CT303 Lecture 17: 19 October 2020

- Lecture 16 review:
- Binary PCM - Matched filter to minimize P_B is $h(t) = s_1(T-t) - s_2(T-t)$.
- With this, $P_B = Q\left(\frac{1}{2}\sqrt{SNR}\right) = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$, where E_d is the energy of the difference between the two symbol waveforms: $s_1 - s_2$.

Equivalent receiver for Binary PCM

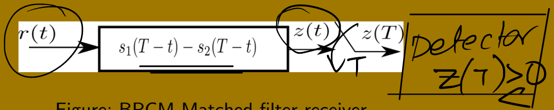


Figure: BPCM Matched filter receiver

$$z(t) = r(t) * \left(s_1(T-t) - s_2(T-t)\right)$$

$$= r(t) * s_1(T-t) - r(t) * s_2(T-t)$$

$$Z(T) = \langle r, s_1 \rangle - \langle r, s_2 \rangle$$

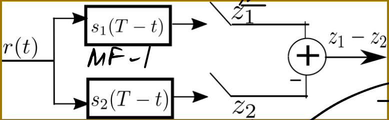
$$r * h_1 \quad - \quad r * h_2$$

$$\downarrow \qquad\qquad \downarrow$$

$$z_1(T) \qquad\qquad z_2(T)$$

Equivalent receiver for Binary PCM

How well r matches (correlates) with s_1



MF-1

MF-2

Detector
$z_1 - z_2 \gtrless 0$

Figure: BPCM Matched filter receiver

How well r matches with s_2

$$z_1 - z_2 \gtrless 0$$
$$\Updownarrow$$
$$z_1 \gtrless z_2$$

- Simplifying the energy of difference between signals:

$P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$

$$E_d = \int_0^T (s_1(t) - s_2(t))^2 \, dt$$
$$= \int_0^T s_1^2(t) \, dt + \int_0^T s_2^2(t) \, dt - 2\int_0^T s_1(t)s_2(t) \, dt$$
$$= E_{s_1} + E_{s_2} - 2\int_0^T s_1(t)s_2(t) \, dt$$

$$\text{Let } \rho = \frac{\int_0^T s_1(t)s_2(t) \, dt}{\sqrt{E_{s_1}E_{s_2}}} = \frac{\langle s_1, s_2 \rangle}{||s_1|| \, ||s_2||} = \cos(\theta_{s_1,s_2})$$

$$\boxed{E_d = ||s_1||^2 + ||s_2||^2 - 2\rho||s_1|| \, ||s_2||}$$

$\Rightarrow$ Assuming $E_{s_1} = E_{s_2} = E_b$

$$E_d = 2E_b - 2\rho E_b = 2E_b(1-\rho) = 2E_b(1-\cos\theta)$$

Thus, $\boxed{P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)}$

$$E_{s_1} = \int_0^T s_1^2(t)\,dt \qquad \|s_1\|_{L_2} = \left(\int_0^T s_1^2(t)\,dt\right)^{1/2}$$

$\underline{\mathbb{R}^2}$: $v \in \mathbb{R}^2$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\quad \|v\| = \sqrt{v_1^2 + v_2^2}$

$\underline{\mathbb{R}^n}$: $v \in \mathbb{R}^n$, $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2}$

$\mathbb{R}^\infty$, $v \in \mathbb{R}^\infty$, $v = (v_i)_{i=-\infty}^{\infty}$ $\quad \|v\| = \sqrt{\sum_{i=-\infty}^{\infty} v_i^2}$

$v : \mathbb{Z} \to \mathbb{R}$ $\qquad v = (1)_{i=-\infty}^{\infty}$ $\qquad \|v\| < \infty$

$\Rightarrow$ v has finite energy

$\{f : \mathbb{R} \to \mathbb{R}\}$

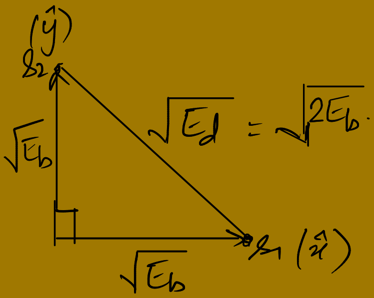
$$\|f\|_{L_2} := \left(\int_{-\infty}^{\infty} |f(t)|^2\,dt\right)^{1/2} \Rightarrow \text{DTFT}$$

