Primal

$$\max Z = 3x_1 + 4x_2$$

$$\begin{aligned}
\text{s.t.}\quad x_1 + x_2 &\le 12 \quad \text{—— } (1) \cdots y_1 \\
2x_1 + 3x_2 &\le 21 \quad \text{—— } (2) \text{ —— } y_2 \\
x_1 \qquad &\le 8 \quad \text{—— } (3) \cdots y_3 \\
x_2 &\le 6 \quad \text{—— } (4) \cdots y_4 \\
x_1, x_2 &\ge 0
\end{aligned}$$

Dual

$$\min w = 12y_1 + 21y_2 + 8y_3 + 6y_4$$

$$\begin{aligned}
\text{s.t.}\quad y_1 + 2y_2 + y_3 \qquad &\ge 3 \\
y_1 + 3y_2 \qquad + y_4 &\ge 4 \\
y_1, y_2, y_3, y_4 &\ge 0.
\end{aligned}$$

Simplex: Primal

we introduce slack variables x_3, x_4, x_5, x_6 to ①, ②, ③, and ④ respectively.

C_B	B	x_B	c_j → b	3 → a_1	4 → a_2	0 → a_3	0 → a_4	0 → a_5	0 → a_6	min ratio	operation
0	a_3	x_3	12	1	1	1	0	0	0	$12/1 = 12$	
0	a_4	x_4	21	2	3	0	1	0	0	$21/3 = 7$	
0	a_5	x_5	8	1	0	0	0	1	0	$8/0$	
0	a_6	x_6	6	0	$\boxed{1}$	0	0	0	1	$6/1 = 6$	
$z_j - c_j$				-3	-4	0	0	0	0		
0	a_3	x_3	6	1	0	1	0	0	-1	$6/1 = 6$	$R_1' = R_1 - R_4'$
0	a_4	x_4	3	$\boxed{2}$	0	0	1	0	-3	$3/2 = 1.5$	$R_2' = R_2 - 3R_4'$
0	a_5	x_5	8	1	0	0	0	1	0	$8/1 = 8$	$R_3' = R_3$
4	a_2	x_2	6	0	1	0	0	0	1	$6/0$ ---	$R_4' = R_4$
$z_j - c_j$				-3	0	0	0	0	4		
0	a_3	x_3	$9/2$	0	0	1	$-1/2$	0	$1/2$	$\frac{9/2}{1/2} = 9$	$R_1' = R_1 - R_2'$
3	a_1	x_1	$3/2$	1	0	0	$1/2$	0	$-3/2$	-----	$R_2' = R_2/2$
0	a_5	x_5	$13/2$	0	0	0	$-1/2$	1	$\boxed{3/2}$	$\frac{13/2}{3/2} = \frac{13}{3}$	$R_3' = R_3 - R_2'$
4	a_2	x_2	6	0	1	0	0	0	1	$6/1 = 6$	$R_4' = R_4$
$z_j - c_j$				0	0	0	$3/2$	0	$-1/2$		
0	a_3	x_3	$7/3$	0	0	1	$-1/3$	$-1/3$	0		$R_1' = R_1 - \frac{1}{2}R_3'$
3	a_1	x_1	8	1	0	0	0	1	0		$R_2' = R_2 + \frac{3}{2}R_3'$
0	a_6	x_6	$13/3$	0	0	0	$-1/3$	$2/3$	1		$R_3' = R_3/3/2$
4	a_2	x_2	$5/3$	0	1	0	$1/3$	$-2/3$	0		$R_4' = R_4 - R_3'$
$z_j - c_j$				0	0	$\boxed{0}$	$\boxed{4/3}$	$\boxed{1/3}$	$\boxed{0}$		

$x_1 = 8$, $x_2 = 5/3$ and $Z = \frac{92}{3}$

For the dual problem, the optimal solution corresponds to $z_j - c_j$ of the columns of the vectors a_3, a_4, a_5, and a_6 ~~thd~~ that are the vectors corresponding to the slack variables are

$$\boxed{\begin{gathered} y_1 = 0,\ y_2 = \frac{4}{3},\ y_3 = \frac{1}{3} \text{ and } y_4 = 0 \\ w = \frac{92}{3} \end{gathered}}$$

Simplex-Dual

Dual. standard form.

$$\max\ w' = -12y_1 - 21y_2 - 8y_3 - 6y_4 + 0y_5 + 0y_6$$

$$\text{s.t.}\quad y_1 + 2y_2 + y_3 - y_5 = 3$$

$$y_1 + 3y_2 + y_4 - y_6 = 4$$

$$y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$$

C_B	B	X_B	b	a_1	a_2	a_3	a_4	a_5	a_6	min ratio	Operations
		C_j		-12	-21	-8	-6	0	0		
-8	a_3	y_3	3	1	2	1	0	-1	0	$3/2 = 1.5$	
-6	a_4	y_4	4	1	$\boxed{3}$	0	1	0	-1	$4/3 = 1.33 \rightarrow$	
$z_j - c_j$				-2	-13 ↑	0	0	8	6		
-8	a_3	y_3	1/3	1/3	0	1	-2/3	-1	2/3		$R_1' = R_1 - 2 \cdot R_2'$
-21	a_2	y_2	4/3	1/3	1	0	1/3	0	-1/3		$R_2' = R_2/3$
$z_j - c_j$				7/3	0	0	13/3	$\boxed{8}$	$\boxed{5/3}$		

all $z_j - c_j \geq 0$, optimal solution is $y_1 = 0, y_2 = \frac{4}{3}, y_3 = \frac{1}{3}, y_4 = 0$

$w' = -\frac{92}{3}$ so $w = \frac{92}{3}$

solution to the primal:

$x_1 = 8,\ x_2 = 5/3,\ Z = \frac{92}{3}$

Example

$$\min z = x_1 - x_2$$

s.t. $2x_1 + x_2 \geq 2 \quad \cdots \quad y_1$

$-x_1 - x_2 \geq 1 \quad \cdots \quad y_2$

$x_1, x_2 \geq 0.$

Solve the dual and show the nature of the solution to the primal.

Solution

Dual:

$$\max w = 2y_1 + y_2$$

s.t. $2y_1 - y_2 \leq 1$ — (1)

$y_1 - y_2 \leq -1$ — (2)

$y_1, y_2 \geq 0$

y_3 slack in (1)

y_4, y_5 surplus and artificial for (2')

$-y_1 + y_2 \geq 1$ — (2')

		c_j		2	1	0	0	$-M$		
B	C_B	X_B	b	a_1	a_2	a_3	a_4	a_5	min ratio	observation
0	a_3	y_3	1	2	-1	1	0	0	----	
$-M$	a_5	y_5	1	-1	$\boxed{1}$	0	-1	1	$1/1=1$	
z_j-c_j				$M-2$	$-M-1$	0	M	0		
0	a_3	y_3	2	$\boxed{1}$	0	1	-1			
1	a_2	y_2	1	-1	1	0	-1			
z_j-c_j				-3	0	0	-1			
2	a_1	y_1	2	1	0	1	-1		----	
1	a_2	y_2	3	0	1	1	-2		----	
z_j-c_j				0	0	3	-4			

Hence from dual table we see that the solution is unbounded

This implies primal has no feasible solution.