Dhirubhai Ambani Institute of Information and Communication Technology (DA-IICT)
First In Semester Examination
CT314 (Statistical Communication Theory)
Date of Examination: Feb 05, 2011
Duration: 2 Hours
Maximum Marks: 20

Instructions:

1. Attempt all questions.
2. Use of scientific non programmable calculator is permitted.
3. Figures in brackets indicate full marks.
4. All the acronyms carry their usual meaning.
5. A one page hand written formula sheet allowed (to be submitted).

Q1. Let two honest (fair) coins marked 1 and 2, be tossed together. The four possible outcomes are $T_1T_2, H_1T_2, T_1H_2, H_1H_2$. ($T_1$ indicates toss of coin 1 resulting in tails; similarly T_2 etc.) All these outcomes are equally likely and mutually exclusive. Let the event A be 'not H_1H_2' and B be the event 'match'. (match comprises of two outcomes T_1T_2, H_1H_2). Find P(B/A), P(B). Are A and B independent? (3 marks)

Q2. A random variable X takes only positive integer values {k=1,2,3…}. Show that $E(X) = \sum_k P\{X \geq k\}$ (3 marks)

Q3. Consider a random variable X with probability density function (pdf) $f_X(x) = e^{-x}, x \geq 0$. Using the concept of function of a random variable find the pdf of the transformed random variable $Y = g(X) = \dfrac{3}{(X+1)^2}$ (4 marks)

Q4. If $f_X(x) = 0$ for $x < 0$, show that $P\{X > \sqrt{m_X}\} \leq \sqrt{m_X}$ (5 marks)

Q5. Which of the following (2 by 2) matrices are valid covariance matrices and why?

a. $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ b. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ (2 marks)

Q6. Consider a 2 by 2 covariance matrix for random variables X and Y. Using a property of the matrix prove that $|\varsigma_{XY}| \leq 1$, ς_{XY} being correlation coefficient
(2 marks)

Q7. Write an example 2 by 2 covariance matrix for which $Var(X_1 + X_2) = Var(X_1)$, ($X_2$ is not a constant.). You need to write the entries like the one given in Q5. (1 mark)

Q paper Solutions
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Q.1

$$P(B/A) = \frac{P(AB)}{P(A)}.$$

AB is the event 'not H_1H_2' and 1 match

i.e. it is the outcome T_1T_2

$\therefore P(AB) = 1/4$, Event A has outcome 'T_1T_2, T_1H_2, H_1T_2' $\therefore P(A) = 3/4$

$$P(B/A) = \frac{\frac{1}{4}}{3/4} = 1/3$$

$P(B) = \frac{1}{2}$ & $P(B/A) = 1/3$,

$P(B) \neq P(B/A)$ hence they are dependent

Q.2.

$$E[X] = 1.P[X=1] + 2P[X=2] + 3.P[X=3] + \cdots$$
$$= P[X=1] + P[X=2] + P[X=3] + \cdots$$
$$+ P[X=2] + P[X=3] + \cdots$$
$$+ P[X=3] + \cdots$$

$$= P[X \geq 1] + P[X \geq 2] + \cdots$$

$$= \sum_k P(X \geq k)$$

Q3. We have $f_X(x) = e^{-x}$, $x \geq 0$ So x takes values $x \geq 0$ in $f_X(x)$.

Now $g(x) = \frac{3}{(x+1)^2} = Z$

Since square term in the denominator & $x \geq 0$

$z \geq 0$

and it has two roots

$x_1 = +\sqrt{\frac{3}{z}} - 1$ and $x_2 = -\sqrt{\frac{3}{z}} - 1$

$$g'(x) = \frac{-6}{(x+1)^3}$$

$$g'(x_1) = \frac{-6}{\left(\sqrt{\frac{3}{z}}\right)^3}, \quad g'(x_2) = \frac{-6}{\left(-\sqrt{3/z}\right)^3}$$

$$\therefore |g'(x_1)| = |g'(x_2)| = \frac{6}{\left(\frac{3}{z}\right)^{3/2}} = \frac{2}{\sqrt{3}} z^{3/2}.$$

$$f_Z(z) = \frac{1}{\frac{2}{\sqrt{3}} z^{3/2}} \left[f_X\left(\sqrt{\frac{3}{z}} - 1\right) + f_X\left(-\sqrt{\frac{3}{z}} - 1\right) \right]$$

The ~~first~~ second f^n. in the square brackets does not exist as it makes argument of $f_X(x)$ i.e. x as $-ve$ for $z > 0$ which is not possible

$\therefore$ The first f^n. exists only when

$\sqrt{\frac{3}{Z}} \geq 1$ or $\frac{3}{Z} \geq 1$ or $3 \geq Z$

$Z \leq 3$.

$\therefore$ So $f_Z(z) = \frac{1}{2/\sqrt{3}\, z^{3/2}} f_X\left(\sqrt{\frac{3}{z}} - 1\right)$ for $0 < z \leq 3$.

i.e.

$f_Z(z) = \frac{\sqrt{3}}{2} z^{-3/2} e^{-\left(\sqrt{\frac{3}{z}} - 1\right)}$

for $0 < z \leq 3$

Q4

$f_x(0) = 0$ for $x < 0$

$$m_x = E(X) = \int_0^\infty x f_x(x)\, dx$$

$$= \int_0^{\sqrt{m_x}} x f_x(x)\, dx + \int_{\sqrt{m_x}}^\infty x f_x(x)\, dx$$

or $m_x \geq \int_{\sqrt{m_x}}^\infty x f_X(x)\, dx \rightarrow k_2$

$\sqrt{m_x}\int_{\sqrt{m_x}}^\infty f_x(x)\, dx \longrightarrow k_1 \qquad k_1 < k_2$

$\therefore m_x \geq \sqrt{m_x}\int_{\sqrt{m_x}}^\infty f_x(x)\, dx$

because $\int_{\sqrt{m_x}}^\infty \sqrt{m_x} f_x(x)\, dx < \int_{\sqrt{m_x}} x f_x(x)\, dx$.

$\therefore \sqrt{m_x} \geq \int_{\sqrt{m_x}}^\infty f_x(x)\, dx$ or $P[X > m_x] \leq \sqrt{m_x}$.

Q5. a. – No det < 0

b. – No diagonal elements –ve

c. – yes satisfies all properties of C matrix.

d. – No not symmetrical matrix.

Q6. det C_{XY} must be ≥ 0.

i.e. $\begin{vmatrix} \sigma_X^2 & cov(X,Y) \\ cov(X,Y) & \sigma_Y^2 \end{vmatrix} \geq 0$

or $\sigma_X^2 \sigma_Y^2 - (cov(X,Y))^2 \geq 0$.

or $(cov(X,Y))^2 \leq \sigma_X^2 \sigma_Y^2$

or $\frac{(cov(X,Y))^2}{\sigma_X^2 \sigma_Y^2} \leq 1$

or $\left|\frac{cov(X,Y)}{\sigma_X \sigma_Y}\right| \leq 1$

i.e. $|\rho_{XY}| \leq 1$

Q.7: Consider the covariance matrix

$C_{\underline{X}} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ This gives $Var(X_1 + X_2) = 1 + 2 - 2 = 1$

$= Var(X_1)$.

but $Var(X_1) = 1$ i.e. X_1 is not constant.

(May be valid for other symmetric +ve semidefinite matrix also)