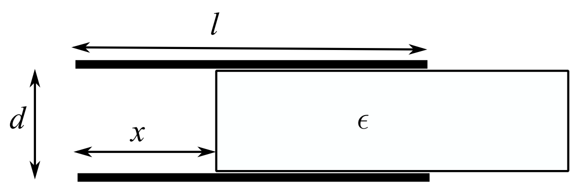
1. A slab of linear dielectric material is partially inserted between the plates of a parallel plate capacitor whose area is l^2 and the distance between the plates is d. Find the force by which the slab is sucked inside the capacitor while it is charged to a potential V.



soln
The width of the capacitor is l and let x be the length over which the capacitor is filled with air while $l-x$ be the length over which it is filled with the dielectric material of permitivity ϵ. The capacitance of this configuration can be shown to be (looked upon as two capacitors connected in parallel):

$$\begin{aligned} C &= \frac{\epsilon_0 l^2}{d}\frac{x}{l} + \frac{\epsilon l^2}{d}\frac{l-x}{d} \\ &= \frac{l}{d}\left[\epsilon l + x(\epsilon_0 - \epsilon)\right] \end{aligned}$$

The work done in charging the capacitor to potential V is

$$W = \frac{1}{2}CV^2 = \frac{lV^2}{2d}\left[\epsilon l + x(\epsilon_0 - \epsilon)\right]$$

The force acting on the dielectric slab is given as

$$F = -\frac{\partial W}{\partial x} = -\frac{V^2}{2}\frac{\partial C}{\partial x} = \frac{V^2}{2}\frac{l}{d}(\epsilon - \epsilon_0)$$

This is the force with which the dielectric slab is pulled in.

2. Consider a charged particle with mass m placed at $(0, -a, 0)$ in a magnetic field $\vec{B} = B\hat{i}$ and has an initial velocity $\vec{v}_0 = v_0\hat{k}$. Derive the equation of trajectory that the particle follows due to the magnetic force and find the condition for which the particle follows a circular motion of radius a.

soln:
The force on a particle moving in a magnetic field is given as $q\vec{v}\times\vec{B}$. So we have by Newton's law

$$\frac{d\vec{v}}{dt} = \frac{q}{m}\vec{v}\times\vec{B} \tag{1}$$

If we take

$$\vec{v} = \vec{r}\times\vec{\alpha} + \vec{C} \tag{2}$$

then we have

$$\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\alpha}$$

Here $\vec{\alpha}$ and $\vec{C}$ are constant vectors. Comparing this equation with Eq. 1 gives

$$\vec{\alpha} = \frac{q\vec{B}}{m} = \vec{\omega}$$

The initial conditions:
When $\vec{r} = \vec{r}_0, \quad \vec{v} = \vec{v}_0$, from Eq. 2 we get

$$\vec{v}_0 = \vec{r}_0 \times \vec{\omega} + \vec{C}$$

Now we take $\vec{r}_0 = -a\hat{j}, \quad \vec{B} = B\hat{i}, \quad \vec{v}_0 = v_0\hat{k}$ in the above equation.

$$v_0\hat{k} = -a\hat{j} \times \omega\hat{i} + \vec{C} = a\omega\hat{k} + \vec{C}$$

This gives $a\omega + C_z = v_0$ and $C_x = C_y = 0$.

Now Equation (2) can be written as

$$\vec{v} = \vec{r} \times \omega\hat{i} + (v_0 - a\omega)\hat{k}$$

It is instructive to look at $(v_0 - a\omega)\hat{k}$ as $-(\frac{v_0}{\omega} - a)\hat{j} \times \omega\hat{i}$. Then the trajectory is described as

$$\vec{v} = \left[\vec{r} - (\frac{v_0}{\omega} - a)\hat{j}\right] \times \omega\hat{i}$$

This describes a motion where the vector $\vec{r} - (\frac{v_0}{\omega} - a)\hat{j}$ rotates about $\hat{i}$ with angular velocity ω. The center of rotation is at $\vec{C} = (v_0/\omega - a)\hat{j}$. To find the radius we subtract the center from the initial position vector. This gives $R = v_0/\omega$.
When $v_0 = a\omega$ the center is at the origin and the radius is a.

3. For a configuraion of charges and currents confined within a volume $\mathcal{V}$, show that

$$\int_{\mathcal{V}} \vec{\mathrm{J}} d\tau = \frac{d\vec{\mathrm{p}}}{dt}$$

where $\vec{\mathrm{p}}$ is the total dipole moment.

