Lecture - 23

A: like insem 1, question paper uploaded, answers written on paper, photo uploaded on Moodle

B: like insem 2, completely online, MCQ, fill in the blanks, T/F, immediate evaluation

C: other

A: if you absolutely cannot take the online exam

Coupon Collecting Problem 2

N types of coupons

X = total no. of items that you purchase in order to get N coupons.

$E[X]$

X_i : i coupons have already been collected. X_i is the no. of additional items that you buy ~~coupo~~ items that you buy inorder to get a coupon of a different type.

X_3 means that you already have 3 types of coupons, the no. of additional ~~coupons~~ items that you buy to get a different coupon.

$X = X_0 + X_1 + X_2 + \cdots + X_{N-1}$

$X_0 = 1$

X_1	Probability	
1	$\frac{N-1}{N}$	B, C or B, E or B, A
2	$\frac{1}{N} \cdot \frac{N-1}{N}$	B, $\boxed{B, A}$ or B, B, C
3	$\frac{1}{N} \cdot \frac{1}{N} \cdot \frac{N-1}{N}$	
4	$\frac{1}{N} \cdot \frac{1}{N} \cdot \frac{1}{N} \cdot \frac{N-1}{N}$	
⋮		

You already have coupon B

$\underbrace{B,}_{X_0 = 1} \underbrace{B, B, B, A}_{X_1 = 4}$

X_1 is a geometric r.v.

$E[X_1] = \frac{1}{p} = \frac{N}{N-1}$

X_2	Probability	
1	$\frac{N-1-1}{N}$	B, A, E A, A, A, B, C
2	$\frac{2}{N} \cdot \frac{N-2}{N}$	B, B, B, A, $\boxed{B, C}$
3.	$\left(\frac{2}{N}\right)^2 \cdot \frac{N-2}{N}$	
$\vdots$	$\ddots$	$\vdots$

X_2 is also a geometric r.v.

