Stationarity: $\rightarrow$ Strict Sense Stationarity
$\hookrightarrow$ Wide-Sense Stationarity (1st / 2nd)

AC : $E[X_{t_1} X_{t_2}]$

① $X(t) = \underline{A} \cos(\omega_c t)$, $t \in [0, 2\pi)$

1 RV

$A \sim U[-1, 1]$

$X(n) = A \cos\left(\frac{\pi}{4} n\right)$, $A \sim U[-1, 1]$.

$m_X(t) = 0$

$R_X(t_1, t_2) = E\left[A\cos(\omega_c t_1)\, A\cos(\omega_c t_2)\right]$

$= \frac{1}{3} \cos(\omega_c t_1) \cos(\omega_c t_2)$.

$\Rightarrow$ Not a function of $(t_1 - t_2)$

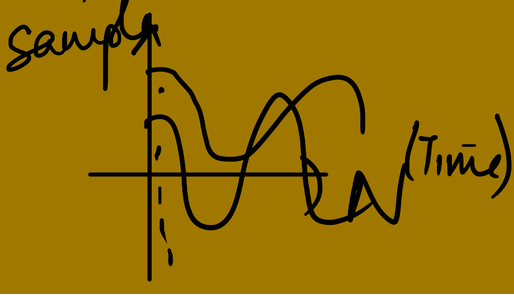
$\therefore$ X is not Stationary

Simulation

$\Rightarrow$ $q = \cos\left(\frac{\pi}{4}(0:N-1)\right)_{1 \times N}$

$r =$ uniform[rand] $(T, 1)$

$r_{T \times 1}$



$\Rightarrow \underline{X} = (r * q)_{T \times N}$

Q-1(c): $X(n) \sim A(n)$, $A(n) \sim \mathcal{N}(0,1)$ Gaussian.

Every time instance n has a diff. RV

Each RV has the same distribution

* Each $A(n)$ is an independent RV.

(2) $X \rightarrow$ Contains a family of "independent and identically distributed" RV's-
[i.i.d]

$$p(X_{t_1+T}, \ldots, X_{t_n+T})$$
$$= \prod_{i=1}^{n} p(X_{t_i+T})$$
$$= \prod_{i=1}^{n} p(X_{t_i}) = p(X_{t_1}, \ldots, X_{t_n})$$

$\Rightarrow$ Strict Sense Stationarity //

(3) Bernoulli Process (Trials)

n-coin tosses in a sequence

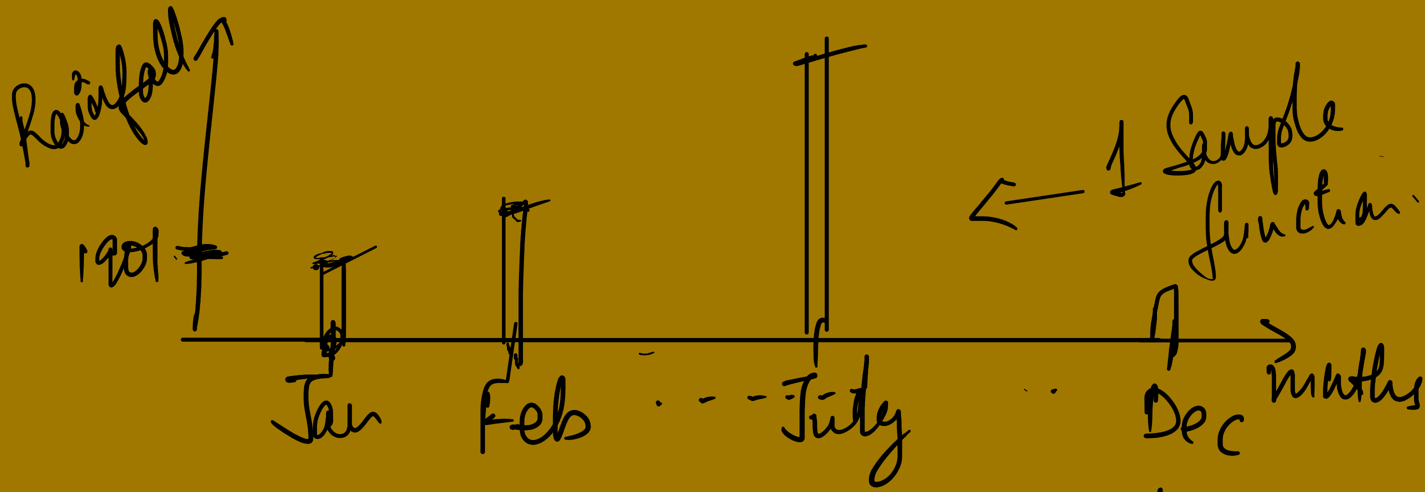
$X_1 \rightarrow 1^{st}$ coin toss
$\vdots$
$X_n \rightarrow n^{th}$ coin toss

