1. Which of the following can be an electrostatic field?
   (a) $x\hat{\text{i}}$, **(b)** $y\hat{\text{i}}$, **(c)** $(1/r)\hat{\theta}$, **(d)** $(1/s)\hat{\phi}$

   soln:
   Electrostatic field is curlless. Only the fields (a) and (c) satisfy this condition. Hence they can be electric fields.

2. A sphere of radius a is maintained at a uniform potential V_0. Find the potential both, inside and outside the sphere.

   soln:
   The problem has spherical symmetry. The potential V is independent of θ and ϕ.

$$\therefore \quad \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)$$

Since there is no charge in the region $r > a$ and $r < a$, we solve

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

in both the region. This gives

$$V = -\frac{c}{r} + d$$

where c and d are constants. The solution will have the same form, both inside and outside the sphere but the constants are different and have to be determined from the boundary conditions. So we have

$$V_{in} = -\frac{c_1}{r} + d_1 \qquad \text{and} \qquad V_{out} = -\frac{c_2}{r} + d_2$$

Inside the sphere $\vec{\nabla}^2 V_{in} = 4\pi c_1 \delta^3(\vec{r})$. This corresponds to a point charge of magnitude $-4\pi\epsilon_0 c_1$ placed at the center of the sphere. Since there is no charge inside the sphere we conclude that $c_1 = 0$.
As $r \to \infty, V_{out} \to d_2$. If we demand the potential far away from the sphere to be 0 then $d_2 = 0$. So now we have

$$V_{out} = -\frac{c_2}{r} \quad \text{and} \quad V_{in} = d_1$$

At $r = a \quad V_{in} = V_{out} = V_0 \quad \Longrightarrow \quad c_2 = -aV_0$ and $d_1 = V_0$.

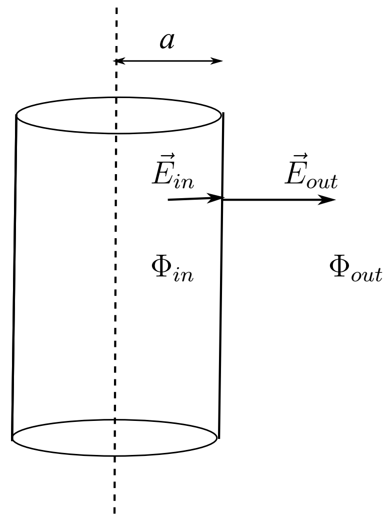
$$\therefore \quad V_{in} = V_0 \quad \text{and} \quad V_{out} = \frac{a}{r}V_0$$

3. A very long cylinder of radius a has a uniform surface charge of density σ. Find the electric potential both inside and outside the cylinder by solving the Laplace's Equation in the chargeless regions.

soln:

Consider the two regions outside and inside the cylinders. $\rho = 0$ in these regions. So the potential satisfies Laplace's Equation. in these regions. We will have

$$\nabla^2 \Phi_{out} = 0 \quad \text{and} \quad \nabla^2 \Phi_{in} = 0$$



Due to cylindrical symmetry the potential function will be only dependent on s. First consider Φ_{out}.

$$\nabla^2 \Phi_{out} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \Phi_{out}}{\partial s}\right) = 0$$

This gives $\Phi_{out} = c_1 \ln(s) + c_2$. where c_1 and c_2 are constants to be determined. Simiarly $\Phi_{in} = d_1 \ln(s) + d_2$.

From these potentials we get the electric field in the two regions as

$$\vec{E}_{out} = -\frac{c_1}{s}\hat{s} \qquad \text{and} \qquad \vec{E}_{in} = -\frac{d_1}{s}\hat{s}$$

But $\vec{E}_{in} = -\frac{d_1}{s}\hat{s}$ corresponds to a line charge along the axis of the cylinder with d_1 proportional to the line charge density. Since this is not present, we conclude $d_1 = 0$. This gives

$$\Phi_{in} = d_2 \quad \text{and} \quad \vec{E}_{in} = 0$$

At the boundary surface $s = a$ we apply the boundary condition for the normal component of the electric field.

$$\begin{aligned}\vec{E}_{out}\cdot\hat{s} - \vec{E}_{in}\cdot\hat{s} &= \frac{\sigma}{\epsilon_0} \\ \therefore\quad -\frac{c_1}{a} &= \frac{\sigma}{\epsilon_0} \\ \therefore\quad c_1 = -\frac{\sigma a}{\epsilon_0} &\end{aligned}$$

