

1. A sphere of radius R , centered at the origin, carries charge density $\rho(r, \theta) = k(R - r) \cos \theta$. Find the approximate potential on the z axis far from the sphere.

soln:

Let us denote the charge distribution as $\rho(\vec{r}')$ and the observation point as \vec{r} . In this notation the multipole expansion of the potential at \vec{r} is given as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int (r')^l P_l(\cos \theta') \rho(\vec{r}') d\tau'$$

The monopole term is the $l = 0$ term. This is given as

$$\begin{aligned} V_{mon}(z) &= \frac{1}{4\pi\epsilon_0 z} \int \rho(\vec{r}') d\tau' \\ &= \frac{1}{4\pi\epsilon_0 z} \int_0^R k(R - r) r'^2 dr' \int_0^\pi \sin \theta' \cos \theta' d\theta' \\ &= 0 \end{aligned} \tag{1}$$

as the angular integral is 0. So we go to the next significant term

$$\begin{aligned} V_{dip}(z) &= \frac{1}{4\pi\epsilon_0 z^2} \int_0^R \int_0^\pi r' \cos \theta' k(R - r') \cos \theta' r'^2 \sin \theta' dr' d\theta' d\phi' \\ &= \frac{1}{2\epsilon_0 z^2} \int_0^R r'^3 k(R - r') dr' \int_0^\pi \cos^2(\theta') \sin \theta' d\theta' \\ &= \frac{1}{2\epsilon_0 z^2} \frac{kR^5}{20} \frac{2}{3} \\ &= \frac{1}{60\epsilon_0 z^2} kR^5 \end{aligned}$$

This is the most significant term in the potential far away on the z axis. The charge distribution has an azimuthal symmetry. So at a general point (r, θ) the potential is

$$V_{dip}(r, \theta) = \frac{1}{60\epsilon_0 r^2} kR^5 \cos \theta$$

2. A dipole \vec{p} is at a distance r from a point charge q and oriented so that \vec{p} makes an angle θ with the vector \vec{r} from q to \vec{p} .
- What is the force on \vec{p} ?
 - What is the force on q ?

soln

In both the parts it is easier if we take the dipole along \hat{z} .

(a) Due to q at the origin the force on the dipole \vec{p} is

$$\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E} = p \frac{\partial \vec{E}}{\partial z}$$

$$\text{where } \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$$

$$\begin{aligned} \therefore \vec{F}_p &= \frac{qp}{4\pi\epsilon_0} \frac{\partial}{\partial z} \left(\frac{\vec{r}}{r^3} \right) \\ &= \frac{qp}{4\pi\epsilon_0} \left[\vec{r} \frac{\partial}{\partial z} \left(\frac{1}{r^3} \right) + \frac{1}{r^3} \frac{\partial \vec{r}}{\partial z} \right] \\ &= \frac{qp}{4\pi\epsilon_0} \left[-\frac{3z\vec{r}}{r^5} + \frac{\hat{z}}{r^3} \right] \\ &= \frac{q}{4\pi\epsilon_0} \left[-\frac{(3\vec{p} \cdot \vec{r})\vec{r}}{r^5} + \frac{\vec{p}}{r^3} \right] \\ &= -\frac{q}{4\pi\epsilon_0 r^3} [(3\vec{p} \cdot \hat{r})\hat{r} - \vec{p}] \end{aligned}$$

(b) For this part we place the dipole at the origin.

The electric field at q due to \vec{p} is

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3} [3(\vec{p} \cdot \hat{r}')\hat{r}' - \vec{p}]$$

Now $\vec{r}' = -\vec{r}$ (used in part (a)).

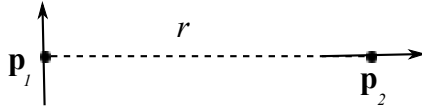
$$\therefore \hat{r}' = -\hat{r}.$$

\therefore force on q is

$$\vec{F}_q = a\vec{E} = \frac{q}{4\pi\epsilon_0 r^3} [3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}]$$

We see that the forces are equal and opposite.

