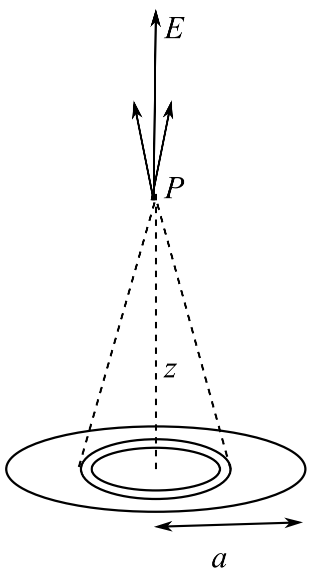
1. Find the electric field a distance z above the center of a flat circular disc of radius a which carries a uniform surface charge density σ. Work out the limits $z << a$ and $z >> a$.

   soln Consider a ring of radius r and thickness dr. The electric field due to this ring at



   the point P is vertically upward due to cylindrical symmetry. Let the magnitude be dE. Then

$$\begin{aligned} dE &= \frac{1}{4\pi\epsilon_0}\frac{\sigma 2\pi r dr}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma z r dr}{(r^2+z^2)^{\frac{3}{2}}} \end{aligned}$$

   Due to the whole disk the field is

$$F = \frac{1}{4\pi\epsilon_0}\int_0^a \frac{2\pi\sigma z r}{(r^2+z^2)^{\frac{3}{2}}}dr = \frac{1}{4\pi\epsilon_0}2\pi\sigma\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)$$

   In the limit $z >> a$,

$$E = \frac{2\pi\sigma}{4\pi\epsilon_0}\left[1-(1-(a^2/2z^2))\right] = \frac{\sigma\pi a^2}{4\pi\epsilon_0 z^2}$$

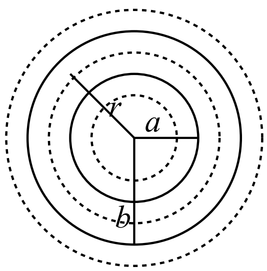
   The disk behaves like a point charge.
   When $z << a$

$$E = \frac{2\pi\sigma}{4\pi\epsilon_0}\left[1-\frac{z}{a}(1-z^2/2a^2)\right] \approx \frac{\sigma}{2\epsilon_0}$$

   In this limit the disc behaves like an infinite plane of uniform charge density σ.

2. If $\vec{\mathrm{E}} = kr^3\hat{\mathrm{r}}$ in a region find the charge density in the region.
   soln
   By Gauss' law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$.
   $\therefore \quad \rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 k r^3) = 5k\epsilon_0 r^2.$

3. A hollow spherical shell carries a charge density $\rho = k/r^2$ in the region $a \leq r \leq b$. Find the electric field in the three regions , $r < a, \quad a < r < b, \quad r > b$.
   soln
   Consider a Gausian surface inside the shell. Then



$$E \times 4\pi r^2 = \frac{q_{enc}}{\epsilon_0} = 0$$

$\therefore E = 0$.
For a gausian surface in the shell we have

$$\begin{aligned} E \times 4\pi r^2 &= \frac{1}{\epsilon_0}\int \rho dV = \frac{1}{\epsilon_0}\int_a^r \frac{k}{r^2} 4\pi r^2 dr \\ &= \frac{4\pi k}{\epsilon_0}(r-a) \end{aligned}$$

$\therefore \quad E = \frac{k(r-a)}{\epsilon_0 r^2}$.
For $r > b$ the enclosed charge is $\frac{4\pi k}{\epsilon_0}(b-a)$.

$$\therefore \quad E = \frac{k(b-a)}{\epsilon_0 r^2}$$

4. Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$. Find the electric field in the three regions, into which the planes partition the space.
   soln
   We have seen the electric field due to an infinite plane of uniform charge density σ. The electric field will be perpendicular to the plane and on either side it will be directed away from the plane. The magnitude of the field is $E = \frac{\sigma}{2\epsilon_0}$. Due to an infinite plane with surface charge density $-\sigma$ the electric field will be equal and opposite everywhere. When these two palnes are placed parallel to each other the electric field outside the region will cancel while between the plates they add up. So the electric field between the planes will be $\frac{\sigma}{\epsilon_0}$. The direction will be perpendicular to the planes and directed from the positively charged plane to the negatively charged plane. Outside the planes the field will be 0.

