1. Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Find $\vec{\nabla} f$. Find the rate of change of f at the point $(1, 1, 0)$ along a direction specified by the unit vector $\frac{1}{\sqrt{2}}(\hat{\mathrm{i}} - \hat{\mathrm{j}})$.

   soln
   $f = \sqrt{x^2 + y^2 + z^2} = r$.

$$\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

$$\begin{aligned} \vec{\nabla} f &= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\frac{\partial f}{\partial z}\hat{k} \\ &= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r} = \hat{r} \end{aligned}$$

   At (1,1,0), $\vec{\nabla} f = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$.
   Let $\hat{n} = \frac{\hat{i}-\hat{j}}{\sqrt{2}}$.
   Let $\vec{dr} = dr\hat{n}$.
   $\therefore\ df = \vec{\nabla} f \cdot \vec{dr} = \vec{\nabla} \cdot \hat{n} dr$.
   $\therefore\ \frac{df}{dr} = \vec{\nabla} f \cdot \hat{n} = \frac{\hat{i}+\hat{j}}{\sqrt{2}} \cdot \frac{\hat{i}-\hat{j}}{\sqrt{2}} = 0$

2. Let $\vec{\mathrm{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z). Show that
   (a) $\vec{\nabla}(1/r) = -\hat{\mathrm{r}}/r^2$
   (b) Evaluate $\vec{\nabla}(r^n)$

   soln
   (a)
   $f(\vec{r}) = 1/r$.
   $\therefore\ \vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$.
   $\frac{\partial f}{\partial x} = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}\left(\frac{x}{r}\right) = -\frac{x}{r^3}$
   Similarly $\frac{\partial f}{\partial y} = -\frac{y}{r^3}$ and $\frac{\partial f}{\partial z} = -\frac{z}{r^3}$
   $\therefore\ \vec{\nabla} f = -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r^3} = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$.

   (b)
   Let $f(\vec{r}) = r^n$. Then

$$\frac{\partial f}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-r}\frac{x}{r} = nr^{n-2}x$$

   $\therefore\ \vec{\nabla} f = nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-1}\hat{r}$.

3. A real square matrix M is orthogonal if $M^{-1} = M^T$. Using the fact that the magnitude of a vector doesn't change under rotation prove that a rotation matrix is orthogonal.

soln:
Let R be a rotation matrix.
Let $\vec{A'} = R\vec{A}$ and $\vec{B'} = R\vec{B}$. Let us denote the matrix representation of $vecA$ as $[A] = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$. Then $[A'] = R[A]$ and $[B'] = R[B]$.
In the matrix representation $\vec{A} \cdot \vec{B} = [A]^T[B]$.
Since $\vec{A} \cdot \vec{B}$ is a scalar we have $\vec{A'} \cdot \vec{B'} = \vec{A} \cdot \vec{B}$, i.e , $[A']^T[B'] = [A]^T[B]$. So we have

$$[A]^T R^T R[B] = [A]^T[B]$$

If this has to be true for any arbitrary vector A and B then the only possibility is $R^T R = \mathbb{I}$, i.e $R^{-1} = R^T$. So R is an orthogonal matrix.

4. *This question tries to give an idea of what a scalar quantity is.*
The electric potential at a point on a horizontal plate with respect to a given coordinate system is given as $V(x, y) = xy$. If someone work with a coordinate system that is rotated by 45°, the new coordinates (x', y') are given in terms of the old ones as $x' = \frac{x+y}{\sqrt{2}}$ and $y' = \frac{y-x}{\sqrt{2}}$. Let's write this as $\vec{r'} = R\vec{r}$. Potential is a scalar quantity. If $V'(x', y')$ is the functional form of the potential function in the new coordinate system then $V'(x', y') = V(x, y)$.

   (a) Find the form of the function $V'(x', y')$.

   (b) Verify that $\vec{\nabla}'V' = R\vec{\nabla}V$, i.e., components of a gradient transform as a vector quantity.

soln:

   (a) The relation between the coordinates (x', y') and (x, y) is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$$

Inverting the above relation we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{aligned} V(x, y) &= xy = \frac{1}{2}(x' - y')(x' + y') = \frac{1}{2}(x'^2 - y'^2) \\ \therefore \quad V'(x', y') &= \frac{1}{2}(x'^2 - y'^2) \end{aligned}$$

(b)

$$\vec{\nabla}V(x,y) = y\hat{i} + x\hat{j} \equiv \begin{pmatrix} A_x \\ A_y \end{pmatrix}, \quad \vec{\nabla}'V'(x',y') = x'\hat{i}' - y'\hat{j}' \equiv \begin{pmatrix} A'_x \\ A'_y \end{pmatrix}$$

