CS374: Practice Sheet 3
Deadline for Submission: 02 Oct. 2020

Prob 1) Prove the following facts:

a) If U is upper triangular and invertible, then U^{-1} is also upper triangular.

b) The inverse of a unit lower triangular matrix is unit lower triangular.

Prob 2) Prove that if A is invertible and has an LU-decomposition, then all principal minors of A are nonsingular.

Prob 3) Prove ir disprove: If a singular matrix has a Doolittle factorization, then that factorization is not unique.

Prob 4) Find the Doolittle, Crout and Cholesky factorizations of the matrix

$$A = \begin{bmatrix} 60 & 30 & 20 \\ 30 & 20 & 15 \\ 20 & 15 & 12 \end{bmatrix}.$$

Prob 5) Solve the following linear system thrice. First, use basic Gauss Elimination and give the factorization $A = LU$. Second, use Gauss Elimination with partial pivoting and finally use Gauss elimination with complete pivoting.

$$\begin{bmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1/3 \end{bmatrix}.$$

Prob 6) Assume that $0 < \epsilon < 2^{-22}$. If the Gaussian algorithm without pivoting is used to solve the system

$$\begin{bmatrix} \epsilon & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

on the Marc-32, what will be the solution vector (x_1, x_2)?

Prob 7) A **Tridiagonal** matrix is characterized by the condition $a_{ij} = 0$ if $\mid i-j \mid \geq 2$. Write an algorithm to solve a system $Ax = b$, where the coefficient matrix $A = [a_{ij}]_{n\times n}$ has tridiagonal structure. Further, apply the same on the following system

$$\begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}.$$

Prob 8) Prove that for any vector norm and its subordinate matrix norm, and for any $n \times n$ matrix A, there corresponds a vector $x \neq 0$ such that $\|Ax\| = \|A\| \, \|x\|$.

Prob 9) Do the subordinate matrix norms satisfy $\|AB\| = \|BA\|$? Explain.

Prob 10) Using the infinity matrix norm, compute the condition number of the matrix

$$A = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}.$$

Prob 11) For any real number $p \geq 1$, the formula

$$\|x\|_p = \left(\sum_{i=1}^{n} | x_i |^p \right)^{1/p}$$

defines a norm. Prove that for each $x \in \mathbb{R}^n$,

$$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty .$$

This explains why the notation $\|.\|_\infty$ is used.

Prob 12) Prove the followings: $(k(A) \equiv \|A\| \, \|A^{-1}\|)$

a) $k(AB) \leq k(A)k(B)$. b) $k(A) = \sup_{\|x\|=\|y\|} \frac{\|Ax\|}{\|Ay\|}$. c) $k(\lambda A) = k(A)$ $(\lambda \neq 0)$.

d) $k(A) \geq \frac{\lambda_{\max}}{\lambda_{\min}}$, where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of A, respectively.

Prob 13) Prove that if a square matrix A satisfies an inequality $\|Ax\| \geq \theta \, \|x\|$ for all x, with $\theta > 0$, then A is nonsingular and $\|A^{-1}\| \leq \theta^{-1}$.

Lab Exercises

Ex 1) Write a code for getting Doolitle's, Crout's and Cholesky factorization of a nonsingular matrix A for which all leading principal minors are nonsingular. Test the result with the problem 4.

Ex 2) Write three separate codes for solving system of linear equations. First, by using Gauss Elimination without pivoting. Second, by using Gauss Elimination with Partial pivoting and finally by using Gauss Elimination with Complete pivoting. Test the result with the problem 4.

Ex 3) Write a programme to solve a system of linear equation $Ax = b$, where A has a Tridiagonal structure and the matrix A is strictly diagonally dominant, i.e.,

$$| a_{ii} | > \sum_{j=1, j \neq i}^{n} | a_{ij} | \quad (1 \leq i \leq n).$$

Apply the code to solve the problem 7.