1. Find the charge distribution that would cause the following electric fields: **(6)**

$$\textbf{(a)} \quad \vec{E}(\vec{r}) = \frac{A\sin\theta\hat{r} + B\hat{\phi}}{r\sin\theta} \qquad\qquad \textbf{(b)} \quad \vec{E}(\vec{r}) = \frac{A\hat{r} + B\sin\theta\cos\theta\hat{\phi}}{r}$$

soln:

(a)

$$\vec{E}(\vec{r}) = \frac{A\sin\theta\hat{r} + B\hat{\phi}}{r\sin\theta}; \quad E_r = \frac{A}{r}, \quad E_\phi = \frac{B}{r\sin\theta}$$

$$\begin{aligned}
\vec{\nabla}\times\vec{E} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\phi)\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi} - \frac{\partial}{\partial r}(rE_\phi)\right]\hat{\theta} + \frac{1}{r}\left[-\frac{\partial E_r}{\partial\theta}\right]\hat{\phi} \\
&= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{B}{r}\right)\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial(A/r)}{\partial\phi} - \frac{\partial}{\partial r}\left(\frac{B}{\sin\theta}\right)\right]\hat{\theta} - \frac{1}{r}\frac{\partial(A/r)}{\partial\theta}\hat{\phi} \\
&= 0
\end{aligned}$$

So $\vec{E}$ is a valid electrostatic field that can be caused by a static charge distribution. By Gause's law

$$\begin{aligned}
\rho &= \epsilon_0\vec{\nabla}\cdot\vec{E} \\
&= \epsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_r\right) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}\right] \\
&= \frac{\epsilon_0}{r^2}\frac{\partial(Ar)}{\partial r} + \frac{\epsilon_0}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{B}{r\sin\theta}\right) \\
&= \frac{\epsilon_0 A}{r^2}
\end{aligned}$$

(b)

$$\vec{E}(\vec{r}) = \frac{A\hat{r} + B\sin\theta\cos\theta\hat{\phi}}{r}: \quad E_r = \frac{A}{r}, \quad E_\phi = \frac{B\sin\theta\cos\theta}{r}$$

Here $\vec{\nabla}\times\vec{E} \neq 0$. Hence no static charge distribution can produce this electric field.

2. Two concentric thin spherical shells of radii a and b, ($a < b$) are maintained at potentials V_a and V_b respectively. By solving the Laplace's equation in the three regions, $r < a, \quad a < r < b, \quad r > b$, and applying appropriate boundary conditions at the interfaces of the three regions, determine the potential due to this configuration everywhere. **(6)**

soln:

The charge distribution mentioned has a spherical symmetry. The potentials in the three regions mentioned will exhibit this symmetry. So the potential functions will only depend upon r. Let $r < a$ be region 1 , $a < r < b$ be region 2 and $r > b$ be region 3. Since these regions are chargeless the Laplace's equation we have is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = 0$$

The general solution to the above equation is $\Phi = \frac{c}{r} + d$.
In region 1, $\Phi_1 = \frac{c_1}{r} + d_1$, in region 2, $\Phi_2 = \frac{c_2}{r} + d_2$ and in region 3, $\Phi_3 = \frac{c_3}{r} + d_3$. Using appropriate boundary conditions we will determine the constants c_i and d_i.
As $r \to 0$, $\frac{c_1}{r} \to \infty$. $\vec{E}_1 = \frac{c_1}{r^2}\hat{r}$ corresponds to a point charge at the origin. Since we don't have any such point charge at the origin $c_1 = 0$. This gives

$$\Phi_1 = d_1$$

In region 3 we demand $\Phi_3 \to 0$ as $r \to \infty$ since the charge distribution is bounded in space. This gives $d_3 = 0$. So we have

$$\Phi_3 = \frac{c_3}{r}$$

At $r = a$, $\Phi_1(a) = \Phi_2(a) = V_a$. This gives $d_1 = V_a$ and $\frac{c_2}{a} + d_2 = V_a$.
At $r = b$, $\Phi_2(b) = \Phi_3(b) = V_b$. This gives $c_3 = bV_b$ and $\frac{c_2}{b} + d_2 = V_b$.
We are left to solve the two equations

$$\frac{c_2}{a} + d_2 = V_a \qquad (1)$$

$$\frac{c_2}{b} + d_2 = V_b \qquad (2)$$

Solving them gives $c_2 = \frac{(V_a - V_b)ab}{b-a}$ and $d_2 = \frac{aV_a - bV_b}{a-b}$.
The potential in the three region is thus given as

$$\Phi_1 = V_a, \quad \Phi_2 = \frac{(V_a - V_b)ab}{(b-a)r} + \frac{aV_a - bV_b}{a-b} \quad \text{and} \quad \Phi_3 = \frac{bV_b}{r}$$

