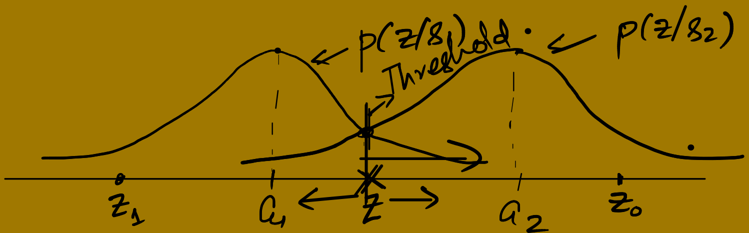
CT303 Lecture 15: 12 October 2020

- Lecture 14 recap.
- Received signal: $r(t) = h_c(t) * s_i(t) + \eta(t)$.
- Output of receiving filter: $z(t) = h_r(t) * h_c(t) * s_i(t) + h_r(t) * \eta(t)$.
- Detection based on samples of z: $z(T) = a_i + \eta_0$.
- Likelihood: $p(z|s_i) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{(z-a_i)^2}{2\sigma_0}\right)$.



- MLE and MAP estimation
  - MLE: $s_k = \arg\max p(z/s_i)$. $\Longleftrightarrow$ $\underset{i}{\arg\min}\, d(z, a_i)$
  - MAP: $s_k = \arg\max p(s_i/z)$. $\rightarrow$ $p(s_1/z) > p(s_2/z)$
  - Baye's rule: $p(s_i|z) = \frac{p(z|s_i)p(s_i)}{p(z)}$

Ap L Prior Evidence.

MAP: $\underset{s_i}{\arg\max}\, p(s_i/z) = \underset{s_i}{\arg\max}\, p(z/s_i)\,p(s_i)$

Uniform dist over s_1, s_2, i.e. $p(s_1) = p(s_2)$

$= \underset{s_i}{\arg\max}\, p(z/s_i) = \text{MLE}$

  - Threshold for MAP: $p(s_1|z) = p(s_2|z)$, or MLE: $p(z|s_1) = p(z|s_2)$.

$p(s_1) = 1/3 \quad p(s_2) = 2/3$

$\frac{1}{\cancel{3}}\, p(z/s_1) = \frac{2}{\cancel{3}}\, p(z/s_2)$

$d(z, a_1) = d(z, a_2)$

$\Rightarrow z = \frac{a_1 + a_2}{2}$

If $z >$ Threshold $\Rightarrow s_2$

$z <$ Threshold $\Rightarrow s_1$

Error Probability

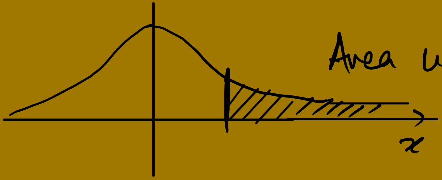
- $P(e|s_1)$. $= \int_{\text{Threshold}}^{\infty} p(z/s_1)\,dz$
- $P(e|s_2) = \int_{-\infty}^{\text{Threshold}} p(z/s_2)\,dz$
- $\boxed{P_B} = P(e|s_1)P(s_1) + P(e|s_2)P(s_2)$. (1) $P(s_1) = P(s_2) = 1/2$
  - Is $P(e|s_1)$ and $P(e|s_2)$ different?

$$P(e/s_1) = P(e/s_2)$$

$$P_B = P(e/s_1) = P(e/s_2)$$

- Probability of bit error (assuming $P(s_1) = P(s_2)$):

$$\left(P(e/s_1)\right) = P_B = \int_{\frac{a_1+a_2}{2}}^{\infty} \underbrace{\frac{1}{\sigma_0\sqrt{2\pi}} \exp\left(-\frac{(z-a_1)^2}{2\sigma_0}\right) dz}_{P(z/s_1)}$$



Area under the tail of the gaussian

- Probability of bit error (assuming $P(s_1) = P(s_2)$):

$$P_B = \int_{\frac{a_1+a_2}{2}}^{\infty} \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left(-\frac{(z-a_1)^2}{2\sigma_0}\right) dz$$

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  - Change of variables: $\boxed{u = \frac{z-a_1}{\sigma_0}} \Rightarrow \underline{\sigma_0 \; du = dz}, \underline{z = \frac{a_1+a_2}{2}} \Rightarrow \underline{u = \frac{a_2-a_1}{2\sigma_0}}$.

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  - Change of variables: $u = \frac{z-a_1}{\sigma_0} \Rightarrow \sigma_0\, du = dz, z = \frac{a_1+a_2}{2} \Rightarrow u = \frac{a_2-a_1}{2\sigma_0}$.
  - Thus, $\boxed{P_B = Q\left(\frac{a_2-a_1}{2\sigma_0}\right)}$.

If $x_1 > x_2$, $Q(x_1) \leq Q(x_2)$

$\sigma_0 \uparrow \Rightarrow P_B \uparrow$

$\underbrace{(a_2 - a_1)} \uparrow \Rightarrow P_B \downarrow$

- Symbol waveforms
- Receiving filter

Receiving filter/Matched filter

- $r(t) = h_c(t) * s_i(t) + \eta(t)$, $\eta \sim$ AWGN, $N_0/2$.

