1. Find the capacitance per unit length of two coaxial metal cylindrical tubes of radii a and b.

   soln
   Let us assume $a < b$. To find capacitance we place equal and opposite charges on the cylinders. Since the cylinders are of infinite length we consider charge per unit length rather than total charge. Let λ be the charge per unit length on the inner cylinder and $-\lambda$ be the charge per unit length over the outer cylinder. The electric field in between the cylinders is radially outward and is given as $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$. The potential difference between the cylinders is

$$
\begin{aligned}
V_b - V_a &= -\int_a^b \vec{E} \cdot \vec{dl} = -\int_a^b \frac{\lambda}{2\pi\epsilon_0 s} ds \\
&= -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{a}{b}\right)
\end{aligned}
$$

   Per unit length the capacitance is

$$
C = \frac{\lambda}{V_b - V_a} = \frac{2\pi\epsilon_0}{\ln(a/b)}
$$

2. A chargeless region is bounded by two conducting surfaces.

   (a) If conductor 1 is maintained at potential V_1 and 2 is grounded the potential in the region is given by the function $\Phi_1(x, y, z)$. If conductor 2 is maintained at potential V_2 and 1 is grounded the potential in the region is given by the function $\Phi_2(x, y, z)$.
   Now if conductor 1 is maintained at potential V_1 and conductor 2 is maintained at potential V_2 prove that the potential in the region will be given by the function $\Phi = \Phi_1 + \Phi_2$.
   soln:
   In the chargeless region

$$
\nabla^2\Phi = \nabla^2\Phi_1 + \nabla^2\Phi_2 = 0 + 0 = 0
$$

   So $\Phi = \Phi_1 + \Phi_2$ satisfies the Laplace's equation in the region. We have to verify whether Φ matches the boundary condition, i.e it matches the potential on the two conductors.
   Over conductor 1, $\Phi_1 = V_1, \Phi_2 = 0$. So $\Phi = V_1$. Over conductor 2, $\Phi_1 = 0, \Phi_2 = V_2$. So $\Phi = V_2$. So this potential satisfies the potentials at the two bounding surfaces. Hence $\Phi = \Phi_1 + \Phi_2$ is a solution to the given electrostatic problem.
   Here we assume that the potential at ∞ is 0.

(b) If a charge Q_1 is placed on conductor 1 while 2 is chargeless the potential in the region is given by the function $\Phi_1(x, y, z)$. If a charge Q_2 is placed on conductor 2 while 1 is chargeless the potential in the region is given by the function $\Phi_2(x, y, z)$. Now if charge Q_1 is placed on conductor 1 and charge Q_2 is placed on 2 prove that the potential in the region will be given by the function $\Phi = \Phi_1 + \Phi_2$.

soln

Here we will work with the electric field. Let $\vec{E}_1 = -\vec{\nabla}\Phi_1$. Then over the surfaces S_1 and S_2 of conductor 1 and 2 we have

$$\oint_{S_1} \vec{E}_1 \cdot \hat{n} da = Q_1/\epsilon_0 \quad \text{and} \quad \oint_{S_2} \vec{E}_1 \cdot \hat{n} da = 0$$

Similarly for $\vec{E}_2 = -\vec{\nabla}\Phi_2$ we have

$$\oint_{S_1} \vec{E}_2 \cdot \hat{n} da = 0 \quad \text{and} \quad \oint_{S_2} \vec{E}_2 \cdot \hat{n} da = Q_2/\epsilon_0$$

Let $\vec{E} = \vec{E}_1 + \vec{E}_2 = -\vec{\nabla}\Phi$. From the above eqns. we can see that

$$\oint_{S_1} \vec{E} \cdot \hat{n} da = Q_1/\epsilon_0 \quad \text{and} \quad \oint_{S_2} \vec{E} \cdot \hat{n} da = Q_2/\epsilon_0$$

So the electric field caused by the potential Φ satisfies the required boundary conditions.

This result can be extended to a region bounded by any number of conductors.

