Lecture-24

P 1

Recap

Expectation of sum = Sum of expectations.

→ Coupon collecting problem

→ drunkard's random walk

Quick Sort Algorithm

Average case $n \log n$

5, 9, 3, 10, 11, 14, 8, 4, 17, 6, 9 ②
3 6 1 7 8 9 5 2 10 4

{5, 9, 3, 8, 4, 6} 10 {11, 14, 17} 5

{5, 3, 4} 6 {9, 8} 10 {11, 14, 17} 2

{3} 4 {5} 6 {9, 8} 10 {11, 14, 17} 1

~~{3} 4 {5} 6 {8}~~

{3} 4 {5} 6 8 {9} 10 {11, 14, 17} 2

{3} 4 {5} 6 8 {9} 10 {11} 14 {17}

{3} 4 {5} 6 8 {9}

no. of comparisons = 19

x = no. of comparisons

For the sake of simplicity let us label the smallest element as 1, next smallest as 2 and so on.

$$I(i,j) = \begin{cases} 1 & \text{if } i \,\&\, j \text{ are ever compared} \\ 0 & \text{otherwise} \end{cases}$$

$I(1,5) = 0$, $I(5,7) = 1$ etc

$$X = \sum_{i \neq j}\sum I(i,j)$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} I(i,j)$$

$$E[X] = \sum\sum E[I(i,j)]$$

How do you compute $E[I(i,j)]$?

$E[I(i,j)] =$
$P(I(i,j) = 1)$

For any <u>indicator</u> random variable

$E[X] =$
$P(X=1)$

$E[X] =$
$1 \cdot P(X=1) +$
$0 \cdot P(X=0)$

i or j must be selected as pivots.

As long as i & j are in the same bracket, there is a possibility that they get compared.

If they are in different brackets, then they will never get compared.

Focus on these $\underline{\underline{j-i+1}}$ values

$i, i+1, i+2, \cdots, j-1, j$

If a pivot is chosen from one of these values, then they will get separated into different brackets.

i & j will get compared to each other iff i or j is the pivot.

$$P(I(ij)=1) = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n-1} \underbrace{\sum_{j=i+1}^{n} \frac{2}{j-i+1}}$$

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$$\sum_{j=i+1}^{n} \frac{2}{j-i+1} \simeq \int_{i+1}^{n} \frac{2}{x-i+1}\,dx$$

$$= 2\log(x-i+1) \Big]_{i+1}^{n}$$

$$= 2\left[\log(n-i+1) - \log(i+1-i+1)\right]$$

$$= 2\log(n-i+1) - \underbrace{2\log 2}_{\text{ignore}}$$

$$\simeq 2\log(n-i+1)$$

$$E[X] \simeq \sum_{i=1}^{n-1} 2\log(n-i+1)$$

$$\simeq \, ?$$

⑥

$$2\int_{1}^{n-1} \log(n-x+1)\,dx$$

$n-x+1=y$
$-dx=dy$

$$= 2\int_{n}^{2} \log y\,(-dy)$$

$$= 2\int_{2}^{n} \log y\,dy$$

$$= 2\left(y\log y - y\right)_{2}^{n}$$

$\cancel{\simeq 2y}$

$\simeq 2n\log n$

$= O(n\log n)$

$$N\left(\frac{N}{1} + \frac{N}{2} + \frac{N}{3} + \dots + \frac{N}{N}\right)$$

$$6\left(\frac{6}{1} + \frac{6}{2} + \dots + \frac{6}{6}\right) \simeq 14.7$$

$$\begin{cases} X_1 = \text{no. of times 1 appears} \\ X_2 = \qquad\qquad\quad 2 \\ \vdots \\ X_6 = \qquad\qquad\quad 6 \end{cases}$$

$$E[X_1] + E[X_2] + \dots + E[X_6] = 14.7$$

$$E[X_1] = \frac{14.7}{6}$$