Receiving filter/Matched filter

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  - Assuming channel does not distort (introduce ISI) symbol waveforms, let us ignore the channel impulse response. (Equalizing filter)

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- Receiving filter: $z(t) = \underline{h_r(t)} * s_i(t) + h_r(t) * \eta(t)$.

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- Receiving filter: $\underline{z(t)} = h_r(t) * s_i(t) + h_r(t) * \eta(t)$.
- Samples: $\boxed{z(T)} = a_i + \eta_0$.
  - What is the aim of the receiving filter?

Rx filter should maximize the SNR at multiples of T secs.

$$z(T) = a_i + \eta_0$$

$$SNR_{z(T)} = \frac{a_i^2}{\|\eta_0\|^2}$$

- We want the SNR at the output of the filter to be maximized every T secs.

$$SNR = \left(\frac{||a_i||^2}{||\eta_0||^2}\right), \quad = a_i^2$$

→ Notation for power

- expression will follow

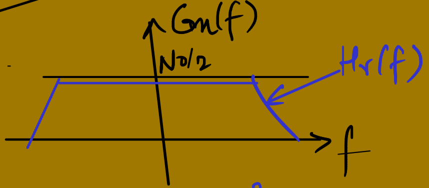
- We want the SNR at the output of the filter to be maximized every T secs.

$$SNR = \left(\frac{||a_i||^2}{||\eta_0||^2} \right),$$

- $a_i = (h_r * s_i)(T) = \int H_r(f) S_i(f) \exp(j2\pi f T) df$, $||\eta_0||^2 = \sigma_0^2 = \frac{N_0}{2} \int |H_r(f)|^2 df$

$t = T$

$\eta \rightarrow$ AWGN, $\frac{N_0}{2}$



$$G_{n_0}(f) = |H_r(f)|^2 \underbrace{G_n(f)}_{N_0/2}$$

$$G_{n_0}(f) = \frac{N_0}{2} |H_r(f)|^2$$

- We want the SNR at the output of the filter to be maximized every T secs.

$$SNR = \left(\frac{||a_i||^2}{||\eta_0||^2}\right),$$

- $a_i = (h_r * s_i)(T) = \int H_r(f) S_i(f) \exp(j2\pi fT) df$, $||\eta_0||^2 = \sigma_0^2 = \frac{N_0}{2} \int |H_r(f)|^2 df$
- SNR:

$$\boxed{SNR = \frac{2|\int H_r(f) [S_i(f) \exp(j2\pi fT)] df|^2}{N_0 \int |H_r(f)|^2 df}}$$

Let $x = H_r(f)$ and $y = \overline{S_i(f)} \exp(-j2\pi fT)$. Then the SNR expression can be written in terms of norms and inner products, as follows:

$$SNR = \frac{2|\langle x, y\rangle|^2}{N_0 ||x||^2}.$$

$\langle x, x\rangle = ||x||^2$

Using Cauchy-Schwarz inequality: $|\langle x, y\rangle| \leq ||x|| \, ||y||$, we get

$$SNR \leq SNR_{max} = \frac{2||y||^2}{N_0} = \frac{2\int |S_i(f)|^2 \, df}{N_0} = \frac{2E}{N_0},$$

E_{s_i}

$E \uparrow \quad SNR_{max} \uparrow$

where E is the signal energy.

$\langle f, g\rangle = \int_{\Omega} f(t) \overline{g(t)} \, dt$

$\langle y, y\rangle = \int \overline{S_i(f)} \cancel{e^{-j2\pi fT}} S_i(f) \cancel{e^{j2\pi fT}} df$

$= \int |S_i(f)|^2 df$

- This maximum SNR is achieved for $x = ky$. Setting $k = 1$ since the value of k does not affect the SNR, we get,

$$SNR = \frac{2|\langle y, y\rangle|^2}{N_0||y||^2} = \frac{2||y||^4}{N_0||y||^2} = \frac{2||y||^2}{N_0} = \frac{2E}{N_0} = SNR_{max}.$$

Thus, $H_r(f) = \overline{S_i(f)} \exp(-j2\pi fT)$. Now, $\overline{S_i(f)}$

$$\overline{S_i(f)} \exp(-j2\pi fT) = \left(\int_{-\infty}^{\infty} s_i(t) \exp(j2\pi ft)\, dt\right) \exp(-j2\pi fT) \quad (\text{since } s_i(t) \in \mathbb{R})$$

$$= \int_{-\infty}^{\infty} s_i(t) \exp(j2\pi f(t - T))\, dt$$

Let $u = T - t$, $t = T - u, du = -dt$

$$= -\int_{\infty}^{-\infty} s_i(T - u) \exp(-j2\pi fu)\, du$$

$$= \int_{-\infty}^{\infty} s_i(T - u) \exp(-j2\pi fu)\, du$$

$$= \mathcal{F}(s_i(T - u)).$$

Thus $\boxed{h_r(t) = s_i(T - t)}, \forall t \in [0, T]$.

Impulse response of the receiving filter