1. Starting from the defination of a Coulomb of charge in the M.K.S system and an e.s.u of charge in the C.G.S system determine how much e.s.u of charge make one coulomb of charge.

   soln
   In the M.K.S system 1C is an amount of charge which when separated by 1m exerts a force of 9×10^9N on each other.

   So in the M.K.S system the electrostatic force law is given as

   $$F = 9 \times 10^9 \frac{q_1 q_2}{r^2}$$

   In the C.G.S system 1.e.s.u of charge exert a force of 1 dyne on each other when separated by 1cm. Hence in the C.G.S system the force law is

   $$F = \frac{q_1 q_2}{r^2}$$

   Ler $1C = x$e.s.u. Then converting the M.K.S into C.G.S units we get

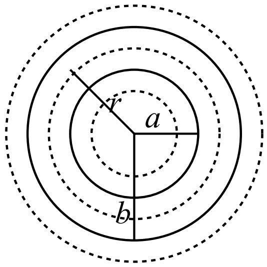
   $$\begin{aligned} F \times 10^5 \text{dynes} &= \frac{q_1 q_2 x^2}{(r \times 10^2)^2} \\ \therefore \quad 9 \times 10^9 \frac{q_1 q_2}{r^2} \times 10^5 &= \frac{q_1 q_2 x^2}{(r \times 10^2)^2} \\ \therefore \quad 9 \times 10^{18} &= x^2 \\ \therefore \quad x = 3 \times 10^9 & \end{aligned}$$

   $\therefore$ 1C $= 3 \times 10^9$e.s.u

2. If $\vec{\mathrm{E}} = kr^3\hat{\mathrm{r}}$ in a region find the charge density in the region.
   soln
   By Gauss' law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$.
   $\therefore \quad \rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} = \epsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 k r^3) = 5k\epsilon_0 r^2$.

3. A hollow spherical shell carries a charge density $\rho = k/r^2$ in the region $a \leq r \leq b$. Find the electric field in the three regions , $r < a, \quad a < r < b, \quad r > b$.
   soln
   Consider a Gausian surface inside the shell. Then

   $$E \times 4\pi r^2 = \frac{q_{enc}}{\epsilon_0} = 0$$



$\therefore E = 0$.
For a gausian surface in the shell we have

$$\begin{aligned} E \times 4\pi r^2 &= \frac{1}{\epsilon_0}\int \rho dV = \frac{1}{\epsilon_0}\int_a^r \frac{k}{r^2}4\pi r^2 dr \\ &= \frac{4\pi k}{\epsilon_0}(r-a) \end{aligned}$$

$\therefore\ E = \frac{k(r-a)}{\epsilon_0 r^2}$.
For $r > b$ the enclosed charge is $\frac{4\pi k}{\epsilon_0}(b-a)$.

$$\therefore\ E = \frac{k(b-a)}{\epsilon_0 r^2}$$

4. Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$. Find the electric field in the three regions, into which the planes partition the space.
   soln
   We have seen the electric field due to an infinite plane of uniform charge density σ. The electric field will be perpendicular to the plane and on either side it will be directed away from the plane. The magnitude of the field is $E = \frac{\sigma}{2\epsilon_0}$. Due to an infinite plane with surface charge density $-\sigma$ the electric field will be equal and opposite everywhere. When these two palnes are placed parallel to each other the electric field outside the region will cancel while between the plates they add up. So the electric field between the planes will be $\frac{\sigma}{\epsilon_0}$. The direction will be perpendicular to the planes and directed from the positively charged plane to the negatively charged plane. Outside the planes the field will be 0.

5. The electric field in a region is cylindrically symmetric, given as follows:

$$\begin{aligned} \vec{\mathrm{E}}(\vec{\mathrm{r}}) &= \frac{c\hat{\mathrm{s}}}{s}; \quad \text{when} \quad s \geq a \\ &= 0; \quad \text{when} \quad s < a \end{aligned}$$

   Find the charge distribution in the region using Gauss' law.
   soln
   The charge density is given by the differential form of Gauss' law $\vec{\nabla}\cdot\vec{E} = \rho/epsilon_0$. Due to cylindrical symmetry of the problem the partial differentiation w.r.t z and ϕ is zero. So we have

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s)$$

For $s > a, E_s = c/s \implies \vec{\nabla} \cdot \vec{E} = 0$.
For $s < a, E_s = 0 \implies \vec{\nabla} \cdot \vec{E} = 0$. So the charge density is 0 outside and inside the cylinder.
At $s = a, sE_s$ is not differentiable.

