1. Find the capacitance per unit length of two coaxial metal cylindrical tubes of radii a and b.

   soln
   Let us assume $a < b$. To find capacitance we place equal and opposite charges on the cylinders. Since the cylinders are of infinite length we consider charge per unit length rather than total charge. Let λ be the charge per unit length on the inner cylinder and $-\lambda$ be the charge per unit length over the outer cylinder. The electric field in between the cylinders is radially outward and is given as $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$. The potential difference between the cylinders is

$$\begin{aligned} V_b - V_a &= -\int_a^b \vec{E}\cdot d\vec{l} = -\int_a^b \frac{\lambda}{2\pi\epsilon_0 s} ds \\ &= -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{a}{b}\right) \end{aligned}$$

   Per unit length the capacitance is

$$C = \frac{\lambda}{V_b - V_a} = \frac{2\pi\epsilon_0}{\ln(a/b)}$$

2. A chargeless region is bounded by two conducting surfaces.

   (a) If a charge Q_1 is placed on conductor 1 while 2 is chargeless the potential in the region is given by the function $\Phi_1(x, y, z)$. If a charge Q_2 is placed on conductor 2 while 1 is chargeless the potential in the region is given by the function $\Phi_2(x, y, z)$. Now if charge Q_1 is placed on conductor 1 and charge Q_2 is placed on 2 prove that the potential in the region will be given by the function $\Phi = \Phi_1 + \Phi_2$.

   soln
   Both Φ_1 and Φ_2 makes the surfaces of the two conductors S_1 and S_2 equipotential. Hence the potential function $\Phi = \Phi_1 + \Phi_2$ also makes the two surfaces equipotential. This is a necessary boundary condition for any solution of the electrostatic problem. Now if this function also gives the given charges on the two conductors then this will satisfy all the required physical boundary conditions of the problem. For this we need the electric field.
   Let $\vec{E}_1 = -\vec{\nabla}\Phi_1$. Then over the surfaces S_1 and S_2 of conductor 1 and 2 we have

$$\oint_{S_1} \vec{E}_1 \cdot \hat{n} da = Q_1/\epsilon_0 \quad \text{and} \quad \oint_{S_2} \vec{E}_1 \cdot \hat{n} da = 0$$

   Similarly for $\vec{E}_2 = -\vec{\nabla}\Phi_2$ we have

$$\oint_{S_1} \vec{E}_2 \cdot \hat{n} da = 0 \quad \text{and} \quad \oint_{S_2} \vec{E}_2 \cdot \hat{n} da = Q_2/\epsilon_0$$

Let $\vec{E} = \vec{E}_1 + \vec{E}_2 = -\vec{\nabla}\Phi$. From the above eqns. we can see that

$$\oint_{S_1} \vec{E} \cdot \hat{n} da = Q_1/\epsilon_0 \quad \text{and} \quad \oint_{S_2} \vec{E} \cdot \hat{n} da = Q_2/\epsilon_0$$

So the electric field caused by the potential Φ satisfies the required boundary conditions. Hence $\Phi = \Phi_1 + \Phi_2$ satisfies all the required physical conditions, i.e, it makes the conducting surfaces equipotential and gives the correct charges over the conductors. The uniqueness theorem suggests that such solutions are unique and hence nature adopts this potential function in the region bounded by the conductors. Hence we have found the required solution for the problem.

(b) If conductor 1 is maintained at potential V_1 and 2 is grounded the potential in the region is given by the function $\Phi_1(x, y, z)$. If conductor 2 is maintained at potential V_2 and 1 is grounded the potential in the region is given by the function $\Phi_2(x, y, z)$.
Now if conductor 1 is maintained at potential V_1 and conductor 2 is maintained at potential V_2 prove that the potential in the region will be given by the function $\Phi = \Phi_1 + \Phi_2$.

soln:
In the chargeless region

$$\nabla^2\Phi = \nabla^2\Phi_1 + \nabla^2\Phi_2 = 0 + 0 = 0$$

So $\Phi = \Phi_1 + \Phi_2$ satisfies the Laplace's equation in the region. We have to verify whether Φ matches the boundary condition, i.e it matches the potential on the two conductors.
Over conductor 1, $\Phi_1 = V_1, \Phi_2 = 0$. So $\Phi = V_1$. Over conductor 2, $\Phi_1 = 0, \Phi_2 = V_2$. So $\Phi = V_2$. So this potential satisfies the potentials at the two bounding surfaces. Hence $\Phi = \Phi_1 + \Phi_2$ is a solution to the given electrostatic problem.
Here we assume that the potential at ∞ is 0.

