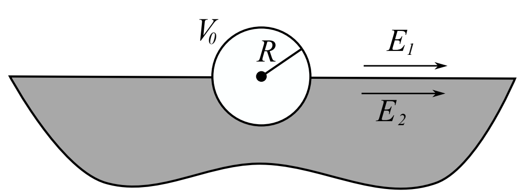
$$W = \frac{1}{2}CV^2 = \frac{lV^2}{2d}\left[\epsilon l + x(\epsilon_0 - \epsilon)\right]$$

The force acting on the dielectric slab is given as

$$F = -\frac{\partial W}{\partial x} = -\frac{V^2}{2}\frac{\partial C}{\partial x} = \frac{V^2}{2}\frac{l}{d}\left(\epsilon - \epsilon_0\right)$$

This is the force with which the dielectric slab is pulled in.

2. A conducting sphere at potential V_0 is half embedded in linear dielectric material of susceptibility χ_e, which occupies the region $z < 0$. Justify that the potential everywhere is exactly the same as it would have been in the absence of the dielectric.



soln
The tangential component of the electric field at the interface of the dielectric and the

air have to be continuous. So $E_1 = E_2 = E$ as shown in the figure. The electric field orginates at the surface of the conducting sphere and is given by $-\vec{\nabla}\Phi$. The boundary condition on Φ is $\Phi = V_0$ on the surface of the sphere. Moreover there is no other charges anywhere. This is because the dielectric is homogeneous. If χ_e is the electric susceptibility of the dielectric material then $\vec{\nabla}\cdot\vec{P} = \frac{\chi_e}{1+\chi_e}\vec{\nabla}\cdot\vec{D} = \frac{\chi_e}{1+\chi_e}\rho_f$. Since there is no free charge anywhere except on the conductor, $\vec{\nabla}\cdot\vec{P} = 0$. So there is no bound charge in the dielectric except at the interface with the sphere. This makes the electric field radial and spheriacally symmetric. This is precisely the solution for the sphere maintained at potential V_0 in free space. So the potential everywhere is the same as in the absence of the dielectric.

Note: The free surface charge on the conducting surface is however not uniformly distributed. In the upper hemisphere $\sigma_1 = \epsilon_0 E$ while in the lower hemisphere $\sigma_2 = D = \epsilon E$. σ_2 will be reduced by the bound surface charges over the interface of the conductor and the dielectric.

3. For a configuraion of charges and currents confined within a volume $\mathcal{V}$, show that

$$\int_{\mathcal{V}} \vec{\mathrm{J}} d\tau = \frac{d\vec{\mathrm{p}}}{dt}$$

where $\vec{\mathrm{p}}$ is the total dipole moment.

[Hint: Evaluate $\int_{\mathcal{V}} \vec{\nabla}\cdot(x\vec{J})d\tau$]

soln

The charges and the currents are confined within the volume $\mathcal{V}$. So no current crosses the surface S. Under these conditions we have to show that

$$\int_{\mathcal{V}} \vec{\mathrm{J}} d\tau = \frac{d\vec{\mathrm{p}}}{dt}$$

where $\vec{\mathrm{p}}$ is the total dipole moment of the charge distribution in the volume $\mathcal{V}$. We start with the R.H.S.

Let $\rho(\vec{r}, t)$ be the charge density in the region.

$$\vec{p}(t) = \int_{\mathcal{V}} \vec{r} d\tau$$

$$\therefore \quad \frac{d\vec{p}}{dt} = \int_{\mathcal{V}} \vec{r}\frac{\partial\rho}{\partial t}d\tau = \hat{i}\int x\frac{\partial\rho}{\partial t}d\tau + \hat{j}\int y\frac{\partial\rho}{\partial t}d\tau + \hat{k}\int z\frac{\partial\rho}{\partial t}d\tau$$

Let us consider the integral $\int x\frac{\partial\rho}{\partial t}d\tau$. By continuity equation $\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$.

$$\therefore \quad \int x\frac{\partial\rho}{\partial t}d\tau = -\int_{\mathcal{V}} x(\vec{\nabla}\cdot\vec{J})d\tau$$

$$\vec{\nabla}\cdot(x\vec{J}) = x\vec{\nabla}\cdot\vec{J} + \vec{\nabla}x\cdot\vec{J} = x\vec{\nabla}\cdot\vec{J} + J_x$$

