# M-ary PSK / FSK - error performance

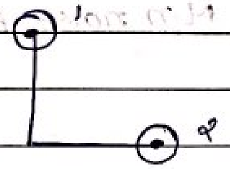
1. M-ary FSK :

$M \uparrow \Rightarrow P_B \downarrow$ (Assuming you are comparing
$BW \uparrow$ for same $\frac{E_b}{N_0}$ ratio.)

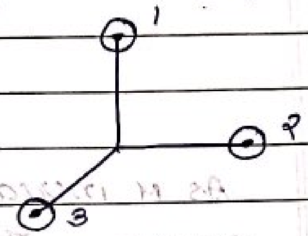
$M \uparrow \Rightarrow P_S \uparrow$

$d(s_i, s_j) = k \quad i \neq j \quad \propto$ symbol Energy

M = 2




M = 3




Distance remains the same, when M increases, there are more no. of ways to misclassify, thus $P_S$ increases if we have the same SNR ~~Eb/No~~ values.

$E_S = k E_B$ (k bits) $M = 2^k$

$E_B = \frac{E_S}{k}$

$k \uparrow \quad E_B \downarrow$ (Assuming same $E_S$)

We want to know how many bits we have incorrectly detected, not the symbols.

If we want to compare for same Eb/No, as k increases we need to increase $E_S$. $d(s_i, s_j) \propto E_S$. Thus more noise would be required for misclassification. Thus $P_B$ decreases as M increases.

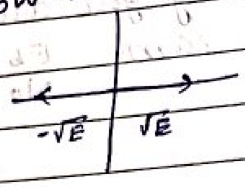
2. M-ary PSK:

$M \uparrow \Rightarrow P_B \uparrow$

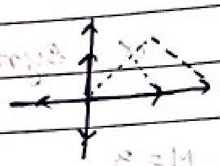
BW = K

Min noise power = E

M = 2




M = 4



Min noise power = $\frac{E}{2}$

As M increases, the distance between classes decreases. Thus less energy would be required for misclassification. Thus $P_B$ increases.

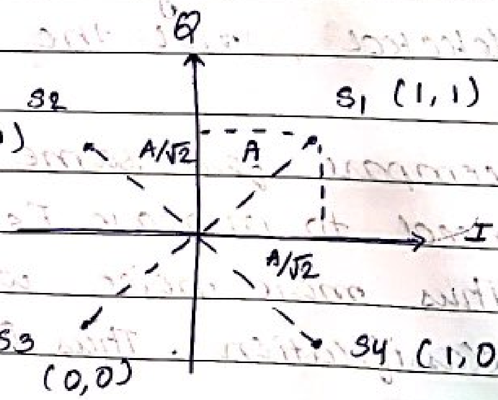
$E_S \uparrow \Rightarrow P_B \downarrow$

If we increase the symbol energy, then more noise power would be required for misclassification. Thus $P_B$ decreases.

$$S_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega t + \frac{\pi}{2}(i-1)\right) \qquad 0 \le t \le T$$

$i = 1, 2, 3, 4$

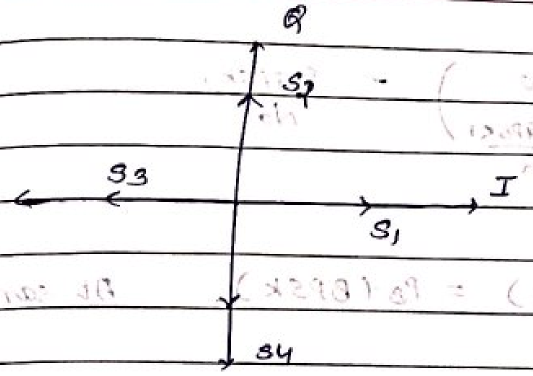
QPSK: (2)