[Hint: Evaluate $\int_{\mathcal{V}} \vec{\nabla} \cdot (x\vec{J}) d\tau$]

soln
The charges and the currents are confined within the volume $\mathcal{V}$. So no current crosses the surface S. Under these conditions we have to show that

$$\int_{\mathcal{V}} \vec{\mathrm{J}} d\tau = \frac{d\vec{\mathrm{p}}}{dt}$$

where $\vec{\mathrm{p}}$ is the total dipole moment of the charge distribution in the volume $\mathcal{V}$. We start with the R.H.S.
Let $\rho(\vec{r}, t)$ be the charge density in the region.

$$\vec{p}(t) = \int_{\mathcal{V}} \vec{r}\rho(\vec{r})d\tau$$

$$\therefore \quad \frac{d\vec{p}}{dt} = \int_{\mathcal{V}} \vec{r}\frac{\partial\rho}{\partial t}d\tau = \hat{i}\int x\frac{\partial\rho}{\partial t}d\tau + \hat{j}\int y\frac{\partial\rho}{\partial t}d\tau + \hat{k}\int z\frac{\partial\rho}{\partial t}d\tau$$

Let us consider the integral $\int x\frac{\partial\rho}{\partial t}d\tau$. By continuity equation $\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$.

$$\therefore \quad \int x\frac{\partial\rho}{\partial t}d\tau = -\int_{\mathcal{V}} x(\vec{\nabla}\cdot\vec{J})d\tau$$

$$\vec{\nabla}\cdot(x\vec{J}) = x\vec{\nabla}\cdot\vec{J} + \vec{\nabla}x\cdot\vec{J} = x\vec{\nabla}\cdot\vec{J} + J_x$$

$$\begin{aligned}\therefore \quad \int_{\mathcal{V}} x\frac{\partial\rho}{\partial t}d\tau &= \int_{\mathcal{V}} J_x d\tau - \int_{\mathcal{V}} \vec{\nabla}\cdot(x\vec{J})d\tau \\ &= \int_{\mathcal{V}} J_x d\tau - \int_{S} x\vec{J}\cdot\hat{n}da\end{aligned}$$

Since no current crosses the surface S we have $\vec{J}\cdot\hat{n} = 0$.

$$\therefore \quad \int_{\mathcal{V}} x\frac{\partial\rho}{\partial t}d\tau = \int_{\mathcal{V}} J_x d\tau$$

$$\therefore \quad \frac{d\vec{p}}{dt} = \hat{i}\int_{\mathcal{V}} J_x d\tau + \hat{j}\int_{\mathcal{V}} J_y d\tau + \hat{k}\int_{\mathcal{V}} J_z d\tau = \int_{\mathcal{V}} \vec{J}d\tau$$

4. Find the magnetic field at point P for each of the steady current configurations shown below

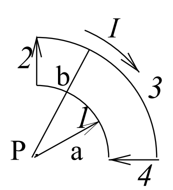


Fig.1

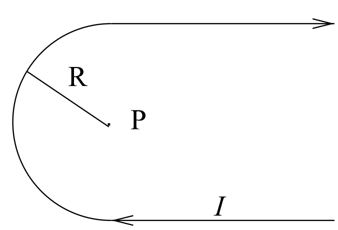


Fig. 2

soln
(a)
By Bio Savart Law, at P

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint\frac{\vec{dl}\times\hat{r}}{r^2}$$

We can break the integral into four pieces. Along arc 1 $\vec{dl} \times \hat{r}$ is a vector comming out orthogonally from the paper. Also $|\vec{dl} \times \hat{r}| = dl$.
So due to arc 1

$$\vec{B}_1 = \hat{n}\frac{\mu_0 I}{4\pi}\int \frac{dl}{a^2} = \frac{\mu_0 I}{8a}\hat{n}$$

By the same argument due to arc 3

$$\vec{B}_3 = -\frac{\mu_0 I}{8b}\hat{n}$$

The contribution from the straight pieces 2 and 4 are 0 since $\vec{dl} \times \hat{r} = 0$ along them.

$$\therefore \quad \vec{B} = \frac{\mu_0 I}{8}\left[\frac{1}{a} - \frac{1}{b}\right]\hat{n}$$

