1. In the spherical polar system:

   (a) Evaluate $\frac{\partial \hat{r}}{\partial \theta}, \ \frac{\partial \hat{\theta}}{\partial \theta}, \ \frac{\partial \hat{\phi}}{\partial \theta}, \ \frac{\partial \hat{r}}{\partial \phi}, \ \frac{\partial \hat{\theta}}{\partial \phi}, \ \frac{\partial \hat{\phi}}{\partial \phi}$

   soln
   In the spherical polar system

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

   In this system $h_r = 1, \ h_\theta = r, \ h_\phi = r\sin\theta$.

$$\begin{aligned} \hat{r} &= \frac{1}{h_r}\left(\frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}\hat{j} + \frac{\partial z}{\partial r}\hat{k}\right) dr \\ &= \left(\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}\right) \end{aligned}$$

   Similarly

$$\begin{aligned} \hat{\theta} &= \left(\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}\right) \\ \hat{\phi} &= -\sin\phi\hat{i} + \cos\phi\hat{j} \end{aligned}$$

   So we have

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}, \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}, \quad \frac{\partial \hat{\phi}}{\partial \theta} = 0$$

$$\frac{\partial \hat{r}}{\partial \phi} = \sin\theta\hat{\phi}, \quad \frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta\hat{\phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\sin\theta\hat{r} - \cos\theta\hat{\theta}$$

   (b) Using the above partial derivatives evaluate $\vec{\nabla}\cdot\hat{r}, \ \ \vec{\nabla}\cdot\hat{\theta}$ and $\vec{\nabla}\cdot\hat{\phi}$ where $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

   soln

$$\begin{aligned} \vec{\nabla}\cdot\hat{r} &= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\hat{r} \\ &= \frac{\hat{\theta}}{r}\cdot\hat{\theta} + \frac{\hat{\phi}}{r\sin\theta}\cdot\sin\theta\hat{\phi} = \frac{2}{r} \\ \vec{\nabla}\cdot\hat{\theta} &= \frac{\hat{\theta}}{r}\cdot(-\hat{r}) + \frac{\hat{\phi}}{r\sin\theta}\cdot\cos\theta\hat{\phi} = \frac{1}{r\tan\theta} \\ \vec{\nabla}\cdot\hat{\phi} &= \frac{\hat{\phi}}{r\sin\theta}\cdot(-\sin\theta\hat{r} - \cos\theta\hat{\theta}) = 0 \end{aligned}$$

Now you can evaluate the expression for $\vec{\nabla}\cdot\vec{A}$ in spherical polar co-ordinates using the result of part (b) and using the product rules.

2. Cylindrical system of co-ordinate is specified by three variables (s, ϕ, z) given by

$$x = s\cos\phi; \qquad y = s\sin\phi; \qquad z = z$$

Find the unit vectors $\hat{s}$, $\hat{\phi}$, $\hat{z}$ in this co-ordinate system. Find h_s, h_ϕ and h_z and write down the expression for $\vec{\nabla}F$ for a scalar function F in this system.

soln

$$\begin{aligned}
\vec{dl}_s &= (\cos\phi\hat{i} + \sin\phi\hat{j})ds \implies \quad h_s = 1 \\
\vec{dl}_\phi &= (-s\sin\phi\hat{i} + s\cos\phi\hat{j})d\phi \implies \quad h_\phi = s \\
\vec{dl}_z &= \hat{k}dz \quad \implies \quad h_z = 1
\end{aligned}$$

$$\therefore \quad \hat{s} = \cos\phi\hat{i} + \sin\phi\hat{j}, \hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}, \hat{z} = \hat{z}$$

$$\begin{aligned}
\vec{\nabla}F &= \frac{1}{h_s}\frac{\partial F}{\partial s}\hat{s} + \frac{1}{h_\phi}\frac{\partial F}{\partial\phi}\hat{\phi} + \frac{1}{h_z}\frac{\partial F}{\partial z}\hat{z} \\
&= \hat{s}\frac{\partial F}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial F}{\partial\phi} + \hat{z}\frac{\partial F}{\partial z}
\end{aligned}$$

3. If $\vec{A} = s\hat{z}$ find $\vec{\nabla}\times\vec{A}$.

soln

$$\vec{\nabla}\times\vec{\mathrm{A}} = \left[\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\mathrm{s}} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial\phi}\right]\hat{\mathrm{z}}$$

Here the only component we have is $A_z = s$.

$$\therefore \quad \vec{\nabla}\times\vec{A} = \hat{s}\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \hat{\phi}\frac{\partial A_z}{\partial s} = -\hat{\phi}$$

4. Find the divergence of $\vec{\mathrm{v}} = (r\cos\theta)\hat{\mathrm{r}} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}$. Check the divergence theorem for this function, using the volume as the inverted hemispherical bowl of radius R, resting on the x-y plane and centred at the origin.