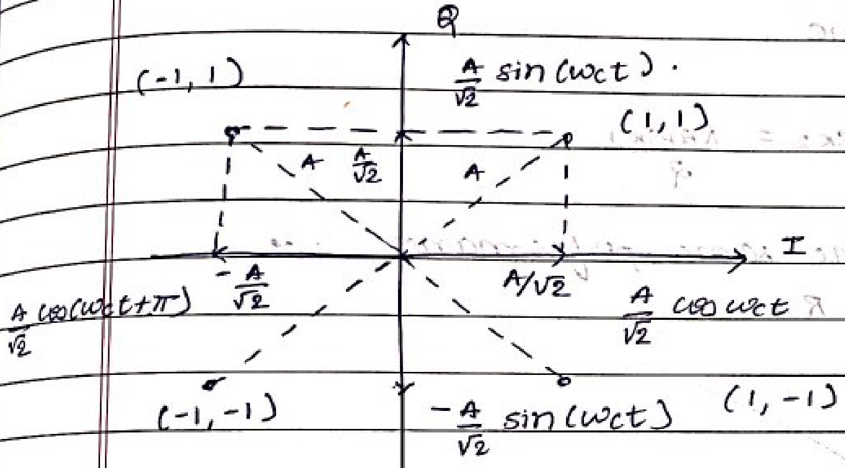



Constellation Diagram

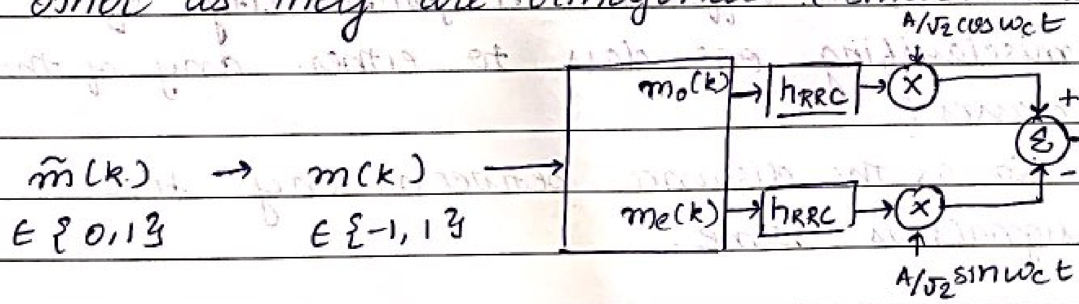
$$S_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega t + \frac{\pi}{4} + \frac{\pi}{2}(i-1)\right)$$

QPSK : (1)




If we only look at I axis, it would either be $\frac{A}{\sqrt{2}}$ or $-\frac{A}{\sqrt{2}}$ time I axis. Similarly for Q axis. Thus, this is ~~a~~ BPSK classification. Thus this QPSK can be considered as 2 BPSK. The 2 BPSKs ~~would~~ won't interfere with each other as they are orthogonal ($\sin\omega_c t$ and $\cos\omega_c t$).




$$\frac{E_{QPSK1}}{N_0} \qquad \frac{E_{QPSK2}}{N_0} = \frac{S_{QPSK2} \times T}{N/W}$$

$$= \frac{S_{QPSK2}}{N}\left(\frac{W}{R_{QPSK_2}}\right)$$

$$\frac{E_{QPSK2}}{N_0} = \frac{S_{QPSK1}}{2N}\left(\frac{W}{\frac{R_{QPSK1}}{2}}\right) = \frac{E_{QPSK1}}{N_0}.$$

$$P_B(QPSK_1) = P_B(QPSK_2) = P_B(BPSK) \quad \text{At same } \frac{E_B}{N_0}$$

Contradicting as $M\uparrow$, $P_B\uparrow$, but here it remains the same.

Over here $R_{QPSK2} = \frac{R_{QPSK1}}{2}$.

Thus, to keep the same performance, we need to sacrifice $R$.

M-ary FSK:

$$s_i(t) = \sqrt{\frac{2E}{T}}\cos(\omega_i t) \quad 0 \le t \le T$$
$$i = 1, 2, \ldots M$$

$$d(s_i, s_j) = k \quad i \neq j$$

Is there a difference in probability of misclassifing one class to either any of the classes?
No as the distance between any two signals is same.

