Lecture - 21 P 1

→ sequential, time management

→ different questions for different roll nos.

→ subjective test is better for evaluating understanding

→ questions were very simple

→ no partial marking, made one small mistake

→ you could see the result almost immediately, got feedback for every question

Recap:

Properties of Expectation

$E[|X-Y|]$

$E[g(X,Y)] = \iint g(x,y) f(x,y)\,dx\,dy$

True even if X & Y are not independent

$g(X,Y) = X+Y$

$E[X+Y] = E[X] + E[Y]$

True even if X & Y are not independent

Expectation of sum = 3
sum of expectations

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discrete as well as
continuous

$E[X] = np$ for Binomial

$E[X] = \frac{mn}{N}$ for hypergeometric

Boole's inequality.

$A_1, A_2, \ldots, A_n$ are events.

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

$$P(A \cup B) \leq P(A) + P(B)$$

Expectation of a sum = sum of expectations.

Let $X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{if } A_i \text{ doesn't occur} \end{cases}$

<u>indicator</u> random variable

$$E[X_i] = P(A_i)$$

$X_i = 1$ with probability $P(A_i)$

$\quad\; 0 \quad$ " $\quad 1 - P(A_i)$

$$E[X_i] = 1(P(A_i)) + 0(1 - P(A_i))$$

$$= P(A_i)$$

$X = \sum_{i=1}^{n} X_i =$ no. of events that occur

$$Y = \begin{cases} 1 & \text{if } X \geq 1 \\ 0 & \text{if } X = 0 \end{cases}$$

$Y = \begin{cases} 1 & \text{if atleast one event takes place} \\ 0 & \text{if no event takes place} \end{cases}$

$X \geq Y$

$$E[X] \geq E[Y]$$

$$\downarrow \qquad \downarrow$$

$$E[X] = E[\sum X_i]$$

$$= \sum \underline{\underline{E[X_i]}}$$

$$= \sum_{i=1}^{n} P(A_i)$$

$$\boxed{\Rightarrow \underline{P(\cup A_i) \leq \sum P(A_i)} \quad \text{Boole's inequality}}$$

$$E[Y] = P(\cup A_i)$$

$$Y = \begin{cases} 1 & \text{with prob. } P(\cup A_i) \\ 0 & \text{with prob. } 1 - P(\cup A_i) \end{cases}$$

e.g. Expected no. of runs of 1 in a bitstring having m 0's & n 1's. (7)

$$\frac{(m+n)!}{m!\,n!} \qquad \frac{7!}{3!\,4!}$$

$m = 3, \quad n = 4,$

1 2 3 4 5 6 7

→ $0\,0\,0\,\underline{1\,1\,1\,1}$ $\quad 1 = X \quad X_4 = 1$

$\underline{1}\,0\,\underline{1}\,0\,\underline{1}\,0\,\underline{1}$ $\quad 4 = X \quad X_1 = 1 \;\; X_3 = 1$, $X_5 = 1, X_7 = 1$

$\underline{1\,1\,1\,1}\,0\,0\,0$ $\quad 1 = X$

$\underline{1}\,0\,\underline{1\,1}\,0\,0\,\underline{1}$ $\quad 3 = X$

$\underline{1\,1}\,0\,0\,0\,\underline{1\,1}$ $\quad 2 = X$

Expectation of a sum = sum of expectations

X = no. of runs of 1's in the bit string

How do you define X_i's s.t. $\sum_{i=1}^{m+n} X_i = X$

X_i = 1 0 0 0 0 1 0 0 0 1 0 0
$\underline{1\,1}\; 0\,0\,0\,0\; \underline{1\,1\,1}\; 0\; \underline{1}\; 0\,0$

$\sum X_i = 3 = X$

$X_i = 1$ if a new run of 1's is starting from the i^{th} position

$E[X] = E[\sum X_i]$ (7)

$= \sum E[X_i]$

$E[X_i] =$

$E[X_1] = \frac{n}{m+n}$

$$X_1 \begin{cases} \to 1 \left(\frac{n}{m+n}\right) \\ \to 0 \left(\frac{m}{m+n}\right) \end{cases}$$

$E[X_2] =$

$$\begin{tikzcd}[row sep=small] & 1 \\ x_2 \arrow[ur] \arrow[dr, bend right] & \\ & 0 \end{tikzcd}$$

$\vdots$

x_{m+n}

when is $x_2 = 1$?

when the 1^{st} bit is 0,

2^{nd} bit is 1

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x_i, $i > 1$

$x_i = 1$ if the preceding bit is 0,

current bit is 1

$$E(X_2) = \frac{mn}{(m+n)(m+n-1)}$$

$$\frac{0}{} \qquad \frac{1}{}$$

$$\frac{m}{(m+n)} \qquad \frac{n}{(m+n-1)}$$

$$E[X] = \sum E[X_i]$$

$$= E[X_1] + \sum_{i=2}^{m+n} E[X_i]$$

$$= \frac{n}{m+n} + \frac{\cancel{(m+n-1)}(mn)}{(m+n)\cancel{(m+n-1)}}$$

$$= \frac{mn+n}{m+n}$$

e.g.:

Coupon collecting problem

You need to collect

N coupons