CT 303 - Digital Communications

Lecture 25 : 9 December 2020

* Performance Analysis of M-ary Bandpass Systems

→ Symbol Error Probability (P_E) & Bit error Probability (P_B)

→ Symbol Energy (E/E_s) & Bit Energy (E_b). $E_b = \frac{E_s}{k}$

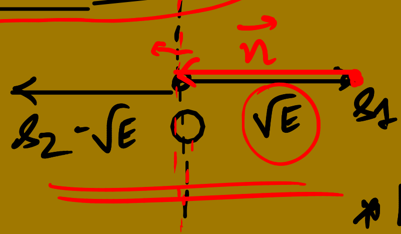
→ How does increasing $M(=2^k)$ affect the performance in terms of error and BW?

1. Coherent MPSK

$\|s_1\| = \sqrt{E}$ $\quad E_s = E_b = E$.

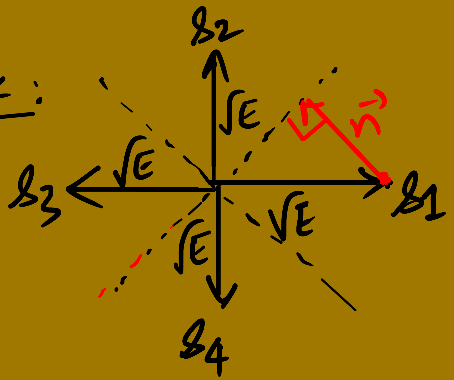
$\|s_2\| = \sqrt{E}$

BPSK :
M = 2

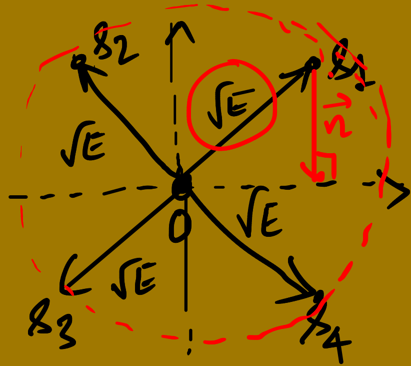


* Minimum Noise power in order to introduce error = E

QPSK :
M = 4



OR



* Minimum noise power to introduce error = $\frac{E}{2}$ //

* As M ↑, the minimum noise power required to introduce error ↓.

* ∴ We expect P_E ↑ as M ↑.

* This comparison holds for same E_s.

but $E_b = E_s/k$, where $M = 2^k$.

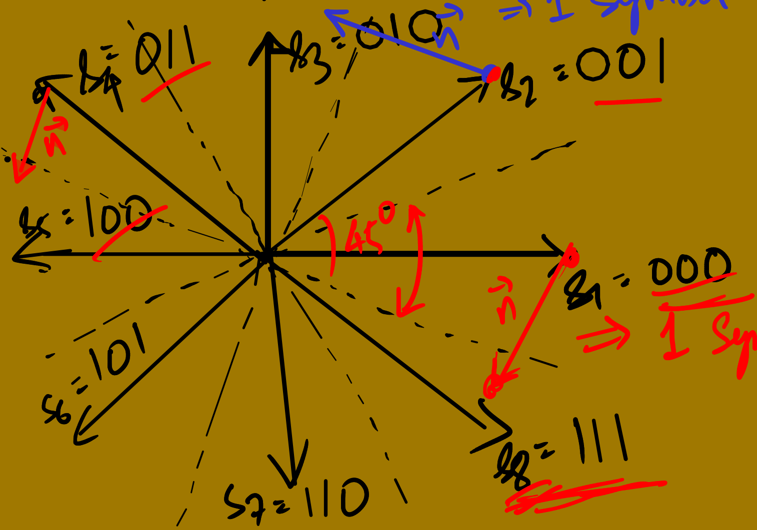
P_E vs E_s deteriorates as M ↑.

* In a DCS we are interested in P_B vs E_b/N_0 behaviour.

* $P_B \longleftrightarrow P_E$?

