1. Verify the divergence theorem with the vector field $\vec{A} = \vec{r}$ over a spherical region bounded by the surface of the sphere $x^2 + y^2 + z^2 = R^2$.

   soln
   We have seen earlier that $\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{r} = 3$. Over the surface of the sphere at every point the normal is radial. So $\hat{n} = \hat{r}$ and $\vec{r} = R\hat{r}$ at each point over the spherical surface. This gives

$$\begin{aligned}\oint_S \vec{A} \cdot \hat{n} da &= \oint_S \vec{r} \cdot \hat{r} da \\ &= R \oint_S da = R(4\pi R^2) = 4\pi R^3\end{aligned}$$

$$\begin{aligned}\int_V \vec{\nabla} \cdot vecA dv &= 3 \int_V dv \\ &= 3\left(\frac{4}{3}\pi R^3\right) = 4\pi R^3\end{aligned}$$

   Hence we see that
$$\oint_S \vec{A} \cdot \hat{n} da = \int_V \vec{\nabla} \cdot \vec{A} dv$$

2. Find the volume of the tetrahedron whose vertices are
   $(0,0,0), \quad (a,0,0), \quad (0,b,0), \quad (0,0,c)$.

   soln
   Let us denote the volume of the tetrahedron as V. The tetrahedron is bounded by the four faces $x = 0, y = 0, z = 0$ and $x/a + y/b + z/c = 1$. The volume will be given by the tripple integral $V = \int\int\int dxdydz$. We first do the z integral at a fixed (x, y). z runs from $z = 0$ upto the plane $x/a+y/b+z/c = 1$ which is characterized by the value $z = c(1 - x/a - y/b)$.

$$\therefore \quad V = \int\int\int_0^{c(1-x/a-y/b)} dzdxdy = \int\int c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dxdy$$

   Now we have a double integral on the xy plane. The region of the integration is a triangle on the plane bounded by the lines $x = 0, y = 0$ and $x/a + y/b = 1$.
   We will first do the y integral at a fixed x. y runs from $y = 0$ upto the line $x/a+y/b = 1$ which is characterized by the value $y = b(1 - x/a)$.

$$\begin{aligned}\therefore \quad V &= \int\int_0^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dydx \\ &= \int \left[c\left(1 - \frac{x}{a}\right) c\left(1 - \frac{x}{a}\right) - \frac{c}{2b} b^2 \left(1 - \frac{x}{a}\right)^2\right] dx\end{aligned}$$

Now we are left with an ordinary integral over x and x runs from $x = 0$ to $x = a$. So

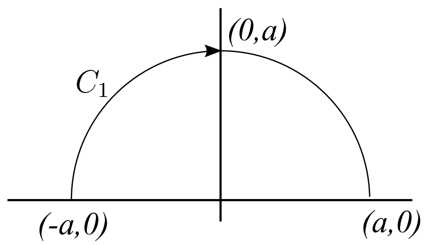
$$\begin{aligned} V &= \int_0^a \frac{bc}{2}\left(1 - \frac{x}{a}\right)^2 dx \\ &= \frac{abc}{6} \end{aligned}$$

Volume of the tetrahedron is one-sixth the volume of the rectangular parallelopiped with sides a, b, c.

3. Evaluate $\int_P^Q \vec{\mathrm{A}} \cdot \vec{\mathrm{dl}}$ for $\vec{\mathrm{A}} = y\hat{\mathrm{i}} - x\hat{\mathrm{j}}$ along the following arcs of a circle of radius a: $P \equiv (-a, 0)$; $Q \equiv (a, 0)$.

