1. In the spherical polar system:

   (a) Evaluate $\frac{\partial \hat{r}}{\partial \theta}, \quad \frac{\partial \hat{\theta}}{\partial \theta}, \quad \frac{\partial \hat{\phi}}{\partial \theta}, \quad \frac{\partial \hat{r}}{\partial \phi}, \quad \frac{\partial \hat{\theta}}{\partial \phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi}$
   soln
   In the spherical polar system

   $$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

   In this system $h_r = 1, \; h_\theta = r, \; h_\phi = r\sin\theta$.

   $$\begin{aligned} \hat{r} &= \frac{1}{h_r}\left(\frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}\hat{j} + \frac{\partial z}{\partial r}\hat{k}\right) dr \\ &= \left(\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}\right) \end{aligned}$$

   Similarly

   $$\begin{aligned} \hat{\theta} &= \left(\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}\right) \\ \hat{\phi} &= -\sin\phi\hat{i} + \cos\phi\hat{j} \end{aligned}$$

   So we have

   $$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}, \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}, \quad \frac{\partial \hat{\phi}}{\partial \theta} = 0$$

   $$\frac{\partial \hat{r}}{\partial \phi} = \sin\theta\hat{\phi}, \quad \frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta\hat{\phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\sin\theta\hat{r} - \cos\theta\hat{\theta}$$

   (b) Using the above partial derivatives evaluate $\vec{\nabla}\cdot\hat{r}, \quad \vec{\nabla}\cdot\hat{\theta}$ and $\vec{\nabla}\cdot\hat{\phi}$ where $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

   (c) Using the above partial derivatives evaluate $\vec{\nabla}\cdot\hat{r}, \quad \vec{\nabla}\cdot\hat{\theta}$ and $\vec{\nabla}\cdot\hat{\phi}$ where $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

   soln

   $$\begin{aligned} \vec{\nabla}\cdot\hat{r} &= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\hat{r} \\ &= \frac{\hat{\theta}}{r}\cdot\hat{\theta} + \frac{\hat{\phi}}{r\sin\theta}\cdot\sin\theta\hat{\phi} = \frac{2}{r} \\ \vec{\nabla}\cdot\hat{\theta} &= \frac{\hat{\theta}}{r}\cdot(-\hat{r}) + \frac{\hat{\phi}}{r\sin\theta}\cdot\cos\theta\hat{\phi} = \frac{1}{r\tan\theta} \\ \vec{\nabla}\cdot\hat{\phi} &= \frac{\hat{\phi}}{r\sin\theta}\cdot(-\sin\theta\hat{r} - \cos\theta\hat{\theta}) = 0 \end{aligned}$$

Now you can evaluate the expression for $\vec{\nabla}\cdot\vec{A}$ in spherical polar co-ordinates using the result of part (b) and using the product rules. Try it and see whether you get the expression for divergence.

2. Cylindrical system of co-ordinate is specified by three variables (s, ϕ, z) given by

$$x = s\cos\phi; \qquad y = s\sin\phi; \qquad z = z$$

Find the unit vectors $\hat{s}$, $\hat{\phi}$, $\hat{z}$ in this co-ordinate system. Find h_s, h_ϕ and h_z and write down the expression for $\vec{\nabla}F$ for a scalar function F in this system.

soln

$$\begin{aligned}
\vec{dl}_s &= (\cos\phi\hat{i} + \sin\phi\hat{j})ds \implies \quad h_s = 1 \\
\vec{dl}_\phi &= (-s\sin\phi\hat{i} + s\cos\phi\hat{j})d\phi \implies \quad h_\phi = s \\
\vec{dl}_z &= \hat{k}dz \quad \implies \quad h_z = 1
\end{aligned}$$

$$\therefore \quad \hat{s} = \cos\phi\hat{i} + \sin\phi\hat{j}, \hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}, \hat{z} = \hat{z}$$

$$\begin{aligned}
\vec{\nabla}F &= \frac{1}{h_s}\frac{\partial F}{\partial s}\hat{s} + \frac{1}{h_\phi}\frac{\partial F}{\partial\phi}\hat{\phi} + \frac{1}{h_z}\frac{\partial F}{\partial z}\hat{z} \\
&= \hat{s}\frac{\partial F}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial F}{\partial\phi} + \hat{z}\frac{\partial F}{\partial z}
\end{aligned}$$

3. (a) Evaluate $\vec{\nabla}\cdot\vec{r}$, $\vec{\nabla}\cdot(r^2\hat{r})$ and $\vec{\nabla}\cdot(\frac{\hat{r}}{r^2})$ for $r \neq 0$.

