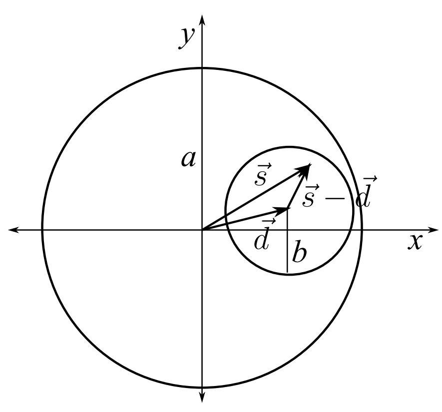
1. An infinitely long cylindrical cavity of radius b is bored into a bigger cylinder of radius a. The axes of the two cylinders are parallel but the cylinders are not concentric. The remaining part of the cylinder has a cosnstant volume charge density ρ. Show that the electric field inside the cavity is uniform and directed along the line joining the center of the two cylinders.

soln
The given configuration of charge is a superposition of two simple charge configuration. One is a cylinder of radius a carrying a uniform charge density ρ. The other is a cylinder of radius b with a charge density $-\rho$. when the negatively charged cylinder is inserted into the bigger cylinder it creates a hollow chargeless region as required. We can easily calculate the electric field due to the two cylinders and then add the electric field caused by them. Let us place the positively charged cylinder coincident with the z axis. Let $E_1(s)$ be the magnitude of the electric field at a distance s from the z axis. Consider a cylindrical Gaussian surface of radius s and height h. The electric field on this Gaussian surface is along $\hat{s}$ which is also the direction of the normal to the cylinder. The charge enclosed by this Gaussian cylinder is $\rho\pi s^2 h$. The flux of the electric field over the Gausian surface is $2\pi s h E_1(s)$. By Gauss' law we have

$$\begin{aligned}
2\pi s h E_1(s) &= \frac{\rho\pi s^2 h}{\epsilon_0} \\
\therefore \quad E_1(s) &= \frac{\rho s}{2\epsilon_0} \\
\therefore \quad \vec{E}_1(\vec{s}) &= \frac{\rho s}{2\epsilon_0}\hat{s} \\
&= \frac{\rho\vec{s}}{2\epsilon_0}
\end{aligned}$$

Let te position vector of the center of the other cylinder be $\vec{d}$ as shown in the figure.



This is also a uniformly charged cylinder with density $-\rho$. We can directly write down the electric field at a point inside this cylinder as

$$\vec{E}_2(\vec{s}) = \frac{-\rho(\vec{s}-\vec{d})}{2\epsilon_0}$$

The total electric field at a point described by the position vector $\vec{s}$ will be

$$\vec{E}(\vec{s}) = \vec{E}_1(\vec{s}) + \vec{E}_2(\vec{s}) = \frac{\rho\vec{d}}{2\epsilon_0}$$

which is a constant electric field directed along the vector connecting the center of the two cylinders.

2. A hollow spherical shell carries a uniform charge density ρ_0 in the region $a \leq r \leq b$. Find the electric potential as a function of r.

soln:
Let us first find the electric field in the three regions, $r > b$, $\quad a < r < b$, $\quad$ and $r < a$. The problem has a spherical symmetry. The electric field everywhere is along $\hat{r}$. Using Gauss' law we can easily calculate these fields. They are given as follows:

$$\begin{aligned} E(r) &= \frac{\rho_0}{3\epsilon_0}\frac{b^3-a^3}{r^2} \qquad ; r \geq b \\ &= \frac{\rho_0}{3\epsilon_0}\left(r - \frac{a^3}{r^2}\right) \qquad ; a \leq r < b \\ &= 0 \qquad ; r < a \end{aligned}$$

To calculate the potential we take a reference at a point $\vec{r}_c$ at a distance c which is outside the outer surface of the shell. The electric potential at a point $\vec{r}$ is given as

$$\Phi(\vec{r}) = -\int_{\vec{r}_c}^{\vec{r}} \vec{E}(\vec{r})\cdot d\vec{l}$$

$d\vec{l} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$ and $\vec{E}(\vec{r}) = E(r)\hat{r}$. So the potential is given as

$$\Phi(\vec{r}) = -\int_{c}^{r} E(r)dr$$

Since the potential only depends on r it is written as $\Phi(r)$. So the potentials in the three regions are

