Duality Theory

Associated with every linear program there is another linear program called its dual.

Motivation: Finding upper bounds.

Consider the following LPP

P_1

$$
\begin{aligned}
\text{Max } Z &= 4x_1 + x_2 + 3x_3 && \text{— (0)} \\
\text{s.t.} \quad & x_1 + 4x_2 \le 1 && \text{— (1)} \\
& 3x_1 - x_2 + x_3 \le 3 && \text{— (2)} \\
& x_1, x_2, x_3 \ge 0.
\end{aligned}
$$

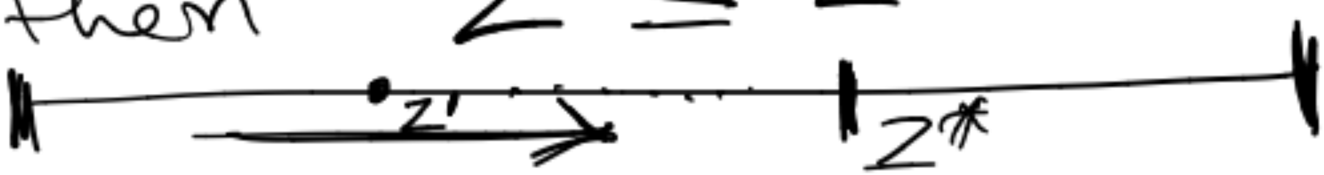
Observation:

Let Z^* be the optimal value of the objective function.

Every feasible solution is a lower bound to the optimal solution.

Z' value of any feasible solution,

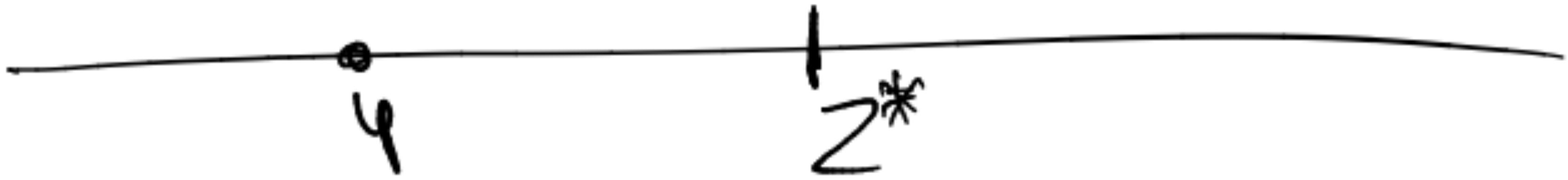
then $Z' \le Z^*$



For example:

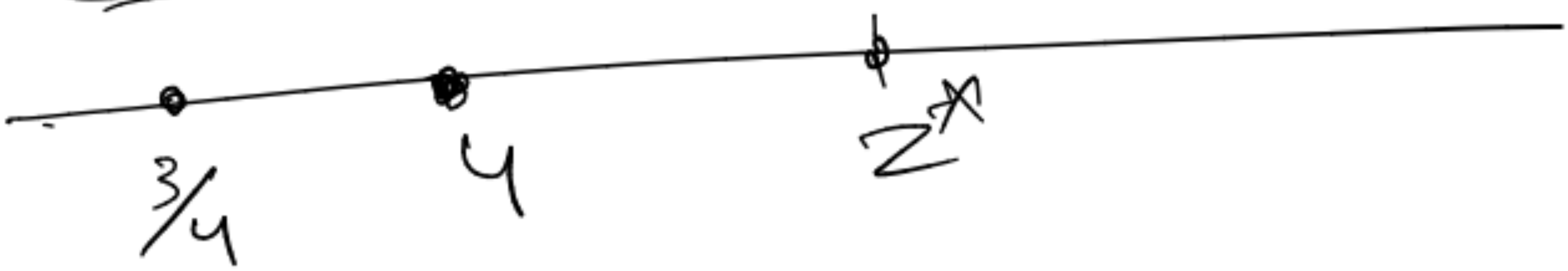
✓ $(x_1, x_2, x_3) = (1, 0, 0)$ is a feasible solution,

How $Z = 4 \Rightarrow 4 \le Z^*$



✓ $(x_1, x_2, x_3) = (0, 0, \frac{1}{4})$ is a feasible solution

$$\boxed{Z = \frac{3}{4}}$$



Question: ✓

How good is this bound?

Is it close to the optimal??

To answer we need to give one upper bound.

