Examples and Applications

Stokes's Law of Terminal Velocity

For a heavy sphere (or any other shape) falling through a long column of viscous liquid, there are three forces acting on it, namely,

i) Gravity, $\boxed{mg}$, ii) buoyancy, $\boxed{\rho_l V g}$, where ρ_l is the liquid density and V is the volume of the sphere, and iii) viscous drag, $\boxed{kv}$, where $\boxed{k = 6\pi\eta r}$, r being the radius of the sphere, η the viscosity and v the velocity.

Hence, $\boxed{m\frac{dv}{dt} = mg - \rho_l V g - kv}$.

Writing $\boxed{m = \rho V}$, where ρ is the density of the sphere, and dividing throughout by m we get, $\boxed{\frac{dv}{dt} = \bar{g} - \frac{k}{m} v}$, in which $\boxed{\bar{g} = g\left(1 - \frac{\rho_l}{\rho}\right)}$. The above equation is in the form $\boxed{\frac{dx}{dt} = a - bx}$, with the equivalence $\boxed{a \longrightarrow \bar{g}}$ and $\boxed{b \longrightarrow k/m}$.

Atomic Waste Disposal

$z \to$ depth of the sea

Following the principle of the problem of Stokes's law of terminal velocity, we write $\boxed{m\dfrac{d^2z}{dt^2} = F = W - B - D}$, where $\boxed{W = mg}$ is the weight, $\boxed{B = (\rho_w V g)}$ is the buoyancy and $\boxed{D = kv}$ is the drag. Here ρ_w is the density of water, and k is the drag coefficient. The drag is proportional to the velocity, $\boxed{D \propto v}$. Noting $\boxed{\dfrac{dz}{dt} = v}$, we get $\boxed{\dfrac{dv}{dt} = g\left(1 - \dfrac{\rho_w V}{m}\right) - \dfrac{k}{m}v}$, in which we further write, $m = \bar{\rho} V$, where $\bar{\rho}$ is the average density of the full drum and V is its volume. Hence, $\boxed{\bar{g} = g\left(1 - \dfrac{\rho_w}{\bar{\rho}}\right)}$ Using which we get $\boxed{\dfrac{dv}{dt} = \bar{g} - \dfrac{k}{m}v}$. The solution of this equation is $\boxed{v = v_T\left(1 - e^{-t/t_0}\right)}$ where $\boxed{v_T = \dfrac{m\bar{g}}{k}}$ and $\boxed{t_0 = \dfrac{m}{k}}$ under the initial condition at $\boxed{t=0,\ v=0}$.

Clearly $\boxed{v_T = \bar{g}\, t_0}$ which is the terminal velocity obtained when $t \to \infty$, in $\boxed{v = v_T\left(1 - e^{-t/t_0}\right)}$. Experimentally $\boxed{k = 0{\cdot}08 \text{ (in fps units)}}$, which gives the value of $\boxed{v_T \simeq 714 \text{ ft s}^{-1}}$. This is far greater than the tolerance velocity $\boxed{v_{tol} \simeq 40 \text{ ft s}^{-1}}$ at which the drums would break upon impact with the sea floor.

Since $v_T > v_{tol}$, the v-t equation does not guarantee that v_{tol} may not be overcome. Hence, we need to look at the v-z equation, which can be obtained from $\frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz}$

$$\therefore \boxed{v\frac{dv}{dz} = \bar{g} - \frac{v}{t_0}} \quad \text{Since } t_0 = m/k$$

$$\Rightarrow \boxed{t_0\, v\frac{dv}{dz} = \bar{g}\, t_0 - v = v_T - v}$$

$$\Rightarrow \quad -\frac{v\,dv}{v_T - v} = -\frac{dz}{t_0}$$

$$\Rightarrow \quad \frac{v_T - v - v_T}{v_T - v}\, dv = -\frac{dz}{t_0}$$

$$\Rightarrow \quad \int dv + \int \frac{v_T\, d(-v)}{v_T - v} = -\int \frac{dz}{t_0}$$

$$\Rightarrow \quad v + v_T \int \frac{d(-v/v_T)}{1 + (-v/v_T)} = -\frac{z}{t_0}$$

$$\Rightarrow \quad \boxed{v + v_T \ln\left(1 - \frac{v}{v_T}\right) = -\frac{z}{t_0} + c}\,.$$

