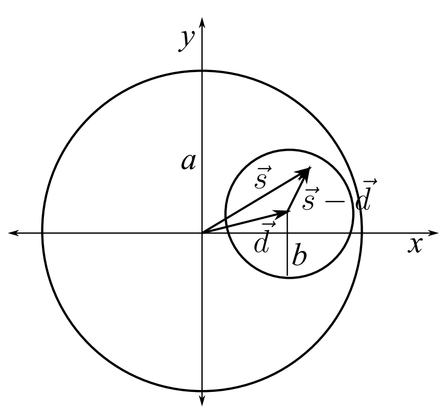
1. An infinitely long cylindrical cavity of radius b is bored into a bigger cylinder of radius a. The axes of the two cylinders are parallel but the cylinders are not concentric. The remaining part of the cylinder has a cosnstant volume charge density ρ. Show that the electric field inside the cavity is uniform and directed along the line joining the center of the two cylinders.

soln
The given configuration of charge is a superposition of two simple charge configuration. One is a cylinder of radius a carrying a uniform charge density ρ. The other is a cylinder of radius b with a charge density $-\rho$. when the negatively charged cylinder is inserted into the bigger cylinder it creates a hollow chargeless region as required. We can easily calculate the electric field due to the two cylinders and then add the electric field caused by them. Let us place the positively charged cylinder coincident with the z axis. Let $E_1(s)$ be the magnitude of the electric field at a distance s from the z axis. Consider a cylindrical Gaussian surface of radius s and height h. The electric field on this Gaussian surface is along $\hat{s}$ which is also the direction of the normal to the cylinder. The charge enclosed by this Gaussian cylinder is $\rho\pi s^2 h$. The flux of the electric field over the Gausian surface is $2\pi s h E_1(s)$. By Gauss' law we have

$$
\begin{aligned}
2\pi s h E_1(s) &= \frac{\rho\pi s^2 h}{\epsilon_0} \\
\therefore \quad E_1(s) &= \frac{\rho s}{2\epsilon_0} \\
\therefore \quad \vec{E}_1(\vec{s}) &= \frac{\rho s}{2\epsilon_0}\hat{s} \\
&= \frac{\rho\vec{s}}{2\epsilon_0}
\end{aligned}
$$

Let te position vector of the center of the other cylinder be $\vec{d}$ as shown in the figure.



This is also a uniformly charged cylinder with density $-\rho$. We can directly write down the electric field at a point inside this cylinder as

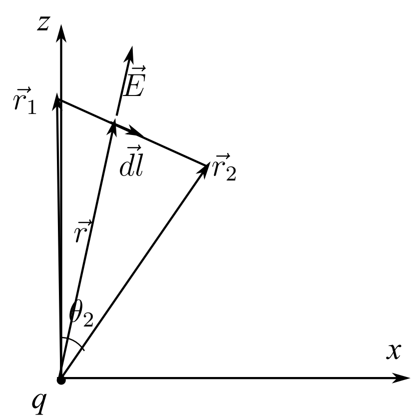
$$\vec{E}_2(\vec{s}) = \frac{-\rho(\vec{s}-\vec{d})}{2\epsilon_0}$$

The total electric field at a point described by the position vector $\vec{s}$ will be

$$\vec{E}(\vec{s}) = \vec{E}_1(\vec{s}) + \vec{E}_2(\vec{s}) = \frac{\rho\vec{d}}{2\epsilon_0}$$

which is a constant electric field directed along the vector connecting the center of the two cylinders.

2. Consider a point charge q at the origin. Find the electric potential at a point $\vec{r}_2 : (r = r_2, \theta = \theta_2, \phi = 0)$ with respect to the potential at $\vec{r}_1 : r = r_1, \theta = 0, \phi = 0$ as reference by evaluating the integral $-\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot\vec{dl}$ along a straight line joining $\vec{r}_1$ to $\vec{r}_2$.



soln:
Since $\phi = 0$ both the points are on the xz plane. The equation of the line joining $\vec{r}_1$ and $\vec{r}_2$ is

$$\frac{r\cos\theta - r_1}{r\sin\theta} = \frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}, \quad \phi = 0$$

