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Solutions Tutorial V

Q.1 $Y = aX + b$, determine r_{xy}

$\eta_y = E[Y] = a\eta_x + b \quad \therefore Y - \eta_y = Y - b - a\eta_x$

$= ax + b - b - a\eta_x$

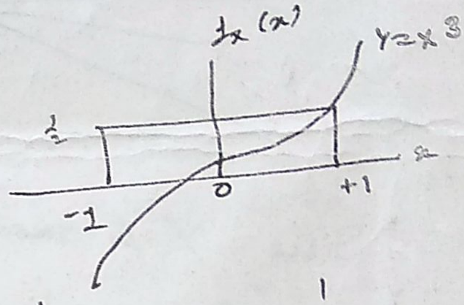
$= a(x - \eta_x)$

$\therefore \mu_{11} = E[(Y-\eta_y)(X-\eta_x)]$

$= a\,E[(X-\eta_x)^2] = a\sigma_x^2 \;; \quad \text{and } \sigma_y^2 = a^2\sigma_x^2$

$\therefore r_{xy} = \dfrac{\mu_{11}}{\sigma_x\sigma_y} = \dfrac{a\sigma_x^2}{\sigma_x \cdot a\sigma_x} = 1$

Q2 $E[X] = 0$



$\therefore \sigma_x^2 = \dfrac{2}{2}\displaystyle\int_0^1 x^2\,dx = \dfrac{x^3}{3} = \dfrac{1}{3}$

If $Y = ax + b$ clearly $b = 0$ as $\eta_y = 0$

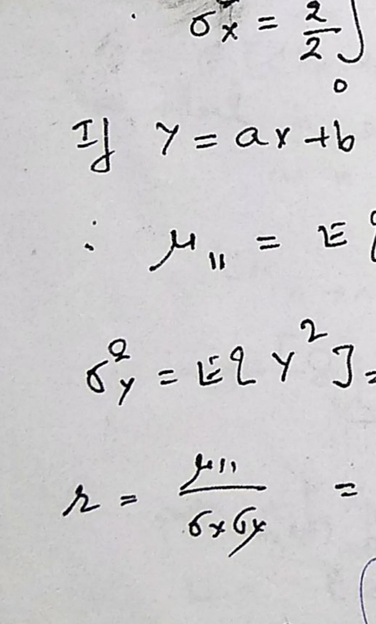
$\therefore \mu_{11} = E[x^3 \cdot x] = E[x^4] = \dfrac{2}{2}\displaystyle\int_0^1 x^4\,dx = \dfrac{x^5}{5}\Big|_0^1 = \dfrac{1}{5}$

$\sigma_y^2 = E[Y^2] = E[x^6] = \dfrac{2}{2}\displaystyle\int_0^1 x^6\,dx = \dfrac{x^7}{7} = \dfrac{1}{7}$

$r = \dfrac{\mu_{11}}{\sigma_x\sigma_y} = \dfrac{1}{5}\sqrt{3}\cdot\sqrt{7} = \dfrac{\sqrt{21}}{5}$

$a = \mu_{11}\dfrac{\sigma_y}{\sigma_x} = \dfrac{1}{5}\,\dfrac{1/\sqrt{7}}{1/\sqrt{3}} = \dfrac{1}{5}\sqrt{\dfrac{3}{7}} = \dfrac{1}{5}\,0.66 = 0.131$

If



$\eta_x = \eta_y = 0 \qquad \sigma_x^2 = 2\displaystyle\int_0^1 x^3\,dx = 2\,\dfrac{x^4}{4}\Big|_0^1 = \dfrac{1}{2}$

$r_{xy} = \dfrac{E[XY]}{\sigma_x\sigma_y} \qquad \text{But } E[XY] = 2\displaystyle\int_0^1 x^4 \cdot x\,dx = 2\,\dfrac{x^6}{6}\Big|_0^1 = \dfrac{2}{6}$

$\text{and } \sigma_y^2 = E[Y^2] = E[x^6] = 2\displaystyle\int_0^1 x^7 = 2\,\dfrac{x^8}{8}\Big|_0^1 = \dfrac{2}{8}$

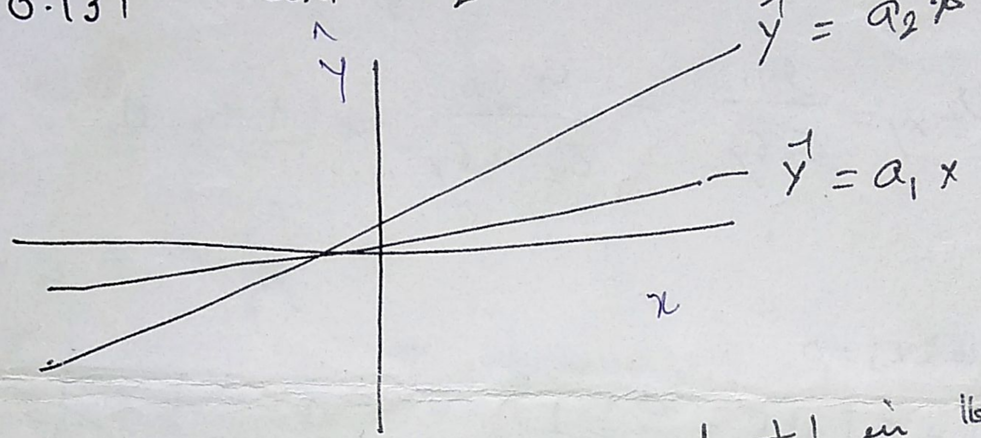
$\therefore \cancel{r_{xy} = \dfrac{2}{5}} \; 2\sqrt{2}$