When $z = 0$ (at the surface of the sea), $v = 0$. For this initial condition $c = 0$.

$$\Rightarrow \quad \boxed{z = -t_0\left[v + v_T \ln\left(1 - \frac{v}{v_T}\right)\right]}\,.$$

This is a transcendental equation and a solution of $v \equiv v(z)$ cannot be found in closed form. Therefore, we invert the problem. First we write $v = v_{tol} = 40\ \mathrm{ft\,s^{-1}}$. The depth at which this velocity is to be reached is z_{tol}. The weight of a drum, $W = 527.4\,\mathrm{lbs}$. Hence, $m = \frac{W}{g} = \frac{527.4}{32.2} = 16.38\,\mathrm{slugs}$.

$\Rightarrow t_0 = \frac{m}{k} = \frac{16.38}{0.08}$ in fps unit, $v_T = 714\ \mathrm{ft\,s^{-1}}$.

Hence, $z_{tol} = -\frac{16.38}{0.08}\left[40 + 714 \ln\left(1 - \frac{40}{714}\right)\right]$.

$$\Rightarrow \quad Z_{tol} = \frac{-16.38}{0.08} \times -1.1644 = 238\,ft.$$

Since the actual sea depth is 300 ft, at the point of impact, $v > v_{tol} \Rightarrow$ Drums will break

To check if the depth, z, is a monotonic function of t,

we consider $v = v_T\left(1 - e^{-t/t_0}\right)$, in which we write $\boxed{v = \frac{dz}{dt} = v_T\left(1 - e^{-t/t_0}\right)}$.

$$\Rightarrow z = v_T t - \frac{v_T\, e^{-t/t_0}}{-1/t_0} + c = v_T t + v_T t_0 e^{-t/t_0} + c$$

When $t = 0$, $z = 0 \Rightarrow \boxed{c = -v_T t_0}$.

$$\Rightarrow \boxed{z = v_T t + v_T t_0\left(e^{-t/t_0} - 1\right)}$$

define $\xi = z / v_T t_0$ and $\tau = t/t_0$.

$\therefore$ We get $\boxed{\xi = (\tau - 1) + e^{-\tau}}$. $\Rightarrow \frac{d\xi}{d\tau} = 1 - e^{-\tau}$

$\frac{d\xi}{d\tau} = 0$ only when $\tau = 0$. Hence ξ (or z) increases monotonically for τ (or t) > 0.

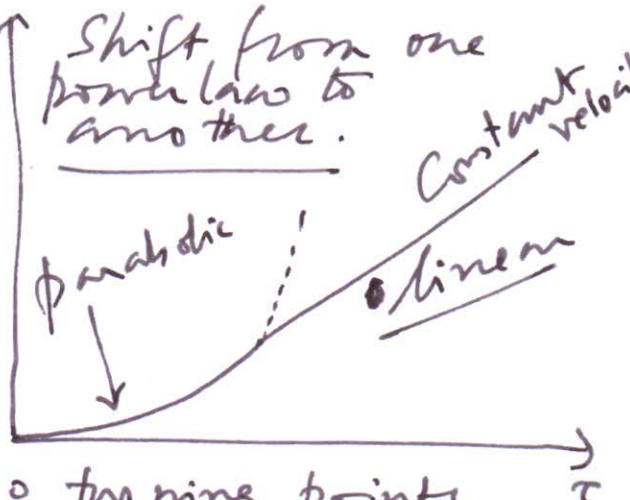
i.) When $\tau \to 0$,

$\xi = \cancel{\tau} - \cancel{1} + (\cancel{1} - \cancel{\tau} + \frac{\tau^2}{2} + \cdots)$

