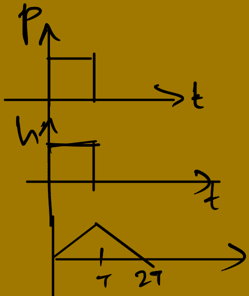
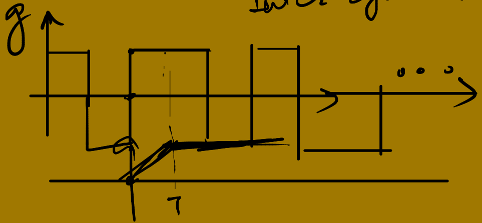
- Lecture 19 Recap:
  - Eye diagram.
- Channel distortion. Delay distortion
  - Amplitude and Phase distortion may lead to ISI.

Inter Symbol Interference





Modeling ISI

$p(t)$

- Let us denote the transmitter pulse waveform by p, the positive and negative impulse that encodes the m^{th} bit by a_m and the transmitted signal by $\boxed{g(t) = \sum_m a_m p(t - mT).}$

$+V$ or $-V$

bit 1 bit 0

T – duration of a symbol.

$g \rightarrow \boxed{\text{channel}} \rightarrow \{g, g+n\}$

$h_c(t)$

Not true

$$r = h_c * g + n$$

$$\cancel{h_c * (g+n)}$$

$p(t) \rightarrow \boxed{\text{Channel}} \rightarrow q(t)$

ignoring noise

Modeling ISI

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- Since the channel is assumed to be an LTI system, if for an input of $p(t)$ to the channel, the output is $q(t)$, the received signal can be written as

$$\boxed{r(t) = \sum_m a_m q(t - mT) + \eta(t),}$$

where η is AWGN.

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Lab 5: $r(t) = \sum_m a_m p(t - mT) + \eta(t)$

- How should we arrive at a decision on what bits were transmitted?

Channel is deterministic (h_c is known), p is in your control

$\therefore$ q is known!

MLE Estimation

$r(t) = \sum_n r_n f_n(t)$

- Assume that in some orthonormal basis $\{f_n, n \in \mathbb{Z}^+\}$, $r = \sum_n r_n f_n$.

$r(t)$

scalar $\in \mathbb{R}$

MLE Estimation

- Assume that in some orthonormal basis $\{f_n, n \in \mathbb{Z}^+\}$, $r = \sum_n r_n f_n$.
- Then, denoting $\langle q(t - mT), f_k \rangle$ by q_{km} and $\langle \eta(t), f_k \rangle$ by η_k, we get

$$q(t-mT) = \sum_k q_{km} f_k$$

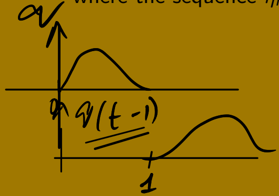
$$\boxed{r_n = \sum_k a_k q_{nk} + \eta_n,}$$

$$\eta_1, \eta_2, \ldots, \eta_n, \ldots$$

$$LT_1, LT_2, \ldots, LT_n$$

$$\eta(t)$$

where the sequence η_n has a white Gaussian distribution.



$$E[\eta_k \eta_m] = E[\langle \eta, f_k \rangle \langle \eta, f_m \rangle] = 0 \text{ if } k \neq m$$

MLE Estimation

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$$r_n = \sum_k a_k q_{nk} + \eta_n,$$

independent|uncorrelated

where the sequence η_n has a ~~white~~ Gaussian distribution.

- Let us assume that L bits were transmitted. We need to find $\{a_0, \ldots, a_{L-1}\}$, from N coefficients $\{r_0, \ldots, r_{N-1}\}$ of the received signal.

$$\eta_n = r_n - \sum_k a_k q_{nk} \sim N(0, N_0)$$

MLE Estimation

$r = r_1 f_1 + r_2 f_2$

$\tilde{r} = r_1 f_1$

- Assume that in some orthonormal basis $\{f_n, n \in \mathbb{Z}^+\}$, $\boxed{r = \sum_n r_n f_n}$
- Then, denoting $\langle q(t - mT), f_k \rangle$ by q_{km}, and $\langle \eta(t), f_k \rangle$ by η_k, we get

$$\boxed{r_n = \sum_k a_k q_{nk} + \eta_n,}$$

$\eta_1 \ldots \eta_N$ — are independent

where the sequence η_n has a ~~white~~ Gaussian distribution.

