# CS374: Assignment 1
# Deadline for Submission: 07 Sep. 2020

Prob 1) Expand the **error function**

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

in a series by using the exponential series and integrating. Obtain the Taylor series of $\mathrm{erf}(x)$ about zero directly. Are the two series the same? Evaluate $\mathrm{erf}(1)$ by adding four terms of the series and compare with the value $\mathrm{erf}(1) \approx 0.8427$, which is correct to four decimal places.
**Hint:** Recall from the Fundamental Theorem of Calculus that

$$\frac{d}{dx} \int_0^x f(t)dt = f(x).$$

Prob 2) What is the least number of terms required to obtain $\pi$ correct up to four decimal places, using the series

$$\pi = 4\left[1 - 2\sum_{k=1}^{\infty}(16k^2 - 1)^{-1}\right].$$

Prob 3) What are the condition numbers of the following functions? Where are they large?

(i) $(x-1)^a$, where $a > 0$. (ii) $x^{-1}e^x$. (iii) $\cos^{-1} x$.

Prob 4) We consider a classic example given by Wilkinson. Let

$$f(x) = (x-1)(x-2)...(x-20) \text{ and } g(x) = x^{19}.$$

The roots of $f$ are obviously the integers 1,2,3,...,20. How is the root $r = 20$ affected by perturbing $f$ to $f + \epsilon g$?

Prob 5) Let the Bisection algorithm is applied to a continuous function $f$ on an interval $[a, b]$ to solve $f(x) = 0$, where $f(a)f(b) < 0$. Denote the successive intervals that arise in the Bisection method by $[a_0, b_0], [a_1, b_1], ..., [a_n, b_n]$ and so on with $a = a_0$ and $b = b_0$. Show that

a) $a_0 \le a_1 \le a_2 \le .....$ and $b_0 \ge b_1 \ge b_2 \ge ......$

b) $b_n - a_n = 2^{-n}(b_0 - a_0)$.

c) After $n-$steps, an approximate root will have been computed with error at most $(b_0 - a_0)/2^{(n)}$.

Further, if $a = 0.1$ and $b = 1.0$, how many steps of the Bisection method are required to determine the root with an error of at most $\frac{1}{2} \times 10^{-8}$.

Prob 6) Using Bisection method, find where the graphs of $y = 3x$ and $y = e^x$ intersect by finding roots of $e^x - 3x = 0$ correct to four decimal digits.

Prob 7) Verify that when Newton's Method is used to compute $\sqrt{N}$ (by solving the equation $x^2 = N$), the sequence of iterates is defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right).$$

Perform three iterations of this scheme for computing $\sqrt{2}$, starting with $x_0 = 1$, and of the Bisection method for $\sqrt{2}$, starting with interval $[1, 2]$. How many iterations are needed for each method in order to obtain $10^{-6}$ accuracy?

Prob 8) Show that if $r$ is a root of $f(x) = 0$ with multiplicity $m > 1$, then Newton's method converges linearly. For this case, we define **modified Newton's method** by the following formula:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}.$$

Show that the modified Newton's method has quadratic convergence. (Hint: Use Taylor series for each of $f(r + e_n)$ and $f'(r + e_n)$).