Tute 11 Sol^n

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Sol^n-1 $X_1, X_2, \ldots, X_n$ independent, ~~equal~~ identically distributed R.V.

Expected value $= \mu$

variance $= \sigma^2$

$$\overline{X} = \sum_{i=1}^{n} \frac{x_i}{n} \rightarrow \text{sample mean.}$$

$$x_i - \overline{x} = \text{deviations}$$

$$S^2 = \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{n-1} \quad \text{sample variance.}$$

- i) $Var(\overline{X}) = ?$

$$Var(\overline{x}) = Var\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var(x_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2$$

$$= \frac{n\sigma^2}{n^2}$$

$$\boxed{= \frac{\sigma^2}{n}}$$

ii) $E[S^2] = E\left[\sum_{i=1}^{n} \frac{(x_i - \overline{X})^2}{n-1}\right]$

$$\therefore (n-1)E[S^2] = \sum_{i=1}^{n} E[x_i - \overline{x}]^2$$

$$\therefore (n-1)E[S^2] = \sum_{i=1}^{n} E(x_i - \mu + \mu - \overline{x}]^2.$$

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$$\therefore (n-1)E[S^2] = \sum_{i=1}^{n} E\left[(X_i-\mu)^2 + (\bar{X}-\mu)^2 - 2\,\cancel{(\bar{X}-\mu)(\bar{X}-\mu)}\ (X_i-\mu)(\bar{X}-\mu)\right]$$

$$= \sum_{i=1}^{n} E[x_i-\mu]^2 + \sum_{i=1}^{n} E\left[(\bar{X}-\mu)^2 - 2\,\cancel{(\bar{X}-\mu)(\bar{X}-\mu)}\ (x_i-\mu)(\bar{X}-\mu)\right]$$

$$= n\sigma^2 + E\left[\sum_{i=1}^{n}(\bar{X}-\mu)^2 - 2(\bar{X}-\mu)\sum_{i=1}^{n}(x_i-\mu)\right]$$

$\left(\because \bar{X} = \frac{\sum x_i}{n}\right)$

$$= n\sigma^2 + E\left[n(\bar{X}-\mu)^2 - 2n(\bar{X}-\mu)(\bar{X}-\mu)\right]$$

$\downarrow$

$n\bar{X} = \sum_i X_i$

$$= n\sigma^2 + E\left[-n(\bar{X}-\mu)^2\right]$$

$$= n\sigma^2 - E[\bar{X}-\mu]^2$$

$$= n\sigma^2 - n\,Var(\bar{X})$$

$$= n\sigma^2 - \sigma^2$$

$$\boxed{=(n-1)\sigma^2}$$

Sol^n 2 X & Y $\rightarrow$ Independent binomial random variables.
$\rightarrow$ identical parameters n & p.

~~$P\{X\}$~~

$k < \min(n,m)$

$$P\{X=k \mid X+Y=m\} = \frac{P\{X=k,\ X+Y=m\}}{P\{X+Y=m\}}$$

$$= \frac{P\{X=k,\ X=m-k\}}{P\{X+Y=m\}}$$

$$= \frac{\binom{n}{k}p^k(1-p)^{n-k}\ \binom{n}{m-k}p^{m-k}(1-p)^{n-m+k}}{\binom{2n}{m}p^m(1-p)^{2n-m}}$$

$$= \boxed{\frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}}$$

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here $X+Y$ is a binomial R.V. with parameters $(2n, p)$. And the conditional distribution of X, given that $X+Y=m$ is hypergeometric ~~R.V.~~ Distribution.

$$E[X \mid X+Y=m] = \frac{m \cdot n}{(2n)}$$

$$\boxed{= \cancel{\frac{m}{2}}} \quad \boxed{= \frac{m}{2}}$$

$m \rightarrow$ here $X+Y=m \rightarrow$ total successes.

$N \rightarrow$ $2n \rightarrow$ total possible trials / "total balls"

$n \rightarrow$ $n \rightarrow$ total trials of X.

<u>Soln 3</u>

N = number of customers enter the store.

X_i = amount of money spent by i^{th} customer.

