# Tute - 1

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Sol$^n$ 1 6 runners & 3 types of medals. ties are possible.

① Runner / Runners completed the race fastest will get Gold medal.

② ~~Sist~~ Runner / Runners finish the race with ~~exed~~ exactly one runner ahead will get silver medal.

③ with exactly 2 runners ahead will get Bronze medal

• In howmany ways we can distribute the medals?

| # Gold | # Silver | # Bronze | # of ways ($w_i$) |
|---|---|---|---|
| 6 | 0 | 0 | $w_1 = {}^6C_6 = 1$ |
| 5 | 0 | 0 | $w_2 = {}^6C_5 = 6$ |
| 4 | 0 | 0 | $w_3 = {}^6C_4 = 15$ |
| 3 | 0 | 0 | $w_4 = {}^6C_3 = 20$ |
| 2 | 0 | 1 | $w_5 = {}^6C_2 \cdot {}^4C_1 = 60$ |
| 2 | 0 | 2 | $w_6 = {}^6C_2 \cdot {}^4C_2 = 90$ |
| 2 | 0 | 3 | $w_7 = {}^6C_2 \cdot {}^4C_3 = 60$ |
| 2 | 0 | 4 | $w_8 = {}^6C_2 \cdot {}^4C_4 = 15$ |
| 1 | 1 | 1 | $w_9 = {}^6C_1 \cdot {}^5C_1 \cdot {}^4C_1 = 120$ |
| 1 | 1 | 2 | $w_{10} = {}^6C_1 \cdot {}^5C_1 \cdot {}^4C_2 = 180$ |
| 1 | 1 | 3 | $w_{11} = {}^6C_1 \cdot {}^5C_1 \cdot {}^4C_3 = 120$ |
| 1 | 1 | 4 | $w_{12} = {}^6C_1 \cdot {}^5C_1 \cdot {}^4C_4 = 30$ |
| 1 | 2 | 0 | $w_{13} = {}^6C_1 \cdot {}^5C_2 = 60$ |
| 1 | 3 | 0 | $w_{14} = {}^6C_1 \cdot {}^5C_3 = 60$ |
| 1 | 4 | 0 | $w_{15} = {}^6C_1 \cdot {}^5C_4 = 30$ |
| 1 | 5 | 0 | $w_{16} = {}^6C_1 \cdot {}^5C_5 = 6$ |

total number of ways = $\sum_{i=1}^{16} w_i = \boxed{873}$

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- Short method.
- $n$ = number of Gold medals.

Case 1: $n=2$, there are $\binom{6}{2} = 15$ ways to choose the Gold medalist. & $\binom{4}{1}+\binom{4}{2}+\dots+\binom{4}{4} = 2^4-1 = 15$ ways to choose a bronze medalist

$\therefore$ total $15 \times 15 = \boxed{225}$ possibilities

Case-2: $n=1$. $\binom{6}{1}$ ways to choose a gold medalist $= 6$

- Sub cases

① → 1 silver medalist: $\binom{5}{1}$ ways to award silver medalist

&

$\binom{4}{1}+\binom{4}{2}+\dots+\binom{4}{4} = 15$ ways to award bronze medalist.

$\Rightarrow$ total ways $= 6 \times 5 \times 15 = \boxed{450}$

② → $\geq 1$ Silver medalist: $\binom{5}{2}+\binom{5}{3}+\dots+\binom{5}{5} = 2^5 - \binom{5}{0} - \binom{5}{1}$

$= \boxed{26}$ ways for silver medalist.

$\therefore$ total ways $= 6 \times 26 = \boxed{156}$

Case-3: $n \geq 3$ there are $\binom{6}{3}+\binom{6}{4}+\dots+\binom{6}{6} = \boxed{42}$ ways to choose gold medalist

$\therefore$ total ways $= 225 + 450 + 156 + 42 = \boxed{873}$ ways

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Sol$^n$ 2 How many positive integers less than 1,000,000 have sum of their digits equal to 19?

Let $d_1, d_2, d_3, d_4, d_5, d_6$ be the digits of such a number.

$\Rightarrow \boxed{d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 19}$ — (1)

$\Rightarrow \boxed{0 \leq d_i \leq 9}$ — (2)

- first off all let's count for equition 1 & then remove the cases where any digit exceeds 9.