So we have

$$\begin{aligned} \begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \begin{pmatrix} x' \\ -y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x+y \\ x-y \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} A_y + A_x \\ A_y - A_x \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \end{aligned}$$

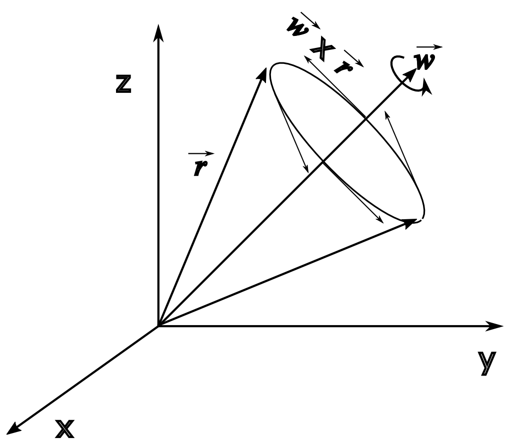
$\therefore\ \vec{\nabla}'V' = R\vec{\nabla}V$

5. Let $\vec{\mathrm{A}} = \vec{\omega} \times \vec{\mathrm{r}}$ where $\vec{\omega}$ is a fixed vector in space. Find $\vec{\nabla} \times \vec{\mathrm{A}}$.

soln:

$$\begin{aligned} \vec{A} = \vec{\omega} \times \vec{r} &= \hat{i}(\omega_y z - \omega_z y) + \hat{j}(\omega_z x - \omega_x z) + \hat{k}(\omega_x y - \omega_y x) \\ \therefore\ \vec{\nabla} \times \vec{A} &= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\ &= \hat{i}\left(\omega_x - (-\omega_x)\right) + \hat{j}\left(\omega_y - (-\omega_y)\right) + \hat{k}\left(\omega_z - (-\omega_z)\right) = 2\vec{\omega} \end{aligned}$$

Interpretation:
The vector field $\vec{\omega} \times \vec{r}$ curls around the vector $\vec{\omega}$. It is the velocity vector of the particles of a rigid body rotating with angular velocity $\vec{\omega}$.



6. Prove that for any vector field $\vec{\mathrm{A}}$, $\vec{\nabla} \cdot \vec{\mathrm{A}}$ is a scalar.

soln:
Let us denote the coordinates before and after rotation as

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad X' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}$$

$X' = RX$ where R is the rotation matrix.

$$\therefore \; x'_i = \sum_j R_{ij} x_j \qquad \text{and} \qquad x_i = \sum_j (R^T)_{ij} x'_j = \sum_j R_{ji} x'_j \tag{1}$$

Here we have used the fact that $R^{-1} = R^T$ i.e. R is an orthogonal matrix. We have to show that $\vec{\nabla}' \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A}$, i.e

$$\sum_i \frac{\partial A'_i}{\partial x'_i} = \sum_i \frac{\partial A_i}{\partial x_i}$$

$$\frac{\partial}{\partial x'_i} = \sum_j \frac{\partial}{\partial x_j} \frac{\partial x_j}{\partial x'_i} = \sum_j R_{ij} \frac{\partial}{\partial x_j} \qquad \text{from Eq.(1)}$$

$A'_i = \sum_k R_{ik} A_k$ since $\vec{A}$ is a vector.

$$\begin{aligned} \vec{\nabla}' \cdot \vec{A}' = \sum_i \frac{\partial A'_i}{\partial x'_i} &= \sum_i \sum_j R_{ij} \frac{\partial}{\partial x_j} \left(\sum_k R_{ik} A_k \right) \\ &= \sum_j \sum_k \left(\sum_i R_{ij} R_{ik} \right) \frac{\partial A_k}{\partial x_j} \end{aligned}$$

Now $\sum_i R_{ij} R_{ik} = \sum_i (R^T)_{ji} R_{ik} = \delta_{jk}$ since $R^T R = \mathbb{I}$.
So only $j = k$ terms survive in the above summation

$$\therefore \quad \vec{\nabla}' \cdot \vec{A}' = \sum_j \frac{\partial A_j}{\partial x_j} = \vec{\nabla} \cdot \vec{A}$$

So the divergence of a vector is invariant under rotation. Hence it is a scalar quantity. This proof is valid for the divergence operator in any dimension since we have only used the orthogonality of R.