$$E[X_2] = \frac{1}{p} = \frac{N}{N-2}$$

$$X = X_0 + \dots + X_{N-1}$$

$$E[X] = \sum_{i=0}^{N-1} E[X_i]$$

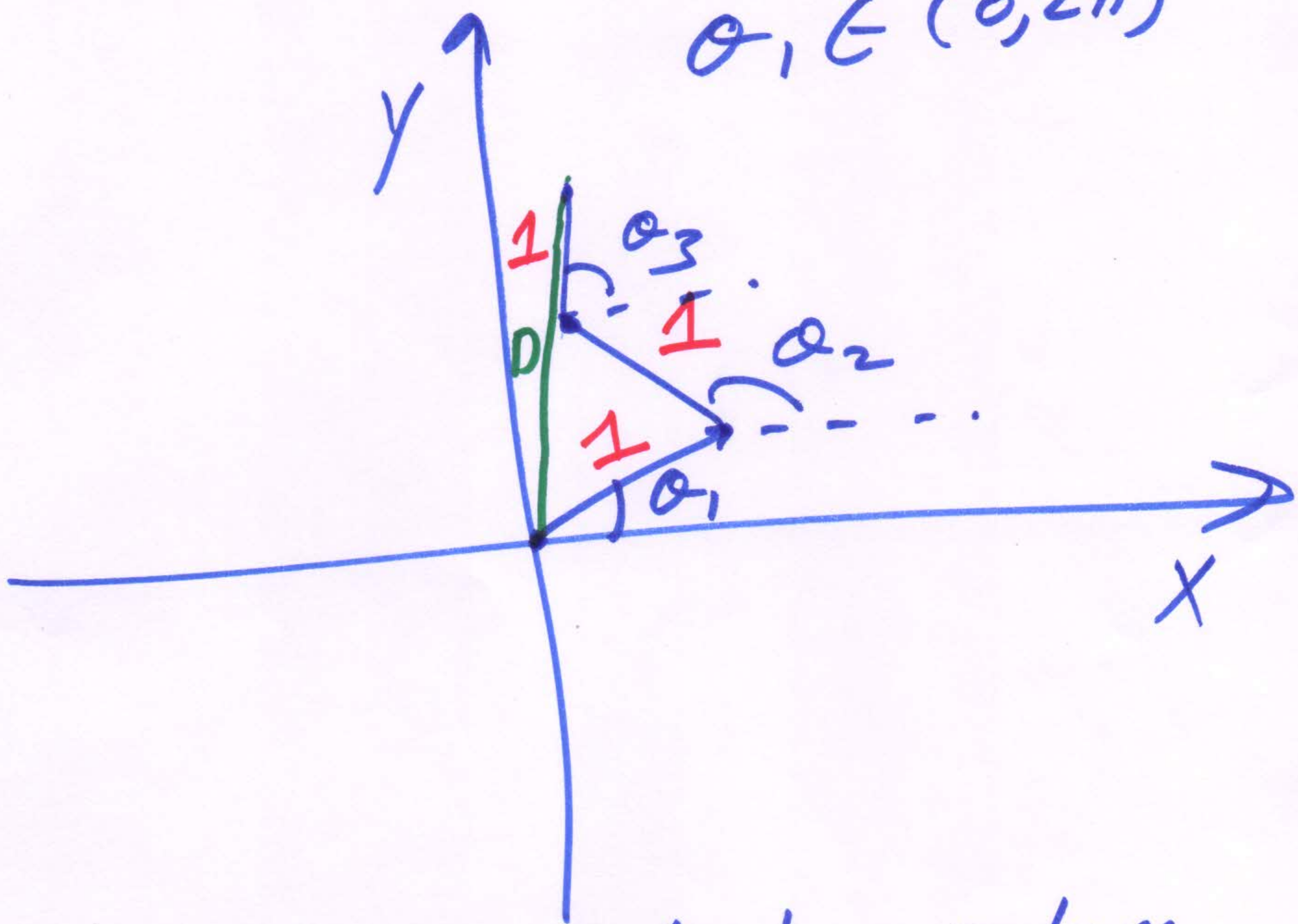
$$= \frac{N}{N} + \frac{N}{N-1} + \frac{N}{N-2} + \dots + \frac{N}{1}$$

$$(\approx N \log N$$

E.g.:

