Second Order Systems: Examples and Applications

Richardson's Theory of Conflict (Lewis Fry Richardson)

(The British Journal of Psychology: Generalised Foreign Politics, 1939)

$x(t) \rightarrow$ War potential or armaments of a nation.
$y(t) \rightarrow$ War potential of an enemy nation.

For x: $\boxed{\frac{dx}{dt} = ky + g - \alpha x}$ in which,

$\boxed{k, g, \alpha > 0}$. k indicates the war readiness of y, g indicates the grievance x feels towards y, and α is the cost of armaments incurred by x (which restrains growth of x)

For y: $\boxed{\frac{dy}{dt} = lx + h - \beta y}$ in which

$\boxed{l, h, \beta > 0}$. l indicates the war readiness of y, h indicates the grievance y feels towards x, and β is the cost of armaments incurred by y.

$k, g, l, h \rightarrow$ ~~hawk~~ hawk parameters, $\alpha\beta \rightarrow$ dove parameters
k, l (Thucydides, Grey), g, h (Leo Amery)

Case I: Mutual Disarmament without Animosity and Grievance.

$\boxed{g = h = 0} \Rightarrow \boxed{\frac{dx}{dt} = -\alpha x + Ky}$ and

$\boxed{\frac{dy}{dt} = lx - \beta y} \Rightarrow$ Equilibrium $\boxed{\frac{dx}{dt} = \frac{dy}{dt} = 0}$

$\Rightarrow \boxed{x_e = 0}$ and $\boxed{y_e = 0}$ are equilibrium solutions.

For a system $\boxed{\frac{dx}{dt} = Ax + By}$ and $\boxed{\frac{dy}{dt} = Cx + Dy}$,

we can get $\boxed{\frac{d^2x}{dt^2} - (A+D)\frac{dx}{dt} + (AD - BC)x = 0}$.

(The same applies for y). Here $\boxed{\tau = A + D}$ and $\boxed{\Delta = AD - BC}$.

Use a solution $\boxed{x = x_0 e^{\omega t}}$, to get,

$\boxed{\frac{dx}{dt} = \omega x}$ and $\boxed{\frac{d^2x}{dt^2} = \omega^2 x}$. From these

we can write $\boxed{\omega^2 - \tau\omega + \Delta = 0}$, which

implies $\boxed{\omega_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$. $\boxed{\begin{array}{l}\text{Now } A = -\alpha \\ B = K,\ C = l,\ D = -\beta\end{array}}$

Hence, $\omega_{1,2} = \frac{-(\alpha+\beta) \pm \sqrt{(\alpha+\beta)^2 - 4(\alpha\beta - Kl)}}{2}$.

If $\boxed{\alpha\beta > Kl}$, then the discriminant of the quadratic above will be less than $(\alpha+\beta)$. Hence both roots of ω will be negative, i.e. $\boxed{x = 0}$ and $\boxed{y = 0}$ will be stable equilibrium solutions. This state represents mutual disarmament.

Hence with $\boxed{x=0}$ and $\boxed{y=0}$ (mutual disarmament) and with both roots of $\boxed{\omega_{1,2} < 0}$, peace prevails for all time.

Example: Canada / U.S., Norway / Sweden.

Case II: Mutual Disarmament without Satisfaction of Grievance.

Initially $\boxed{x = y = 0}$ (mutual disarmament) but $\boxed{g, h \neq 0}$ (Grievance continues)

Hence $\boxed{\frac{dx}{dt} = g}$ and $\boxed{\frac{dy}{dt} = h}$. Since both $\boxed{g, h > 0}$, x and y will grow in time.

Case III: Unilateral disarmament.

Initially $\boxed{y = 0}$ but $\boxed{x \neq 0}$ (Unilateral disarmament)

Hence, $\boxed{\frac{dy}{dt} = lx + h}$ Since, x, l and h are all positive.

$\boxed{\frac{dy}{dt} > 0}$ $\Rightarrow$ y will grow again.

Example: German rearmament before World War II

Can be reduced by reducing grievance and building confidence. Eg. Germany and Japan after World War II.

