Lecture-5 Permutation groups

A permutation of a set X is a bijection (one-one and onto mapping) from X onto itself. One can easily check that the set of permutations S_X is a group under the operation function composition. If α, β are permutations of X then we define the element $\alpha\beta(x) = \alpha((\beta(x))$ for all $x \in X$. If the set X is the first n positive integers then S_X is written as S_n and is called the Symmetric group. For example the elements of the group S_3 are

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

If $\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$ then $\alpha\beta$ (applying β first)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$ whereas $\beta\alpha$ is given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

So $\alpha\beta \neq \beta\alpha$ in general and S_n is a non-commutative group for all $n \geq 3$.

One can write any permutation in cycle notation as discussed before for example

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 6 & 1 & 2 \end{bmatrix} \in S_6$$ can be written as $(15)(246)$

where a cycle is a permutation of the form $(a_1 a_2 a_3 \dots a_k)$ where $a_1 \to a_2$, $a_2 \to a_3, \dots, a_k \to a_1$ while leaving all other elements fixed. $(a_1 a_2 \dots a_k)$ is called a k-cycle. The procedure to write a permutation in cycle notation leads to a product of disjoint cycles, and we have the following theorem

<u>Theorem</u>: Every permutation in S_n can be written as a product of disjoint cycles.

One can see that disjoint cycles commute. $(1\,3\,5)(2\,4) = (2\,4)(1\,3\,5)$

A 2-cycle is $(a_1 a_2)$ is called a transposition.

<u>Theorem</u>: The transpositions generate S_n.

An arbitrary k-cycle $(a_1 a_2 \dots a_k)$ can be written as

$(a_1 a_2 \dots a_k) = (a_1 a_k)(a_1 a_{k-1}) \dots (a_1 a_3)(a_1 a_2)$ (check!)

Since every permutation of S_n is a product of disjoint cycles we can write any permutation as a product of transpositions

The decompositions of a permutation as a product of transpositions may not be unique For example

$$(15)(246) = (15)(26)(24) = (15)(46)(26)$$

<u>Theorem</u>: (i) The transpositions $(12), (13), \dots, (1n)$ generate S_n

(ii) The transpositions $(12), (23), \dots (n-1\ n)$ generate S_n

Proof: (i) $(ab) = (1a)(1b)(1a)$ then use previous theorem

(ii) $(1k) = (k-1\ k)(k-2\ k-1) \dots (34)(23)(12)(23) \dots (k-1\ k)$ then use part (i)

We already saw that given an element of S_n it can be decomposed as a product of transpositions in many different ways. However the number of transpositions that occur will always be either even or odd.

Define a polynomial $P = \prod_{x_i < x_j} (x_i - x_j) = (x_1 - x_2)(x_1 - x_3) \cdots (x_{n-1} - x_n)$

If α is a permutation then $\alpha P = \prod_{x_i < x_j} (x_{\alpha(i)} - x_{\alpha(j)})$. Clearly αP is either P or $-P$. If $\alpha P = P$ then we say that the sign of the permutation α $sgn(\alpha)$ is $+1$, otherwise if $\alpha P = -P$ then $sgn(\alpha) = -1$.

One also observes that if α, β are two permutations then $sgn(\alpha\beta) = sgn(\alpha)\, sgn(\beta)$.

Since a permutation can be written as a product of transpositions the sign of a permutation is the product of the signs of the transpositions (the sign of a transposition is -1). Therefore the sign of a permutation is $+1$, if it can be written as a product of even number of transpositions or odd if it is written as a product of odd number of transpositions.

Theorem: The even permutations in S_n form a subgroup of order $n!/2$.

Proof: (i) e is an even permutation $e = (12)(12)$

(ii) If α and β are even permutations then $\alpha\beta$ is an even permutation. (sum of two even numbers is even).

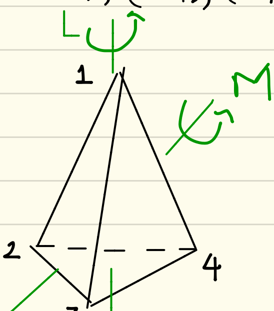
(iii) If $\alpha = (a_1 a_2)(a_3 a_4) \cdots (a_{n-1} a_n)$ is an even permutation then $\alpha^{-1} = (a_{n-1} a_n)(a_{n-2} a_{n-3}) \cdots (a_1 a_2)$ is also even.

There are exactly $n!/2$ even permutations since the mapping Φ: Even Perm. $\to$ Odd Perm given by $\Phi(\alpha) = (12)\alpha$ is bijective.

We denote the subgroup of even permutations of S_n as A_n.

For eg. subgroup A_4 of S_4 has the following elements.

$\{e, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$



Notice similarities with the group of rotational symmetries of the tetrahedron.