- So, the problem is like to distribute 19 things (identical) among 6 people completely!

→ 19 things & 6 partitions ⇒ 6−1 walls.

⇒ number of ways = choose 6−1 places for wall among 19+6−1 slotes.

- - - - | - - - - | - - - - - - | - - - - - | - - | - - - - - -

↳ Like that!

$\therefore$ Number of ways $= \binom{19+6-1}{6-1} = \binom{24}{5} = \boxed{42504}$

- Now we have to remove the cases where ~~th~~ any digit exceeds 19.

here since total is 19, only one digit (at most) can exceed 9. So there are 6 ways to choose a digit that will exceed 9. → Let's say that is $d_1$

$\Rightarrow d_1' = d_1 - 10$

$\therefore d_1' + d_2 + d_3 + d_4 + d_5 + d_6 = 19 - 10 = 9.$

⇒ from above theurm/result number of ways in which digit will exceed 9 $= 6 \times \binom{6+9-1}{9} = \boxed{12012}$.

$\Rightarrow$ Total possible such numbers $= 42504 - 12012$

$$\boxed{= 30492}$$

The final answer!

<u>Sol$^n$ 3</u> Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with initial condition $a_0 = 2$, $a_1 = 5$, $a_2 = 15$.

the characteristic polynomial is

$$\cancel{x^3 +} \; x^3 - 6x^2 + 11x - 6 = 0$$

here $1 + 11 - 6 - 6 = 0 \Rightarrow x = 1$ is the sol$^n$

$\therefore$ devide it with $(x-1)$

$$
\begin{array}{r|l}
 & x^2 - 5x + 6 \\
\hline
x - 1 & x^3 - 6x^2 + 11x - 6 \\
 & x^3 - x^2 \\
\hline
 & -5x^2 + 11x \\
 & -5x^2 + 5x \\
\hline
 & 6x - 6 \\
 & 6x - 6 \\
\hline
 & 0 \quad 0
\end{array}
$$

$\therefore (x-1)(x^2 - 5x + 6) = 0$

$\therefore \cancel{(x-1)(x^2 - 2x - 3x - 6) = 0}$ $\therefore (x-1)(x^2 - 2x - 3x + 6) = 0$

$\therefore (x-1)(x(x-2) - 3(x-2)) = 0$

$\therefore \boxed{(x-1)(x-2)(x-3) = 0}$

roots $\Rightarrow \boxed{x = 1, 2, 3}$

$\therefore \boxed{a_n = C_1 + C_2 2^n + C_3 3^n}$ —— (1)

$a_0 = 2$, $a_1 = 5$, $a_2 = 15$

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Now using the innitial condition.

$a_n = c_1 + c_2 \cdot 2^n + c_3 \cdot 3^n$

$n=0$

$\therefore \boxed{2 = c_1 + c_2 + c_3}$ — (1)

$n=1$

$\boxed{5 = c_1 + 2c_2 + 3c_3}$ — (2)

$n=2$

$\boxed{15 = c_1 + 4c_2 + 9c_3}$ — (3)

now solving for $c_1, c_2$ & $c_3$

(2 − 1) $\therefore \boxed{c_2 + 2c_3 = 3}$ — (4)

(3 − 1) $\therefore \boxed{3c_2 + 8c_3 = 13}$ — (5)

multiply (4) with 3 • (5) − (4×3)

$3c_2 + 8c_3 = 13$

$3c_2 + 6c_3 = 9$

$2c_3 = 4$

$\boxed{\therefore c_3 = 2}$

$\therefore c_2 = 3 - 4$

$\therefore \boxed{c_2 = (-1)}$

$\therefore c_1 = 2 - (2 - 1)$ (from eq$^n$ 1)

$\therefore \boxed{c_1 = 1}$

$\therefore a_n = c_1 + c_2 \cdot 2^n + c_3 \cdot 3^n$

$\therefore \boxed{a_n = 1 - 2^n + 2 \cdot 3^n}$

cc.

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Soln-4 Find the soln for the recurrence relation.

$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

Non homogeneous

this is the ~~homogeneous~~ Linear Non homogeneous recurrence relation.