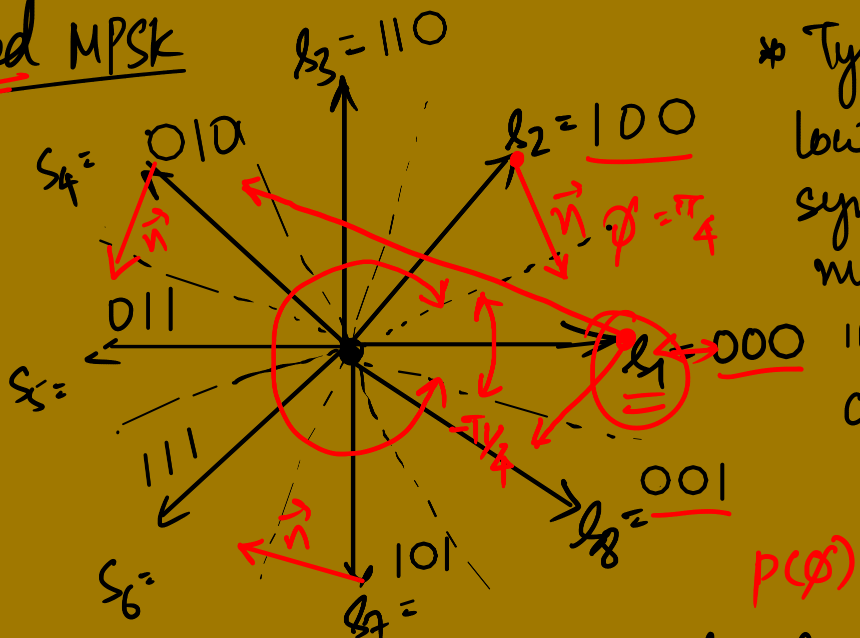
Example 8-PSK : $M = 8, k = 3$.

⇒ 1 symbol error ⇒ 2 bit errors



⇒ 1 Symbol Error → 3 bit errors.

Gray Coded MPSK

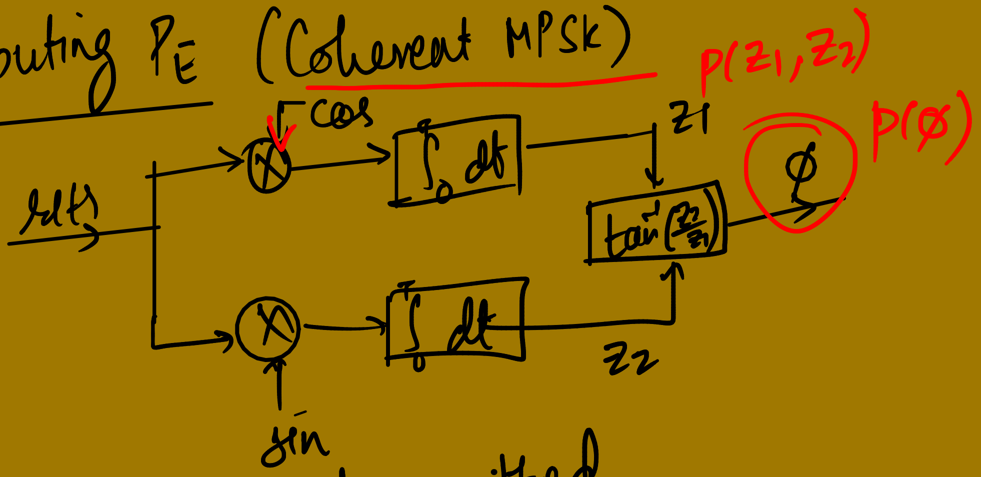


* Typically, under low noise conditions symbols may be misclassified into adjacent classes only

* In such cases, if Gray code is used, for every symbol error, only 1 bit error occurs.

