1. Given a region of space in which the electric field is everywhere directed parallel to the x axis. Prove that the electric field is independent of the y and the z co-ordinates.
soln:
$\vec{\nabla} \times \vec{E} = 0$ gives in this case $\frac{\partial E_x}{\partial y} = 0$ and $\frac{\partial E_x}{\partial z} = 0$. So the electric field is independent of y and z coordinates.

2. The plane $x = 0$ has a constant surface charge density σ_1 and the plane $x = a$ has a constant charge density σ_2. Find the electric field in the three regions $x < 0,\ \ 0 < x < a$ and $x > a$.
soln:
By symmetry along the y and the z coordinates, the potential varies only along x. In the three regions, the volume charge density is 0. Let the potential in region 1, $x < 0$, be Φ_1, region 2, $0 < x < a$, be Φ_2 and in region 3, $x > 0$, be Φ_3.
Then we have
$$\frac{\partial^2 \Phi_1}{\partial x^2} = 0, \quad \frac{\partial^2 \Phi_2}{\partial x^2} = 0, \quad \frac{\partial^2 \Phi_3}{\partial x^2} = 0$$
The solutions are of the same type,
$$\Phi_1 = c_1 x + d_1, \quad \Phi_2 = c_2 x + d_2, \quad \Phi_3 = c_3 x + d_3$$
The 6 constants have to be determined from the boundary conditions at the interfaces and infinities. The electric fields in the three regions are
$$E_{1x} = -c_1, \quad E_{2x} = -c_2, \quad E_{3x} = -c_3$$
Using the boundary condition on electric fields at he interface we get
$$\begin{aligned} E_{2x} - E_{1x} &= \frac{\sigma_1}{\epsilon_0} \\ \therefore \quad c_1 - c_2 &= \frac{\sigma_1}{\epsilon_0} \\ \text{similarly} \quad c_2 - c_3 &= \frac{\sigma_2}{\epsilon_0} \end{aligned} \tag{1}$$
This gives
$$c_2 = c_3 + \frac{\sigma_2}{\epsilon_0} \quad \text{and} \quad c_1 = c_3 + \frac{\sigma_2 + \sigma_1}{\epsilon_0}$$
Equating the potentials at $x = 0$ gives $d_1 = d_2$. At $x = a$ $\Phi_2(a) = \Phi_3(a)$ gives
$c_2 a + d_2 = c_3 a + d_3$
$\therefore \quad (c_2 - c_3)a = d_3 - d_2$
$\therefore \quad d_1 = d_3 - \frac{\sigma_2}{\epsilon_0} a.$
All the constants are obtained in terms of c_3 and d_3. They correspond to a constant background potential and electric field which can be arbitrary. So the potential in the three region is
$$\Phi_1 = \left(c_3 + \frac{\sigma_1 + \sigma_2}{\epsilon_0} \right) x + d_3 - \frac{\sigma_2}{\epsilon_0} a, \quad \Phi_2 = \left(c_3 + \frac{\sigma_2}{\epsilon_0} \right) x + d_3 - \frac{\sigma_2}{\epsilon_0} a, \quad \Phi_3 = c_3 x + d_3$$

3. A point charge q is placed a distance d in front of an infinite grounded conducting plane.

   (a) Find the force acting on the plane.
   soln:
   The electrostatic force on the infinite plane by the point charge is equal and opposite to the force exerted by the plane on the point charge. This is easily obtained by the force exerted on the given charge q by the effective image charge $-q$. Hence the force is given as

$$F = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2}$$

   (b) What is the electrostatic energy stored in this configuration? The electrostatic energy stored is equal to the work done in bringing the charge from infinity to the point where it is placed by the electrostatic force between the plane and the charge.

$$\begin{aligned} W &= \frac{1}{4\pi\epsilon_0}\int_{\infty}^{d}\frac{q^2}{4x^2}dx \\ &= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d} \end{aligned}$$

   Note that this is half of the electrostatic energy between q and $-q$ separated by $2d$.

4. A point charge q is placed a distance a from the center of a grounded conducting sphere of radius R, $\quad a > R$.

   (a) In the method of images find the quantity and the position of the image charge. Justify that this image charge makes the potential of the whole conductor 0.
   soln
   If we consider the z- axis passing through the center of the sphere and the point charge q, then the problem has azimuthal symmetry. The potential at any point on the sphere can be specified as a function $V(r,\theta)$. We assume the image charge of magnitude q' placed at a distance a' from the center of the sphere so that the two poles are at 0 potential. This means $V(R,0) = V(R,\pi) = 0$. This gives us

$$q' = -\frac{R}{a}q \qquad \text{and} \qquad a' = \frac{R^2}{a}$$

   We have to make sure that this makes the potential over the sphere equal to zero.

$$\begin{aligned} V(R,\theta) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2+R^2-2aR\cos\theta}} + \frac{q'}{\sqrt{a'^2+R^2-2a'R\cos\theta}}\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2+R^2-2aR\cos\theta}} - \frac{qR}{a\sqrt{(R^2/a)^2+R^2-2(R^2/a)R\cos\theta}}\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{a^2+R^2-2aR\cos\theta}} - \frac{q}{\sqrt{R^2+a^2-2aR\cos\theta}}\right] \\ &= 0 \end{aligned}$$