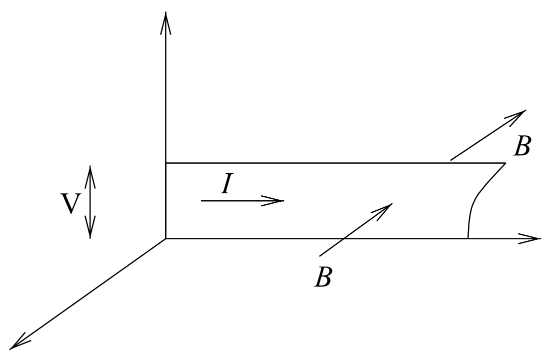
$$\therefore \quad \int_{\mathcal{V}} x\frac{\partial\rho}{\partial t}d\tau = \int_{\mathcal{V}} J_x d\tau - \int_{\mathcal{V}} \vec{\nabla}\cdot(x\vec{J})d\tau$$
$$= \int_{\mathcal{V}} J_x d\tau - \int_S x\vec{J}\cdot\hat{n}da$$

Since no current crosses the surface S we have $\vec{J}\cdot\hat{n} = 0$.

$$\therefore \quad \int_{\mathcal{V}} x\frac{\partial\rho}{\partial t}d\tau = \int_{\mathcal{V}} J_x d\tau$$

$$\therefore \quad \frac{d\vec{p}}{dt} = \hat{i}\int_{\mathcal{V}} J_x d\tau + \hat{j}\int_{\mathcal{V}} J_y d\tau + \hat{k}\int_{\mathcal{V}} J_z d\tau = \int_{\mathcal{V}} \vec{J}d\tau$$

4. A current I flows through a long rectangular strip of conductor of width a. The surface electron density on the strip is n. If the strip is placed in a magnetic field B perpendicular to its plane, the moving charges constituting the current experiences a force along the width of the strip.
(a) Calculate this force.
(b) As a result of this force charges get accumulated along the two edges of the strip. This produces an electric force which balances the magnetic force and an equilibrium sets in. This phenomenon is called the Hall effect and the electric potential difference between the two edges is called the Hall voltage. Find the Hall voltage in terms of B, I and the electron charge e. This effect is used in the Gaussmeter which measures the magnetic field.



soln:

(a) The given current constitutes of electron moving with velocity v in opposite direction. The force on the electron is upward given by

$$\vec{F}_{mag} = e\vec{v}\times\vec{B} = evB\hat{z}$$

$I = \sigma va$ where a is the width of the strip and $\sigma = ne$, the charge density.
$\therefore \quad I = neva$
$\therefore \quad F_{mag} = evB = IB/(na)$.
This is the force on the electron in the strip.

(b) Let V be the Hall voltage developed across te edges of the strip. Then the electric field in the strip is $E = V/a$.
The force on an electron due to this field is $F_{elect} = eV/a$.
At equilibrium $F_{elect} = F_{mag}$.

$$\therefore \quad \frac{eV}{a} = \frac{IB}{na} \qquad \Longrightarrow \qquad V = \frac{IB}{ne}$$

5. Find the magnetic field at point P for each of the steady current configurations shown below

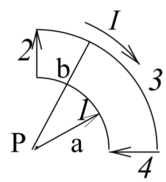


Fig.1

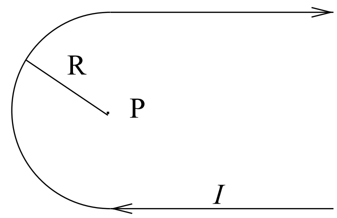


Fig. 2

soln
(a)
By Bio Savart Law, at P

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{\vec{dl} \times \hat{r}}{r^2}$$

We can break the integral into four pieces. Along arc 1 $\vec{dl} \times \hat{r}$ is a vector comming out orthogonally from the paper. Also $|\vec{dl} \times \hat{r}| = dl$.
So due to arc 1

$$\vec{B}_1 = \hat{n}\frac{\mu_0 I}{4\pi} \int \frac{dl}{a^2} = \frac{\mu_0 I}{8a}\hat{n}$$

By the same argument due to arc 3

$$\vec{B}_3 = -\frac{\mu_0 I}{8b}\hat{n}$$

The contribution from the straight pieces 2 and 4 are 0 since $\vec{dl} \times \hat{r} = 0$ along them.

$$\therefore \quad \vec{B} = \frac{\mu_0 I}{8}\left[\frac{1}{a} - \frac{1}{b}\right]\hat{n}$$

(b)
Due to the semicircular arc the field at P is

$$\vec{B}_1 = -\frac{\mu_0 I}{4R}\hat{n}$$

Due to the lower straight piece the field at P is downward $(-\hat{n})$. This field will be half of the field due to an infinite straight wire. This is equal to $\frac{\mu_0 I}{4\pi R}$. We have two such straight pieces producing field in the same direction $-\hat{n}$. The sum of these are

$$\vec{B} = \left(\frac{\mu_0 I}{4R} + \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{4\pi R}\right)(-\hat{n}) = \frac{\mu_0 I}{4R}\left(1 + \frac{2}{\pi}\right)(-\hat{n})$$