Now by the continuity of the potentials at the boundary $s = a$ gives

$$d_2 = -\frac{\sigma a}{\epsilon_0}\ln(a) + c_2$$

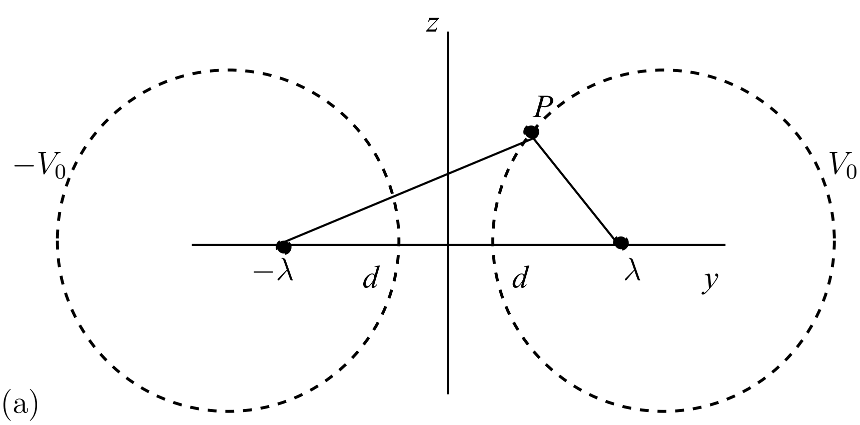
So we have

$$\Phi_{in} = -\frac{\sigma a}{\epsilon_0}\ln(a) + c_2 \quad \text{and} \quad \Phi_{out} = -\frac{\sigma a}{\epsilon_0}\ln(s) + c_2$$

c_2 is an arbitrary constant which can be reset to 0.

4. Two infinitely long wires running parallel to the x axis carry uniform charge densities $+\lambda$ and $-\lambda$.
(a) Find the potential at any point using the origin as the reference.
(b) Show that the equipotential surfaces are circular cylinders. Locate the axis and radius of the cylinder corresponding to a given potential V_0.
soln:
The two line charges $+\lambda$ and $-\lambda$, parallel to the x axis cuts the y axis at $y = -d$ and $y = d$ respectively. Consider a point $P(x, y, z)$. The distance of P from the line charges are s_1 and s_2 given by

$$s_1 = \sqrt{(y+d)^2 + z^2} \qquad \text{and} \qquad s_2 = \sqrt{(y-d)^2 + z^2}$$



(a)

The potential at point P is

$$V = \frac{-\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_1}{k_1}\right) + \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_2}{k_2}\right) = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\left(\frac{s_2}{s_1}\right) + \ln\left(\frac{k_1}{k_2}\right)\right]$$

If we want $V = 0$ at the origin where $s_1 = s_2 = d$ then $k_1 = k_2$. So

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\sqrt{(y-d)^2+z^2}}{\sqrt{(y+d)^2+z^2}}\right) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{(y-d)^2+z^2}{(y+d)^2+z^2}\right)$$

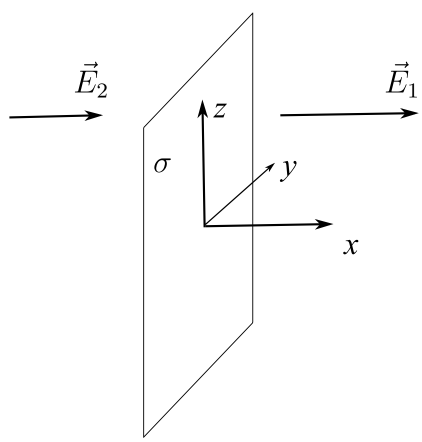
(b) Consider a surface of constant potential V_0. Then from the above eqn. we have

$$(y-d)^2 + z^2 = \left[(y+d)^2 + z^2\right] K \quad \text{where} \quad K = \exp\left[\frac{4\pi\epsilon_0 V_0}{\lambda}\right]$$

$$\begin{aligned} \therefore \quad y^2(1-K) + z^2(1-K) - 2yd(1+K) + d^2(1-K) &= 0 \\ \therefore \quad y^2 + z^2 - 2yd\left(\frac{1+K}{1-K}\right) + d^2 &= 0 \end{aligned}$$

This is the equation of a circle $(y - y_0)^2 + z^2 = R^2$ with center at $(y_0, 0) = (d\left(\frac{1+K}{1-K}\right), 0)$ and radius $R = \frac{2d\sqrt{K}}{1-K}$.

5. An infinite plane has a uniform charge with surface density σ. Solve the Laplace's Eqn on the two sides of the plane and evaluate the electric field using the boundary conditions on the fields at the interface.
soln:



Let the plane be along the yz plane. The plane divides the space into two regions, region 1, $x > 0$ and region 2, $x < 0$.
By symmetry we will have no variation of any field along the y and z directions. In region 1 we have $\nabla^2\Phi_1 = 0$, which simplifies to
$\frac{\partial^2\Phi_1}{\partial x^2} = 0$.
This gives $\Phi_1 = k_1 x + c_1$.
Similarly in region 2 $\Phi_2 = k_2 x + c_2$.
From the potential we get the electric fields

$$\vec{E}_1 = -k_1\hat{\imath} \quad \vec{E}_2 = -k_2\hat{\imath}$$

Using the boundary condition on the normal component of the electric field at he interface we get

$$E_1 - E_2 = k_2 - k_1 = \frac{\sigma}{\epsilon_0}$$

$$\therefore \quad k_2 = k_1 + \frac{\sigma}{\epsilon_0}$$

Equating Φ_1 and Φ_2 at $x = 0$ gives $c_1 = c_2$.
The potentials in the two regions then is

$$\Phi_1 = k_1 x + c_1 \quad \text{and} \quad \Phi_2 = \left(k_1 + \frac{\sigma}{\epsilon_0}\right) x + c_1$$

