Taylor Expansion in Multiple Variables

I/. **One variable:** $\boxed{f \equiv f(x)}$ expanded about $x = x_c$.

$$\Rightarrow \boxed{f = f(x_c) + \left.\frac{df}{dx}\right|_{x_c}(x - x_c) + \frac{1}{2!}\frac{d^2 f}{dx^2}(x - x_c)^2 + \cdots}$$

II/. **Two Variables:** $\boxed{f \equiv f(x,y)}$ about (x_c, y_c).

$$\Rightarrow f = f(x_c, y_c) \longrightarrow \underline{1 \text{ zero-order term}}\ (2^0)$$

$$+ \left.\frac{\partial f}{\partial x}\right|_{x_c,y_c}(x - x_c) + \left.\frac{\partial f}{\partial y}\right|_{x_c,y_c}(y - y_c) \longrightarrow \underline{2 \text{ first-order terms}}\ (2^1)$$

$$+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c,y_c}(x - x_c)^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{x_c,y_c}(x - x_c)(y - y_c)$$

$$+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x_c,y_c}(y - y_c)(x - x_c) + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial y^2}\right|_{x_c,y_c}(y - y_c)^2 + \cdots$$

$$\longrightarrow \underline{4 \text{ second-order}} \text{ terms } (2^2)$$

III/. **Three Variables:** $\boxed{f \equiv f(x,y,z)}$ about (x_c, y_c, z_c).

$$\Rightarrow f = f(x_c, y_c, z_c) \longrightarrow \underline{1 \text{ zero-order term}}\ (3^0)$$

$$+ \left.\frac{\partial f}{\partial x}\right|_{x_c,y_c,z_c}(x - x_c) + \left.\frac{\partial f}{\partial y}\right|_{x_c,y_c,z_c}(y - y_c) + \left.\frac{\partial f}{\partial z}\right|_{x_c,y_c,z_c}(z - z_c) \longrightarrow \underline{3 \text{ first-order terms}}\ (3^1)$$

$$+ \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c,y_c,z_c}(x - x_c)^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial y^2}\right|_{x_c,y_c,z_c}(y - y_c)^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial z^2}\right|_{x_c,y_c,z_c}(z - z_c)^2$$

$$+ \frac{2}{2!}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x_c,y_c,z_c}(x - x_c)(y - y_c) + \frac{2}{2!}\left.\frac{\partial^2 f}{\partial y \partial z}\right|_{x_c,y_c,z_c}(y - y_c)(z - z_c)$$

$$+ \frac{2}{2!}\left.\frac{\partial^2 f}{\partial z \partial x}\right|_{x_c,y_c,z_c}(z - z_c)(x - x_c) + \cdots \longrightarrow \underline{9 \text{ second-order}} \text{ terms } 3^2, \text{ with } \underline{6 \text{ mixed terms}}.$$

Additional Discussions on the Spread of Industrial Innovations (E. Mansfield)

$\boxed{\lambda = f\left(p, s, \frac{x}{N}\right)}$. Following a Taylor expansion we are able to write $\boxed{\lambda = (a_4 + a_8 p + a_9 s)\frac{x}{N}}$.

In $\underline{\lambda}$, we have <u>p and s as variables</u>,

Writing $\boxed{\lambda = k\, x/N}$, where $\boxed{k = a_4 + a_8 p + a_9 s}$.

We use it in $\boxed{\frac{dx}{dt} = k\frac{x}{N}(N - x)}$. In this equation, $\boxed{k = k(p, s)}$ has <u>p and s as free parameters</u>, with their values <u>fixed at the beginning</u>.

Nonlinear Time Scale in Mansfield's Equation

Given $\boxed{x = \frac{N}{1 + (N-1)e^{-k(t-t_0)}}}$, which is the solution of the logistic equation, we set $\boxed{x = N/2}$, the scale of nonlinearity in time, $(t - t_0)|_{nl}$.

$$\therefore \frac{N}{2} = \frac{N}{1 + (N-1)e^{-k(t-t_0)|_{nl}}} \Rightarrow 2 = 1 + (N-1)e^{-k(t-t_0)|_{nl}}$$

$$\Rightarrow (N-1)e^{-k(t-t_0)|_{nl}} = 1 \Rightarrow \boxed{(N-1) = e^{k(t-t_0)|_{nl}}}.$$

$$\therefore k(t-t_0)|_{nl} = \ln(N-1) \Rightarrow \boxed{(t-t_0)|_{nl} = \frac{1}{k}\ln(N-1)}.$$

The nonlinear time.