Assignment Problem

Structural observation of an AP

Suppose all $c_{ij} \geq 0$ and a feasible assignment exists. for which all corresponding c_{ij} are equal to zero

Then this assignment will make the objective function value zero and this solution is optimal.

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Theorem

If a constant be added to any row and/or any column of the cost matrix of an AP, then the resulting AP has the same optimal solution as the original Problem.

Proof: Let the cost matrix be $C = [c_{ij}]$

suppose we add α_i to row i

for $i = 1, 2, \cdots, m$

suppose we add β_j to column j

for $j = 1, 2, \cdots, m$.

$$\begin{array}{ccc|c} c_{11} & c_{12} & c_{13} & \alpha_1 \\ & & & \alpha_2 \\ & & & \alpha_3 \\ \hline \beta_1 & \beta_2 & \beta_3 & \end{array}$$

The new cost matrix will be.

$$\bar{C} = [\bar{c}_{ij}]$$

where,

$$\bar{c}_{ij} = c_{ij} + \alpha_i + \beta_j$$

Let Z and $\overline{Z}$ be the values of the objective function of the original and the new problem respectively.

then,

$$\overline{Z} = \sum_{i=1}^{m} \sum_{j=1}^{m} \overline{c_{ij}}\, x_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \underline{\underline{(c_{ij} + \alpha_i + \beta_j)}}\, x_{ij}$$

$$= \underline{\underline{\sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}\, x_{ij}}} + \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i\, x_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{m} \beta_j\, x_{ij}$$

$$= Z + \sum_{i=1}^{m} \alpha_i \underline{\underline{\left[\sum_{j=1}^{m} x_{ij}\right]}} + \sum_{j=1}^{m} \beta_j \left[\sum_{i=1}^{m} x_{ij}\right]$$

$$= Z + \sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{m} \beta_j$$

We see that Z and $\overline{Z}$ differs by a constant that is independent of x_{ij}

Hence the optimal solution of the original problem must be the optimal solution of the new problem.

$\underline{\underline{Th^m}}$ If all $c_{ij} \geq 0$ and we find a set $x_{ij} = x^*_{ij}$ such that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x^*_{ij} = 0 \text{ for a}$$

minimisation AP, then this solution is optimal.

$\underline{\underline{\text{Proof}}}$: $c_{ij} \geq 0$ and $x_{ij} \geq 0$

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij} \text{ must be}$$

non-negative.

The minimum value Z attains is zero.

x^*_{ij} is a solution that cause value of Z as zero.

Some observations

$\Rightarrow$ we can introduce sufficient number of zeros into the cost matrix by proper choice of α_i and β_j

$\Rightarrow$ Then we shall find a feasible solution/assignment for which corresponding c_{ij}'s are zero

$\Rightarrow$ This gives an optimal solution.

$\Rightarrow$ A feasible solution of an AP will be such that there will be exactly one assigned cell in each row and each column.

Finding a solution of an AP

Hungarian method (König and Egervary)

Step 1: Subtract the minimum element of each row in the cost matrix from other elements of the respective row.

This results at least one zero in each row.

If a column does not have a zero, then subtract the minimum element in each column from the other elements of the respective column.

This results at least one zero in each column.

Example

	f_1	f_2	f_3	f_4
J_1	8	26	17	11
J_2	13	28	4	26
J_3	38	19	18	15
J_4	19	26	24	10

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Step 2: Draw the minimum possible number of horizontal and vertical lines to cover all zeros in the cost matrix returned by Step 1.

Now there are two cases.

Case i)

The number of lines is equal to the order of the cost matrix

In this case an optimal assignment has been reached.

Case ii) The number of lines is less than the order of the matrix.

we need to do something more to reach the optimality.

Example

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

number of lines = 4

order of matrix = 4

So case (1) occures.

step-3

How to get the optimal assignment ??

$$
\begin{array}{c|cccc}
 & f_1 & f_2 & f_3 & f_4 \\
\hline
J_1 & \boxed{0} & 14 & 9 & 3 \\
J_2 & 9 & 20 & \boxed{0} & 22 \\
J_3 & 23 & \boxed{0} & 3 & 0 \\
J_4 & 9 & 12 & 14 & \boxed{0}
\end{array}
$$

The optimal assignment is,

$J_1 \rightarrow f_1$, $J_2 \rightarrow f_3$, $J_3 \rightarrow f_2$

$J_4 \rightarrow f_4$

The cost of the solution is,

$$8 + 4 + 19 + 10$$

$$= \underline{\underline{41}}$$

case ii)

number of lines < m ie order of the matrix.

we move to step 4

Step 4:

→ Find the smallest element say α among the uncovered elements in the matrix after drawing vertical and horizontal lines.

→ Subtract α from all the uncovered elements of the current matrix.

→ add α to the elements lying at the intersections of horizontal and vertical lines.

→ Again we proceed with step 2 with this modified matrix.

Example

160	130	175	190	200
135	120	130	160	175
140	110	155	170	185
50	50	80	80	110
55	35	70	80	105

After step 1

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

min = 15

no. of lines = 3 < 5

	f_1	f_2	f_3	f_4	f_5
J_1	15	0	20	15	$\boxed{0}$
J_2	15	15	$\boxed{0}$	10	0
J_3	15	$\boxed{0}$	20	15	5
J_4	$\boxed{0}$	15	20	0	5
J_5	5	0	10	$\boxed{0}$	0

number of lines $= 5 = m$.

we get an optimal assignment.

The optimal solution is.

$J_1 \to f_5$, $J_2 \to f_3$, $J_3 \to f_2$

$J_4 \to f_1$, $J_5 \to f_4$

$$\text{cost} = 200 + 130 + 110 + 50 + 80$$
$$= 570$$