1. Evaluate

   (a) $\int(r^2 + \vec{\mathrm{r}}\cdot\vec{\mathrm{a}} + a^2)\delta^3(\vec{\mathrm{r}}-\vec{\mathrm{a}})dV$ over the whole space where $\vec{\mathrm{a}}$ is a fixed vector.

   soln

$$\int_V (r^2 + \vec{\mathrm{r}}\cdot\vec{\mathrm{a}} + a^2)\delta^3(\vec{\mathrm{r}}-\vec{\mathrm{a}})dV = 3a^2$$

   (b) $\int_V |\vec{\mathrm{r}}-\vec{\mathrm{b}}|^2\delta^3(5\vec{\mathrm{r}})dV$ over a cube of side 2, centered at the origin, and $\vec{\mathrm{b}} = 4\hat{\mathrm{y}} + 3\hat{\mathrm{z}}$

   soln

   $dV = r^2 dr \sin\theta d\theta d\phi$.

   Let $5\vec{r} = \vec{r'}$. Then

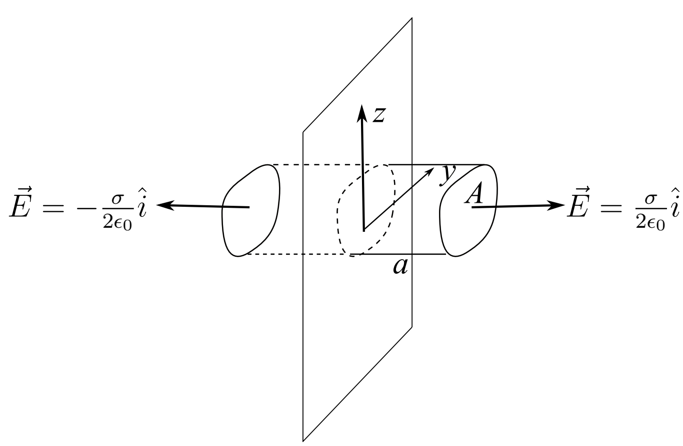
$$\begin{aligned} dv' &= r'^2 dr' \sin\theta d\theta d\phi = 5^3 r^2 dr \sin\theta d\theta d\phi \\ &= 5^3 dV \\ \therefore \int_V |\vec{r}-\vec{b}|\delta^3(5\vec{r})dV &= \int_{V'} |\frac{\vec{r'}}{5} - \vec{b}|\delta^3(\vec{r'})\frac{1}{5^3}dV' \\ &= \frac{1}{5^3}|\vec{b}| = \frac{1}{25} \end{aligned}$$

2. The electric field in a region is given as

$$\begin{aligned} \vec{E} &= \frac{\sigma}{2\epsilon_0}\hat{i}; \qquad \text{for } \ x > 0 \\ &= -\frac{\sigma}{2\epsilon_0}\hat{i}; \quad \text{for } \ x < 0 \end{aligned}$$

   Find the charge distribution in the region using the differential form of Gauss's law.

   soln:



   The electric field has only the x component and it is independent of the y and the z coordinates.

$\vec{\nabla} \cdot \vec{E} = 0$ for both $x > 0$ and $x < 0$. Consider a cylindrical volume with a cross sectional area A parallel to the yz plane and length spanning from $x = -a$ to $x = a$. Applying divergence theorem over the region enclosed by this cylinder we get

$$\begin{aligned} \int_V \vec{\nabla} \cdot \vec{E} dV &= \oint_S \vec{E} \cdot \hat{n} da \\ \therefore \quad A \int_{-a}^{a} (\vec{\nabla} \cdot \vec{E}) dx &= A(E_x(a) - E_x(-a)) \\ \therefore \quad \int_{-a}^{a} (\vec{\nabla} \cdot \vec{E}) dx &= (\frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0}) = \frac{\sigma}{\epsilon_0} \end{aligned}$$

