CT303 - Lecture 22: 23 November 2020

- Lecture 21 Recap
  - Discrete time system as a model for ISI:

$$z_k = \sum_m a_m x_{k-m} + v_k,$$

Uncorrelated

where $x_k = \int_{-\infty}^{\infty} q(t)q(t+kT)\,dt$, and $\mathbb{E}[v_k v_m] = N_0 x_{k-m}$.

  - Equalization filter to counter ISI: $H_{eq}(f) = \frac{1}{H_c(f)}$.
  - Let $z_k, k = -N, \ldots, N$ denote the $2N+1$ sampled output of the matched filter. Assume z (i.e., ISI) is zero elsewhere.
  - Determine $c_k, k = -N, \ldots, N$ such that $x = c * z$ is the desired ISI-less sequence, i.e., $x = \delta$.
  - $x(n) = \sum_{k=-N}^{N} c(k)z(n-k), n = -2N, \ldots, 2N.$ (4N+1)

$n=-2N$

$$\begin{array}{c} N \\ \\ \\ \\ N \end{array}
\begin{bmatrix} z(-N) & 0 & \ldots & 0 \\ z(-N+1) & z(-N) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ z(N) & z(N-1) & \ldots & z(-N) \\ 0 & z(N) & \ldots & z(-N+1) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & z(N) \end{bmatrix}_{(4N+1)\times(2N+1)}
\underbrace{\begin{bmatrix} c_{-N} \\ \vdots \\ c_N \end{bmatrix}}_{c}
= \underbrace{\begin{bmatrix} x_{-2N} \\ \vdots \\ x_{2N} \end{bmatrix}}_{x}
\begin{array}{c} \}N \\ \\ \}N \end{array}$$

$$Z_{(4N+1)\times(2N+1)}\, c_{2N+1} = x_{4N+1} = \delta$$

Over determined system of eq

$$\begin{bmatrix} z_{-N} & z_{-N+1} & \cdots & z_N & 0 & \cdots & 0 \\ 0 & z_{-N} & \cdots & & z_N & 0 \cdots & 0 \\ \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} z_{-N} & 0 & 0 & \cdots & 0 \\ z_{-N+1} & z_{-N} & & & \vdots \\ \vdots & \vdots & & & z(-N) \\ z(N) & z(N) & & & z(-N+1) \\ 0 & 0 & & & \vdots \\ \vdots & & & & z(N) \\ 0 & 0 \end{bmatrix}$$

$$\left(Z^T Z\right)\underset{1,2}{(i,j)} = \sum_k z(k)\, z(k+(i-j)) \leftarrow \text{?}$$

$$= \underline{\underline{E\left[z(i)\, z(j)\right]}} \text{ ?}$$

$Ax = b$

$x^* = \arg\min ||Ax - b||$

- Solution 1: Ignore the first N and the last N equations:

Ignoring noise $\Rightarrow$

$$\tilde{Z}_{(2N+1)\times(2N+1)} c_{2N+1} = x_{2N+1} = \delta$$

$$\boxed{c = \tilde{Z}^{-1} x}$$

- ▶ This is called the *Zero-forcing solution*.
- Solution 2: Minimize the mean square error: (MMSE)
- ▶ Let x be the desired sequence and let $\hat{z}$ denote the sequence obtained as the output of the equalization filter.
- ▶ Error sequence $e(k) = x(k) - \hat{z}(k)$. Minimize $\mathbb{E}[||e||^2] = \mathbb{E}[(x - \hat{z})^T (x - \hat{z})]$, with $\hat{z} = Cz$.

$||e||^2$

- ▶ Given only one sequence z, and C is the matrix representation for convolution, i.e., $Cz = c * z$, since $Cz = c * z = z * c = Zc$, we have $\mathbb{E}[||e||^2] \sim ||x - Zc||^2$.
- ▶ Solution: $c^* = (Z^T Z)^{-1} Z^T x$.

$Z^T Z$

$\hat{z} = Zc$

$c^* \rightarrow$ Statistics of Stochastic process

Equalizers: Nonlinear Eq filters: Decision FB Eq (DFE)

Adaptive Eq filters

- Why is modulation needed?
  - Antenna size $\sim \lambda$
  - Multiplexing (Freq.)