This is the most general potential we can obtain by solving the laplace's equation.
From these potentials $\vec{E}_1 = -k_1 \hat{i}$ and $\vec{E}_2 = -(k_1 + \sigma/\epsilon_0)\hat{i}$.
If we demand by symmetry of the problem that $\vec{E}_1$ and $\vec{E}_2$ are equal and opposite then $-k_1 = k_1 + \sigma/\epsilon_0$
$\implies \quad k_1 = -\sigma/(2\epsilon_0)$. This gives

$$\vec{E}_1 = \frac{\sigma}{2\epsilon_0}\hat{i} \quad \text{and} \quad \vec{E}_2 = -\frac{\sigma}{2\epsilon_0}\hat{i}$$

This problem highlights the point that though we expect a specific charge distribution to produce a unique field, there might be unspecified configuration outside our region of interest which can produce its fields. For example in this problem if we add a uniform electric field $\vec{E}_0$ everywhere, i.e, $\vec{E} \to \vec{E} + \vec{E}_0$ then this also satisfies the two required equation $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ and $\vec{\nabla} \times \vec{E} = 0$ and hence a valid solution. This reflects the fact that solutions of a differential equation inherrently has an arbitrariness which gets fixed by the boundary conditions.

6. A chargeless region is bounded by two conducting surfaces.

   (a) If conductor 1 is maintained at potential V_1 and 2 is grounded the potential in the region is given by the function $\Phi_1(x, y, z)$. If conductor 2 is maintained at potential V_2 and 1 is grounded the potential in the region is given by the function $\Phi_2(x, y, z)$.
   Now if conductor 1 is maintained at potential V_1 and conductor 2 is maintained at potential V_2 prove that the potential in the region will be given by the function $\Phi = \Phi_1 + \Phi_2$.
   soln:
   In the chargeless region

   $$\nabla^2\Phi = \nabla^2\Phi_1 + \nabla^2\Phi_2 = 0 + 0 = 0$$

   So $\Phi = \Phi_1 + \Phi_2$ satisfies the Laplace's equation in the region. We have to verify whether Φ matches the boundary condition, i.e it matches the potential on the two conductors.
   Over conductor 1, $\Phi_1 = V_1, \Phi_2 = 0$. So $\Phi = V_1$. Over conductor 2, $\Phi_1 = 0, \Phi_2 = V_2$. So $\Phi = V_2$. So this potential satisfies the potentials at the two bounding surfaces. Hence $\Phi = \Phi_1 + \Phi_2$ is a solution to the given electrostatic problem.
   Here we assume that the potential at ∞ is 0.

(b) If a charge Q_1 is placed on conductor 1 while 2 is chargeless the potential in the region is given by the function $\Phi_1(x, y, z)$. If a charge Q_2 is placed on conductor 2 while 1 is chargeless the potential in the region is given by the function $\Phi_2(x, y, z)$. Now if charge Q_1 is placed on conductor 1 and charge Q_2 is placed on 2 prove that the potential in the region will be given by the function $\Phi = \Phi_1 + \Phi_2$.

soln

Here we will work with the electric field. Let $\vec{E}_1 = -\vec{\nabla}\Phi_1$. Then over the surfaces S_1 and S_2 of conductor 1 and 2 we have

$$\oint_{S_1} \vec{E}_1 \cdot \hat{n} da = Q_1/\epsilon_0 \quad \text{and} \quad \oint_{S_2} \vec{E}_1 \cdot \hat{n} da = 0$$

Similarly for $\vec{E}_2 = -\vec{\nabla}\Phi_2$ we have

$$\oint_{S_1} \vec{E}_2 \cdot \hat{n} da = 0 \quad \text{and} \quad \oint_{S_2} \vec{E}_2 \cdot \hat{n} da = Q_2/\epsilon_0$$

Let $\vec{E} = \vec{E}_1 + \vec{E}_2 = -\vec{\nabla}\Phi$. From the above eqns. we can see that

$$\oint_{S_1} \vec{E} \cdot \hat{n} da = Q_1/\epsilon_0 \quad \text{and} \quad \oint_{S_2} \vec{E} \cdot \hat{n} da = Q_2/\epsilon_0$$

So the electric field caused by the potential Φ satisfies the required boundary conditions.

This result can be extended to a region bounded by any number of conductors.