The Logistic Equation (Second Order of Nonlinearity)

Write an equation $\boxed{c\frac{dx}{dt} = Ax - Bx^2}$

$\Rightarrow \boxed{\frac{dx}{dt} = ax - bx^2}$, $a, b > 0$ $\quad a = A/c$, $b = B/c$

When $x \to 0$ $\boxed{\frac{dx}{dt} \simeq ax}$ (Rate $\propto$ State)

$\Rightarrow \boxed{x \simeq x_0 e^{at}} \Rightarrow$ Early growth is exponential

When x is large, $-bx^2$ inhibits and saturates growth (as in population growth)

Rescaling of variables: $\frac{dx}{dt} = ax\left(1 - \frac{bx}{a}\right)$

Define $\boxed{K = a/b} \to$ (Carrying capacity)

$$\Rightarrow \frac{dx}{dt} = ax\left(1 - \frac{x}{K}\right)$$

$$\Rightarrow \frac{d}{d(at)}\left(\frac{x}{K}\right) = \left(\frac{x}{K}\right)\left(1 - \frac{x}{K}\right)$$

Define $\boxed{X = x/K}$ and $\boxed{T = at = \frac{t}{1/a}}$

$\Rightarrow \boxed{\frac{dX}{dT} = X(1-X)}$ The rescaled logistic equation

Integral Solution: (Separation of variables)

$$\int \frac{dx}{x(1-x)} = \int dT$$

Now, by the method of partial fractions,

$$\boxed{\frac{1}{x(1-x)} \equiv \frac{A}{x} + \frac{B}{1-x}} \Rightarrow 1 \equiv A(1-x) + Bx$$

i.) When $x = 1$, $B = 1$, ii) When $x = 0$, $A = 1$.

$$\Rightarrow \int \frac{dx}{x(1-x)} = \int \frac{dx}{x} + \int \frac{dx}{1-x} = \int dT$$

$$\Rightarrow \int \frac{dx}{x} - \int \frac{d(-x)}{1-x} = \int dT$$

$$\Rightarrow \ln x - \ln(1-x) = \ln e^{T} + \ln c$$

$$\Rightarrow \frac{x}{1-x} = ce^{T} \Rightarrow x = ce^{T} - xce^{T}$$

$$\Rightarrow x(1 + ce^{T}) = ce^{T}$$

$$\Rightarrow \boxed{x = \frac{ce^{T}}{1+ce^{T}} = \frac{1}{1+c^{-1}e^{-T}}}$$

When $T = 0$ (i.e. $t = 0$), $x = x_0$ (or $x = x_0$).
(The initial value must NOT be zero)

$$\Rightarrow x_0 = \frac{1}{1+c^{-1}} \Rightarrow 1 + c^{-1} = \frac{1}{x_0}$$

$$\Rightarrow \frac{1}{C} = \frac{1}{X_0} - 1 \Rightarrow \frac{1}{C} = \frac{1-X_0}{X_0}$$

$$\Rightarrow \boxed{C = \frac{X_0}{1-X_0} = \frac{x_0/k}{1-x_0/k} = \frac{x_0}{k-x_0}}.$$

Returning to variables x and t we get,

$$X = \frac{x}{k} = \frac{1}{1+c^{-1}e^{-T}} = \frac{1}{1+c^{-1}e^{-at}} \Rightarrow \boxed{x = \frac{k}{1+c^{-1}e^{-at}}}$$

i.) When $t \to \infty$, $x \to k$ (The limiting carrying capacity).
(OR $X \to 1$) (for ANY initial value)

Further, $x = \dfrac{k\,e^{at}}{\left(\frac{k-x_0}{x_0}\right) + e^{at}} = \dfrac{x_0 k\, e^{at}}{(k-x_0) + x_0 e^{at}}$

$$\Rightarrow \boxed{x = \frac{x_0 k\, e^{at}}{k + x_0(e^{at}-1)} = \frac{x_0\, e^{at}}{1 + \frac{x_0}{k}(e^{at}-1)}}.$$

ii.) When $t \ll a$, $(t \to 0)$ in the early growth stage.

$$e^{at} - 1 \simeq \cancel{1} + at - \cancel{1} \simeq at \to 0$$

Hence, e^{at} in the numerator determines the dynamics, compared to $e^{at}-1$ in the denominator.

