1. Some particles carry electrical charge. Experiments suggest that there are two kinds of charges. Suppose there were three kinds of charges in nature which we call red, blue and green. Charges of the same kind repel while charges of different kind attract. Ofcourse there are particles which neither repel nor attract other particles. You can't see the colors on the particles. You can only observe the repulsion and attraction between the particles. Treating these repulsion and attraction as relations on the set of particles how will you partition the particles and thus discover the existence of the three kinds of charges.

   soln:
   Let us define relations on the set of particles. Let R_A be the relation such that a particle is related to another if they attract each other. It is obvious that R_A is symmetric due to the third law of mechanics. We can't test whether this relation is reflexive, i.e, whether a particle attracts itself. So we will assume that the relation is reflexive. Now we investigate whether R_A is transitive. So consider three particles A, B and C. Suppose A attracts B and B attracts C. Then A and B are of different colors. Say A is red and B is blue. Now B attracts C. So B and C are of different colors. It is possible that C is red. So A and C has the same color. Hence they repel each other. So R_A is not an equivalence relation on the set of particles. Hence we can't partition the particles based on this relation.

   Now let R_R be the repulsive relation on the set of particle. Again this relation is obviously symmetric and we assume it to be reflexive. Now suppose A repels B and B repels C. Then A and B are of the same color. Also B and C has the same color. So all the particles A, B and C have the same color. Hence R_R is an equivalence relation on the set of particles. The set of particles get partitioned into particles of three colors, red, blue and green.

   There may be particles which neither attract, nor repel other particles. Each of these particles make a class of itself under R_A and R_R. So we have three classes of particles each characterized by their color, red, blue and green, that form equvalence classes under R_R and several classes, each containing a single colorless particle.

   How will this change if particles with same color were attracting while particles of different color repelled ?

2. Find the volume of the tetrahedron whose vertices are
   $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

   soln
   Let us denote the volume of the tetrahedron as V. The tetrahedron is bounded by the

four faces $x = 0, y = 0, z = 0$ and $x/a + y/b + z/c = 1$. The volume will be given by the tripple integral $V = \int\int\int dxdydz$. We first do the z integral at a fixed (x, y). z runs from $z = 0$ upto the plane $x/a+y/b+z/c = 1$ which is characterized by the value $z = c(1 - x/a - y/b)$.

$$\therefore \quad V = \int\int\int_0^{c(1-x/a-y/b)} dzdxdy = \int\int c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dxdy$$

Now we have a double integral on the xy plane. The region of the integration is a triangle on the plane bounded by the lines $x = 0, y = 0$ and $x/a + y/b = 1$.
We will first do the y integral at a fixed x. y runs from $y = 0$ upto the line $x/a+y/b = 1$ which is characterized by the value $y = b(1 - x/a)$.

$$\begin{aligned}
\therefore \quad V &= \int\int_0^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dydx \\
&= \int \left[c\left(1 - \frac{x}{a}\right) c\left(1 - \frac{x}{a}\right) - \frac{c}{2b} b^2 \left(1 - \frac{x}{a}\right)^2\right] dx
\end{aligned}$$

Now we are left with an ordinary integral over x and x runs from $x = 0$ to $x = a$. So

$$\begin{aligned}
V &= \int_0^a \frac{bc}{2}\left(1 - \frac{x}{a}\right)^2 dx \\
&= \frac{abc}{6}
\end{aligned}$$

Volume of the tetrahedron is one-sixth the volume of the rectangular parallelopiped with sides a, b, c.