soln

$$\begin{aligned}
\vec{\nabla}\cdot\vec{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \\
&= 5\cos\theta - \sin\phi
\end{aligned}$$

$$\therefore \quad \int_V \vec{\nabla}\cdot\vec{v}dV \;=\; \int_0^R\int_0^{\frac{\pi}{2}}\int_0^{2\pi}(5\cos\theta-\sin\phi)r^2\sin\theta drd\theta d\phi$$
$$= 2\pi\frac{R^3}{3}\int_0^{\frac{\pi}{2}}5\cos\theta\sin\theta d\theta$$
$$= 5\pi R^3/3$$

$$\oint_S \vec{v}\cdot\hat{n}da == \int_{hemisphere}\vec{v}\cdot\hat{n}da+\int_{basecircle}\vec{v}\cdot\hat{n}da$$

Over the hemisphere $\hat{n}=\hat{r}$ and $da=R^2\sin\theta d\theta d\phi$

$$\therefore \quad \int_{hem}\vec{v}\cdot\hat{n}da=\int_{hem}(R\cos\theta)R^2\sin\theta d\theta d\phi=\pi R^3$$

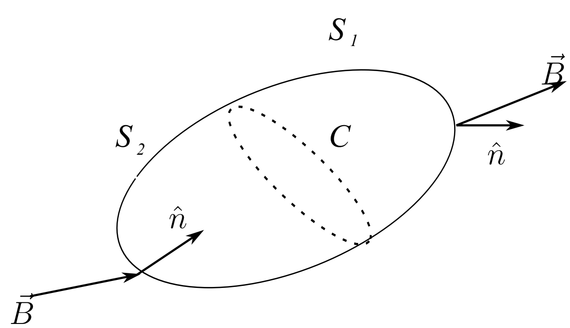
Over the circle $\hat{n}=\hat{\theta}$ and $da=r\sin\theta drd\phi rdrd\phi$ since $\theta=\pi/2$.

$$\therefore \quad \int_{basecircle}\vec{v}\cdot\hat{n}da=\int_0^R\int_0^{2\pi}(r\sin\frac{\pi}{2})rdrd\phi=\frac{2\pi R^3}{3}$$

$$\therefore \quad \oint_S\vec{v}\cdot\hat{n}da=\pi R^3+\frac{2\pi R^3}{3}=\frac{5\pi R^3}{3}$$

5. (a) If $\vec{\nabla}\cdot\vec{\mathrm{B}}=0$ show that there exists a vector function $\vec{\mathrm{A}}$ such that $\vec{\nabla}\times\vec{\mathrm{A}}=B$

soln



$$\vec{\nabla}\cdot\vec{B}=0$$
$$\therefore \quad \oint_S\vec{B}\cdot\hat{n}da=\int_V(\vec{\nabla}\cdot\vec{B})dV=0$$
$$\therefore \quad \int_{S_1}\vec{B}\cdot\hat{n}da=\int_{S_2}\vec{B}\cdot\hat{n}da$$

On the surface S_1 $\hat{n}$ is outward to the volume V. On the surface S_2 $\hat{n}$ is inward to the volume V as shown in the figure.
The surface integral will be same over any surface bounded by the dotted curve (loop) C. So these integrals are related to the values of certain fields along the

curve C. This can be obtained by a line integral along C. There are two possibilities for these to be scalars:

$$\oint_C \phi dl \quad \text{and} \oint_C \vec{A}\cdot\vec{dl}$$

In the first possibility ϕ is a scalar function integrated over C with length element $dl = |\vec{dl}|$. In the second possibility $\vec{A}$ is a vector function integrated over C with vector length element $\vec{dl}$.
We choose the second possibility because if we reverse the direction of the normals then the surface integrals reverses the sign. This should be accompanied by reversing the way we are traversing C thus replacing $\vec{dl}$ by $-\vec{dl}$. We can see that the first integral will have the same result whether we traverse C clockwise or anticlockwise. So that can't be equal to our surface integral.

$$\therefore \quad \int_{S_1} \vec{B}\cdot\hat{n}da = \oint_C \vec{A}\cdot\vec{dl}$$

By Stokes' theorem

$$\therefore \quad \int_{S_1} \vec{B}\cdot\hat{n}da = \int_{S_1} (\vec{\nabla}\times\vec{A})\cdot\vec{n}da$$

Since this is true for any arbitrary surface bounded by the curve C we conclude that

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

We have proved that $\exists$ a vector field $\vec{A}$ such that $\vec{B} = \vec{\nabla}\times\vec{A}$. This $\vec{A}$ is not unique. For e.g if $\vec{A}' = \vec{A} + \vec{\nabla}\Phi$ then

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times(\vec{\nabla}\Phi) = \vec{\nabla}\times\vec{A} = \vec{B}$$

(b) Show that any vector field $\vec{\mathrm{F}}$ can be expressed as

$$\vec{\mathrm{F}} = \vec{\nabla}\Phi + \vec{\nabla}\times\vec{\mathrm{A}}$$

where Φ is a scalar field and $\vec{\mathrm{A}}$ is a vector field.
Justify
soln
We note that $\vec{\nabla}\Phi$ is a curlless field and $\vec{\nabla}\times\vec{A}$ is a divergenceless field. So we are trying to justify that every field can be expresses as a sum of a curlless field and a divergenceless field.

