suppose an LPP is given

$$\text{Max } Z = 3x_1 + 4x_2$$

$$\text{s.t.} \quad x_1 + 2x_2 \le 4 \quad \text{①}$$

$$3x_1 + 2x_2 \le 6 \quad \text{②}$$

$$x_1, x_2 \ge 0$$

1. Graphical method
2. Try for all basic feasible solutions and find the solution that gives the maximum value.

- We know how to solve the system of equations.
- Convert the constraints into equalities.

Introducing slack variables x_3 and x_4 to ① and ② respectively

Then the problem becomes.

$$\left\{\begin{array}{ll} \max & Z = 3x_1 + 4x_2 + 0x_3 + 0x_4 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \qquad\quad = 4 \\ & 3x_1 + 2x_2 \qquad + x_4 = 6 \end{array}\right.$$

$$x_1, x_2, x_3, x_4 \geq 0$$

no. of variables is 4

no. of constraints is 2

Total no. of basic solutions is $^4C_2 = 6$

Try all basic solutions and See which of them are feasible

Finally return the solution with maximum value.

$$A = \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$B_1 = [a_1\ a_2]$ $\quad B_2 = \{a_1, a_3]$ $\cdots$

$B_6 = [a_3\ a_4]$

$\det(B_1) = -4$

$\det(B_2) = -3$

$\vdots$

$x_{B_i} = B_i^{-1} b$ for $i = 1, 2, \ldots 6$.

$\boxed{\text{H.W} \rightarrow \text{Solve the problem and find optimal solution}}$

Limitations

Graphical method

- Difficult to visualise the feasible region.
- Solving is difficult.

Exhaustive search

Try all possible solutions.

$^{n}C_{m}$ no. of solution.

Simplex method

Algebraic form of simplex algorithm

we have problem as,

$$\max Z = 3x_1 + 4x_2 + 0x_3 + 0x_4 \quad \text{--- (0)}$$

$$\text{s.t.} \quad x_1 + 2x_2 + x_3 = 4 \quad \text{--- (1)}$$

$$3x_1 + 2x_2 + x_4 = 6 \quad \text{--- (2)}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Iteration 1

We start with a basic feasible solution by choosing x_3 and x_4 as basic variables.

From (1) we get,

$$x_3 = 4 - x_1 - 2x_2 \quad \text{--- (3)}$$

From (2) we get,

$$x_4 = 6 - 3x_1 - 2x_2 \quad \text{--- (4)}$$

From (0) we get,

$$Z = 0 + 3x_1 + 4x_2 \quad \text{--- (5)}$$

clearly $x_1 = 0$ and $x_2 = 0$ because they are non-basic variables.
Then from 5 we have $Z = 0$.

Now ur target is to increase Z.
This is possible by increasing either x_1 or x_2 one at a time.

we make either x_1 or x_2 as basic variable and one of x_3 and x_4 as non-basic variable.

we increase x_2 which has the highest rate of increase as the coefficient of x_2 is higher (4) than x_1.

x_2 will be next basic variable

consider equation 3.
we can increase x_2 to a value 2 and beyond that x_3 becomes negative.

From equation 4
we can increase x_2 to a value 3 and beyond that x_4 becomes negative.

From above two cases we can say that x_2 can increase upto value 2. based on equation 3

New basic variable is x_2

New non-basic variable is x_3

Iteration 2

x_2 is basic
x_3 is non-basic

From (3) we have,

$$x_3 = 4 - x_1 - 2x_2$$

$$\Rightarrow 2x_2 = 4 - x_1 - x_3$$

$$\Rightarrow x_2 = \boxed{2 - \frac{1}{2}x_1} - \frac{1}{2}x_3 \quad \text{---} \quad (6)$$

Substitute x_2 from (6) in (4)

$$x_4 = 6 - 3x_1 - 2x_2$$

$$\Rightarrow x_4 = 6 - 3x_1 - 2\left[2 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right]$$

$$= 6 - 3x_1 - 4 + x_1 + x_3$$

$$= \boxed{2 - 2x_1} + x_3 \quad \text{---} \quad (7)$$

Now express Z from (5) as,

$$Z = 0 + 3x_1 + 4x_2$$

$$= 0 + 3x_1 + 4\left[2 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right]$$

$$= 0 + 3x_1 + 8 - 2x_1 - 2x_3$$

$$= 8 + x_1 - 2x_3 \quad \text{---} \quad (8)$$

Our target is to increase Z further

$$Z = \underline{\underline{8}} + x_1 - 2x_3$$

increase x_1

decrease x_3

not possible

why??

currently x_3 is a non-basic variable and its value is 0

The only possiblity is to increase x_1

x_1 will be the next basic variable

Now find which of x_2 & x_4 will be non-basic.

From equation 6 we can increase x_1 to a value upto 4

From 7 we can increase x_1 to a value upto 1.

The limiting value is 1 and it is based on equation 7.

Therefore,

$\Rightarrow$ x_1 is my new basic variable

$\Rightarrow$ x_4 is new non-basic variable.

Iteration 3

$x_1 \leftarrow$ basic variable

$x_4 \leftarrow$ non-basic variable

consider equation (7) and rewrite it in terms of x_1 as follows.

$$x_4 = 2 - 2x_1 + x_3$$

$$\Rightarrow 2x_1 = 2 + x_3 - x_4$$

$$\Rightarrow x_1 = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4 \quad \text{—— (9)}$$

Substituting x_1 from (9) in (6) we have,

$$x_2 = 2 - \frac{1}{2}x_1 - \frac{1}{2}x_3$$

$$= 2 - \frac{1}{2}\left[1 + \frac{1}{2}x_3 - \frac{1}{2}x_4\right] - \frac{1}{2}x_3$$

$$= 2 - \frac{1}{2} - \frac{1}{4}x_3 + \frac{1}{4}x_4 - \frac{1}{2}x_3$$

$$= \frac{3}{2} - \frac{3}{4}x_3 + \frac{1}{4}x_4 \quad \text{—— (10)}$$

Express Z from ⑧ as follows

$$Z = 8 + x_1 - 2x_3$$
$$= 8 + \left[1 + \tfrac{1}{2}x_3 - \tfrac{1}{2}x_4\right] - 2x_3$$
$$= \underline{\underline{9}} - \tfrac{3}{2}\underline{\underline{x_3}} - \tfrac{1}{2}\underline{\underline{x_4}} \quad \text{—— (11)}$$

The target is to increase Z.

If we want to increase Z

$$Z = \underline{\underline{9}} - \tfrac{3}{2}x_3 - \tfrac{1}{2}x_4$$

constant fixed.

decrease x_3

not possible

decrease x_4

not possible

because x_3 and x_3 are non basic variables and their current value is 0.

$\boxed{\text{As a result, we can not increase Z further.}}$

we reached the maximum value.

we have x_1 & x_2 as basic variables

x_3 and x_4 are non basic variables.

Therefore, $x_3 = 0$, $x_4 = 0$

From (9) we have,

$$x_1 = 1$$

From (10) we have,

$$x_2 = \frac{3}{2}$$

From (11) we have,

$$Z = 9$$

The solution is
$x_1 = 1$, $x_2 = \frac{3}{2}$, $x_3 = 0$, $x_4 = 0$
and the optimum value is $Z = 9$

$\boxed{\text{H.W}}$

verify this solution using

1) Graphical method ✓✓

2) Exhaustive search ✓✓

$\boxed{\text{Algebraic method}}$