soln:
All the above fields have only the radial component $A_r\hat{r}$. So using the expression for divergence in spherical polar coordinates we get $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r)$.
For $\vec{A} = \vec{r} = r\hat{r}$,

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r) = 3$$

For $\vec{A} = r^2\hat{r}$,

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2) = 4r$$

For $\vec{A} = \frac{\hat{r}}{r^2}$ we have ,

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \times \frac{1}{r^2}) = 0$$

(b) Evaluate $\vec{\nabla}\cdot\vec{r}$, $\vec{\nabla}\cdot(r^2\hat{r})$ and $\vec{\nabla}\cdot(\frac{\hat{r}}{r^2})$ at $r = 0$.

soln:
For this we calculate divergence using the limiting procedure

$$\vec{\nabla}\cdot\vec{A}\Big|_{\vec{r}=0} = \lim_{r\to 0}\frac{1}{\frac{4}{3}\pi r^3}\oint_S \vec{A}\cdot\hat{r}da$$

where the surface S is a spherical surface of radius r.
For $\vec{A} = \vec{r}$

$$\vec{\nabla} \cdot \vec{r} = \lim_{r \to 0} \frac{1}{\frac{4}{3}\pi r^3} \; r \times 4\pi r^2 = 3$$

So divergence is well defined for this field at $\vec{r} = 0$.

For $\vec{A} = r^2 \hat{r}$

$$\vec{\nabla} \cdot r^2 \hat{r} = \lim_{r \to 0} \frac{1}{\frac{4}{3}\pi r^3} \; r^2 \times 4\pi r^2 = 0$$

Here too the vector field and thee divergence is well defined at $\vec{r} = 0$.

For $\vec{A} = \frac{\hat{r}}{r^2}$

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = \lim_{r \to 0} \frac{1}{\frac{4}{3}\pi r^3} \; \frac{1}{r^2} \times 4\pi r^2 \to \infty$$

So divergence is not well defined for this field at $\vec{r} = 0$. It is not surprising since the field itself is not well defined at $\vec{r} = 0$

4. (a) Evaluate $\vec{\nabla} \times \hat{\phi}$, $\quad \vec{\nabla} \times \frac{1}{r \sin\theta} \hat{\phi}$ and $\vec{\nabla} \times r \sin\theta \hat{\phi}$ for $0 < \theta < \pi$.

soln:

$$\vec{\nabla} \times \vec{\mathrm{A}} = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi} \right] \hat{\mathrm{r}} + \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi) \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta} \right] \hat{\phi}$$

Since all the fields have only the ϕ component we can work with the following shorter expression for $\vec{\nabla} \times \vec{A}$

$$\vec{\nabla} \times \vec{\mathrm{A}} = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{\mathrm{r}} - \frac{1}{r} \frac{\partial}{\partial r}(r A_\phi)\hat{\theta}$$

When $\vec{A} = \hat{\phi}$, $\quad A_\phi = 1$. Then

$$\vec{\nabla} \times \vec{A} = \frac{1}{r \sin\theta} \cos\theta \; \hat{r} - \frac{1}{r}\hat{\theta} = \frac{\cos\theta \hat{r} - \sin\theta \hat{\theta}}{r \sin\theta} = \frac{\hat{z}}{r \sin\theta}$$

When $\vec{A} = r \sin\theta \hat{\phi}$, $\quad A_\phi = r \sin\theta$. Then

$$\vec{\nabla} \times \vec{A} = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}(r \sin^2\theta)\hat{r} - \frac{1}{r} \frac{\partial}{\partial r}(r^2 \sin\theta)\hat{\theta} = 2(\cos\theta \hat{r} - \sin\theta \hat{\theta}) = 2\hat{z}$$

When $\vec{A} = \frac{1}{r \sin\theta}\hat{\phi}$, $\quad A_\phi = \frac{1}{r \sin\theta}$. Then

$$\vec{\nabla} \times \vec{A} = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}(\frac{1}{r})\hat{r} - \frac{1}{r} \frac{\partial}{\partial r}(\frac{1}{\sin\theta})\hat{\theta} = 0$$

(b) Evaluate $\vec{\nabla} \times \hat{\phi}$, $\quad \vec{\nabla} \times \frac{1}{r\sin\theta}\hat{\phi}$ and $\vec{\nabla} \times r\sin\theta\hat{\phi}$ for $\theta = 0$.

soln:
When $\theta = 0$ we are on the z axis. We consider a small circular loop of radius $a = r\sin\theta$ around the z axis. Using the limiting process we get the component of the curl along the z axis.
For $\vec{A} = \hat{\phi}$

$$(\vec{\nabla} \times \vec{A}) \cdot \hat{z} = \lim_{a\to 0} \frac{1}{\pi a^2} 2\pi a = \lim_{a\to 0} \frac{2}{a} \to \infty.$$

Hence $\vec{\nabla} \times \hat{\phi}$ is not defined on the z axis. The vector field $\hat{\phi}$ is also not defined on the z axis.