- Examples:

$$P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

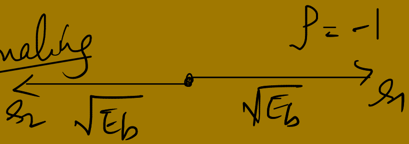
① Orthogonal Signaling

$$P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$



② Antipodal Signaling

$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$



$$\boxed{\frac{E_b}{N_0}}$$

SNR ratio

$\frac{E_b}{N_0}$ - Bit energy to Noise PSD ratio

- Let S denote the signal power, and N denote the total noise power over bandwidth W Hz.

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  - Then, $\boxed{E_b = S \cdot T}$, and $\boxed{N = N_0 \cdot W}$. W

$$\frac{E_b}{N_0} = \frac{S \cdot T}{N/W} \qquad T = \frac{1}{R} \rightarrow \text{symbol rate}$$

$$\boxed{\frac{E_b}{N_0} = \left(\frac{S}{N}\right)\left(\frac{W}{R}\right)}$$

$\frac{E_b}{N_0}$ - Bit energy to Noise PSD ratio

- Let S denote the signal power, and N denote the total noise power over bandwidth W Hz.
  - Then, $E_b = S \cdot T$, and $N = N_0 \cdot W$.
  - $\frac{E_b}{N_0} = \frac{S \cdot T}{N/W} = \left(\frac{S}{N}\right)\left(\frac{W}{R}\right)$, where R is the bit (symbol) rate.

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- Let S denote the signal power, and N denote the total noise power over bandwidth W Hz.
  - Then, $E_b = S \cdot T$, and $N = N_0 \cdot W$.
  - $\frac{E_b}{N_0} = \frac{S \cdot T}{N/W} = \left(\frac{S}{N}\right)\left(\frac{W}{R}\right)$, where R is the bit (symbol) rate.
  - $\boxed{E_b/N_0}$ is a more suitable parameter for DCS.

(1) All symbol wf used for DC are energy signals

(2) 8-ary PCM signalling → $s_1, \ldots, s_8$ → 3 bits per symbol

4-ary PCM signalling → $q_1, \ldots, q_4$ → 2 bits per symbol.

Compare P_B At the same $\boxed{E_b/N_0}$ value

$E_s \longrightarrow E_b$

8 ary signaling $E_b = \frac{E_s}{3}$ | 4 ary case $E_b = E_s/2$

M-ary signaling

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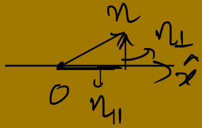
  - Received signal $r = s_i + \underline{\eta}$ for some i and $\eta \sim$ AWGN.

$s_i \in S$

η?



$$\eta = \eta_\parallel + \eta_\perp$$

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  - $\eta = \eta_\perp + \eta_{||}$, where $\eta_\perp \in S^\perp$ and $\eta_{||} \in S$.

$\in S^\perp$ $\in S$

$$r = s_i + \eta \longrightarrow \boxed{h_i} \longrightarrow a_i + \tilde{\eta}$$

$\langle r, s_i \rangle$

Projection

$\tilde{\eta} \in S.$

Assume that $\eta \in S.$

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  - Since matched filters can be seen as projectors to $\{s_1, \ldots, s_M\}$, the perpendicular component $\eta_\perp$ will be filtered out by the matched filters.

ignore

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  - Received signal $r = s_i + \eta$ for some i and $\eta \sim$ AWGN.
  - $\eta = \eta_\perp + \eta_{||}$, where $\eta_\perp \in S^\perp$ and $\eta_{||} \in S$.
  - Since matched filters can be seen as projectors to $\{s_1, \dots, s_M\}$, the perpendicular component $\eta_\perp$ will be filtered out by the matched filters.
  - Hence the received signal r can be assumed to belong to S.

$$r = s_i + \eta \in S.$$

Since $S = span(\{s_1, \dots, s_M\})$, $\dim(S) < \infty$

Orthonormal basis

- Let $B = \{\Psi_1, \ldots, \Psi_N\}$ denote an ONB of S. Thus $s_i = \sum_{k=1}^{N} a_{ik} \Psi_k$.

