Some definitions and theorems

Hyperplane:

In E^n, a set of points whose coordinate satisfy the linear equation of the form

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = Z \equiv CX = Z$$

This equation $cx = Z$ is called a hyperplane for fixed Z and $c_i \ \forall\ i = 1, 2, \ldots, n$

$Z = 0$ means hyperplane passes through origin.

- $CX = Z$ in E^n divides the whole plane into 3 mutually disjoint sets

$$X_1 = \{X \mid cx < Z)$$
$$X_2 = \{X \mid cX = Z)$$
$$X_3 = \{X \mid cX > Z)$$

X_1 and X_3 are called open half-spaces.

$$\left.\begin{array}{l} X_4 = \{x : cx \leq z\} \\ X_5 = \{x : cx \geq z\} \end{array}\right\} \rightarrow \text{closed half spaces.}$$

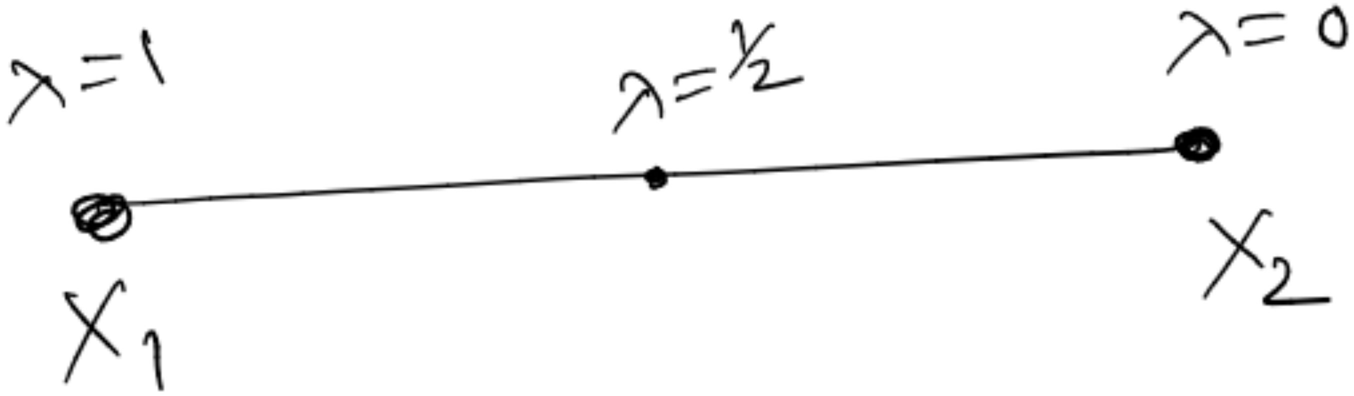
Line Line passes through x_1 and x_2.

$$X = \{x \mid x = \lambda x_1 + (1-\lambda) x_2, \lambda \text{ real}\}$$

Line segment

Line segment joining x_1 and x_2 is the set of points

$$X = \{x : x = \lambda x_1 + (1-\lambda) x_2, 0 \leq \lambda \leq 1\}$$



convex combination and convex set

Convex combination

A point x is said to be the convex combination of the points.
$x_1, x_2, \ldots, x_p$, if x can be expressed as

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p \quad \text{for } \lambda_i \geq 0$$

and $$\sum_{i=1}^{p} \lambda_i = 1$$

For 2 points

$$\boxed{\begin{gathered} x = \lambda_1 x_1 + \lambda_2 x_2 \\ \lambda_1 + \lambda_2 = 1 \end{gathered}}$$

Convex set:

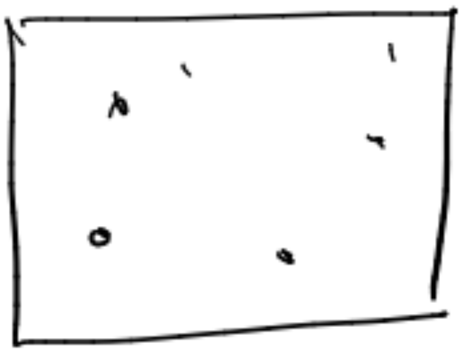
A set X is said to be a convex set if any two points $x_1, x_2 \in X$ the line segment joining x_1 and x_2 must be also in the set X.