Let us multiply (1) by 2 and
(2) by 3 then add.

$$
\begin{array}{rl}
2(x_1 + 4x_2) & \le 2\cdot 1 \\
+\ 3(3x_1 - x_2 + x_3) & \le 3\cdot 3 \\
\hline
11x_1 + 5x_2 + 3x_3 & \le 11 \quad \text{--- (3)}
\end{array}
$$

we already have $x_1, x_2, x_3 \geq 0$

Now compare (3) with (0) we have.

$$Z = \underbrace{4x_1 + x_2 + 3x_3}_{(0)} \le \underbrace{11x_1 + 5x_2 + 3x_3}_{(3)} \overset{\text{by (3)}}{\le} 11$$

we can say

$$Z^* \le 11$$

we can localise our search to find optimal solution between 4 and 11

Question

Can we improve these bounds??

Can we improve the upper bound??
If so then How??

We apply the same upper bound technique but with variables instead of two magical numbers 2 and 3

And then we try to find the values of the variables. that gives the best upper bound.

we multiply ① by y_1 and
② by y_2 and then add.

H.W. Why we need to take y_1 and $y_2 \geq 0$?

$$
\begin{aligned}
& y_1(x_1+4x_2) \leq y_1 \cdot 1 \\
+\ & y_2(3x_1-x_2+x_3) \leq y_2 \cdot 3 \\
\hline
& (y_1+3y_2)x_1+(4y_1-y_2)x_2+y_2x_3 \leq y_1+3y_2
\end{aligned}
$$

④

We compare ④ with ⓪

maintain each coefficient of ④ is at least as large as its corresponding coefficients in ⓪ $\left[Z=4x_1+x_2+3x_3\right]$

Thus we have,

$$
\left.\begin{aligned}
y_1+3y_2 &\geq 4 \\
4y_1-y_2 &\geq 1 \\
y_2 &\geq 3
\end{aligned}\right\} \text{condition ①}
$$

$$Z = 4x_1 + x_2 + 3x_3 \le (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3$$

$$\le y_1 + 3y_2$$

Now we have an upper bound of $\boxed{y_1 + 3y_2}$

Question: What is our intution??

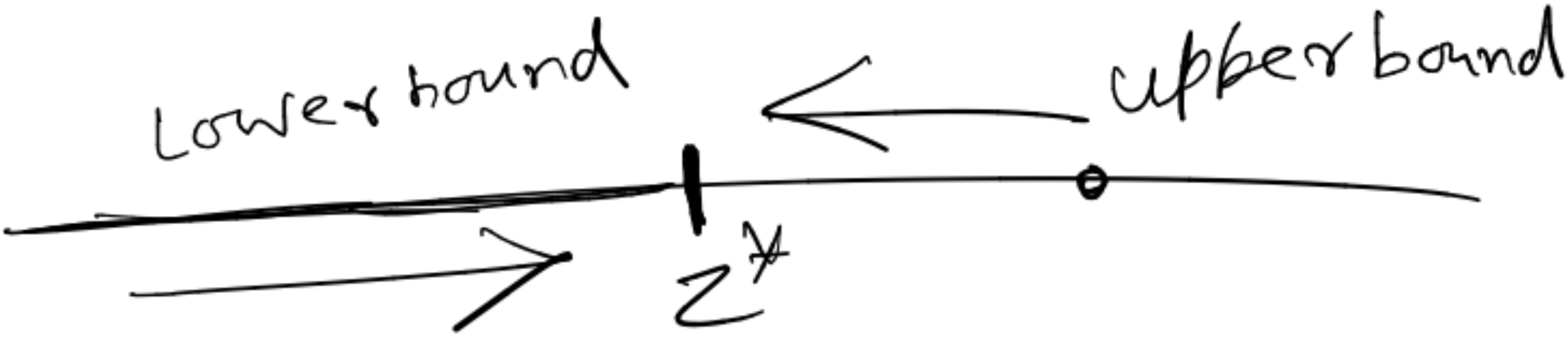
We need to minimise this upper bound to obtain the best possible upper bound.

$$\min\ w = y_1 + 3y_2$$

$$\text{s.t.}\quad y_1 + 3y_2 \geq 4$$
$$4y_1 - y_2 \geq 1$$
$$y_2 \geq 3$$
$$y_1, y_2 \geq 0$$

$\boxed{P_2}$

P_2 is the dual linear program of P_1



Dual linear program

Given a linear program in standard form as: called Primal

$$\max \ Z = \sum_{j=1}^{n} c_j x_j$$

$\boxed{P_1}$ s.t. $\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \forall \ i = 1, 2, \cdots m$ $\quad \boxed{y_i}$

$x_j \geq 0 \ \forall \ j = 1, 2, \cdots, n.$

The associated dual of P_1 is.

$$\min \ w = \sum_{i=1}^{m} b_i y_i$$

$\boxed{D_1}$ s.t. $\sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad \forall j = 1, 2, \cdots n$

$y_i \geq 0 \quad \forall \ i = 1, 2, \cdots, m$

Example:

Primal $\max Z = 5x_1 + 3x_2 - 4x_3$

s.t. $2x_1 + x_2 - x_3 \le 6 \quad \cdots \quad y_1$

$x_1 - 2x_2 + 3x_3 \le 9 \quad \cdots \quad y_2$

$x_1, x_2, x_3 \ge 0$

$\boxed{P_1}$

Dual

$\min w = 6y_1 + 9y_2$

s.t. $2y_1 + y_2 \ge 5$

$y_1 - 2y_2 \ge 3$

$-y_1 + 3y_2 \ge -4$

$y_1, y_2 \ge 0$.