Case IV : Arms Race

Initially set $\boxed{\alpha = \beta = 0}$ $\Rightarrow$ No restraint on armament.

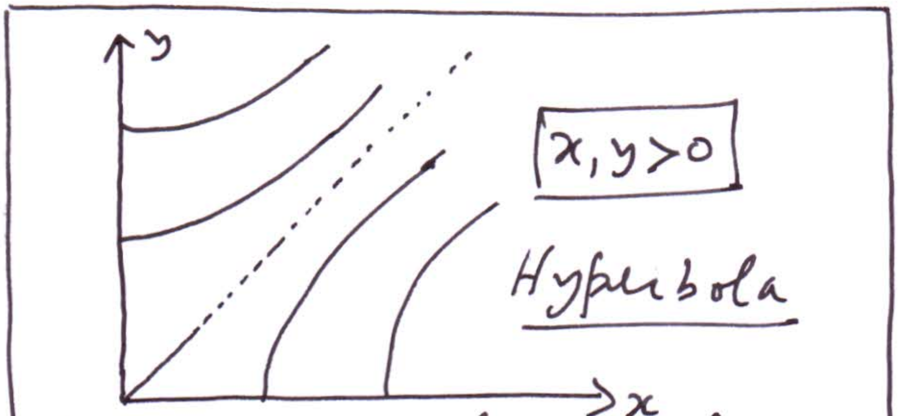
Also $\boxed{g = h = 0}$ $\Rightarrow$ No history of animosity.

$\Rightarrow$ $\boxed{\dfrac{dx}{dt} = ky}$ and $\boxed{\dfrac{dy}{dt} = lx}$. Equilibrium is obtained for $\boxed{\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0}$ $\Rightarrow$ $\boxed{x_c = y_c = 0}$.

Now $\boxed{\dfrac{dy/dt}{dx/dt} = \dfrac{dy}{dx} = \dfrac{lx}{ky}}$ $\Rightarrow \int ky\,dy = \int lx\,dx$

$$\Rightarrow \quad \frac{lx^2}{2} - \frac{ky^2}{2} = \underset{(c)}{\text{Constant}} \quad \Rightarrow \boxed{\frac{x^2}{2kc} - \frac{y^2}{2lc} = 1}$$

A hyperbola



Curves in the first quadrant

Whenever, x grows, y will also grow, and vice-versa – ~~An~~ an arms race

Example: USA/Soviet Union

Now $\boxed{\dfrac{d^2x}{dt^2} = k\dfrac{dy}{dt} = klx = \omega^2 x}$ $\boxed{\text{where } \omega = \sqrt{kl}}$

$\Rightarrow$ $\boxed{x = Ae^{\sqrt{kl}\,t} + Be^{-\sqrt{kl}\,t}}$. Since $y = \dfrac{1}{k}\dfrac{dx}{dt}$,

we get $\boxed{y = A\sqrt{\dfrac{l}{k}}\, e^{\sqrt{kl}\,t} + B\sqrt{\dfrac{l}{k}}\, e^{-\sqrt{kl}\,t}}$ after differentiating.

As $t \to \infty$, $x \to \infty$ and $y \to \infty$ (Uncontrolled growth)

The General Condition ($\alpha, \beta, k, l, h, g$ are all non-zero)

$$\boxed{\frac{dx}{dt} = -\alpha x + ky + g} \text{ and } \boxed{\frac{dy}{dt} = lx - \beta y + h}$$

Equilibrium is obtained for $\boxed{\frac{dx}{dt} = \frac{dy}{dt} = 0}$.

$\Rightarrow$ $\boxed{\begin{array}{l} -\alpha x_c + ky_c + g = 0 \\ lx_c - \beta y_c + h = 0 \end{array}}$ (and) $\Rightarrow$ $-l\alpha x_c + kly_c + lg = 0$ and $l\alpha x_c - \alpha\beta y_c + \alpha h = 0$

$\Rightarrow$ $\boxed{y_c = \frac{\alpha h + lg}{\alpha\beta - lk}}$.