3. A point charge q is placed a distance a from the center of a grounded conducting sphere of radius R, $\quad a > R$.

(a) In the method of images find the quantity and the position of the image charge. Justify that this image charge makes the potential of the whole conductor 0.

soln

If we consider the z- axis passing through the center of the sphere and the point charge q, then the problem has azimuthal symmetry. The potential at any point on the sphere can be specified as a function $V(r, \theta)$. We assume the image charge of magnitude q' placed at a distance a' from the center of the sphere so that the two poles are at 0 potential. This means $V(R, 0) = V(R, \pi) = 0$. This gives us

$$q' = -\frac{R}{a}q \qquad \text{and} \qquad a' = \frac{R^2}{a}$$

We have to make sure that this makes the potential over the sphere equal to zero.

$$\begin{aligned}
V(R,\theta) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2+R^2-2aR\cos\theta}} + \frac{q'}{\sqrt{a'^2+R^2-2a'R\cos\theta}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2+R^2-2aR\cos\theta}} - \frac{qR}{a\sqrt{(R^2/a)^2+R^2-2(R^2/a)R\cos\theta}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2+R^2-2aR\cos\theta}} - \frac{q}{\sqrt{R^2+a^2-2aR\cos\theta}}\right] \\
&= 0
\end{aligned}$$

(b) Find the force of attraction between the point charge and the sphere.

soln
The force of attraction between the point charge and the sphere is equivalent to the attraction between the charge q and the image charge. This is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-a')^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}$$

(c) If the sphere was not grounded, what would be the potential of the sphere when the charge q is placed at the distance a from the center of the sphere?

soln
If the sphere is not grounded, the sphere will acquire a uniform potential over it. Let us say this potential is V_0. In the absence of the external charge a potential of V_0 on the surface of the sphere indicates a total charge $Q = 4\pi\epsilon_0 RV_0$ uniformly spread over the sphere.
The given problem is a linear superposition of two charge distributions. One, due to the grounded sphere in the presence of the external charge q in front of it. Here the sphere acquires a negative charge equal to $q' = -qR/a$ over it. The other is a sphere charged to potential V_0 but no charge outside it. Here a charge Q is spread uniformly over it. But in the combined situation, since the sphere is insulated, the total charge on the sphere is 0.

$$\therefore\ Q + q' = 0, \implies \quad Q = qR/a$$

$$\therefore\ V_0 = \frac{Q}{4\pi\epsilon_0 R} = \frac{q}{4\pi\epsilon_0 a}$$

This will be the potential of an insulated spherical conductor.

4. A metal sphere of radius R carrying a charge q is surrounded by a thick concentric metal shell of inner radius a and outer radius b. The shell carries no net charge.

(a) Find the surface charge density σ at radius R, a and b.

soln

$$\sigma_R = \frac{q}{4\pi R^2}, \quad \sigma_a = -\frac{q}{4\pi a^2}, \quad \sigma_b == \frac{q}{4\pi b^2}$$

(b) Find the potential at the center, using infinity as the reference point.

soln
The potential at the center is equal to the sum of the potentials due to the three charged surfaces.

$$\therefore\ V_{center} = \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 a} + \frac{q}{4\pi\epsilon_0 b}$$

(c) If the outer surface $r = b$ is grounded how do the answers to part (a) and (b) change ?

soln

The potential of the outer sphere, when insulated , is $q/4\pi\epsilon_0 b$. When the outer sphere is grounded, it acquires a negative charge so as to cancel this potential. So it has to acquire an amount of charge $-q$ from the ground. This will make the charge density on the outer sphere 0. The inner sphere's don't feel this change on the outer sphere. The electric fields don't change, neither the charge densities on the inner surfaces ,i.e, at $r = R$ and $r = a$. But the potentials readjust. This will be given as

$$V_{center} = \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 a}$$

5. The points on the xy plane is maintained at potential $V_0 \sin(\alpha x + \beta)$. The potential goes to 0 as $z \to \pm\infty$. Find the potential at all the points above and below the xy plane.

soln:

