Phase Plots of First-Order Autonomous Systems

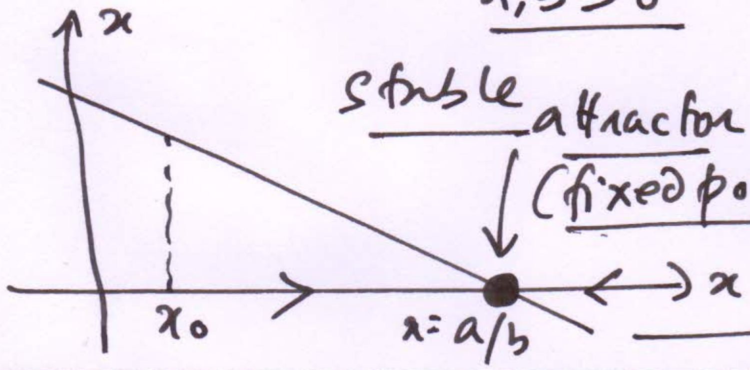
$$\boxed{\frac{dx}{dt} = \dot{x} = f(x)}$$

First-order ordinary autonomous differential equation.

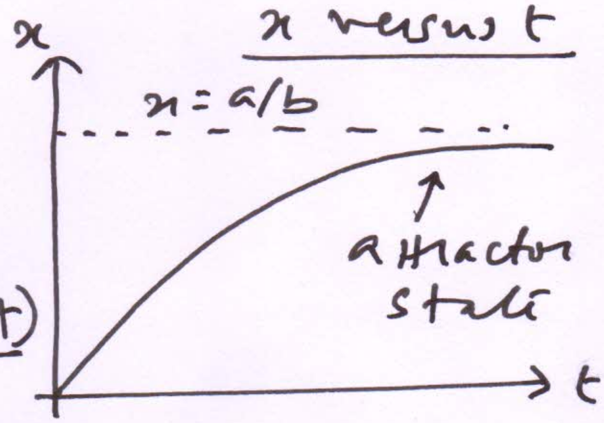
Plot of $\dot{x}$ versus x $\rightarrow$ Phase Diagram.

1/. $\boxed{\dot{x} = f(x) = a - bx}$

$a, b > 0$



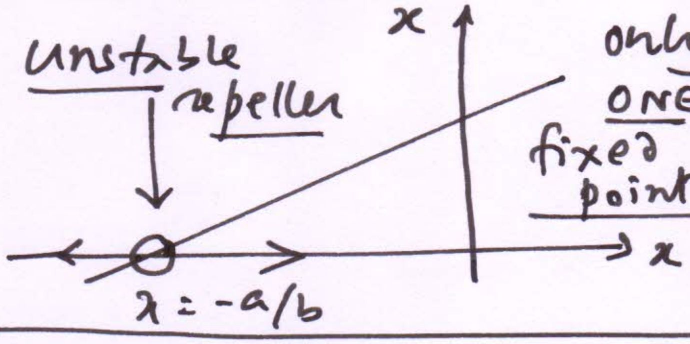
$\longrightarrow$ linear differential equation



2/. $\boxed{\dot{x} = f(x) = a + bx}$

$a, b > 0$

Linear equations have only ONE root for $\boxed{\dot{x} = 0}$



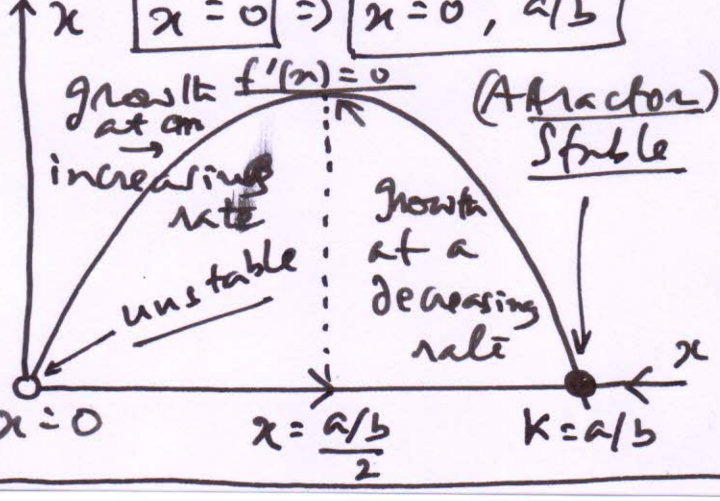
only ONE fixed point.

