Groups and linear algebra (SC220) Autumn 2019
In Sem -I Time: 1hr 30 min

Name: ______________________

Student I.D.: ______________________

Section 1. True/False (2 pts. each)

Print "T" if the statement is true, otherwise print "F". In either case give a very brief (One or Two line) justification or a counter example.

F $A_4 =< (12)(34), (13)(24) >$

Let $a = (12)(34)$ and $b = (13)(24)$. Note that $a^2 = b^2 = e$ and $ab = ba = (14)(23)$. Hence any words $a^k b^l$ can generate e, a, b or ab.

F The group $(\mathbb{Q}, +)$ is isomorphic to $(\mathbb{Q}^+, \times)$

Suppose $\Phi : (\mathbb{Q}, +) \to (\mathbb{Q}^+, \times)$ was an isomorphism then $\exists\, x \in \mathbb{Q}$ s.t. $\Phi(x) = 2$, then $\Phi(\frac{x}{2} + \frac{x}{2}) = 2 \Rightarrow$ $\Phi(\frac{x}{2}) \cdot \Phi(\frac{x}{2}) = 2 \Rightarrow (\Phi(\frac{x}{2}))^2 = 2 \Rightarrow \Phi(\frac{x}{2}) = \sqrt{2}$ a contradiction.

F S_3 is isomorphic to $Z_2 \times Z_3$

S_3 is not abelian but $\mathbb{Z}_2 \times \mathbb{Z}_3$ is abelian.

T There are exactly 6 automorphisms from $\mathbb{Z}_9$ to $\mathbb{Z}_9$

Generators of $\mathbb{Z}_9$ are $1, 2, 4, 5, 7, 8$. Since $\mathbb{Z}_9$ is cyclic mapping any one of the generators completely determines the isomorphism. Since any one generator can be sent to six possible generators, there are 6 isomorphisms

T In S_4 let $\sigma = (123)(134)$ then σ^{2019} is e

$\sigma = (123)(134) = (234) \therefore \sigma^3 = e$

$\therefore \sigma^{2019} = (\sigma^3)^{673} = e$

T If G is a group of order p^k where p is a prime then it has a subgroup of order p^m for some positive integer $m \leq k$.

Let $H \leq G$ then by Lagrange's thm $|H| \mid |G|$
Since $|G| = p^k$ then $|H|$ has to be p^m for
Some $m \leq K$

F The number of 4 cycles in S_5 is 24

The number of 4 cycles in S_5 is
$$\frac{5 \times 4 \times 3 \times 2}{4} = 30$$

T The elements r^2s and r^3s generate D_5

$r^3 s\, r^2 s = r^3 s s r^{-2} = r^3 . r^{-2} = r$ and
$r . r . r\, r^2 s = r^5 s = s$ and since $D_5 = \langle r, s \rangle$
$\therefore$ $r^2 s$ and $r^3 s$ generate D_5

T The remainder when 13^{36} is divided by 18 is 1

Applying Eulers them $a^{\Phi(n)} \equiv 1 \bmod 18$ if
$\gcd(a,n)=1$ with $a = 13$ $n=18$ we get $13^{\Phi(18)} \equiv 1 \bmod 18$
$\therefore$ $13^6 \equiv 1 \bmod 18 \Rightarrow (13^6)^2 = 13^{36} \equiv 1 \bmod 18$.

F D_6 (Group of Symmetries of a hexagon) is isomorphic to A_4 (Group of even permutations of 4 letters)

D_6 has an element r of order 6
Whereas A_4 has no element of order 6

Section 2. Short Answer (10 pts each)

Answer all problems in as thorough detail as possible.

1. Prove that if H and K are proper subgroups of a finite group G such that $gcd(|H|, |K|) = 1$, then $H \cap K = \{e\}$. Is the converse true ? That is, if H and K are proper subgroups of G such that $H \cap K = \{e\}$ then is it necessary that $gcd(|H|, |K|) = 1$.

a) First note that $H \cap K \leq G$ (No proof needed)

Since $H \cap K \subseteq H$ and $H \cap K \subseteq K$ this implies that $H \cap K \leq H$ and $H \cap K \leq K$

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Now by Lagrange's theorem

$|H \cap K| \mid |H|$ and $|H \cap K| \mid |K|$

But since $gcd(|H|, |K|) = 1$ $\therefore$

$|H \cap K| = 1 \implies H \cap K = \{e\}$

b) No

Consider $D_4 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$.