3. \vec{p}_1 and \vec{p}_2 are perfect dipoles a distance r apart. \vec{p}_2 is along \vec{r} while \vec{p}_1 is orthogonal to \vec{r} . Calculate the torque on the dipoles. Are they equal and opposite?



soln

To calculate torque on \vec{p}_2 we consider \hat{z} along \vec{p}_1 . So at \vec{p}_2 the electric field is

$$\begin{aligned} \vec{E}_{p1} &= \frac{p_1}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}) \\ &= \frac{p_1}{4\pi\epsilon_0 r^3} \hat{\theta} \quad \text{since } \theta = \frac{\pi}{2} \\ \therefore \vec{\tau}_{p2} &= \vec{p}_2 \times \vec{E}_{p1} = \frac{p_1 p_2}{4\pi\epsilon_0 r^3} (-\hat{n}) \end{aligned}$$

where \hat{n} is a normal to the paper outward.

To calculate the torque on \vec{p}_1 due to \vec{p}_2 we consider the origin at \vec{p}_2 with \hat{z} along \hat{p}_2 .

$$\begin{aligned}\vec{E}_{p_2} \text{ at } \vec{p}_1 &= \frac{p_2}{4\pi\epsilon_0 r^3} (2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \\ &\quad - \frac{p_2}{4\pi\epsilon_0 r^3}(-2\hat{r}) \\ \therefore \vec{\tau}_{p_1} &= \vec{p}_1 \times \vec{E}_{p_2} = \frac{2p_1 p_2}{4\pi\epsilon_0 r^3}(-\hat{n})\end{aligned}$$

Note that the torques are not equal and opposite. Did you expect them to be so?

4. A sphere of radius R carries a polarization $\vec{P}(\vec{r}) = k\vec{r}$
- (a) Calculate the bound charges ρ_b and σ_b and the electric field caused due to them inside and outside the sphere.
- (b) Find the electric field using the Gauss' law for the displacement vector \vec{D} given as $\oint_S \vec{D} \cdot \hat{n} da = Q_{f(enc)}$.

soln:

(a)

The bound volume charge density is given as

$$\begin{aligned}\rho_b &= -\vec{\nabla} \cdot \vec{P} = -3k \\ \sigma_b &= \vec{P} \cdot \hat{n} = kR\end{aligned}$$

The given electrostatic configuration has a spherical symmetry. So by Gauss's law inside the sphere $r < R$ we have

$$E_{in} 4\pi r^2 = -\frac{3k}{\epsilon_0} \frac{4}{3} \pi r^3$$

This gives $\vec{E}_{in} = -\frac{k\vec{r}}{\epsilon_0}$.

Outside the sphere $r > R$ we have

$$E_{out} 4\pi r^2 = \frac{1}{\epsilon_0} \left[-3k \frac{4}{3} \pi R^3 + kR \times 4\pi R^2 \right] = 0$$

This gives $\vec{E}_{out} = 0$.

(b)

Since there is no free charges anywhere we have $Q_{f(enc)} = 0$. So using the Gauss' law for the displacement vector we get $\vec{D} = 0$ everywhere.

Since $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ we have $\vec{E} = -\frac{\vec{P}}{\epsilon_0}$.

This directly gives

$$\vec{E}_{in} = -\frac{\vec{P}}{\epsilon_0} = -\frac{k\vec{r}}{\epsilon_0}, \quad \text{and} \quad \vec{E}_{out} = 0$$

5. A point charge q is imbedded at the center of a sphere of linear dielectric material with susceptibility χ_e and radius R . Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b . What is the total bound charge on the surface? Where is the compensating negative bound charge located?

soln

The problem has a spherical symmetry.