This result can be extended to a region bounded by any number of conductors.

3. A conducting sphere of radius a is concentrically surrounded by another conducting spherical shell of radius b.

(a) A charge Q is placed on the inner conducting sphere. What will be the potential over the outer sphere.

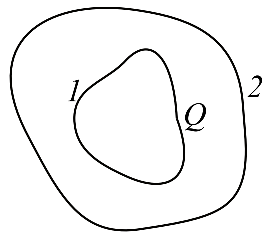
soln:
For points outside the inner sphere the potential is $\frac{Q}{4\pi\epsilon_0 r}$. So the potential over the outer sphere will be $\frac{Q}{4\pi\epsilon_0 b}$.

(b) Instead if the charge Q is placed over the outer shell, what will be the potential of the inner sphere?

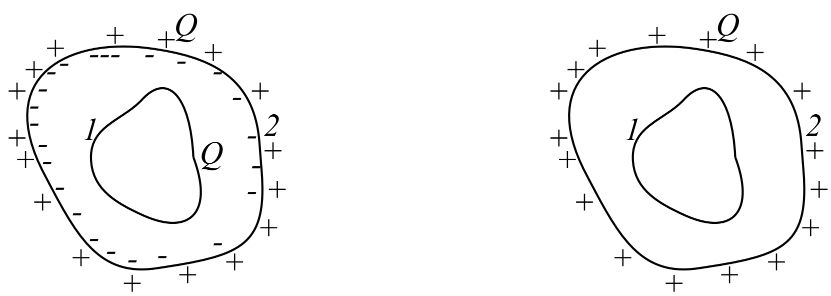
soln:

If the charge is placed on the outer sphere the potential of the outer sphere will be $\frac{Q}{4\pi\epsilon_0 b}$. Since there is no charge within the outer sphere, the electric field within the sphere everywhere will be 0 and the potential will be constant. This potential will be same as that of the outer sphere and hence it will be $\frac{Q}{4\pi\epsilon_0 b}$.

(c) How will your answer change if the shapes of the conductors were not spherical but arbitrary.



soln:



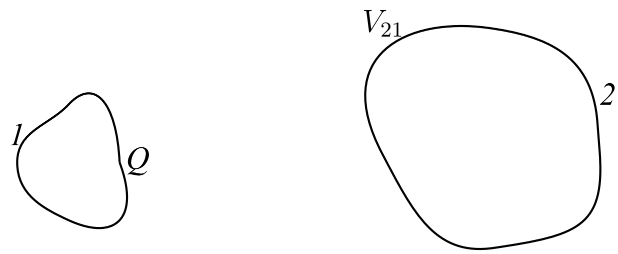
Configuration 1 Configuration 2

In part (a) and (b) we see that whether the charge Q is placed on the inner or the outer sphere, the potential on the other sphere happens to be the same viz. $\frac{Q}{4\pi\epsilon_0 b}$. We could explicitly calculate the potential in the case of spherical conductors. When the shape of the conductors is arbitrary, we can't calculate the potential but we can make certain statements about the potentials on the two conducting surfaces in these configurations.

First let us consider the charge Q placed on the outer conductor. This is shown in configuration 2 in the figure. This will make the potential over the surface of the outer conductor V. As there is no charge within this surface, there is no electric field within and hence this region is equipotential with the value V. So the potential over the surface of the inner conductor is also V.

