## An Equation of the form : $\boxed{\dfrac{dx}{dT} = -x(1-x)}$

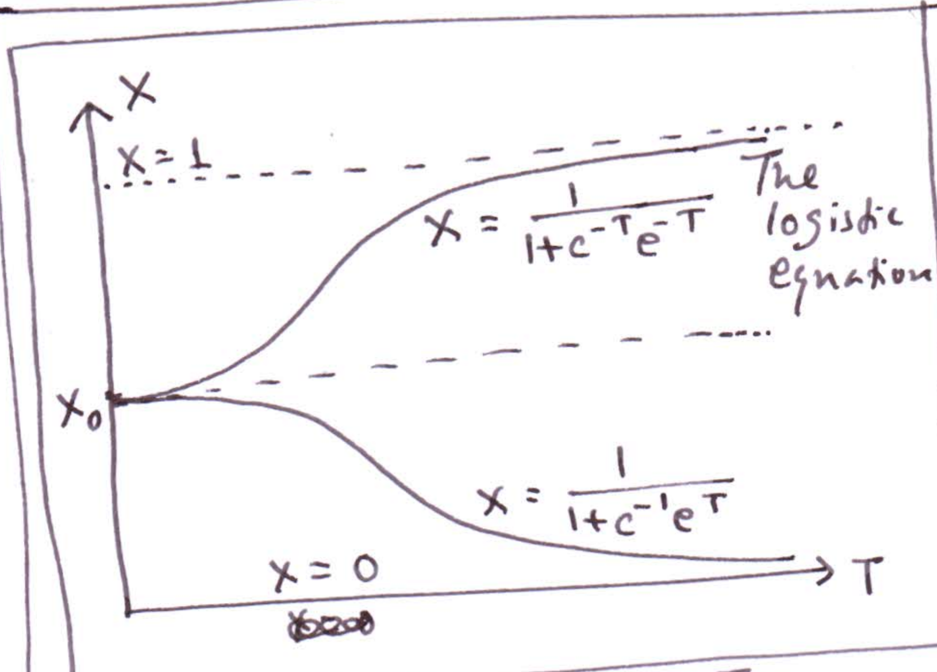
$$\Rightarrow \boxed{\frac{dx}{d(-T)} = x(1-x)}$$

This is the equivalent of $\boxed{T \longrightarrow -T}$ in $\boxed{\dfrac{dx}{dT} = x(1-x)}$.

Since the solution of the last equation is $\boxed{x = \dfrac{1}{1+c^{-1}e^{-T}}}$, we simply transform $T \longrightarrow -T$ to get $\boxed{x = \dfrac{1}{1+c^{-1}e^{T}}}$.

Hence, when $T \longrightarrow \infty$, $x \longrightarrow 0$.

At $T = 0$, $x = x_0$ (the usual initial condition)




For a temperature T, Compare with the Fermi-Dirac Distribution function.

$$\boxed{f(\epsilon) = \frac{1}{1+e^{(\epsilon-\epsilon_F)/k_B T}}}$$

$$\boxed{f(\epsilon) = \frac{1}{1+e^{-\epsilon_F/k_B T}\, e^{\epsilon/k_B T}}}$$

$\epsilon_F \rightarrow$ Parameter.

## An Equation of the form : $\boxed{\dfrac{dx}{dt} = a - bx^2}$

We write $\dfrac{1}{a}\dfrac{dx}{dt} = 1 - \dfrac{x^2}{a/b}$ and

define $\boxed{X = \dfrac{x}{\sqrt{a/b}}}$ $\Rightarrow$ $\dfrac{\sqrt{a/b}}{a}\dfrac{dX}{dt} = 1 - X^2$

Now also define $\boxed{T = \sqrt{ab}\, t}$, to get

$\boxed{\dfrac{dX}{dT} = 1 - X^2}$ $\Rightarrow \displaystyle\int \frac{dX}{(1-X)(1+X)} = \int dT$

Using the method of partial fractions,

$$\frac{1}{(1-X)(1+X)} = \frac{A}{1-X} + \frac{B}{1+X} \Rightarrow 1 = A(1+X) + B(1-X)$$

i.) When $X = 1$. $\Rightarrow 1 = A.2 \Rightarrow \boxed{A = 1/2}$.

ii.) When $X = -1$. $\Rightarrow 1 = B.2 \Rightarrow \boxed{B = 1/2}$.

$$\Rightarrow \int \frac{dX}{(1-X)(1+X)} = \frac{1}{2}\int \frac{dX}{1-X} + \frac{1}{2}\int \frac{dX}{1+X} = \int dT$$

$$\Rightarrow \int \frac{dX}{1+X} - \int \frac{d(-X)}{1+(-X)} = 2\int dT$$

$$\Rightarrow \ln(1+X) - \ln(1-X) = 2T + c$$

When $\boxed{t = 0, \text{ i.e., } T = 0}$ and $\boxed{x = 0, \text{ i.e. } X = 0}$,

