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CS374: Practice Sheet 6

Prob 1) Write the Cubic Hermite interpolating polynomial for the following data. Further, find the error bound if $\left\|f^{(4)}\right\|_\infty = 24$.

a.)

x	0	1
f(x)	1	3
f'(x)	1	5

b.)

x	0	1	2
f(x)	1	3	19
f'(x)	1	5	

Show that the data recovers the function $f(x) = x^4 + 1$.

Prob 2) Find the piecewise linear and quadratic interpolating polynomials for the function $x^4 + 1$ on $[0, 2]$. Determine the step size h for achieving accuracy 10^{-6} in each case.

Prob 3) In Prob 1, determine the step size h for piecewise Cubic Hermite Interpolation polynomial to achieve accuracy of 10^{-6} on $[0, 2]$. Finally, write the piecewise Cubic Hermite Interpolating polynomial.

Prob 4) Let f be a real-valued function defined on $[a, b]$. Consider the partition

$$a = x_0 \le x_1 \le x_2 \le ... \le x_n = b.$$

We say that the function f is

(i) **linear** spline function if f is continuous at $x_1,\ x_2, ...,\ x_{n-1}$.

(ii) **quadratic** spline function if f is continuously differentiable at $x_1,\ x_2, ...,\ x_{n-1}$.

(iii) **cubic** spline function if f is continuously differentiable 2-times at $x_1,\ x_2, ...,\ x_{n-1}$.

Following above definition, prove the followings:

1) Determine whether this is a quadratic spline function:

$$f(x) = \begin{cases} x & x \in [0,1] \\ -\frac{1}{2}(2-x)^2 + \frac{3}{2} & x \in [1,2] \\ \frac{3}{2} & x \in [2,3]. \end{cases}$$

2) Is the function in the preceding problem a cubic spline function?

3) Determine all the values of a, b, c, d, e for which the following function is a cubic spline:

$$f(x) = \begin{cases} a(x-2)^2 + b(x-1)^3 & x \in [0,1] \\ c(x-2)^2 & x \in [1,3] \\ d(x-2)^2 + e(x-3)^3 & x \in [3,4]. \end{cases}$$

Next, determine the values of the parameters so that the cubic spline interpolates this table:

x	0	1	4
f(x)	26	7	25

Prob 5) Determine Cubic Spline Interpolating function for the problem 1. For solving the system of equations obtained in the process, use Thomas algorithm.

Prob 6) Give the explicit form of numerical differentiation based on Lagrange interpolating polynomial for the following cases:

(i) $n = 0,\ i = 0$.

(ii) $n = 1,\ i = 0$ and $i = 1$.

(iii) $n = 2,\ i = 0$ and $i = 2$.

Solve at least one example in each case.

Prob 7) Derive the Newton-Cotes formula for $\displaystyle\int_0^1 f(x)dx$ based on the nodes $0, \frac{1}{3}, \frac{2}{3}$, and 1. Use this formula for evaluating the integral for the following functions e^{x^2}, x^2+x+1, and $\frac{1}{x+1}$.

Prob 8) Verify that the following formula is exact for polynomials of degree $\leq\ 4$:

$$\int_0^1 f(x)dx \approx \frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right].$$

Prob 9) From the formula in the preceding problem, obtain a formula for $\displaystyle\int_a^b f(x)dx$ that is exact for polynomials of degree $\leq\ 4$. Further, calculate $\ln 2$ approximately by applying the formula to $\displaystyle\int_1^2 \frac{dx}{x}$. Compare your answer to the correct value and compute the error.

Prob 10) Find a formula of the form

$$\int_0^{2\pi} f(x)dx = A_1 f(0) + A_2 f(\pi)$$

that is exact for any function of the form $a + b\cos x$. Prove that the resulting formula is exact for any function of the form

$$f(x) = \sum_{k=0}^{n} \left[a_k \cos(2k+1)x + b_k \sin kx\right].$$

Prob 11) Determine values for $A,\ B$, and C that makes the formula

$$\int_0^2 x f(x)dx \approx Af(0) + Bf(1) + Cf(2)$$

exact for all polynomials of degree as high as possible. What is the maximum degree?

Prob 12) The **mid-point rule** over the interval $[a, b]$ is given by

$$\int_a^b f(x)dx \approx (b-a) f\left(\frac{a+b}{2}\right).$$

Derive the composite mid-point rule over the interval $[a, b]$ with uniform spacing of $h = \frac{b-a}{n}$ such that $x_i = a + ih$ for $i = 0, 1, 2, ..., n$ (n is even). Further, derive the error term.

Prob 13) There are two Newton-Cotes formulas for $n = 2$ and $[a, b] = [0, 1]$; namely,

$$\int_0^1 f(x)dx \approx af(0) + bf\left(\frac{1}{2}\right) + cf(1).$$
$$\int_0^1 f(x)dx \approx \alpha f\left(\frac{1}{4}\right) + \beta f\left(\frac{1}{2}\right) + \gamma f\left(\frac{3}{4}\right).$$

Which is better?

Prob 14) Determine the minimum number of subintervals needed to approximate

$$\int_1^2 \left(x + e^{-x^2}\right) dx$$

to an accuracy of at least $\frac{1}{2} \times 10^{-7}$, using the Trapezoidal rule and the Simpson's 1/3rd rule.