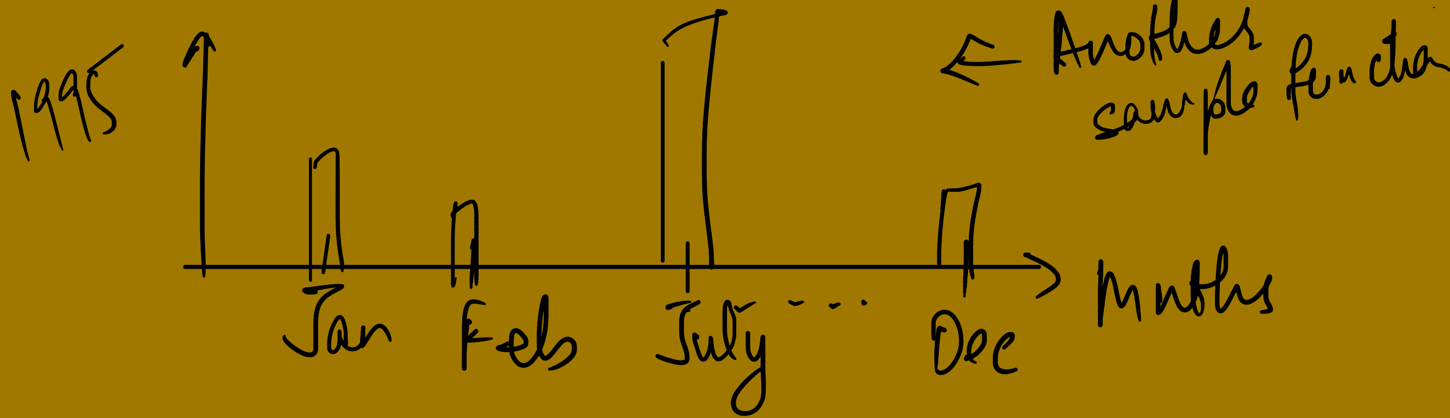
$X_k(P=1) = a$
$X_k(P=0) = 1-a$

⇒ Independent & Identically distributed.

⇒ Stationary in the Strict Sense.

Q.2 Lab 2: Monthly rainfall data for GNR.





$$\mathbb{E}\left[X_{t_1+T}, X_{t_2+T}\right] = \int\int x_{t_1+T}\, x_{t_2+T}\, p(x_{t_1+T}, x_{t_2+T})\, dx_{t_1+T}\, dx_{t_2+T}$$

$$p(x_{t_1}, x_{t_2})$$

$$= \mathbb{E}\left[X_{t_1}, X_{t_2}\right]$$

$$R_X(t_1, t_2) = R_X(t_1 - t_2)$$

SSS & WSS.

4 Gaussian Random Process

$\{X_i\}_{i \in I}$ $\quad$ $\{X_t\}_{t \in T}$ $\quad$ $T = [0,1] \subset \mathbb{R}$.

