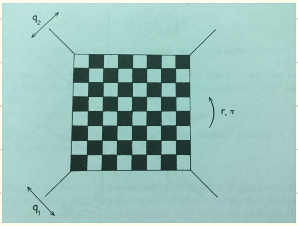
Lecture 6: Isomorphism



Consider the symmetries of the chessboard as shown in the figure. There are four symmetries $\{e, r, q_1, q_2\}$. Here r is the rotation by angle π about the axis going through the centre and perpendicular to the plane in which the chessboard lies. q_1 and q_2 are reflections about the two diagonals. The group multiplication table is given below

	e	r	q_1	q_2
e	e	r	q_1	q_2
r	r	e	q_2	q_1
q_1	q_1	q_2	e	r
q_2	q_2	q_1	r	e

Now consider the set $\{1,3,5,7\}$ with operation $a*b = ab \bmod 8$. This set forms a group and its group multiplication table is

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

If we define a mapping $\Phi : \{e, r, q_1, q_2\} \to \{1,3,5,7\}$ $\Phi(e) = 1$, $\Phi(r) = 3$, $\Phi(q_1) = 5$, $\Phi(q_2) = 7$ then we see that the bijective map Φ obeys $\Phi(a*b) = \Phi(a) * \Phi(b)$. The two groups are structurally identical.

Defn: (Isomorphism)

Two groups G and K are said to be isomorphic if there exists a bijective map $\Phi: G \to K$ such that $\Phi(a *_G b) = \Phi(a) *_K \Phi(b)$ for all $a, b \in G$.

Example 1:

$(\mathbb{R}, +)$ is isomorphic to $(\mathbb{R}^{pos}, \times)$

The map $\Phi(x) = e^x$ is a bijection from $\mathbb{R}$ to $\mathbb{R}^{pos}$. Moreover $\Phi(x+y) = e^{x+y} = e^x . e^y =$ $\Phi(x)\Phi(y)$. Hence Φ is an isomorphism.

Example 2: The group of rotational symmetries of a tetrahedron is isomorphic to A_4

Example 3: Every infinite cyclic group is isomorphic to $\mathbb{Z}$

Let G be an infinite cyclic group and let x be the generator of G then the mapping $\Phi(x^m) = m$ is an isomorphism (check!)

Example 4: If G is a finite cyclic group of order m then it is isomorphic to $\mathbb{Z}_m$.

$\Phi(x^k) = k \bmod m$ is the isomorphism. (check!)

Example 5: $\{1,-1,i,-i\}$ under multiplication is a group. Notice that it is a cyclic group. i and $-i$ are generators. It is a group of order 4. Hence by example 4 it must be isomorphic to $\mathbb{Z}_4$. $1 \to 0$, $-1 \to 2$, $i \to 1$, $-i \to 3$.

Example 6: $\mathbb{Q}$ is not isomorphic to $\mathbb{Q}^{pos}$.

If there was an isomorphic mapping from $\mathbb{Q}$ to $\mathbb{Q}^{pos}$ then there exists $x \in \mathbb{Q}$ s.t. $\Phi(x) = 2$. $\therefore \Phi(\frac{x}{2} + \frac{x}{2}) = 2 \Rightarrow \Phi(\frac{x}{2})\Phi(\frac{x}{2}) = 2$ (Since Φ is an isomorphism), which implies that $\Phi(\frac{x}{2}) = \sqrt{2}$, which is a contradiction. Therefore there is no isomorphic mapping from $\mathbb{Q}$ to $\mathbb{Q}^{pos}$.

If $\Phi: G \to K$ is an isomorphism then the following are true

1. $\Phi(e) = e$.
2. $\Phi(x^{-1}) = (\Phi(x))^{-1}$
3. $|x| = |\Phi(x)|$
4. If G is cyclic then K is abelian.

5. If G is abelian then K is abelian.

6. If $H \leq G$ then $\Phi(H) = \{\Phi(h) : h \in H\}$ is a subgroup of K

Proofs: easy try it!

Consider the three groups A_4, D_6 and $\mathbb{Z}_{12}$. They are all groups of order 12. Are they isomorphic?

$\mathbb{Z}_{12}$ is cyclic so it is not isomorphic to either A_4 or D_6 neither of which are cyclic. Moreover D_6 has an element of order 6 (r) but A_4 has no element of order 6. Therefore D_6 and A_4 are not isomorphic.