This gives the relation between r and θ as

$$r\left[\cos\theta - \sin\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)\right] = r_1 \tag{1}$$

$$\therefore \quad r\left[-\sin\theta - \cos\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)\right]d\theta + dr\left[\cos\theta - \sin\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)\right] = 0$$

$$\therefore \quad \frac{dr}{d\theta} = \frac{r\left[\sin\theta + \cos\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)\right]}{\cos\theta - \sin\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)} = -r\frac{f'(\theta)}{f(\theta)} \tag{2}$$

where $f(\theta) = \cos\theta - \sin\theta \left(\frac{r_2\cos\theta_2 - r_1}{r_2 \sin\theta_2}\right)$. The potential difference between $\vec{r}_1$ and $\vec{r}_2$ is now

$$\begin{aligned}
-\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{l} &= -\frac{q}{4\pi\epsilon_0}\int_{\vec{r}_1}^{\vec{r}_2} \frac{1}{r^2}\hat{r}\cdot(\hat{r}dr + rd\theta\hat{\theta} + r\sin\theta d\theta\hat{\phi}) \\
&= -\frac{q}{4\pi\epsilon_0}\int_0^{\theta_2} \frac{1}{r^2}dr \\
&= \frac{q}{4\pi\epsilon_0}\int_0^{\theta_2} \frac{1}{r^2} r\frac{f'(\theta)}{f(\theta)}d\theta \quad \text{from eq.(2)} \\
&= -\frac{q}{4\pi\epsilon_0}\int_0^{\theta_2} \frac{f'(\theta)}{rf(\theta)}d\theta
\end{aligned}$$

From Eq.(1) we see that $rf(\theta) = r_1$.

$$\begin{aligned}
-\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{l} &= \frac{1}{r_1}\int_0^{\theta_2} f'(\theta)d\theta \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{r_1}\left[f(\theta)\right]_0^{\theta_2} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{r_1}\left(f(\theta_2) - f(0)\right) = \frac{q}{4\pi\epsilon_0}\frac{1}{r_1}\left(\frac{r_1}{r_2} - 1\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)
\end{aligned}$$

If the reference point $\vec{r}_1$ is at ∞ then the potential at $\vec{r}_2$ is $\frac{q}{4\pi\epsilon_0 r_2}$. We see that the potential is independent of θ_2. It can be shown to be also independent of ϕ_2. It only depends on r_2, the distance from the origin.

3. A hollow spherical shell carries a uniform charge density ρ_0 in the region $a \le r \le b$. Find the electric potential as a function of r.

soln:
Let us first find the electric field in the three regions, $r > b$, $a < r < b$, and $r < a$. The problem has a spherical symmetry. The electric field everywhere is along $\hat{r}$. Using Gauss' law we can easily calculate these fields. They are given as follows:

$$\begin{aligned}
E(r) &= \frac{\rho_0}{3\epsilon_0}\frac{b^3 - a^3}{r^2} \quad ; r \ge b \\
&= \frac{\rho_0}{3\epsilon_0}\left(r - \frac{a^3}{r^2}\right) \quad ; a \le r < b \\
&= 0 \quad ; r < a
\end{aligned}$$

To calculate the potential we take a reference at a point $\vec{r}_c$ at a distance c which is outside the outer surface of the shell. The electric potential at a point $\vec{r}$ is given as

$$\Phi(\vec{r}) = -\int_{\vec{r}_c}^{\vec{r}} \vec{E}(\vec{r})\cdot d\vec{l}$$

$d\vec{l} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$ and $\vec{E}(\vec{r}) = E(r)\hat{r}$. So the potential is given as

$$\Phi(\vec{r}) = -\int_c^r E(r)dr$$

Since the potential only depends on r it is written as $\Phi(r)$. So the potentials in the three regions are

$$\begin{aligned}
\Phi(r>b) &= -\int_c^r E(r)dr = -\int_c^r \frac{\rho_0}{3\epsilon_0}\frac{b^3-a^3}{r^2}dr \\
&= \frac{\rho_0(b^3-a^3)}{3\epsilon_0}\left(\frac{1}{r}-\frac{1}{c}\right) \\
\Phi(a<r<b) &= -\int_c^b E(r)dr - \int_b^r E(r)dr \\
&= \frac{\rho_0(b^3-a^3)}{3\epsilon_0}\left(\frac{1}{b}-\frac{1}{c}\right) - \int_b^r \frac{\rho_0}{3\epsilon_0}\left(r-\frac{a^3}{r^2}\right)dr \\
&= \frac{\rho_0(b^3-a^3)}{3\epsilon_0}\left(\frac{1}{b}-\frac{1}{c}\right) - \frac{\rho_0}{3\epsilon_0}\left(\frac{r^2-b^2}{2}+\frac{a^3}{r}-\frac{a^3}{b}\right) \\
\Phi(r<a) &= -\int_c^b E(r)dr - \int_b^a E(r)dr - \int_a^r E(r)dr \\
&= \frac{\rho_0(b^3-a^3)}{3\epsilon_0}\left(\frac{1}{b}-\frac{1}{c}\right) - \frac{\rho_0}{3\epsilon_0}\left(\frac{a^2-b^2}{2}+a^2-\frac{a^3}{b}\right) \\
&= \frac{\rho_0}{2\epsilon_0}(b^2-a^2) - \frac{\rho_0}{3\epsilon_0}\frac{b^3-a^3}{c}
\end{aligned}$$

As $r \to \infty$ $\Phi \to -\frac{\rho_0}{3\epsilon_0}\frac{b^3-a^3}{c}$. If we take the reference point c to ∞ we will have to add this amount of potential to each of the regions.