(b)
Due to the semicircular arc the field at P is

$$\vec{B}_1 = -\frac{\mu_0 I}{4R}\hat{n}$$

Due to the lower straight piece the field at P is downward $(-\hat{n})$. This field will be half of the field due to an infinite straight wire. This is equal to $\frac{\mu_0 I}{4\pi R}$. We have two such straight pieces producing field in the same direction $-\hat{n}$. The sum of these are

$$\vec{B} = \left(\frac{\mu_0 I}{4R} + \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{4\pi R}\right)(-\hat{n}) = \frac{\mu_0 I}{4R}\left(1 + \frac{2}{\pi}\right)(-\hat{n})$$

5. Two parallel, infinite line charges λ, a distance d apart are moving at a constant velocity $\vec{v}$. The direction of $\vec{v}$ is along the line charges. How great would v have to be in order for the magnetic attraction to balance the electrical repulsion?

soln
The moving line charges constitute a current $I = \lambda v$. The charges on one wire experience a force of attraction due to the magnetic field caused due to this current.

$$\begin{aligned} F_{mag} &= B\lambda v \quad \text{per unit length} \\ &= \frac{\mu_0 I}{2\pi d}\lambda v = \frac{\mu_0 \lambda^2 v^2}{2\pi d} \end{aligned}$$

The electric field due to one line charge on the other at a distance d is

$$E = \frac{\lambda}{2\pi\epsilon_0 d}$$

The repulsive force per unit length on the other wire is

$$F_{elect} = E\lambda = \frac{\lambda^2}{2\pi\epsilon_0 d}$$

When $F_{elect} = F_{mag}$ we have

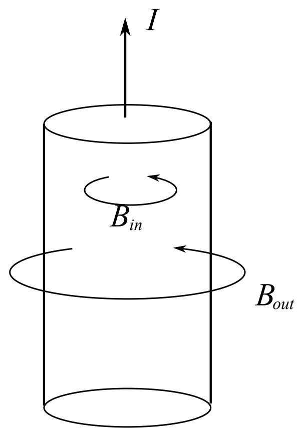
$$\frac{\lambda^2}{2\pi\epsilon_0 d} = \frac{\mu_0 \lambda^2 v^2}{2\pi d}$$

$$\therefore \quad v^2 = \frac{1}{\mu_0 \epsilon_0} = c^2$$

where c is the velocity of light.
The two forces balance only when they move with the velocity of light.

6. A steady current I flows through a long cylindrical wire of radius a. Find the magnetic field both inside and outside the wire, if

   (a) The current is uniformly distributed over the outside surface of the wire.
   soln



The current is only over the surface

$$\therefore \quad K = \frac{I}{2\pi a} \qquad \Longrightarrow \qquad \vec{K} = \frac{I}{2\pi a}\hat{z}$$

By Ampere's law, over a loop concentric with the axis of the wire,

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$$

If the loop is inside the cylinder i.e, the radius of the loop $s < a$ then
$B \times 2\pi s = \mu_0 \times 0$. So $B_{in} = 0$.
To find B_{out} we have

$$B_{out} \times 2\pi s = \mu_0 I \quad \Longrightarrow \quad B_{out} = \frac{\mu_0 I}{2\pi s}$$

This magnetic field is along $\hat{\phi}$. $\therefore \quad \vec{B}_{out} = \mu_0 I/2\pi s \hat{\phi}$

   (b) The current is distributed in such a way that J is proportional to s.
   soln:
   $\vec{J} = \alpha s \hat{z}$ where α is a proportionality constant.

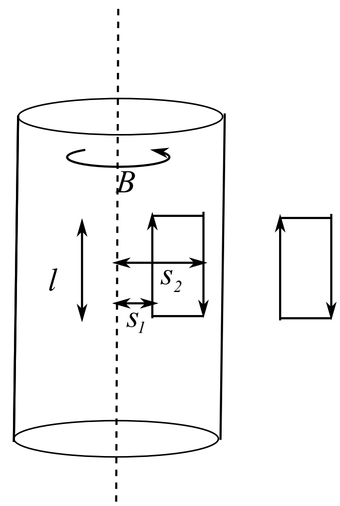
$$\int_0^a \alpha s(2\pi s) ds = I \quad \Longrightarrow \quad \alpha = \frac{3I}{2\pi a^3}$$

Inside the wire

$$
\begin{aligned}
B_{in} 2\pi s &= \mu_0 I_{enc} = \mu_0 \int_0^s \alpha s 2\pi s ds \\
&= \mu_0 \alpha 2\pi s^3/3 = \mu_0 I \frac{s^3}{a^3} \\
\vec{B}_{in} &= \frac{\mu_0 I}{2\pi a^3} s^2
\end{aligned}
$$

The field outside remains the same as in part (a).