In this problem it is clear that the potential doesn't vary along y.
Using the Variable separation method we get. $V(x, z) = X(x)Z(z)$ where

$$\begin{aligned} X(x) &= A\cos kx + B\sin kx \\ Z(z) &= Ce^{kz} + De^{-kz} \end{aligned}$$

We assume $k > 0$.
As $z \to \infty \quad V \to 0 \qquad \Longrightarrow \qquad C = 0$.
So for $z > 0$ we have $Z(z) = De^{-kz}$. Similarly for $z < 0$ we have $Z(z) = Ce^{kz}$.
Since the twopotentials match at $z = 0$ we have $C = D$.
So we have

$$\begin{aligned} V &= Ce^{-kz}(A\cos kx + B\sin kx) \quad \text{for} \quad z > 0 \\ &= Ce^{kz}(A\cos kx + B\sin kx) \quad \text{for} \quad z < 0 \end{aligned} \tag{1}$$

The constant C can be absorbed into A and B.
At $z = 0$ we equate V to the given potential function.

$$\begin{aligned} \therefore \quad A\cos kx + B\sin kx &= V_0\sin(\alpha x + \beta) \\ &= V_0(\sin\beta\cos\alpha x + \cos\beta\sin\alpha x) \end{aligned} \tag{2}$$

$\therefore \quad A = V_0 \sin\beta, \quad B = V_0\cos\beta$ and $k = \alpha$.

$$\begin{aligned} \therefore \quad V &= V_0 e^{-kz}(\sin\beta\cos kx + \cos\beta\sin kx) = V_0 e^{-kz}\sin(\alpha x + \beta) \quad ; \quad z > 0 \\ &= V_0 e^{kz}\sin(\alpha x + \beta) \quad ; \quad z < 0 \end{aligned} \tag{3}$$

6. Find the electric potential for $z \neq 0$ due to the infinite xy plane carrying a uniform charge density σ.
How can we get this potential by solving the two dimensional Laplace's equation
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} = 0$$
by the variable separation method ?
(We can also start with the three dimensional Laplace's equation. But two is good enough to demonstrate the process.)

soln
The electric field due to an infinite plane of uniform charge density is
$$\begin{aligned} \vec{E}(z > 0) &= \frac{\sigma}{2\epsilon_0}\hat{z} \\ \vec{E}(z < 0) &= -\frac{\sigma}{2\epsilon_0}\hat{z} \end{aligned}$$
The electric potential for $z > 0$ is
$$V = -\frac{\sigma z}{2\epsilon_0} + c$$
If we choose the xy plane as reference then $c = 0$. So $V = -\frac{\sigma z}{2\epsilon_0}$.
Similarly
$V = \frac{\sigma z}{2\epsilon_0}$ for $z < 0$.

Solving the Laplace's equation in two dimension we get the potential as
$$V(x, z) = X(x)Z(z)$$
where
$$\begin{aligned} X(x) &= A\cos kx + B\sin kx \\ Z(z) &= Ce^{kz} + De^{-kz} \end{aligned} \tag{4}$$
For the uniformly charged plate we expect no variation of potential along the x axis. This limit is realized with $k \to 0$. In this limit, $X(x) = A$ and
$$\begin{aligned} Z &\approx C(1 + kz) + D(1 - kz) \\ &= (C + D) + (C - D)kz \end{aligned}$$
$$\begin{aligned} \therefore \quad V(x, z) &= A(C + D) + A(C - D)kz \\ &= C' + D'z \end{aligned}$$
Comparing with the form of the potential we know for the infinitely charged sheet we see that $C' = 0$ and $D' = -\sigma/2\epsilon_0$ for $z > 0$ and $D' = \sigma/2\epsilon_0$ for $z < 0$.

Note: There is no such thing as an infinite plane with uniform charge. All such planes are finite. However when x and z are small the boundary of the plane appears at infinity and z is small compared to the size of the plane. In such limit kx, ky and kz is small. This limit is realized by taking $k \to 0$ in the general solution given by Eq.(4).