4. (a) A charge distribution $\rho_1(\vec{r})$ produces a potential $\phi_1(\vec{r})$ in a region τ and another charge distribution $\rho_2(\vec{r})$ produces a potential $\phi_2(\vec{r})$ in the region. Prove that

$$\int_\tau \rho_1\phi_2 d^3\vec{r} = \int_\tau \rho_2\phi_1 d^3\vec{r}$$

How do you interpret this result.

soln:
Poisson's equation gives $\frac{\rho_1}{\epsilon_0} = -\vec{\nabla}^2\phi_1$.

$$\therefore \quad \int_\tau \rho_1\phi_2 d^3\vec{r} = -\epsilon_0\int_\tau (\vec{\nabla}^2\phi_1)\phi_2 d^3\vec{r}$$

Using the product rule $\vec{\nabla}\cdot(\phi_2\vec{\nabla}\phi_1) = \phi_2\vec{\nabla}^2\phi_1 + \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2$ we get

$$\begin{aligned}
\int_\tau \rho_1\phi_2 d^3\vec{r} &= -\epsilon_0\int_\tau \vec{\nabla}\cdot(\phi_2\vec{\nabla}\phi_1)d^3\vec{r} + \epsilon_0\int_\tau \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2 d^3\vec{r} \\
&= -\epsilon_0\int_S \phi_2\vec{\nabla}\phi_1\cdot\hat{n}da + \epsilon_0\int_\tau \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2 d^3\vec{r}
\end{aligned}$$

We can extend the integral on the l.h.s beyond the region τ. This doesn't change the value of the integral on the l.h.s as $\rho_1 = 0$ outside this region. But on the r.h.s both the surface and the volume integral contributes and their contribution changes as we extend the region of integration. But for a charge configuration which is confined in a finite region τ, both ϕ_2 and $\vec{\nabla}\phi_1$ goes to 0 as the surface S tends to infinity. So we get

$$\int_\tau \rho_1\phi_2 d^3\vec{r} = \epsilon_0 \int_{\text{all space}} \vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2 d^3\vec{r}$$

Similarly we can prove

$$\int_\tau \rho_2\phi_1 d^3\vec{r} = \epsilon_0 \int_{\text{all space}} \vec{\nabla}\phi_2 \cdot \vec{\nabla}\phi_1 d^3\vec{r}$$

So we have

$$\int_\tau \rho_1\phi_2 d^3\vec{r} = \int_\tau \rho_2\phi_1 d^3\vec{r}$$

The integral $\int_\tau \rho_1\phi_2 d^3\vec{r}$ gives the work done in bringing the charge configuration specified by $\rho_1(\vec{r})$ in the electric field created by the charge distribution $\rho_2(\vec{r})$. Similarly the other integral gives the work done in bringing the charge configuration specified by $\rho_2(\vec{r})$ in the electric field created by the charge distribution $\rho_1(\vec{r})$. This two work done must be the same since the final charge configuration at the end of the two processes is the same, i.e a combined charge configuration given by ρ_1 and ρ_2.

(b) The interaction energy of two point charges q_1 and q_2 placed at $\vec{r}_1$ and $\vec{r}_2$ is given as $\epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d^3\vec{r}$ where the integration is done over the whole space. Prove that this is equal to $\frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$ where $r_{12} = |\vec{r}_2 - \vec{r}_1|$ as is expected.

soln:

$$\begin{aligned}
\epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d^3\vec{r} &= \epsilon_0 \int \vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2 d^3\vec{r} \\
&= \epsilon_0 \int \vec{\nabla} \cdot (\phi_1 \vec{\nabla}\phi_2) d^3\vec{r} - \epsilon_0 \int \phi_1 \vec{\nabla}^2\phi_2 d^3\vec{r} \\
&= \epsilon_0 \int_S (\phi_1 \vec{\nabla}\phi_2) \cdot \hat{n} da - \epsilon_0 \int \phi_1 \vec{\nabla}^2\phi_2 d^3\vec{r}
\end{aligned}$$

The surface S of the surface integral is at infinity since we do the integral over the whole space. ϕ_1 goes as $1/r$ and $\vec{\nabla}\phi_2$ goes as $1/(r^2)$. So the integrand goes as $\frac{1}{r^3}$ while the surface area goes as r^2. So the surface integral goes to 0. In the other integral we know $\phi_1 = \frac{q_1}{4\pi\epsilon_0|\vec{r}-\vec{r}_1|}$ and $\vec{\nabla}^2\phi_2 = -\frac{q_2\delta^3(\vec{r}-\vec{r}_2)}{\epsilon_0} d^3\vec{r}$

