Two person zero sum strictly determined game.

$$\begin{array}{cc}
 & B \\
A & \begin{array}{c|cccccc}
 & B_1 & B_2 & \cdots & B_j & \cdots & B_n \\
\hline
A_1 & a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
A_2 & a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & & & & & \\
A_i & a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & & & & & \\
A_m & a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{array}
\end{array}$$

1. calculate the value of row minimum for each row and write down the m values in a column under the heading row minimum.

   The find the maximum value of this column.

   Let it be $\boxed{\alpha}$ and it is corresponding to the k-th row in pay-off matrix.

2. Calculate the value of column maximum for each column and write down the n values in a row under the heading column maximum.

Find the minimum value of this row.

Let it be $\boxed{\beta}$ and it is corresponding to the l-th column of pay-off matrix.

3. optimal strategy for player A is A_k and optimal strategy for player B is B_l and the value of the game is a_{kl}

Remark: i) $\alpha = \beta = a_{kl}$ then the game is zero sum.

(ii) If $\alpha = \beta = 0$ then the game is called fair game.

iii) If $\alpha_{kl} > 0$ the game is in favour of A

$\alpha_{kl} < 0$ " " " " " B

iv) If $A = (a_{ij})_{m \times n}$ be a pay-off matrix then we are basically finding

