Lecture -15

Recap:

Normal distribution

$N(\mu, \sigma) \rightarrow N(0,1)$

$X \rightarrow \dfrac{X - \mu}{\sigma}$

we have tables for $\Phi(a)$

Normal approx. to Binomial

$\mu = np$

$\sigma = \sqrt{np(1-p)}$

Continuity correction

$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}$

$P(i - \frac{1}{2} \leq X \leq i + \frac{1}{2})$ for normal

Exponential random variables

Usually used to model time: time until next earthquake or next war or the next accident on S.G. Highway.

let $\lambda > 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Prove that it is a probability density function.

$$\int_{-\infty}^{\infty} f(x)dx = 1 = \int_{0}^{\infty} f(x)dx$$

H.W.

$$E[X] = \frac{1}{\lambda}$$ H.W.

$$Var[X] = \frac{1}{\lambda^2}$$ H.W.

$$F(a) = P(X \leq a)$$

$$= \int_0^a f(x)\,dx = 1 - e^{-\lambda a}$$

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e.g. Suppose that the length of a phone call, in minutes, is an exponential random variable with $\lambda = \frac{1}{10}$. Suppose someone arrives at a PCO just before you. What is the probability that you need to wait for more than 10 mins?

X = no. of minutes that you need to wait (4)

$$P(X > 10) = 1 - P(X \leq 10)$$

$$= 1 - F(10)$$

$$= 1 - \left(1 - e^{-\frac{1}{10} \cdot 10}\right)$$

$$= e^{-1} = \frac{1}{e}$$

Memorylessness property 5

$P(X > s+t \mid X > t) = P(X > s)$

X : battery life

$t = 50$

$s = 50$

$$\frac{P(X > s+t \cap X > t)}{P(X > t)}$$

$$= \frac{P(X > s+t)}{P(X > t)} = \frac{1 - F(s+t)}{1 - F(t)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s)$$

Functions of continuous random variable 6

e.g. X is uniformly distributed over $(0,1)$.

$$f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{otherwise.} \end{cases}$$

$$Y = X^3$$

How is Y distributed?

$$F_a(x) = \int_0^a f(x)dx = a$$

$$P(Y \leq y) = P(X^3 \leq y)$$

$$= P(X \leq y^{1/3}) = F_X(y^{1/3})$$

$$P(Y \leq y) = y^{1/3} = F_Y(y)$$

$$F_Y(y) = y^{1/3}$$

$$f_Y(y) = \frac{d}{dy}\left(y^{1/3}\right)$$

$$= \frac{1}{3} y^{-2/3}$$

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e.g. X is c.r.v. f_X

$$Y = X^2$$

$$P(X < \sqrt{y})$$

$$\equiv$$

$$P(Y \leq y) = P(X^2 \leq y)$$

$$= P(-\sqrt{y} < X < \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) - \frac{-1}{2\sqrt{y}} f_X(-\sqrt{y})$$

$$= \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$$

e.g: $X : f_X$

$$Y = |X|$$

$$F_Y(y) = P(Y \le y)$$

$$= P(|X| \le y)$$

$$= P(-y \le X \le y)$$

$$= F_X(y) - F_X(-y)$$

$$f_Y(y) = f_X(y) - (-1)\, f_X(-y)$$

$$= f_X(y) + f_X(-y)$$

Theorem:

X, $f_X(x)$

$g(x)$ is strictly monotonic (increasing or decreasing), differentiable function of x.

Then $Y = g(X)$ has the probability density function

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \dfrac{d}{dy}(g^{-1}(y)) \right| & \text{if } y = g(x) \\ 0 & \text{if } y \neq g(x) \ \forall x \end{cases}$$

$$g(x) = x^3$$

$$f_{(Y)}(y)$$

$$= f_X\left(g^{-1}(y)\right) \left| \frac{d}{dy}\left(g^{-1}(y)\right) \right|$$

$$= f_X\left(y^{1/3}\right) \left| \frac{d}{dy}\left(y^{1/3}\right) \right|$$

$$= \frac{1}{3\,y^{2/3}} \underbrace{f_X\left(y^{1/3}\right)}_{=1}$$

$$= \frac{1}{3\,y^{2/3}}$$