6. Two parallel, infinite line charges λ, a distance d apart are moving at a constant velocity $\vec{v}$. The direction of $\vec{v}$ is along the line charges. How great would v have to be in order for the magnetic attraction to balance the electrical repulsion?

   soln
   The moving line charges constitute a current $I = \lambda v$. The charges on one wire experience a force of attraction due to the magnetic field caused due to this current.

$$\begin{aligned} F_{mag} &= B\lambda v \quad \text{per unit length} \\ &= \frac{\mu_0 I}{2\pi d}\lambda v = \frac{\mu_0 \lambda^2 v^2}{2\pi d} \end{aligned}$$

   The electric field due to one line charge on the other at a distance d is

$$E = \frac{\lambda}{2\pi\epsilon_0 d}$$

   The repulsive force per unit length on the other wire is

$$F_{elect} = E\lambda = \frac{\lambda^2}{2\pi\epsilon_0 d}$$

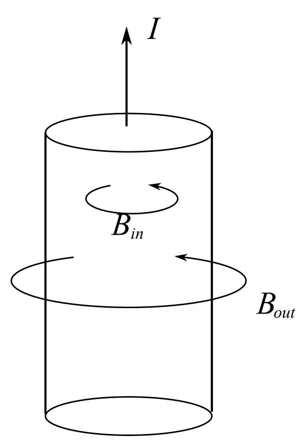
   When $F_{elect} = F_{mag}$ we have

$$\begin{aligned} \frac{\lambda^2}{2\pi\epsilon_0 d} &= \frac{\mu_0 \lambda^2 v^2}{2\pi d} \\ \therefore \quad v^2 &= \frac{1}{\mu_0\epsilon_0} = c^2 \end{aligned}$$

   where c is the velocity of light.
   The two forces balance only when they move with the velocity of light.

7. A steady current I flows through a long cylindrical wire of radius a. Find the magnetic field both inside and outside the wire, if

   (a) The current is uniformly distributed over the outside surface of the wire.
   soln



The current is only over the surface

$$\therefore \quad K = \frac{I}{2\pi a} \quad \Longrightarrow \quad \vec{K} = \frac{I}{2\pi a}\hat{z}$$

By Ampere's law, over a loop concentric with the axis of the wire,

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$$

If the loop is inside the cylinder i.e, the radius of the loop $s < a$ then $B \times 2\pi s = \mu_0 \times 0$. So $B_{in} = 0$.
To find B_{out} we have

$$B_{out} \times 2\pi s = \mu_0 I \quad \Longrightarrow \quad B_{out} = \frac{\mu_0 I}{2\pi s}$$

This magnetic field is along $\hat{\phi}$. $\therefore \quad \vec{B}_{out} = \mu_0 I/2\pi s\hat{\phi}$

(b) The current is distributed in such a way that J is proportional to s.
soln:
$\vec{J} = \alpha s\hat{z}$ where α is a proportionality constant.

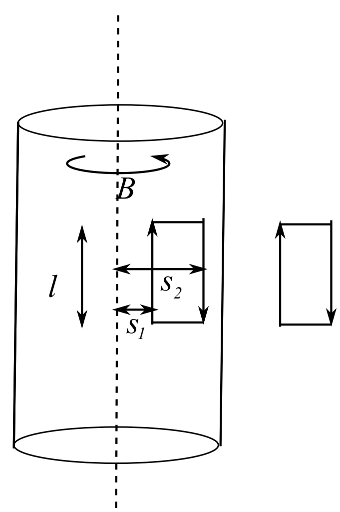
$$\int_0^a \alpha s(2\pi s)ds = I \quad \Longrightarrow \quad \alpha = \frac{3I}{2\pi a^3}$$

Inside the wire

$$\begin{aligned} B_{in}2pis &= \mu_0 I_{enc} = \mu_0 \int_0^s \alpha s 2\pi s ds \\ &= \mu_0 \alpha 2\pi s^3/3 = \mu_0 I\frac{s^3}{a^3} \\ \vec{B}_{in} &= \frac{\mu_0 I}{2\pi a^3}s^2 \end{aligned}$$

The field outside remains the same as in part (a).

(c) Find the vector potential in both the cases above.
soln
(for case (a))



$\vec{\nabla} \times \vec{A} = \vec{B}$
$\therefore \oint_C \vec{A} \cdot \vec{dl} = \int_S \vec{B} \cdot \hat{n} da$ where S is the surface enclosed by C.
For inside the wire consider a loop as shown. We consider a vector potential along z axis

$$(A_z(s_1) - A_z(s_2))l = \int_S \vec{B}_{in} \cdot \hat{n} da = 0$$

$\therefore \quad A_z(s_1) = A_z(s_2)$.
$\therefore \quad A_z^{in} = c_1$, a constant.
For outside

$$(A_z(s_1) - A_z(s_2))l = \int_{s1}^{s2} B_{out} dsl$$

$$\therefore \quad A_z(s_1) - A_z(s_2) = \int_{s1}^{s2} \frac{\mu_0 I}{2\pi s} ds \frac{\mu_0 I}{2\pi} \ln\left(\frac{s_2}{s_1}\right)$$

$$\therefore \quad - A_z^{out}(s) = \frac{\mu_0 I}{2\pi} \ln\left(\frac{s}{s_0}\right)$$

