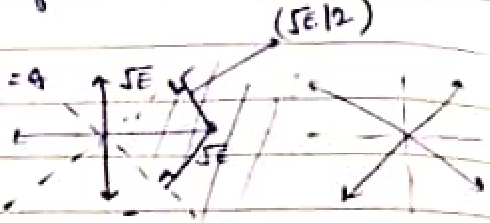
M-ary signalling

1. M-ary PSK

$M = 2^k$ — k is the no of bits encoded per waveform

Binary $M=2$

$M=4$



min noise power needed to cause error

ⓧ Add a vector so that it goes to shaded region.
min len. $= \sqrt{E}$ to touch the decision boundary

$\min(\text{Power}) = (\sqrt{E})^2 = E$

min noise power when $M=4$ is $\frac{E}{2}$

project (orthogonally) $\sqrt{E}$ on the perpendicular bisector.

$d(s_i, s_j) \downarrow\downarrow$ as $M\uparrow\uparrow$
noise power goes down
∴ more errors
but bandwidth needed is less i.e. it is same

$$\boxed{\text{As } M\uparrow\uparrow \quad P_B \uparrow\uparrow}$$

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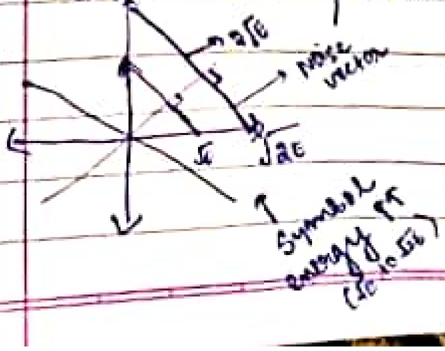
Performance is degrading then why do we ↑ M ?

[Bit Rate Goes up
Every waveform encodes more than 1 bit.

[Same bandwidth as M ↑↑ but data rate goes up
but also probability of bit error goes up.



Doubling the energy i.e.

$(\sqrt{2E})^2 + (\sqrt{2E})^2 = 2E + 2E = 4E = E_b^2$

$E_b = 2\sqrt{E}$ half of it $= \sqrt{E}$

same frequency ⇒ same bandwidth (PSK)

Large M, Large errors, same Bandwidth (PSK)

In PSK if we ↑↑ Es, $d(s_i, s_j)$ b/w vectors goes up, noise power goes up ∴ less errors.

If distance b/w signals ↑ P_B ↓
M increases in OFSK 2 orthogonal unit vectors distance is always $\sqrt{2}$ P_B same.

M-ary OFSK

P_B is majorly dependent on the distance b/w the signal vectors.

Claim: As M↑↑ P_B↓↓ Why?

[As M↑↑ the bit energy goes down.
Bit energy = Symbol energy / bits

If we are comparing 2 different M, we should compare same $\frac{E_b}{N_0}$, then P_B will go down.

[Comparing symbol energy $P_{\text{symbol error}}$ go down as M ↑↑
min noise power to cause error still remains same.
~~symbol error ↑↑ as M↑↑ but P_S ↓~~

M = 2

1 → 2

2 → 1

M = 4

1 → 2

1 → 3

1 → 4

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$\boxed{M\uparrow\uparrow \quad P_B \downarrow\downarrow \quad \text{same } E_b/N_0}$

As M goes up symbol error probability goes up.
M↑↑ P_S↑↑

$$d(s_i, s_j) = k \quad i \neq j$$

the amount of noise to be added is same for misclassification.
As M↑↑, there are more no of classes, the distance b/w 2 classes is same. The same noise vector can misclassify even if m is increased.