* Thus $\boxed{P_B \simeq \frac{P_E}{k}}$

* Computing P_E (Coherent MPSK)



* Assume $s_1(t)$ was transmitted
* $r(t) = s_1(t) + n_1(t)$.

$$E[z_1 | s_1] = \sqrt{E} \qquad E[z_2 | s_2] = 0.$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E} \\ 0 \end{pmatrix}$$

* Joint density of z_1, z_2 is:

$$p(z_1, z_2) = \frac{1}{2\pi\sigma_0^2} \exp\left(-\frac{\left((z_1 - \sqrt{E})^2 + z_2^2\right)}{2\sigma_0^2}\right)$$

* Let $m := (z_1^2 + z_2^2)^{1/2}$ & $\phi = \tan^{-1}\left(\frac{z_2}{z_1}\right)$

$$\Rightarrow \boxed{\begin{aligned} z_1 &= m\cos\phi \\ z_2 &= m\sin\phi \end{aligned}}$$

* Change of variables $(z_1, z_2) \longrightarrow (m, \phi)$

$$p(m, \phi) = p(z_1, z_2)\,|J|$$

$J \rightarrow$ Jacobian for change of variables

* $|J| = m$

$$p(m, \phi) = \frac{m}{2\pi\sigma_0^2}\, e^{\left(-\frac{(z_1^2 + E - 2z_1\sqrt{E} + z_2^2)}{2\sigma_0^2}\right)}$$

* $z_1^2 + z_2^2 = m^2, \; z_1 = m\cos\phi$

$$\Rightarrow p(m, \phi) = \frac{m}{2\pi\sigma_0^2}\, e^{\left(-\frac{(m^2 - 2m\sqrt{E}\cos\phi + E)}{2\sigma_0^2}\right)}$$

$$\boxed{p(m,\phi) = \frac{m}{2\pi\sigma_0^2} e^{-\left(\frac{(m-\sqrt{E}\cos\phi)^2 + E\sin^2\phi}{2\sigma_0^2}\right)}}$$

$$p(\phi) = \frac{1}{2\pi\sigma_0^2}\int_0^\infty m\, e^{-\left(\frac{(m-\sqrt{E}\cos\phi)^2+E\sin^2\phi}{2\sigma_0^2}\right)} dm$$

$$\boxed{P_E = 1 - \int_{-\pi/M}^{\pi/M} p(\phi)\, d\phi.}$$ → Closed form solution is not available.

* Either Numerical computations or approximations

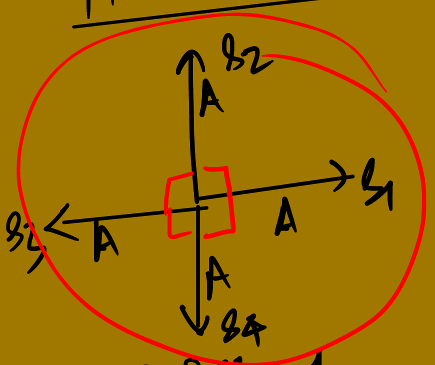
* For $E_s/N_0 >> 1$, $P_E \approx 2Q\left(\sqrt{\frac{2E_b k}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$ — P_E vs. $\frac{E_b}{N_0}$

As $M\uparrow$, $P_E\uparrow$, ⇒ $P_B\uparrow$

* Conclusion: As $M\uparrow$ $P_B\uparrow$ for the same E_b/N_0, but Bandwidth efficiency Improves!

* Difference between QPSK and BPSK.