7. Find the divergence of the following:

   (a) $\vec{\mathrm{A}} = \hat{\mathrm{r}}$,

   (b) $\vec{\mathrm{A}} = \frac{\hat{\mathrm{r}}}{r}$ in 2 dimension

   (c) $\vec{\mathrm{A}} = \frac{\hat{\mathrm{r}}}{r}$ in 3 dimension

   (d) $\vec{\mathrm{A}} = \frac{\hat{\mathrm{r}}}{r^2}$ in 3 dimension. Plot this field.

   (e) $\vec{\mathrm{A}} = \frac{\hat{\mathrm{r}}}{r^3}$ in 3 dimension

   (f) $\vec{\mathrm{A}} = y\hat{\mathrm{i}} - x\hat{\mathrm{j}}$

   soln:

(a) $\vec{A} = \hat{r} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}; r = \sqrt{x^2+y^2+z^2}$
$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
$\frac{\partial A_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = -\frac{x^2}{r^3} + \frac{1}{r}$
Similarly $\frac{\partial A_y}{\partial y} = -\frac{y^2}{r^3} + \frac{1}{r}$ and $\frac{\partial A_z}{\partial z} = -\frac{z^2}{r^3} + \frac{1}{r}$.
$\therefore\ \vec{\nabla}\cdot\vec{A} = -\frac{x^2+y^2+z^2}{r^3} + 3/r = 2/r$.

For the remaining parts we evaluate the following:

$$\begin{aligned}
\vec{\nabla}\cdot(r^n\hat{r}) &= r^n\vec{\nabla}\cdot\hat{r} + \vec{\nabla}(r^n)\cdot\hat{r} \\
&= r^n\left(\frac{2}{r}\right) + nr^{n-1}\hat{r}\cdot\hat{r} \quad \text{(in 3 dim.)} \\
&= (n+2)r^{n-1} \\
\vec{\nabla}\cdot(r^n\hat{r}) &= r^n\left(\frac{1}{r}\right) + nr^{n-1}\hat{r}\cdot\hat{r} \quad \text{(in 2 dim.)} \\
&= (n+1)r^{n-1}
\end{aligned}$$

(b) $\vec{A} = \frac{\hat{r}}{r}$ in 2 dim. $(n=-1)$
$\vec{\nabla}\cdot\vec{A} = (-1+1)r^{-2} = 0$.
(This is true only for $r \neq 0$. At $r = 0, \vec{\nabla}\cdot\vec{A} \to \infty$)

(c) $\vec{A} = \frac{\hat{r}}{r}$ in 3 dim. $(n=-1)$
$\vec{\nabla}\cdot\vec{A} = (-1+2)r^{-2} = 1/r^2$

(d) $\vec{A} = \frac{\hat{r}}{r^2}$ in 3 dim. $(n=-2)$
$\vec{\nabla}\cdot\vec{A} = (-2+2)r^{-3} = 0$.
(This is true only for $r \neq 0$. At $r = 0, \vec{\nabla}\cdot\vec{A} \to \infty$)

(e) $\vec{A} = \frac{\hat{r}}{r^3}$ in 3 dim. $(n=-3)$
$\vec{\nabla}\cdot\vec{A} = (-3+2)r^{-4} = -1/r^4$

8. Let

$$D = \begin{pmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} \\ \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} \end{pmatrix}$$

Under a rotation of the coordinate system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = R\begin{pmatrix} x \\ y \end{pmatrix}$$

show that

$$D' = \begin{pmatrix} \frac{\partial A'_x}{\partial x'} & \frac{\partial A'_y}{\partial x'} \\ \frac{\partial A'_x}{\partial y'} & \frac{\partial A'_y}{\partial y'} \end{pmatrix} = RDR^T$$

soln
D can be wrtten as

$$D = \begin{pmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} \\ \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}(A_x\ A_y)$$

The first column matrix in the above product is the $\vec{\nabla}$ operator which we have seen transforms as a vector under the rotation R. So

$$\vec{\nabla}' = \begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{pmatrix} = R \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

Also the vector $\vec{A}$ transforms as $\vec{A}' = R\vec{A}$.

$$\begin{aligned} \therefore \quad D' &= \begin{pmatrix} \frac{\partial A'_x}{\partial x'} & \frac{\partial A'_y}{\partial x'} \\ \frac{\partial A'_x}{\partial y'} & \frac{\partial A'_y}{\partial y'} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{pmatrix} (A'_x \ A'_y) \\ &= R \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} (A_x \ A_y) R^T \\ &= R \begin{pmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} \\ \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} \end{pmatrix} R^T = RDR^T \end{aligned}$$