1. Prove that for any vector field $\vec{A}$, $\vec{\nabla} \cdot \vec{A}$ is a scalar.

soln:
Let us denote the coordinates before and after rotation as

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad X' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}$$

$X' = RX$ where R is the rotation matrix.

$$\therefore \; x'_i = \sum_j R_{ij} x_j \quad \text{and} \quad x_i = \sum_j (R^T)_{ij} x'_j = \sum_j R_{ji} x'_j \tag{1}$$

Here we have used the fact that $R^{-1} = R^T$ i.e. R is an orthogonal matrix. We have to show that $\vec{\nabla}' \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A}$, i.e

$$\sum_i \frac{\partial A'_i}{\partial x'_i} = \sum_i \frac{\partial A_i}{\partial x_i}$$

$$\frac{\partial}{\partial x'_i} = \sum_j \frac{\partial}{\partial x_j} \frac{\partial x_j}{\partial x'_i} = \sum_j R_{ij} \frac{\partial}{\partial x_j} \quad \text{from Eq.(1)}$$

$A'_i = \sum_k R_{ik} A_k$ since $\vec{A}$ is a vector.

$$\begin{aligned} \vec{\nabla}' \cdot \vec{A}' = \sum_i \frac{\partial A'_i}{\partial x'_i} &= \sum_i \sum_j R_{ij} \frac{\partial}{\partial x_j} \left(\sum_k R_{ik} A_k \right) \\ &= \sum_j \sum_k \left(\sum_i R_{ij} R_{ik} \right) \frac{\partial A_k}{\partial x_j} \end{aligned}$$

Now $\sum_i R_{ij} R_{ik} = \sum_i (R^T)_{ji} R_{ik} = \delta_{jk}$ since $R^T R = \mathbb{I}$.
So only $j = k$ terms survive in the above summation

$$\therefore \quad \vec{\nabla}' \cdot \vec{A}' = \sum_j \frac{\partial A_j}{\partial x_j} = \vec{\nabla} \cdot \vec{A}$$

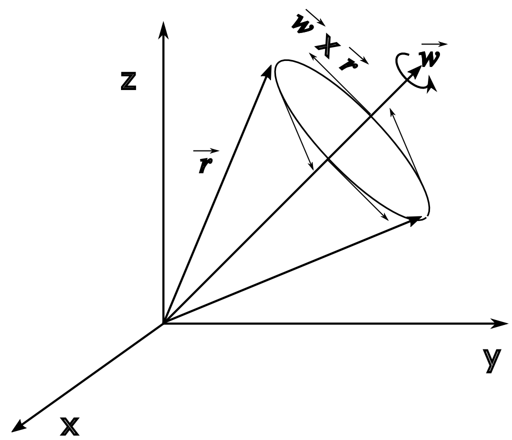
So the divergence of a vector is invariant under rotation. Hence it is a scalar quantity. This proof is valid for the divergence operator in any dimension since we have only used the orthogonality of R.

2. Let $\vec{\mathrm{A}} = \vec{\omega} \times \vec{\mathrm{r}}$ where $\vec{\omega}$ is a fixed vector in space. Find $\vec{\nabla} \times \vec{\mathrm{A}}$.

soln:

$$\begin{aligned}
\vec{A} = \vec{\omega} \times \vec{r} &= \hat{i}(\omega_y z - \omega_z y) + \hat{j}(\omega_z x - \omega_x z) + \hat{k}(\omega_x y - \omega_y x) \\
\therefore\ \vec{\nabla} \times \vec{A} &= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \hat{i}\left(\omega_x - (-\omega_x)\right) + \hat{j}\left(\omega_y - (-\omega_y)\right) + \hat{k}\left(\omega_z - (-\omega_z)\right) = 2\vec{\omega}
\end{aligned}$$

Interpretation:
The vector field $\vec{\omega} \times \vec{r}$ curls around the vector $\vec{\omega}$. It is the velocity vector of the particles of a rigid body rotating with angular velocity $\vec{\omega}$.



3. Find the divergence of the following:

   (a) $\vec{\mathrm{A}} = \hat{\mathrm{r}}$,

   (b) $\vec{\mathrm{A}} = \frac{\hat{\mathrm{r}}}{r}$ in 2 dimension

   (c) $\vec{\mathrm{A}} = \frac{\hat{\mathrm{r}}}{r}$ in 3 dimension

   (d) $\vec{\mathrm{A}} = \frac{\hat{\mathrm{r}}}{r^2}$ in 3 dimension. Plot this field.