$E_b/N_0 \downarrow\downarrow$ Large error

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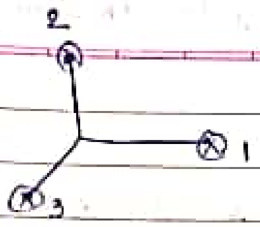


M = 2
1 − 2
2 − 1

M = 3
1 → 2
1 → 3

$d(1,2) = d(1,3) = d(2,3)$

The ways to introduce misclassification with same noise power increases as M increases when looking at same ~~Eb/No~~ ratio.
SNR

that is $P_s \uparrow\uparrow$

Each symbol is encoded by k bits
M different symbols, $k = \log_2 M$ bits, every symbol (duration T),
One with high value of M needs to have a higher value of K for keeping E_b/N_0 same, each symbol energy encodeds for K bits,

$$E_s \longrightarrow K E_{bits}$$

$$E_{bit} \rightarrow \frac{E_s}{K}$$

As $K\uparrow\uparrow$ bit energy $\downarrow\downarrow$

$$\boxed{K\uparrow\uparrow \quad E_b\downarrow\downarrow \quad E_s \text{ (constant)}}$$

∴ comparing @ same E_b/N_0, we get an increase in symbol energy, $d(s_i, s_j)$ between symbols dependent on symbol energy goes up, ∴ noise power needed for wrong classification $\uparrow\uparrow$, and hence error (bit) $\downarrow\downarrow$ as $M\uparrow\uparrow$

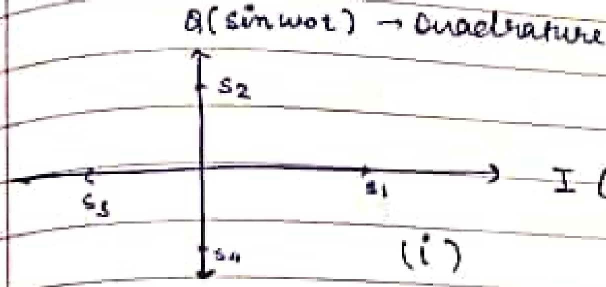
Large value of M gives better performance, that is lower error but higher bandwidth.

M waveforms
$w_{i+1} - w_i = \pi/T$

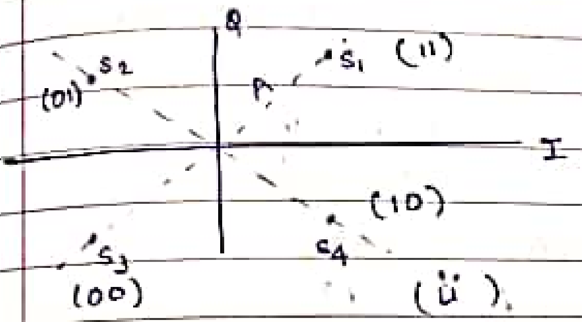
In QPSK

$$S_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_0 t + \frac{\pi}{2}(i-1)\right) \qquad 0 \le t \le T$$

$i = 1, 2, 3, 4.$



(i) Constellation Diagram

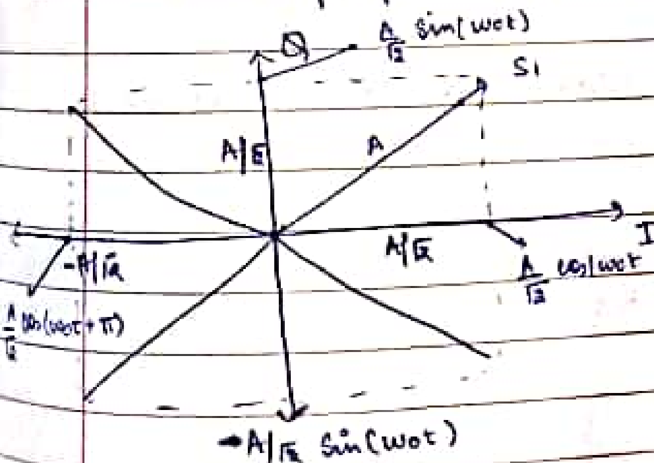


(ii).