3. When an amount of charge Q is placed on a conductor it attains a potential V.
Find the electrostatic energy stored in the conductor by evaluating the integral

$$\frac{\epsilon_0}{2}\int_{\text{all space}} |E|^2 d\tau$$

where $\vec{E}$ is the electric field created in the region surrounding the conductor due to the charge Q on it. **(6)**

soln:

The electric field within the conductor is 0. So the energy will be given by the integral becomes

$$\mathcal{E} = \frac{\epsilon_0}{2}\int_{\text{outside}} |E|^2 d\tau$$

$$E^2 = \vec{E}\cdot\vec{E} = \vec{\nabla}\Phi\cdot\vec{\nabla}\Phi = \vec{\nabla}\cdot(\Phi\vec{\nabla}\Phi) - \Phi\nabla^2\Phi$$

This gives

$$\mathcal{E} = \frac{\epsilon_0}{2}\int_{\text{outside}} \vec{\nabla}\cdot(\Phi\vec{\nabla}\Phi)d\tau - \frac{\epsilon_0}{2}\int_{\text{outside}} \Phi\nabla^2\Phi d\tau$$

Since $\nabla^2\Phi = \frac{\rho}{\epsilon_0} = 0$ outside the second integral doesn't contribute to the energy. Using divergence theorem the first integral can be converted to a surface integral over the surface bounding the region. This surface is the surface of the conductor S and the surface at infinity. At infinity we expect the fields go to zero if the given conductor is finite. So we have

$$\begin{aligned}
\mathcal{E} &= \frac{\epsilon_0}{2}\int_{\text{outside}} \vec{\nabla}\cdot(\Phi\vec{\nabla}\Phi)d\tau \\
&= \frac{\epsilon_0}{2}\oint_S \Phi\vec{\nabla}\Phi\cdot\hat{n}da
\end{aligned}$$

Here the normal $\hat{n}$ is directed into the closed surface of the conductor since the region we are integrating over is the outside.

$$\begin{aligned}
\therefore\ \mathcal{E} &= \frac{\epsilon_0}{2}V\oint_S(-\vec{E})\cdot\hat{n}da \\
&= \frac{\epsilon_0}{2}V\oint_S \vec{E}\cdot(-\hat{n})da \\
&= \frac{\epsilon_0}{2}V\frac{Q}{\epsilon_0} \\
&= \frac{1}{2}QV
\end{aligned}$$

4. A point charge q is situated at $z = 2$ on the z axis.

   (a) Find the average potential over the surface of a sphere $x^2 + y^2 + z^2 = 1$. **(2)**

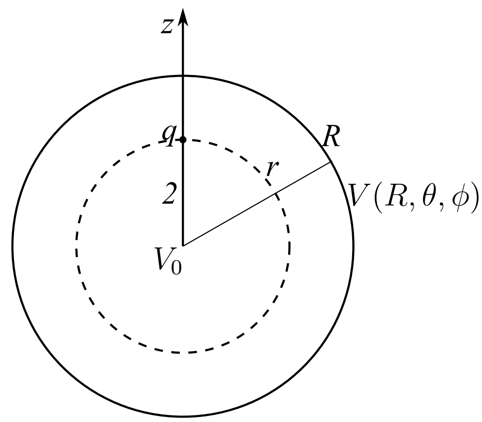
   (b) Find the average potential over the surface of a sphere $x^2 + y^2 + z^2 = 9$. **(5)**

soln:

(a) Since the charge q lies outside the sphere of radius 1, the region inside the sphere is chargeless. We can directly apply the mean value theorem over this sphere. The average potential over this surface will be the potential at the center V_0. V_0 is caused only due to the charge q and given as $V_0 = \frac{q}{8\pi\epsilon_0}$.
$\therefore\ V_{avg} = \frac{q}{8\pi\epsilon_0}$.

(b) Now the sphere encloses the charge q and hence the region within the sphere is not chargeless. The mean value theorem is not applicable. But the average potential can be worked out by the same method used to prove the mean value theorem. Consider a sphere of radius R. The average value of potential over this sphere is

$$V_{avg} = \frac{1}{4\pi R^2}\int_0^{2\pi}\int_0^{\pi} V(R,\theta,\phi)R^2\sin\theta d\theta d\phi$$



