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CS374: Practice Sheet 5

Prob 1) Write the Newton and Lagrange forms of the interpolating polynomial for the following data:

a.)

x	1	2	0	3
y	3	2	-4	5

b.)

x	1	3	2	6
y	-2	-22	-1	-37

c.)

x	-2	0	1
y	0	1	-1

Write both polynomials in the form $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ to verify that they are identical as functions.

Prob 2) The polynomial p of degree $\leq n$ that interpolates a given function f at $n+1$ prescribed nodes is uniquely defined. Hence, there is a mapping $f \mapsto p$. Denote this mapping by L and show that

$$Lf = \sum_{i=0}^{n} f(x_i)l_i.$$

Further, show that L is **linear**; that is, $L(af + bg) = aL(f) + bL(g)$.

Prob 3) Prove that the mapping L, in Problem 2, has the property that $Lq = q$ for every polynomial of degree at most n.

Prob 4) Prove that $\sum_{i=0}^{n} l_i(x) = 1$ for all x.

Prob 5) Prove that if g interpolates the function f at $x_0, x_1, ..., x_{n-1}$ and if h interpolates f at $x_1, x_2, ..., x_n$, then the function

$$g(x) + \frac{x_0 - x}{x_n - x_0}[g(x) - h(x)]$$

interpolates f at $x_0, x_1, ..., x_n$. Notice that h and g need not be polynomials.

Prob 6) Prove that if we take any set of 23 nodes in the interval $[-1, 1]$ and interpolate the function $f(x) = \cosh x$ with a polynomial p of degree 22, then the relative error

$$\frac{|\ p(x) - f(x)\ |}{|\ f(x)\ |} < 5 \times 10^{-6} \quad \text{on } [-1, 1].$$

Prob 7) Compute a divided difference table and then write the Newton interpolating polynomial for the function values prescribed in Problem 1.

Prob 8) Let $f \in C^n[a, b]$. Prove that if $x_0 \in (a, b)$ and if $x_1, x_2, ..., x_n$ all converge to x_0, then $f[x_0, x_1, ..., x_n]$ will converge to $\frac{f^{(n)}(x_0)}{n!}$.

Prob 9) Prove that if f is a polynomial of degree k, then for $n > k$,

$$f[x_0, x_1, ..., x_n] = 0.$$

Prob 10) Prove the following formula:

$$f[x_0, x_1, ..., x_n] = \sum_{i=0}^{n} f(x_i)\Pi^{n}_{j=0,j\neq i}(x_i - x_j)^{-1}.$$

Prob 11) The polynomial $p(x) = 2 - (x+1) + x(x+1) - 2x(x+1)(x-1)$ interpolates the first four points in the table:

x	-1	0	1	2	3
y	2	1	2	-7	10

By adding one additional term to p, find a polynomial that interpolates the whole table.

Prob 12) Prove that for $h > 0$,

$$f(x+2h) - 2f(x+h) + f(x) = h^2 f''(\xi)$$

for some ξ in the interval $(x, x+2h)$.

Lab Exercises

Ex 1) Consider these five data points: (0,8), (1,12), (3,2), (4,6) and (8,0).

(i) Construct and plot the interpolation polynomial using the two outermost points.

(ii) Repeat this process by adding one additional point at a time until all the points are included.

(iii) What conclusions can you draw?

Ex 2) Consider the equation $x - 9^{-x} = 0$. The equation has a root in an interval $[0, 1]$. Construct the Lagrange's and Newton's forms of interpolating polynomial using the equally spaced nodes with step size 0.1 and plot the same. By setting the interpolating polynomials equal to 0 and solving the equation, find an approximate solution to the equation through root finding method.

Ex 3) **(Runge Phenomenon)** Consider the function

$$f(x) = \frac{1}{1+x^2}$$

on the interval $[-5, 5]$. For $n = 5$, 10 and 15, find the Newton interpolating polynomial p_n for this function using equally spaced nodes. In each case, compute $f - p_n$ for 30 equally spaced nodes in $[-5, 5]$? Do experiment on computer to see this behavior and conclude.