Forms of modulation

- In general, the carrier can be written as $c(t) = A\cos(2\pi ft + \phi)$.
  - Vary A - Amplitude modulation
  - Vary f - Frequency modulation
  - Vary ϕ - Phase modulation
  - Vary A and ϕ - Amplitude Phase modulation
- For the carrier $c(t) = A\cos(2\pi ft + \phi)$, if A_{rms} denotes its rms value, then $A = \sqrt{2}A_{rms}$, and power $P = A_{rms}^2 = A^2/2$ and therefore the energy $E = P \cdot T = \frac{A^2 T}{2}$. Thus $A = \sqrt{\frac{2E}{T}}$.
  - The transmitted waveform will typically be represented in terms of energy: $\sqrt{\frac{2E}{T}}\cos(\omega t + \phi), 0 \leq t \leq T$, or in its phasor form:

$$Re\left\{\sqrt{\frac{2E}{T}}\exp\left(j\omega t + \phi\right)\right\}, 0 \leq t \leq T$$

Bandpass modulation

- Phase Shift Keying: $s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \phi_i)$, $i = 1, \ldots, M-1$, with $\phi_i = \frac{2\pi(i-1)}{M}$.
  - Binary Phase Shift Keying (BPSK): $s_1(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t)$ and $s_2(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \pi) = -s_1(t)$.
  - Quadrature Phase Shift Keying (QPSK): $(M=4)$ $s_1(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t)$, $s_2(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \frac{\pi}{2})$, $s_3(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \pi)$, $s_4(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \frac{3\pi}{2})$.

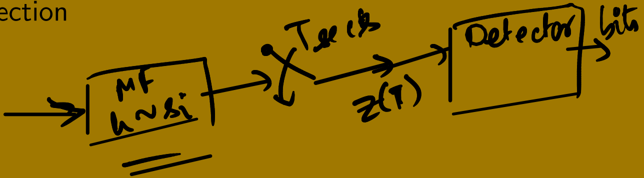
$s_2 \longleftarrow 0 \longrightarrow s_1$

↳ Antipodal signals

- Frequency Shift Keying (FSK): $s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \phi)$, $i = 1, \ldots, M$.
- Amplitude Shift Keying (ASK): $s_i(t) = \sqrt{\frac{2E_i}{T}} \cos(\omega_0 t + \phi)$, $i = 1, \ldots, M$.
- Amplitude Phase Keying (APK): $s_i(t) = \sqrt{\frac{2E_i}{T}} \cos(\omega_0 t + \phi_i)$, $i = 1, \ldots, M$.

(E_i, ϕ_i), $i = 1, \ldots, M$

Demodulation and Detection



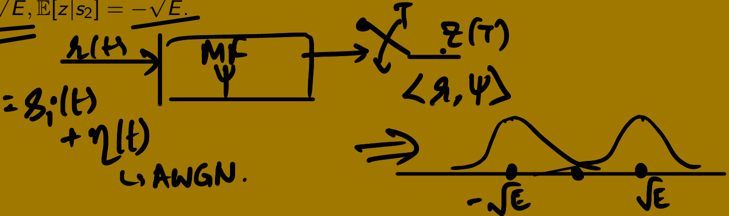
- The receiving filter to increase the SNR depends only on the signal energy, and not on the particular waveform, thus we can use analogous receiving filters to what we studied earlier in Chapter 3.

(Matched filters to Tx symbol)

Coherent Rx

- $\boxed{\text{Coherent PSK:}}$
  - Consider $\boxed{\text{BPSK}}$ - $s_1(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t)$ and $s_2(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \pi) = -s_1(t)$.
  - Let $\psi(t) = \sqrt{\frac{2}{T}} \cos(\omega_0 t)$ and using a matched filter with impulse response h matched to ψ.

    → basis for s_1, s_2.
  - $s_1 = \sqrt{E}\psi, s_2 = -\sqrt{E}\psi$.
  - $\mathbb{E}[z|s_1] = \sqrt{E}, \mathbb{E}[z|s_2] = -\sqrt{E}$.