$\Rightarrow \boxed{\xi \simeq \frac{\tau^2}{2}}$ (parabolic)

ii.) When $\tau \to \infty$,

$\boxed{\xi \simeq \tau}$ (linear)



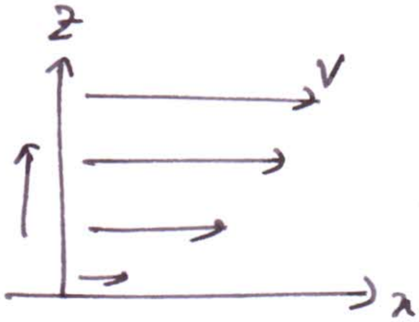
No turning points.
Also velocity increases with time

Kelvin's Viscoelastic Deformation of Rocks

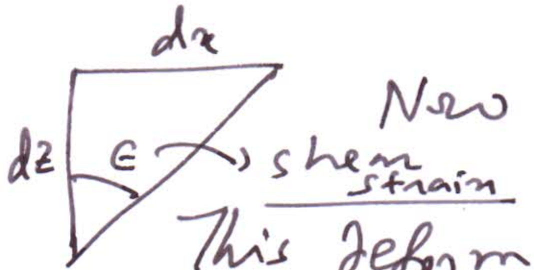
$\boxed{\sigma \rightarrow \text{Stress}}$, $\boxed{\epsilon \rightarrow \text{Strain}}$. For a **Solid** $\boxed{\sigma \propto \epsilon} \Rightarrow \boxed{\sigma = Y\epsilon}$ where Y is the Young's modulus (an elastic property)

For a liquid $\boxed{\sigma = \eta \frac{dv}{dz}}$

where $\eta \rightarrow$ coefficient of viscosity.



Now $\sigma = \eta \frac{d}{dz}\left(\frac{dx}{dt}\right) = \eta \frac{d}{dt}\left(\frac{dx}{dz}\right)$



Now $\boxed{\frac{dx}{dz} = \tan\epsilon \approx \epsilon}$ for small deformation.

This deformation of a highly viscous liquid is named as FUGITIVE ELASTICITY by Maxwell. $\Rightarrow \boxed{\eta = \frac{d\epsilon}{dt}}$. Hence for a constant stress, σ, we can write

$\boxed{\sigma = Y\epsilon + \eta \frac{d\epsilon}{dt}} \rightarrow$ Viscoelastic (Both viscosity and elasticity)

$\Rightarrow \boxed{\frac{d\epsilon}{dt} = \frac{\sigma}{\eta} - \frac{Y}{\eta}\epsilon}$ like $\boxed{\frac{dx}{dt} = a - bx}$

$a \rightarrow \sigma/\eta$, $b \rightarrow Y/\eta$.

Solid rocks FLOW OUT under the weight of the Earth matter above it.

Duckworth - Lewis Method

$$\boxed{Z(u, w) = Z_0(w)\left[1 - e^{-b(w)u}\right]}$$

$w \rightarrow$ No. of wickets lost. $u \rightarrow$ No. of overs left.

$Z(u,w) \longrightarrow$ No. of runs obtainable.

(Compare with $\boxed{x = x_0\left(1 - e^{-t/\tau}\right)}$).

w is to be treated as a parameter.