Similarly we also get

$-\alpha\beta x_c + \beta k y_c + \beta g = 0$
and $kl x_c - k\beta y_c + kh = 0$ $\Rightarrow$ $\boxed{x_c = \frac{kh + \beta g}{\alpha\beta - lk}}$.

If $\boxed{\alpha\beta > lk}$, then $\boxed{x_c, y_c > 0}$. This is a permanent fixed state of war preparedness.

Example: India / Pakistan, North Korea / South Korea

Estimation of the Parameters:

1/. α, β, k, l all have the dimension of inverse time.

2/. α^{-1} and β^{-1} $\rightarrow$ Life time of policy implementation. (Example in life time of the parliament $\simeq$ 5 years).
$\Rightarrow$ $\alpha^{-1} \simeq 5$ yrs $\therefore$ $\alpha = 0.2$ unit.

3/. k and l $\longrightarrow$ Depends on the industrial capacity.

4/. g and h $\longrightarrow$ Historical grievances are not constant in time but can change suddenly.

Lanchester's Combat Models (For battlefield tactics)

(Frederick William Lanchester, 1916)

An "x-force" and a "y-force" are engaged in combat. Strength is the number of combatants.

$x \equiv x(t) \rightarrow$ Number of combatants in x.

$y \equiv y(t) \rightarrow$ Number of combatants in y.

$t \rightarrow$ Measured in days from the start.

$\frac{dx}{dt}$ = reinforcement rate − operational loss rate − combat loss rate.

Same principle applies for dy/dt.

Operational loss : Due to disease, accidents, desertions etc.

Operational loss rate $\propto$ strength (Lanchester)

We assume zero operational loss.

I. Conventional - Conventional Combat :

In modern combat, where x is a conventional force, all of x is within the kill range of the enemy y.

i.) $\frac{dx}{dt} \propto$ fraction of x exposed to $y \equiv 1$ (in modern combat)

ii.) $\frac{dx}{dt} \propto y$ (strength of the enemy)

The fraction of x exposed to y is 1 in modern conventional combat, since enemy fire is concentrated on all of x.

Hence, $\boxed{\dfrac{dx}{dt} = -Ay}$ $A \to$ Combat effectiveness of y $(A>0)$

Similarly $\boxed{\dfrac{dy}{dt} = -Bx}$ $B \to$ Combat effectiveness of x. $(B>0)$

If the reinforcement rate of x is $f(t)$ and for y it is $g(t)$, we get.

$\boxed{\dfrac{dx}{dt} = f(t) - Ay}$ and $\boxed{\dfrac{dy}{dt} = g(t) - Bx}$.

II/. Conventional - Guerilla Combat:

x is the guerilla force. Not all of it is exposed to the enemy fire of y, which is a conventional force.

$\therefore$ fraction of x exposed to $y < 1$. We write this fraction $\propto x$. $(C, D > 0)$

Hence, $\boxed{\dfrac{dx}{dt} \propto x}$ and $\boxed{\dfrac{dx}{dt} \propto y}$.

Jointly, $\boxed{\dfrac{dx}{dt} = -Cxy}$ and $\boxed{\dfrac{dy}{dt} = -Dx}$

With reinforcements. $\boxed{\dfrac{dx}{dt} = f(t) - Cxy}$ and $\boxed{\dfrac{dy}{dt} = g(t) - Dx}$ C and D $\to$ Combat effectiveness.

Isolated Combat (Special Case):

No reinforcement in isolated battle formation

In this case $\boxed{f(t) = g(t) = 0}$.

1/. Conventional - Conventional Combat:

We have $\boxed{\frac{dx}{dt} = -Ay}$ and $\boxed{\frac{dy}{dt} = -Bx}$.

$$\therefore \boxed{\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{+Bx}{+Ay}} \Rightarrow \int Ay\,dy = \int Bx\,dx$$

Integration gives $A\frac{y^2}{2} - \frac{Bx^2}{2} = \text{Constant}$.

$\Rightarrow \boxed{Ay^2 - Bx^2 = K}$. Initially $\boxed{x = x_0,\ y = y_0}$.

