# Tute 9 Sol$^n$.

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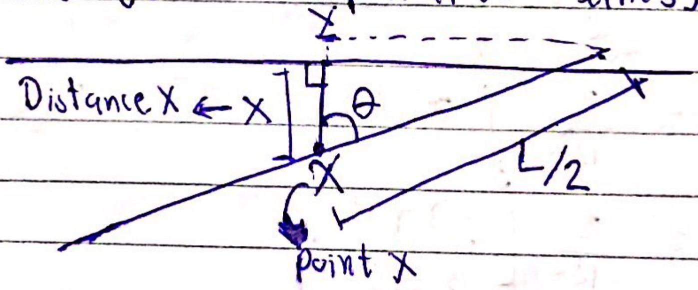
**Sol$^n$ 1** This problem is known as "Buffon's needle problem."

- A table has equidistant parallel lines a distant D apart.
- Length of a needle = L , $L \leq D$.
- Needle is thrown on table. What is the probability that it will intersect with line?

- Here the position of needle in the table can be shown by using 2 parameters.

 X = Distance between middle point of needle and the nearest line.

 $\theta$ = Angle between needle and the perpendicular line (to the parallel lines).




- The needle will intersect if the projection of that half portion (of length L/2) in that perpendicular line is greater than of equal to the perpendicular distance X.

$$\therefore \frac{L}{2}\cos\theta = |\overrightarrow{XY}| \geq X$$

$$\therefore \boxed{X \leq \frac{L}{2}\cos\theta}$$

→ $X$ is uniformly distributed between 0 and $D/2$

$$\therefore f_X(x) = \begin{cases} \frac{2}{D} & 0 < x < \frac{D}{2} \\ 0 & \text{otherwise.} \end{cases}$$

→ $\theta$ is uniformly distributed between $0$ & $\pi/2$

$$\therefore f_\theta(y) = \begin{cases} 2/\pi & 0 < y < \pi/2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\therefore P\left\{X \leq \frac{L}{2}\cos\theta\right\} = \iint_{X \leq \frac{L}{2}\cos\theta} f_X(x)\, f_\theta(y)\, d\theta\, dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{L}{2}\cos y} \frac{2}{D} \times \frac{2}{\pi}\, dx\, dy$$

$$= \frac{4}{D\pi} \int_0^{\frac{\pi}{2}} \frac{L}{2} \cos y\, dy$$

$$= \frac{4}{\pi D}\left[\frac{L}{2} \sin y\right]_0^{\frac{\pi}{2}}$$

$$\boxed{= \frac{2L}{\pi D}}$$

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**Solⁿ -2** First of all Let's prove the convolution function.

- X & Y are independent random variables.
  - $f_X(x) \to f_Y(y) \to$ Density $f^n$s.

$\therefore F_{X+Y}(a) = P[(X+Y) \leq a]$

$$= \iint_{x+y\leq a} f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{a-y} f_X(x)\, f_Y(y)\, dx\, dy$$

$$\therefore F_{X+Y}(a) = \int_{-\infty}^{\infty} F_X(a-y)\, f_Y(y)\, dy$$

$$\therefore f_{X+Y}(a) = \frac{d}{da}\left[F_{X+Y}(a)\right]$$

$$\therefore \boxed{f_{X+Y}(y) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy}$$

- $X_i$, $i = 1, 2, \ldots, n$ are normally distributed independent random variables. With parameters $\mu_i$ & $\sigma_i^2$, $i = 1, 2, \ldots, n$.

- Let's count for 2 R.V.

$X_1 \to \sigma_1^2, \mu_1$

$X_2 \to \sigma_2^2, \mu_2$

$$f_{X_1}(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$$

$$f_{X_2}(x) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}$$

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Now $Z = X_1 + X_2$ (Using convolution function)

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_{X_2}(z-x)\, f_{X_1}(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{(z-x-\mu_2)^2}{2\sigma_2^2}} \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(z-\mu_1)^2}{2\sigma_1^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} \exp\left[-\frac{(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)}\right] \frac{1}{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}} \exp\left[-\frac{\left(x - \frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right] \cdot dx$$

Multiply.

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \exp\left(-\frac{(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)}\right) * \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}} \exp\left[-\frac{\left(x - \frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right] dx$$

$$\boxed{= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \exp\left(-\frac{(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)}\right)} \quad (1)$$

(Because the term (integral term) will be 1 as the density $f^n$ of $X$ is there in integral).

- So we are getting $Z$ as a ~~Std~~ Normal Random variable with parameters $\boxed{\mu_1+\mu_2}$ & $\boxed{\sigma_1^2+\sigma_2^2}$

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* To prove for $n$ normal random variables we will use the induction hypothesis.

- $P(a)$: $X_i$, $i = 1, 2, \ldots, a$ are normal random variables with parameters $\mu_i$ & $\sigma_i^2$, $i = 1, 2, \ldots, a$. $\Rightarrow \sum_{i=1}^{a} x_i$ is a normal random variable with parameters $\sum_{i=1}^{a} \mu_i$ & $\sum_{i=1}^{a} \sigma_i^2$.

- $P(1)$ ÷ "Normal random variable it self is a normal random variable"
$\Rightarrow P(1)$ is true.

- $P(2)$ is also true according to result (1).