The non-zero contribution to the integral on the l.h.s can come only from the plane $x = 0$ since $\vec{\nabla} \cdot \vec{E} = 0$ for $x > 0$ and $x < 0$.
So $\vec{\nabla} \cdot \vec{E}$ here behaves like a delta function firing at $x = 0$. And since the volume integral of $\vec{\nabla} \cdot \vec{E}$ entangling the plane $x = 0$ is $\frac{\sigma}{\epsilon_0}$ we conclude

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho(x)}{\epsilon_0} = \frac{\sigma}{\epsilon_0} \delta(x)$$

$$\therefore \quad \rho(x) = \sigma \delta(x)$$

This is an infinite volume charge density over the yz plane and zero charge density everywhere else. This is a surface charge over the plane $x = 0$. The surface charge density is obtained by integrating ρ over a cylinder with unit cross sectional area parallel to the yz plane and length along x from $x = -a$ to $x = a$.

$$\int_{-a}^{a} \sigma \delta(x) dx = \sigma$$

This is independent of a and hence the value is σ in the limit $a \to 0$. So the electric field corresponds to a uniform surface charge density σ over the yz plane.

3. The electric field in a region is cylindrically symmetric, given as follows:

$$\begin{aligned} \vec{\mathrm{E}}(\vec{\mathrm{r}}) &= \frac{c\hat{\mathrm{s}}}{s}; \quad \text{when} \quad s \geq a \\ &= 0; \quad \text{when} \quad s < a \end{aligned}$$

Find the charge distribution in the region using Gauss' law.

soln
The charge density is given by the differential form of Gauss' law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$.
Due to cylindrical symmetry of the problem the partial differentiation w.r.t z and ϕ is zero. So we have

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{s} \frac{\partial}{\partial s}(s E_s)$$

For $s > a, E_s = c/s \implies \vec{\nabla} \cdot \vec{E} = 0$.
For $s < a, E_s = 0 \implies \vec{\nabla} \cdot \vec{E} = 0$. So the charge density is 0 outside and inside the

cylinder.
At $s = a, sE_s$ is not differentiable.

Consider a cylinder of radius $b_1 < a$ and height h. By divergence theorem

$$\int_{V_1} \vec{\nabla} \cdot \vec{E} dV = \oint_{S_1} \vec{E} \cdot \hat{n} da$$

Since $\vec{\nabla} \cdot \vec{E} = 0$ and $\vec{E} = 0$ everywhere within this cylinder of radius less than a, the divergence theorem is obviously satisfied, both sides being 0. When we do the same procedure over a cylinder of radius $b_2 > a$ the divergence theorem is not satisfied as

$$\begin{aligned} \int_{V_2} \vec{\nabla} \cdot \vec{E} dV &= \oint_{S_2} \vec{E} \cdot \hat{n} da \\ \therefore \quad 2\pi h \int_0^{b_2} (\vec{\nabla} \cdot \vec{E}) s ds &= h \frac{c}{b_2} \times 2\pi b_2 = 2\pi hc \end{aligned}$$

The l.h.s is apparently 0 as $\vec{\nabla} \cdot \vec{E} = 0$ for both, $s < a$ and $s > a$. This ambiguity is removed if we realize that the contribution to the integral on the l.h.s comes from the cylindrical surface $s = a$. So the integral on the l.h.s is a δ function firing at $s = a$.

$$\therefore \quad \vec{\nabla} \cdot \vec{E} s = c\delta(s - a) \quad \Longrightarrow \quad \vec{\nabla} \cdot \vec{E} = \frac{c}{s} \delta(s - a)$$

So the charge density at the surface $s = a$ is given as

$$\rho = \epsilon_0 \frac{c}{a} \delta(s - a)$$

This is an infinite volume charge density. This is a finite amount of charge smeared over the surface $s = a$ whose thickness is zero. hence we must specify this density as a surface charge density. This will be given as

$$\sigma = \epsilon_0 \frac{c}{a}$$

4. We have seen that $\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$. In a similar manner justify that

$$\vec{\nabla} \cdot \left(\frac{\hat{s}}{s}\right) = 2\pi\delta^2(\vec{s})$$

Here s is the distance from the z axis in cylindrical coordinates and $\delta^2(\vec{s})$ is a two dimensional delta function on the xy plane.