(c) Find the vector potential in both the cases above.
soln
(for case (a))



$\vec{\nabla} \times \vec{A} = \vec{B}$
$\therefore \ \oint_C \vec{A} \cdot \vec{dl} = \int_S \vec{B} \cdot \hat{n} da$ where S is the surface enclosed by C.
For inside the wire consider a loop as shown. We consider a vector potential along z axis

$$
(A_z(s_1) - A_z(s_2)) l = \int_S \vec{B}_{in} \cdot \hat{n} da = 0
$$

$\therefore \ \ A_z(s_1) = A_z(s_2)$.
$\therefore \ \ A_z^{in} = c_1$, a constant.
For outside

$$
(A_z(s_1) - A_z(s_2)) l = \int_{s1}^{s2} B_{out} dsl
$$

$$
\therefore \ \ A_z(s_1) - A_z(s_2) = \int_{s1}^{s2} \frac{\mu_0 I}{2\pi s} ds \frac{\mu_0 I}{2\pi} \ln\left(\frac{s_2}{s_1}\right)
$$

$$
\therefore \quad - A_z^{out}(s) = \frac{\mu_0 I}{2\pi} \ln\left(\frac{s}{s_0}\right)
$$

Matching the vector potential on the surface of the cylinder we have

$$
c_1 = \frac{\mu_0 I}{2\pi} \ln\left(\frac{a}{s_0}\right)
$$

$$
\begin{aligned}
\therefore\ \vec{A}^{in} &= -\frac{\mu_0 I}{2\pi}\ln\left(\frac{a}{s_0}\right)\hat{z} \\
\vec{A}^{out} &= -\frac{\mu_0 I}{2\pi}\ln\left(\frac{s}{s_0}\right)\hat{z}
\end{aligned}
$$

s_0 is arbitrary.

(for case (b))
Since $\vec{B}$ is along $\hat{\phi}$ everywhere, $\vec{A}$ will be along $\hat{z}$. So let $\vec{A} = A_z(s)\hat{z}$.

$$
\vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = \vec{B}
$$

For inside $-\frac{\partial A_z}{\partial s} = \frac{\mu_0 I}{2\pi a^3}s^2$

$$
\therefore\ A_z^{in} = -\frac{\mu_0 I}{6\pi a^3}s^3 + c_1
$$

For outside $-\frac{\partial A_z}{\partial s} = \frac{\mu_0 I}{2\pi s}$

$$
\therefore\ A_z^{out} = -\frac{\mu_0 I}{2\pi}\ln(s) + c_2
$$

By matching the two vector potential at the cylindrical surface we have

$$
-\frac{\mu_0 I}{6\pi} + c_1 = -\frac{\mu_0 I}{2\pi}\ln(a) + c_2
$$

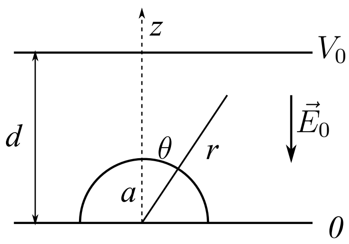
If we take $c_2 = 0$ then

$$
\begin{aligned}
c_1 &= -\frac{\mu_0 I}{2\pi}\ln(a) + \frac{\mu_0 I}{6\pi} \\
&= -\frac{\mu_0 I}{6\pi}(\ln(a^3) - 1)
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \vec{A}_{in} &= -\frac{\mu_0 I}{6\pi}\left[\frac{s^3}{a^3} + \ln(a^3) - 1\right]\hat{z} \\
\vec{A}_{out} &= -\frac{\mu_0 I}{2\pi}\ln s
\end{aligned}
$$

All the vector potentials found are along $\hat{z}$ and they are functions of s, i.e, they are of the type $A_z(s)\hat{z}$. So they are all divergenceless. $\vec{\nabla}\cdot\vec{A} = \frac{\partial A_z(s)}{\partial z=0}$.

