The Threshold Theorem of Epidemiology

1. A small group of people introduce an ~~...~~ infections disease in a large population.
2. The disease has a short incubation period.
3. Recovered individuals gain permanent immunity.

There are three classes of population. They are:
i) $x \longrightarrow$ The infected class, ii) $y \longrightarrow$ The susceptible class.
iii) $z \longrightarrow$ The removed class (recovered class).

Rule 1: $\boxed{x(t) + y(t) + z(t) = N}$, where N is the fixed total number of population. (Conserved condition)

Rule 2: $\boxed{\dfrac{dy}{dt} \propto xy} \Rightarrow \boxed{\dfrac{dy}{dt} = -Axy}$ $A \rightarrow$ The infection rate.

Rule 3: $\boxed{\dfrac{dz}{dt} \propto x} \Rightarrow \boxed{\dfrac{dz}{dt} = Bx}$ $B \longrightarrow$ The removal rate.

Using Rule 1 we can write $\boxed{\dfrac{dx}{dt} = -\dfrac{dy}{dt} - \dfrac{dz}{dt}}$, which gives $\boxed{\dfrac{dx}{dt} = Axy - Bx}$. In all the three $\boxed{dx/dt}$, $\boxed{dy/dt}$ and $\boxed{dz/dt}$ equations, the right hand side does not depend on z. This a pseudo-third-order system.

I.) <u>The x-y equation</u>:

$$\boxed{\frac{dx/dt}{dy/dt} = \frac{dx}{dy} = \frac{Axy - Bx}{-Axy} = -1 + \frac{B}{Ay}}$$

$$\Rightarrow \int dx = \int \left(\frac{B}{Ay} - 1\right) dy \Rightarrow \boxed{x = \frac{B}{A} \ln y - y + c_1}$$

<u>Initial condition</u>: At $\boxed{t=0}$ (initially), $\boxed{x = x_0}$, $\boxed{y = y_0}$ and $\boxed{z=0}$. $\Rightarrow$ $\boxed{x_0 + y_0 = N}$.

$$\therefore \boxed{c_1 = x_0 + y_0 - \frac{B}{A}\ln y_0 = N - \frac{B}{A}\ln y_0}.$$

$$\Rightarrow \boxed{x = (x_0 + y_0) - y + \frac{B}{A}\ln(y/y_0)} \quad \boxed{x \equiv x(y)}$$

in close form.

II.) <u>The y-z equation</u>:

$$\boxed{\frac{dy/dt}{dz/dt} = \frac{dy}{dz} = \frac{-Axy}{Bx} = -\frac{Ay}{B}}$$

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{A}{B} dz \Rightarrow \boxed{\ln y = -\frac{Az}{B} + c_2}.$$

At $\boxed{t=0}$, $\boxed{y = y_0}$ and $\boxed{z=0}$ $\Rightarrow$ $\boxed{c_2 = \ln y_0}$

$\Rightarrow \boxed{\ln\left(\frac{y}{y_0}\right) = -\frac{A}{B} z}$. For $t>0$, $\boxed{y < y_0}$

$\Rightarrow \boxed{\ln\left(\frac{y}{y_0}\right) < 0}$ for <u>$t>0$</u> $\Rightarrow$ $\boxed{z = -\frac{B}{A}\ln\left(\frac{y}{y_0}\right) > 0}$ for <u>$t>0$</u>.

And $\boxed{y = y_0 \exp\left(-\frac{A}{B} z\right)}$ $\boxed{y = y(z)}$ in close form.

III.) <u>The z - x equation</u> :

$$\boxed{\frac{dx/dt}{dz/dt} = \frac{dx}{dz} = \frac{Axy - Bx}{Bx} = \frac{Ay}{B} - 1}.$$

Now $\boxed{y = y_0 e^{-\frac{Az}{B}}} \Rightarrow \boxed{\frac{dx}{dz} = \frac{A}{B} y_0 e^{-\frac{Az}{B}} - 1}.$

$$\Rightarrow x = \int \frac{A}{B} y_0 e^{-\frac{A}{B}z}\, dz - z + C_3 .$$

$$\Rightarrow \quad x = \frac{A}{B} y_0 \frac{e^{-\frac{Az}{B}}}{-A/B} - z + C_3 .$$

$$\Rightarrow \quad \boxed{x = -y_0 e^{-Az/B} - z + C_3}.$$

when (at $t=0$), $\boxed{x = x_0}$ and $\boxed{z = 0}$,

$\boxed{C_3 = x_0 + y_0} \Rightarrow \boxed{x = x_0 + y_0\left(1 - e^{-Az/B}\right) - z}$

OR $\boxed{x = (x_0 + y_0) - y_0 \exp\left(-\frac{Az}{B}\right) - z}$ But z is <u>not</u> written in a <u>closed form for x</u>.