Matching the vector potential on the surface of the cylinder we have

$$c_1 = \frac{\mu_0 I}{2\pi} \ln\left(\frac{a}{s_0}\right)$$

$$\begin{aligned} \therefore \ \vec{A}^{in} &= -\frac{\mu_0 I}{2\pi} \ln\left(\frac{a}{s_0}\right) \hat{z} \\ \vec{A}^{out} &= -\frac{\mu_0 I}{2\pi} \ln\left(\frac{s}{s_0}\right) \hat{z} \end{aligned}$$

s_0 is arbitrary.

(for case (b))
Since $\vec{B}$ is along $\hat{\phi}$ everywhere, $\vec{A}$ will be along $\hat{z}$. So let $\vec{A} = A_z(s)\hat{z}$.

$$\vec{\nabla} \times \vec{A} = -\frac{\partial A_z}{\partial s} \hat{\phi} = \vec{B}$$

For inside $-\frac{\partial A_z}{\partial s} = \frac{\mu_0 I}{2\pi a^3} s^2$

$$\therefore \quad A_z^{in} = -\frac{\mu_0 I}{6\pi a^3} s^3 + c_1$$

For outside $-\frac{\partial A_z}{\partial s} = \frac{\mu_0 I}{2\pi s}$

$$\therefore \ A_z^{out} = -\frac{\mu_0 I}{2\pi} \ln(s) + c_2$$

By matching the two vector potential at the cylindrical surface we have

$$-\frac{\mu_0 I}{6\pi} + c_1 = -\frac{\mu_0 I}{2\pi} \ln(a) + c_2$$

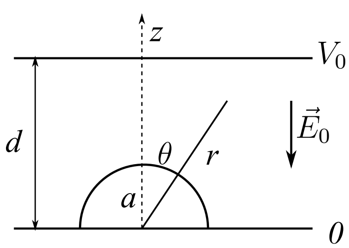
If we take $c_2 = 0$ then

$$\begin{aligned} c_1 &= -\frac{\mu_0 I}{2\pi}\ln(a) + \frac{\mu_0 I}{6\pi} \\ &= -\frac{\mu_0 I}{6\pi}(\ln(a^3) - 1) \end{aligned}$$

$$\begin{aligned} \therefore \quad \vec{A}_{in} &= -\frac{\mu_0 I}{6\pi}\left[\frac{s^3}{a^3} + \ln(a^3) - 1\right]\hat{z} \\ \vec{A}_{out} &= -\frac{\mu_0 I}{2\pi}\ln s \end{aligned}$$

All the vector potentials found are along $\hat{z}$ and they are functions of s, i.e, they are of the type $A_z(s)\hat{z}$. So they are all divergenceless. $\vec{\nabla}\cdot\vec{A} = \frac{\partial A_z(s)}{\partial z = 0}$.

8. Two very large metal plates are held a distance d apart, one at potential 0, the other at potential V_0. A small metal hemisphere (radius $a << d$) is placed on the grounded plate, so that its potential is likewise 0. If the region between the plates is filled with weakly conducting material of uniform conductivity σ, what current flows to the hemisphere?



soln:
In the absence of the hemisphere the electric field is

$$\vec{E}_0 = -\frac{V_0}{d}\hat{z}$$

So this problem is same as placing a grounded spherical conductor in an uniform electric field $\vec{E}_0$.
The potential at a point (r, θ) in this problem is given by

$$V(r,\theta) = E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

The electric field is given by

$$\vec{E} = -\vec{\nabla}V = -\left[\hat{r}\frac{\partial V}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta}\right]$$

The current density is

$$\vec{J} = \sigma\vec{E}$$

On the surface of the hemisphere $r = a$. $\vec{E}$ and $\vec{J}$ only has $\hat{r}$ component given by

$$\vec{J} = -3\sigma E_0 \cos\theta \hat{r}$$

The total current into the hemisphere is

$$\begin{aligned}
I &= \int_S \vec{J} \cdot \hat{r} da = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} 3\sigma E_0 \cos\theta a^2 \sin\theta d\theta d\phi \\
&= 3\sigma E_0 (2\pi) a^2 0^{\frac{\pi}{2}} \cos\theta \sin\theta d\theta \\
&= 3\pi a^2 \sigma E_0 \\
&= -3\pi\sigma a^2 \frac{V_0}{d}
\end{aligned}$$

The negative sign indicates that the current flows into the hemisphere.