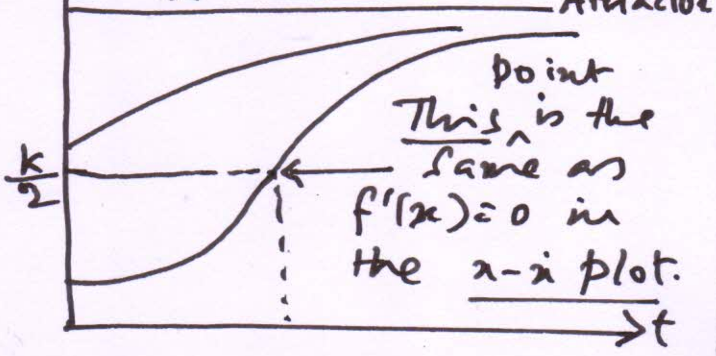
3/. $\boxed{\dot{x} = f(x) = ax - bx^2}$

$a, b > 0$ $\quad f'(x) = a - 2bx = 0$

$k = a/b$ $\quad \Rightarrow x = \frac{a/b}{2} = \frac{k}{2}$ maximum

$\boxed{\dot{x} = 0} \Rightarrow \boxed{x = 0, a/b}$

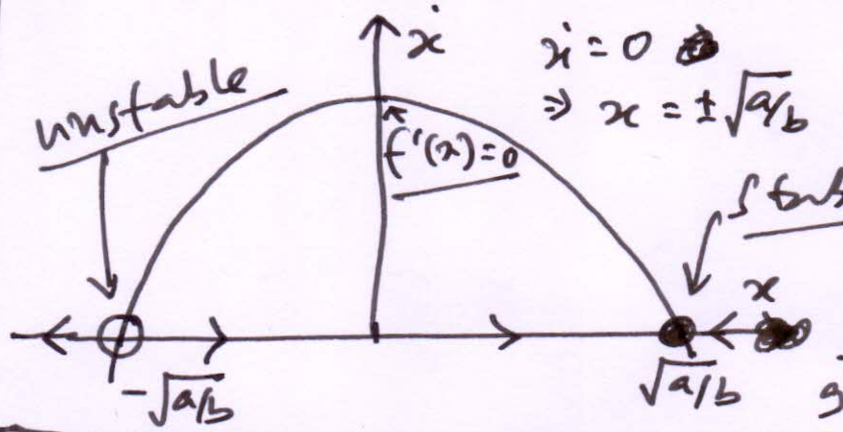


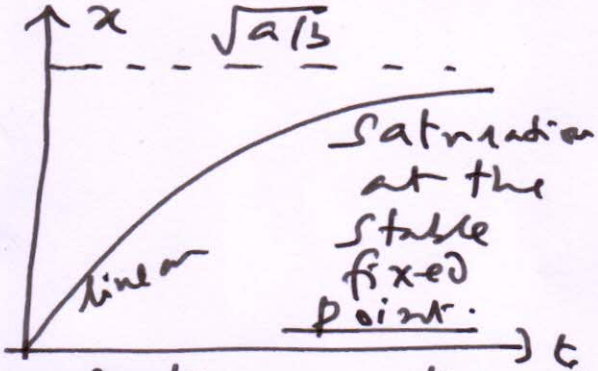


This point is the same as $f'(x) = 0$ in the x-$\dot{x}$ plot.

4/. $\boxed{\dot{x} = f(x) = a - bx^2}$ $a, b > 0$ $f'(x) = -2bx = 0 \Rightarrow x = 0$

$\dot{x} = 0 \Rightarrow x = \pm\sqrt{a/b}$

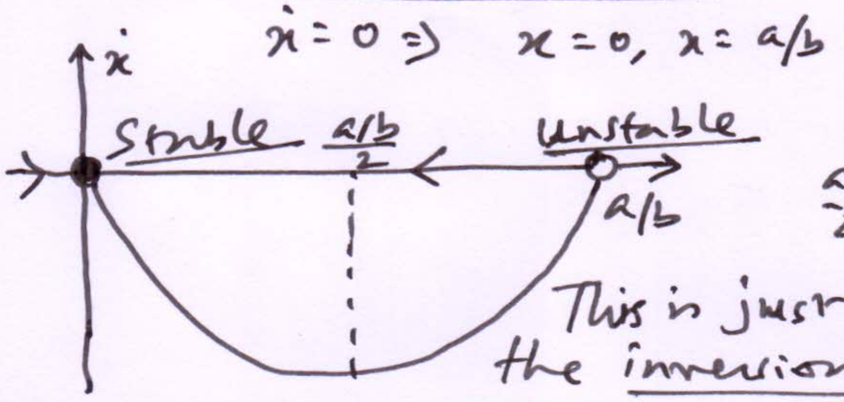


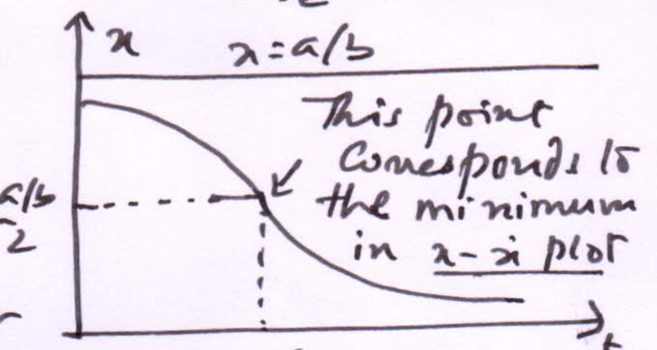


Second-degree nonlinearity gives two fixed points.