$\boxed{c = 0}$ under this initial condition.

$$\Rightarrow \ln\left(\frac{1+x}{1-x}\right) = 2T = \ln e^{2T}.$$

$$\Rightarrow \frac{1+x}{1-x} = e^{2T} \Rightarrow 1+x = e^{2T} - xe^{2T}$$

$$\Rightarrow x(1+e^{2T}) = e^{2T}-1$$

$$\Rightarrow x = \frac{e^{2T}-1}{e^{2T}+1} = \frac{(e^{T}-e^{-T})/2}{(e^{T}+e^{-T})/2}$$

Now $\boxed{\sinh(T) = \frac{e^{T}-e^{-T}}{2}}$, $\boxed{\cosh(T) = \frac{e^{T}+e^{-T}}{2}}$

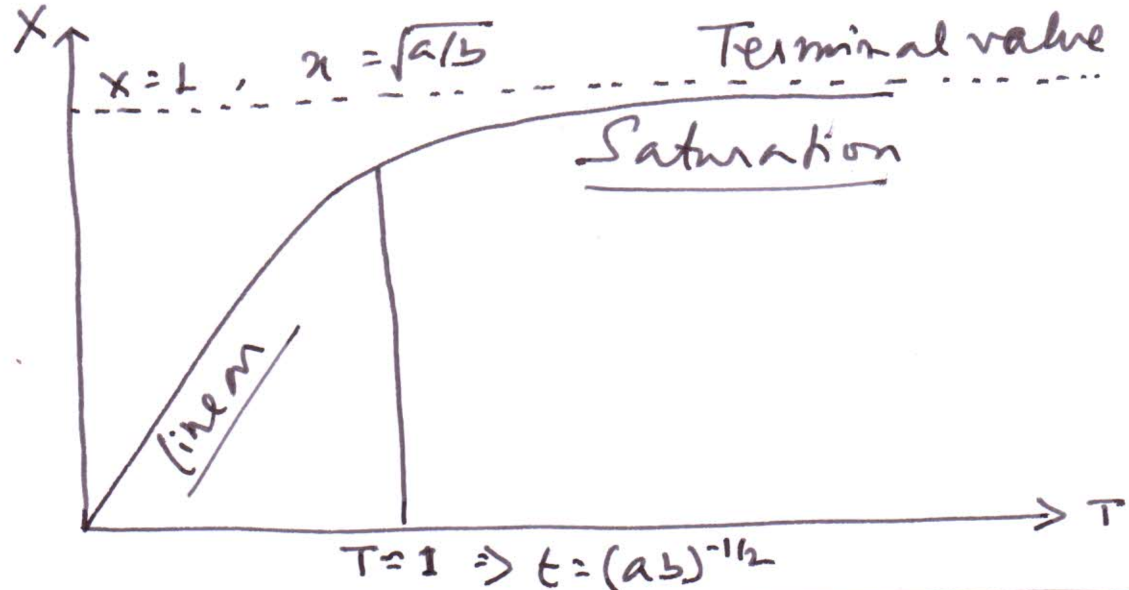
Hence $\boxed{x = \tanh(T)} \Rightarrow \boxed{x = \sqrt{\frac{a}{b}}\tanh(\sqrt{ab}\,t)}$.

i.) When $T \ll 1$, $\boxed{e^{T} \simeq 1+T}$ and $\boxed{e^{-T} \simeq 1-T}$.

$$\therefore x \simeq \frac{(1+T)-(1-T)}{(1+T)+(1-T)} \simeq \frac{2T}{2} \simeq T \quad (\underline{\text{linear}})$$

ii.) When $T \to \infty$, $x = \frac{1-e^{-2T}}{1+e^{-2T}} \longrightarrow 1$

i.e $x \longrightarrow \sqrt{a/b}$.



# Modifications of the Logistic Equation

$$\boxed{\frac{dx}{dt} = ax - bx^2 + c} \quad \text{where } a, b, c > 0$$

$$\Rightarrow \frac{dx}{dt} = -(\sqrt{b}\,x)^2 + 2\sqrt{b}\,x\,\frac{a}{2\sqrt{b}} + c + \frac{a^2}{4b} - \frac{a^2}{4b}$$

$$\Rightarrow \frac{dx}{dt} = -\left[(\sqrt{b}\,x)^2 - 2(\sqrt{b}\,x)\left(\frac{a}{2\sqrt{b}}\right) + \frac{a^2}{4b}\right] + \left(\frac{a^2}{4b} + c\right)$$

$$\Rightarrow \frac{dx}{dt} = \left(\frac{a^2}{4b} + c\right) - \left(\sqrt{b}\,x - \frac{a}{2\sqrt{b}}\right)^2$$

$$\Rightarrow \frac{dx}{dt} = \left(c + \frac{a^2}{4b}\right) - b\left(x - \frac{a}{2b}\right)^2$$

define $\boxed{\alpha^2 = \frac{a^2}{4b} + c}$ and $\boxed{y = x - \frac{a}{2b}}$,

to get, $\boxed{\frac{dy}{dt} = \alpha^2 - by^2}$ $\boxed{\text{Since, } \frac{dx}{dt} = \frac{dy}{dt}}$