Let V_0 be the potential at the center of the sphere. Then

$$V(R,\theta,\phi) = V_0 + \int_0^R \vec{\nabla} V \cdot \hat{r} dr$$

The integral in the above step is independent of the path since $\vec{\nabla} V = -\vec{E}$ is a curlless field. So we do the integral along the radial direction from 0 to R, θ, ϕ.
So average potential is

$$\begin{aligned} V_{avg} &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left[V_0 + \int_0^R \vec{\nabla} V \cdot \hat{r} dr \right] \sin\theta d\theta d\phi \\ &= \frac{1}{4\pi} V_0 \int_0^{2\pi} \int_0^{\pi} \sin\theta d\theta d\phi + \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^R \vec{\nabla} V \cdot \hat{r} dr \sin\theta d\theta d\phi \\ &= V_0 + \frac{1}{4\pi} \int_0^R \left[\int_0^{2\pi} \int_0^{\pi} \vec{\nabla} V \cdot \hat{r} \sin\theta d\theta d\phi \right] dr \end{aligned} \tag{3}$$

The integral over θ and ϕ is at a constant r. We can write this as a surface integral over a sphere of radius r as follows:

$$\int_0^{2\pi} \int_0^{\pi} \vec{\nabla} V \cdot \hat{r} \sin\theta d\theta d\phi = \frac{1}{r^2} \int_0^{2\pi} \int_0^{\pi} (\vec{\nabla} V \cdot \hat{r}) r^2 \sin\theta d\theta d\phi = \frac{1}{r^2} \oint_S -\vec{E} \cdot \hat{r} da$$

This surface integral is equal to the total charge enclosed inside the sphere of radius r. As long as $r < 2$ this is 0. But when $r > 2$ the surface integral is $\frac{q}{\epsilon_0}$ by Gauss' Law.

So we divide the radial integral in Eq.(3) in two parts.

$$\begin{aligned}
\int_0^R \left[\int_0^{2\pi}\int_0^{\pi} \vec{\nabla} V \cdot \hat{r} \sin\theta d\theta d\phi\right] dr &= \int_0^2 \left[\int_0^{2\pi}\int_0^{\pi} \vec{\nabla} V \cdot \hat{r} \sin\theta d\theta d\phi\right] dr \\
&\quad + \int_2^R \left[\int_0^{2\pi}\int_0^{\pi} \vec{\nabla} V \cdot \hat{r} \sin\theta d\theta d\phi\right] dr \\
&= 0 + \int_2^R \frac{1}{r^2}\frac{(-q)}{\epsilon_0} dr \\
&= -\frac{q}{\epsilon_0}\left(\frac{1}{2} - \frac{1}{R}\right) \\
&= \frac{q}{\epsilon_0}\left(\frac{1}{R} - \frac{1}{2}\right)
\end{aligned}$$

From Eq.(3) we get

$$V_{avg} = V_0 + \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{2}\right)$$

Due to the given charge q at $z = 2$ the potential at the center of the sphere is $V_0 = \frac{q}{8\pi\epsilon_0}$. Using this we get

$$V_{avg} = \frac{q}{4\pi\epsilon_0 R}$$

For $R = 3, \quad V_{avg} = \frac{q}{12\pi\epsilon_0}$

There is an alternate method which is equally interesting.

Alternate Method

Let us denote the position of the charge q as $\vec{r}_o$. Then the average potential over a sphere of radius $R > r_0$ is given as

$$V_{avg} = \frac{1}{4\pi R^2}\int \frac{q}{4\pi\epsilon_0(|\vec{r} - \vec{r}_0|)} da$$

where $\vec{r}$ is a point over the sphere of radius R and da is an area element over the surface of the sphere.

The above integral can be looked upon as that of calculating the potential at $\vec{r}_0$ due to a uniform charge density $\sigma = \frac{q}{4\pi R^2}$ spread over the sphere of radius R.

It is well known that for such a charge distribution the potential inside the sphere is constant and is equal to the potential at the center. The potential at the center due to the uniform charge on the sphere is ridiculously easy. It is ofcourse $\frac{q}{4\pi\epsilon_0 R}$. So we have

$$V_{avg} = \frac{q}{4\pi\epsilon_0 R}$$

Gradient, divergence and curl

Spherical polar system

$$\vec{\nabla}F = \frac{\partial F}{\partial r}\hat{\mathrm{r}} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\hat{\phi} \qquad \vec{\nabla}\cdot\vec{\mathrm{A}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\vec{\nabla}\times\vec{\mathrm{A}} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathrm{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

Cylindrical System

$$\vec{\nabla}F = \frac{\partial F}{\partial s}\hat{\mathrm{s}} + \frac{1}{s}\frac{\partial F}{\partial\phi}\hat{\phi} + \frac{\partial F}{\partial z}\hat{\mathrm{z}} \qquad \vec{\nabla}\cdot\vec{\mathrm{A}} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\times\vec{\mathrm{A}} = \left[\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\mathrm{s}} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial\phi}\right]\hat{\mathrm{z}}$$