7. Two very large metal plates are held a distance d apart, one at potential 0, the other at potential V_0. A small metal hemisphere (radius $a << d$) is placed on the grounded plate, so that its potential is likewise 0. If the region between the plates is filled with weakly conducting material of uniform conductivity σ, what current flows to the hemisphere?



soln:
In the absence of the hemisphere the electric field is

$$\vec{E}_0 = -\frac{V_0}{d}\hat{z}$$

So this problem is same as placing a grounded spherical conductor in an uniform electric field $\vec{E}_0$.
The potential at a point (r, θ) in this problem is given by

$$V(r,\theta) = E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

The electric field is given by

$$\vec{E} = -\vec{\nabla}V = -\left[\hat{r}\frac{\partial V}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta}\right]$$

The current density is

$$\vec{J} = \sigma\vec{E}$$

On the surface of the hemisphere $r = a$. $\vec{E}$ and $\vec{J}$ only has $\hat{r}$ component given by

$$\vec{J} = -3\sigma E_0\cos\theta\hat{r}$$

The total current into the hemisphere is

$$\begin{aligned}
I &= \int_S \vec{J}\cdot\hat{r}da = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} 3\sigma E_0\cos\theta a^2\sin\theta d\theta d\phi \\
&= 3\sigma E_0(2\pi)a^2 0^{\frac{\pi}{2}}\cos\theta\sin\theta d\theta \\
&= 3\pi a^2\sigma E_0 \\
&= -3\pi\sigma a^2\frac{V_0}{d}
\end{aligned}$$

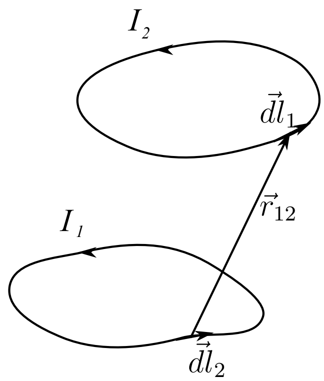
The negative sign indicates that the current flows into the hemisphere.

8. Magnetostatics treats the source current (the one that sets up the field) and the recipient current (the one that experiences the force) so asymmetrically that it is by no means obvious that the magnetic force between the current loops is consistent with the Newton's third law. Show that the force on loop 2 due to loop 1 can be written as

$$\vec{\mathrm{F}}_2 = -\frac{\mu_0}{4\pi}I_1I_2\oint\oint\frac{\hat{\mathrm{r}}_{12}}{r_{12}^2}\vec{\mathrm{dl}}_1\cdot\vec{\mathrm{dl}}_2$$

where $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$

soln



$$\vec{F}_{21} = I_2 \int_{C2} \vec{dl}_2 \times \vec{B}_1$$

where $\vec{B}_1$ is the magnetic field created due to the current in the loop 1.

$$\begin{aligned}
\therefore \quad \vec{F}_{21} &= I_2 \int_{C2} \vec{dl}_2 \times \left(\frac{\mu_0}{4\pi} I_1 \int_{C1} \frac{\vec{dl}_1 \times \vec{r}_{12}}{r_{12}^2} \right) \\
&= \frac{\mu_0 I_1 I_2}{4\pi} \int_{C2} \int_{C1} \frac{\vec{dl}_2 \times (\vec{dl}_1 \times \hat{r}_{12})}{r_{12}^2} \\
&= \frac{\mu_0 I_1 I_2}{4\pi} \left[\int_{C2} \int_{C1} \frac{\vec{dl}_1 (\vec{dl}_2 \cdot \hat{r}_{12})}{r_{12}^2} - \int_{C2} \int_{C1} \frac{\hat{r}_{12}}{r_{12}^2} (\vec{dl}_1 \cdot \vec{dl}_2) \right]
\end{aligned}$$

The first integral can be written as

$$\oint_{C1} \vec{dl}_1 \oint_{C2} \frac{\hat{r}_{12}}{r_{12}^2} \cdot \vec{dl}_2$$

The integral along C_2 is an integral of a vector function $\frac{\hat{r}_{12}}{r_{12}^2}$ along a closed path C_2. This is like the integral of an electric field at 2 due to a point charge at $\vec{r}_1$. This integral is 0 since static electric field is curlless. So the first integral is 0.

$$\therefore \quad \vec{F}_2 = -\frac{\mu_0 I_1 I_2}{4\pi} \int_{C2} \int_{C1} \frac{\hat{r}_{12}}{r_{12}^2} (\vec{dl}_1 \cdot \vec{dl}_2)$$

When we calculate $\vec{F}_1$, the force on loop 1, everything remains the same but $\hat{r}_{12}$ is repalced by $\hat{r}_{21}$ which has an opposite direction. So $\vec{F}_1 = -\vec{F}_2$. This is consistent with Newton's third law of motion.