They both are equivalent as everything depends on distance & the energy

(ii) $S_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_0 t + \frac{\pi}{4} + \frac{\pi}{2}(i-1)\right)$

$S_1(t)$ projection on I and Q axis will be $A/\sqrt{2}$



$S_1 \quad \frac{A}{\sqrt{2}}$ (I axis) $\quad \frac{A}{\sqrt{2}}$ (Q axis)

$S_2 \quad -\frac{A}{\sqrt{2}}$ (I axis) $\quad \frac{A}{\sqrt{2}}$ (Q axis)

{ only looking @ I axis we say that it is BPSK $0 \to -ve$, $1 \to +ve$

Basically QPSK can be looked as 2 BPSK modulations, there is no interference b/w the 2 BPSK schemes.

Implement a QPSK as 2 BPSK.

-1 1   1 $\boxed{h_T}$ = pulse shaping root raised cosine

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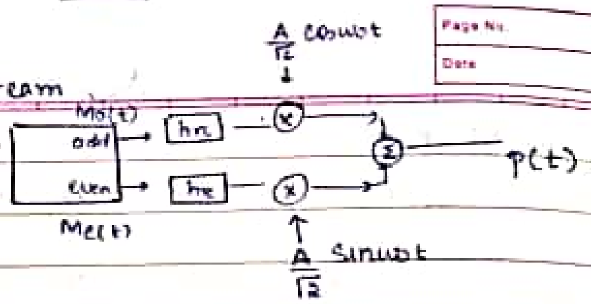
Divide into even odd stream
-1 -1   1 -1

Modulator $\tilde{m}(k) \rightarrow m(k) \rightarrow$
$\{0,1\} \rightarrow \{-1,1\}$



Top arm 1 BPSK
Bottom arm 1 BPSK   <implement demodulator > <HW>

$\frac{E_{QPSK1}}{N_0}$ where you transmit S1, S2, S3, S4

$\frac{E_{QPSK2}}{N_0}$ where QPSK implemented as 2 BPSK schemes.

$$\frac{E_{QPSK2}}{N_0} = \frac{S_{QPSK2} \times T}{N/W} = \frac{S_{QPSK2}}{N}\left(\frac{W}{R}\right)$$

In QPSK1 signal power $(A)^2 = A^2$
In QPSK2 signal power $\left(\frac{A}{\sqrt{2}}\right)^2 = \frac{A^2}{2}$

$$S_{QPSK2} = \frac{S_{QPSK1}}{2}$$

$$\frac{E_{QPSK2}}{N_0} = \frac{S_{QPSK1}}{2N}\left(\frac{W}{R}\right)$$

even stream / odd stream arriving at half the rate of original stream.

$$\frac{E_{QPSK2}}{N_0} = \frac{S_{QPSK1}}{2N}\left(\frac{2W}{R_{QPSK1}}\right)$$
$$= \frac{E_{QPSK1}}{N_0}$$

we are going to do coherent detection

$P_B(QPSK2) = P_B(QPSK1) = P_B(BPSK)$ @ the corresponding E_b/N_0 ratio.

Just for M=2 & M=4, the fact that M↑ P_B↑ is not true because we reduced the rate by 2.
when

000 001
010 011
100 101
110 111

M-ary FSK

$$S_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t) \qquad 0 \le t \le T, \quad i = 1 \ldots M$$

$$d(S_i, S_j) = k, \quad i \ne j$$

How many ways are there to misclassify?
which of them is more likely?

- Distance b/w symbols are same ∴ all misclassifications are equally likely.

There are $M-1$ ways to misclassify a certain symbol

Let us say you misclassify a symbol (k bits).
now how many bits were misclassified.
= Every symbol error, means all bits were wrong.

$M = 2^k$

One symbol can be misclassified $2^k - 1$ ways.

0000 … 1111

{ 000$\boxed{0}$ ⟶ 0106
{ 2 bit errors -

$\boxed{0}$ bit misclassification → by 0.5 probability
If I take any of these bits → half of the bits give correct detection & half of them give incorrect.