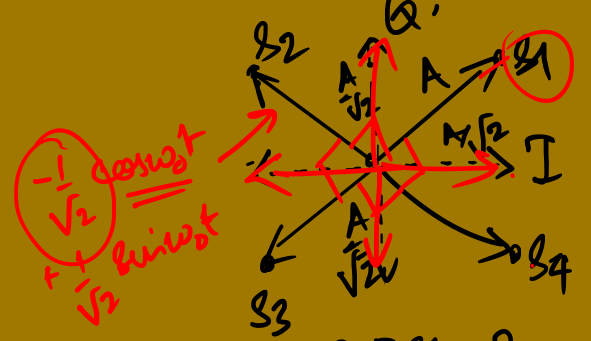
$E_s = kE_b$



QPSK-1

$-\frac{1}{\sqrt{2}}\cos\omega_0 t$

$+\frac{1}{\sqrt{2}}\sin\omega_0 t$



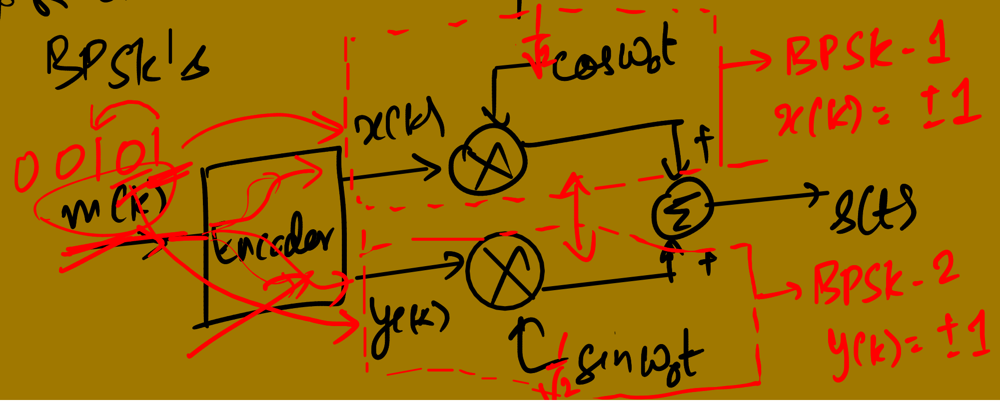
QPSK-2

$\cos(\omega_0 t)\cos(\pi/4)$

$+\sin\frac{\pi}{4}\sin(\omega_0 t)$

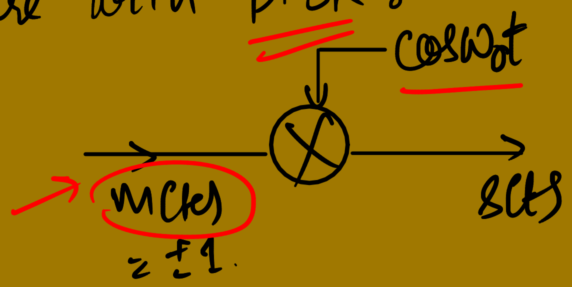
$+\frac{1}{\sqrt{2}}\cos\omega_0 t + \frac{1}{\sqrt{2}}\sin\omega_0 t$

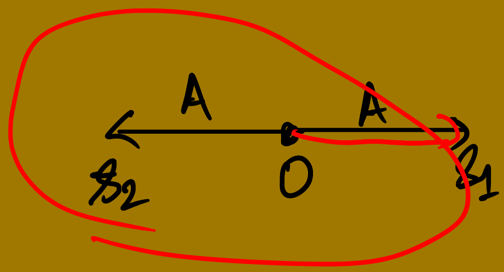
* QPSK-2 can be implemented as 2 orthogonal/Quadrature BPSK's



Since these are orthogonal, they do not interfere & can be treated independently.

* Compare with BPSK :





1. If R_{BPSK} is the rate at which the BPSK operates then $R_{QPSK\text{-}2} = ?$ $\frac{R_{BPSK}}{2}$

2. Signal Power used in BPSK is $S = A^2$.
   Signal Power used in QPSK-2 is $(A/\sqrt{2})^2 = \frac{A^2}{2} \therefore S/2$.

3. $\therefore \left(\frac{E_b}{N_0}\right)_{QPSK\text{-}2} = \frac{S \cdot T}{N/W} = \frac{S}{N} \cdot \left(\frac{W}{R}\right) = \frac{\cancel{S/2}}{N} \cdot \frac{W}{\cancel{R/2}}$

$$= \frac{S}{N} \cdot \left(\frac{W}{R}\right) = \left(\frac{E_b}{N_0}\right)_{BPSK}.$$

$\therefore$ P_B vs E_b/N_0 for BPSK and QPSK are same !

* How about P_E for BPSK and QPSK ?