All symbol misclassifications are equally likely.

$M = 2^k$

$\left.\begin{array}{l} 0000 \\ 0001 \\ \vdots \\ 1111 \end{array}\right\}$ 0000 can be equally likely be misclassified in any of the 15.

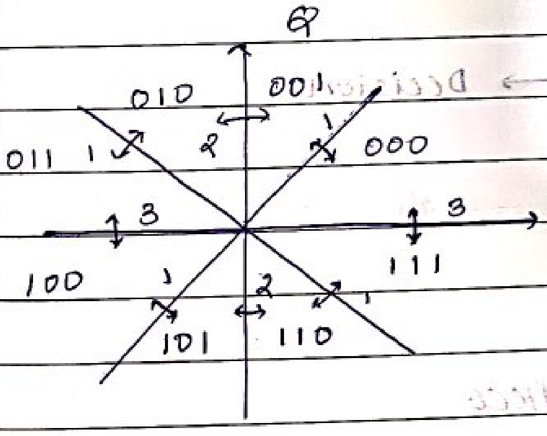
For any bit for a number, half classes would have correct detection and half would have incorrect detection.

$$\frac{P_B}{P_S} = \frac{2^{k-1}}{2^k - 1}$$

$k=1 \quad P_B = P_S$.

M-ary PSK:

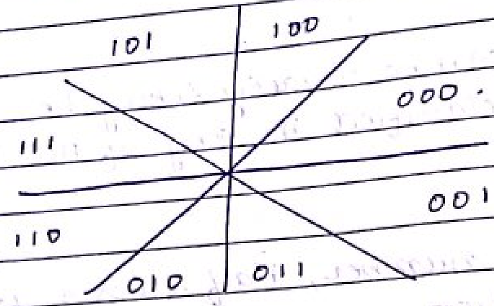
$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega t + \frac{\pi}{M}(i-1)\right) \quad 0 \le t \le T$$
$$i = 1, 2, \dots M$$




If 000 is transmitted, it is more likely it gets missclassified to 001 or 111

Thus symbol misclassification is more likely to get classified as near classes.

Assuming $P_S$ is small



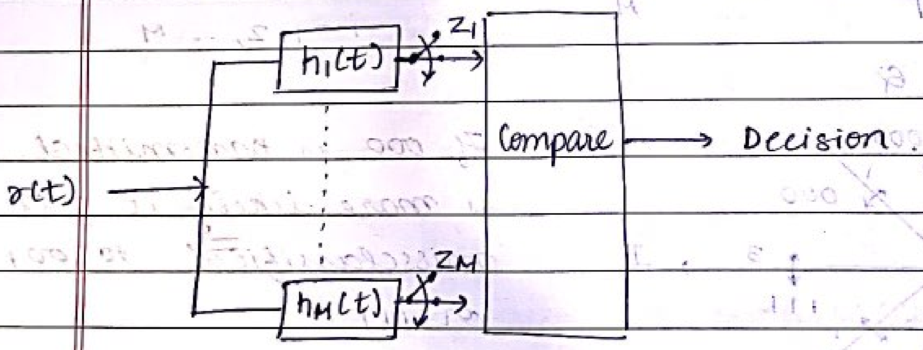

Gray code.
Every adjacent encoding is misclassified by only 1 bit error.

$$\frac{P_B}{P_S} = \frac{1}{k}$$

Once the symbol error has occurred what are the number of bits are incorrectly detected.