$$E[XY] = \frac{1}{3}\Big|^{2\sqrt{2}} = \sqrt{8/9} = \frac{E[XY]}{\sigma_x \sigma_y} = r$$

$$\therefore \; a = r\,\sigma_y^2 = \sqrt{8/9} \cdot \frac{1}{4} = 0.236$$

Compare the slopes of the estimate in the two case

$a_1 = 0.131$ and $a_2 = 0.236$



Since X is more probable around ± 1 in the second case the estimator adjusts to become a better estimator around that region.

Q3 We have $f_{Y/X\le 0}(y) = \dfrac{P[y < Y \le y+dy,\; X \le 0]}{F_X(0)}$

numerator $= \int_{-\infty}^{0} f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{0} f_{Y/X}(y)\, f_X(x)\,dx\,dy$

By $E[Y/X\le 0] = \int y\, f_{Y/X\le 0}(y)\,dy$ $\therefore$

$= \frac{1}{F_X(0)} \int y\,dy \int f_{Y/X}(y) \int^{0} f_X(x)\,dx$ But $\int f_{Y/X}(y,x)\, y\, dy = E[Y/X]$

$E[Y/\le 0] := \frac{1}{F_X(0)} \int_{-\infty}^{0} E[Y/X]\, f_X(x)\, dx$ $\square$

Q4 We need η_Z and σ_Z^2. Then $\hat{Z} = \eta_Z$ and $MSE = \sigma_Z^2$

Since X and Y are independent - (show) $f_X(x) = f_Y(y) = 1$, $0 \le x, y \le 1$

$$E[Z] = E[X]\,E[Y] = 1/4$$

$$E[Z^2] = \int_0^1 x^2\,dx \int_0^1 y^2\,dy = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$$

$$\therefore \sigma_Z^2 = E[Z^2] - \eta_x^2 = \frac{1}{9} - \frac{1}{16} = 7/144 = MSE$$

$$\hat{Z} = 1/4$$

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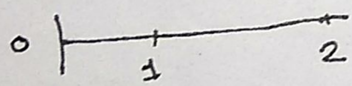
Q5 Consider $E[(X_n - a)^2]$, We have to show that $E[(X_n - a)^2] \to 0$ as $n \to \infty$

Rewrite as $E[\{(X_n - a_n) + (a_n - a)\}^2]$

$$= \underbrace{E[(X_n - a_n)^2]}_{\to 0 \text{ as } n\to\infty} + \underbrace{E[(a_n - a)^2]}_{\to 0 \text{ as } n\to\infty} + 2\underbrace{E[(X_n - a_n)(a_n - a)]}_{\to 0 \text{ as } n\to\infty}$$

$$\therefore \lim_{n\to\infty} [(X_n - a)^2] = 0$$

Example: 

Consider Interval $[0, 1+\frac{1}{n})$ $n = 1, 2, 3, \ldots$

Let $\{X_n\}$ be uniform in $[0, 1+\frac{1}{n})$

Then $E[X_n] = \frac{n+1}{2n}$

Let $\{a_n\} = \frac{n+1}{2n}$ Clearly $\lim_{n\to\infty} \{a_n\} = \frac{1}{2}$

Define $Y_m \triangleq \frac{1}{m} \sum_{m=1}^{m} X_n$ Clearly $E[(Y_n - a_n)^2] \to 0$ as $n \to \infty$

and $\{a_n\} \to \frac{1}{2}$.

Q6 $\eta_z = \eta_x + \eta_y = 0 + 1/2 = 1/2$

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2$$

$\hat{Z} = aX + b$, where $b = \eta_y - a\eta_x = \eta_y = 1/2$

$$a = \frac{\mu_{11}}{\sigma_x^2}.$$

$$\mu_{11} = E[(Z-\eta_z)X] = E[ZX]$$

$$= E[(X+Y)X] = E[X^2] = \sigma_x^2.$$

$$\therefore a = \frac{\sigma_x^2}{\sigma_x^2} = 1$$

$$\therefore \boxed{\hat{Z} = X + 1/2}$$

for $MSE = \sigma_z^2(1-r^2)$, $r = \frac{\mu_{xz}}{\sigma_x\sigma_z} = \frac{\sigma_x^2}{\sigma_x\sigma_z} = \frac{\sigma_x}{\sigma_z}$

$$r^2 = \frac{\sigma_x^2}{\sigma_z^2} = \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2} \quad \therefore 1-r^2 = 1 - \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2} = \frac{\sigma_y^2}{\sigma_x^2+\sigma_y^2} = \frac{\sigma_y^2}{\sigma_z^2}$$

$$\therefore MSE = \sigma_y^2 = \frac{1}{12} \quad \square$$