→ For BPSK , $P_B = P_E$

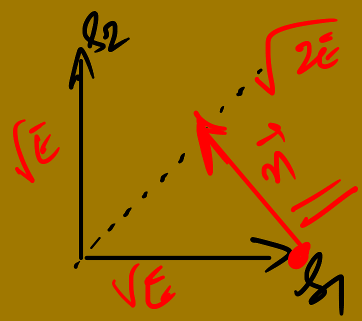
→ For QPSK , $P_E = (1 - P_B)^2$

(2) Coherent MFSK (OFSK)

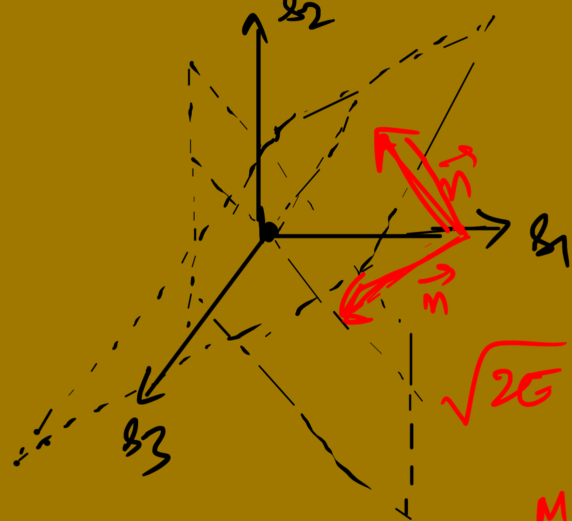
* Irrespective of M, $d(s_i, s_j) = \sqrt{2E}$

  ∴ Minimum Noise power required to introduce symbol error $= E/2$ (irrespective of M).

* But as M ↑, the # of classes into which a received signal can get misclassified, for the same noise power also ↑, thus we expect $P_E \uparrow$ as $M \uparrow$ for the same E_s/N_0.



M = 2



M = 3

→ P_B vs E_b/N_0 as $M \uparrow$

k = 3

$s_i \to k$ bits ○ – –

* $\boxed{\dfrac{E_b}{N_0} \approx \dfrac{1}{k}\dfrac{E_s}{N_0}}$

* $P_E \longleftrightarrow P_B$?

* For $M = 2^k$, there are $2^k - 1$ $(M-1)$ ways for a symbol error to occur, but for any of the symbol errors 'k' bits, approximately half of these symbol errors will lead to an error in that bit.

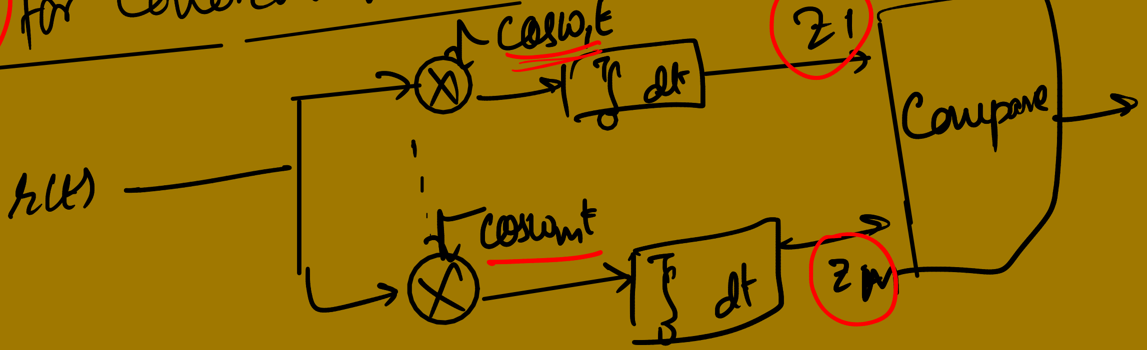
$$\frac{P_B}{P_E} = \frac{2^{(k-1)}}{2^k - 1} \qquad \frac{P_B}{P_E} = \frac{4}{7}$$

4 { 000, 001, 010, 011 }
{ 100, 101, 110 } 8

* Since every symbol error is equally likely (since $d(s_i, s_j) = \sqrt{2E}, \forall i,j, i \neq j$)

$$\boxed{\frac{P_B}{P_E} = \frac{2^{k-1}}{2^k - 1}}, \quad \lim_{k \to \infty} \frac{P_B}{P_E} = \frac{1}{2}.$$

* P_E for Coherent MFSK:



P

Given that $r(t) = s_1(t) + n(t)$

$$z_1 \sim N(\sqrt{E}, \sigma_0)$$
$$z_2 \sim N(0, \sigma_0)$$
$$\vdots$$
$$z_M \sim N(0, \sigma_0)$$