$N \leq M$

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  - Let $r = \sum_{i=1}^{N} \bar{r}_i \Psi_i$, $\bar{r} = [\bar{r}_1, \ldots, \bar{r}_N]^T$, and $\eta_{||} = \sum_{k=1}^{N} \eta_k \Psi_k$.

$[\bar{r}]_B$

$r = s_i + \eta_{||}$

$\hookrightarrow$ Gaussian

$\eta_i \Psi_i$, $\eta_j \Psi_j$

$\eta \to 0$ mean $\eta_{||} \to 0$ mean

$$\Sigma_{\eta_{||}} = \left[\sigma^2 Id \right]_{N \times N}$$

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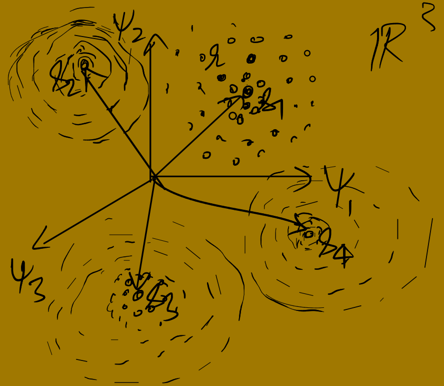
  - $\eta_{||} \sim \mathcal{N}(0, \sigma I)$.



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  - $\eta_{||} \sim \mathcal{N}(0, \sigma I)$.
  - Likelihood $p(r|s_i) = \mathcal{N}(a_i, \sigma I)$, $i = 1, \ldots, M$, i.e.,

$$a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{iN} \end{bmatrix}$$

$$r = s_i + \eta_{||}$$

$$p(r|s_i) = \frac{1}{(2\pi)^{\frac{N}{2}} \sigma} \exp\left(-\frac{1}{2} (\bar{r} - a_i)^T \Sigma^{-1} (\bar{r} - a_i) \right)$$

$$p(r|s_i) = \frac{1}{(2\pi)^{\frac{N}{2}} \sigma} \exp\left(-\frac{1}{2} \frac{||\bar{r} - a_1||^2}{\sigma^2} \right)$$

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$$= \frac{1}{(2\pi)^{\frac{N}{2}} \sigma} \underbrace{\exp\left(-\frac{1}{2} \frac{||\bar{r} - a_1||^2}{\sigma^2} \right)}$$

ln

  - $\ln\, p(r|s_i) = k - \frac{||\bar{r} - a_i||^2}{2\sigma^2}$

- Let $B = \{\Psi_1, \dots, \Psi_N\}$ denote an ONB of S. Thus $s_i = \sum_{k=1}^{N} a_{ik}\Psi_k$.
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p(r|s_i) &= \frac{1}{(2\pi)^{\frac{N}{2}}\sigma} \exp\left(-\frac{1}{2}(\bar{r} - a_i)^T \Sigma^{-1} (\bar{r} - a_i)\right) \\
&= \frac{1}{(2\pi)^{\frac{N}{2}}\sigma} \exp\left(-\frac{1}{2}\frac{||\bar{r} - a_1||^2}{\sigma^2}\right)
\end{aligned}
$$

   - $ln\ p(r|s_i) = \cancel{k} - \frac{||\bar{r} - a_i||^2}{2\sigma^2}$
   - MAP:

$$
\begin{aligned}
ln(p(r|s_i)p(s_i)) &\propto 2\sigma^2 ln\ p(s_i) - ||\bar{r} - a_i||^2 \\
&\propto 2\sigma^2\ ln\ p(s_i) - (\cancel{||\bar{r}||^2} + ||a_i||^2 - 2\bar{r}^T a_i) \\
&\propto \boxed{2\sigma^2\ ln\ p(s_i)} - \boxed{||a_i||^2} + 2\bar{r}^T a_i \\
&\propto k_i + \cancel{2}\bar{r}^T a_i \simeq k_i + \bar{r}^T a_i
\end{aligned}
$$

posterior distribution

$\rightsquigarrow k_i + \bar{r}^T a_i$

- Let $B = \{\Psi_1, \ldots, \Psi_N\}$ denote an ONB of S. Thus $s_i = \sum_{k=1}^{N} a_{ik} \Psi_k$.
- Let $r = \sum_{i=1}^{N} \bar{r}_i \Psi_i, \bar{r} = [\bar{r}_1, \ldots, \bar{r}_N]^T$, and $\eta_{||} = \sum_{k=1}^{N} \eta_k \Psi_k$.
- $\eta_{||} \sim \mathcal{N}(0, \sigma I)$.
- Likelihood $p(r|s_i) = \mathcal{N}(a_i, \sigma I), i = 1, \ldots, M$, i.e.,