   (e) $\vec{\mathrm{A}} = \frac{\hat{\mathrm{r}}}{r^3}$ in 3 dimension

soln:

   (a) $\vec{A} = \hat{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}; r = \sqrt{x^2 + y^2 + z^2}$
   $\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
   $\frac{\partial A_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = -\frac{x^2}{r^3} + \frac{1}{r}$
   Similarly $\frac{\partial A_y}{\partial y} = -\frac{y^2}{r^3} + \frac{1}{r}$ and $\frac{\partial A_z}{\partial z} = -\frac{z^2}{r^3} + \frac{1}{r}$.
   $\therefore\ \vec{\nabla} \cdot \vec{A} = -\frac{x^2+y^2+z^2}{r^3} + 3/r = 2/r.$

For the remaining parts we evaluate the following:

$$\begin{aligned}
\vec{\nabla}\cdot(r^n\hat{r}) &= r^n\vec{\nabla}\cdot\hat{r}+\vec{\nabla}(r^n)\cdot\hat{r} \\
&= r^n\left(\frac{2}{r}\right)+nr^{n-1}\hat{r}\cdot\hat{r} \quad \text{(in 3 dim.)} \\
&= (n+2)r^{n-1} \\
\vec{\nabla}\cdot(r^n\hat{r}) &= r^n\left(\frac{1}{r}\right)+nr^{n-1}\hat{r}\cdot\hat{r} \quad \text{(in 2 dim.)} \\
&= (n+1)r^{n-1}
\end{aligned}$$

(b) $\vec{A}=\frac{\hat{r}}{r}$ in 2 dim. $(n=-1)$
$\vec{\nabla}\cdot\vec{A}=(-1+1)r^{-2}=0.$
(This is true only for $r\neq 0$. At $r=0, \vec{\nabla}\cdot\vec{A}\to\infty$)

(c) $\vec{A}=\frac{\hat{r}}{r}$ in 3 dim. $(n=-1)$
$\vec{\nabla}\cdot\vec{A}=(-1+2)r^{-2}=1/r^2$

(d) $\vec{A}=\frac{\hat{r}}{r^2}$ in 3 dim. $(n=-2)$
$\vec{\nabla}\cdot\vec{A}=(-2+2)r^{-3}=0.$
(This is true only for $r\neq 0$. At $r=0, \vec{\nabla}\cdot\vec{A}\to\infty$)

(e) $\vec{A}=\frac{\hat{r}}{r^3}$ in 3 dim. $(n=-3)$
$\vec{\nabla}\cdot\vec{A}=(-3+2)r^{-4}=-1/r^4$

All the fields above are radially outward. But the divergence changes from positive to 0 to negative. Explain this.

4. Find the curl of the following:

(a) $\vec{\mathrm{A}}=y\hat{\mathrm{i}}-x\hat{\mathrm{j}}$

(b) $\vec{\mathrm{A}}=\frac{1}{\sqrt{x^2+y^2}}(y\hat{\mathrm{i}}-x\hat{\mathrm{j}})$

(c) $\vec{\mathrm{A}}=\frac{1}{x^2+y^2}(y\hat{\mathrm{i}}-x\hat{\mathrm{j}})$

(d) $\vec{\mathrm{A}}=(x^2+y^2)\hat{\mathrm{k}}$

soln:

(a) $\vec{\mathrm{A}}=y\hat{\mathrm{i}}-x\hat{\mathrm{j}}$
soln
$\vec{\nabla}\times\vec{A}=\hat{k}(-1-1)=-2\hat{k}$

(b) $\vec{\mathrm{A}}=\frac{1}{\sqrt{x^2+y^2}}(y\hat{\mathrm{i}}-x\hat{\mathrm{j}})$
soln:

We first evaluate $\vec{\nabla} \times \left[(x^2+y^2)^n(y\hat{i}-x\hat{j})\right]$ which will help us work out the other parts.

$$\begin{aligned}
\vec{\nabla} \times \left[(x^2+y^2)^n(y\hat{i}-x\hat{j})\right] &= (x^2+y^2)^n\vec{\nabla}\times(y\hat{i}-x\hat{j}) + \vec{\nabla}(x^2+y^2)^n \times (y\hat{i}-x\hat{j}) \quad \text{product rules} \\
&= (x^2+y^2)^n(-2\hat{k}) + n(x^2+y^2)^{n-1}(2x\hat{i}+2y\hat{j})\times(y\hat{i}-x\hat{j}) \\
&= -2(x^2+y^2)^n\hat{k} + n(x^2+y^2)^{n-1}(-2x^2-2y^2)\hat{k} \\
&= -2(x^2+y^2)^n(1+n)\hat{k}
\end{aligned}$$

For $\vec{A} = (y\hat{i}-x\hat{j})/\sqrt{x^2+y^2}, \quad n=-1/2.$

$$\vec{\nabla}\times\vec{A} = -\frac{1}{\sqrt{x^2+y^2}}\hat{k}$$

This is not differentiable at $x=y=0$, i.e along the z axis.

(c) $\vec{\mathrm{A}} = \frac{1}{x^2+y^2}(y\hat{\mathrm{i}}-x\hat{\mathrm{j}})$
soln
Here $n=-1$.

$$\therefore \quad \vec{\nabla}\times\vec{A} = 0$$

This is not valid along the z axis.