Let $\vec{F}$ be a vector field. Consider the matrix

$$D = \begin{pmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} & \frac{\partial F_x}{\partial z} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} & \frac{\partial F_y}{\partial z} \\ \frac{\partial F_z}{\partial x} & \frac{\partial F_z}{\partial y} & \frac{\partial F_z}{\partial z} \end{pmatrix}$$

Every matrix can be expressed as a sum of a symmetric and an antisymmetric matrix. So the matrix D can be expressed as $S+A$ where $S=(D+D^T)/2$ and $A=(D-D^T)/2$. S is symmetric while A is antisymmetric. We have

$$S=\begin{pmatrix} \frac{\partial F_x}{\partial x} & \frac{1}{2}\left(\frac{\partial F_x}{\partial y}+\frac{\partial F_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial F_x}{\partial z}+\frac{\partial F_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial F_y}{\partial x}+\frac{\partial F_x}{\partial y}\right) & \frac{\partial F_y}{\partial y} & \frac{1}{2}\left(\frac{\partial F_y}{\partial z}+\frac{\partial F_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial F_z}{\partial x}+\frac{\partial F_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial F_z}{\partial y}+\frac{\partial F_y}{\partial z}\right) & \frac{\partial F_z}{\partial z} \end{pmatrix}$$

and

$$A=\begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial F_x}{\partial y}-\frac{\partial F_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial F_y}{\partial z}-\frac{\partial F_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right) & 0 \end{pmatrix}$$

We can see that S almost corresponds to a vector field that is curlless while A corresponds to a vector field that is divergenceless. But they are too restrictive in the sense that all the diagonal elements of A are 0. We can play around with the diagonal elements without disturbing the curlless and the divergenceless requirement as follows:

$$D_B=\begin{pmatrix} \frac{\partial F_x}{\partial x}+\alpha & \frac{1}{2}\left(\frac{\partial F_x}{\partial y}+\frac{\partial F_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial F_x}{\partial z}+\frac{\partial F_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial F_y}{\partial x}+\frac{\partial F_x}{\partial y}\right) & \frac{\partial F_y}{\partial y}+\beta & \frac{1}{2}\left(\frac{\partial F_y}{\partial z}+\frac{\partial F_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial F_z}{\partial x}+\frac{\partial F_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial F_z}{\partial y}+\frac{\partial F_y}{\partial z}\right) & \frac{\partial F_z}{\partial z}+\gamma \end{pmatrix}$$

and

$$D_C=\begin{pmatrix} -\alpha & \frac{1}{2}\left(\frac{\partial F_x}{\partial y}-\frac{\partial F_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right) & -\beta & \frac{1}{2}\left(\frac{\partial F_y}{\partial z}-\frac{\partial F_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right) & -\gamma \end{pmatrix}$$

with $\alpha+\beta+\gamma=0$. Note that D_B continues to be symmetric while D_C is not strictly antisymmetric. But the matrices are now less restrictive and we can find a vector field $\vec{B}$ and $\vec{C}$ such that D_B corresponds to the partial derivatives of the components of $\vec{B}$ and D_C corresponds to the partial derivatives of the components of $\vec{C}$. Here $\vec{B}$ is curlless and $\vec{C}$ is divergenceless. Also $\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot\vec{F}$.

$\therefore \quad \vec{\nabla}\cdot(\vec{F}-\vec{B})=0 \quad \Longrightarrow \quad \vec{F}-\vec{B}=\vec{\nabla}\times\vec{A}$

$\therefore \quad \vec{F}=\vec{B}+\vec{\nabla}\times\vec{A}$

Since $\vec{\nabla}\times\vec{B}=0, \quad \vec{B}=\vec{\nabla}\Phi$

$$\therefore \quad \vec{F}=\vec{\nabla}\Phi+\vec{\nabla}\times\vec{A}$$

Note:

(i) The fields Φ and $\vec{A}$ are not unique. They depend on the choice of α, β and γ.

(ii) The matrix D_C corresponds to a field $\vec{C}$ such that $\vec{\nabla}\times\vec{C}=\vec{\nabla}\times\vec{F}$.