1. Calculate the laplacian of the following:
   (i) $F = x^2 + 2xy + 3z + 4$ (ii) $F = \sin(\hat{\mathrm{k}} \cdot \vec{\mathrm{r}})$ (iii) $F = \frac{1}{r}$

   soln:
   (i) $\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 2$
   (ii)

$$\begin{aligned}
\nabla^2 F &= \frac{\partial^2}{\partial x^2}\sin(\vec{k}\cdot\vec{r}) + \frac{\partial^2}{\partial y^2}\sin(\vec{k}\cdot\vec{r}) + \frac{\partial^2}{\partial z^2}\sin(\vec{k}\cdot\vec{r}) \\
&= -k_x^2\sin(\vec{k}\cdot\vec{r}) - k_y^2\sin(\vec{k}\cdot\vec{r}) - k_z^2\sin(\vec{k}\cdot\vec{r}) \\
&= -k^2\sin(\vec{k}\cdot\vec{r})
\end{aligned}$$

   (iii)

$$\begin{aligned}
\nabla^2\left(\frac{1}{r}\right) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{r}\right) \\
\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) &= \frac{\partial}{\partial x}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(-\frac{1}{r^2}\frac{1}{2r}\cdot 2x\right) \\
&= \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) \\
&= -x\left(-\frac{3}{r^4}\frac{1}{2r}\cdot 2x\right) - \frac{1}{r^3} \\
&= \frac{3x^2}{r^5} - \frac{1}{r^3}
\end{aligned}$$

$$\therefore\ \nabla^2\left(\frac{1}{r}\right) = \frac{3}{r^5}\left(x^2+y^2+z^2\right) - \frac{3}{r^3} = 0$$

   This is valid only for $r \neq 0$. At $r = 0$ the function is not differentiable.

2. Find the curl of the following:

   (a) $\vec{\mathrm{A}} = y\hat{\mathrm{i}} - x\hat{\mathrm{j}}$
   soln
   $\vec{\nabla} \times \vec{A} = \hat{k}(-1-1) = -2\hat{k}$

   (b) $\vec{\mathrm{A}} = \frac{1}{\sqrt{x^2+y^2}}(y\hat{\mathrm{i}} - x\hat{\mathrm{j}})$
   soln:
   We first evaluate $\vec{\nabla} \times \left[(x^2+y^2)^n(y\hat{i} - x\hat{j})\right]$ which will help us work out the other

parts.

$$\begin{aligned}\vec{\nabla}\times\left[(x^2+y^2)^n(y\hat{\imath}-x\hat{\jmath})\right] &= (x^2+y^2)^n\vec{\nabla}\times(y\hat{\imath}-x\hat{\jmath})+\vec{\nabla}(x^2+y^2)^n\times(y\hat{\imath}-x\hat{\jmath}) \quad \text{product rules}\\ &= (x^2+y^2)^n(-2\hat{k})+n(x^2+y^2)^{n-1}(2x\hat{\imath}+2y\hat{\jmath})\times(y\hat{\imath}-x\hat{\jmath})\\ &= -2(x^2+y^2)^n\hat{k}+n(x^2+y^2)^{n-1}(-2x^2-2y^2)\hat{k}\\ &= -2(x^2+y^2)^n(1+n)\hat{k}\end{aligned}$$

For $\vec{A}=(y\hat{\imath}-x\hat{\jmath})/\sqrt{x^2+y^2}, \quad n=-1/2.$

$$\vec{\nabla}\times\vec{A}=-\frac{1}{\sqrt{x^2+y^2}}\hat{k}$$

This is not differentiable at $x=y=0$, i.e along the z axis.

(c) $\vec{\mathrm{A}}=\frac{1}{x^2+y^2}(y\hat{\mathrm{i}}-x\hat{\mathrm{j}})$
soln
Here $n=-1$.

$$\therefore \quad \vec{\nabla}\times\vec{A}=0$$

This is not valid along the z axis.

(d) $\vec{\mathrm{A}}=(x^2+y^2)\hat{\mathrm{k}}$
soln

$$\begin{aligned}\vec{\nabla}\times\vec{A} &= \hat{\imath}\left(\frac{\partial A_z}{\partial y}\right)+\hat{\jmath}\left(-\frac{\partial A_z}{\partial x}\right)\\ &= 2y\hat{\imath}-2x\hat{\jmath}=2(y\hat{\imath}-x\hat{\jmath})\end{aligned}$$

3. For any vector field $\vec{\mathrm{A}}$ and any scalar field F show that
(i) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathrm{A}})=0$; (ii) $\vec{\nabla}\times(\vec{\nabla}F)=0$.

soln:
(i) $\vec{\nabla}\times\vec{A}=\hat{\imath}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{\jmath}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{k}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$
$\therefore \ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)=0.$

(ii)
$\vec{\nabla}\times\vec{\nabla}F=\hat{\imath}\left(\frac{\partial^2 F}{\partial y\partial x}-\frac{\partial^2 F}{\partial x\partial y}\right)+.....=0$