$\Rightarrow \frac{1}{\alpha^2}\frac{dy}{dt} = 1 - \frac{y^2}{\alpha^2/b}$, Now define $\boxed{X = \frac{y}{\alpha/\sqrt{b}}}$

$\Rightarrow \frac{1}{\alpha^2}\cdot\frac{\alpha}{\sqrt{b}}\,\frac{dX}{dt} = 1 - X^2 \Rightarrow \boxed{\frac{dX}{dT} = 1 - X^2}$,

when $\boxed{T = \alpha\sqrt{b}\,t}$. The solution of this equation, as earlier

is $\boxed{\frac{1+X}{1-X} = Ae^{2T}} \Rightarrow \boxed{X = \frac{Ae^{2T} - 1}{Ae^{2T} + 1}}$ A is an integration constant

# Power Laws in Non-Autonomous Systems

Consider a non-autonomous equation $\boxed{\dfrac{dx}{dt} = \alpha\dfrac{x}{t}}$.

Integral Solution: $\displaystyle\int\frac{dx}{x} = \alpha\int\frac{dt}{t} \Rightarrow \boxed{\ln x = \alpha\ln t - \alpha\ln c}$

$\therefore \boxed{x = \left(\dfrac{t}{c}\right)^{\alpha}}$ when $\alpha < 0$, for $t \to \infty$, $x \to 0$ and for $t \to 0$, $x \to \infty$.

To prevent this divergence we translate $\boxed{t \to t + t_0}$.

Hence $\boxed{T = t + t_0} \Rightarrow \boxed{dT = dt}$. We write an equation as $\boxed{\dfrac{dx}{dt} = \alpha\dfrac{x}{t+t_0}}$, which we transform as $\boxed{\dfrac{dx}{dT} = \alpha\dfrac{x}{T}}$. The integral solution of this equation is $\boxed{x = \left(\dfrac{t+t_0}{c}\right)^{\alpha}}$, in which when $t \to 0$ (for $\alpha < 0$), the divergence on $x$ is contained by $\boxed{x \to (t_0/c)^{\alpha}}$.

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A Nonlinear Generalisation: Consider now $\boxed{(t+t_0)\dfrac{dx}{dt} = \alpha x - b x^{\mu+1}}$, which is a nonlinear, non-autonomous equation.

Substitute $\boxed{T = t + t_0} \Rightarrow \boxed{dT = dt}$, and $\boxed{\xi = x^{\mu}}$.

$\therefore$ We get, $\boxed{T\dfrac{dx}{dT} = \alpha x\left(1 - \dfrac{x^{\mu}}{\alpha/b}\right)}$. $\boxed{k = \dfrac{\alpha}{b}}$

Now $\dfrac{d\xi}{dT} = \mu\dfrac{x^{\mu}}{x}\dfrac{dx}{dT} \Rightarrow \boxed{\dfrac{dx}{dT} = \dfrac{x}{\mu\xi}\dfrac{d\xi}{dT}}$

$$T\frac{dx}{dT} = \frac{T\cancel{x}}{\mu\xi_s}\frac{d\xi_s}{dT} = \alpha\cancel{x}\left(1-\frac{\xi_s}{k}\right)$$

$$\Rightarrow \boxed{\frac{d\xi_s}{dT} = \alpha\mu\,\frac{\xi_s}{T}\left(1-\frac{\xi_s}{k}\right)}$$ . Now rescale $\boxed{X = \xi_s/K}$.

$$\Rightarrow \frac{d(\xi_s/k)}{dT} = \alpha\mu\,\frac{(\xi_s/k)}{T}\left(1-\frac{\xi_s}{k}\right)$$

$$\Rightarrow \boxed{\frac{dX}{dT} = \alpha\mu\,\frac{X}{T}(1-X)}$$ . We integrate this equation by the <u>method of separation of variables</u> and <u>partial fractions</u>.

$$\Rightarrow \int\frac{dX}{X(1-X)} = \alpha\mu\int\frac{dT}{T}$$ . Now $\frac{1}{X(1-X)} \equiv \frac{A}{X} + \frac{B}{1-X}$

$\Rightarrow 1 \equiv A(1-X) + BX$. Now when $X=0$, $A=1$. and when $X=1$, $B=1$.