∴ Incorrect / misclassify $= \frac{2^k}{2}$

$$\boxed{\frac{P_B}{P_S} = \frac{2^{k-1}}{2^k - 1}}$$

$< \frac{1}{2} >$ min

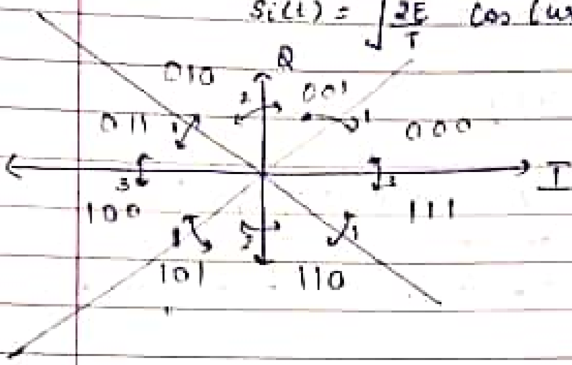
Relating P_B with P_S is assumed if the misclassification has already occured.
Every symbol error gives a bit error.

$k=2$ 00 01 11 10. $\quad \frac{2^{2-1}}{2^2-1} = \frac{2}{3}$

2/3

M-ary FSK

$$S_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_0 t + \frac{2\pi}{M}(i-1)\right) \quad 0 \le t \le T$$
$$i = 1 \cdots M$$



Higher probability of misclassification in close by class (Adjacent)

Assuming P_s is small.
If misclassification occurs only over adjacent -
Given that symbols are misclassified, then how much does symbol error contributes to bit error
Instead of above encoding scheme, as there are a lot of bit errors we can use Gray Code

000
001
011
010
110
111
101
100

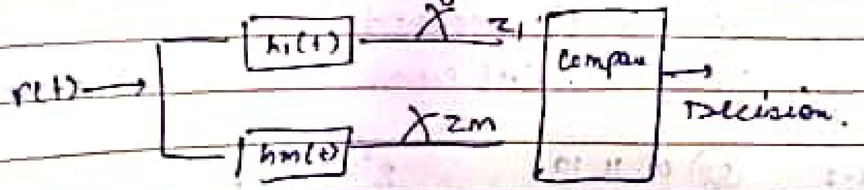
Anticlockwise.

$$\boxed{\frac{P_b}{P_s}(\text{Gray}) = \frac{1}{K}}$$

← Once the symbol error occurred how many bits were misclassified.

Symbol error rate (P_s)

1. FSK - coherent demodulator - using Matched filter detection

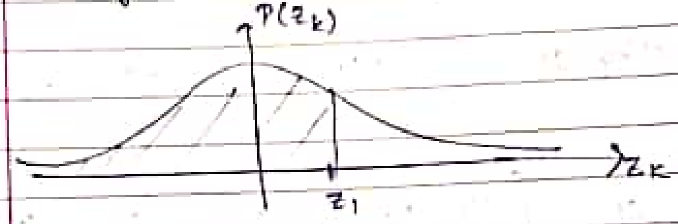


$Z_K \sim N(0, \sigma_0)$ $K \neq 1$ if s_1 was tx.

$$p(Z_1 > Z_K / Z_1) = \int_{-\infty}^{Z_1} N_{Z_K}(0, \sigma_0)\, dZ_K$$

= Integrate out Z_K which have less value than Z_1

For a given Z_1 what is the Probability $Z_K < Z_1$ integrate out the pdf of Z_K.



$Z_1 \sim N(\sqrt{E}, \sigma_0)$ $p(Z_1) =$

Evaluate $P_S = 1 - P_C$

closed form solution does not exist.