   (a) $(-a, 0) \to (0, a) \to (a, 0)$

   soln



$$\begin{aligned} \int_{C_1} \vec{A} \cdot \vec{dl} &= \int_{C_1} (y\hat{i} - x\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) \\ &= \int_{C_1} ydx - xdy \end{aligned}$$

Let $x = a\cos\theta,\ y = a\sin\theta$.
Then $dx = -a\sin\theta d\theta,\quad dy = a\cos\theta d\theta$. Over the curve C_1 θ goes from π to 0.

$$\begin{aligned} \therefore \quad \int_{C_1} \vec{A} \cdot \vec{dl} &= \int_\pi^0 -a^2 d\theta \\ &= \pi a^2 \end{aligned}$$

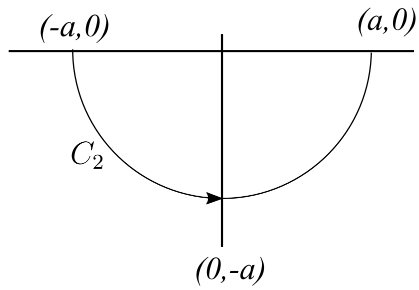
Along the curve C_1 $x^2 + y^2 = a^2$.
$\therefore\ 2xdx + 2ydy = 0$.
$\therefore\ dy = -\frac{x}{y}dx$

   (b) $(-a, 0) \to (0, -a) \to (a, 0)$

   soln
   soln

Along the lower curve we follow the same procedure. Here the only change will be that θ goes from π to 2π. This gives

$$\int_{C_2} \vec{A} \cdot \vec{dl} = -\pi a^2$$



(c) a loop, forward along (a) and backward along (b)
soln:
When we go backward along (b) θ goes from 2π to π or from 0 to $-\pi$. This will make the value of the integral as πa^2. So the total integral when we make one complete circle clockwise will be

$$\int_{C_1} \vec{A} \cdot d\vec{l} - \int_{C_2} \vec{A} \cdot d\vec{l} = \pi a^2 + \pi a^2 = 2\pi a^2$$

(d) Let I be the value of the loop integral evaluated in (c). Verify that at the origin

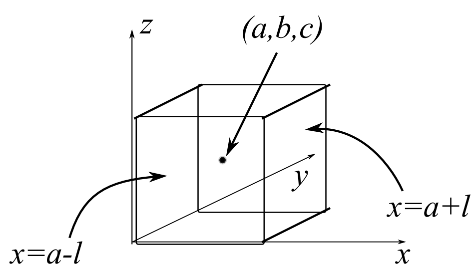
$$|\vec{\nabla} \times \vec{A}| = \lim_{a \to 0} I/(\pi a^2)$$

soln
$\vec{\nabla} \times \vec{A} = \hat{k}(-1-1) = -2\hat{k}$ which is constant everywhere. So $\vec{\nabla} \times \vec{A} = -2\hat{k}$ at the origin. We have $I = 2\pi a^2$.
$\therefore \ \frac{I}{\pi a^2} = 2.$
$\therefore \ \lim_{a \to 0} I/(\pi a^2) = 2$ which is the magnitude of $\vec{\nabla} \times \vec{A}$ at the origin.

4. Consider $\vec{\mathrm{A}} = x^2\hat{\mathrm{i}} + y^2\hat{\mathrm{j}} + z^2\hat{\mathrm{k}}$

(a) Evaluate $\oint_S \vec{\mathrm{A}} \cdot \vec{\mathrm{da}}$ where S is a cubical surface given by the planes $x = a \pm l$; $\quad y = b \pm l$; $\quad z = c \pm l$.

soln:
The surface of the cube consists of 6 planes. Let S_1 be the surface $x = a + l$.



Over S_1, $\quad \vec{A} = (a+l)^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ and $\vec{da} = \hat{i}dydz$.

$$\begin{aligned} \therefore \quad \int_{S_1} \vec{A} \cdot \vec{da} &= \int_{c-l}^{c+l} \int_{b-l}^{b+l} (a+l)^2 dydz \\ &= 4l^2(a+l)^2 \end{aligned}$$

Over the surface $S_2: \quad x = a - l$,
$\vec{A} = (a-l)^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ and $\vec{da} = -\hat{i}dydz$

$$\begin{aligned} \therefore \quad \int_{S_2} \vec{A} \cdot \vec{da} &= -\int_{c-l}^{c+l}\int_{b-l}^{b+l} (a-l)^2 dydz \\ &= -4l^2(a-l)^2 \end{aligned}$$

$\therefore$ net flux from S_1 and S_2 is $4l^2(a+l)^2 - (a-l)^2 = 16al^3$.
Similarly from the other two pair of surfaces we will have $16bl^3$ and $16al^3$.
So the total flux of the vector field $\vec{A}$ through the given cube is $16l^3(a+b+c)$.