$\therefore \boxed{K = Ay_0^2 - Bx_0^2}$ (Lanchester's Square Law).

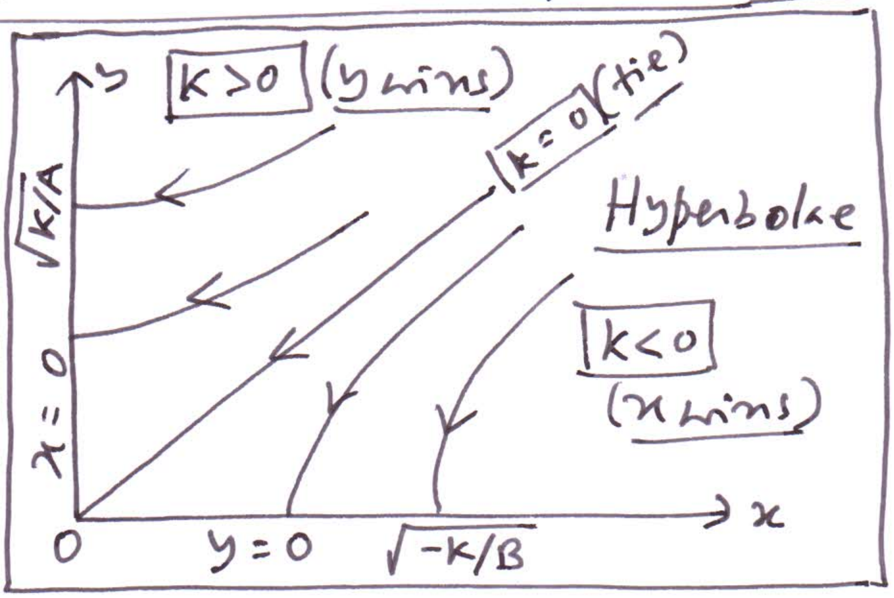
Victory criterion: One force wins the battle if the other force vanishes completely.

i) If $x = 0$, y wins.

$\boxed{y = \sqrt{K/A}}$.

ii) If $y = 0$, x wins

$\boxed{x = \sqrt{K/B}}$.



a) For y to win, $\boxed{K > 0} \Rightarrow \boxed{Ay_0^2 - Bx_0^2 > 0} \Rightarrow y$ wins.

b) For x to win, $\boxed{K < 0} \Rightarrow \boxed{Ay_0^2 - Bx_0^2 < 0}$

$\Rightarrow x$ wins.

c.) When $\boxed{K = 0}$, the battle is tied.

$\Rightarrow \boxed{A y_0^2 - B x_0^2 = 0} \Longrightarrow$ A tie.

Lanchester's Square Law: $\boxed{K = A y_0^2 - B x_0^2}$

$A, B \rightarrow$ Controlled by weaponry and equipment.

$x_0, y_0 \rightarrow$ Initial troops concentration.

The decisiveness of well-trained and concentrated troops is quadratic.

Decisiveness of weaponry is linear.

IV. Conventional - Guerilla Combat:

We have $\boxed{\frac{dx}{dt} = -Cxy}$ and $\boxed{\frac{dy}{dt} = -Dx}$.

$$\therefore \frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{\cancel{-}Dx}{\cancel{-}Cxy} \Rightarrow \boxed{\frac{dy}{dx} = \frac{D}{Cy}}$$

$$\Rightarrow \int Cy\,dy = \int D\,dx \Rightarrow \boxed{Cy^2 - 2Dx = M} \text{ Parabola}$$

when $\boxed{x = x_0,\ y = y_0}$, $\boxed{M = Cy_0^2 - 2Dx_0}$

i.) If $x = 0$, y wins.