Reduce the Duckworth - Lewis equation to an autonomous system. We write

$$\frac{dz}{du} = -z_0 e^{-bu} \times -b = \left(z_0 e^{-bu}\right) b .$$

But $\boxed{z_0 e^{-bu} = z_0 - z}$. Therefore,

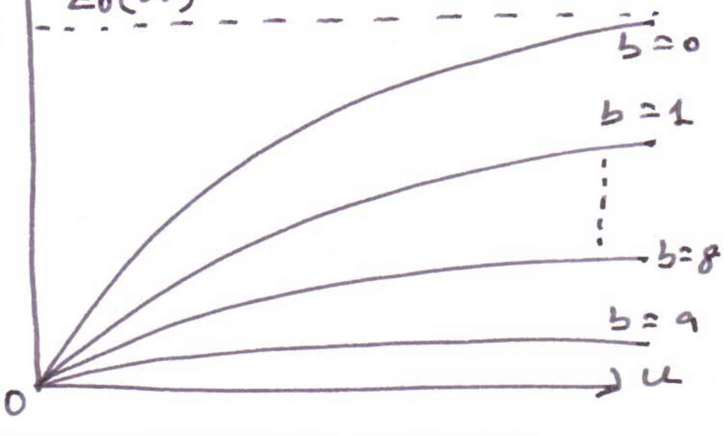
$$\boxed{\frac{dz}{du} = b(z_0 - z) = bz_0 - bz} .$$

Now compare with $\boxed{\frac{dx}{dt} = a - bx}$.

We see $a \rightarrow bz_0$ and $b \rightarrow b(w)$.

The limiting value is $a/b \rightarrow bz_0/b = z_0(w)$

As b increases more wickets are lost. Hence less will become the gettable runs.



Van Meegeren Art Forgery Case

Radioactivity: $\boxed{\text{Rate} \propto \text{State}}$

$$\boxed{\frac{dN}{dt} = -\lambda N}$$

$\lambda \to$ Decay constant
$\lambda > 0$, radioactive DECAY

Integrate: $\Rightarrow$ $\boxed{\ln N = -\lambda t + c}$.

Initial condition is when $t = t_0$, $N = N_0$.

$\therefore\ c = \ln N_0 + \lambda t_0 \Rightarrow \ln N - \ln N_0 = -\lambda(t - t_0)$

$$\Rightarrow \boxed{N = N_0 e^{-\lambda(t-t_0)}}.$$

Half-life: $\Rightarrow\ N = N_0/2$

Time taken to decay to half the initial amount.

$$\Rightarrow \frac{N}{N_0} = 2^{-1} = e^{-\lambda(t-t_0)}$$

$$\Rightarrow -\lambda(t - t_0) = -\ln 2$$

Write $t - t_0 = T_{1/2}$

$$\Rightarrow \boxed{t - t_0 = T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}}$$

Eg.
$T_{1/2}$ (Carbon C14) = 5568 years, $T_{1/2}$ (Uranium U238) = 4.5×10^9 years

Actual Age: $\boxed{t - t_0 = \frac{1}{\lambda} \ln(N_0/N)}$

OR $\boxed{t - t_0 = \frac{T_{1/2}}{\ln 2} \ln(N_0/N)}$

1/. N and λ can be measured.
2/. The difficulty is in knowing N_0 (the initial amount)

All paints contain white lead (lead oxide).
White lead contains radioactive Pb-210, with a half life of approximately 22 years in which it decays to Pb-206 (non-radioactive).

Let $\boxed{x_0 = x(t_0)}$ be the amount of Pb-210 contained per gram of white lead, at the time of manufacture of the pigment.

The decay rate of Pb-210 is given by

$$\boxed{\frac{dx}{dt} = -\lambda x + r(t)}$$

, in which $r(t)$ is the rate at which Pb 210 is replenished due to the radioactive decay of Ra-226 per minute per gram of white lead. If R is the amount of radium (Ra-226) at time t, with a half life of $T_{R1/2} = 1600$ years, we write the decay equation of Ra-226 as $\boxed{R = R_0 e^{-\lambda_R (t-t_0)}}$.

We expand this as $R = R_0 \left[1 - \lambda_R (t-t_0) + \cdots\right]$.

No $t - t_0 = 300$ years at most, which is the age of the original painting. Further $\lambda_R = \frac{\ln 2}{T_{R1/2}}$

Hence $\boxed{\lambda_R (t-t_0) = \frac{\ln 2}{T_{R1/2}} (t-t_0) \simeq 0.13 \ll 1}$.

Therefore, we neglect all the higher powers in the expansion and retain only.

