Examples of Stochastic Processes

* Gaussian Random Process

Any finite subset from $\{X_t\}_{t\in I}$ must have the joint pdf described as a M.V. Gaussian

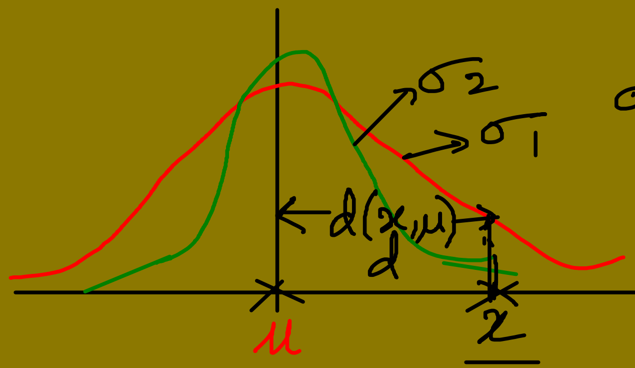
$$p(x_1,\ldots,\cancel{x_n}\,x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}}\, e^{-\frac{(x-\mu)^T\Sigma^{-1}(x-\mu)}{2}}$$

$$\mu = \begin{pmatrix} m_{x_1} \\ \vdots \\ m_{x_n} \end{pmatrix} \qquad \Sigma_{ij} = Cov(X_i, X_j)$$

$$(x-\mu)^T\Sigma^{-1}(x-\mu)$$

$\rightarrow \left.\dfrac{(x-\mu)^2}{\sqrt{\sigma}}\right\}$ 1 RV with Gaussian distribution