Upper bounds for P_S

M waveforms $d(s_i, s_j) = K$ $i \neq j$

all misclassification equally likely

$\Downarrow$

$\exists\, k \in (2, \ldots m)$ s.t $Z_K > Z_1$

E_K : event for $Z_K > Z_1$, $K = 2 \ldots m$

misclassification: any one of these $m-1$ events occur

$1 - (\text{none } Z_K > Z_1)$

$P_S = P$(one of E_K occured) @ least one of E_K =

$$P\left(\bigcup_{K=2}^{M} E_K\right) \leq \sum_{K=2}^{M} P(E_K)$$

Z_2 & Z_3 both can be > than Z_1

any event in union of $E_K \Rightarrow$ one of the event occur

$0 \le t \le T$
$i = 1 \dots M$
probability
classification
close by classes
adjacent)

ent -
n how
o but error
o there
Gray Code

rror
bits

m.

Assume one is Tx, it gets misclassify into M-1 classes

Assume S_1 was transmitted

$$P_{z_1}(z_1) \sim N(\sqrt{E}, \sigma_0)$$

$$P_{z_k}(z_k) \sim N(0, \sigma_0) \quad , k \neq 1 \quad k = 2 \dots M$$

$$P(\text{correct detecting } s_1) = P(z_1 > z_2, z_1 > z_3 \dots z_1 > z_m / z_1) \times P(z_1)$$

$$P(\text{correct} / s_1) = \int_{-\infty}^{+\infty} P(z_1 > z_2 \dots z_1 > z_m / z_1) \cdot P(z_1)\, dz_1$$

$$p(z_1 > z_2 \dots z_1 > z_m / z_1) = \prod_{k=2}^{M} P(z_1 > z_k / z_1)$$

07/11/19 Symbol error rates (P_s)

1. MFSK (coherent demodulation)

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t) \quad 0 \le t \le T \quad i = 1 \dots M$$

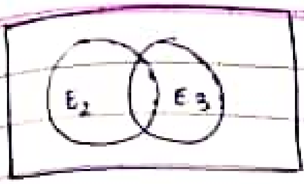
o/p of M correlators $\rightarrow z_1, \dots z_m$

- Assume S_1 was transmitted

(a) - $P_s = 1 - P_c$

(b) - $$P_c = \int_{-\infty}^{+\infty} P(z_1 > z_2, z \dots z_1 > z_m / z_1) P(z_1)\, dz_1$$

$$P(z_1 > z_2 \dots z_1 > z_m / z_1) = \prod_{k=2}^{M} p(z_1 > z_k / z_1).$$



$P(E_2 \cup E_3) = P(E_2) + P(E_3) - P(E_2 \cap E_3)$

$P(E_k)$ = each event is worried about s_k & s_1 so what is the probability that E_k happens. it is exactly same as Binary FSK.

$$P(E_k) = Q\left(\sqrt{\frac{E}{N_0}}\right)$$

$$\sum_{k=2}^{M} P(E_k) = (M-1)\, Q\left(\sqrt{\frac{E}{N_0}}\right)$$

E – symbol energy

$\underline{E_s = k E_b}$

$$\therefore \; P_s \le (M-1)\, Q\left(\sqrt{\frac{kE_b}{N_0}}\right).$$

$k \to \infty \quad P_s \to 0$

$$P_s < M Q\left(\sqrt{\frac{kE_b}{N_0}}\right)$$

$$P_s < M e^{-\frac{kE_b}{2N_0}}$$

$$P_s < 2^k e^{-\frac{kE_b}{2N_0}}$$

$$Q(x) \le e^{-x^2/2}$$

$$P_s < e^{k\left(\ln 2 - \frac{E_b}{2N_0}\right)}$$

$$\boxed{P_s < e^{\frac{k}{2}\left[2\ln 2 - \frac{E_b}{N_0}\right]}}$$

we want : $\frac{E_b}{N_0} > 2\ln 2$

- as $k\uparrow$ error will goup if $2\ln 2 > \frac{E_b}{N_0}$

But by $\uparrow k$ $P_s \downarrow$.