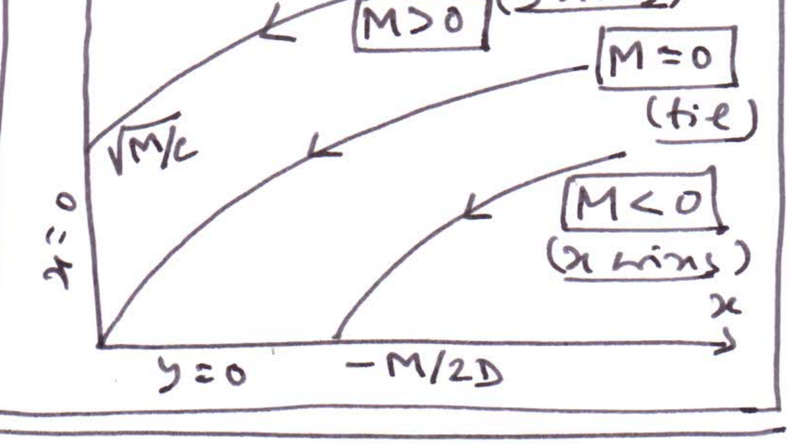
$\Rightarrow \boxed{y = \sqrt{M/c}}$

ii.) If $y = 0$, x wins

$\Rightarrow \boxed{x = -M/2D}$

iii.) When $x = y = 0$, $\boxed{M = 0}$.

Condition for a tie.



Lanchester's Linear Law:

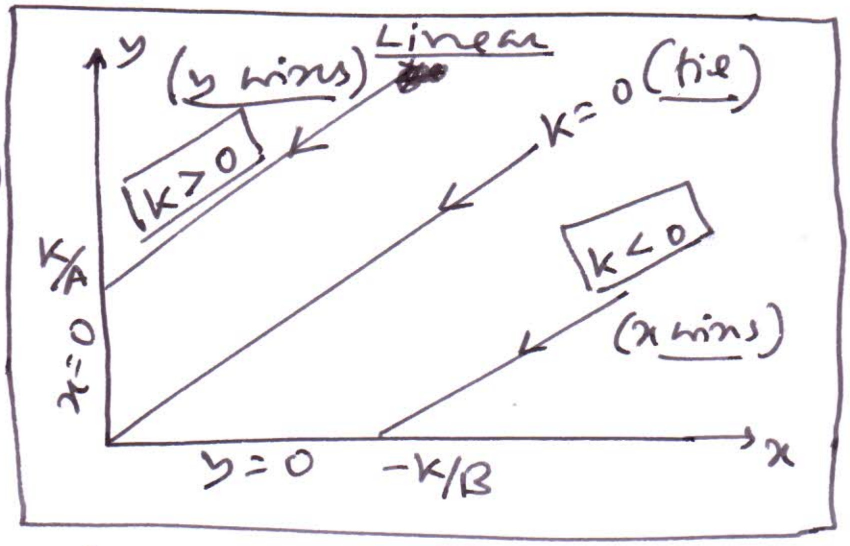
Appropriate for ancient combat

$\boxed{\frac{dx}{dt} = -Axy}$ and $\boxed{\frac{dy}{dt} = -Bxy}$

$\therefore \frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{-Bxy}{-Axy} \Rightarrow \boxed{\frac{dy}{dx} = \frac{B}{A}}$.

$\Rightarrow \boxed{Ay - Bx = k}$

(The Linear equation)



i) When $x = 0$, $\boxed{y = \frac{K}{A}}$
y wins.

ii) When $y = 0$, $\boxed{x = -K/B}$
x wins.

iii) $\boxed{K = 0}$, when $x = y = 0$. (condition for a tie)

Initially $\boxed{x = x_0, y = y_0} \Rightarrow \boxed{K = Ay_0 - Bx_0}$.

Bracken's generalisation of Lanchester's laws

$\boxed{\frac{dx}{dt} = -Ax^p y^q}$ and $\boxed{\frac{dy}{dt} = -Bx^q y^p}$.

$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{-Bx^q y^p}{-Ax^p y^q} = \frac{B}{A}\frac{x^{q-p}}{y^{q-p}}$.

Integration gives $\boxed{Ay^\alpha - Bx^\alpha = \text{constant}}$

in which $\boxed{\alpha = q - p + 1}$. In the context of modern combat a power law of $\boxed{1.5}$ is used (between 1 and 2)

Models are used for battles of Iwo Jima, Kursk and Ardennes (all from World War II).

