<u>Transportation Problem (TP)</u>

A transportation problem is completely defined by a table of the following type.

	D_1	D_2	$\cdots$	D_n	availability
O_1	c_{11}	c_{12}	$\cdots$	c_{1n}	a_1
O_2	c_{21}	c_{22}		c_{2n}	a_2
$\vdots$					$\vdots$
O_m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	a_m
requirement	b_1	b_2	$\cdots$	b_n	

$O_i \leftarrow$ i-th origin, there are m origins

$D_j \leftarrow$ j-th destination, there are n destinations

$a_i \leftarrow$ availability at origin i, $a_i \geq 0$

$b_j \leftarrow$ requirement a destination j, $b_j \geq 0$

$c_{ij} \leftarrow$ cost of shipping one unit of product from origin i to destination j

$c_{ij} \geq 0$.

Example: A company has three factories A, B, and C that supply products to the werehouses x, y, z, and w. The monthly capacities of different factories are 120, 160, 230 units respectively. Monthly werehouse requirements are 80, 120, 200, and 70 respectively. The unit transportation costs from factories to were houses are given below.

	x	y	z	w	
A	20	30	15	32	120
B	12	8	95	36	160
C	42	25	18	45	230
	80	120	200	70	

Question: Determine the optimum Distribution for this company to minimise total transportation cost.

A solution to the transportation problem can be written as the following solution matrix:

$$
X = \begin{array}{c|ccccc|c}
 & D_1 & D_2 & \cdots & D_j & \cdots \; D_n & \\
\hline
O_1 & x_{11} & x_{12} & \cdots & x_{1j} & x_{1n} & a_1 \\
O_2 & x_{21} & x_{22} & \cdots & x_{2j} & \cdots \; x_{2n} & a_2 \\
\vdots & \vdots & & & & & \\
O_i & x_{i1} & x_{i2} & \cdots & x_{ij} & \cdots \; x_{in} & a_i \\
\vdots & \vdots & & & & & \\
O_n & x_{m1} & x_{m2} & \cdots & x_{mj} & \cdots \; x_{mn} & a_n \\
\hline
 & b_1 & b_2 & & b_j & b_m &
\end{array}
$$

→ Each x_{ij} is constrained to be non-negative.

→ $x_{ij} \leftarrow$ The quantity to be shifted from origin O_i to the destination D_j.

- There are a_i units of product to be shifted from the origin O_i to different destination.

  Then the solution matrix must satisfy:

$$\boxed{\sum_{j=1}^{n} x_{ij} = a_i \quad \forall\, i = 1, 2, \ldots, m.} \quad \text{—— (1)}$$

- There are b_j units must be shifted to destination D_j from different origins.

  Then the solution matrix must satisfy.

$$\boxed{\sum_{i=1}^{m} x_{ij} = b_j \quad ; \forall\, j = 1, 2, \ldots, n.} \quad \text{—— (2)}$$

- The objective to minimize the transportation cost subject to satisfying (1) and (2) is,

$$\boxed{\begin{aligned} \min Z &= c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{mn}x_{mn}. \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij} \end{aligned}}$$

The problem becomes.

$$\min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = a_i \quad \forall\, i = 1, 2, \dots, m \qquad (3)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \forall\, j = 1, 2, \dots, n \qquad (4)$$

$$x_{ij} \geq 0.$$

TP in standard LP form

A general TP involving m origins and n destinations can be written as

$$\min Z = cx$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

where,

$c = (c_{11}, c_{12}, \ldots, c_{mn})$ is an mn-component row vector.

$x = (x_{11}, x_{12}, \ldots, x_{mn})$ is an mn-component column vector.

$b = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$ is an $(m+n)$-component column vector.

$$A = \underbrace{\begin{bmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & & & & & & & & & & & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & & & & & & & & & & & & \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1
\end{bmatrix}}_{m+n \times mn} = \begin{bmatrix}
1' & 0' & \cdots & 0' \\
0' & 1' & \cdots & 0' \\
\vdots & & & \\
0' & 0' & \cdots & 1' \\
I_n & I_n & \cdots & I_n
\end{bmatrix}$$

where

$0' = (0, 0, \ldots, 0)_{1 \times n}$

$1' = (1, 1, \ldots, 1)_{1 \times n}$

I_n = identity matrix.

A transportation problem is $\boxed{\text{balanced}}$ if the total supply from the origins must equal to the total requirement at the destinations. i.e.,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The sum of the shipments from a source can not exceeds its availability

$$\sum_{j=1}^{n} x_{ij} \leq a_i \quad \forall\, i = 1, 2, \ldots m$$

If the sum of the shipments to a destination cannot exceeds its demand, then.

$$\sum_{i=1}^{m} x_{ij} \leq b_j \quad j = 1, 2, \ldots n$$

Some features of TP

Th^m: The number of basic variables in a TP is at most $(m+n-1)$

Proof We show that a basic for the TP consists of at most $(m+n-1)$ variables although there are $(m+n)$ constraints given by (3) and (4).

We sum the first m constraints of (3), then we have,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \underline{\underline{\sum_{i=1}^{m} a_i}} = \sum_{j=1}^{n} b_j \quad \text{——— } (**)$$

We sum the first $(n-1)$ constraints of (4), then we have,

$$\sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \quad \text{——— } (*)$$

Subtract (*) from (**), we have,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j$$

$$\Rightarrow \sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij}\right) = b_n$$

$$\Rightarrow \sum_{i=1}^{m} x_{in} = b_n \quad \text{——— } (***)$$

*** is the n-th constraint of type 4. Thus one of the $(m+n)$ constraints is redundant and may be removed from the set of constraints.

<u>Note:</u> A non-degenerate basic feasible solution to the problem consists of $(m+n-1)$ positive variables and the rest being zero.

Thm: The TP always has a feasible solution.

Proof: Assume that $T = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

we show that,

$$\boxed{x_{ij} = \frac{a_i b_j}{T}}$$ is a feasible solution

i.e., it satisfies (3) and (4)

$a_i, b_j \geq 0$, then clearly $x_{ij} \geq 0$ $\forall i, j$.

Summing over j we have,

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{a_i b_j}{T} = \frac{a_i \sum_{j=1}^{n} b_j}{T}$$

$$= a_i$$

Summing over i we have,

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{a_i b_j}{T} = \frac{b_j \sum_{i=1}^{m} a_i}{T} = b_j$$

Hence there exists at least one feasible solution.

$\underline{\underline{Th^m}}$ The solution of a TP is never unbounded.

$\underline{\underline{Proof}}$ Each x_{ij} appears exactly in two constraints of (3) & (4)

Both times the coefficient is +1

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad \sum_{i=1}^{m} x_{ij} = b_j$$

$$0 \le x_{ij} \le \max\{a_i, b_j\}$$

a_i and b_j are finite.