Drunkard's walk in the plane. Steplength = 1

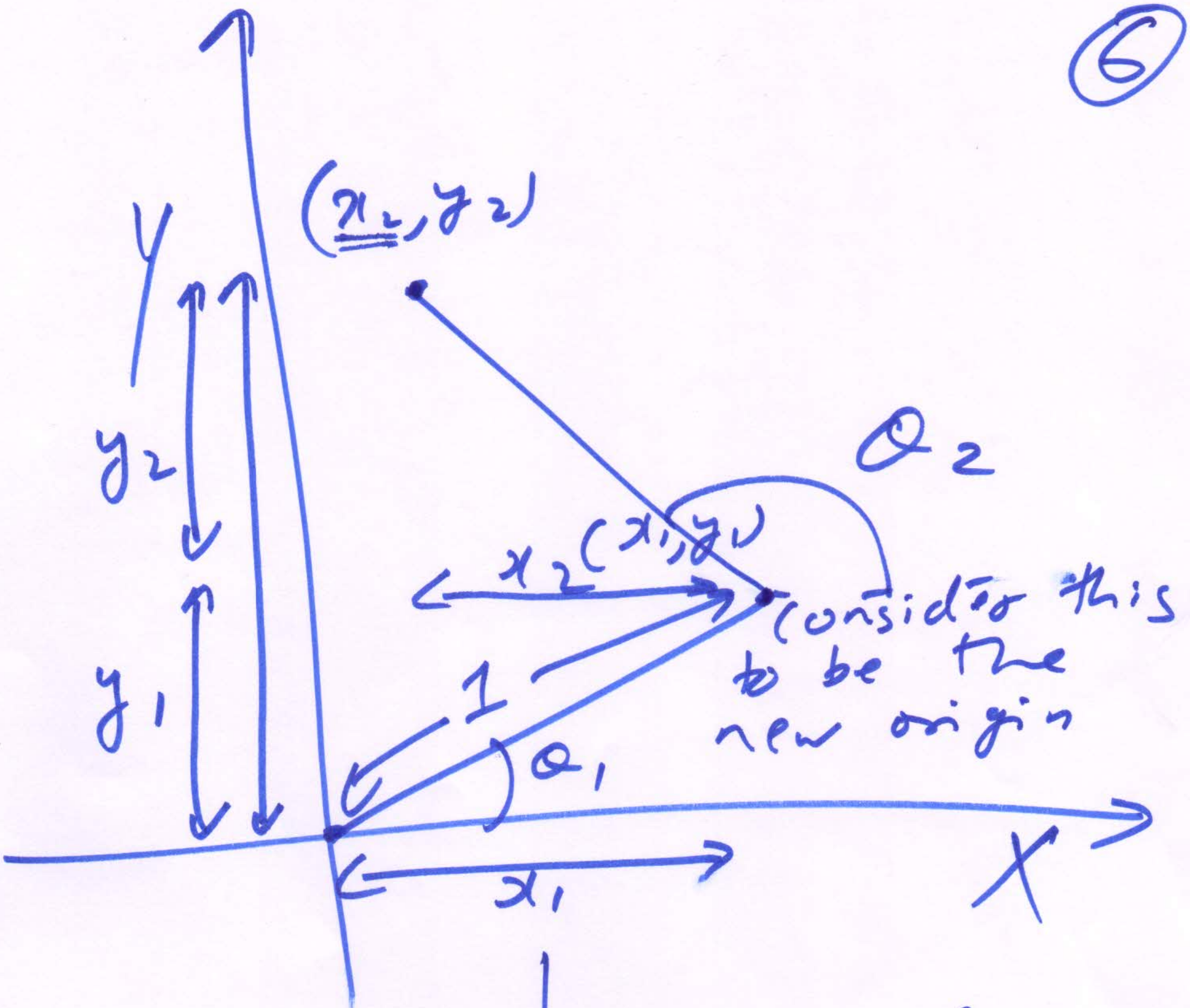
$\theta_1 \in (0, 2\pi)$



the angle that he makes with the positive x axis is uniformly distributed over $(0, 2\pi)$.

After n steps, he is D units away from the origin.

Compute $E[D^2]$.



$x_1 = \cos\theta_1$

$y_1 = \sin\theta_1$

$x_2 = \cos\theta_2$

$y_2 = \sin\theta_2 \ldots$

$(x_1 + x_2, \ y_1 + y_2)$

Position of the person after n steps

$(x_1 + x_2 + \cdots + x_n, \ y_1 + y_2 + \cdots + y_n)$

$$D^2 = (x_1 + x_2 + \cdots + x_n)^2 + (y_1 + y_2 + \cdots + y_n)^2$$

$$= (\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n)^2 + (\sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_n)^2$$

$$= \underbrace{1 + 1 + \cdots + 1}_{n} + \sum_{i \neq j}\sum \cos\theta_i \cos\theta_j + \sum_{i \neq j}\sum \sin\theta_i \sin\theta_j$$

$$D^2 = n + \boxed{\sum_{i \neq j}\sum \cos\theta_i \cos\theta_j} + \boxed{\sum_{i \neq j}\sum \sin\theta_i \sin\theta_j}$$

$$E[D^2] = n + E\left(\sum\sum \cos\theta_i \cos\theta_j\right) + E\left(\sum\sum \sin\theta_i \sin\theta_j\right)$$

$$E[D^2] = n$$

$$+ \sum_{i \neq j}\sum E[\cos\theta_i \cos\theta_j] +$$

$$\sum_{i \neq j}\sum E[\sin\theta_i \sin\theta_j]$$

θ_i and θ_j are independent.

$$E[\cos\theta_i \cos\theta_j] = E[\cos\theta_i] * E[\cos\theta_j]$$

$$E[\cos\theta_i] = \int_0^{2\pi} \frac{\cos\theta_i \cdot d(\theta_i)}{2\pi}$$

$$= 0$$

$$E[\sin\theta_i] = \int_0^{2\pi} \frac{\sin\theta_i \, d(\theta_i)}{2\pi}$$

$$= 0$$