DA-IICT

Dhirubhai Ambani Institute of Information and Communication Technology (DA-IICT)
First In Semester Examination
CT314 (Statistical Communication Theory)
Date of Examination: Feb 08, 2012
Duration: 2 Hours
Maximum Marks: 20

Instructions:

1. Attempt all questions.
2. Use of scientific non programmable calculator is permitted.
3. Figures in brackets indicate full marks.
4. All the acronyms carry their usual meaning.

Q1. Prove that for events A, B, C, which are not necessarily mutually exclusive, $p(A \cup B / C) = p(A/C) + p(B/C) - p(A \cap B / C)$ (2 marks)

Q2. A random variable X takes only positive integer values {k=1,2,3...}. Show that $E(X) = \sum_k p\{X \geq k\}$ (3 marks)

Q3. Suppose the radius of a sphere is a continuous random variable R with the pdf as $f_R(r) = 6r(1-r), \quad 0 \leq r \leq 1$. (a) Find the CDF of volume $V = \frac{4}{3}\pi R^3$ by using the definition of CDF. (b) Use this CDF to find the pdf. (c) Now use the formula (do not derive) for finding the pdf of a function of one random variable and verify your answer (for pdf). (4 marks)

Q4: Consider an random variable $Z = X + Y$, where X and Y are statistically independent random variables.
(a) Show that $\phi_Z(w) = \phi_X(w)\phi_Y(w)$.
(b) Using the definition of characteristic function find the pdf of Z.
(c) Now consider the following uniform pdfs for X and Y:
(8 marks)

$f_X(x) = \frac{1}{2}, \quad for \ -1 \leq x \leq 1, \ 0$ otherwise and

$f_Y(y) = \frac{1}{3}, \quad for \ -2 \leq x \leq 1, 0$ otherwise. Find $p[Z \leq -2]$

Q5: Let X be an random variable and $0 < \text{var}(X) < \infty$, Show that

$$p\left\{-3.2 < \frac{X - E(X)}{\sqrt{Var(X)}} < 3.2\right\} > 0.9$$ (3 marks)

ANSWERS

In Sem I 8-2-2012

Ans 1.

$$P(A \cup B / C) = \frac{P((A \cup B) \cap C)}{P(C)}$$

$$= \frac{P((A+B)C)}{P(C)}$$

$$= \frac{P(AC + BC)}{P(C)} \quad , AC \text{ \& } BC \text{ NonME}$$

$$= \frac{P(AC) + P(BC) - P(ABC)}{P(C)}$$

$$= P(A/C) + P(B/C) - P(AB/C)$$

Ans 2.

$$E[X] = \sum_i x_i P(X = x_i)$$

$$= 1 \cdot P(X=1) + 2P(X=2) + 3P(X=3) + \cdots$$

$$= P(X=1) + P(X=2) + P(X=2) + P(X=3) + P(X=3)$$
$$+ P(X=3) + 4 \text{ times } P(X=4)$$

$$= P(X=1) + P(X=2) + P(X=3) + P(X=4) + \cdots$$
$$+ P(X=2) + P(X=3) + P(X=4) + P(X=5) + \cdots$$
$$+ P(X=3) + P(X=4) + P(X=5) + P(X=6) + \cdots$$

$$= \sum_k P(X \geq k)$$

Ans 3. Given $f(r) = 6r(r-1)$, $0 < r < 1$

Now V is another random variable (R.V) (transformed R.V).

V takes values betn 0 and $\frac{4\pi}{3}$ corresponding to $r=0$ & $r=1$

To find $F_V(v)$, for $v > 0$

$$v = g(r) = \frac{4}{3}\pi r^3$$

$$r = \left(\frac{3v}{4\pi}\right)^{1/3}$$

Now $\{V \leq v\} = \left\{R \leq \left(\frac{3v}{4\pi}\right)^{1/3}\right\}$

r & r are same

$$F_V(v) = P(V \leq v)$$

$$= P\left(R \leq \left(\frac{3v}{4\pi}\right)^{1/3}\right)$$

$$= \int_0^{\left(\frac{3v}{4\pi}\right)^{1/3}} f_R(r)\,dr$$

$$= \left(3r^2 - 2r^3\right)\Big|_0^{\left(\frac{3v}{4\pi}\right)^{1/3}}$$

ie. $$F_V(v) = \left\{3\cdot\left(\frac{3v}{4\pi}\right)^{2/3} - 2\left(\frac{3v}{4\pi}\right)\right\} \quad \ldots \; 0 \leq v \leq \frac{4\pi}{3}$$