Consider a Gaussian sphere of radius r . We have $D4\pi r^2 = q$.

$$\therefore D = q/4\pi r^2.$$

$$D = \epsilon_0 E + P = \epsilon_0 E + \epsilon_0 \chi_e E = \epsilon_0 (1 + \chi_e) E$$

$$\therefore E = \frac{D}{\epsilon_0 (1 + \chi_e)} = \frac{q}{4\pi \epsilon_0 (1 + \chi_e) r^2}$$

$$\text{Polarization } \vec{P} = \epsilon_0 \chi_e \vec{E}.$$

$$\therefore P = \frac{\chi_e}{1 + \chi_e} \frac{q}{4\pi r^2}$$

$$\text{Hence } \vec{\nabla} \cdot \vec{P} = 0 \text{ for } r > 0$$

$$\therefore \rho_b = -\vec{\nabla} \cdot \vec{P} = 0$$

On the surface of the sphere

$$\sigma_b = \vec{P} \cdot \hat{r} = \frac{\chi_e}{(1 + \chi_e)} \frac{q}{4\pi R^2}$$

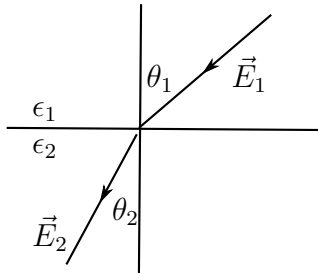
Total bound charge on the surface of the sphere is $\frac{\chi_e}{1 + \chi_e} q$.

Since the total bound charge has to be 0, the remaining bound charge is concentrated at the center surrounding the point charge. We can write

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{\chi_e}{1 + \chi_e} q \delta^{(3)}(\vec{r})$$

Inside the dielectric the charge q is screened by ρ_b and reduces the electric field

6. At the interface between one linear dielectric and another the electric field lines bend. Show that $\tan \theta_2 / \tan \theta_1 = \epsilon_2 / \epsilon_1$ assuming there is no free charge at the boundary. Refer to fig.1 below.



soln:

$$\vec{D}_1 = \epsilon_0 \epsilon_1 \vec{E}_1 \quad \text{and} \quad \vec{D}_2 = \epsilon_0 \epsilon_2 \vec{E}_2$$

Since there are no free charges at the interface

$$\begin{aligned} D_1^\perp &= D_2^\perp \\ \therefore \epsilon_0 \epsilon_1 E_1 \cos \theta_1 &= \epsilon_0 \epsilon_2 E_2 \cos \theta_2 \\ \therefore \epsilon_1 E_1 \cos \theta_1 &= \epsilon_2 E_2 \cos \theta_2 \end{aligned} \quad (2)$$

The parallel component of electric field must be equal.

$$\therefore E_1 \sin \theta_1 = E_2 \sin \theta_2 \quad (3)$$

From 2 and 3 we have

$$\frac{\tan \theta_1}{\epsilon_1} = \frac{\tan \theta_2}{\epsilon_2} \implies \frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_2}{\epsilon_1}$$

7. Suppose the field inside a large piece of dielectric is \vec{E}_0 , so that the electric displacement is $\vec{D}_0 = \epsilon_0 \vec{E}_0 + \vec{P}$.

- (a) If we have a narrow cylindrical (needle-like) cavity inside the material running parallel to \vec{P} find the field near the center of the cavity in terms of \vec{E}_0 and \vec{P} . Also find the displacement at the center of the cavity in terms of \vec{D}_0 and \vec{P} .
- (b) Do the same for a thin wafer shaped cavity perpendicular to \vec{P} .

soln:

- (a) The tangential component of the electric field along the cylindrical walls of the cavity must be continuous.

$$\begin{aligned} \therefore \vec{E} &= \vec{E}_0 \\ \vec{D} &= \epsilon_0 \vec{E}_0 = \vec{D}_0 - \vec{P} \end{aligned}$$

- (b) Here we use the boundary condition on the perpendicular component of \vec{D} since there are no free charges.

Near the center of the cavity

$$\begin{aligned} \vec{D} &= \vec{D}_0 \\ \vec{E} &= \frac{1}{\epsilon_0} \vec{D}_0 = \vec{E}_0 + \frac{1}{\epsilon_0} \vec{P} \end{aligned}$$