$$\begin{aligned}
\Phi(r > b) &= -\int_c^r E(r)dr = -\int_c^r \frac{\rho_0}{3\epsilon_0}\frac{b^3-a^3}{r^2}dr \\
&= \frac{\rho_0(b^3-a^3)}{3\epsilon_0}\left(\frac{1}{r}-\frac{1}{c}\right) \\
\Phi(a < r < b) &= -\int_c^b E(r)dr - \int_b^r E(r)dr \\
&= \frac{\rho_0(b^3-a^3)}{3\epsilon_0}\left(\frac{1}{b}-\frac{1}{c}\right) - \int_b^r \frac{\rho_0}{3\epsilon_0}\left(r-\frac{a^3}{r^2}\right)dr \\
&= \frac{\rho_0(b^3-a^3)}{3\epsilon_0}\left(\frac{1}{b}-\frac{1}{c}\right) - \frac{\rho_0}{3\epsilon_0}\left(\frac{r^2-b^2}{2}+\frac{a^3}{r}-\frac{a^3}{b}\right) \\
\Phi(r < a) &= -\int_c^b E(r)dr - \int_b^a E(r)dr - \int_a^r E(r)dr \\
&= \frac{\rho_0(b^3-a^3)}{3\epsilon_0}\left(\frac{1}{b}-\frac{1}{c}\right) - \frac{\rho_0}{3\epsilon_0}\left(\frac{a^2-b^2}{2}+a^2-\frac{a^3}{b}\right) \\
&= \frac{\rho_0}{2\epsilon_0}(b^2-a^2) - \frac{\rho_0}{3\epsilon_0}\frac{b^3-a^3}{c}
\end{aligned}$$

As $r \to \infty \quad \Phi \to -\frac{\rho_0}{3\epsilon_0}\frac{b^3-a^3}{c}$. If we take the reference point c to ∞ we will have to add this amount

3. The electric field in a region is cylindrically symmetric, given as follows:

$$\begin{aligned}
\vec{\mathrm{E}}(\vec{\mathrm{r}}) &= \frac{c\hat{\mathrm{s}}}{s}; \quad \text{when} \quad s \geq a \\
&= 0; \quad \text{when} \quad s < a
\end{aligned}$$

Find the charge distribution in the region using the differential form of Gauss' law $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$.

soln

The charge density is given by the differential form of Gauss' law $\vec{\nabla}\cdot\vec{E} = \rho/epsilon_0$. Due to cylindrical symmetry of the problem the partial differentiation w.r.t z and ϕ is zero. So we have

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s)$$

For $s > a, E_s = c/s \quad \Longrightarrow \quad \vec{\nabla}\cdot\vec{E} = 0$.
For $s < a, E_s = 0 \quad \Longrightarrow \quad \vec{\nabla}\cdot\vec{E} = 0$. So the charge density is 0 outside and inside the cylinder.
At $s = a, sE_s$ is not differentiable.

Consider the annular region enclosed by two cylindrical gausian surfaces of radius s_1 and s_2 such that $s_1 < a < s_2$. We confine the height of the cylinder between z_1 and z_2. $z_2 - z_1 = h$.
We will calculate the flux of $\vec{E}$ across the surface of this annular region. Over the outer cylinder $\hat{n} = \hat{s}$ and $\vec{E} = c\hat{s}/s_2$.

Over the inner cylinder $\hat{n} = -\hat{s}$ and $\vec{E} = 0$. The flux of $\vec{E}$ over the top and the bottom of the cylinders is 0 since the electric field is orthogonal to the normals there. So over our Gaissian surface we have

$$\oint_S \vec{E} \cdot \vec{da} = \int_{z_1}^{z_2} \int_0^{2\pi} \frac{c\hat{s}}{s_2}(s_2 d\phi dz)\hat{s} = 2\pi hc$$

If V is the volume enclosed by the annular region then

$$\int_V \vec{\nabla} \cdot \vec{E} dV = 2\pi hc \quad \text{by divergence theorem}$$

$$\therefore \quad \int_{s_1}^{s_2} \int_{z_1}^{z_2} \int_0^{2\pi} (\vec{\nabla} \cdot \vec{E}) s ds d\phi dz = 2\pi h \int_{s_1}^{s_2} (\vec{\nabla} \cdot \vec{E}) s ds = 2\pi hc$$

$$\therefore \quad \int_{s_1}^{s_2} (\vec{\nabla} \cdot \vec{E}) s ds = c$$

If the annular region doesn't enclose the surface $s = a$ then this integral is zero. But if $s_1 < a < s_2$ then the value of the integral is c. So we conclude

$$\vec{\nabla} \cdot \vec{E} s = c\delta(s-a) \quad \Longrightarrow \quad \vec{\nabla} \cdot \vec{E} = \frac{c}{s}\delta(s-a)$$