Consider the annular region enclosed by two cylindrical gausian surfaces of radius s_1 and s_2 such that $s_1 < a < s_2$. We confine the height of the cylinder between z_1 and z_2. $z_2 - z_1 = h$.
We will calculate the flux of $\vec{E}$ across the surface of this annular region. Over the outer cylinder $\hat{n} = \hat{s}$ and $\vec{E} = c\hat{s}/s_2$.
Over the inner cylinder $\hat{n} = -\hat{s}$ and $\vec{E} = 0$. The flux of $\vec{E}$ over the top and the bottom of the cylinders is 0 since the electric field is orthogonal to the normals there. So over our Gaissian surface we have

$$\oint_S \vec{E} \cdot \vec{da} = \int_{z_1}^{z_2} \int_0^{2\pi} \frac{c\hat{s}}{s_2}(s_2 d\phi dz)\hat{s} = 2\pi hc$$

If V is the volume enclosed by the annular region then

$$\int_V \vec{\nabla} \cdot \vec{E} dV = 2\pi hc \quad \text{by divergence theorem}$$

$$\therefore \quad \int_{s_1}^{s_2} \int_{z_1}^{z_2} \int_0^{2\pi} (\vec{\nabla} \cdot \vec{E}) s ds d\phi dz = 2\pi h \int_{s_1}^{s_2} (\vec{\nabla} \cdot \vec{E}) s ds = 2\pi hc$$

$$\therefore \quad \int_{s_1}^{s_2} (\vec{\nabla} \cdot \vec{E}) s ds = c$$

If the annular region doesn't enclose the surface $s = a$ then this integral is zero. But if $s_1 < a < s_2$ then the value of the integral is c. So we conclude

$$\vec{\nabla} \cdot \vec{E} s = c\delta(s-a) \quad \implies \quad \vec{\nabla} \cdot \vec{E} = \frac{c}{s}\delta(s-a)$$

So the charge density at the surface $s = a$ is given as

$$\rho = \epsilon_0 \frac{c}{a} \delta(s-a)$$

This is an infinite volume charge density. This is a finite amount of charge smeared over the surface $s = a$ whose thickness is zero. hence we must specify this density as a surface charge density. This will be given as

$$\sigma = \epsilon_0 \frac{c}{a}$$

Alternatively one can directly calculate this surface charge density by considering a small pillbox shaped gaussian surface enenclosing a part of the surface $s = a$. The flux contributed by the surface just outside the cylinder will be $c/a \times \Delta a$ where Δa is the area of the pillbox. There is no contribution the flux from the inner surface since the field is zero there. There will be no contribution from the side walls since the normals there will be orthogonal to the field. All the charge enclosed by the pill box is over the surface $s = a$. The amount of enclosed charge is $\sigma \Delta a$. So by Gauss' law we have

$$\frac{\sigma \Delta a}{\epsilon_0} = \frac{c}{a} \Delta a \quad \implies \quad \sigma = \epsilon_0 \frac{c}{a}$$

6. Evaluate
   (a) $\int(r^2+\vec{\mathrm{r}}\cdot\vec{\mathrm{a}}+a^2)\delta^3(\vec{\mathrm{r}}-\vec{\mathrm{a}})dV$ over the whole space where $\vec{\mathrm{a}}$ is a fixed vector.
   soln

$$\int_V (r^2+\vec{\mathrm{r}}\cdot\vec{\mathrm{a}}+a^2)\delta^3(\vec{\mathrm{r}}-\vec{\mathrm{a}})dV = 3a^2$$

   (b) $\int_V |\vec{\mathrm{r}}-\vec{\mathrm{b}}|^2\delta^3(5\vec{\mathrm{r}})dV$ over a cube of side 2, centered at the origin, and $\vec{\mathrm{b}} = 4\hat{\mathrm{y}}+3\hat{\mathrm{z}}$
   soln
   $dV = r^2 dr\sin\theta d\theta d\phi$.
   Let $5\vec{r} = \vec{r'}$. Then

$$\begin{aligned} dv' &= r'^2 dr'\sin\theta d\theta d\phi = 5^3 r^2 dr\sin\theta d\theta d\phi \\ &= 5^3 dV \\ \therefore \quad \int_V |\vec{r}-\vec{b}|\delta^3(5\vec{r})dV &= \int_{V'} |\frac{\vec{r'}}{5}-\vec{b}|\delta^3(\vec{r'})\frac{1}{5^3}dV' \\ &= \frac{1}{5^3}|\vec{b}| = \frac{1}{25} \end{aligned}$$