<u>Plot of x - y:</u> $\boxed{x = (x_0 + y_0) - y + \frac{B}{A} \ln\left(\frac{y}{y_0}\right)}.$

i.) when $y \longrightarrow 0$, $x \longrightarrow -\infty$.

ii.) $\boxed{\frac{dx}{dt} = x(Ay - B)} \Rightarrow \boxed{\frac{dx}{dt} = 0}$, when either $\boxed{x = 0}$ OR $\boxed{y = B/A}$.

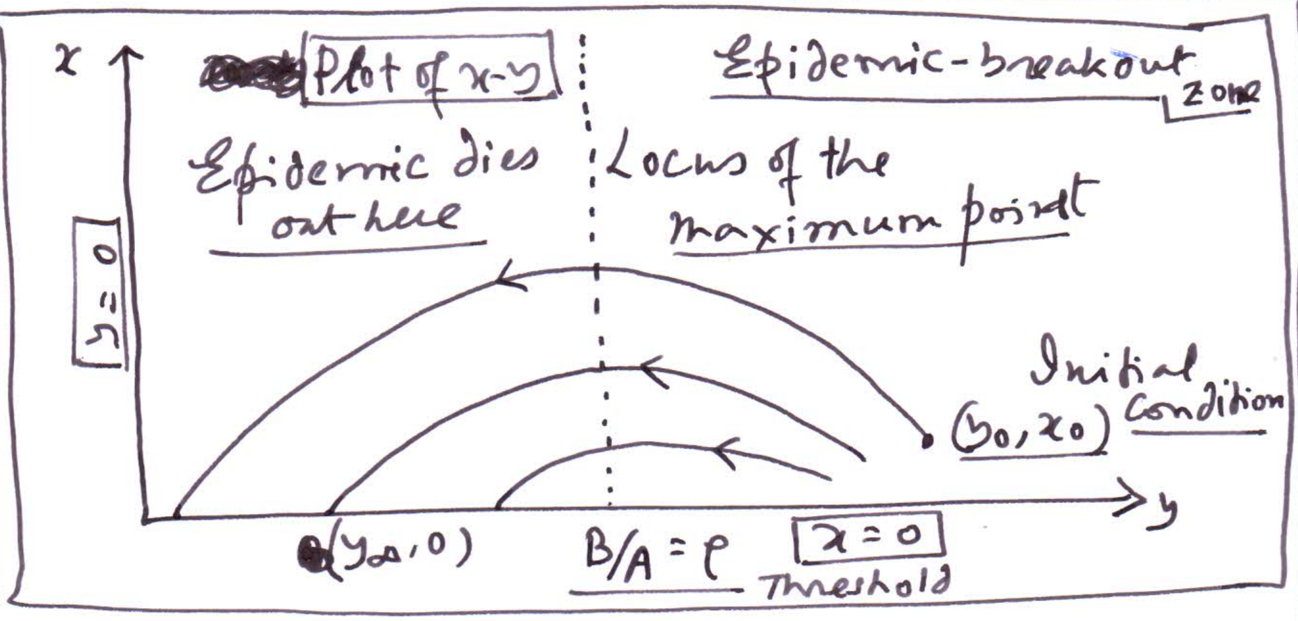
iii.) $\boxed{\frac{dx}{dy} = \frac{B}{Ay} - 1} \Rightarrow \boxed{\frac{dx}{dy} = 0}$, when $\boxed{y = \frac{B}{A} = \rho \text{ (say)}}$

iv.) $\boxed{\frac{d^2x}{dy^2} = -\frac{B}{Ay^2}}$ At $\boxed{y = \frac{B}{A}}$, $\boxed{\frac{d^2x}{dy^2} = -\frac{A}{B} < 0}$. Hence, <u>$y = B/A$ is a maximum.</u>

v.) When $\boxed{x = 0}$, write $\boxed{y = y_\infty}$ (say).

With $x=0$, the point becomes an equilibrium point, since both $\boxed{\frac{dx}{dt} = \frac{dy}{dt} = 0}$.