$$f_V(v) = \frac{d}{dv}F_V(v) = \frac{3}{2\pi}\left[\left(\frac{3v}{4\pi}\right)^{-1/3} - 1\right], \quad 0 \leq v \leq \frac{4\pi}{3}$$

$$\therefore f_V(v) = \frac{6r(1-r)}{4\pi r^2}$$

$$= \frac{3}{2\pi}\left(\frac{1}{r} - 1\right)$$

$$= \frac{3}{2\pi}\left[\left(\frac{3v}{4\pi}\right)^{-1/3} - 1\right]$$

$0 < v \leq \frac{4\pi}{3}$

Now using the concept of transformation of one r.v (to find the pdf) ie. $Y = g(X)$, $V = g(R)$

$f_V(v) = \frac{f_R(r)}{g'(r)}$ (one root at $r_1 = \left(\frac{3v}{4\pi}\right)^{1/3}$) $\therefore f_V(v) = (6r^2 - 6r) / \frac{4}{3}\cdot 3\pi r^2$

Q4. $Z = X + Y$

Given X and Y are statistically independent.

Any f^n s of X and Y are also statistically independent.

ie $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

Now Characteristic f^n of Z is given by

$$E[e^{j\omega Z}] = \int_{-\infty}^{\infty} e^{j\omega z} f_Z(z)\,dz = \phi_Z(\omega)$$

$$= \int_{-\infty}^{\infty} e^{j\omega(\cancel{x}+y)} f_Z(z)\,dz$$ ⇒ Not required to be written.

& $$E[e^{j\omega Z}] = E[e^{j\omega(X+Y)}]$$

$$= E[e^{j\omega X} \cdot e^{j\omega Y}]$$

Use statistical independence of X & Y

$$\therefore E[e^{j\omega Z}] = E[e^{j\omega X}] \cdot E[e^{j\omega Y}]$$

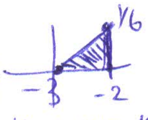
$$\therefore \phi_Z(\omega) = \phi_X(\omega)\,\phi_Y(\omega)$$

taking the inverse FT gives $f_Z(z)$

But we know that

$$f_X(z) * f_Y(z) \leftrightarrow \phi_X(\omega)\,\phi_Y(\omega)$$

$\therefore f_Z(z)$ = convolution of $f_X(z)$ and $f_Y(z)$

The convolution extends from −3 to 2 so $P[Z \le -2]$ = Area of 

Triangle

$= 1/6 \times 1/2 \times 1 = 1/12$

Q5. $P\left[\left| \frac{X - m_x}{\sigma_x} \right| < 3.2 \right\}$ is required.

ie. $P[\, |X - m_x| < 3.2\sigma \,]$ is required.

We know that

$$P[\, |X - m_x| \geq k\sigma \,] \leq \frac{1}{k^2}$$

$$\therefore \cancel{P[\, |X - m_x| \leq k\sigma \,] \geq \left(1 - \frac{1}{k^2}\right)}$$ ~~$k = 3.2$~~ in this ~~example~~

$$\cancel{\leq \left(1 - \frac{1}{(3.2)^2}\right)}$$

$$\therefore \cancel{P[\, |X - m_x| \leq 3.2\sigma \,]} >$$

$$P[\, |X - m_x| < k\sigma_x \,] > 1 - 1/k^2, \quad k = 3.2$$

$$\therefore P\left[\frac{|X - m_x|}{\sigma_x} < 3.2 \right] > 1 - \frac{1}{(3.2)^2}$$

$$> 0.9023438 \quad \text{(exact)}$$

~~d, think~~

ie. > 0.90