Symbol Error Rate ($P_S$)

1. FSK (Coherent Matched Filter Detector)




Assume $s_1$ was transmitted

$$\underset{z_1}{P(z_1)} \sim \mathcal{N}(\sqrt{E}, \sigma_0)$$

$$\underset{z_k}{P(z_k)} \sim \mathcal{N}(0, \sigma_0) \qquad k = 2, \ldots M \;/\; k \neq 1$$

~~$P_c = P(z_1 > z_2, z_3, \dots z_M)$~~

$$P_c = \int_{-\infty}^{\infty} P(z_1 > z_2,\ z_1 > z_3, \dots z_1 > z_M | z_1) P(z_1) \cdot dz_1$$

$$P(z_1 > z_2,\ z_1 > z_3, \dots z_1 > z_M | z_1) = \prod_{k=2}^{M} P(z_1 > z_k | z_1).$$

(As $z_k$ are independent)

Symbol error rates ($P_s$):

1. MFSK (Coherent).

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t) \qquad 0 \le t \le T$$
$$i = 1, 2, \dots M.$$

Output of M correlation: $z_1, z_2, \dots z_M$

Assume $s_1$ was transmitted.

$$P_s = 1 - P_c$$

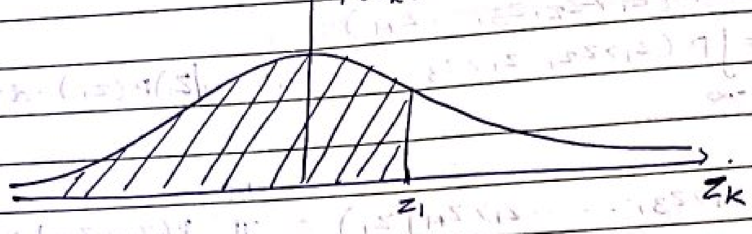
$$P_c = \int_{-\infty}^{\infty} P(z_1 > z_2, \dots z_1 > z_M | z_1)\, p(z_1)\, dz_1 =$$

$$P(z_1 > z_2, \dots z_1 > z_M | z_1) = \prod_{k=2}^{M} P(z_1 > z_k | z_1)$$

$P(z_1 > z_k | z_1)$

$$z_k \sim \mathcal{N}(0, \sigma_0) \qquad k \neq 1.$$

$$P(z_1 > z_k | z_1) = \int_{-\infty}^{z_1} \mathcal{N}_{z_k}(0, \sigma_0)\, dz_k$$




$z_1 \sim \mathcal{N}(\sqrt{E}, \sigma_0)$. As we know that $s_1$ was transmitted.

Upper bounds for $P_s$:

$s_1, s_2, \dots\dots s_M$.

Classification is done by using coherant demodulators.
Assume $s_1$ was transmitted
Misclassification $\Rightarrow \exists k \in (2, 3, \dots M)$ s.t. $z_k > z_1$

$E_k$ event for $z_k > z_1$ $k = 2, 3, \dots M$

$P_s = P($~~any~~ atleast one of $E_k$ has occured$)$

$= P\left(\bigcup_{k=2}^{M} E_k\right)$

$\leq \sum_{k=2}^{M} P(E_k)$

$\leq (M-1)\, Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

$\leq (M-1)\, Q\left(\sqrt{\frac{kE_b}{N_0}}\right)$ $E_s = kE_b$

$k \uparrow \Rightarrow P_s \downarrow$

$k \to \infty \quad P_s \to 0$

$$P_s \leq M\, Q\left(\sqrt{\frac{kE_b}{N_0}}\right) \qquad Q(x) \leq e^{-x^2/2}$$