$\Rightarrow \boxed{x \simeq x_0 e^{at}}$ in the early growth, but this also appears as if due to $k \to \infty$, i.e. an infinite carrying capacity.

Going back to ~~dear~~ $\frac{dx}{dT} = x(1-x) = f(x)$

we see that starting from $x = x_0$ and tending towards $x \to 1$ (the upper limit), $\frac{dx}{dT} > 0$, i.e. there is always growth.

Now $\boxed{\frac{d^2x}{dT^2} = \frac{df}{dT} = \frac{df}{dx}\cdot\frac{dx}{dT}}$.

$\boxed{f(x) = x(1-x) = x - x^2} \Rightarrow \boxed{\frac{df}{dx} = 1 - 2x}$.

i.) when $x < 1/2$, $\frac{df}{dx} > 0$,

ii.) when $x > 1/2$, $\frac{df}{dx} < 0$.

f(x) has a TURNING POINT at $x = 1/2$

Since $\frac{dx}{dT} > 0$ for any FINITE value of T,

we see that $\boxed{\text{for } x < 1/2,\ \frac{d^2x}{dT^2} > 0}$, i.e.

Growth occurs at an increasing rate. On the other hand $\boxed{\text{for } x > 1/2,\ \frac{d^2x}{dT^2} < 0}$,

i.e. growth occurs at a decreasing rate. This means that before $x = 1/2$, the growth is exponential, and beyond $x = 1/2$, the growth starts slowing down towards the carrying capacity.

Hence, $\boxed{X = 1/2}$ is the point where the <u>NONLINEAR</u> effect starts to be functional.

The corresponding time scale T_{nl} (the <u>nonlinear time scale</u>) can be obtained by

$$\boxed{X = \frac{1}{2} = \frac{1}{1 + c^{-1} e^{-T_{nl}}}} \Rightarrow 2 = 1 + c^{-1} e^{-T_{nl}}$$

$$\Rightarrow c^{-1} e^{-T_{nl}} = 1 \Rightarrow c\, e^{T_{nl}} = 1.$$

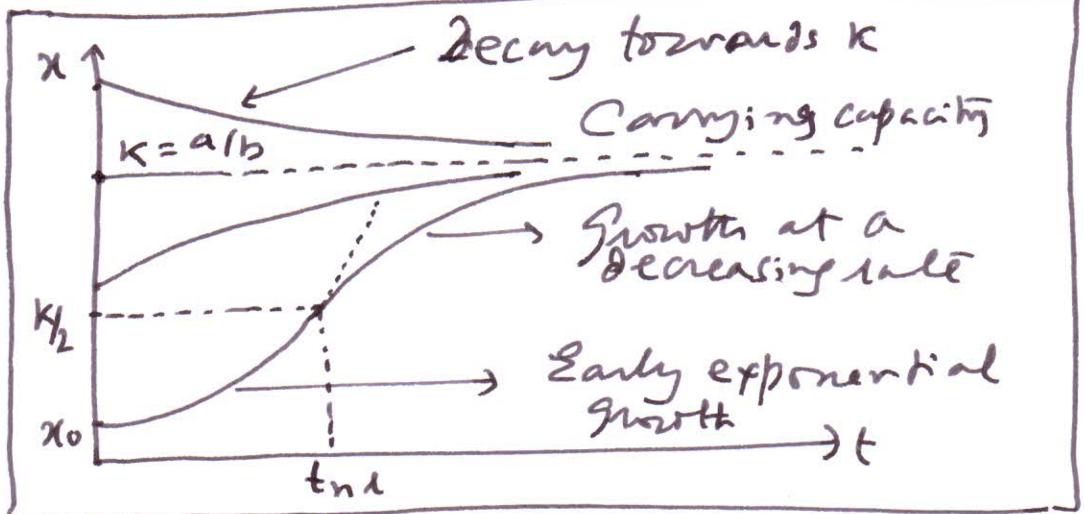
$$\Rightarrow \boxed{T_{nl} = \ln\left(\frac{1}{c}\right) = \ln\left(\frac{1 - X_0}{X_0}\right)}$$