$$\therefore \int\frac{dX}{X(1-X)} = \int\frac{dX}{X} - \int\frac{d(-X)}{1-X} = \alpha\mu\int\frac{dT}{T}$$

$$\Rightarrow \ln X - \ln(1-X) = \alpha\mu\ln T - \alpha\mu\ln c \quad \Big|\ c>0$$

$$\Rightarrow \ln\left(\frac{X}{1-X}\right) = \ln\left(\frac{T}{c}\right)^{\alpha\mu} \Rightarrow \boxed{\frac{X}{1-X} = \left(\frac{T}{c}\right)^{\alpha\mu}}.$$

$$\Rightarrow X = \left(\frac{T}{c}\right)^{\alpha\mu} - X\left(\frac{T}{c}\right)^{\alpha\mu} \Rightarrow X\left[1+\left(\frac{T}{c}\right)^{\alpha\mu}\right] = \left(\frac{T}{c}\right)^{\alpha\mu}$$

$$\Rightarrow \boxed{X = \frac{(T/c)^{\alpha\mu}}{1+(T/c)^{\alpha\mu}}} \Rightarrow \boxed{X = \frac{1}{1+(T/c)^{-\alpha\mu}}}$$

$$\boxed{X = \frac{x^M}{k}} \Rightarrow \boxed{x^M = \frac{k\,(T/c)^{\alpha\mu}}{1+(T/c)^{\alpha\mu}}}$$ in which $\boxed{T = t+t_0}$.

**Case I:** $\boxed{\mu = 1}$ and $\boxed{\alpha > 0}$ and $\boxed{t_0 = 0}$.

$$\therefore \quad \boxed{x = \frac{k(t/c)^{\alpha}}{1+(t/c)^{\alpha}}}$$

i.) When $t \to 0$, $\boxed{1 + \left(\frac{t}{c}\right)^{\alpha} \simeq 1}$

$$\Rightarrow \quad \boxed{x \simeq k(t/c)^{\alpha}}$$ for small values of $t$.

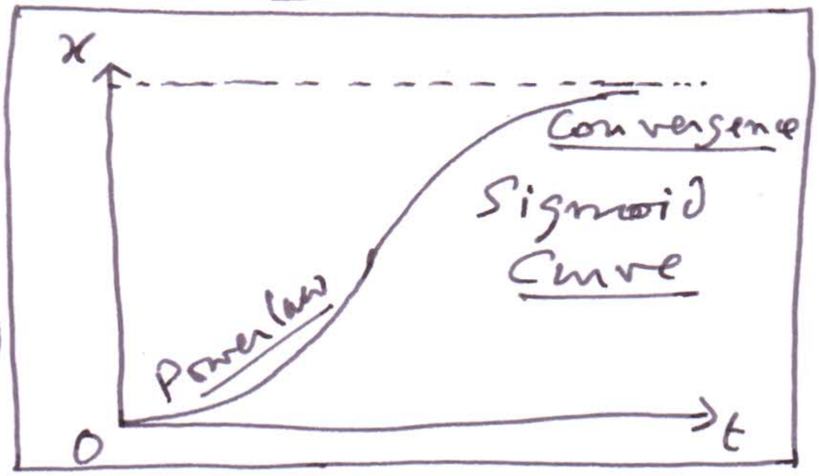
$\Rightarrow$ when $\boxed{t = 0, \ x = 0}$.

ii.) When $t \to \infty$,

$$\boxed{x = \frac{k}{1+(t/c)^{-\alpha}}}.$$

$\Rightarrow \ x \to k$ (limiting value)

($x$ starts at $x = 0$)




**Case II:** $\boxed{\mu = -1}$ and $\boxed{\alpha < 0}$ and $\boxed{t_0 \neq 0}$.

We write $\boxed{k^{-1} = \eta}$ in $\boxed{x^{\mu} = \frac{1}{k^{-1} + k^{-1}(T/c)^{-\alpha\mu}}}$

and $\boxed{\frac{1}{k} \cdot \frac{1}{c^{-\alpha\mu}} = \frac{1}{c_1^{-\alpha\mu}}}$, to get,

$$\boxed{x = \left[\frac{1}{\eta + \left(\frac{t+t_0}{c_1}\right)^{-\alpha\mu}}\right]^{1/\mu}} \Rightarrow \boxed{x = \left[\eta + \left(\frac{t+t_0}{c_1}\right)^{-\alpha\mu}\right]^{-1/\mu}}$$

When $\mu = -1$, $\boxed{x = \eta + \left(\frac{t+t_0}{c_1}\right)^{\alpha}}$

We know $\alpha < 0$. For the special case of $\boxed{\alpha = -2}$ (Zipf's law), (GEORGE KINGSLEY ZIPF)

$$\boxed{x = \eta + \left(\frac{c_1}{t+t_0}\right)^{2}}$$

When $t \to \infty$, $x \to \eta$.