The Principle of Competitive Exclusion in Population Biology

In nature, the struggle for existence between two similar species, competing for the same limited food supply and living space, nearly always ends in the complete extinction of one species.

— The Principle of Competitive Exclusion
Charles Darwin (1859)

The Logistic Equation: $\boxed{\frac{dx}{dt} = ax - bx^2}$

$$\Rightarrow \boxed{\frac{dx}{dt} = a\left(1 - \frac{x}{a/b}\right)x = a\left(1 - \frac{x}{K}\right)x} \quad \underline{a, b > 0}$$

We write $\boxed{\frac{dx}{dt} = rx}$ where $\boxed{r = a\left(1 - \frac{x}{K}\right)}$

$K \rightarrow$ Carrying capacity. $\boxed{\text{When } x \rightarrow K,\ r \rightarrow 0}$.

When $x \ll K$, $\boxed{r \simeq a} \Rightarrow \boxed{\frac{dx}{dt} \simeq ax}$ $ax \rightarrow$ Biotic potential.

$\boxed{1 - \frac{x}{K} = \frac{K - x}{K}} \rightarrow$ Fraction of the environment that can still be accessed.

Represents competition WITHIN the species.

Now consider two similar species with population size x and y. They intrude on each other's space. As a result <u>competition arises</u> between x and y.

We write $\boxed{\frac{dx}{dt} = a_x\left(\frac{K_x - x - \alpha y}{K_x}\right)x}$ and

$\boxed{\frac{dy}{dt} = a_y\left(\frac{K_y - y - \beta x}{K_y}\right)y}$ $\underline{\alpha, \beta} \rightarrow$ Parameters of competition. $(\alpha, \beta > 0)$

When $\boxed{\alpha = \beta = 0}$ $\Rightarrow$ <u>No competition</u> $\underline{(a_x, a_y > 0)}$

$\boxed{\begin{array}{l} K_x \rightarrow \text{Carrying capacity of species } x. \\ K_y \rightarrow \text{Carrying capacity of species } y. \end{array}}$

If $\boxed{\alpha = \beta}$ $\Rightarrow$ <u>Intense competition</u>.

Expanding the $\frac{dx}{dt}$ and $\frac{dy}{dt}$ equations,

$\boxed{\frac{dx}{dt} = a_x x - \frac{a_x}{K_x}x^2 - \frac{\alpha a_x}{K_x}xy}$ and,

$\boxed{\frac{dy}{dt} = a_y y - \frac{a_y}{K_y}y^2 - \frac{\beta a_y xy}{K_y}}$. When $\alpha = \beta = 0$,

we get <u>uncoupled logistic equations</u> for x and y.

The <u>competitive interaction</u> is only <u>active</u> through the <u>coupling</u> of x and y in the <u>cross terms</u> $\boxed{\alpha \frac{a_x}{K_x}xy}$ and $\boxed{\beta \frac{a_y}{K_y}xy}$.

$\boxed{\begin{array}{l} \text{i.) With more species, more such equations can be set down.} \\ \text{ii.) The effect of seasons and natural calamities has been ignored.} \end{array}}$

The Predator - Prey Model | Vito Volterra

(To solve the question of why the predator shark population rose ~~arose~~ disproportionately when fishing declined in the Mediterranean Sea)

$$\boxed{\begin{array}{l} x(t) \to \text{Population of prey (food fish)} \\ y(t) \to \text{Population of predators (sharks)} \end{array}}$$

In the absence of fishing : $(A, B, C, D > 0)$

i.) Food supply for the prey fish is abundant.

ii.) Their population is not very dense.

iii.) Competition among the prey fish is, therefore, not very intense, and their population grows by the Malthusian law,

$\boxed{\dfrac{dx}{dt} = Ax}$. (The growth is exponential).

iv.) In contact with the predators, however, the population of prey declines. This interaction is modelled as $\boxed{-Bxy}$. This gives

$$\boxed{\frac{dx}{dt} = Ax - Bxy}$$ for the prey fish population.

v.) For the predators, a large population is unsustainable. Hence, with growing numbers, their growth rate decreases, as per $\boxed{dy/dt = -Cy}$.

vi.) The predator population increases with their contact with the prey, an interaction that is modelled as $\boxed{Dxy}$.