Hence $a\, t_{nl} = \ln\left(\frac{1 - x_0/K}{x_0/K}\right) = \ln\left(\frac{K - x_0}{x_0}\right)$

$$\Rightarrow \boxed{t_{nl} = \frac{1}{a} \ln\left(\frac{K}{x_0} - 1\right)}$$ <u>Realistically $t_{nl} > 0$.</u>

This can only happen if $\frac{K}{x_0} - 1 > 1$

$\Rightarrow \frac{K}{x_0} > 2 \Rightarrow \boxed{x_0 < K/2}$ <u>Needed for strong growth</u>



i) For $\boxed{\frac{K}{2} < x_0 < K}$ there will be ONLY growth at a decreasing rate.

ii) For $\boxed{x_0 > K}$, there will be ONLY DECAY

Higher Orders of Nonlinearity : Logistic-Type Equation

$$\boxed{\frac{dx}{dt} = ax - bx^{\alpha+1}} \quad \alpha \geq 2, \ \alpha \in \mathbb{Z}$$

$$\Rightarrow \frac{dx}{dt} = ax\left(1 - \frac{bx^\alpha}{a}\right) = ax\left(1 - \frac{x^\alpha}{a/b}\right)$$

Now transform $\boxed{\xi = x^\alpha} \Rightarrow d\xi = \frac{\alpha x^\alpha}{x}\, dx$

$$\Rightarrow \frac{d\xi}{dt} = \alpha \frac{\xi}{x} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{d\xi}{dt} \cdot \frac{x}{\alpha \xi}$$

Hence $\frac{d\xi}{dt} \frac{\cancel{x}}{\alpha \xi} = a\cancel{x}\left(1 - \frac{\xi}{k}\right)$ $\quad \boxed{k = a/b}$

$$\Rightarrow \boxed{\frac{d\xi}{dt} = a\alpha\, \xi \left(1 - \frac{\xi}{k}\right)}$$

Now rescale $\boxed{X = \xi/k}$ and $\boxed{T = a\alpha t}$,

$$\Rightarrow \boxed{\frac{d}{d(a\alpha t)}\left(\frac{\xi}{k}\right) = \frac{\xi}{k}\left(1 - \frac{\xi}{k}\right)}$$

$$\Rightarrow \boxed{\frac{dX}{dT} = X(1-X)}$$

in a familiar rescaled form.

$$\Rightarrow \boxed{X = \frac{\xi}{k} = \frac{1}{1 + c^{-1}e^{-T}} = \frac{1}{1 + c^{-1}e^{-a\alpha t}}}$$

$$\boxed{c = \frac{X_0}{1 - X_0} = \frac{\xi_0/k}{1 - \xi_0/k} = \frac{\xi_0}{k - \xi_0} = \frac{x_0^\alpha}{k - x_0^\alpha}}$$

$$\Rightarrow \xi = \frac{k e^{a\alpha t}}{c^{-1} + e^{a\alpha t}} \quad \Big| \quad c^{-1} = \frac{k - x_0^\alpha}{x_0^\alpha} = \frac{k - \xi_0}{\xi_0}$$

$$\Rightarrow \quad x^{\alpha} = \frac{k e^{a\alpha t}}{\left(\frac{k - x_0^{\alpha}}{x_0^{\alpha}}\right) + e^{a\alpha t}} = \frac{k x_0^{\alpha} e^{a\alpha t}}{(k - x_0^{\alpha}) + x_0^{\alpha} e^{a\alpha t}}$$

$$\Rightarrow \quad \boxed{x^{\alpha} = \frac{x_0^{\alpha} e^{a\alpha t}}{1 + \frac{x_0^{\alpha}}{k}\left(e^{a\alpha t} - 1\right)}}$$

$$\Rightarrow \quad \boxed{x = \frac{x_0 e^{at}}{\left[1 + \frac{x_0^{\alpha}}{k}\left(e^{a\alpha t} - 1\right)\right]^{1/\alpha}}}$$