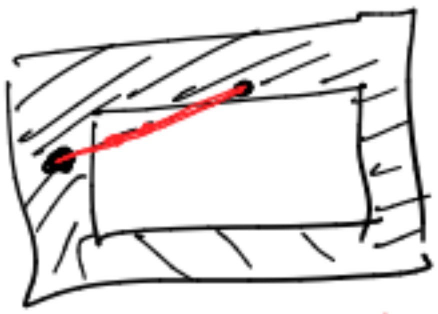
X is a convex set if every point x_1 and x_2

$$y = \lambda x_1 + (1-\lambda) x_2 , \quad 0 \leq \lambda \leq 1, \quad x_1, x_2 \in X$$

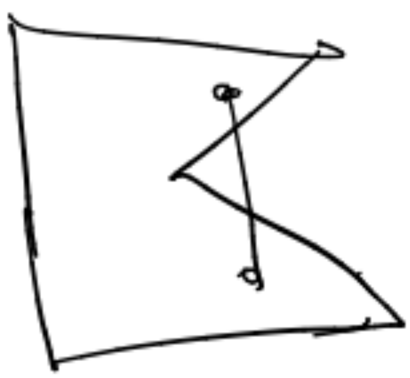
must be also in the set X

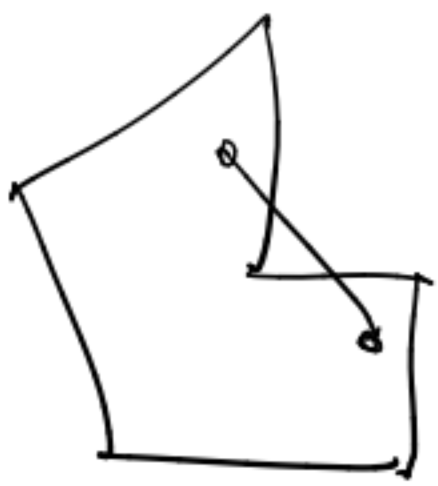
i.e, $y \in X$.





not convex





Few results

- A hyperplane is a convex set.
- A half space either open or closed is also a convex set.
- Intersection of two convex sets is also a convex set.

Relation with LPP

Consider an LPP

$$\left.\begin{aligned} &\text{optimise } Z = CX \\ &\text{s.t. } AX \ (\leq = \geq)\, b \\ &X \geq 0 \end{aligned}\right\}$$

- Each constraint is a hyperplane or closed half-space.
- Hence the intersection of all the constraints that forms a feasible region is a convex set.

further it is a closed convex set.

Extreme point/corner point

Extreme point:

A point y of a convex set X is an extreme point if it cannot be expressed as a convex combination of any two other points in X.

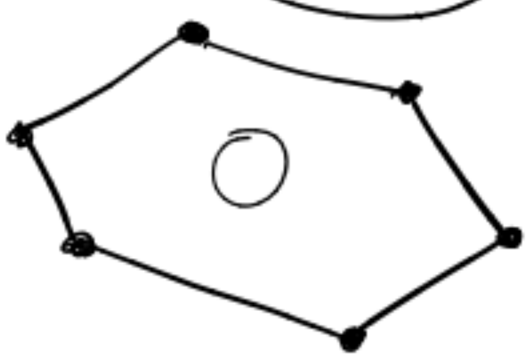
Geometrically we can say, a point y in a convex set X is said to be an extreme point if it does not lie on the line segment joining any two other point in X.

Example consider a convex set

i) of point on and inside a circle.

Then all points on the circumference are extreme points.

ii)



All vertices are corner points.

Standard form of an LPP

Let we are given a LPP whose first r constraints are "$\leq$" type, next $(s-r)$ constraints are "$\geq$" type and remaining $(m-s)$ constraints are "$=$" type.

$\boxed{\text{LPP-1}}$

optimise $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

$$
\begin{array}{l}
r \left[\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
\vdots \\
a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n \leq b_r
\end{array}\right. \\
(s-r) \left[\begin{array}{l}
a_{(r+1)1}x_1 + a_{(r+1)2}x_2 + \cdots + a_{(r+1)n}x_n \geq b_{r+1} \\
\vdots \\
a_{s1}x_1 + a_{s2}x_2 + \cdots + a_{sn}x_n \geq b_s
\end{array}\right. \\
m-s \left[\begin{array}{l}
a_{(s+1)1}x_1 + a_{(s+1)2}x_2 + \cdots + a_{(s+1)n}x_n = b_{s+1} \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{array}\right.
\end{array}
$$

$$x_1, x_2, \ldots, x_n \geq 0$$

Standard form

$$
\left.\begin{array}{l}
\text{optimise } Z = CX \\
\text{s.t. } AX = b \\
\qquad X \geq 0.
\end{array}\right]
$$

- Suppose we have a "$\leq$" type constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

we add one non-negative variable x_{n+i} to the left hand side.

we call such a variable $\boxed{\text{slack}}$ variable

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + x_{n+i} = b_i$$

- For "$\geq$" type constraint.