$k \to \infty$ $P_s \downarrow 0$ $\qquad \frac{E_b}{N_0} > 1.386$

Tighter upper bound

$$P_s \sim e^{-K\left(\sqrt{\frac{E_b}{N_0}} - \sqrt{\ln 2}\right)^2}$$

expression is valid only if $\sqrt{\frac{E_b}{N_0}} > \sqrt{\ln 2}$

$$\boxed{\frac{E_b}{N_0} > \ln 2}$$

Shannon's limit

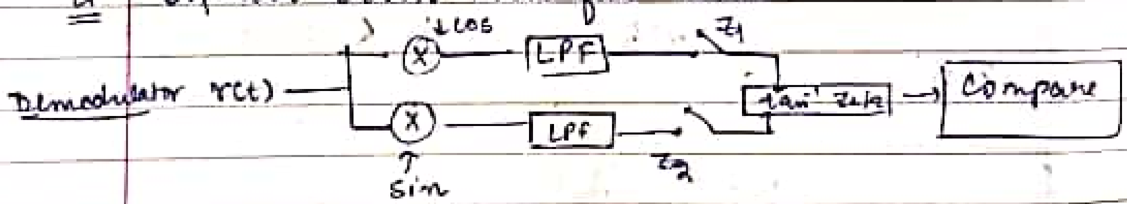
Greater noise power compared to bit energy $\leftarrow \left\{ \frac{E_b}{N_0} > 0.69 \right.$

? $K\uparrow$ $P_s\downarrow$ (upper bound) (In theory)

If $\frac{E_b}{N_0} > 0.69$ $K \to \infty$ $P_s \downarrow 0$

If you increase SNR increasing M is going to decrease P_s.

2. symbol error rate for MPSK



S_1 was transmitted

~~$z_1 \sim N(\sqrt{E}, \sigma_0)$~~

$$p(z_1, z_2) \longrightarrow p(m, \phi)$$

$$m = \sqrt{z_1^2 + z_2^2}$$

$$\phi = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

for BPSK dec. boundary $-\pi/2$ to $\pi/2$
QPSK $-\pi/4$ to $\pi/4$

$$P_s = 1 - \int_{-\pi/M}^{\pi/M} \int_{0}^{\infty} P(m, \phi)\, dm\, d\phi$$

$$L(\hat{s}) = \| \bar{r} - \bar{h}s \|^2$$

$$= \bar{r}^T r - 2r^T \bar{h} s + \bar{h}^T h s^2$$

let us somehow assume that s^2 remain const
BPSK constellation dist(s) same.

What I need?

$$\min (L(\hat{s})) \quad \text{or} \quad \max (r^T \bar{h})$$

max projection / or
correlation / or
Matched filter receiver

What if I

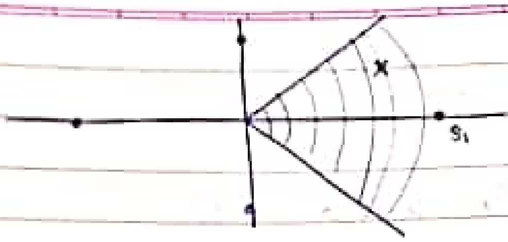
$$P(\hat{s}/r) = \alpha\, \underbrace{e^{-L(\hat{s}).K}}_{\text{Gaussian PDF}}$$

{Multivariate as s is a vector}

Conditional PDF / likelihood

To minimize the cost; you maximize the likelihood.

Max. likelihood estimate (MLE) of parameter symbol vector.



$\{ d(x, s_1)$ is minimum euclidian

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M-PSK

$T \rightarrow$ Symbol duration

$$R_s = \frac{1}{T}$$

$$W_{min} = R_s = \frac{R_b}{\log_2 M}$$

R_b - bits/sec

$$\text{Bandwidth efficiency} : \frac{R_b}{W} = \text{bits/sec/Hz}$$

$$= \log_2 M$$

For every Hz of channel bandwidth you are able to transmit R_b bits

The idea to transfer more & more bits

As you $\uparrow$ M, channel bandwidth efficiency $\uparrow$

M-FSK

coherent $W = \frac{(M-1)}{2T} \approx \frac{M}{2T}$

$\frac{1}{2T}$ comes from seperation

$$\approx \frac{M \cdot R_s}{2}$$

$$R_s = \frac{1}{T}$$

The aim is to compute $\frac{R_b}{W}$

$$W \approx \frac{M \cdot R_b}{2\log_2 M}$$

$$\frac{R_b}{W} = \frac{2\log_2 M}{M}$$

As M ↑↑ the bandwidth efficiency decreases.