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\begin{aligned}
p(r|s_i) =& \frac{1}{(2\pi)^{\frac{N}{2}} \sigma} \exp\left(-\frac{1}{2} (\bar{r} - a_i)^T \Sigma^{-1} (\bar{r} - a_i)\right) \\
=& \frac{1}{(2\pi)^{\frac{N}{2}} \sigma} \exp\left(-\frac{1}{2} \frac{||\bar{r} - a_1||^2}{\sigma^2}\right)
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- $ln\ p(r|s_i) = k - \frac{||\bar{r} - a_i||^2}{2\sigma^2}$
- MAP:

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\begin{aligned}
ln(p(r|s_i)p(s_i)) \propto& 2\sigma^2 ln\ p(s_i) - ||\bar{r} - a_i||^2 \\
\propto& 2\sigma^2\ ln\ p(s_i) - (||\bar{r}||^2 + ||a_i||^2 - 2\bar{r}^T a_i) \\
\propto& 2\sigma^2\ ln\ p(s_i) - ||a_i||^2 + 2\bar{r}^T a_i \\
\propto& k_i + 2\bar{r}^T a_i \simeq k_i + \bar{r}^T a_i
\end{aligned}
$$

- Decision: s_k was transmitted if $k_k + \bar{r}^T a_k > k_i + \bar{r}^T a_i, \forall i \neq k$.

MAP estimation

- M matched filters with impulse responses $\{\underline{s_1(T-t), \ldots, s_M(T-t}\}$

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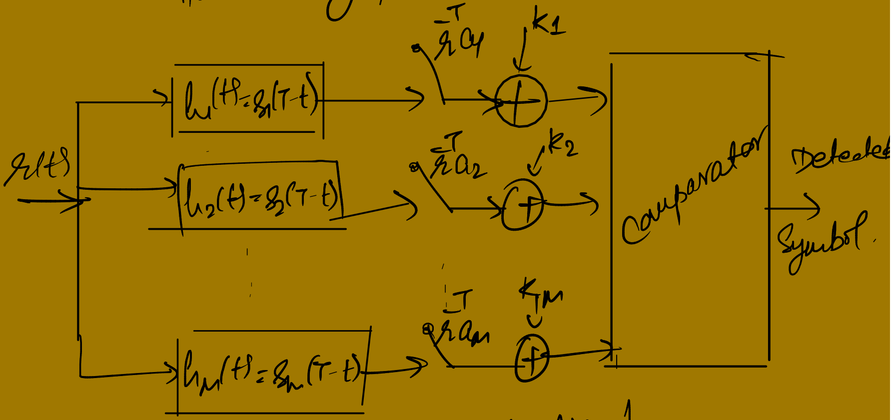
- M matched filters with impulse responses $\{s_1(T-t), \ldots, s_M(T-t)\}$
  - The output of the matched filter $h_i(t) = s_i(T-t)$ is

$$
\begin{aligned}
\int_0^T r(t)s_i(t)\ dt &= \int_0^T \sum_{k=1}^{N} \bar{r}_k \Psi_k(t) \sum_{p=1}^{N} a_{ip}\Psi_p(t)\ dt \\
&= \sum_{k=1}^{N}\sum_{p=1}^{N} \bar{r}_k a_{ip} \int_0^T \Psi_k(t)\Psi_p(t)\ dt \\
&= \sum_{k=1}^{N} \bar{r}_k a_{ik} \\
&= \bar{r}^T a_i
\end{aligned}
$$

1 if $k = p$

$= \langle \bar{r}, a_i \rangle$

Rx Arch using M matched filters for M-ary PCM.



$$Y = AX + b$$

$$\Sigma_Y = A \Sigma_X A^T + \Sigma_b.$$