$\langle r \rangle = \{e, r, r^2, r^3\}$

4

$\langle s \rangle = \{e, s\}$

$\langle r \rangle \cap \langle s \rangle = e$

but $gcd(|\langle r \rangle|, |\langle s \rangle|) = 2$

3

2. Show that if G and H are groups and $A \leq G$ and $B \leq H$ then $A \times B$ is a subgroup of $G \times H$. Does every subgroup of $G \times H$ has to be of the form $A \times B$ where $A \leq G$ and $B \leq H$?

a) $A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$

we will use the subgroup critereon, that is $H \subseteq G$ non-empty is a subgroup of a group $G \iff xy^{-1} \in H \ \forall \ x, y \in H$

$A \times B$ is non-empty since $(e,e) \in A \times B$

6 Let (a_1, b_1) and $(a_2, b_2) \in A \times B$ then

$$(a_1, b_1) (a_2, b_2)^{-1} = (a_1, b_1)(a_2^{-1}, b_2^{-1})$$
$$= (a_1 a_2^{-1}, b_1 b_2^{-1})$$

Since A and B are subgroups of G and H respectively $\therefore$ $a_1 a_2^{-1} \in A$ and $b_1 b_2^{-1} \in B$

$\therefore$ $(a_1 a_2^{-1}, b_1 b_2^{-1}) \in A \times B$ $\therefore$ $A \times B \leq G \times H$

b) No. Consider $G = \mathbb{Z}$ and $H = \mathbb{Z}$

then $\langle (1,1) \rangle \leq \mathbb{Z} \times \mathbb{Z}$ is

4 not of that form

3. Let G be a finite group.
i)Show that if $x \in G$ then the map $\lambda_x : G \to G$ given by $\lambda_x(g) = xg$ is a 1-1 and onto map from G to G (that is the map λ_x is a permutation of the elements of G).
ii)Let S_G be the set of all permutations of G. Show that the map $\Phi : G \to S_G$ given by $\Phi(g) = \lambda_g$ is a 1-1 mapping and satisfies $\Phi(x * y) = \Phi(x) * \Phi(y)$. Hence conclude that every finite group is isomorphic to a subgroup of the permutation group S_n.

i) 1-1

Let $\lambda_x(g_1) = \lambda_x(g_2)$ then

$x g_1 = x g_2 \Rightarrow g_1 = x^{-1} x g_2 = g_2$

4 $\therefore \lambda_x : G \to G$ is 1-1

onto

Let $g \in G$, then $x^{-1} g \in G$ s.t

$\lambda_x(x^{-1}g) = x x^{-1} g = g$ $\therefore \lambda_x$ is onto

ii) $\Phi : G \to S_G$ is given by

$g \mapsto \lambda_g$. To show that Φ is 1-1

Let $\Phi(g_1) = \Phi(g_2) \Rightarrow \lambda_{g_1} = \lambda_{g_2}$

Note λ_{g_1} and λ_{g_2} are permutatons of G, since ~~to~~ ~~show~~ ~~that~~ they are equal ~~we~~ ~~need~~ ~~to~~ ~~show~~ ~~that~~ they act the same on every element

2 $a \in G$. $\lambda_{g_1}(a) = g_1 a$ and $\lambda_{g_2}(a) = g_2 a$

$\therefore g_1 a = g_2 a \quad \forall a \in G$

$\therefore g_1 = g_2 \quad \therefore \Phi$ is 1-1

$\Phi(xy) = \lambda_{xy}$

P. T. O.

Now $\lambda_{xy}(g) = xyg$

$$(\lambda_x \circ \lambda_y)(g) = \lambda_x(\lambda_y(g))$$

composition of permutations λ_x and λ_y

$$= \lambda_x(yg)$$

$$= xyg$$

4

$$\therefore \lambda_{xy}(g) = \lambda_x \circ \lambda_y(g)$$

$$\therefore \underbrace{\Phi(x \cdot y)}_{\text{group mult.}} = \underbrace{\Phi(x) \circ \Phi(y)}_{\text{composition of permutations}}$$

Now $\Phi(G) \leq S_G$ (No need for proof)

$\therefore$ Every group G is isomorphic to a subgroup of S_G (and hence of S_n for some n).