Coherent MPSK

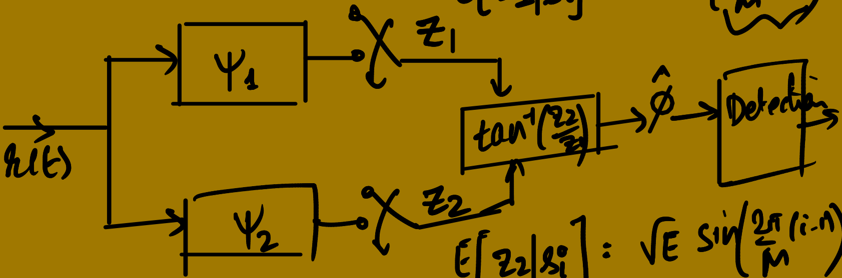
- MPSK: $s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0(t) + \frac{2\pi}{M}(i-1)), i = 1, \ldots, M.$

$$\psi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_0 t), \quad \psi_2(t) = \sqrt{\frac{2}{T}} \sin(\omega_0 t)$$

$$s_i(t) = \sqrt{\frac{2E}{T}} \left[\cos\omega_0 t \cos\left(\frac{2\pi}{M}(i-1)\right) - \sin\omega_0 t \sin\left(\frac{2\pi}{M}(i-1)\right)\right]$$

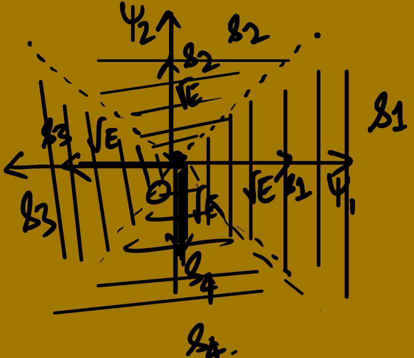
$$s_i(t) = \sqrt{E}\left[\cos\left(\frac{2\pi}{M}(i-1)\right)\psi_1(t) - \sin\left(\frac{2\pi}{M}(i-1)\right)\psi_2(t)\right]$$

$$E[z_1|s_i] = \sqrt{E}\cos\left(\frac{2\pi}{M}(i-1)\right)$$



$$E[z_2|s_i] : \sqrt{E}\sin\left(\frac{2\pi}{M}(i-1)\right)$$

M = 4



Frequency Shift Keying (Coherent)

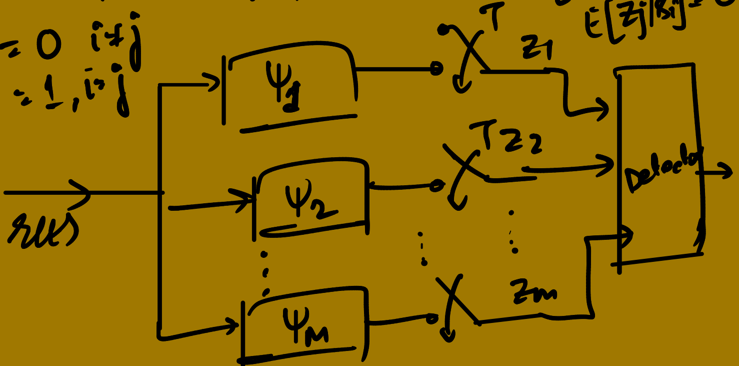
$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t), \quad i = 1, \dots, M.$$

$$\psi_i(t) = \sqrt{\frac{2}{T}} \cos(\omega_i t)$$

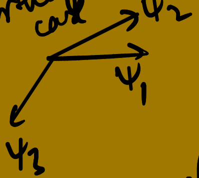
$E[z_i \mid s_i] = \sqrt{E}$

$E[z_j \mid s_i] = 0$

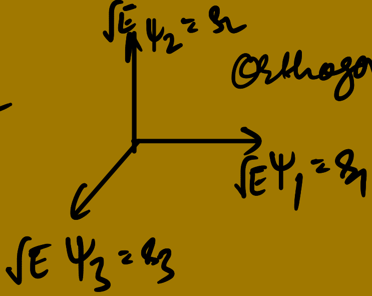
$$\langle \psi_i, \psi_j \rangle = 0 \quad i \neq j$$
$$= 1, \quad i = j$$



Non orthogonal case



Orthogonal case.



$$\int_{-\infty}^{\infty} \cos(\omega_i t)\cos(\omega_j t)\,dt = 0 \;?$$