When the charge Q is placed on the inner conductor as shown in configuration 1, surface charge $-Q$ is induced over the inner surface of the outer conductor. This shields the outer world of the charge Q placed on the inner conductor. Since the total charge on the outer conductor is 0, an amount of charge Q spreads itself over the outer surface of the outer conductor. Since this charge can't see the existence of the charge Q on the inner conductor due to shielding by the negative charges, it spreads itself over the outer surface exactly in the same fashion as in the first case where wee placed an amount of charge Q on the outer conductor. So the potential on the outer conductor will be V. Note however that now the potential on the inner conductor is not V as the whole region within the outer conductor is not equipotential.

(d) In general if we have two conducting surfaces S_1 and S_2, when a charge Q is placed on conductor 1, the potential on conductor 2 is found to be V_{21}. Whereas when the charge Q is placed on conductor 2 the potential on conductor 1 is found to be V_{12}. Prove that $V_{12} = V_{21}$.



soln:
Let Φ_1 be the potential function in the region bounded by the two conductors when the charge Q is placed on the first conductor and Φ_2 be the potential function when the charge is placed on the second conductor. If we denote the region bounded by the two conductors as τ then

$$\begin{aligned}\int_\tau \vec{\nabla}\cdot(\Phi_1\vec{\nabla}\Phi_2)d\tau &= \int_\tau \vec{\nabla}\Phi_1\cdot\vec{\nabla}\Phi_2 d\tau + \int_\tau \Phi_1\vec{\nabla}^2\Phi_2 d\tau \\ \therefore \quad \oint_{S_1+S_2} \Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da &= \int_\tau \vec{\nabla}\Phi_1\cdot\vec{\nabla}\Phi_2 d\tau + \int_\tau \Phi_1\vec{\nabla}^2\Phi_2 d\tau\end{aligned}$$

Since there are no volume charge density in the region τ, the second integral on the r.h.s is 0 as $\vec{\nabla}^2\Phi_2 = 0$ by Gauss' law. The integral on the l.h.s is easy to evaluate since over the surfaces S_1 and S_2 the potentials are constant they being surfaces of conductors.

$$\begin{aligned}\oint_{S_1+S_2} \Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da &= \oint_{S_1} \Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da + \oint_{S_2} \Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da \\ &= 0 + V_{12}Q\end{aligned}$$

The first integral n r.h.s is 0 since $\oint_{S_1}(\vec{\nabla}\Phi_2)\cdot\hat{n}da$ gives the total charge on the first conductor. Since the charge Q is placed on the second conductor this integral is 0. Over the surface S_2 of the second conductor, $\oint_{S_2}\Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da = V_{12}Q$ as $\Phi_1 = V_{12}$ is the potential on the second conductor when Q is placed on the first conductor.

$$\therefore \quad \oint_{S_2} \Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da = V_{12}\oint_{S_2}\vec{\nabla}\Phi_2\cdot\hat{n}da = V_{12}Q$$

Putting everything together we see that

$$V_{12}Q = \int_\tau \vec{\nabla}\Phi_1 \cdot \vec{\nabla}\Phi_2 d\tau$$

Similarly considering $\int_\tau \vec{\nabla} \cdot (\Phi_2 \vec{\nabla}\Phi_1) d\tau$ we can show that

$$V_{21}Q = \int_\tau \vec{\nabla}\Phi_2 \cdot \vec{\nabla}\Phi_1 d\tau$$

As the integrals on the r.h.s are the same we conclude

$$V_{12}Q = V_{21}Q \quad \Longrightarrow V_{12} = V_{21}$$

4. A metal sphere of radius R carrying a charge q is surrounded by a thick concentric metal shell of inner radius a and outer radius b. The shell carries no net charge.