Note: x is plotted along the vertical axis, and y along the horizontal



Conclusion 1: An epidemic will breakout, if $(y_0 > \rho)$, i.e. the initial number of susceptibles are above the threshold ρ. This is the Threshold Theorem of Epidemiology (Kermack & McKendrick)

Conclusion 2: The spread of the disease stops $(x=0)$ because the infective population is reduced to zero, even though there may be some susceptibles left.

Practically speaking epidemics breakout due to overcrowding of a susceptible population in an unhygienic environment.

Case of Initial Number of Susceptibles Slightly Higher than the Threshold

$\boxed{\dfrac{dx}{dt} = x(Ay - B)}$ and $\boxed{x = (x_0 + y_0) - y + \dfrac{B}{A}\ln\left(\dfrac{y}{y_0}\right)}$.

When $\boxed{\dfrac{dx}{dt} = 0}$ Either $\boxed{y = B/A = \rho}$ or $\boxed{x = 0}$.

i.) The former case is a turning point in which $y = \rho$ is a maximum $\Rightarrow \boxed{d^2x/dt^2 < 0}$.

ii.) $\boxed{\dfrac{d^2x}{dt^2} = \dfrac{dx}{dt}(Ay - B) + x\dfrac{d}{dt}(Ay - B)}$. When $\boxed{x = 0}$ at $\boxed{\dfrac{dx}{dt} = 0} \Rightarrow \boxed{\dfrac{d^2x}{dt^2} = 0}$. This is an equilibrium point. (at $x = 0$).

When $\boxed{x = 0}$, we write $\boxed{y = y_\infty}$. With these, we get $\boxed{0 = x_0 + y_0 - y_\infty + \dfrac{B}{A}\ln\left(\dfrac{y_\infty}{y_0}\right)}$. Usually $\boxed{x_0 \ll y_0}$, and so we neglect x_0 above.

Now y_0 is the initial value of the susceptibles, and y_∞ is the final value of the susceptibles.

$\therefore$ $y_0 - y_\infty$ is the number of people who have contracted the infectious disease. Further, $\boxed{y_\infty < y_0}$, which implies that $\boxed{y_0 - y_\infty > 0}$.

Hence we can write $\boxed{(y_0 - y_\infty) + \dfrac{B}{A}\ln\left(\dfrac{y_0 - y_0 + y_\infty}{y_0}\right) \simeq 0}$

$\Rightarrow \boxed{(y_0 - y_\infty) + \rho\ln\left(1 - \dfrac{y_0 - y_\infty}{y_0}\right) \simeq 0}$ in which we have neglected x_0.

We now consider the case where the initial number of susceptibles is slightly greater than the threshold, i.e. $\boxed{y_0 = \rho + \epsilon}$, in which $\boxed{\epsilon \ll \rho}$. $\therefore \boxed{\frac{(y_0 - \rho)}{\rho} = \frac{\epsilon}{\rho} \ll 1}$.

From the plot of x-y we can see that in this ~~case~~ case $\boxed{(y_0 - y_\infty) \ll y_0}$. Hence using the formula $\boxed{\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \dots}$, we can write $\boxed{\ln\left[1 - \left(\frac{y_0 - y_\infty}{y_0}\right)\right] = -\left(\frac{y_0 - y_\infty}{y_0}\right) - \frac{1}{2}\left(\frac{y_0 - y_\infty}{y_0}\right)^2}$

going only up to the second order term.

Hence, $(y_0 - y_\infty) + \rho\left[-\frac{(y_0 - y_\infty)}{y_0} - \frac{1}{2}\left(\frac{y_0 - y_\infty}{y_0}\right)^2\right] \simeq 0$

$$\Rightarrow \boxed{(y_0 - y_\infty)\left[1 - \frac{\rho}{y_0} - \frac{\rho}{2}\frac{(y_0 - y_\infty)}{y_0^2}\right] \simeq 0} \quad (\text{Now, } y_0 - y_\infty \neq 0)$$

$$\Rightarrow \quad y_0 - y_\infty \simeq \frac{2y_0^2}{\rho}\left(1 - \frac{\rho}{y_0}\right) \simeq 2y_0\left(\frac{y_0}{\rho} - 1\right)$$

But $\boxed{y_0 = \rho + \epsilon}$ and $\boxed{\left(\frac{y_0}{\rho} - 1\right) = \frac{\epsilon}{\rho}}$. Using these

we get, $\quad y_0 - y_\infty \simeq 2y_0\epsilon = 2(\rho + \epsilon)\epsilon/\rho$

Neglecting ϵ in $\boxed{\rho + \epsilon \simeq \rho}$, we finally get.

$$\boxed{y_0 - y_\infty \simeq 2\rho\,\epsilon/\rho} \Rightarrow \boxed{y_0 - y_\infty = 2\epsilon}.$$

$\therefore \boxed{y_0 - y_\infty \simeq 2(y_0 - \rho)}$, an approximate result that is valid only when y_0 is slightly greater than ρ.