Hence, $\boxed{\dfrac{dy}{dt} = -Cx + Dxy}$ for the predator population.

Equilibrium conditions are $\boxed{\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0}$.

$\Rightarrow$ $\boxed{Ax_c - Bx_c y_c = 0}$ and $\boxed{-Cy_c + Dx_c y_c = 0}$.

We have solutions $\boxed{x_c = 0}$, $\boxed{y_c = A/B}$, $\boxed{y_c = 0}$ and $\boxed{x_c = C/D}$. An optimal equilibrium solution is both $\boxed{x_c \neq 0}$ and $\boxed{y_c \neq 0}$ to maintain Nature's balance.

In the presence of fishing : $(A, B, C, D > 0)$

Both prey and predators are affected in the same way (to the same extent).

Hence, $\boxed{\dfrac{dx}{dt} = Ax - Bxy - \epsilon x}$ and | $\epsilon > 0$

$\boxed{\dfrac{dy}{dt} = -Cy + Dxy - \epsilon y}$. Fishing decreases both populations.

Equilibrium solutions for $\boxed{\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0}$ are

$\boxed{\dfrac{dx}{dt} = (A - \epsilon)x_c - Bx_c y_c = 0}$

and $\boxed{\dfrac{dy}{dt} = -(C+\epsilon)y_c + Dx_c y_c = 0}$. Two trivial Solutions are $\boxed{x_c = 0}$

(Trivial implies uninteresting) and $\boxed{y_c = 0}$. ...

But two other solutions are $\boxed{x_c = \frac{C+\epsilon}{D}}$ and $\boxed{y_c = \frac{A-\epsilon}{B}}$. This implies that fishing boosts the population of the prey fish.

A more general model is ($E, F > 0$)

$$\boxed{\frac{dx}{dt} = Ax - Bxy - Ex^2 = x(A - By - Ex)}$$

and $$\boxed{\frac{dy}{dt} = -Cy + Dxy - Fy^2 = y(-C + Dx - Fy)}.$$

Equilibrium solutions are $\boxed{x_c = y_c = 0}$. Also $\boxed{x_c \neq 0}$ and $\boxed{y_c \neq 0}$. The two latter solutions are optimal, because non-zero populations of both prey and predator maintain Nature's balance.

Comparison with the Equilibrium Solutions in the Principle of Competitive Exclusion.

$$\boxed{\frac{dx}{dt} = x\left(a_x - \frac{a_x}{k_x}x - \frac{\alpha a_x y}{k_x}\right)}$$ When $\frac{dx}{dt} = 0$ $\boxed{x_c = 0}$.

and $$\boxed{\frac{dy}{dt} = y\left(a_y - \frac{a_y}{k_y}y - \frac{\beta a_y x}{k_y}\right)}$$ When $\frac{dy}{dt} = 0$ $\boxed{y_c = 0}$.

Two other solutions are $\boxed{x_c \neq 0}$ and $\boxed{y_c \neq 0}$.

The ~~opti~~ optimal combinations are either $\boxed{x_c = 0}$ and $\boxed{y_c \neq 0}$ or $\boxed{x_c \neq 0}$ and $\boxed{y_c = 0}$.

A Love Affair (of Romeo and Juliet)

$x(t) \rightarrow$ Romeo's feelings for Juliet.

$\boxed{x > 0 \Rightarrow \text{Love}, \quad x < 0 \Rightarrow \text{aversion}}$.

$y(t) \rightarrow$ Juliet's feelings for Romeo.

$\boxed{y < 0 \Rightarrow \text{aversion}, \quad y > 0 \Rightarrow \text{love}}$.

Conditions:

i.) The more Romeo loves Juliet, the more Juliet wants to run away.

ii.) When Romeo backs off, Juliet finds him attractive (a negative feedback)

iii.) Romeo echoes Juliet's feelings (in a positive feedback).