soln

We have seen that $\vec{\nabla} \cdot \left(\frac{\hat{s}}{s}\right) = 0$ for $s > 0$. It tends to ∞ as $s \to 0$. Let us calculate the integral of this function over a cylindrical volume of radius a and height h enclosing the z axis.

$$\begin{aligned} h \int_0^a \int_0^{2\pi} \vec{\nabla} \cdot \left(\frac{\hat{s}}{s}\right) s ds d\phi &= h \int_0^{2\pi} \frac{\hat{s}}{a} \cdot \hat{s} a d\phi \qquad \text{by divergence theorem} \\ \therefore \quad \int_0^a \int_0^{2\pi} \vec{\nabla} \cdot \left(\frac{\hat{s}}{s}\right) s ds d\phi &= 2\pi \end{aligned}$$

This is true for any cylinder with radius $a > 0$ around the z axis. So we have

$$\vec{\nabla} \cdot \left(\frac{\hat{s}}{s} \right) = 2\pi \delta^2(\vec{s})$$

5. Prove that $\delta(r) = 4\pi r^2 \delta^3(\vec{r})$ and $\delta(s) = 2\pi s \delta^2(\vec{s})$.
Here $\int_0^\epsilon \delta(r) dr = 1$ for any $\epsilon > 0$. The integral is 0 otherwise. $\delta(s)$ is defined likewise.

soln
Consider a sphere of radius ϵ around the origin

$$\begin{aligned} \int_V \delta^3(\vec{r}) dV &= \int_0^\epsilon \delta^3(\vec{r}) 4\pi r^2 dr \\ \therefore \quad 1 &= \int_0^\epsilon \delta^3(\vec{r}) 4\pi r^2 dr \end{aligned}$$

So $\delta^3(\vec{r}) 4\pi r^2$ behaves as a one dimensional δ function $\delta(r)$.

In 2-dimension consider a circular disc of radius ϵ.

$$\begin{aligned} \int_S \delta^2(\vec{s}) da &= \int_0^\epsilon \delta^2(\vec{s}) 2\pi s ds \\ \therefore \quad 1 &= \int_0^\epsilon \delta^2(\vec{s}) 2\pi s ds \end{aligned}$$

So $\delta^2(\vec{s}) 2\pi s$ behaves as a one dimensional δ function $\delta(s)$.

6. The electric field in a region is given as

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s}$$

Find the charge densities in the region.

soln
The volume charge density is given by the differential form of Gauss' law.

$$\rho(\vec{s}) = \epsilon_0 \vec{\nabla} \cdot \vec{E} = \frac{\lambda}{2\pi} \vec{\nabla} \cdot \left(\frac{\hat{s}}{s} \right)$$

It was proved earlier that $\vec{\nabla} \cdot \left(\frac{\hat{s}}{s} \right) = 2\pi \delta^{(2)}(\vec{s})$. So $\rho(\vec{s}) = \lambda \delta^{(2)}(\vec{s})$. This is a charge disdribution which is 0 every where except at $s = 0$, i.e along the z axis. We can get the linear charge density by integrating this volume charge density $\rho(\vec{s})$ over a thin cylinder of radius ϵ and height 1 unit.

$$\begin{aligned} \int_0^1 \int_0^\epsilon \rho(\vec{s}) 2\pi s ds dz &= 1 \times \int_0^\epsilon \lambda \delta^2(\vec{s}) 2\pi s ds \\ &= \lambda \int_0^\epsilon \delta(s) ds \\ &= \lambda \end{aligned}$$

So we have a line charge with linear density λ along the z axis.