Stochastic Processes (Random Processes)

$\{X_i\}_{i \in I}$, $I \sim$ time

- Characterize a RP: $p(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$

$\forall\, t_i \in I, \ \forall n \in \mathbb{Z}^+$.

- Simplifying assumptions

1. Stationarity: (Shift Invariance)

$$p(X_{t_1}, \ldots, X_{t_n}) = p(X_{t_1+T}, \ldots, X_{t_n+T})$$

$$\forall\, t_i \in I, \ \forall n \in \mathbb{Z}^+.$$

$$\forall\, T$$

Such a Random process is called "Stationary in the strict sense".

$\Rightarrow$ Mean: of RVs from the SP X

$$m_X(t) = E[X_t] = \int_{-\infty}^{\infty} x_t \, p(x_t) \, dx_t$$

If X is a stationary (in the strict sense) RP

then
$$E[X_t] = \int_{-\infty}^{\infty} x_t \, p(x_{t+T}) \, dx_t = m_X(t+T) \quad \forall\, T$$

$$= m_X$$

② (Auto-) Correlation: $R(X_{t_1}, X_{t_2})$

$$R_X(t_1,t_2) := \mathbb{E}[X_{t_1}X_{t_2}]$$

$$= \mathbb{E}[X_{t_1+T}X_{t_2+T}], \forall T$$

$$T = -t_2 \Rightarrow \mathbb{E}[X_{t_1-t_2}\,X_0] = R_{X}(t_1-t_2, 0)$$

$$R_X(t_1,t_2) = R_{X}(t_1-t_2, 0), \forall t_1, t_2.$$

$\Rightarrow$ Autocorrelation depends only on the index (time) difference, and not on the absolute values of the indices!

☆ "Stationarity in Wide-Sense": If a Random process $\{X_t\}_{t\in I}$ satisfies:

a. $m_X(t) = m_X$.

b. $R_{X}(t_1,t_2) = R_{X}(t_1-t_2, 0) = R_X(t_1-t_2)$
$\forall t_1, t_2 \in I$.

then X_t is said to be stationary in the Wide-Sense OR Wide-Sense Stationary

$R_X(\tau)$, $\tau = t_1 - t_2$.

* Strict-Sense Stationary $\Rightarrow$ Wide-Sense Stationary

Not TRUE!

$X(t) = \cos(t+\theta)$, $\theta \sim U[0, 2\pi)$

Examples. (1) $X(t) = A\cos(\omega_c t + \theta)$, $A \sim U[-1,1]$

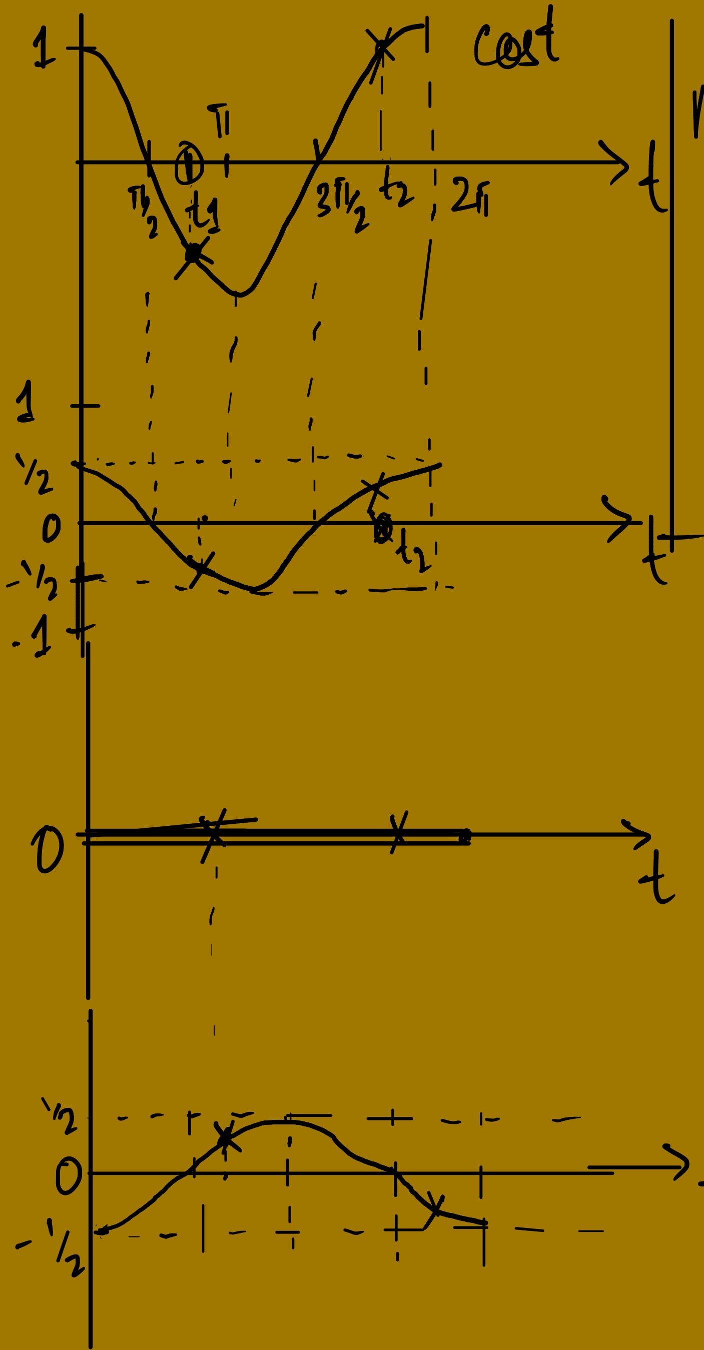
$\theta \in [0, 2\pi)$

$X_{t_1} = A\cos(\omega_c t_1 + \theta)$, $X_{t_2} = A\cos(\omega_c t_2 + \theta)$

$\ldots\ldots\quad X_{t_n} = A\cos(\omega_c t_n + \theta)$

$X(t) = A\cos t \quad \Big|\ \theta = 0,\ \omega_c = 1.\ \ I = [0, 2\pi).\ \Big|\ A \sim U[-1,1].$

$X_{t_1} = A\cos t_1 \quad X_{t_2} = A\cos t_2, \ldots, X_{t_n} = A\cos t_n.$



$A \to 1/2$
Sample func. $\to 1$

$A \to 0$
Sample func $\to 2$

$A \to -1/2$
Sample func $\to 3$

$$m_X(t) = \mathbb{E}[A\cos t]$$
$$= \mathbb{E}[A]\cos t$$
$$= 0 = m_X$$

$R_X(t_1, t_2)$
$$= \mathbb{E}[A\cos t_1 \cdot A\cos t_2]$$
$$= \mathbb{E}[A^2]\cos t_1 \cdot \cos t_2$$
$$= \frac{1}{3}\cos t_1 \cdot \cos t_2$$

$X(t) = A\cos t$ is not a Stationary RP.