Hence, $\boxed{\dfrac{dx}{dt} = Ay}$ and $\boxed{\dfrac{dy}{dt} = -Bx}$ | $A, B > 0$

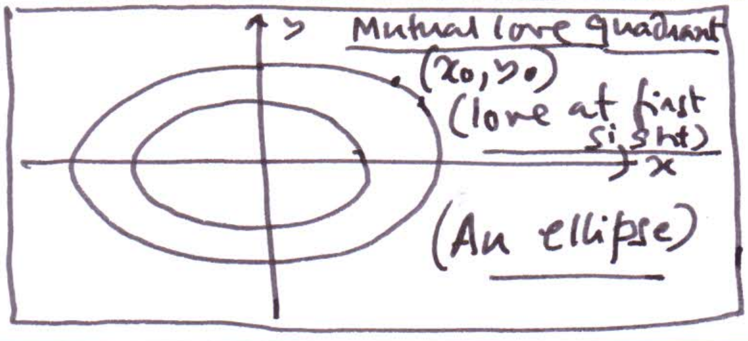
Equilibrium condition is $\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0$

$\Rightarrow$ $\boxed{x_c = 0}$ and $\boxed{y_c = 0}$ (No love)

Now $\boxed{\dfrac{dy/dt}{dx/dt} = \dfrac{dy}{dx} = \dfrac{-Bx}{Ay}}$ On integrating

$\boxed{A\dfrac{y^2}{2} + B\dfrac{x^2}{2} = \text{Constant}}$ $\Rightarrow$ $\boxed{\dfrac{x^2}{2AC} + \dfrac{y^2}{2BC} = 1}$

(An ellipse)



i.) They find mutual love one-fourth of the time when $\boxed{x, y > 0}$.

ii.) If $\boxed{x, y > 0}$ at $t = 0$ ($x = x_0$, $y = y_0$) $\Rightarrow$ Love at first sight

An Oscillator System: $\boxed{\dfrac{dx}{dt} = Ay}$ $\boxed{\dfrac{dy}{dt} = -Bx}$

$$\Rightarrow \frac{d^2x}{dt^2} = A\frac{dy}{dt} = -ABx = -\omega^2 x \quad (A, B > 0)$$

(where $\omega^2 = AB$).

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

$\because \omega^2 > 0$, this system is like the simple harmonic oscillator.

Equally Cautious Lovers: $A, B > 0$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= -Ax + By \\ \frac{dy}{dt} &= Bx - Ay \end{aligned}} \Rightarrow \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -A & B \\ B & -A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We can write this set of equations (both of the first order) as a single second-order equation)

$$\boxed{\frac{d^2x}{dt^2} - \tau\frac{dx}{dt} + \Delta x = 0} \quad \boxed{\tau = -2A}, \boxed{\Delta = A^2 - B^2}$$

(Same for y as well)

$A \rightarrow$ Cautiousness parameter, $B \rightarrow$ Responsiveness parameter.

Equilibrium condition, $\boxed{\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0}$ gives $\boxed{x_c = y_c = 0}$ (No love)

Use a trial solution $\boxed{x = x_0 e^{\omega t}}$ to get,

$\boxed{\dfrac{d^2x}{dt^2} = \omega^2 x}$ and $\boxed{\dfrac{dx}{dt} = \omega x}$, from which we get

$$(\omega^2 - \tau\omega + \Delta)x = 0 \Rightarrow \boxed{\omega^2 - \tau\omega + \Delta = 0} \text{ quadratic}$$

$$\omega_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{-2A \pm \sqrt{4A^2 - 4(A^2 - B^2)}}{2}$$

$\Rightarrow \boxed{\omega_{1,2} = -A \pm B}$. Using $\boxed{\omega_1 = -A - B < 0}$, in the trial solution, $x \rightarrow 0$ as $t \rightarrow \infty$. Using $\boxed{\omega_2 = -A + B}$, x and y may grow (love flourishes) when $B > A$. Otherwise love dies out.