Tutorial - 10

* Properties of autocorrelation function.

Mean square value $R_x(0) = E[x$

$R_x(\tau) = R_x(-\tau)$

$R_x(\tau) \leq R_x(0)$.

1. Which of the following cannot be valid autocorrelation function.

(a) $e^{-\tau^2}$

$R_x(0) = 1$. $R_x(\tau) \leq R_x(0)$

$R_x(\tau) = R_x(-\tau)$

Valid.

(b). $|\tau| e^{-|\tau|}$

$R_x(0) < R_x(\tau)$

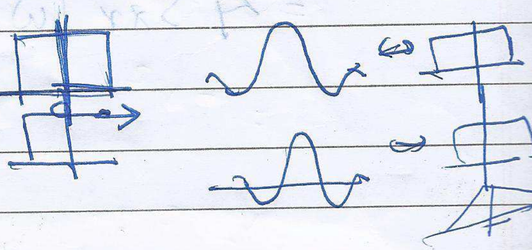
Not valid.

(c). $10 e^{-(\tau+2)}$

$R_x(\tau) \neq R_x(-\tau)$

Not valid.

(d). $\left(\frac{\sin \pi f}{\pi f}\right)^2$



Valid.

(e) $\frac{\tau^2+4}{\tau^2+9}$

$R_X(0) < R_X(\tau)$.

Not valid.

(f) $R_X(\tau) = 2$, for $|\tau| < 2$ & 0 elsewhere

– Not valid.

– PSD is a sinc funⁿ. which goes negative making it an invalid PSD.

* Properties of PSD.

$S_X(f) = S_X(-f)$. PSD is real & even. $R_X(\tau)$ is also real & even.

$S_X(f) \geq 0$ for all f.

2. Determine which of the following can & cannot be valid power spectral densities.

(a) $S_X(\omega) = \frac{\omega^2}{\omega^6 + 3\omega^2 + 3}$.

Valid.

(b) $S_X(\omega) = e^{-(\omega-1)^2}$

$S_X(f) \neq S_X(-f)$ Not valid.

(c) $S_X(\omega) = \frac{\omega^2}{\omega^4 + 1} - \delta(\omega)$.

Not valid.

(d) $S_X(\omega) = \frac{\omega^4}{1 + \omega^2 + j\omega^6}$.

Complex. Not valid.

3. A WSS random process has PSD

$$S_X(f) = \begin{cases} 1 + \frac{1}{4}|f| & |f| \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

Find the mean-square value of this process.

Solⁿ = $R_X(0)$ = Mean Square value.

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f\tau}\, df$$

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df.$$

$$= \int_{-\infty}^{\infty} 1 + \frac{1}{4}|f|\, df.$$

$$= \int_{-4}^{0} 1 - \frac{1}{4}\cdot f\, df + \int_{0}^{4} 1 + \frac{1}{4} f\, df.$$

$$= \left[f - \frac{1}{4}\cdot\frac{f^2}{2}\right]_{-4}^{0} + \left[f + \frac{1}{4}\frac{f^2}{2}\right]_{0}^{4}$$

$$= \left[-\left(-4 - \frac{1}{4}\frac{(-4)^2}{2}\right) + \left(4 + \frac{1}{4}\cdot\frac{(4)^2}{2}\right)\right]$$

$$= [6 + 6] = 12.$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$$

4) $Y(t) = X(t)\,A\cos(\omega_c t+\phi)$. $X(t)$ & ϕ are independent. ϕ is uniformly distributed betn 0 to 2π.

soln

$$E[X(t)Y(t+\tau)] = E[X(t)A\cos(\omega_c t+\phi)\cdot X(t+\tau)A\cos(\omega(t+\tau)+\phi)]$$

$$= A^2E[X(t)X(t+\tau)\cdot\cos(\omega_c t+\phi)\cos(\omega(t+\tau)+\phi)]$$

$$= A^2E[X(t)X(t+\tau)]\,E[\cos(\omega t+\phi)\cos(\omega(t+\tau)+\phi)]$$

$$= A^2R_X(\tau)E[(\cos\omega t\cdot\cos\phi - \sin\omega t\cdot\sin\phi)\cdot(\cos(\omega(t+\tau))\cdot\cos\phi - \sin(\omega(t+\tau))\sin\phi)]$$

$$= A^2R_X(\tau)E[\cos(\omega t+\phi)\cdot\{\cos(\omega t+\phi)\cos(\omega\tau) - \sin(\omega t+\phi)\cdot\sin(\omega\tau)\}].$$

$$= A^2R_X(\tau)E[\cos^2(\omega t+\phi)\cos(\omega\tau) - \sin(\omega t+\phi)\cos(\omega t+\phi)\cdot\sin(\omega\tau)].$$

$$= A^2R_X(\tau)E[\cos^2(\omega t+\phi)\cdot\cos(\omega\tau)] - E[\sin(\omega t+\phi)\cdot\cos(\omega t+\phi)\cdot\sin\omega\tau].$$

$$= A^2R_X(\tau)E\left[\frac{\cos(\omega t+\phi+\omega(t+\tau)+\phi)}{2} + \cos(\omega t+\phi-\omega(t+\tau)-\phi)\right]$$

FT $\cos 2\pi f_c t \longleftrightarrow \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$.

$$= A^2 R_x(\tau)\left\{E\left[\frac{\cos(2\omega t + 2\phi + \omega\tau)}{2}\right] + E\left[\frac{\cos(-\omega\tau)}{2}\right]\right\}$$

$$= \frac{A^2}{2} R_x(\tau)\left\{E[\cos(2\omega t + 2\phi + \omega\tau)] + E[\cos(\omega\tau)]\right\}.$$

$$= \frac{A^2}{2} R_x(\tau)\{1 \cdot \cos\omega\tau + 0\}.$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos(2\omega_c t + 2\phi + \omega\tau)\, d\phi = 0.$$

$$\boxed{R_Y(\tau) = \frac{A^2}{2} R_x(\tau) \cdot \cos\omega\tau.}$$

$$S_Y(f) = \frac{A^2}{4}[S_x(f - f_c) + S_x(f + f_c)].$$

(3). ~~let~~ $x(t) = A\cos(w_c t + \theta)$, where A and w_c are constants and θ is a r v with the pdf

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

~~let us~~ Compute a) $m_x(t)$ and b) $R_x(t_1, t_2)$.

Sol^n = $m_x(t) = E[A\cos(w_c t + \theta)]$

$$= \frac{A}{2\pi}\int_0^{2\pi} \cos(w_c t + \theta)\, d\theta$$

$$= 0.$$

$$R_x(t_1, t_2) = E[A\cos(w_c t_1 + \theta) \cdot A\cos(w_c t_2 + \theta)]$$

$$= \frac{A^2}{2}\left\{ \underbrace{E[\cos(w_c(t_1+t_2) + 2\theta)]}_{=0?} + E[\cos[w_c(t_1 - t_2)]] \right\}$$

$$R_x(t_1, t_2) = \frac{A^2}{2}\cos[w_c(t_1 - t_2)]$$

$$= \frac{A^2}{2}\cos[w_c \tau].$$

$$E[\cos(w_c(t_1+t_2) + 2\theta)] = \int_0^{2\pi} \cos(w_c(t_1+t_2) + 2\theta) \cdot \frac{1}{2\pi}\, d\theta$$

$$= 0.$$

$$E[X] = \int x\, f_X(x)\, dx$$

$$E[g(x)] = \int g(x)\, \underbrace{f_X(x)}_{\text{pdf of } X}\, dx.$$