(b) Verify that at the point (a, b, c),

$$\vec{\nabla} \cdot \vec{A} = \lim_{l \to 0} \frac{1}{8l^3} \oint_S \vec{\mathrm{A}} \cdot \vec{\mathrm{da}}$$

soln:
The volume of the cube is $8l^3$.

$$\begin{aligned} \therefore \quad \lim_{l \to 0} \frac{1}{V} \oint_S \vec{A} \cdot \vec{da} &= \lim_{l \to 0} \frac{1}{8l^3} 16l^3(a+b+c) \\ &= 2(a+b+c) \end{aligned}$$

This is same as the value of $\vec{\nabla} \cdot \vec{A}$ at (a, b, c).
This limit will be true for volume of any shape enclosing the point (a, b, c).

5. Let $\vec{A} = \hat{r}$. Evaluate $\int_S \vec{A} \cdot \vec{da}$ over the surface of a sphere given by the equation $x^2 + y^2 + z^2 = a^2$.

soln:
The normal to the surface of the sphere $x^2 + y^2 + z^2 = a^2$ is along $\hat{r}$. So $\vec{da} = \hat{r}da$.

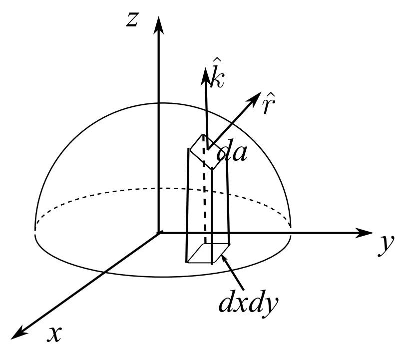
$$\therefore \quad \int_S \vec{A} \cdot \vec{da} = \int_S \hat{r} \cdot \hat{r} da = \int_S da$$

This integral gives the surface area of the sphere. We will find the surface area of the upper hemisphere as shown in the figure. The element of area da on the surface S has a projection $dydx$ on the xy plane as shown in the figure. The surface element $dxdy$ is along $\hat{k}$. So we have the following relation between da and $dxdy$:

$$dxdy = da\hat{r} \cdot \hat{k}$$

The sphere cuts the xy plane along a circle of radius a which forms the region of our integration over x and y. So over the upper hemisphere the integral becomes

$$\int_{hemisphere} da = \int_{hemisphere} \frac{dxdy}{\hat{r} \cdot \hat{k}}$$



Over the surface of the sphere $\hat{r} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$. So $\hat{r}\cdot\hat{k} = z/a$. So the above integral is

$$\int_{hemisphere} \frac{a}{z} dxdy$$

Over this hemisphere $z = \sqrt{a^2 - x^2 - y^2}$. For a given value of y, x runs from $-\sqrt{a^2-y^2}$ to $\sqrt{a^2-y^2}$. Putting all these together we have

$$\int_{hemisphere} da = \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \frac{a}{\sqrt{a^2-x^2-y^2}} dxdy$$

Substituting $x = \sqrt{a^2-y^2}\sin\theta$ we get

$$\int_{hemisphere} da = a\int_{-a}^{a}\int_{-\pi/2}^{\pi/2} d\theta dy = a\int_{-a}^{a} \pi dy = 2\pi a^2$$

The lower hemisphere will also contribute $2\pi a^2$ to the integral. Hence

$$\int_S \vec{A}\cdot\vec{da} = 4\pi a^2$$

which is the surface area of the sphere.