$E[N] = 50$

$E[X_i] = 8\$$ $\quad i = 1, 2, \ldots, N.$

$$\sum_{i=1}^{N} E[X_i] = ? \quad = E\left[E\left[\sum_{i=1}^{N}(X_i \mid N)\right]\right]$$

but $$E\left[\sum_{i=1}^{N} X_i \mid N=n\right] = E\left[\sum_{i=1}^{n} X_i \mid N=n\right]$$

$$= E\left[\sum_{i=1}^{n} X_i\right] \quad (\because X_i \text{ is independent of } N)$$

$$\boxed{= nE[X]} \quad (E[X] = E[X_i])$$

$$\therefore E\left[\sum_{i=1}^{N} X_i \mid N\right] = NE[X]$$

$$\therefore E\left[\sum_{i=1}^{N} X_i\right] = E[NE[X]] = E[N]\,E[X]$$

$$= 50 \times 8\$$$

$$\boxed{= 400\$}$$

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Sol^n 4 Gambling situation.

r players. Player i has initially n_i amount unit.

$n_i > 0$, $i = 1, \dots, r$.

each stage → 2 players are choosen, ~~winning~~ winning probability is equal for both ($1/2$)

→ looser will give 1 unit to winner.

→ Player with 0 unit will get out of the game.

→ Game ends when a player will have all the amount $n = \sum_{i=1}^{r} n_i$ → He will be victor.

X = number of stages required to play

$E[X] = ?$

- First of all lets assume that there are only 2 players

Player 1 $n_1 = j$

player 2 $n_2 = n - j$

X_j = number of stages played.

$E[X_j] = ? = a_j$

$X_j = 1 + A_j$

A_j = number of stages played after the first stage.

$\therefore E[X_j] = 1 + E[A_j]$

$= 1 + E[A_j \mid 1 \text{ wins the first stage}] \cdot \frac{1}{2} +$

$E[A_j \mid 1 \text{ looses the first stage}] \cdot \frac{1}{2}$

$= 1 + E[X_{j+1}]\frac{1}{2} + E[X_{j-1}]\frac{1}{2}$

$$\therefore \boxed{a_j = 1 + \frac{a_{j+1}}{2} + \frac{a_{j-1}}{2}}$$

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$\therefore a_{j+1} = 2a_j - a_{j-1} - 2 \quad\ldots\quad j = 1, 2, \ldots, n-1$

$a_0 = 0$

$a_2 = 2a_1 - 2$

$a_3 = 2a_2 - a_1 - 2 = 3a_1 - 6 = 3(a_1 - 2)$

$a_4 = 2a_3 - a_2 - 2 = 4a_1 - 12 = 4(a_1 - 3)$

$\vdots$

$\therefore \boxed{a_i = i(a_1 - i + 1)}$ — ① (Mathematical Induction)

$a_j = j(a_1 - j + 1)$

$\therefore a_{j+1} = 2j(a_1 - j + 1) - (j-1)(a_1 - j + 2) - 2$

$= (j+1)a_1 - 2j^2 + 2j + j^2 - 3j + 2 - 2$

$= (j+1)a_1 - j^2 - j$

$\boxed{= (j+1)(a_1 - j)}$

→ Proof of the Induction hypothesis.

• Now putting $i = n$ in eqn 1

$\therefore a_n = n(a_1 - n + 1) \qquad (a_n = 0)$

$\therefore 0 = n(a_1 - n + 1)$

$\therefore \boxed{a_1 = n - 1}$

$\therefore \boxed{a_i = i(n-i)}$ — ②

• ~~Int~~ It means if there are 2 players with amount (i) & $(n-i)$ there are $i(n-i)$ stages played (Expectingly).

• Now if there are 'r' players with initial amount of n_i, $\sum_{i=1}^{r} n_i = n$.

X = number of stages need to be played.

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player 'i'

- From a perspective of a single player with initial amount n_i, The game can bee seen as a game between 2 players with initial amount of n_i & $n - n_i$.

  $\hookrightarrow$ Initial amount of whole team except that player.

- Now there are only 2 possible outcomes
① that single player will end up with 0 fortune.
② " " " " " " n fortune.

X_i = number of stages played.

$E[X_i] = n_i(n-n_i)$

each stage involves 2 players

$$\therefore E[X] = \frac{1}{2}\sum_{i=1}^{r} E[X_i]$$

$$= \frac{1}{2}\sum_{i=1}^{r} n_i(n-n_i)$$

$$= \frac{1}{2}\left(n \cdot n - \sum_{i=1}^{r} n_i^2\right)$$

$$\boxed{= \frac{1}{2}\left[n^2 - \sum_{i=1}^{r} n_i^2\right]}$$