$$\max_i \min_j a_{ij} = \underline{v}$$

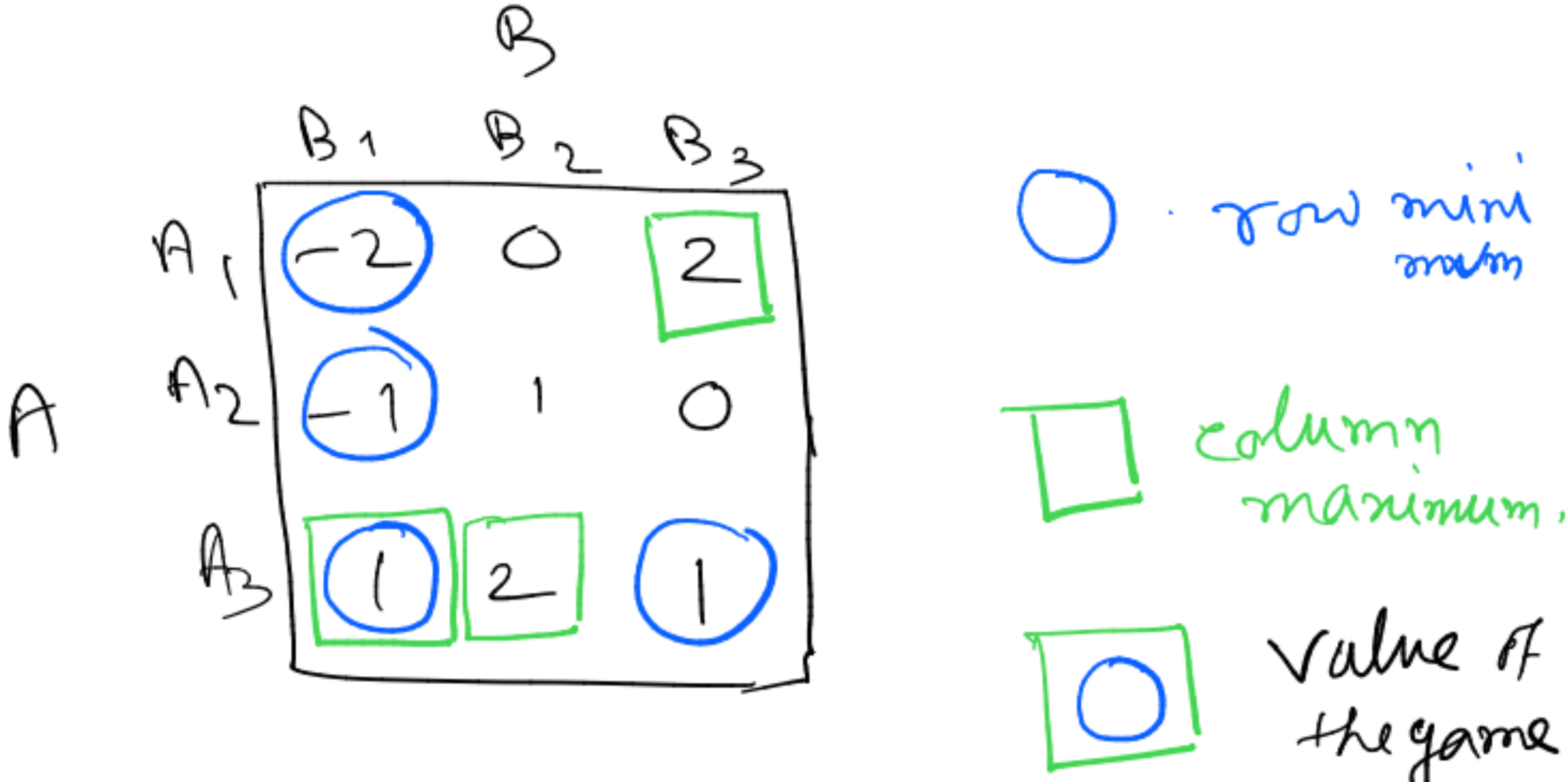
$$\min_j \max_i a_{ij} = \overline{v}$$

Then if $\underline{v} \neq \overline{v}$ then the value of the game is v and

$$\boxed{\underline{v} \geqslant v \geqslant \overline{v}}$$

Example:

$$\begin{array}{cc|ccc|c} & & & B & & \\ & & B_1 & B_2 & B_3 & \text{rowmin} \\ \hline & A_1 & -2 & 0 & 2 & -2 \\ A & A_2 & -1 & 1 & 0 & -1 \\ & A_3 & 1 & 2 & 1 & 1 \\ \hline & \text{columnmax.} & 1 & 2 & 2 & \end{array}$$



value of the game v = 1

optimal strategy for A is A_3

optimal strategy for B is B_1

and the corresponding rows and columns are the optimal strategies for A and B.

Example

$$
\begin{array}{cc|cccc|}
 & & & B & & \\
 & & B_1 & B_2 & B_3 & B_4 \\
\hline
 & A_1 & \boxed{4} & \boxed{6} & \enclose{circle}{-2} & 1 \\
A & A_2 & 3 & 3 & 4 & \boxed{\enclose{circle}{2}} \\
 & A_3 & \boxed{4} & 5 & \boxed{5} & \enclose{circle}{1} \\
\hline
\end{array}
$$

value of the game is 2

optimal strategy is (A_2, B_4)

Note: The cell $\boxed{\bigcirc}$ is called saddle point.

The value at the cell is called saddle value.

Result: There can be more than one saddle point in a pay-off matrix but the saddle value remains same.

Proof Assume there are two saddle points.

Let a_{rs} and a_{kl} be two saddle values.

we need to prove $\boxed{a_{rs} = a_{kl}}$

$a_{rs} \leq a_{rl}$ [since a_{rs} is the minimum in the r-th row] —①

$a_{rs} \geq a_{ks}$ since a_{rs} is the maximum value in the s-th column. —②

$a_{kl} \leq a_{ks}$ a_{ks} is the min in k-th row. —③

$a_{kl} \geq a_{rl}$ —④ a_{kl} is the max in l-th row.

$$\underline{\underline{a_{kl}}} \overset{\textcircled{1}}{\leq} a_{ks} \overset{\textcircled{2}}{\leq} a_{rs} \overset{\textcircled{1}}{\leq} a_{rl} \overset{\textcircled{4}}{\leq} \underline{\underline{a_{kl}}}$$

$\Rightarrow a_{kl} = a_{ks} = a_{rs} = a_{rl} = a_{kl}$.

$\Rightarrow \boxed{a_{kl} = a_{rs}}$.

Example

$$\begin{array}{cc|cc} & & & B \\ & & B_1 & B_2 \\ \hline A & A_1 & \boxed{4} & \enclose{circle}{-1} \\ & A_2 & \enclose{circle}{2} & \boxed{3} \end{array}$$

No saddle exists.

$$2 \leq v \leq 3$$

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 0 & \boxed{2} & \enclose{circle}{-3} & 0 \\ A_2 & \enclose{circle}{-2} & 0 & 0 & \boxed{3} \\ A_3 & \boxed{3} & 0 & 0 & \enclose{circle}{-4} \\ A_4 & 0 & \enclose{circle}{-3} & \boxed{4} & 0 \end{array}$$

No saddle point

$$-2 \leq v \leq 2$$

- These examples show that maximin or minimax may not fetch solution of a game.
- when there is no existence of a saddle point.
- No pure strategy exists for the above situations.
- one needs to adopt the idea of <u>mixed strategy</u>.

