Lecture - 17

Recap

Jointly distributed random variables

$$F(a,b) = P(X \leq a, Y \leq b)$$

$$P(X = a) = \sum_y P(X=a, Y=y)$$

$$f_X(x) = \int_y f(x,y)\,dy.$$

Today's tutorial is 2-3 pm

e.g.

$$f(x,y) = \begin{cases} e^{-x} e^{-y} & 0 < x < \infty \\ & 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z = \frac{X}{Y}$

Compute $E[Z]$.

i) Compute cumulative for Z

ii) " density "

iii) " $E[Z]$

i) $P(Z \leq a)$

$= P\left(\frac{X}{Y} \leq a\right)$

$$P(Z \leq a) = P\left(\frac{X}{Y} \leq a\right)$$

③

$$= P(X \leq aY)$$

$$= \iint_{X \leq aY} f(x,y)\, dx\, dy$$

$$= \int_0^{\infty} \int_0^{aY} f(x,y)\, dx\, dy$$

$$= \int_0^{\infty} \boxed{\int_0^{ay} e^{-x}\, dx} \cdot e^{-y}\, dy$$

$$= \frac{a}{a+1}$$

ii) $$f_Z(a) = \frac{d}{da}\left(\frac{a}{a+1}\right)$$

$$= \frac{1}{(a+1)^2}$$

iii) $$\int_0^{\infty} f_Z(a) \cdot a \cdot da =$$

Independence of random variables

$$\boxed{\begin{array}{l}\text{Events} \\ A \,\&\, B \text{ are independent} \\ \text{iff} \quad P(A \cap B) = P(A)\,P(B)\end{array}}$$

X & Y are independent random variables if for any two sets $A, B \subseteq \mathbb{R}$

$$P(\overline{X \in A, Y \in B}) = P(X \in A)\,P(Y \in B)$$

In particular,

$$P(X \leq a, Y \leq b) = P(X \leq a)\,P(Y \leq b)$$

$$F_{X,Y}(a,b) = F_X(a)\,F_Y(b)$$

$$p(x,y) = p(x)\,p(y)$$

$$f(x,y) = f_X(x)\,f_Y(y)$$

e.g.:

$X \backslash Y$	0	1	
0	0.1	0.2	0.3
1	0.2	0.2	0.4
2	0.2	0.1	0.3
	0.5	0.5	

$A = \{0, 1, 2\}$

$B = \{0, 1\}$

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$$

$$1 = 1 * 1$$

$X = 1$, $Y = 1$

$A = \{1\}$, $B = \{1\}$

$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$ 7

$P(X = 1, Y = 1) = P(X = 1)\, P(Y = 1)$

$0.2 = 0.4 * 0.5$

$0.2 = 0.2$ ✓

$A = \{0\} \qquad B = \{0\}$

$P(X = 0, Y = 0) = P(X = 0)\, P(Y = 0)$

$0.1 \neq 0.3 * 0.5$

$\therefore$ X & Y are not independent

e.g:

X \ Y	1	2	3	
0	0.1	0.2	0.2	0.5
1	0.1	0.2	0.2	0.5
	0.2	0.4	0.4	

X & Y are independent

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e.g: $f(x,y) = \begin{cases} 4xy & 0<x<1 \\ & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

$$f(x,y) = f_x(x) \; f_y(y)$$

$$= \int_0^1 f(x,y)\,dy \; \int_0^1 f(x,y)\,dx$$

$$4xy = 2x \qquad 2y \checkmark$$

e.g.

$$f(x,y) = \begin{cases} x+y & 0<x<1 \\ & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x,y) = f_x(x)\, f_y(y)$$

$$x+y \neq \left(x+\tfrac{1}{2}\right)\left(y+\tfrac{1}{2}\right)$$

Not independent

e.g.