(b) Find the force of attraction between the point charge and the sphere.
soln
The force of attraction between the point charge and the sphere is equivalent to the attraction between the charge q and the image charge. This is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-a')^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}$$

(c) If the sphere was not grounded, what would be the potential of the sphere when the charge q is placed at the distance a from the center of the sphere?
soln
If the sphere is not grounded, the sphere will acquire a uniform potential over it. Let us say this potential is V_0. In the absence of the external charge a potential of V_0 on the surface of the sphere indicates a total charge $Q = 4\pi\epsilon_0 RV_0$ uniformly spread over the sphere.
The given problem is a linear superposition of two charge distributions. One, due to the grounded sphere in the presence of the external charge q in front of it. Here the sphere acquires a negative charge equal to $q' = -qR/a$ over it. The other is a sphere charged to potential V_0 but no charge outside it. Here a charge Q is spread uniformly over it. But in the combined situation, since the sphere is insulated, the total charge on the sphere is 0.

$$\therefore\ Q + q' = 0, \implies \quad Q = qR/a$$

$$\therefore\ V_0 = \frac{Q}{4\pi\epsilon_0 R} = \frac{q}{4\pi\epsilon_0 a}$$

This will be the potential of an insulated spherical conductor.

(d) Get the potential of the ungrounded sphere in part (c) using the meanvalue theorem.
soln:
When the sphere is insulated from the ground, the total charge on the sphere is 0. As the charge q is placed near it, the electrons in the conducting sphere move around and readjust to the electric field of the nearby charge q. The sphere attains a non zero potential due to the influence of the charge q. Though the total charge of the sphere is 0 the local surface charge densities are not 0. All these charges, the charge q and the induced surface charges on the sphere will be responsible to create a potential at the center of the sphere which is easy to calculate. Since the surface charges all are at the same distance R from the center, the total potential at the center due to them is 0 as the total surface charge is 0. So the only potential at the center is due to the charge q which is given as $\frac{q}{4\pi\epsilon_0 a}$. Since the region within the sphere is chargeless, the potential at the center is the average of the potential over the surface of the sphere by the mean value theorem. As the sphere is an equipotential surface, the average is same as this uniiform potential. Hence the potential of the sphere is same as the potential at the center of the sphere. Hence

$$V_0 = \frac{q}{4\pi\epsilon_0 a}$$

5. A uniform line charge with linear charge density λ and parallel to the x axis is placed in front of the an infinite grounded xz plane parallel to the line at a distance d. Find the potential at all points $y > 0$.

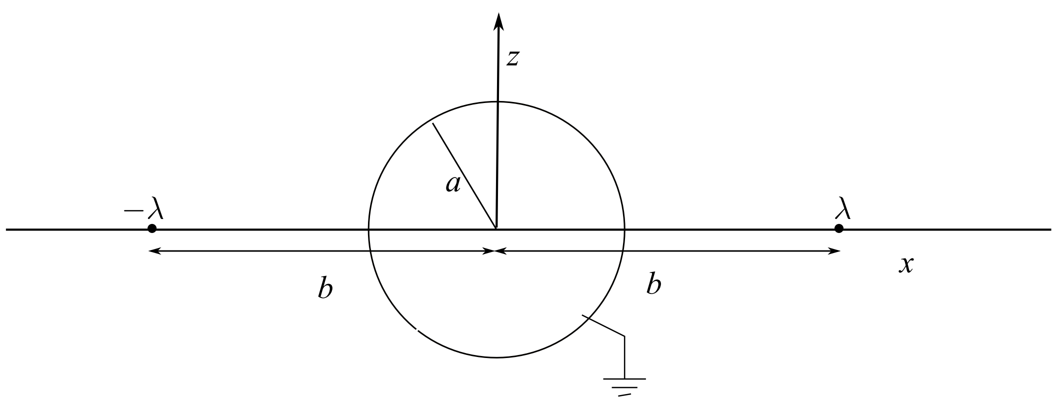
soln:

We will use the method of images. On account of symmetry the image charge distribution is a line charge of linear density $-\lambda$ parallel to the x axis and cutting the y axis at $y = -d$. Then it is easy to calculate the potential at all points as done in a previous assignment.

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\sqrt{(y-d)^2+z^2}}{\sqrt{(y+d)^2+z^2}}\right) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{(y-d)^2+z^2}{(y+d)^2+z^2}\right)$$

Note that this potential is valid only for $y > 0$. For $y < 0$, $V = 0$ everywhere.