$A_1, A_2, \ldots, A_m$: Strategies of A

$X = x_1, x_2, \ldots, x_m$: probabilities $\sum x_i = 1$

$B_1, B_2, \ldots, B_n$: Strategies of B

$Y = y_1, y_2, \ldots, y_n$: probabilities $\sum y_i = 1$

$A = (a_{ij})_{m \times n}$: pay-off matrix.

Expected pay-off to player-A, when he chooses strategy A_r is.

$$E(\xi_r, Y) = \sum_{j=1}^{n} a_{rj} y_j$$

Expected pay-off to player-A.

$$\sum_{r=1}^{m} \left(\sum_{j=1}^{n} a_{rj} y_j \right) x_r$$

$$= \sum_{r=1}^{m} E(\xi_r, Y)\, x_r$$

$$= E(X, Y)$$

Expected pay-off to player-A when player-B chooses B_s is

$$E(X, \eta_s) = \sum_{i=1}^{m} a_{is} x_i$$

Expected pay-off to player-A is

$$\sum_{s=1}^{n} \left(\sum_{i=1}^{m} a_{is} x_i \right) y_s$$

$$= \sum_{s=1}^{n} E(X, \eta_s) y_s$$

$$= E(X, Y)$$

Thus, the expected pay-off to player-A is

$$\boxed{E(X, Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j}$$

To find strategies

Let (X_0, Y_0) be the strategy if

$$E(X_0, Y_0) \geq E(X, Y_0), \forall X \neq X_0$$
$$\leq E(X_0, Y), \forall Y \neq Y_0$$

or,

$$E(X_0, Y_0) = \max_{X} \min_{Y} E(X, Y)$$
$$= \min_{Y} \max_{X} E(X, Y)$$

Here (X_0, Y_0) is called the strategic saddle point.

Example:

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{B} \\ & & B_1 & B_2 \\ \hline A & A_1 & \boxed{5} & \textcircled{1} \\ & A_2 & \textcircled{3} & \boxed{4} \end{array}$$

This is a 2x2 game of chance.

Here no saddle point exist.

Let x be the probability attached with A_1 then $1-x$ is the probability attached with A_2

Let y be the probability attached with B_1 then $1-y$ is the probability attached with B_2.

Let (X_0, Y_0) be the strategic saddle point

Then
$$E(X_0, Y_0) = \max_X \min_Y E(X,Y)$$
$$= \min_Y \max_X E(X,Y)$$

$$\begin{array}{cc|cc} & & y & 1-y \\ & & B_1 & B_2 \\ \hline x & A_1 & 5 & 1 \\ 1-x & A_2 & 3 & 4 \end{array}$$

$$E(x,Y) = E(x,y)$$

$$= 5xy + x(1-y) + 3\cdot y(1-x) + 4(1-x)(1-y)$$

$$\boxed{= C + D(x-K)(y-l)}$$

$$= C + D(xy - Ky - lx + Kl)$$

Equating coefficients:

$$5xy + x - xy + 3y - 3xy + 4 - 4x - 4y + 4xy$$

$$= (5-1-3+4)xy + (1-4)x + (3-4)y + 4$$

Thus.

$$D = 5-1-3+4 = 5$$

$$C + DKl = 4$$

$$C = 4 - 5\cdot\frac{1}{5}\cdot\frac{3}{5}$$

$$= 4 - \frac{3}{5}$$

$$= \frac{17}{5}$$

$$-DK = 3-4$$

$$K = \frac{-1}{-D}$$

$$= \frac{1}{D}$$

$$= \frac{1}{5}$$

$$-Dl = 1-4$$

$$l = \frac{3}{5}$$

$$E(X,Y) = \frac{17}{5} + 5\left(x - \underline{\underline{\frac{1}{5}}}\right)\left(y - \frac{3}{5}\right)$$

<u>Note:</u>

- If A chooses $x = \frac{1}{5}$ then whatever B chooses $E(X,Y) = \frac{17}{5}$
- If A chooses other than $\frac{1}{5}$ then B can choose a value of y such that $E(X,Y)$ become < 0 and it become the loss of A.

- If B chooses $y = \frac{3}{5}$ then whatever A chooses $E(X,Y) = \frac{17}{5}$
- If B chooses other than $\frac{1}{5}$ then A can choose a value of x such that $E(X,Y) > \frac{17}{5}$

Then the optimal strategy is

for A $= \left(\frac{1}{5}, \frac{4}{5}\right) \equiv (x, 1-x)$

for B $= \left(\frac{3}{5}, \frac{2}{5}\right) = (y, 1-y)$

$v = \frac{17}{5}$