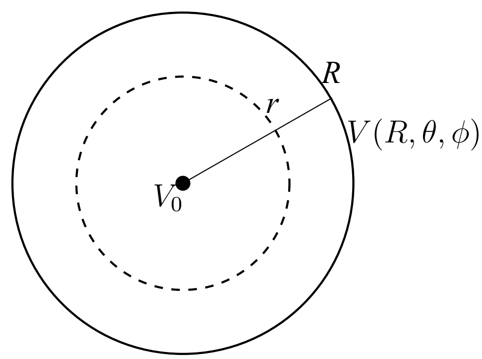
$$\begin{aligned}
\epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d^3\vec{r} &= -\epsilon_0 \int \frac{q_1}{4\pi\epsilon_0|\vec{r}-\vec{r}_1|} \frac{(-q_2\delta^3(\vec{r}-\vec{r}_2))}{\epsilon_0} d^3\vec{r} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_2 - \vec{r}_1|}
\end{aligned}$$

5. Prove the mean value theorem in electrostatics which states that in a chargeless region, the average of the potential over the surface of any sphere is equal to the potential at the center of the sphere.
   This is true for any regular polyhedron. If the faces of a regular polyhedron having n faces are maintained at potentials $V_1, V_2, ..., V_n$ then the potential at the center of the polyhedron is $(V_1 + V_2 + ... + V_n)/n$. How many such regular polyhedron do you think are possible? Look for platonic solids. Tetrahedron, cube, octahedron, dodecahedron and icosahedron.

   soln

   Consider a sphere of radius R in a chargeless region. The average value of potential over this sphere is

$$V_{avg} = \frac{1}{4\pi R^2}\int_0^{2\pi}\int_0^{\pi} V(R,\theta,\phi)R^2 \sin\theta d\theta d\phi$$



Let V_0 be the potential at the center of the sphere. Then

$$V(R,\theta,\phi) = V_0 + \int_0^R \vec{\nabla}V \cdot \hat{r}dr$$

The integral in the above step is independent of the path since $\vec{\nabla}V = -\vec{E}$ is a curlless field. So we do the integral along the radial direction from 0 to R, θ, ϕ.
So average potential is

$$\begin{aligned}
V_{avg} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[V_0 + \int_0^R \vec{\nabla}V \cdot \hat{r}dr\right]\sin\theta d\theta d\phi \\
&= \frac{1}{4\pi}V_0\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi + \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^R \vec{\nabla}V \cdot \hat{r}dr \sin\theta d\theta d\phi \\
&= V_0 + \frac{1}{4\pi}\int_0^R\left[\int_0^{2\pi}\int_0^{\pi}\vec{\nabla}V \cdot \hat{r}\sin\theta d\theta d\phi\right]dr
\end{aligned} \tag{3}$$

The integral over θ and ϕ is at a constant r. We can write this as a surface integral over a sphere of radius r as follows:

$$\int_0^{2\pi}\int_0^{\pi}\vec{\nabla}V \cdot \hat{r}\sin\theta d\theta d\phi = \frac{1}{r^2}\int_0^{2\pi}\int_0^{\pi}(\vec{\nabla}V \cdot \hat{r})r^2 \sin\theta d\theta d\phi = \frac{1}{r^2}\oint_S -\vec{E}\cdot\hat{r}da$$

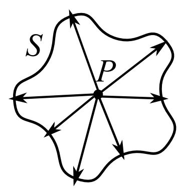
This surface integral is equal to the total charge enclosed inside the sphere of radius r. In a chargeless region this is 0. So

$$V_{avg} = V_0$$

6. Prove that in a chargeless region electrostatic potential cannot have a maxima or a minima.

soln:
In a chargeless region the potential at any point is equal to the average potential over any sphere around it. Since the average can't be a maxima or a minima in a region we conclude that the electrostatic potential can't have a maxima or a minima.

A proof independent of the mean value theorem is physically more interesting.
Suppose there is a maxima of the potential at a point P. Then there exist a neighbourhood of the point over which the potential is lower than that at the point. Let S be the closed surface of this neighbourhood. The potential at every point over the surface S is lower than the potential at P. The electric field lines over the surface is thus directed outward every where as shown in the figure. So $\oint_S \vec{E} \cdot \hat{n} da > 0$ since $\vec{E}$



is directed outward everywhere over the surface. By Gauss' law this implies there is a non-zero charge enclosed within the surface S. This contradicts the fact that the region is chargeless. So we can't have a maxima at the point P. Similarly we can't have a minima at P.