Q Suppose we want to maintain same P_b (in both case) what effects occur in MPSK & MFSK

MPSK :- As M↑↑ ; ↑ Bandwidth efficiency

If M↑↑ P_b ↑ maintain same Eb/No ratio
but inorder to maintain constant P_b we will increase $\frac{E_b}{N_0}$ (Power ↑).

MFSK :- As M↑↑ ; ↓ Bandwidth efficiency
↓ $\frac{E_b}{N_0}$ ratio

If M↑↑ P_b ↓ maintain same Eb/No ratio

If Bandwidth is a constraint you use MPSK ; i.e
if B/W is large/(i.e efficiency is high)

You need to choose one of the 2
trade off b/w Bandwidth efficiency & $\frac{E_b}{N_0}$ (Power ratio).

Shannon - Hartley Capacity Theorem.

AWGN channel
Max. data rate that channel can support i.e demodulation errors are ignorable (or are very very small), theoretically?

It gives an upper bound to the idea of max data rate.
upper bound on data rate = channel capacity = C

$$\boxed{C = W \cdot \log_2\left(1 + \frac{S}{N}\right)}$$ upper bound

$\frac{S}{N}$: Signal to noise ratio (SNR)

: Avg SNR recieved over the W Hz

W : Bandwidth of the channel

C : Max. Bit Rate

Shannon believed there exists some scheme of demodulation with $\left(W \text{ \& } \frac{S}{N}\right)$ as parameters.

In terms of $\frac{E_b}{N_0}$ ratio

$$C = W \cdot \log_2\left(1 + \frac{SR}{N/w}\right)$$

$$\boxed{\frac{E_b}{N_0} = \frac{SW}{NR}}$$

Let us assume that channel rate & Bit rate are same. (C = R)

$$\frac{E_b}{N_0}\frac{R}{W} = \frac{S}{N}$$

$$\frac{R}{W} = \log_2\left(1 + \frac{E_b}{N_0}\left(\frac{R}{W}\right)\right)$$

$$\boxed{\frac{E_b}{N_0} = \frac{2^{R/W} - 1}{R/W}}$$ upper bound.

Semester Recap

(a) Analog = Infinite symbols
Digital - finite symbols

(b) Stochastic processes
- Generating noise
- random bits generation

Stationarity Ergodicity

Analog To Digital
(c) Quantization : uniform / non-uniform

Analog To Digital
(d) Sampling : Anti-aliasing filter

(e) Mapping Digital to analog waveforms ie modulation
- Baseband
- Bandpass :- Antenna
:- Multiplexing

(f) Bandpass demodulation
- Matched Filter (Maximize SNR)
- Detection :- MLE
:- MAP
- Performance Criteria : Pb
Eb/No
ISI : Nyquist pulse shaping & Eye diagram
- RC, RRC
- Equalization filters : Zero forcing
MMSE

(g) Bandpass Mod / Demod
- PSK, FSK, ASK, APK
- PSK / FSK demodulation :- coherent
:- non coherent
- DPSK

What if $\frac{R}{W} \rightarrow 0$ Bandwidth ↑↑ (FSK)

$$\frac{E_b}{N_0} = \ln 2 \quad \text{Shannon's limit}$$

Increasing Bandwidth ⇒ Increasing M in FSK

- Quadrature Mod / Demod (non coherent)
- Error performance P_b & P_s
  :- what happens as M ↑
- Bandwidth efficiency