Not enough $\rightarrow$ Each X_{t_i} is Normally distributed.

$\Rightarrow$ Any finite subset of $\{X_t\}_{t \in T}$ has a normal distribution.

$$\Rightarrow \{X_{t_1}, \ldots, X_{t_n}\} \begin{matrix} \forall n \in \mathbb{Z}^+ \\ \forall t_i \in T \end{matrix} \sim \underline{\underline{\mathcal{N}(\mu, \Sigma)}}$$

X_1, X_2

$$p(X_1, X_2) = \frac{1}{\sqrt{(2\pi)^2 \det(\Sigma)}} e^{-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}}$$

$x \rightarrow 2 \times 1$

$$\mu = \begin{pmatrix} \mu_{x_1} \\ \mu_{x_2} \end{pmatrix}_{2 \times 1}$$

$$\Sigma(i,j) = C(X_i, X_j)$$

$\hookrightarrow$ 2x2 Matrix.

$\{X_1, \ldots, X_n\}$ are Normally distributed if the joint density is

$$p(X_1, \ldots, X_n) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}}$$

$\rightarrow$ Normalization.

$$\Sigma_{i,j} = Cov(X_i, X_j), \quad 1 \le i,j \le n$$

N R.V.'s $X_1, \cdots, X_N$

$\Rightarrow$ Covariance of any 2 of these RV's.

$$\Sigma_{N \times N} \qquad \mu = \begin{pmatrix} m_{x_1} \\ \vdots \\ m_{x_n} \end{pmatrix} \in \mathbb{R}^n$$

$$x = \begin{pmatrix} X_1 = x_1 \\ X_2 = x_2 \\ \vdots \\ X_n = x_n \end{pmatrix} \in \mathbb{R}^n$$

$$p(X_1, \cdots, X_n) = \frac{1}{\sqrt{(2\pi)^N \det(\Sigma)}} e^{-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}}$$

$\Sigma^{-1} \Rightarrow$ (1) Why should Σ be invertible?

$$\mathrm{Cov}(X_i, X_j) = \mathrm{Cov}(X_j, X_i)$$

$\Rightarrow \Sigma$ is symmetric. $(\Sigma = \Sigma^T)$

$$\Sigma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \Sigma^{-1} \text{ does not exist.}$$

Spectral theorem: Any symmetric $n \times n$ matrix is diagonalizable. $\exists$ n orthogonal eigenvectors

$\therefore \exists\, V \in O(n)$, $V^T \Sigma V = \Lambda$ (diag.)

A sq. mtx is not invertible if $\exists\, x \neq \bar{0}$ s.t. $Ax = \bar{0}$

$$\Downarrow$$

$Ax = 0 \cdot x \Rightarrow 0$ is an $\boxed{\text{eigenvalue}}$ $\Rightarrow$ A is not invertible

$A\bar{0} = \bar{0}$
$Ax = \bar{0}$ $\Rightarrow$ A is not injective

$$y^T \Sigma y \geq 0 \quad \forall y \in \mathbb{R}^n$$

$x \neq 0$, $x^T \Sigma x = 0 \Rightarrow \Sigma$ is not invertible.

$$x^T V^T \Lambda V x = 0$$

Let $Vx = w \Rightarrow w^T \Lambda w = 0$

$\Rightarrow$ Some eigenvalue has to be zero!!

A symmetric has only real eigenvalues & if all of them are positive, we say that the matrix is Symmetric Positive Definite (SPD).

$\Rightarrow$ The matrix is invertible.

We will assume that the Cov. Mtx Σ is SPD!! This ensures that Σ^{-1} exists.

2) $(x-\mu)^T \Sigma^{-1} (x-\mu) \Leftarrow$ Interpretation.

1 RV: $\dfrac{(x-\mu)^2}{2\sigma} \Leftarrow$ Interpretation.

* Linear Alg & its applcns
— David Lay