5. Evaluate
(a) $\int (r^2 + \vec{\mathrm{r}} \cdot \vec{\mathrm{a}} + a^2)\delta^3(\vec{\mathrm{r}} - \vec{\mathrm{a}})dV$ over the whole space where $\vec{\mathrm{a}}$ is a fixed vector.
soln

$$\int_V (r^2 + \vec{\mathrm{r}} \cdot \vec{\mathrm{a}} + a^2)\delta^3(\vec{\mathrm{r}} - \vec{\mathrm{a}})dV = 3a^2$$

(b) $\int_V |\vec{\mathrm{r}} - \vec{\mathrm{b}}|^2 \delta^3(5\vec{\mathrm{r}})dV$ over a cube of side 2, centered at the origin, and $\vec{\mathrm{b}} = 4\hat{\mathrm{y}} + 3\hat{\mathrm{z}}$
soln
$dV = r^2 dr \sin\theta d\theta d\phi$.
Let $5\vec{r} = \vec{r'}$. Then

$$\begin{aligned} dV' &= r'^2 dr' \sin\theta d\theta d\phi = 5^3 r^2 dr \sin\theta d\theta d\phi \\ &= 5^3 dV \\ \therefore \int_V |\vec{r} - \vec{b}|^2 \delta^3(5\vec{r})dV &= \int_{V'} |\frac{\vec{r'}}{5} - \vec{b}|^2 \delta^3(\vec{r'})\frac{1}{5^3}dV' \\ &= \frac{1}{5^3}|\vec{b}|^2 = \frac{1}{5} \end{aligned}$$

6. We have seen that $\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$. In a similar manner justify that

$$\vec{\nabla} \cdot \left(\frac{\hat{s}}{s}\right) = 2\pi\delta^2(\vec{s})$$

Here s is the distance from the z axis in cylindrical coordinates and $\delta^2(\vec{s})$ is a two dimensional delta function on the xy plane.

soln
We have seen that $\vec{\nabla} \cdot \left(\frac{\hat{s}}{s}\right) = 0$ for $s > 0$. It tends to ∞ as $s \to 0$. Let us calculate the integral of this function over a cylindrical volume of radius a and height h enclosing the z axis.

$$\begin{aligned} h\int_0^a \int_0^{2\pi} \vec{\nabla} \cdot \left(\frac{\hat{s}}{s}\right) s ds d\phi &= h\int_0^{2\pi} \frac{\hat{s}}{a} \cdot \hat{s} a d\phi \qquad \text{by divergence theorem} \\ \therefore \int_0^a \int_0^{2\pi} \vec{\nabla} \cdot \left(\frac{\hat{s}}{s}\right) s ds d\phi &= 2\pi \end{aligned}$$

This is true for any cylinder with radius $a > 0$ around the z axis. So we have

$$\vec{\nabla} \cdot \left(\frac{\hat{s}}{s}\right) = 2\pi\delta^2(\vec{s})$$

7. Prove that $\delta(r) = 4\pi r^2 \delta^3(\vec{r})$ and $\delta(s) = 2\pi s \delta^2(\vec{s})$.
Here $\int_0^\epsilon \delta(r)dr = 1$ for any $\epsilon > 0$. The integral is 0 otherwise. $\delta(s)$ is defined likewise.

soln
Consider a sphere of radius ϵ around the origin

$$\begin{aligned}
\int_V \delta^3(\vec{r})dV &= \int_0^\epsilon \delta^3(\vec{r})4\pi r^2 dr \\
\therefore \quad 1 &= \int_0^\epsilon \delta^3(\vec{r})4\pi r^2 dr
\end{aligned}$$

So $\delta^3(\vec{r})4\pi r^2$ behaves as a one dimensional δ function $\delta(r)$.

In 2-dimension consider a circular disc of radius ϵ.

$$\begin{aligned}
\int_S \delta^2(\vec{s})da &= \int_0^\epsilon \delta^2(\vec{s})2\pi s ds \\
\therefore \quad 1 &= \int_0^\epsilon \delta^2(\vec{s})2\pi s ds
\end{aligned}$$

So $\delta^2(\vec{s})2\pi s$ behaves as a one dimensional δ function $\delta(s)$.

8. The electric field in a region is given as

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

Find the charge densities in the region.

soln
The volume charge density is given by the differential form of Gauss' law.

$$\rho(\vec{s}) = \epsilon_0 \vec{\nabla} \cdot \vec{E} = \frac{\lambda}{2\pi}\vec{\nabla} \cdot \left(\frac{\hat{s}}{s}\right)$$

From Q.6 we have $\rho(\vec{s}) = \lambda\delta^2(\vec{s})$. This is a charge disdribution which is 0 every where except at $s = 0$, i.e along the z axis. We can get the linear charge density by integrating this volume charge density $\rho(\vec{s})$ over a thin cylinder of radius ϵ and height 1 unit.

$$\begin{aligned}
\int_0^1 \int_0^\epsilon \rho(\vec{s})2\pi s ds dz &= 1 \times \int_0^\epsilon \lambda\delta^2(\vec{s})2\pi s ds \\
&= \lambda \int_0^\epsilon \delta(s) ds \\
&= \lambda
\end{aligned}$$

So we have a line charge with linear density λ along the z axis.