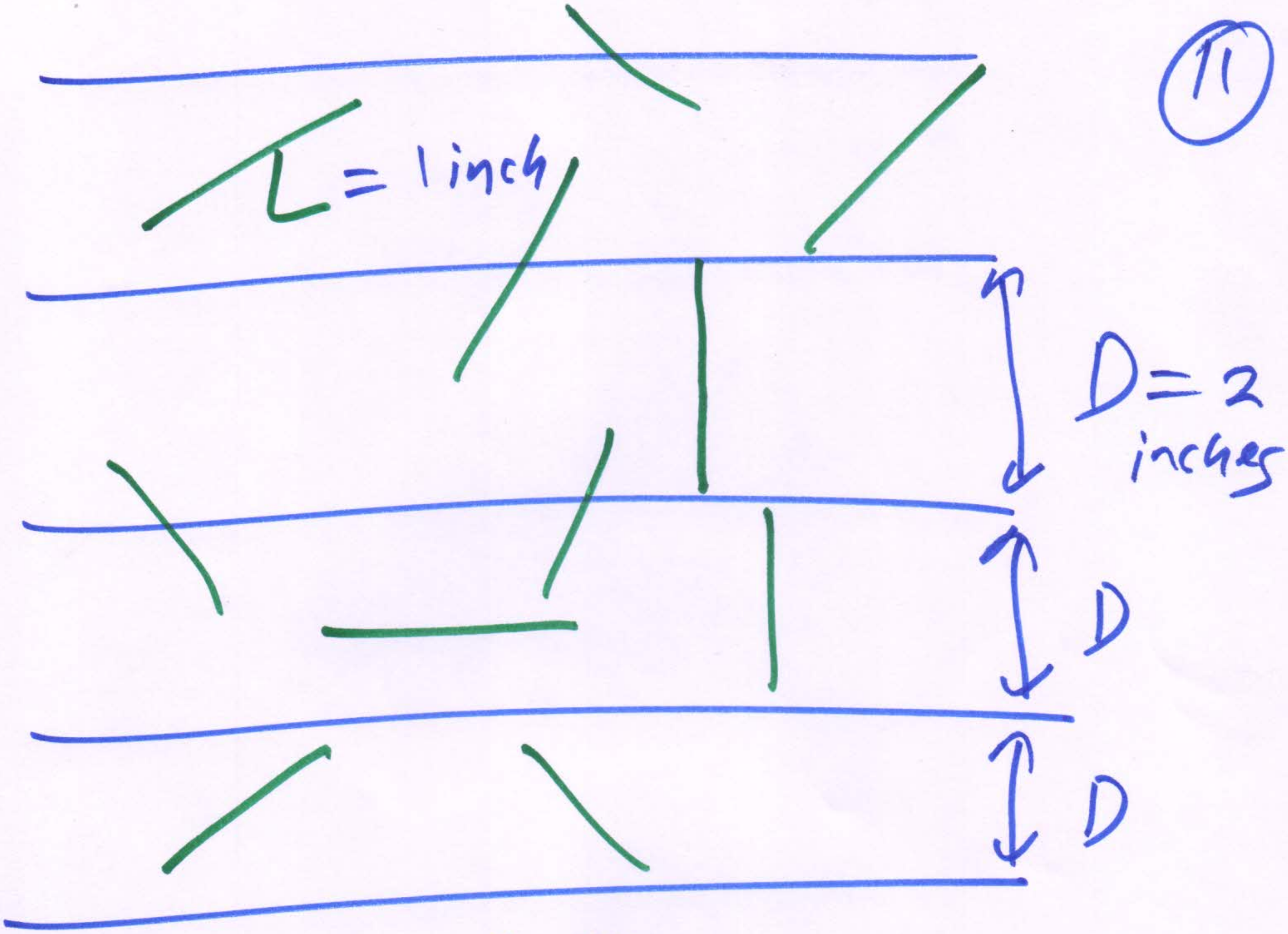
$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_x^1 f(x,y)\,dy = 2 - 2x$$