(d) $\vec{\mathrm{A}} = (x^2+y^2)\hat{\mathrm{k}}$
soln

$$\begin{aligned}
\vec{\nabla}\times\vec{A} &= \hat{i}\left(\frac{\partial A_z}{\partial y}\right) + \hat{j}\left(-\frac{\partial A_z}{\partial x}\right) \\
&= 2y\hat{i} - 2x\hat{j} = 2(y\hat{i}-x\hat{j})
\end{aligned}$$

5. For any vector field $\vec{\mathrm{A}}$ and any scalar field F show that
(i) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathrm{A}}) = 0$; (ii) $\vec{\nabla}\times(\vec{\nabla}F) = 0$.

soln:
(i) $\vec{\nabla}\times\vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$
$\therefore \ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 0.$

(ii)
$\vec{\nabla}\times\vec{\nabla}F = \hat{i}\left(\frac{\partial^2 F}{\partial y\partial x} - \frac{\partial^2 F}{\partial x\partial y}\right) + \ldots.. = 0$

6. Can we find a scalar function F such that $\vec{\nabla}F = y\hat{\mathrm{i}} - x\hat{\mathrm{j}}$?
What about $\vec{\nabla}F = \frac{1}{x^2+y^2}(y\hat{\mathrm{i}}-x\hat{\mathrm{j}})$?

soln
$\vec{\nabla}\times\vec{\nabla}F = \vec{\nabla}\times(y\hat{\mathrm{i}}-x\hat{\mathrm{j}}) = -2\hat{k}$
But curl of a the gradient of any scalar field must be zero. So there exist no such F

such that $\vec{\nabla} F = y\hat{\text{i}} - x\hat{\text{j}}$

If $\vec{\nabla} F = \frac{1}{x^2+y^2}(y\hat{\text{i}} - x\hat{\text{j}})$ it can be shown that $\vec{\nabla} \times (\vec{\nabla} F) = 0$ at almost all places but the result is not applicable when $x = y = 0$ which is the z axis. In fact it can be shown that the $\vec{\nabla} \times (\vec{\nabla} F) \neq 0$ along the z axis. The scalar function F, that we find, steadily increases along $\vec{\nabla} F$. Hence along a circle around the origin, the plot of given $\vec{\nabla} F$ shows that F steadily increases. After completing a full circle when we come to the same point we arrive at a different value of F. This makes F discontinuous, and hence not differentiable. Let us find a F and demonstrate this phenomenon. We have the following system of differential equations

$$\frac{\partial F}{\partial x} = \frac{y}{x^2 + y^2} \qquad ; \qquad \frac{\partial F}{\partial y} = \frac{-x}{x^2 + y^2}$$

The first one gives $F(x, y) = \tan^{-1}(x/y) + g(y)$ while the second equation gives $F(x, y) = -\tan^{-1}(y/x) + h(x) = -(\pi/2 - \tan^{-1}(x/y)) + h(x)$.
For consistency we need $g(y)$ and $h(x)$ to be constants, say, c_1 and c_2.
$c_2 - c_1 = \pi/2$. We can take $c_1 = 0$ and $c_2 = \pi/2$. This gives
$F(x, y) = \tan^{-1}(x/y)$.
So it appears that This function is not continuous. To see this we reparametrize $x = r\sin\theta,\ y = r\cos\theta$. Then we get $F(x, y) = \theta$. As we go around a circle of radius 1, starting at $(0, 1)$ and come back to the same point the value of the function changes from 0 to 2π. So it is discontinuous at $(0, 1)$, hence, not differentiable at this point which lies on the y axis. In fact it is not differentiable at any point on the y axis. Hence the given vector field cannot be the gradient of a scalar function.

7. Find the equation of the tangent plane to the ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) on the ellipsoid.

soln:
Consider a scalar function $f(x, y, z)$. Consider a surface over which f is constant, say, $f(x, y, z) = k$. $\vec{\nabla} f$ at a point on this surface is normal to this surface. This normal will also be normal to the tangent plane at this point.
We have $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\vec{\nabla} f = 2\left(\hat{i}\frac{x}{a^2} + \hat{j}\frac{y}{b^2} + \hat{k}\frac{z}{c^2}\right) \tag{2}$$

At (x_0, y_0, z_0), $\vec{\nabla} f = 2\left(\hat{i}\frac{x_0}{a^2} + \hat{j}\frac{y_0}{b^2} + \hat{k}\frac{z_0}{c^2}\right)$.
Every vector on the tangent plane is perpendicular to this vector. So the equation of the tangent plane is given by

$$\begin{aligned} (x - x_0)\frac{x_0}{a^2} + (y - y_0)\frac{y_0}{b^2} + (z - z_0)\frac{z_0}{c^2} &= 0 \\ \text{i.e} \qquad \frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z &= 1 \end{aligned}$$