$$P_s < 2^k e^{-\frac{kE_b}{2N_0}}$$

$$P_s < e^{k\ln 2}\, e^{-\frac{kE_b}{2N_0}}$$

$$P_s < e^{\frac{k}{2}(2\ln 2 - E_b/N_0)}$$

If $\frac{E_b}{N_0} > 2\ln 2 = 1.386$

$k \to \infty \quad P_s \to 0$

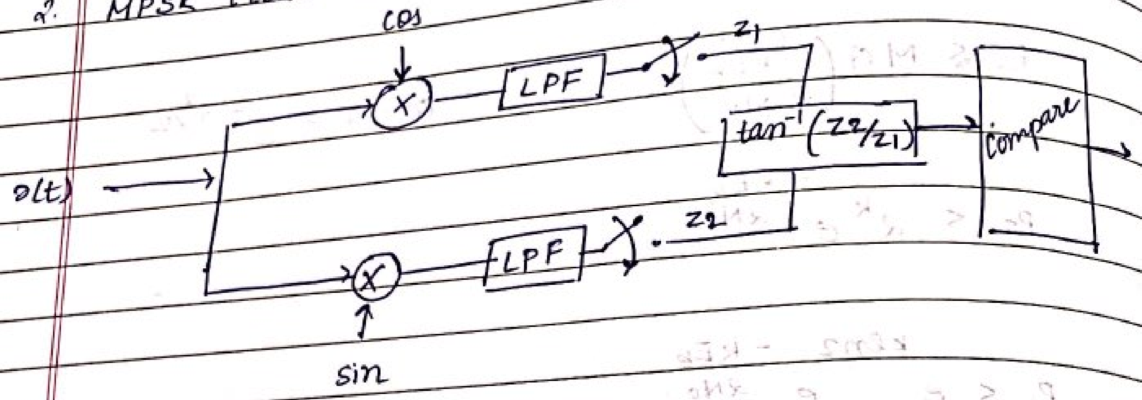
Tighter upper bound for $P_s$ :

$$P_s \sim e^{-k\left(\sqrt{\frac{E_b}{N_0}} - \ln 2\right)^2}$$ and If $\frac{E_b}{N_0} > \ln 2 \sim 0.69$

Shannon's Limit

then $k \to \infty \quad P_s \to 0$

2. MPSK (coherent).




$s_1$ was transmitted

$z_1 \sim \mathcal{N}(\sqrt{E}, \sigma_0^2)$

$z_1, z_2$

$P(z_1, z_2)$

$\downarrow$

$P(m, \phi)$

$m = \sqrt{z_1^2 + z_2^2}$

$\phi = \tan^{-1}\left(\frac{z_2}{z_1}\right)$

$$P_S = 1 - \int_{-\pi/M}^{\pi/M} \left( \int_0^\infty P(m, \phi)\, dm \right) d\phi$$

- Gradient Descent (GD)
- Mean square Error (MSE)

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Yash Vasawada

$$r_j = sh_j + n_j$$

r : output
h : input
n : noise

$$\cancel{\tilde{r}_j = sh_j} \quad \tilde{r}_s(h) = sh$$

Lost/cost Function

$$L(\hat{s}) = \sum_{j=1}^{N} (r_j - \tilde{r}_s(h))^2$$

$$= \sum_{j=1}^{N} (r_j - sh_j)^2$$

Minimize $L(S)$ and find the $\hat{s}$ for which $L(\hat{s})$ is minimum. Then, we get a good estimate for our points.

$$\frac{\partial}{\partial s} L(S) = \sum_{j=1}^{N} 2(r_j - sh_j)(-h_j)$$

$$s = \frac{\sum h_j r_j}{\sum h_j^2}$$

$$h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{bmatrix} \qquad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix}$$

$$s = \frac{h^T r}{h^T h}$$

Least square Solution

$$\therefore s = (H^T H)^{-1} H^T r \qquad r = Hs + u \quad \text{(Matrix)}.$$

$$L(\hat{s}) = \| \bar{r} - \bar{h} s \|^2$$

$$= \bar{r}^T \bar{r} - 2\bar{r}^T \bar{h} s + \bar{h}^T \bar{h} s^2$$

$s$ does not depend on $\bar{r}^T r$ and as we assumed M ary PSK, $s^2$ would be constant.

$$\text{Min}(L(S)) \Rightarrow \text{Max}(r^T h)$$

Minimum/ Projection OR Correlation OR Matched Filter.

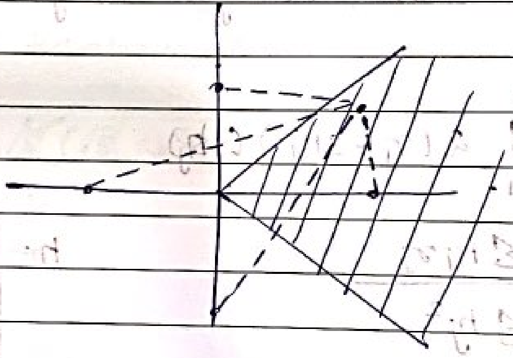
$$P(\hat{s}|\bar{r}) = \alpha e^{-L(\hat{s}).K}$$

↓ Gaussian PDF (Multivariate).