$\rightarrow \dfrac{d(x,\mu)^2}{\sigma}$



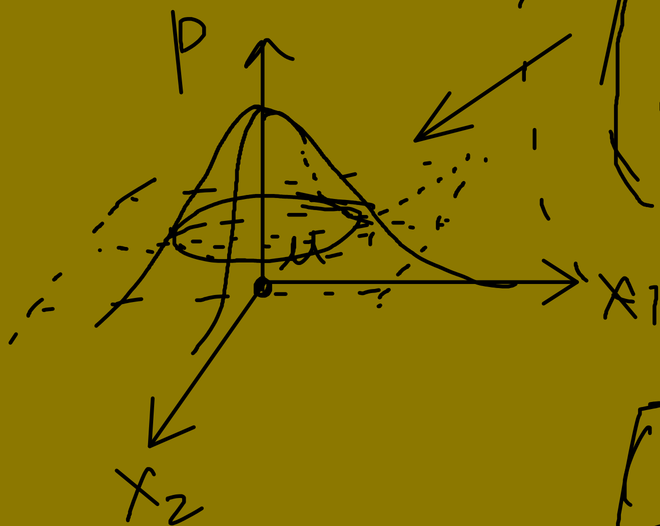
$\sigma_1 > \sigma_2$

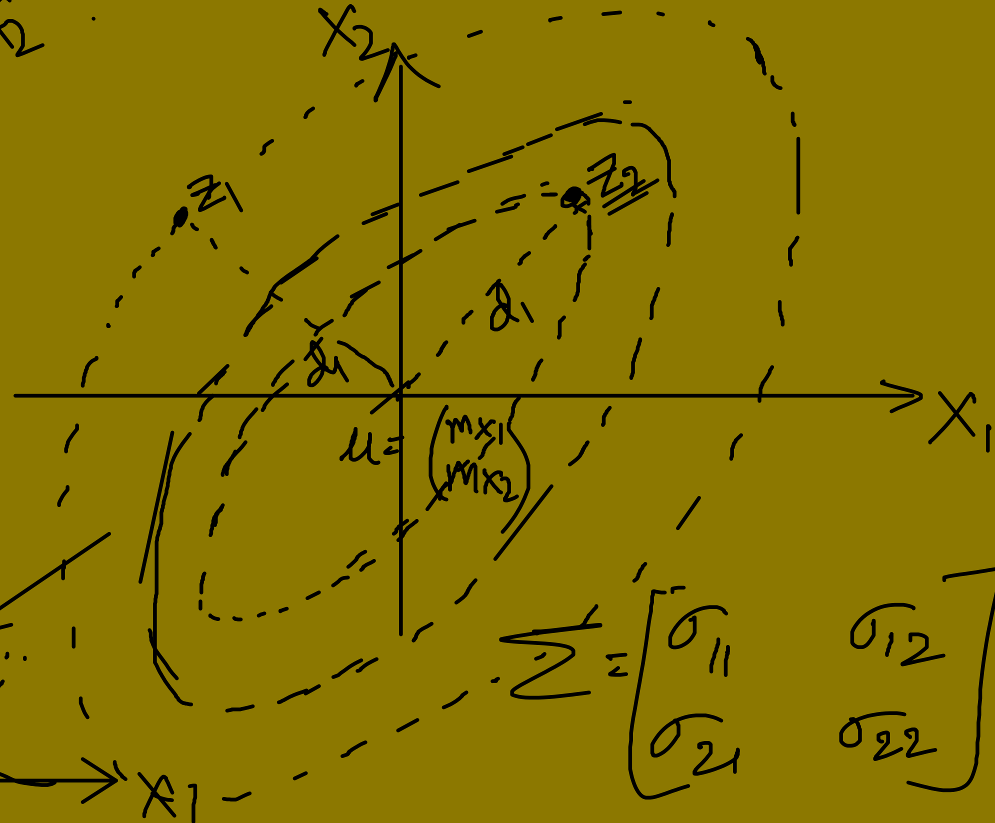
$$\frac{d}{\sigma_1} < \frac{d}{\sigma_2}$$

Euc. dist : d

Dist. from the perspective of the distribution $\dfrac{d}{\sigma}$

2 RV: X_1, X_2





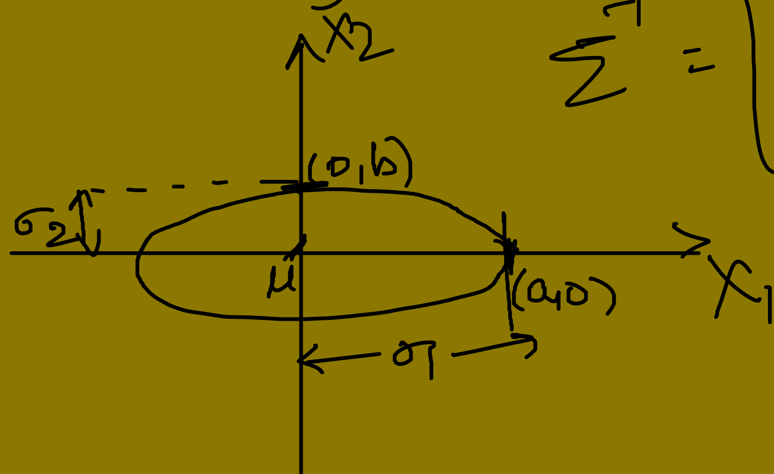
$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}\right]^T \Sigma^{-1} \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}\right]$$

① $\Sigma = Id \Rightarrow \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right\|^2 = d(x,\mu)^2$

② $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \quad \sigma_1 > \sigma_2$

$$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 \\ 0 & 1/\sigma_2 \end{bmatrix}$$



$$\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{}$$

$$= \frac{(x_1-\mu_1)^2}{\sigma_1} + \frac{(x_2-\mu_2)^2}{\sigma_2}$$

★ Equi-probable points ⇒ Same distance from mean

$(x-\mu)^T \Sigma^{-1} (x-\mu) \Rightarrow$ distance between x and μ.

⇒ Mahalanobis distance

* Wide-Sense Stationary Gaussian Random Process

$$\mu = \begin{pmatrix} m_{x_1} \\ \vdots \\ m_{x_n} \end{pmatrix} = \begin{pmatrix} m_x \\ \vdots \\ m_x \end{pmatrix}$$

$$E\left[(X_{t_i} - m_{t_i})(X_{t_j} - m_{t_j})\right]$$

$$\Sigma_{ij} = Cov(X_i, X_j) = Cov(X_{t_i}, X_{t_j})$$

$$= R_x(t_i - t_j) - m^2 \quad \text{discrete convolution}$$

$$\begin{bmatrix} \sigma^2 & a & b \\ a & \sigma^2 & a \\ b & a & \sigma^2 \end{bmatrix}$$

same

same

same.