$\therefore \{a_n\} = \{a_n^{(p)} + a_n^{(h)}\}$ → homogeneous soln.

particular soln. here $F(n) = 7^n$

Let's assume $a_n^{(p)} = C \cdot 7^n$

$\therefore C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$

$\therefore$ (taking $n=2$)

$\therefore C \cdot 49 = 5 \times C \times 7 - 6 \times C \times 1 + 49$

$\therefore C \cdot 49 = 35C - 6C + 49$

$\therefore 49 = 20C$

$\therefore \boxed{C = \frac{49}{20}}$

$\therefore \boxed{a_n^{(p)} = \frac{49}{20} 7^n}$

solving for homogeneous equition. ⇒ $F(n) = 0$

$\therefore a_n = 5a_{n-1} + (-6a_{n-2})$ ↳ particular soln

$\therefore$ Characteristic equition:

$x^2 - 5x + 6 = 0$

$\therefore x^2 - 2x - 3x + 6 = 0$

$\therefore (x-2)(x-3) = 0$

$\therefore$ roots → 2 & 3.

$\therefore \boxed{a_n^{(p)} = \alpha_1 2^n + \alpha_2 3^n}$ ($\alpha_1$ & $\alpha_2$ are constants)

$\therefore a_n = a_n^{(h)} + a_n^{(p)}$

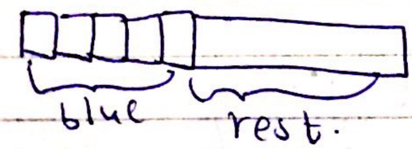
$\therefore \boxed{a_n = \alpha_1 2^n + \alpha_2 3^n + \frac{49}{20} \cdot 7^n}$

Sol^n-5 4 red balls
8 blue balls
5 green balls

} Randomly arranged in a line.

- The balls with same colours are identical.

(a) What is the probability that first 5 balls are blue?




→ This kind of possible arrangements

$$= \binom{8}{5} \times \frac{5! \times 12!}{4! \times 5! \times 8!}$$

total arrangements $= \dfrac{17!}{5! \cdot 8! \cdot 4!}$

$$\therefore \text{Probability} = \frac{\binom{8}{5} \times \dfrac{5! \times 12!}{\cancel{5!} \cdot \cancel{8!} \cdot \cancel{4!}}}{\dfrac{17!}{\cancel{5!} \times \cancel{8!} \times \cancel{4!}}}$$

$$= \frac{\binom{8}{5}}{\dfrac{17!}{12! \cdot 5!}}$$

$$\therefore P(A) \boxed{= \frac{\binom{8}{5}}{\binom{17}{5}}}$$

↳ So, this is the same as the probability of choosing 5 balls randomly and both are blue.

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- (b) What is the probability that none of the first 5 balls are blue?

→ We are not allowed to put blue ball in first 5 places ⇒ we can put 4 blue in any place from rest of the 12 places

⇒ total ways of this arrangement

$$= \binom{12}{8}\frac{8!}{8!} \times \frac{9!}{4! \times 5!}$$

→ total arrangements $= \frac{17!}{8! \cdot 5! \cdot 4!}$

$$\therefore P(B) = \frac{\binom{12}{8} \cdot \frac{9!}{\cancel{4!\,5!}}}{\frac{17!}{8!\,\cancel{5!\,4!}}}$$

$$= \frac{\binom{12}{8}}{\frac{17!}{8!\,9!}}$$

$$\boxed{= \frac{\binom{12}{8}}{\binom{17}{8}}}$$

- (c) What is the probability that final 3 balls are differently colored?

total this kind of possible arrangements

$$\boxed{= \binom{8}{1} \cdot \binom{5}{1} \binom{4}{1} \cdot \frac{3! \times 14!}{8! \times 5! \times 4!}}$$

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• total possible arrangements = $\frac{17!}{8! \cdot 5! \cdot 4!}$

$\therefore P(C) = \frac{\binom{8}{1}\binom{5}{1}\binom{4}{1} \frac{3!\,14!}{\cancel{8!\,5!\,4!}}}{\frac{17!}{\cancel{8!\,5!\,4!}}}$

$= \frac{\binom{8}{1}\binom{5}{1}\binom{4}{1}}{\frac{17!}{14!\,3!}}$

$\boxed{= \frac{\binom{8}{1}\binom{5}{1}\binom{4}{1}}{\binom{17}{3}}}$ Cc

(d) What is the probability that all the red balls are together?

