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CS374: Practice Sheet 4

Prob 1) For any $n \times n$ matrix A, define

$$\|A\|' = \max_{1 \leq i,j \leq n} | a_{ij} | \,.$$

Prove that $\|.\|'$ defines norm on the vector space of all $n \times n$ matrices. Is this a subordinate/induced matrix norm?

Prob 2) In solving the system of equations $Ax = b$ with matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2.01 \end{bmatrix},$$

predict how slight changes in b will affect the solution x. Test your prediction in the concrete case when $b = (4, 4)$ and $\widetilde{b} = (3, 5)$.

Prob 3) If A and B are $n \times n$ matrices such that $\|I - AB\| < 1$, then A and B are invertible. Furthermore, we have

$$A^{-1} = B \sum_{k=0}^{\infty} (I - AB)^k \text{ and } B^{-1} = \sum_{k=0}^{\infty} (I - AB)^k A.$$

Prob 4) Prove that if $\|A\| < 1$, then $I - A$ is invertible and

$$\left\|(I - A)^{-1}\right\| \geq \frac{1}{1 + \|A\|}.$$

Prob 5) If A is diagonally dominant, then the **Gauss-Siedel method** converges for any starting vector $x^{(0)}$.

Prob 6) The **Richardson method** converges if either of the followings is true.

(i) The matrix A has the property (unit row diagonally dominant)

$$a_{ii} = 1 > \sum_{j=1, j \neq i}^{n} | a_{ij} | \quad (1 \leq i \leq n).$$

(ii) The matrix A has the property (unit column diagonally dominant)

$$a_{jj} = 1 > \sum_{i=1, i \neq j}^{n} | a_{ij} | \quad (1 \leq j \leq n).$$

Prob 7) Prove that $\rho(A) < 1$ if and only if $A^k x \to 0$ as $k \to \infty$ for every x.

Prob 8) Compute at least 3 iterations on the following problem, using the Richardson method, Jacobi method and Gauss-Siedel method, starting with $x = (0,0,0)^T$:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \\ \frac{1}{3} & 1 & \frac{1}{2} \\ \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \\ \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{11}{18} \\ \\ \frac{11}{18} \\ \\ \frac{11}{18} \end{bmatrix}.$$

Prob 9) Consider the system

$$\begin{bmatrix} 10 & 2 \\ 4 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 16 \end{bmatrix}.$$

Show that the **Gauss-Seidel method** converges two times faster than **Jacobi method**. Further, determine the minimum number of iterations to get the accuracy at least up to two decimal places in each method.

Lab Exercises

Ex 1) Using a test matrix of order 3, compute $B = \sum_{j=0}^{20} A^j$ and see whether $(I - A)B \approx I$. Be sure that $\|A\| < 1$ for some subordinate/induced matrix norm.

Ex 2) Solve the following linear system, and apply three steps of iterative refinement (or improvement). Print r, e, and x after each iteration.

$$\begin{bmatrix} 60 & 30 & 20 \\ \\ 30 & 20 & 15 \\ \\ 20 & 15 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \\ \\ x_3 \end{bmatrix} = \begin{bmatrix} 110 \\ \\ 65 \\ \\ 47 \end{bmatrix}.$$

Ex 3) Write a code to solve a system $Ax = b$ by using Richardson, Jacobi and Gauss-Seidel methods separately or in a single one. Test your codes on the Problems 8 and 9.

Ex 4) By taking the different values of $w \in (0,1)$ from the user, write a code to solve $Ax = b$ using SOR method and plot the solutions. Further, show graphically that as $w \to 1$ SOR$\to$ Gauss-Seidel method.