⇒ Toeplitz Matrix

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \Rightarrow H \text{ is Toeplitz}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, H(n,k)$$

$$\underline{H x(n)}$$

$$p(X_{t_1}, \ldots, X_{t_n}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}}\, e^{-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}}$$

$$\underline{\mu} = \begin{pmatrix} m \\ \vdots \\ m \end{pmatrix} \quad \Sigma$$

$$X_{t_1}, \ldots, X_{t_n} \longrightarrow X_{t_1+T}, \ldots, X_{t_n+T}$$

$$\Sigma \qquad \underline{\Sigma}$$

is shift invariant!

$$p(X_{t_1}, \ldots, X_{t_n}) = p(X_{t_1+T}, \ldots, X_{t_n+T})$$

$$\forall\, t_1, \ldots, t_n \in I$$

$$\forall\, n \in \mathbb{Z}^+$$

A WSS GRP $\not\Rightarrow$ SSS GRP !!

② Ergodicity :

$\{X_t\}_{t \in I}$.

$$m(X_t) = \mathbb{E}[X_t]$$

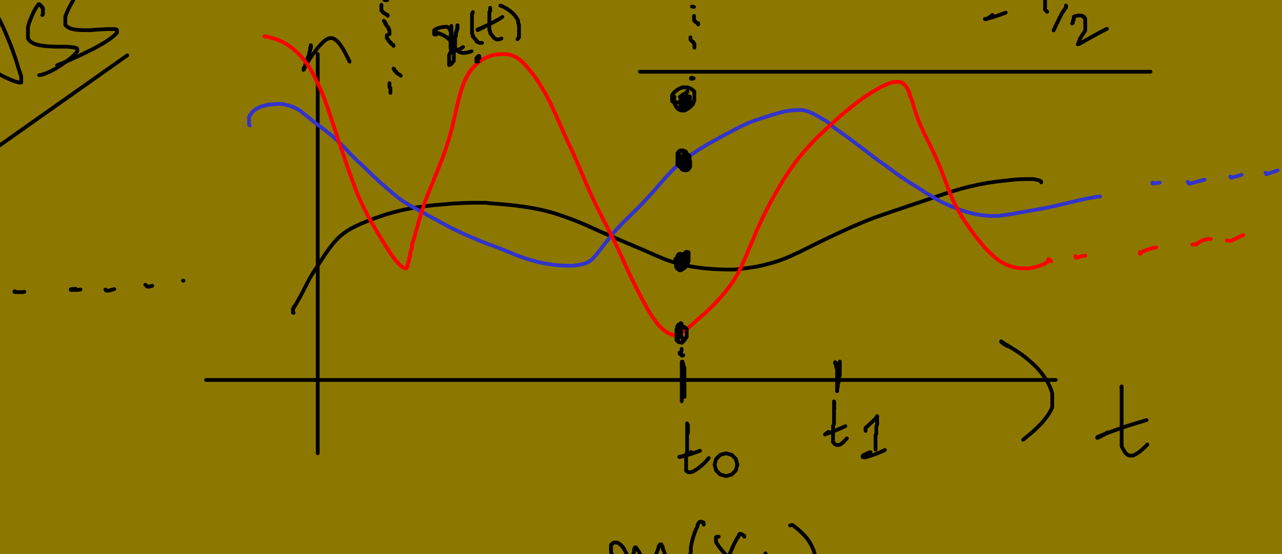
$$R_{XX}(X_{t_i}, X_{t_j}) = \mathbb{E}[X_{t_i} X_{t_j}]$$

Ensemble statistics

Ensemble statistics $\Rightarrow$ Index statistics

$I = (-\infty, \infty)$ $\quad M(X_t) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt$

~~WSS~~



$m(X_{t_0})$

$$\underline{\underline{R_X(\tau)}} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x(t+\tau)\,dt$$

WSS

(atleast WSS)

1 sample function

A ~~s~~tochastic process is called "ERGODIC" if the statistics over the index set yields the ensemble statistics.

1. We say that a (atleast WSS) SP is ergodic in the mean if

$$m_X = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt.$$

2. We say that a (at least WSS) SP is ergodic in the Autocorrelation func. if

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x(t+\tau)\,d\tau.$$

Ergodicity is usually a "SIMPLIFYING ASSUMPTION"