Random Processes

- Stationary RP — WSS / SSS
- Ergodicity

- Stationary (WSS) RP:

↳ Autocorrelation function

$$R_X(\tau) = E[X_t X_{t+\tau}]$$

1. $R_X(0) = E[X_t^2] \rightarrow$ Energy $\rightarrow \infty$ (Not an energy signal)

↳ Average Power

$$\int_{-\infty}^{\infty} x^2(t) \cdot P_{X(t)} \, dx(t)$$

$$\int_{-T/2}^{T/2} E[X_t^2] \, dt = T \times E[X_t^2]$$

T (Time duration)

$E[X_t^2] \rightarrow$ does not depend on 't' ($\because$ WSS)

$R_X(0) \rightarrow$ Average Power

2. $G_X(\omega) := \mathcal{F}\{R_X\}$?

a. $R_X(0) \geq 0$

b. $R_X(\tau) = R_X(-\tau), \forall \tau$
$\Rightarrow$ even function

c. If $\{X_t\}_{t \in I}$ is real valued, then
$R_X(\tau) \in \mathbb{R}, \forall \tau$.

$$G_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$G_x(\omega)^{*} = \int_{-\infty}^{\infty} R_x(\tau)\, e^{j\omega\tau}\, d\tau$$

$$s = -\tau, \quad ds = -d\tau$$

$$\tau = -\infty, \; s = \infty$$

$$\tau = \infty, \; s = -\infty$$

$$= -\int_{\infty}^{-\infty} R_x(-s)\, e^{-j\omega s}\, ds$$

$$= \int_{-\infty}^{\infty} R_x(s)\, e^{-j\omega s}\, ds = G_x(\omega).$$

$$\Rightarrow \; G_x(\omega) \in \mathbb{R}.$$

2. $G_x(\omega) = G_x(-\omega), \; \forall\, \omega \in \mathbb{R}$

3. $G_x(\omega) \geq 0, \; \forall\, \omega \in \mathbb{R}$. [Pending].

$G_x(\omega)$ →
- even
- $\mathbb{R}$.
- non-negative
- func of ω (frequency)

} $G_x(\omega)$ is the Power Spectral Density of X_t

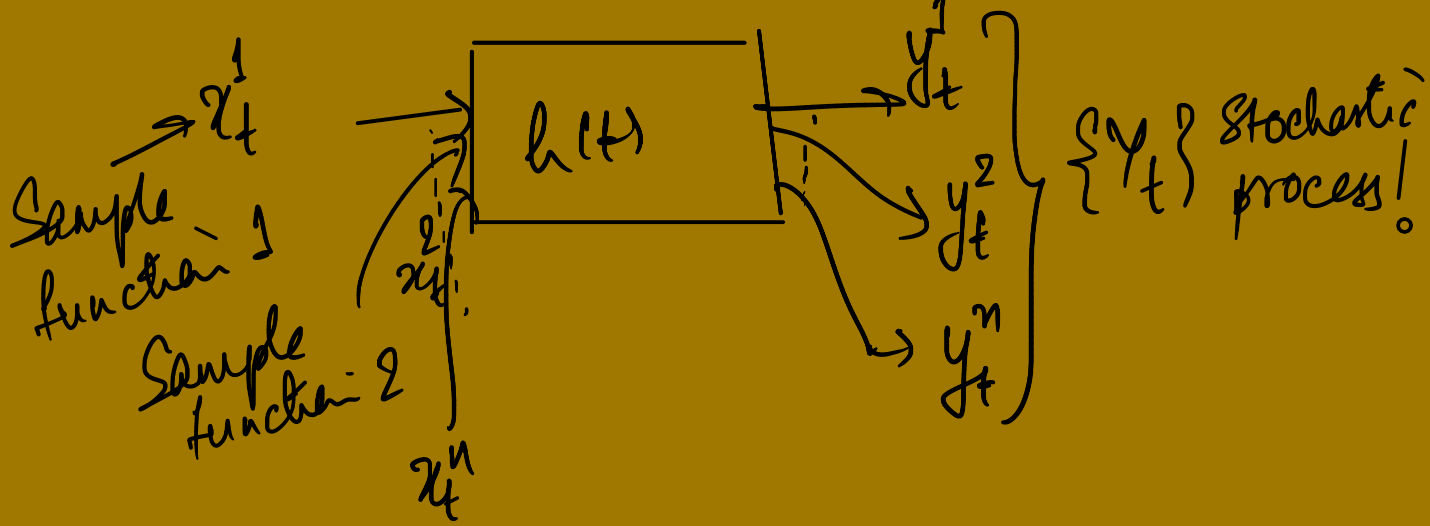
$G_x(\omega)$ → Quantify the amount of noise present in a communication channel.