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n \geq b_j$$

we subtract a non-negative variable x_{n+j} to the left-hand side.

we call such a variable $\boxed{\text{surplus}}$ variable

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n - x_{n+j} = b_j$$

LPP-1 becomes

optimise $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + c_{n+1}x_{n+1} + \cdots + c_{n+s}x_{n+s}$

s.t.

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} &&= b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \quad + x_{n+2} &&= b_2 \\
&\vdots \\
&a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n \quad + x_{n+r} &&= b_r \\
&a_{(r+1)1}x_1 + a_{(r+1)2}x_2 + \cdots + a_{(r+1)n}x_n \quad - x_{n+r+1} &&= b_{r+1} \\
&\vdots \\
&a_{s1}x_1 + a_{s2}x_2 + \cdots + a_{sn}x_n \quad - x_{n+s} &&= b_s \\
&a_{(s+1)1}x_1 + a_{(s+1)2}x_2 + \cdots + a_{(s+1)n}x_n &&= b_{s+1} \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &&= b_m
\end{aligned}$$

$\underline{x_1, x_2, \ldots, x_n}, \underline{x_{n+1}, \ldots, x_{n+r}, x_{n+r+1}, \ldots, x_{n+s}} \geq 0$

LPP in standard form

In short we can write

$$\text{optimize } Z = CX$$
$$\text{s.t. } AX = b$$
$$X \geq 0$$

where,

$$X = [\underline{x_1, x_2, \cdots, x_n}, \underline{x_{n+1}, \cdots, x_{n+r}}, \underline{x_{n+r+1}, \cdots x_{n+s}}]$$

$$= [X_{original}, X_{slack}, X_{surplus}]$$

$$C = [c_1, c_2, \cdots, c_n, c_{n+1}, \cdots, c_{n+r}, c_{n+r+1}, \cdots, c_{n+s}]$$

$$= [C_{original}, C_{slack}, C_{surplus}]$$

$$Z = C_{original} \cdot X_{original} + C_{slack} \cdot X_{slack} + C_{surplus} \cdot X_{surplus}.$$

$$b = [b_1, b_2, \cdots, b_r, b_{r+1}, \cdots, b_s, b_{s+1}, \cdots \underline{b_m}]$$

$$A=\begin{array}{c}
\begin{array}{ccccccccccccc} x_1 & x_2 & & x_n & x_{n+1} & & & & x_{n+r} & x_{n+r+1} & & & x_{n+1} \end{array} \\
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & & & & & & & & & & & \\
a_{r1} & a_{r2} & \cdots & a_{rn} & 0 & 0 & 0 & & 1 & 0 & 0 & \cdots & 0 \\
a_{(r+1)1} & a_{(r+1)2} & \cdots & a_{(r+1)n} & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\vdots & & & & & & & & & & & & \\
a_{s1} & a_{s2} & & a_{sn} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
a_{(s+1)1} & a_{(s+1)2} & \cdots & a_{(s+1)n} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & & & & & & & & & & & \\
a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\end{array}$$

We always assume that the b values ie, the right hand side is always non-negative.

If not we can make it non-negative by multiplying (-1) both sides of the constraint and change the type if needed.

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq -b_i$$

$$-a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \leq b_i$$

Recasting a LPP

- All constraints are equations.
  (except non-negativity restrictions which are ≥ 0)
- The right hand side is non-negative (for each constraint)
- All variables are non-negative.

Question:

What is the relation between the optimal solution values of the original problem and the problem in standard form. ??

Theorem: There is a one-to-one correspondence between the optimal solution of the original problem and the optimal solution of the new problem (where we introduced slack and surplus variables)

if

both $C_{slack} = 0$ and

$C_{surplus} = 0$

Theorem: Let X^* be a solution of the LPP minimum $Z = CX$

$$\text{s.t. } AX = b$$
$$X \geq 0$$

then X^* is also a solution of the LPP

$$\text{maximise } W = (-C)X$$
$$\text{s.t. } AX = b$$
$$X \geq 0$$

we have an LPP

$$\text{maximize } Z = cx$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$

Theorem: The set of all feasible solutions of a LPP is a convex set.

Theorem: The objective function of an LPP attains its optimal value at an extreme point of the convex set of feasible solutions.

Theorem: A basic feasible solution to an LPP corresponds to an extreme point of the convex set of feasible solutions.

Theorem: Each extreme point of the convex set of all feasible solutions of the system

$$Ax = b, \; x \geq 0$$

corresponds to a basic feasible solution.

$$\boxed{Ax = b, \; x \geq 0}$$

Theorem:

If the objective function attains its optimal value at more than one extreme point, then every convex combination of these extreme points also gives the optimal value of the objective function.