4. Can we find a scalar function F such that $\vec{\nabla}F=y\hat{\mathrm{i}}-x\hat{\mathrm{j}}$?
What about $\vec{\nabla}F=\frac{1}{x^2+y^2}(y\hat{\mathrm{i}}-x\hat{\mathrm{j}})$?

soln
$\vec{\nabla}\times\vec{\nabla}F=\vec{\nabla}\times(y\hat{\mathrm{i}}-x\hat{\mathrm{j}})=-2\hat{k}$
But curl of a the gradient of any scalar field must be zero. So there exist no such F such that $\vec{\nabla}F=y\hat{\mathrm{i}}-x\hat{\mathrm{j}}$

If $\vec{\nabla}F = \frac{1}{x^2+y^2}(y\hat{\text{i}} - x\hat{\text{j}})$ it can be shown that $\vec{\nabla} \times (\vec{\nabla}F) = 0$ at almost all places but the result is not applicable when $x = y = 0$ which is the z axis. In fact it can be shown that the $\vec{\nabla} \times (\vec{\nabla}F) \neq 0$ along the z axis. Hence we can't find a scalar function F satisfying the given condition. One may verify that there is not consistent solution to the system of differential equations

$$\frac{\partial F}{\partial x} = \frac{y}{x^2+y^2} \qquad ; \qquad \frac{\partial F}{\partial y} = \frac{-x}{x^2+y^2}$$

The first one gives $F(x,y) = \tan^{-1}(x/y) + g(y)$ while the second equation gives $F(x,y) = -\tan^{-1}(y/x) + h(x) = -(\pi/2 - \tan^{-1}(x/y)) + h(x)$.
For consistency we need $g(y)$ and $h(x)$ to be constants, say, c_1 and c_2.
$c_2 - c_1 = \pi/2$. We can take $c_1 = 0$ and $c_2 = \pi/2$. This gives
$F(x,y) = \tan^{-1}(x/y)$.
This function is not continuous. To see this we reparametrize $x = r\cos\theta,\ y = r\sin\theta$. Then we get $F(x,y) = \theta$. As we go around a circle of radius 1, starting at $(0,1)$ and come back to the same point the value of the function changes from 0 to 2π. So it is discontinuous at $(0,1)$, hence, not differentiable at this point which lies on the y axis. In fact it is not differentiable at any point on the y axis. Hence the given vector field cannot be the gradient of a scalar function.

5. Using the expressions for $\vec{\nabla}\cdot(\vec{A}\times\vec{B})$ and $\vec{\nabla}\times(\vec{A}\times\vec{B})$ evaluate $\vec{\nabla}\cdot(\vec{\omega}\times\vec{r})$ and $\vec{\nabla}\times(\vec{\omega}\times\vec{r})$ where $\vec{\omega}$ is a constant vector.

soln:
$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$
Here $A = \vec{\omega}$ and $\vec{B} = \vec{r}$.
This gives $\vec{\nabla}\cdot(\vec{\omega}\times\vec{r}) = -\vec{\omega}\cdot(\vec{\nabla}\times\vec{r}) = 0$.

$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$
Here $A = \vec{\omega}$ and $\vec{B} = \vec{r}$.
This gives $\vec{\nabla}\times(\vec{\omega}\times\vec{r}) = -(\vec{\omega}\cdot\vec{\nabla})\vec{r} + \vec{\omega}(\vec{\nabla}\cdot\vec{r}) = -(\vec{\omega}\cdot\vec{\nabla})\vec{r} + 3\vec{\omega}$

$$(\vec{\omega}\cdot\vec{\nabla})\vec{r} = \left(\omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z}\right)(\hat{i}x + \hat{j}y + \hat{k}z) = \hat{i}\omega_x + \hat{j}\omega_y + \hat{k}\omega_z = \vec{\omega}$$

$\therefore \quad \vec{\nabla}\times(\vec{\omega}\times\vec{r}) = 2\vec{\omega}$

6. Find the equation of the tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) on the ellipsoid.

soln:
Consider a scalar function $f(x,y,z)$. Consider a surface over which f is constant, say, $f(x,y,z) = k$. $\vec{\nabla}f$ at a point on this surface is normal to this surface. This normal will also be normal to the tangent plane at this point.
We have $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\vec{\nabla}f = 2\left(\hat{i}\frac{x}{a^2} + \hat{j}\frac{y}{b^2} + \hat{k}\frac{z}{c^2}\right) \tag{1}$$

At (x_0, y_0, z_0), $\vec{\nabla} f = 2\left(\hat{i}\frac{x_0}{a^2} + \hat{j}\frac{y_0}{b^2} + \hat{k}\frac{z_0}{c^2}\right)$.
Every vector on the tangent plane is perpendicular to this vector. So the equation of the tangent plane is given by

$$\begin{aligned} (x - x_0)\frac{x_0}{a^2} + (y - y_0)\frac{y_0}{b^2} + (z - z_0)\frac{z_0}{c^2} &= 0 \\ \text{i.e} \qquad \frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z &= 1 \end{aligned}$$