$z(t)$ as a Function of Time : $z \equiv z(t)$.

$\boxed{\dfrac{dz}{dt} = Bx}$, $\boxed{x = (x_0 + y_0) - y + \dfrac{B}{A} \ln\left(\dfrac{y}{y_0}\right)}$ and

$\boxed{y = y_0 \exp\left(-\dfrac{Az}{B}\right)}$. $\Rightarrow$ $\boxed{z = -\dfrac{B}{A} \ln\left(y/y_0\right)}$.

Hence, $\boxed{\dfrac{dz}{dt} = B\left[x_0 + y_0 - y_0 \exp\left(-\dfrac{A}{B} z\right) - z\right]}$,

an equation in an autonomous form $\boxed{\dfrac{dz}{dt} = f(z)}$.

Noting $\boxed{\dfrac{B}{A} = \rho}$, we consider the case of $\boxed{z \ll \rho}$.

$\therefore$ $\boxed{e^{-z/\rho} \simeq 1 - \dfrac{z}{\rho} + \dfrac{1}{2}\left(\dfrac{z}{\rho}\right)^2 + \cdots}$, going up to the second order.

$$\Rightarrow \frac{dz}{dt} \simeq B\left[x_0 + \cancel{y_0} - \cancel{y_0} + y_0 \frac{z}{\rho} - \frac{y_0}{2}\left(\frac{z}{\rho}\right)^2 - z\right]$$

$$\Rightarrow \frac{dz}{dt} \simeq B\left[x_0 + \left(\frac{y_0}{\rho} - 1\right) z - \frac{y_0}{2}\left(\frac{z}{\rho}\right)^2\right].$$

We define $\boxed{\zeta = z/\rho}$, by which we get,

$$\boxed{\frac{\rho}{B}\frac{d\zeta}{dt} = x_0 + (y_0 - \rho)\zeta - \frac{y_0}{2}\zeta^2}.$$

Multiplying throughout by $2/y_0$ we get.

$$\boxed{\frac{2}{y_0}\frac{\rho}{B}\frac{d\zeta}{dt} = \frac{2x_0}{y_0} + 2\left(1 - \frac{\rho}{y_0}\right)\zeta - \zeta^2}.$$

We further define $\boxed{\alpha^2 = 2x_0/y_0}$ and $\boxed{\beta = 1 - \rho/y_0}$.

Hence $\frac{2}{y_0}\frac{\ell}{B}\frac{d\delta}{dt} = -(\delta^2 - 2\beta\delta + \beta^2 - \beta^2) + \alpha^2$

$$\Rightarrow \boxed{\frac{2}{y_0 A}\frac{d\delta}{dt} = (\beta^2+\alpha^2) - (\delta-\beta)^2} \quad \because \frac{\ell}{B} = \frac{\ell}{A}\cdot\frac{1}{\ell}$$

Further define $\boxed{k^2 = \alpha^2+\beta^2}$ and $\boxed{\xi_s = \delta - \beta}$,

which gives $\boxed{\frac{d\delta}{dt} = \frac{d\xi_s}{dt}}$. Using this, we get,

$$\boxed{\frac{2}{Ay_0}\frac{d\xi_s}{dt} = k^2 - \xi_s^2} \Rightarrow \frac{2}{Ay_0 k}\frac{d}{dt}\left(\frac{\xi_s}{k}\right) = 1 - \left(\frac{\xi_s}{k}\right)^2$$

Again, defining $\boxed{\psi = \xi_s/k}$ and $\boxed{\tau = \frac{Ay_0 k t}{2}}$,

we obtain finally, $\boxed{\frac{d\psi}{d\tau} = 1-\psi^2}$. This

equation can be integrated to obtain.

$$\boxed{\ln\left(\frac{1+\psi}{1-\psi}\right) = 2\tau + 2C}$$, where C is an integration constant.