- Let us assume that L bits were transmitted. We need to find $\{a_0, \ldots, a_{L-1}\}$, from N coefficients $\{r_0, \ldots, r_{N-1}\}$ of the received signal.
- Since $\eta_n = r_n - \sum_k a_k q_{nk} \sim \mathcal{N}(0, N_0)$, the likelihood is

$\eta_1, \eta_2, \eta_3 - \ldots$

$p(\eta_1, \ldots, \eta_N)$

$$= p(\{r_0, \ldots, r_{N-1}\} | \{a_0, \ldots, a_{L-1}\}) = K \exp\left(-\frac{1}{2N_0} \sum_{n=0}^{N-1} \left| r_n - \sum_{k=0}^{L-1} a_k q_{nk} \right|^2 \right)$$

$= p(\eta_1) \cdot p(\eta_2) \cdots p(\eta_N)$

and the task boils down to

$$\min_{\{a_0, \ldots, a_{L-1}\}} \sum_{n=0}^{N-1} \left| r_n - \sum_{k=0}^{L-1} a_k q_{nk} \right|^2$$

Let $N \to \infty$

- Also as the number of coefficients used $N \to \infty$, we get

$$\lim_{N\to\infty} \boxed{\sum_{n=0}^{N-1} \left| r_n - \sum_{k=0}^{L-1} a_k q_{nk} \right|^2} = J := \int_{-\infty}^{\infty} \left(r(t) - \sum_k a_k q(t-kT) \right)^2 dt$$

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  - Then,

$$J = \int_{-\infty}^{\infty} \underline{r^2(t)\, dt} - 2 \sum_k a_k \int_{-\infty}^{\infty} r(t) q(t-kT) \, dt$$
$$+ \sum_k \sum_l \underline{a_k a_l} \int_{-\infty}^{\infty} q(\underline{t-kT}) q(\underline{t-lT}) \, dt$$

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$z_k = z(kT)$

- Let $z_k = \int_{-\infty}^{\infty} r(t)q(t-kT)\, dt$ and $x_k = \int_{-\infty}^{\infty} q(t)q(t+kT)\, dt$.

$$z_k = z(kT) = \underbrace{\int_{-\infty}^{\infty} r(t)q(t-kT)\,dt}_{\text{MF response matched to } q(t)}$$

Sampled

$$R_q(\tau) = \int_{-\infty}^{\infty} q(t)q(t+\tau)\,dt$$

$$x_k = R_q(\tau = kT)$$

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- Let $z_k = \int_{-\infty}^{\infty} r(t) q(t-kT)\, dt$ and $x_k = \int_{-\infty}^{\infty} q(t)q(t+kT)\, dt$.
- Note that z_k can be obtained by sampling the output of the matched filter matched to q every T secs, and x is the sampled autocorrelation function of q.

$r(t) = \sum_m a_m q(t-mT) + n(t)$

$z_k = ?$

- Also as the number of coefficients used $N \to \infty$, we get

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- Note that z_k can be obtained by sampling the output of the matched filter matched to q every T secs, and x is the sampled autocorrelation function of q.
- Since $r(t) = \sum_m a_m q(t-mT) + \eta(t)$,
  $z_k = \sum_m a_m \underbrace{\int_{-\infty}^{\infty} q(t-mT)q(t-kT)\, dt}_{x_{k-m}} + \int_{-\infty}^{\infty} \eta(t) q(t-kT)\, dt.$

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  $z_k = \sum_m a_m \int_{-\infty}^{\infty} q(t-mT)q(t-kT) + \int_{-\infty}^{\infty} \eta(t)q(t-mT)\, dt$.
- $\boxed{z_k = \sum_m a_m x_{k-m} + v_k.}$ ← Sampled o/p of Matched filter

Lab 5: $q = p \Rightarrow x(0) \neq 0,\ x(1) = x(-1) = x(2) = x(-2) \cdots = 0$

$z_k = \boxed{a_k} + v_k$

If $x_k \neq \delta_k \Rightarrow$ ISI!!

- If q is such that $x_{k-m} = \int_{-\infty}^{\infty} q(t - mT)q(t - kT) = 0$ for $|k - m| > L$, $z_k = \sum_{m=-L}^{L} a_m x_{k-m} + v_k$.