$$f_Y(y) = \int_0^y f(x,y)\,dx = 2y$$

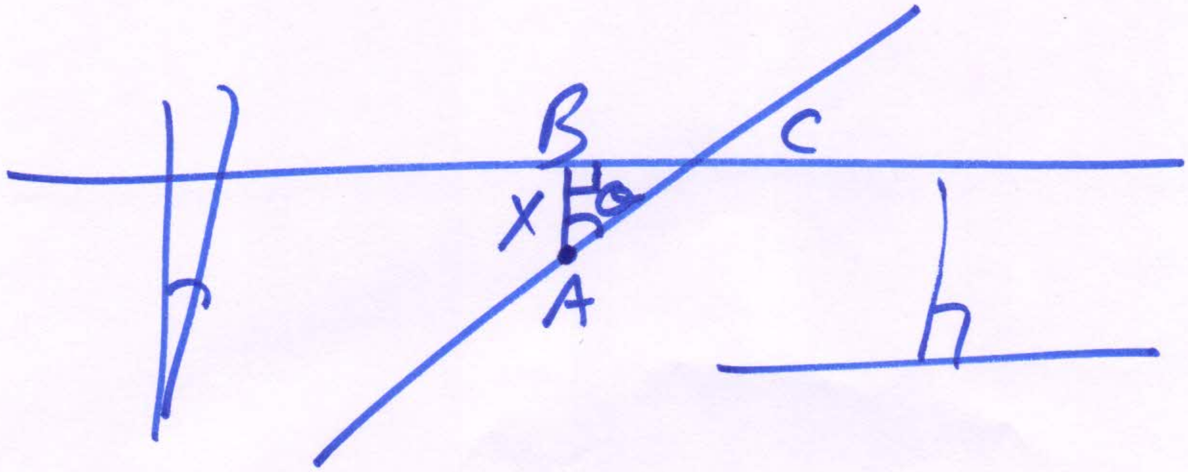
$$f(x,y) \neq f_X(x)\, f_Y(y)$$

Not independent



$L \leq D$

what is the probability that the needle intersects one of the lines?



$$AB = x \in \left(0, \frac{D}{2}\right)$$

$$f_X(x) = \begin{cases} \frac{2}{D} & 0 < x < D/2 \\ 0 & \text{otherwise} \end{cases}$$

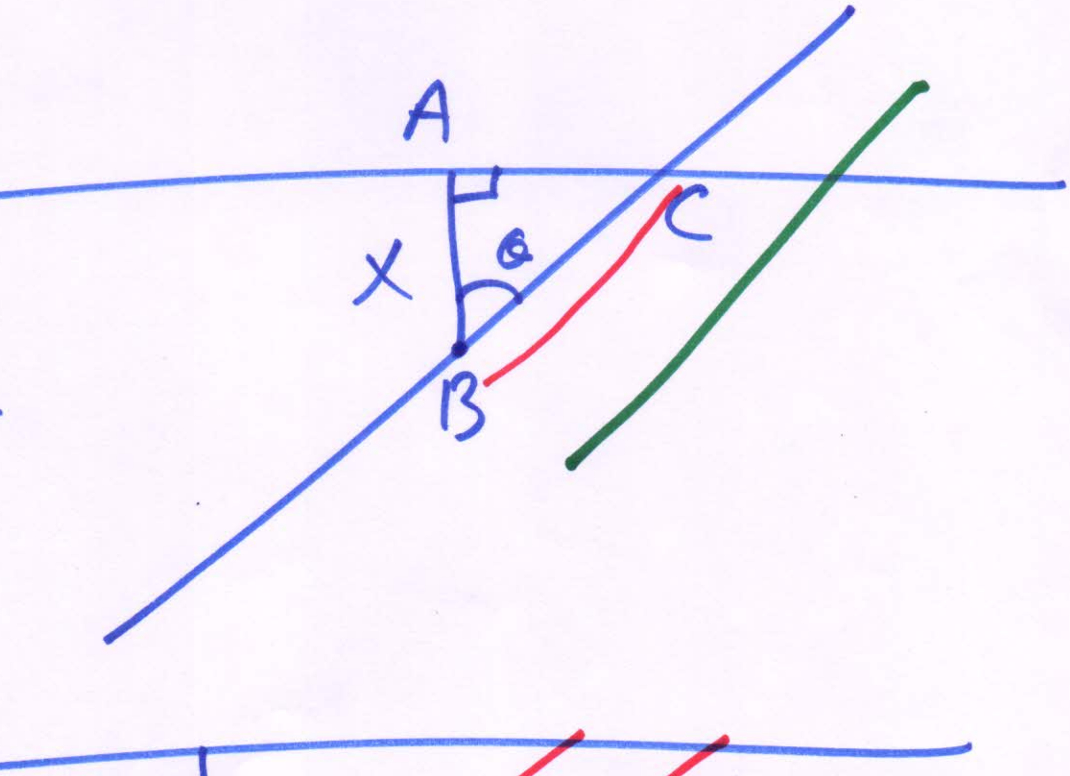
X = distance of center of needle from the nearest line

θ = acute angle between the $\perp$ & the needle

$$g_\Theta(\theta) = \begin{cases} \dfrac{2}{\pi} & 0 < \theta < \dfrac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

X & Θ are independent

when will the needle intersect a line?



$$\frac{X}{\cos\theta} < \frac{L}{2}$$