So the charge density at the surface $s = a$ is given as

$$\rho = \epsilon_0 \frac{c}{a}\delta(s-a)$$

This is an infinite volume charge density. This is a finite amount of charge smeared over the surface $s = a$ whose thickness is zero. hence we must specify this density as a surface charge density. This will be given as

$$\sigma = \epsilon_0 \frac{c}{a}$$

Alternatively one can directly calculate this surface charge density by considering a small pillbox shaped gaussian surface enenclosing a part of the surface $s = a$. The flux contributed by the surface just outside the cylinder will be $c/a \times \Delta a$ where Δa is the area of the pillbox. There is no contribution the flux from the inner surface since the field is zero there. There will be no contribution from the side walls since the normals there will be orthogonal to the field. All the charge enclosed by the pill box is over the surface $s = a$. The amount of enclosed charge is $\sigma \Delta a$. So by Gauss' law we have

$$\frac{\sigma \Delta a}{\epsilon_0} = \frac{c}{a}\Delta a \quad \Longrightarrow \quad \sigma = \epsilon_0 \frac{c}{a}$$

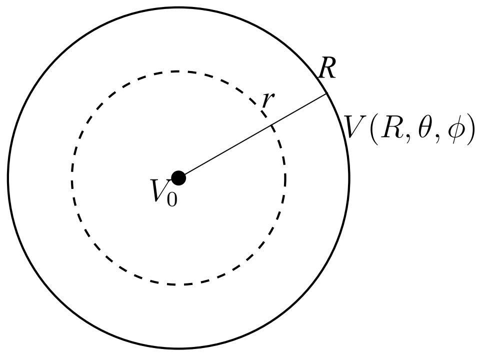
4. Prove the mean value theorem in electrostatics which states that in a chargeless region, the average of the potential over the surface of any sphere is equal to the potential at the center of the sphere.
   This is true for any regular polyhedron. If the faces of a regular polyhedron having n faces are maintained at potentials $V_1, V_2, ..., V_n$ then the potential at the center of the

polyhedron is $(V_1 + V_2 + ... + V_n)/n$. How many such regular polyhedron do you think are possible? Look for platonic solids. Tetrahedron, cube, octahedron, dodecahedron and icosahedron.

soln

Consider a sphere of radius R in a chargeless region. The average value of potential over this sphere is

$$V_{avg} = \frac{1}{4\pi R^2}\int_0^{2\pi}\int_0^{\pi} V(R,\theta,\phi)R^2\sin\theta d\theta d\phi$$



Let V_0 be the potential at the center of the sphere. Then

$$V(R,\theta,\phi) = V_0 + \int_0^R \vec{\nabla}V\cdot\hat{r}dr$$

The integral in the above step is independent of the path since $\vec{\nabla}V = -\vec{E}$ is a curlless field. So we do the integral along the radial direction from 0 to R, θ, ϕ.
So average potential is

$$\begin{aligned} V_{avg} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[V_0 + \int_0^R \vec{\nabla}V\cdot\hat{r}dr\right]\sin\theta d\theta d\phi \\ &= \frac{1}{4\pi}V_0\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi + \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^R \vec{\nabla}V\cdot\hat{r}dr\sin\theta d\theta d\phi \\ &= V_0 + \frac{1}{4\pi}\int_0^R\left[\int_0^{2\pi}\int_0^{\pi}\vec{\nabla}V\cdot\hat{r}\sin\theta d\theta d\phi\right]dr \end{aligned} \tag{1}$$

The integral over θ and ϕ is at a constant r. We can write this as a surface integral over a sphere of radius r as follows:

$$\int_0^{2\pi}\int_0^{\pi}\vec{\nabla}V\cdot\hat{r}\sin\theta d\theta d\phi = \frac{1}{r^2}\int_0^{2\pi}\int_0^{\pi}(\vec{\nabla}V\cdot\hat{r})r^2\sin\theta d\theta d\phi = \frac{1}{r^2}\oint_S -\vec{E}\cdot\hat{r}da$$

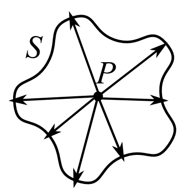
This surface integral is equal to the total charge enclosed inside the sphere of radius r. In a chargeless region this is 0. So

$$V_{avg} = V_0$$

5. Prove that in a chargeless region electrostatic potential cannot have a maxima or a minima.

soln:
In a chargeless region the potential at any point is equal to the average potential over any sphere around it. Since the average can't be a maxima or a minima in a region we conclude that the electrostatic potential can't have a maxima or a minima.

A proof independent of the mean value theorem is physically more interesting.
Suppose there is a maxima of the potential at a point P. Then there exist a neighbourhood of the point over which the potential is lower than that at the point. Let S be the closed surface of this neighbourhood. The potential at every point over the surface S is lower than the potential at P. The electric field lines over the surface is thus directed outward every where as shown in the figure. So $\oint_S \vec{E} \cdot \hat{n} da > 0$ since $\vec{E}$



is directed outward everywhere over the surface. By Gauss' law this implies there is a non-zero charge enclosed within the surface S. This contradicts the fact that the region is chargeless. So we can't have a maxima at the point P. Similarly we can't have a minima at P.