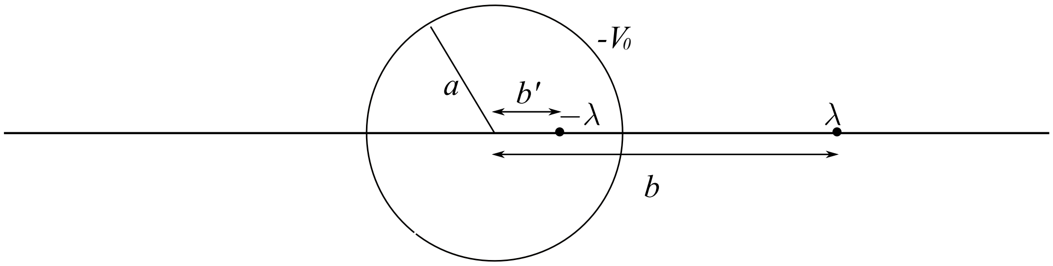
6. Two infinitely long straight wires carrying a uniform line charge density λ and $-\lambda$ are separated by distance 2b (refer Figure 1). An infinitely long cylinder with radius a is placed in between the two wires with its center being at the mid point of the line segment representing their separation. Find the electric potential everywhere outside the cylinder.

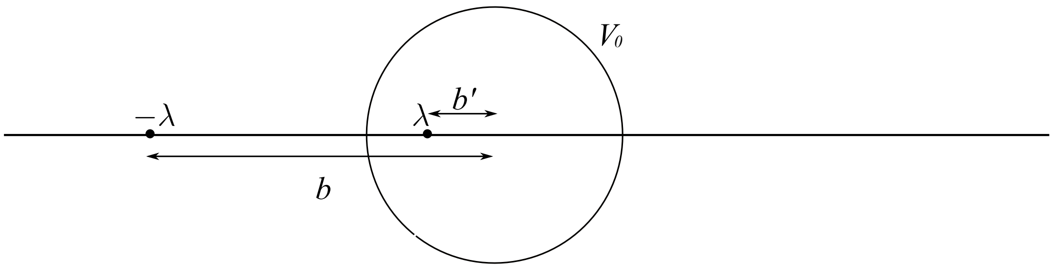


soln:

We will use the result of an earlier problem where we find the potential function for two parallel, equal and opposite infinite line charges. Here we need the cylinder in the middle of the two equal and opposite line charges at 0 potential. Now in the earlier problem we had worked out the equipotential surfaces. They were cylinders enclosing the line charges. But none of the cylinders were at 0 potential. The ones enclosing the positive line charge had positive potential while the ones enclosing the negative line charge had negative potential. The central plane midway between the two line charges will have 0 potential. The positive and the negative potential cylinders are symmetrically placed on either side of the central grounded plane.

Here we will imagine the central grounded cylinder to be a superimposition of two identical cylinders but with opposite potential, $-V_0$ and V_0 as shown in the figure. The cylinder with potential $-V_0$ is imagined to be caused by the given line charge λ





at $x = b$ and an image $-\lambda$ within the given cylinder at a distance $b' < a$ on the +ve x axis.

Similarly the given line charge $-\lambda$ at $x = -b$ and an imaginary image line charge λ at $x = -b'$ within the cylinder will cause the cylinder to have a potential V_0. Since the two configuration produces equal and opposite potential on the cylinder, the superposition of these two configuration will make the cylinder grounded.

So we will take just one of the configuration and find b' so that the given cylinder of radius a is equipotential at $-V_0$. If (x, y, z) is any point on the cylinder then

$$\begin{aligned} -V_0 &= -\frac{\lambda}{4\pi\epsilon_0} \ln \frac{(x-b)^2 + y^2}{(x-b')^2 + y^2} \\ \therefore \frac{(x-b)^2 + y^2}{(x-b')^2 + y^2} &= \exp\left(\frac{4\pi\epsilon_0 V_0}{\lambda}\right) = c \\ \therefore (x-b)^2 + y^2 &= c\left[(x-b')^2 + y^2\right] \end{aligned}$$

This gives us the equation of a circle as follows:

$$\left(x - \frac{b - b'c}{1-c}\right)^2 + y^2 - \left(\frac{b - b'c}{1-c}\right)^2 + \frac{b^2 - cb'^2}{1-c} = 0$$

The center of this circle is $(\frac{b-b'c}{1-c}, 0)$ and the radius is $\left[\left(\frac{b-b'c}{1-c}\right)^2 - \frac{b^2-cb'^2}{1-c}\right]^{\frac{1}{2}}$. Equating the center as $(0,0)$ and the radius as a, we get

$$\frac{b - b'c}{1-c} = 0 \quad \text{and} \quad \left(\frac{b-b'c}{1-c}\right)^2 - \frac{b^2 - cb'^2}{1-c} = a^2, \;\; \text{i.e} \;\; \frac{b^2 - cb'^2}{1-c} = -a^2$$

This gives $c = \frac{b^2}{a^2}$ and $b' = \frac{b}{c} = \frac{a^2}{b}$.

Now we can write down the potential at any point outside the cylinder as

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\lambda \ln\left(\frac{s_-^2 {s'_-}^2}{s_+^2 {s'_+}^2}\right)$$

where $s_-^2 = (x+b)^2+y^2$, ${s'_-}^2 = (x-b')^2+y^2$, $s_+^2 = (x+b')^2+y^2$ and ${s'_+}^2 = (x-b)^2+y^2$.