• Number of this kind of arrangements

$\boxed{= \frac{14! \times 4!}{8! \times 5! \times 4!}}$

• $\therefore P(D) = \frac{\frac{14!}{\cancel{8! \times 5!}}}{\frac{17!}{\cancel{8! \times 5!} \times 4!}}$

$= \frac{14!}{\frac{17!}{4!}} = \frac{\frac{14!}{13! \cdot 1!}}{\frac{17!}{13! \cdot 4!}}$ $\boxed{= \frac{\binom{14}{1}}{\binom{17}{4}}}$ Cc

=

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Sol$^n$. 6 Balls are ~~a~~ randomly removed from an urn that initially contains 20 red & 10 Blue balls

(a) What is the probability that all of the red balls are removed before all of the blue ones have been removed.

→ here we need to find number of possible arrengements where last ball is blue.

As last ball removed must be blue.

# of Possible such arrengements $= \binom{10}{1} \cdot \frac{29!}{10! \cdot 20!}$

& # of total arrengements $= \frac{30!}{20! \cdot 10!}$

$$\therefore P(A) = \frac{\binom{10}{1} \cdot \frac{29!}{\cancel{10!\,20!}}}{\frac{30!}{\cancel{20!\cdot 10!}}}$$

$$= \binom{10}{1} \cdot \frac{\cancel{29!}}{30 \times \cancel{29!}}$$

$$= \frac{10}{30}$$

$$\boxed{= \frac{1}{3}}$$

* Now suppose that the urn initially contains 20 red, 10 blue & 8 green balls.

b] What is the probability that all of the red balls are removed before all of the blue ones have been removed?

→ Answer is the same as last one because this question can be seen as "Among 30 balls What is the probability that last removed ball will be Blue?". Because it is equally likely to be any of the 30 red or blue balls, The probability that it is blue is $\boxed{1/3}$ a.

c] What is the probability that the colors are depleted in the order blue, red, green?

$B_1$ : First removed color is blue.
$R_2$ : Second removed color is red.
$G_3$ : Third removed color is green.

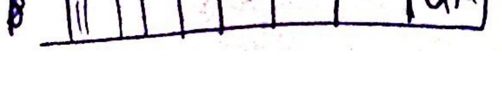
$\therefore P(B_1 R_2 G_3) = ?$

$$= P(G_3)\, P(R_2 | G_3)\, P(B_1 | R_2 G_3)$$

$P(G_3)$ → probability that the very last ball is the green. $= \binom{8}{1} / \binom{38}{1} = \boxed{\dfrac{8}{38}}$

$P(R_2 | G_3)$ → probability that among blue and red last one is red and given that very last one is green. ⇒ we have to ignore rest of the Green colors ⇒ last one can be a blue (from 10) or ~~Gree~~ Red (from 20)

$\therefore P(R_2 | G_3) = \boxed{\dfrac{20}{30}}$



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$P(B_1 | R_2 G_3) \rightarrow$ Probability that last removed (relatively) ball is blue when very last 2 balls are Green & Red ~~relative~~ by respectively.

- So we have to ignore red & green balls
  $\Rightarrow$ the ball which can come last is only blue! (~~80~~ ~~form~~

$\boxed{\therefore P(B_1 | R_2 G_3) = 1}$ $\Rightarrow$ $\boxed{P(B_1 R_2 G_3) = \frac{8}{38} \times \frac{20}{30} = \boxed{\frac{8}{57}}}$

D] What is the probability that the ~~blue~~ group of blue ball is first of the three groups to be removed?

- $B_1$ : First removed is blue.

$\rightarrow$ 2 cases possible $\rightarrow$ $B_1 G_2 R_3$ or $B_1 R_2 G_3$.
(see 6-C)

$$\therefore P(B_1) = P(B_1 G_2 R_3) + P(B_1 R_2 G_3)$$
$$= P(R_3) P(G_2 | R_3) P(B_1 | G_2 R_3) + P(G_3) P(R_2 | G_3) P(B_1 | R_2 G_3)$$
$$= \frac{8}{37} \times \frac{20}{37} \times \frac{8}{18} + \frac{8}{57}$$
$$\boxed{= \frac{64}{171}}$$

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