- Let's assume that $P(K)$ is true
$\therefore \sum_{i=1}^{K} x_i$ is a normal random variable

with parameters $\sum_{i=1}^{K} \mu_i$ & $\sum_{i=1}^{K} \sigma_i^2$.

to prove for $P(K+1)$

$$\sum_{i=1}^{K+1} X_i = \sum_{i=1}^{K} X_i + X_{K+1}$$

now in result (1) puting $\sigma_1^2 = \sum_{i=1}^{K} \sigma_i^2$ &

$$\mu_1 = \sum_{i=1}^{K} \mu_i.$$

$(\sigma_2^2 = \sigma_{K+1}^2)$

$(\mu_2 = \mu_{K+1})$

$X_1 = \sum_{i=1}^{K} x_i$ & $X_2 = X_{K+1}$

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So we will get the $\sum_{i=1}^{k+1} x_i$ also as a normal random variable with parameters $\sum_{i=1}^{k+1} \mu_i$ & $\sum_{i=1}^{k+1} \sigma_i^2$.

$\therefore$ P(k) is true $\Longrightarrow$ P(k+1) is true.

$\Rightarrow$ P(n) is true for all n.

$\Rightarrow$ $\sum_{i=1}^{n} x_i$ is a normal R.V. with parameters $\sum_{i=1}^{n} \mu_i$ & $\sum_{i=1}^{n} \sigma_i^2$.

Sol$^n$ 3 A Basket ball team has to play 44 games.

- 26 against team A. Winning probability = 0.4

  $X_A$ = number of games won by a team against team A.

  $n_1 = 26$ $p_1 = 0.4$

- 18 game against tem B.

  $X_B$ = number of games won against team B.

  $n_2 = 18$ $p_2 = 0.7$

By the normal appoximation of the binomial r.v., $X_A$ & $X_B$ can be seen as a normal random variables.

$X_A \rightarrow \mu_A = n_1 p_1 = 24 \times 0.4 = \boxed{10.4}$

$\sigma_A^2 = 26 \times 0.4 \times 0.6 = \boxed{6.24}$

$X_B \rightarrow \mu_B = n_2 p_2 = 18 \times 0.7 = \boxed{12.6}$

$\sigma_B^2 = n_2 p_2 (1-p_2) = 18 \times 0.7 \times 0.3 = \boxed{3.78}$.

$X_A + X_B$ is a normal random variable with parameters $\mu_A + \mu_B$ & $\sigma_A^2 + \sigma_B^2$.

$= 23$ & $= 10.02$

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(a) $P\{\text{The team wins 25 or more games}\}$

$= P\{X_A + X_B \geq 25\} = P\{X_A + X_B \geq 24.5\}$ (continuity correction)

$$= P\left\{\frac{X_A + X_B - 23}{\sqrt{10.02}} \geq \frac{24.5 - 23}{\sqrt{10.02}}\right\}$$

$$= P\left\{Z \geq \frac{1.5}{\sqrt{10.02}}\right\}$$

$$= 1 - P\{Z < 0.4739\}$$

$$\boxed{= 0.3178}$$

(b) The team wins more games against team A than it does against team B.

$\rightarrow X_A - X_B \rightarrow$ normal R.V. with parameters

$\mu_1 - \mu_2 = -2.2$ & variance

$= \sigma_1^2 + (-\sigma_2)^2 = \sigma_1^2 + \sigma_2^2 = 10.02.$

$\therefore P\{X_A > X_B\} = P\{X_A - X_B > 0\}$

$= P\{X_A - X_B \geq 0.5\}$ (continuity correction)

$$= P\left\{\frac{X_A - X_B - \mu'}{\sqrt{\sigma'^2}} \geq \frac{0.5 + 0.22}{\sqrt{10.02}}\right\}$$

$$= P\left\{Z \geq \frac{2.7}{\sqrt{10.02}}\right\}$$

$$= 1 - P\{Z < 0.8530\}$$

$$\boxed{= 0.1968}$$

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Sol^n 4 $S(n)$ = Price of a certain security after n additional weeks. $n \geq 1$

$\frac{S(n)}{S(n-1)}$, $n \geq 1$, are independent and identically distributed lognormal random variables.

$\mu = 0.0165$

$\sigma = 0.0730$

here $\log\left(\frac{S(n)}{S(n-1)}\right)$ is a normal random variable.

(a) P{The price of security increases over each of the next two weeks}

$\left(P\left\{\frac{S(1)}{S(0)} > 1\right\}\right)^2$

Here $\Rightarrow$ $P\left\{\frac{S(1)}{S(0)} > 1\right\} = P\left\{\log\left(\frac{S(1)}{S(0)}\right) > 0\right\}$

for next 1 week

$= P\left\{Z > \frac{0 - 0.0165}{0.0730}\right\}$

$= P\{Z < 0.2260\}$

$\boxed{= 0.5894}$

here $\frac{S(1)}{S(0)} > 1$ denotes that the price has been increased over one week. for 2 weeks it will be $(0.5894)^2 = \boxed{0.3474}$

(as the events are independent)

(b) P{The price is higher than today at the end of 2 weeks}

$= P\left\{\frac{S(2)}{S(0)} > 1\right\} = P\left\{\frac{S(2)\,S(1)}{S(1)\,S(0)} > 1\right\}$

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$$= P\left\{\log\left(\frac{S(2)}{S(1)}\right) + \log\left(\frac{S(1)}{S(0)}\right) > 0\right\}$$

here $\log\left(\frac{S(2)}{S(1)}\right) + \log\left(\frac{S(1)}{S(0)}\right)$ is a sum of 2

independent normal random variables.

$\mu_1 = \mu_2$ $\therefore \mu = \mu_1 + \mu_2 = 2\times 0.0165 = \boxed{0.0330}$

$\cancel{\sigma_1 = \mu_2}$ $\therefore \sigma = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{2}\times 0.0730 = \boxed{0.1032}$

$\sigma_1 = \sigma_2$ $\therefore P\left\{\log\left(\frac{S(2)}{S(1)}\right) > 1\right\} = P\left\{Z > \frac{-0.0330}{0.1032}\right\}$

$= P\{Z < 0.3195\}$

$\boxed{= 0.6254}$ a.