5/. $\boxed{\dot{x} = f(x) = -ax + bx^2}$ $a, b > 0$ $f'(x) = 2bx = 0 \Rightarrow x = 0$ minimum

$\dot{x} = 0 \Rightarrow x = 0,\ x = a/b$





This is just the inversion of the logistic equation.

Stability of Fixed Points (when $\boxed{\dot{x} = 0}$)

For an autonomous system $\boxed{\dot{x} = f(x)}$ the fixed point condition is $\boxed{\dot{x} = 0} \Rightarrow \boxed{f(x_c) = 0}$ at $x = x_c$.

$\Rightarrow$ x_c is the fixed point coordinate.

Now perturb about the fixed point $\boxed{x = x_c + \epsilon}$ in which $\epsilon \ll x_c$. Now $\boxed{\dot{x} = \dot{\epsilon}}$ ($\because \dot{x}_c = 0$)

$\Rightarrow \dot{x} = \dot{\epsilon} = f(x) = f(x_c + \epsilon) \Rightarrow \boxed{\dot{\epsilon} = f(x_c + \epsilon)}$.

By a Taylor expansion, we can write

$$\boxed{\dot{\epsilon} = f(x_c) + f'(x_c)\epsilon + \frac{1}{2!} f''(x_c)\epsilon^2 + \cdots}$$

(P.T.O.)

In the expansion $\boxed{f(x_c) = 0}$ and we neglect the ϵ^2 term as very small. Hence,

$$\boxed{\dot{\epsilon} \simeq f'(x_c)\,\epsilon} \Rightarrow \frac{d\epsilon}{dt} = f'(x_c)\epsilon \Rightarrow \int \frac{d\epsilon}{\epsilon} = \int f'(x_c)\,dt$$

$$\Rightarrow \ln\epsilon = \ln A + f'(x_c)t \Rightarrow \boxed{\epsilon \simeq A e^{f'(x_c)t}}$$

$$\because \epsilon = x - x_c \Rightarrow \boxed{x \simeq x_c + A e^{f'(x_c)t}}$$ [A is constant]

For a stable fixed point, $\boxed{\text{as } t \to \infty,\ x \to x_c}$.
This happens only when $\boxed{f'(x_c) < 0}$ (stability condition).
If $\boxed{f'(x_c) > 0}$, the fixed point is unstable.

Critical Condition: When both $\boxed{f(x_c) = 0}$ and also $\boxed{f'(x_c) = 0} \Rightarrow \boxed{\dot{\epsilon} \simeq \frac{1}{2!} f''(x_c)\epsilon^2}$ in which the ϵ^2 term is no longer neglected.

$$\Rightarrow \boxed{\frac{d\epsilon}{dt} = \frac{f''(x_c)}{2!}\epsilon^2} \Rightarrow \int \epsilon^{-2}\,d\epsilon = \frac{f''(x_c)}{2}\int dt$$

$$\Rightarrow \frac{\epsilon^{-1}}{-1} = \frac{f''(x_c)}{2}(t - A)$$ [A is integration constant]

$$\Rightarrow \boxed{\epsilon = -\frac{2}{f''(x_c)}\cdot\frac{1}{t-A}} \Rightarrow \boxed{x \simeq x_c - \frac{2}{f''(x_c)}\cdot\frac{1}{t-A}}$$

When $t \to \infty$, $x \to x_c$ (slow power-law convergence)

Examples:
1/. $\boxed{f(x) = a \pm bx} \Rightarrow \boxed{f'(x) = \pm b}$ If $\boxed{f'(x) = b}$ then unstable, and if $\boxed{f'(x) = -b} \Rightarrow$ stable.

2/. $\boxed{f(x) = ax - bx^2} \Rightarrow \boxed{f'(x) = a - 2bx}$ when $x = 0$, $f'(0) = a$ (unstable)
when $x = a/b$, $f'(a/b) = -a$ (stable)

3/. $\boxed{f(x) = a - bx^2}$ $\boxed{f'(x) = -2bx}$. If $x = \sqrt{a/b}$, $f'(\sqrt{a/b}) = -2b\sqrt{\frac{a}{b}}$ (stable).
unstable $\rightarrow$ If $x = -\sqrt{a/b}$, $f'(-\sqrt{a/b}) = 2b\sqrt{a/b}$