For $\vec{A} = r\sin\theta\hat{\phi}$

$$(\vec{\nabla} \times \vec{A}) \cdot \hat{z} = \lim_{a\to 0} \frac{1}{\pi a^2} 2\pi a \times a = 2$$

Since loop integrals of $\vec{A}$ will be 0 along any curve contained in a plane perpendicular to the z axis, we have $\vec{\nabla} \times (r\sin\theta\hat{\phi}) = 2\hat{z}$.

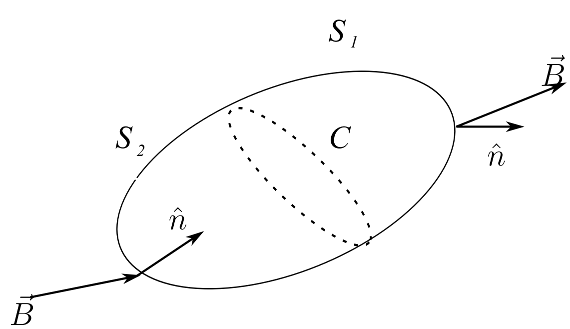
For $\vec{A} = \frac{1}{r\sin\theta}\hat{\phi}$

$$(\vec{\nabla} \times \vec{A}) \cdot \hat{z} = \lim_{a\to 0} \frac{1}{\pi a^2} 2\pi a \times \frac{1}{a} = \lim_{a\to 0} \frac{2}{a^2} \to \infty.$$

So $\vec{\nabla} \times (\frac{\hat{\phi}}{r\sin\theta})$ is not defined on the z axis. Also the vector field is not well defined on the z axis.

5. If $\vec{\nabla} \cdot \vec{\mathrm{B}} = 0$ show that there exists a vector function $\vec{\mathrm{A}}$ such that $\vec{\nabla} \times \vec{\mathrm{A}} = \vec{B}$

soln



$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\therefore \quad \oint_S \vec{B} \cdot \hat{n} da = \int_V (\vec{\nabla} \cdot \vec{B}) dV = 0$$

$$\therefore \quad \int_{S_1} \vec{B} \cdot \hat{n} da = \int_{S_2} \vec{B} \cdot \hat{n} da$$

On the surface S_1 $\hat{n}$ is outward to the volume V. On the surface S_2 $\hat{n}$ is inward to the volume V as shown in the figure.
The surface integral will be same over any surface bounded by the dotted curve (loop) C. So these integrals are related to the values of certain fields along the curve C. This can be obtained by a line integral along C. There are two possibilities for these to be scalars:

$$\oint_C \phi dl \quad \text{and} \oint_C \vec{A} \cdot \vec{dl}$$

In the first possibility ϕ is a scalar function integrated over C with length element $dl = |\vec{dl}|$. In the second possibility $\vec{A}$ is a vector function integrated over C with vector length element $\vec{dl}$.
We choose the second possibility because if we reverse the direction of the normals then the surface integrals reverses the sign. This should be accompanied by reversing the way we are traversing C thus replacing $\vec{dl}$ by $-\vec{dl}$. We can see that the first integral will have the same result whether we traverse C clockwise or anticlockwise. So that can't be equal to our surface integral.

$$\therefore \quad \int_{S_1} \vec{B} \cdot \hat{n} da = \oint_C \vec{A} \cdot \vec{dl}$$

By Stokes' theorem

$$\therefore \quad \int_{S_1} \vec{B} \cdot \hat{n} da = \int_{S_1} (\vec{\nabla} \times \vec{A}) \cdot \vec{n} da$$

Since this is true for any arbitrary surface bounded by the curve C we conclude that

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

We have proved that $\exists$ a vector field $\vec{A}$ such that $\vec{B} = \vec{\nabla} \times \vec{A}$. This $\vec{A}$ is not unique. For e.g if $\vec{A}' = \vec{A} + \vec{\nabla}\Phi$ then

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times (\vec{\nabla}\Phi) = \vec{\nabla} \times \vec{A} = \vec{B}$$