Conditional PDF/
Likelihood Function

To minimize the $L(s)$, we need to maximize the likelihood function.

Maximum Likelihood Estimate.



Minimum Eucleidian Distance.

M-PSK / M-FSK :

1. $P_B$
2. Bandwidth Efficiency

M-PSK :

$T \rightarrow$ symbol duration.

$R_S = \frac{1}{T}$

$W = R_S = \frac{R_B}{\log_2 M}$

Bandwidth Efficiency : $\frac{R_B}{W} = \log_2 M$ — For every Hz of channel bandwidth, you transmit $R_B$ bits/sec.

More the higher ratio, better the system.

$M \uparrow \longrightarrow$ BW-efficiency $\uparrow$

M-FSK :

$W_c = \frac{M}{2T} = \frac{M R_S}{2} = \frac{M}{2} \frac{R_B}{\log_2 M}$

M symbols, min separation is $\frac{1}{2T}$.

$\frac{R_B}{W} = \frac{2\log_2 M}{M}$

$M\uparrow \longrightarrow$ BW-efficiency $\downarrow$

-1.6dB

We want to maintain the same bit error rate i.e. $P_B$

MPSK :

As M ↑, BW efficiency ↑

As M ↑, $P_B$ ↑ for same $E_b/N_o$. So, now to maintain same $P_B$,

M ↑, $E_b/N_o$ ↑ same $P_B$.

MFSK :

As M ↑, BW - Efficiency ↓

$E_b/N_o$ ↓

If channel has sufficient BW, then you can save the power by using FSK.

If channel has BW constraints, then you would have to use PSK at the cost of higher power.

Shannon Hartly Capacity Theorem :

- AWGN channel.
- Channel Capacity (C)

Given that channel is affected by AWGN noise with x Hertz and some noise power. then what is the maximum bit rate that the channel can support theoritically.

$$C = W \log\left(1 + \frac{S}{N}\right)$$

$\frac{S}{N}$ : Avg SNR received over W Hz

W : Bandwidth of channel.

Any practical data rate must be upper bounded by C.

There exists a coding scheme to decode the signal under some S/N and W. so, it is theoritical proof.

$$\frac{E_b}{N_0} = \frac{ST}{N/W} = \frac{S}{N}\left(\frac{W}{R}\right)$$

Assume that $C = R$. (As C is the upper bound)

$$\frac{R}{W} = \log_2\left(1 + \frac{E_b}{N_0}\left(\frac{R}{W}\right)\right)$$

$$\frac{E_b}{N_0} = \frac{2^{(R/W)}-1}{R/W} \quad \text{(Theoritical Limit)}$$

$$\frac{E_b}{N_0} = \lim_{R/W \to 0} = \ln 2 \quad \text{Shannon's Limit}$$

$M \to \infty$ (in FSK)

Recap:

1. Analog vs Digital comparison.
2. Stochastic Processes.
   - Stationarity
   - Ergodicity.
3. Quantization (Analog to Digital)
   - Uniform
   - Non-uniform.

   Sampling
   - Anti aliasing filter.
4. Modulation
   - Baseband Modulation.
   - Bandpass Modulation.
5. Bandpass Demodulation
   - Matched filter (Maximize SNR)
   - Detection (MLE and MAP).
   - Performance criteria $P_B \propto \frac{E_b}{N_0}$.
   - ISI Nyquist Pulse shaping. (RC, RRC, etc); Eye diagram
   - Equalization filters : Zero forcing, MMSE
6. Bandpass Modulation / Demodulation
   - PSK, FSK, ASK, APK
   - PSK, FSK, demodulation : coherent; DPSK : non-coherent
   - QPSK Mod./Demodulation.
   - Error Performance : $P_B$, $P_S$. (what happens as $M \uparrow$?)
   - BW efficiency