7. We have seen that $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$. In a similar manner justify that

$$\vec{\nabla}\cdot\left(\frac{\hat{s}}{s}\right) = 2\pi\delta^2(\vec{s})$$

   Here s is the distance from the z axis in cylindrical coordinates and $\delta^2(\vec{s})$ is a two dimensional delta function on the xy plane.

   soln
   We have seen that $\vec{\nabla}\cdot\left(\frac{\hat{s}}{s}\right) = 0$ for $s>0$. It tends to ∞ as $s\to 0$. Let us calculate the integral of this function over a cylindrical volume of radius a and height h enclosing the z axis.

$$\begin{aligned} h\int_0^a\int_0^{2\pi}\vec{\nabla}\cdot\left(\frac{\hat{s}}{s}\right)sdsd\phi &= h\int_0^{2\pi}\frac{\hat{s}}{a}\cdot\hat{s}ad\phi \qquad \text{by divergence theorem} \\ \therefore \quad \int_0^a\int_0^{2\pi}\vec{\nabla}\cdot\left(\frac{\hat{s}}{s}\right)sdsd\phi &= 2\pi \end{aligned}$$

   This is true for any cylinder with radius $a>0$ around the z axis. So we have

$$\vec{\nabla}\cdot\left(\frac{\hat{s}}{s}\right) = 2\pi\delta^2(\vec{s})$$

8. Prove that $\delta(r) = 4\pi r^2\delta^3(\vec{r})$ and $\delta(s) = 2\pi s\delta^2(\vec{s})$.
   Here $\int_0^\epsilon \delta(r)dr = 1$ for any $\epsilon>0$. The integral is 0 otherwise. $\delta(s)$ is defined likewise.

soln

Consider a sphere of radius ϵ around the origin

$$\begin{aligned}\int_V \delta^3(\vec{r})dV &= \int_0^\epsilon \delta^3(\vec{r})4\pi r^2 dr\\ \therefore \quad 1 &= \int_0^\epsilon \delta^3(\vec{r})4\pi r^2 dr\end{aligned}$$

So $\delta^3(\vec{r})4\pi r^2$ behaves as a one dimensional δ function $\delta(r)$.

In 2-dimension consider a circular disc of radius ϵ.

$$\begin{aligned}\int_S \delta^2(\vec{s})da &= \int_0^\epsilon \delta^2(\vec{s})2\pi s ds\\ \therefore \quad 1 &= \int_0^\epsilon \delta^2(\vec{s})2\pi s ds\end{aligned}$$

So $\delta^2(\vec{s})2\pi s$ behaves as a one dimensional δ function $\delta(s)$.

9. The electric field in a region is given as

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

Find the charge densities in the region.

soln

The volume charge density is given by the differential form of Gauss' law.

$$\rho(\vec{s}) = \epsilon_0 \vec{\nabla}\cdot\vec{E} = \frac{\lambda}{2\pi}\vec{\nabla}\cdot\left(\frac{\hat{s}}{s}\right)$$

It can be shown that $\vec{\nabla}\cdot\left(\frac{\hat{s}}{s}\right) = 2\pi\delta^2(\vec{s})$. This gives $\rho(\vec{s}) = \lambda\delta^2(\vec{s})$. This is a charge disdribution which is 0 every where except at $s = 0$, i.e along the z axis. We can get the linear charge density by integrating this volume charge density $\rho(\vec{s})$ over a thin cylinder of radius ϵ and height 1 unit.

$$\begin{aligned}\int_0^1\int_0^\epsilon \rho(\vec{s})2\pi s ds dz &= 1\times\int_0^\epsilon \lambda\delta^2(\vec{s})2\pi s ds\\ &= \lambda\int_0^\epsilon \delta(s)ds\\ &= \lambda\end{aligned}$$

So we have a line charge with linear density λ along the z axis.