* Assuming equally likely symbols,

$$P_E = P(e/s_1) = 1 - P_{correct}(s_1).$$

$$P_{correct}(s_1) = \int_{-\infty}^{\infty} p(z_1 > z_2, z_1 > z_3, \ldots, z_1 > z_M \mid z_1)\, p(z_1)\, dz_1$$

* Upper bound for P_E: Assuming s_1 was transmitted,

Error occurs if any $z_i > z_1$, $i = 2, \ldots, M$

- Let E_i denote the event that $z_i > z_1$, $i = 2, \ldots, M$.
- Then $P_E = P\left(\bigcup_{i=2}^{M} E_i\right) \leq \sum_{i=2}^{M} P(E_i)$ $\quad P_{E,BFSK}$ (z_i with z_1)

BFSK Coherent

→ E_i is an event that happens in comparing 2 orthogonal signals; i.e., Binary Coherent FSK.

$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

$\therefore \; P(E_i) = P_{E,\text{Binary FSK}} = \underline{Q}\left(\sqrt{\frac{E_s}{N_0}}\right)$

$\therefore \; \underline{P_{E,MFSK} \leq (M-1)\, Q\left(\sqrt{\frac{E_s}{N_0}}\right)} = (M-1)\, Q\left(\sqrt{\frac{kE_b}{N_0}}\right)$

$< \underline{M\, Q\left(\sqrt{\frac{kE_b}{N_0}}\right)}$

* A tighter upper bound is

$$P_{E,MFSK} < \underline{2e^{-k\left(\sqrt{\frac{E_b}{N_0}} - \sqrt{\ln 2}\right)^2}}$$

$\Leftarrow \sqrt{\frac{E_b}{N_0}} \geq \sqrt{\ln(2)}$

* Observe that, as k (or M) ↑, for the same E_b/N_0, P_E and thus P_B ↓.

$\sqrt{\frac{E_b}{N_0}} = \sqrt{\ln 2} \Leftarrow$ Shannon's limit for reliable comm.

* Conclusion: For MFSK, error performance improves as M↑, but Bandwidth efficiency ↓.

P_B vs E_b/N_0

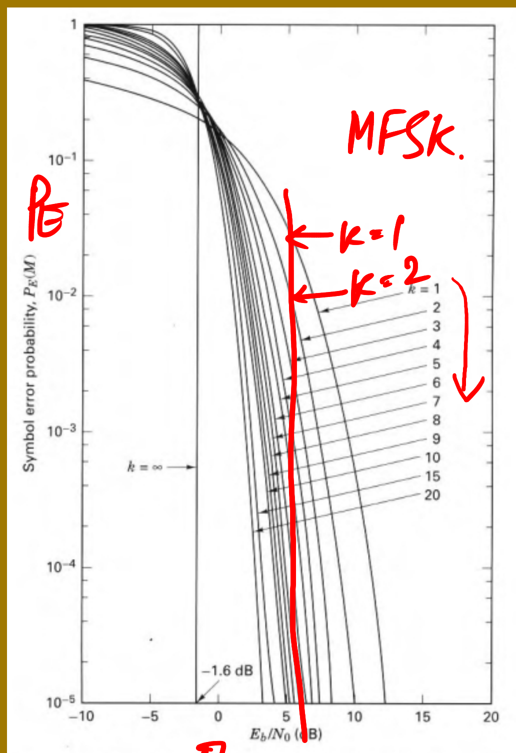


Figure 4.36 Symbol error probability for coherently detected M-ary orthogonal signaling. (Reprinted from W. C. Lindsey and M. K. Simon, *Telecommunication Systems Engineering,* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973, courtesy of W. C. Lindsey and Marvin K. Simon.)