Stochastic Processes and LTI systems

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

$$= (h * x)(t) = (x * h)(\tau).$$

What happens if the i/p to the LTI system is a Stochastic Process $\{X_t\}$?



Characterize $\{Y_t\}$ in terms of $\{X_t\}$ and $h(t)$.

① Assume that $\{X_t\}$ is WSS.

$$m_Y(t) = E[Y_t] = E\left[\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \underbrace{E[x(t-\tau)]}_{m_X}\, d\tau$$

$$= m_X \int_{-\infty}^{\infty} h(\tau) d\tau$$

$$m_Y(t) = m_X H(0) \quad [H = \text{freq. response}].$$

$\Rightarrow$ m_Y does not depend on 't'.

$$R_Y(t_1, t_2) = E[Y_{t_1} Y_{t_2}]$$

$$= E\left[\int_{-\infty}^{\infty} h(p) x(t_1 - p) dp \int_{-\infty}^{\infty} h(q) x(t_2 - q) dq\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(p) h(q) \underbrace{E[x(t_1 - p) x(t_2 - q)]} dp\, dq$$

$$R_Y(t_1, t_2) = \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(p) h(q) R_X(t_1 - t_2 - p + q) dp\, dq}_{f(t_1 - t_2)}$$

R_Y is a function only of $t_1 - t_2$ rather than the absolute values of t_1 & t_2.

$\therefore$ $\{Y_t\}$ is also a WSS SP!!!

$$R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(p) h(q) R_X(\tau - p + q) dp\, dq.$$

$$G_Y(\omega) \sim G_X(\omega)$$

In case of deterministic signals x & y,

$$\underline{\underline{Y(\omega)}} = H(\omega)\,\underline{\underline{X(\omega)}}, \quad \forall\, \omega \in \mathbb{R}.$$

$$G_y(\omega) = \mathcal{F}\{R_y\}.$$

$$= \int_{-\infty}^{\infty} \left(\underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}}\, h(p)\, h(q)\, R_x(\tau - p + q)\, dp\, dq \right) e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(p)\, h(q) \underbrace{\left(\int_{-\infty}^{\infty} R_x(\tau - p + q)\, e^{-j\omega\tau}\, d\tau \right)}\, dp\, dq$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(p)\, h(q)\, G_x(\omega)\, e^{j(p-q)\omega}\, dp\, dq$$

$$= \underbrace{\int_{-\infty}^{\infty} h(p)\, e^{jp\omega}\, dp}_{H^*(\omega)} \; \underbrace{\int_{-\infty}^{\infty} h(q)\, e^{-jq\omega}\, dq}_{H(\omega)} \cdot G_x(\omega)$$

$$= H^*(\omega) \quad H(\omega) \cdot G_x(\omega)$$

(if h is real valued)

$$\boxed{G_y(\omega) = |H(\omega)|^2\, G_x(\omega)}$$

$$0 \leq R_y(0) = E[Y_t^2] = \int_{-\infty}^{\infty} G_y(\omega)\, d\omega = \int_{-\infty}^{\infty} |H(\omega)|^2\, G_x(\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} |H(\omega)|^2 G_{x}(\omega)\, d\omega \geq 0 \longrightarrow (1)$$

$H(\omega)$ is the Freq. response of any real valued LTI system.

$$H(\omega) = 1, \quad \omega_0 \leq \omega \leq \omega_0 + \Delta\omega \longrightarrow (2)$$
$$= 0, \quad \text{else}.$$

Substitute (2) in (1) to get

$$\int_{\omega_0}^{\omega_0+\Delta\omega} G_x(\omega)\, d\omega \geq 0$$ for any arbitrary ω_0 and arbitrarily small $\Delta\omega$.

$$\Rightarrow G_x(\omega) \geq 0, \ \forall \omega \in \mathbb{R}.$$

[Complete the justification for interpreting $G_x(\omega) := \mathcal{F}\{R_x\}$ as the PSD of X_t]

* Ideal filters:

Low pass filter: $H(\omega) = 1 \quad \omega \leq \omega_0$
$= 0 \quad \omega > \omega_0.$

Are ideal filters realizable?