(Check for the solution of $\boxed{\frac{dx}{dT} = 1-x^2}$ equation)

Writing $\boxed{T = \tau + C}$, we get, $\boxed{\frac{1+\psi}{1-\psi} = e^{2T}}$, from

which we get $1+\psi = (1-\psi)e^{2T} \Rightarrow \psi(1+e^{2T}) = e^{2T}-1$

$$\Rightarrow \boxed{\psi = \frac{e^{2T}-1}{e^{2T}+1}} \Rightarrow \boxed{\psi = \frac{e^{T}-e^{-T}}{e^{T}+e^{-T}} = \tanh(T)}.$$

$$\therefore \boxed{\frac{\xi_s}{k} = \tanh(\tau + c)} \Rightarrow \boxed{\frac{\delta-\beta}{k} = \tanh(\tau+c)}.$$

$$\Rightarrow \quad \zeta = \beta + k \tanh(\tau + c)$$

$$\Rightarrow \quad \frac{z}{\rho} = \beta + k \tanh\left(\frac{A y_0 k}{2} t + c\right)$$

$$\Rightarrow \quad \boxed{z = \beta\rho + k\rho \tanh\left(\frac{A y_0 k}{2} t + c\right)} \quad \boxed{z = z(t)}$$

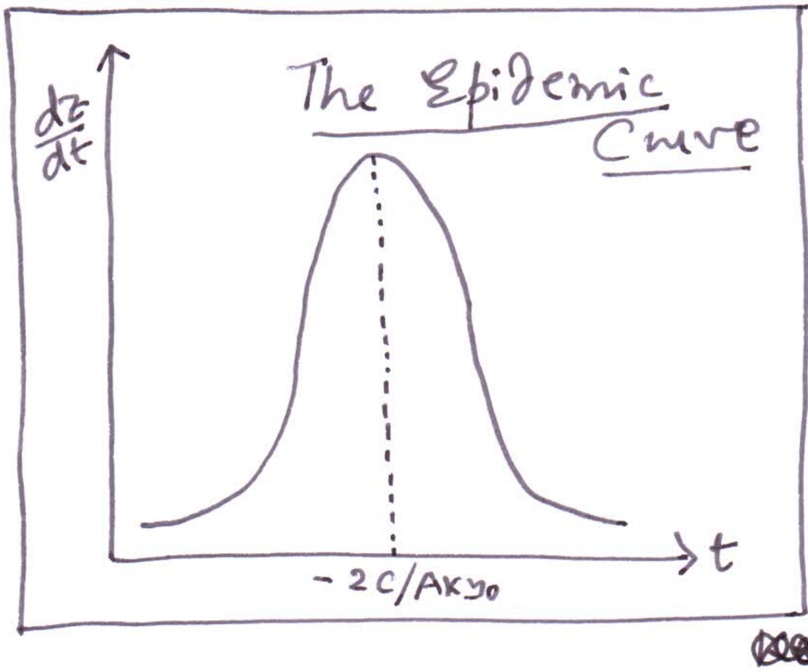
Hence, $\dfrac{dz}{dt} = k\rho \cdot \left(\dfrac{A y_0 k}{2}\right) \operatorname{sech}^2\left(\dfrac{A y_0 k}{2} t + c\right)$

$$\Rightarrow \quad \boxed{\frac{dz}{dt} = \frac{B y_0 k^2}{2} \operatorname{sech}^2\left(\frac{A y_0 k}{2} t + c\right)}.$$

The integration constant c can be fixed from the initial condition, $\boxed{z = 0}$ at $\boxed{t = 0}$.

$$\Rightarrow \quad \boxed{0 = \beta\rho + k\rho \tanh(c)} \quad \Rightarrow \boxed{\tanh(c) = -\frac{\beta}{k}}$$

Now $\boxed{\beta = 1 - \dfrac{\rho}{y_0}}$ $\therefore$ $\boxed{\tanh(c) = \dfrac{1}{k}\left(\dfrac{\rho}{y_0} - 1\right)}$.



i) The epidemic curve reaches a maximum when $\boxed{\dfrac{A y_0 k}{2} t + c = 0}$.

$\Rightarrow \boxed{t = -2c/AKy_0}$ $(c < 0)$

ii) The removal rate dz/dt climbs to a peak value and dies out.

iii) Data matched by the Bombay plague of 1905.