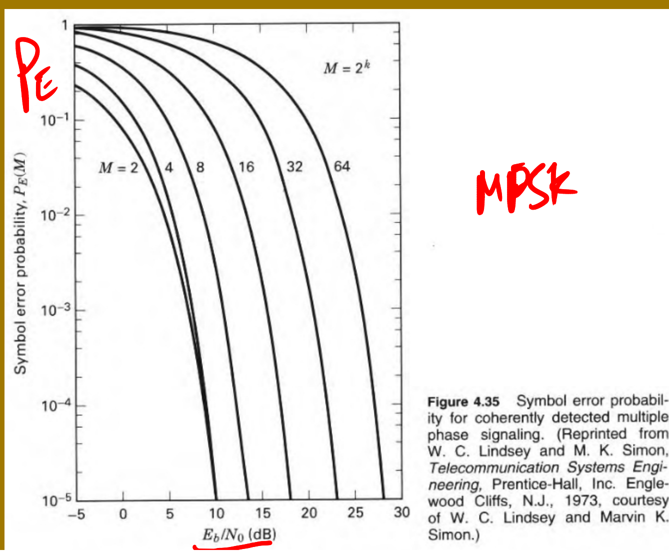


Figure 4.35 Symbol error probability for coherently detected multiple phase signaling. (Reprinted from W. C. Lindsey and M. K. Simon, *Telecommunication Systems Engineering,* Prentice-Hall, Inc. Englewood Cliffs, N.J., 1973, courtesy of W. C. Lindsey and Marvin K. Simon.)

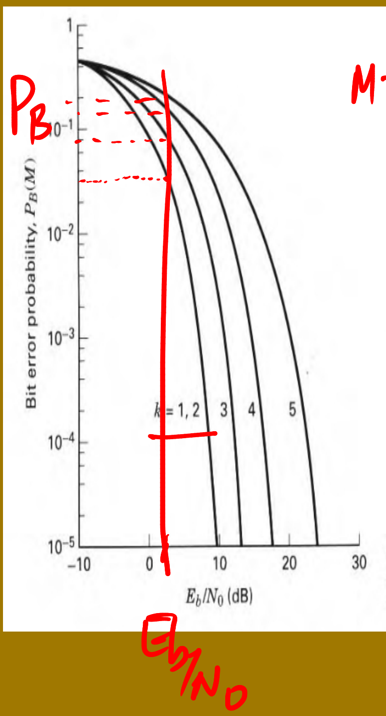


Figure 4.29 Bit error probability for coherently detected multiple phase signaling.

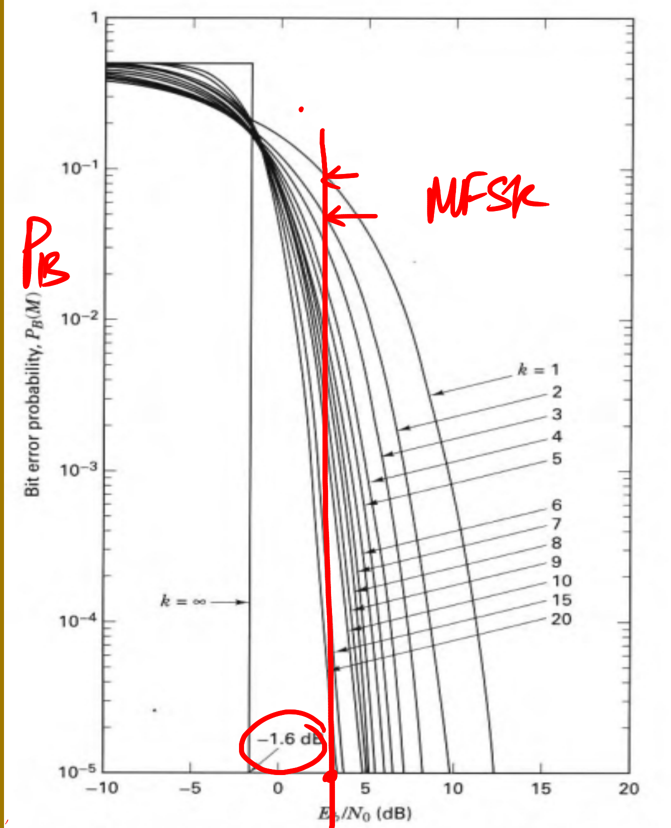


Figure 4.28 Bit error probability for coherently detected *M*-ary orthogonal signaling. (Reprinted from W. C. Lindsey and M. K. Simon, *Telecommunication Systems Engineering*, Prentice Hall, Inc., Englewood Cliffs, N.J., 1973, courtesy of W. C. Lindsey and Marvin K. Simon).

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