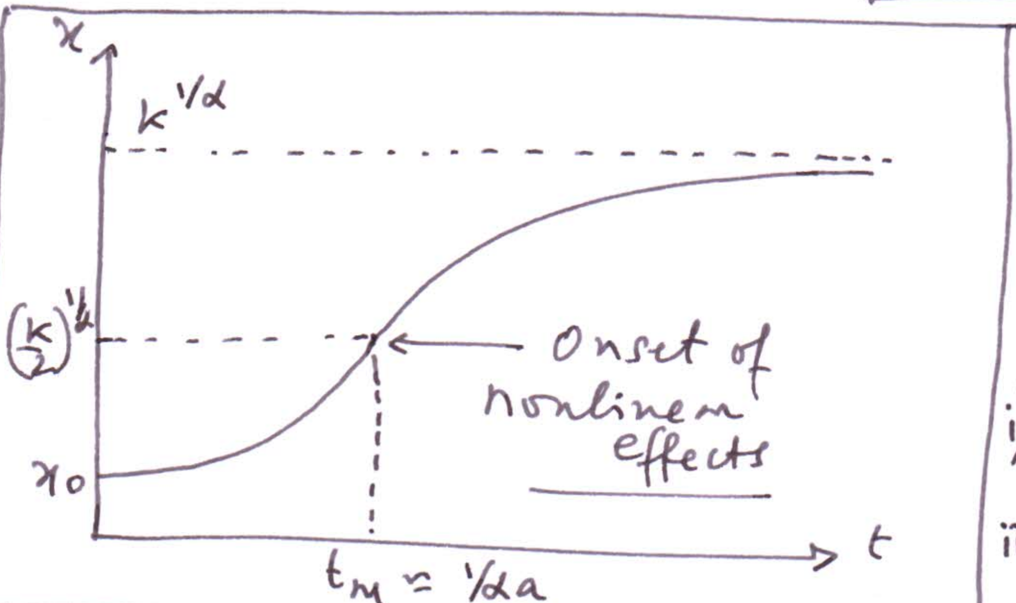
For $t \to 0$ Exponential Early Growth as $\boxed{x = x_0 e^{at}}$

From $\boxed{x^{\alpha} = \frac{k}{1 + c^{-1} e^{-a\alpha t}}}$, we see that for $t \to \infty$, $\boxed{x \to k^{1/\alpha}}$, i.e. the Carrying capacity has been reduced to $k^{1/\alpha}$.

Nonlinear Time Scale: $T_{nl} = \ln\left(\frac{1}{c}\right)$.

$$\Rightarrow \quad \boxed{t_{nl} = \frac{1}{\alpha a} \ln\left(\frac{k - x_0^{\alpha}}{x_0^{\alpha}}\right) = \frac{1}{a\alpha} \ln\left(\frac{k}{x_0^{\alpha}} - 1\right)}$$

Realistically for $t_{nl} > 0$, $\boxed{\frac{k}{x_0^{\alpha}} - 1 > 1} \Rightarrow \boxed{x_0 < \left(\frac{k}{2}\right)^{1/\alpha}}$



For $\alpha \geq 2$, the Carrying Capacity is $k^{1/\alpha}$ in x. In ξ it is κ, and in X it is 1.

i/. The Carrying Capacity is reduced.

ii/. The nonlinear time is ~~also~~ also reduced.