$\boxed{D_1}$

A more complex example

minimise $Z = 5x_1 + 2x_2 + 7x_3$

s.t. $3x_1 + 5x_2 + 2x_3 \geq 12$

$\boxed{P_1}$ $x_1 + 2x_2 + 3x_3 \leq 8$

$2x_1 + x_2 + 4x_3 = 5$

$x_1 \geq 0,\ x_2 \leq 0,\ x_3$ unrestricted in sign.

replace x_2 with $-x_4$, with $x_4 \geq 0$

replace x_3 with $(x_5 - x_6)$ with $x_5, x_6 \geq 0$

$$\min Z = 5x_1 - x_4 + 7(x_5 - x_6) \quad \text{— } ⓪$$

s.t.

$$3x_1 - 5x_4 + 2(x_5 - x_6) \geq 12 \quad \text{— } ①$$

$$x_1 - 2x_4 + 3(x_5 - x_6) \leq 8 \quad \text{— } ②$$

$$2x_1 - x_4 + 4(x_5 - x_6) = 5 \quad \text{— } ③$$

$$x_1, x_4, x_5, x_6 \geq 0.$$

multiply ① with -1, replace $=$ in 3 by two equations "$\geq$" "$\leq$" and "$\geq$" multiply -1

multiply -1 with ⓪

$$\max w = -z = -5x_1 + 2x_4 - 7x_5 + 7x_6$$

s.t.

$$-3x_1 + 5x_4 - 2x_5 + 2x_6 \le -12 \quad — \; y_1$$

$$x_1 - 2x_4 + 3x_5 - 3x_6 \le 8 \quad \cdots \; y_2$$

$$2x_1 - x_4 + 4x_5 - 4x_6 \le 5 \quad \cdots \; y_3$$

$$-2x_1 + x_4 - 4x_5 + 4x_6 \le -5 \quad \cdots \; y_4$$

$$x_1, x_4, x_5, x_6 \ge 0$$

The dual is

$$\min z' = -12y_1 + 8y_2 + 5y_3 - 5y_4$$

s.t.

$$-3y_1 + y_2 + 2y_3 - 2y_4 \ge -5$$

$$5y_1 - 2y_2 - y_3 + y_4 \ge 2$$

$$-2y_1 + 3y_2 + 4y_3 - 4y_4 \ge -7$$

$$2y_1 - 3y_2 - 4y_3 + 4y_4 \ge 7$$

$$y_1, y_2, y_3, y_4 \ge 0.$$

First convert it to a maximization problem by multiplying by -1

$$\max\ w' = -z' = 12y_1 - 8y_2 - 5y_3 + 5y_4$$

define $y_5 = y_4 - y_3$ then y_5 becomes unrestricted.

$$\max\ w' = 12y_1 - 8y_2 + 5y_5$$

s.t.
$$-3y_1 + y_2 - 2y_5 \geq -5$$
$$5y_1 - 2y_2 + y_5 \geq 2$$
$$-2y_1 + 3y_2 - 4y_5 \geq -7$$
$$2y_1 - 3y_2 + 4y_5 \geq 7$$

$y_1, y_2 \geq 0$ $\quad y_5$ unrestricted.

$y_2 = -y_6$ so that $y_6 \leq 0$.

$$\max\ w' = 12y_1 + 8y_6 + 5y_5$$
$$-3y_1 - y_6 - 2y_5 \geq -5$$
$$5y_1 + 2y_6 + y_5 \geq 2$$
$$-2y_1 + 3y_6 - 4y_5 \geq -7$$
$$+ \qquad + \qquad + \qquad \leq +$$
$$2y_1 + 3y_6 + 4y_5 \geq 7$$

$$2y_1 + 3y_6 + 4y_5 = 7$$

$y_1 \geq 0$, $y_6 \leq 0$, y_5 unrestricted.

The final program is

$$\begin{cases} \max z' = 12y_1 + 8y_6 + 5y_5 \\ \text{s.t.} \quad 3y_1 + y_6 + 2y_5 \le 5 \\ \quad 5y_1 + 2y_6 + y_5 \ge 2 \\ \quad 2y_1 + 3y_6 + 4y_5 = 7 \\ \quad y_1 \ge 0,\ y_6 \le 0,\ y_5 \text{ unrestricted} \end{cases}$$