3. Evaluate $(\hat{r} \cdot \vec{\nabla})r$ and $(\hat{r} \cdot \vec{\nabla})\hat{r}$

soln:

$$\begin{aligned}
(\hat{r} \cdot \vec{\nabla})r &= \left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right) r \\
&= \frac{x}{r}\frac{\partial r}{\partial x} + \frac{y}{r}\frac{\partial r}{\partial y} + \frac{z}{r}\frac{\partial r}{\partial z} \\
&= \frac{x}{r}\frac{2x}{2r} + \frac{y}{r}\frac{2y}{2r} + \frac{z}{r}\frac{2z}{2r} \\
&= \frac{x^2 + y^2 + z^2}{r^2} = 1
\end{aligned}$$

$$\begin{aligned}
(\hat{r} \cdot \vec{\nabla})\hat{r} &= \left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right)\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right] \\
&= \hat{i}\left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right)\left(\frac{x}{r}\right) + \hat{j}\left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right)\left(\frac{y}{r}\right) + \hat{k}\left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right) \\
&= \hat{i}0 + \hat{j}0 + \hat{k}0 \\
&= 0
\end{aligned}$$

4. Find the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using the double integral $\int\int dxdy$ with appropriate limits.

soln:
We will calculate the area of the ellipse only in the first quadrant. This will be 1/4 of the required area. The region is bounded by the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the x-axis and the y-axis. We will first do the y integral at a fixed x. The lower limit on the integral is 0 while the upper limit is a function of x given as $b\sqrt{1 - x^2/a^2}$. Then the limit on x goes from 0 to a.

$$\begin{aligned} A &= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} dydx \\ &= \int_0^a b\sqrt{1 - x^2/a^2}dx \\ &= \frac{\pi ab}{4} \end{aligned}$$

This is the area of a quarter of an ellipse. So the area of the ellipse is πab.

5. Find the volume of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the tripple integral $\int\int\int dxdydz$ with appropriate limits.

soln:
We will calculate the volume in the first quadrant of the coordinate system which is 1/8 the volume of the ellipsoid. We will first integrate over z at a fixed (x, y). The lower limit is 0 while the upper limit is decided by the equation of the ellipsoid and is given as a function of (x, y). This will be $c\sqrt{1 - x^2/a^2 - y^2/b^2}$. The ellipsoid cuts the xy plane along an ellipse whose equation is obtained by putting $z = 0$ in the equation of the ellipsoid. This gives the equation of the ellipse as $\frac{x^2}{a^2 + \frac{y^2}{b^2} = 1}$. Next we do the integration over y. The limits will be given as 0 and $b\sqrt{1 - x^2/a^2}$. Finally the limits on x will be from 0 to a.

$$
\begin{aligned}
V &= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dzdydx \\
&= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-x^2/a^2-y^2/b^2}dydx \\
&= c\int_0^a \int_0^{b\alpha} \sqrt{\alpha^2-y^2/b^2}dydx \quad \text{where} \quad \alpha=\sqrt{1-x^2/a^2} \\
&= c\int_0^a \alpha \int_0^{b\alpha} \sqrt{1-\left(\frac{y}{b\alpha}\right)^2}dydx \\
&= bc\int_0^a \alpha^2\frac{\pi}{4}dx \\
&= \frac{\pi bc}{4}\int_0^a (1-x^2/a^2)dx \\
&= \frac{\pi bc}{4}(a-a/3) \\
&= \frac{\pi abc}{6}
\end{aligned}
$$

The volume of the whole ellipsoid is $8\times\frac{\pi abc}{6}=\frac{4}{3}\pi abc.$

6. Let $\vec{E}=\frac{\rho\vec{r}}{3\epsilon_0}$ where ρ and ϵ_0 are constants. Evaluate $\int(\vec{\nabla}\cdot\vec{E})dxdydz$ over the volume of a sphere of radius a centered at the origin.

   soln:
   $\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0$. This is independent of position. Hence

$$
\int(\vec{\nabla}\cdot\vec{E})dxdydz=\int(\rho/e\epsilon_0)dxdydz
$$

   Sphere is a special case of an ellipsoid with $a=b=c$ whose volume we just calculated. So the given integral is $(\rho/\epsilon_0)(4/3)\pi r^3=\frac{4\pi\rho r^3}{3\epsilon_0}$.