   (a) Find the surface charge density σ at radius R, a and b.

   soln

$$\sigma_R = \frac{q}{4\pi R^2}, \quad \sigma_a = -\frac{q}{4\pi a^2}, \quad \sigma_b == \frac{q}{4\pi b^2}$$

   (b) Find the potential at the center, using infinity as the reference point.

   soln
   The potential at the center is equal to the sum of the potentials due to the three charged surfaces.

$$\therefore \; V_{center} = \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 a} + \frac{q}{4\pi\epsilon_0 b}$$

   (c) If the outer surface $r = b$ is grounded how do the answers to part (a) and (b) change ?

   soln
   The potential of the outer sphere, when insulated , is $q/4\pi\epsilon_0 b$. When the outer sphere is grounded, it acquires a negative charge so as to cancel this potential. So it has to acquire an amount of charge $-q$ from the ground. This will make the charge density on the outer sphere 0. The inner sphere's don't feel this change on the outer sphere. The electric fields don't change, neither the charge densities on the inner surfaces ,i.e, at $r = R$ and $r = a$. But the potentials readjust. This will be given as

$$V_{center} = \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 a}$$

5. Two infinitely long wires running parallel to the x axis carry uniform charge densities $+\lambda$ and $-\lambda$.

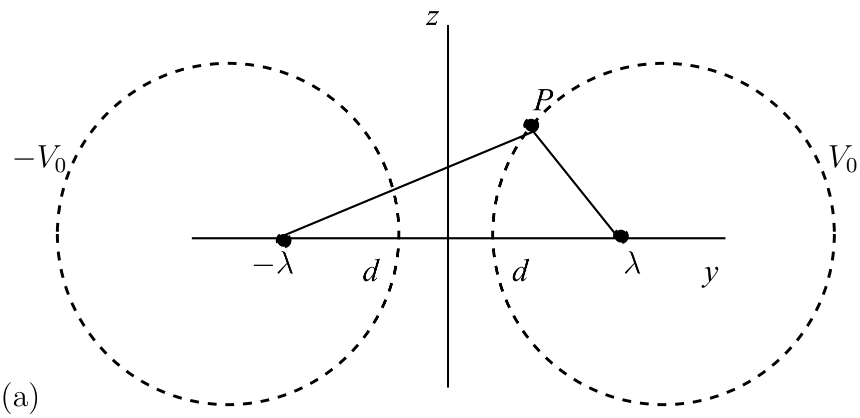
   (a) Find the potential at any point using the origin as the reference.

(b) Show that the equipotential surfaces are circular cylinders. Locate the axis and radius of the cylinder corresponding to a given potential V_0.

soln

The two line charges $+\lambda$ and $-\lambda$, parallel to the x axis cuts the y axis at $y = -d$ and $y = d$ respectively. Consider a point $P(x, y, z)$. The distance of P from the line charges are s_1 and s_2 given by

$$s_1 = \sqrt{(y+d)^2 + z^2} \qquad \text{and} \qquad s_2 = \sqrt{(y-d)^2 + z^2}$$



(a)

The potential at point P is

$$V = \frac{-\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_1}{k_1}\right) + \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_2}{k_2}\right) = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\left(\frac{s_2}{s_1}\right) + \ln\left(\frac{k_1}{k_2}\right)\right]$$

If we want $V = 0$ at the origin where $s_1 = s_2 = d$ then $k_1 = k_2$. So

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\sqrt{(y-d)^2 + z^2}}{\sqrt{(y+d)^2 + z^2}}\right) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{(y-d)^2 + z^2}{(y+d)^2 + z^2}\right)$$

(b) Consider a surface of constant potential V_0. Then from the above eqn. we have

$$(y-d)^2 + z^2 = \left[(y+d)^2 + z^2\right] K \qquad \text{where} \qquad K = \exp\left[\frac{4\pi\epsilon_0 V_0}{\lambda}\right]$$

$$\begin{aligned} \therefore \quad y^2(1-K) + z^2(1-K) - 2yd(1+K) + d^2(1-K) &= 0 \\ \therefore \quad y^2 + z^2 - 2yd\left(\frac{1+K}{1-K}\right) + d^2 &= 0 \end{aligned}$$

This is the equation of a circle $(y - y_0)^2 + z^2 = R^2$ with center at $(y_0, 0) = (d\left(\frac{1+K}{1-K}\right), 0)$ and radius $R = \frac{2d\sqrt{K}}{1-K}$.