$\boxed{R \simeq R_0 \left[1 - \frac{\ln 2}{T_{R1/2}}(t - t_0)\right]}$. The decay rate of Ra-226 is

$\boxed{\frac{dR}{dt} \simeq -\frac{\ln 2}{T_{R1/2}} = -\iota(t)}$, which is constant. Hence, the rate of replenishment of Pb 210, $\iota(t)$ is also constant

$\Rightarrow \boxed{\iota(t) = \frac{R_0 \ln 2}{T_{R1/2}}}$. The decay rate of Pb 210 is given now as

$\boxed{\frac{dx}{dt} = \iota - \lambda x}$, which, with $\iota, \lambda > 0$, is now in the form $\boxed{\frac{dx}{dt} = a - bx}$.

Integration:

$$\frac{dx}{\iota - \lambda x} = dt \quad \Big| \text{ Separation of variables.}$$

$$\Rightarrow \int \frac{d(-\lambda x)}{1 - \lambda x} = -\lambda \int dt \Rightarrow \boxed{\ln(\iota - \lambda x) = -\lambda t + c}$$

The initial condition is when $t = t_0$, $x = x_0$.

$\Rightarrow \boxed{c = \lambda t_0 + \ln(1 - \lambda x_0)}$. Using this we get

$$\ln\left(\frac{1 - \lambda x}{1 - \lambda x_0}\right) = -\lambda(t - t_0)$$

$$\Rightarrow \iota - \lambda x = (\iota - \lambda x_0)\, e^{-\lambda(t - t_0)}$$

$$\Rightarrow \iota - \lambda x_0 = (1 - \lambda x)\, e^{-\lambda(t - t_0)}$$

$$\Rightarrow x_0 = \frac{\iota}{\lambda} - \left(\frac{\iota}{\lambda} - x\right) e^{\lambda(t - t_0)}.$$

$$\boxed{x_0 = \frac{1}{\lambda} + \left(x - \frac{1}{\lambda}\right) e^{\lambda(t-t_0)}}$$

. In this equation, both λ and 1 are fixed known quantities. x can be measured. For a new painting x is large and $t-t_0$ is small, and for an old painting, x is small and $t-t_0$ is large. x_0 is ALWAYS fixed.

i/. When $t-t_0 = 300$ years, $\lambda(t-t_0) = 9.45$

ii/. When $t-t_0 = 20$ years, $\lambda(t-t_0) = 0.63$.

For measured values of x, using $t-t_0 = 300$ yrs makes the value of x_0 absurdly high. x_0 is acceptably small when $t-t_0 = 20$ years.

Hence, the painting is a forgery.

Radio-Carbon Dating: Age of Ancient Cultures.

$$\boxed{{}^1n + {}^{14}N \longrightarrow {}^{14}C + {}^1p} \longrightarrow \text{Willard Libby}$$

$$N = N_0 e^{-\lambda(t-t_0)} \Rightarrow \boxed{\frac{N_0}{N} = e^{\lambda(t-t_0)}}.$$

$$\frac{dN}{dt} = \dot{N} = N_0 e^{-\lambda(t-t_0)} \times -\lambda = -\lambda N. \left(\text{rate} \propto \text{state}\right)$$

At $t = t_0$, $\boxed{\frac{dN}{dt} = \dot{N}(t_0) = -\lambda N_0}$, $(N_0 = N(t_0))$

$$\Rightarrow t - t_0 = \frac{1}{\lambda} \ln\left(\frac{N_0}{N}\right) = \frac{1}{\lambda} \ln\left[\frac{\dot{N}(t_0)}{\dot{N}(t)}\right]$$

$$\Rightarrow \boxed{t - t_0 = \frac{T_{1/2}}{\ln 2} \ln\left[\frac{\dot{N}(t_0)}{\dot{N}(t)}\right]}$$

$T_{1/2} = 5568$ years

Exercise 1: For living wood $\dot{N}(t_0) = 6.68$ unit

For a charcoal sample $\dot{N}(t) = 4.09$ unit

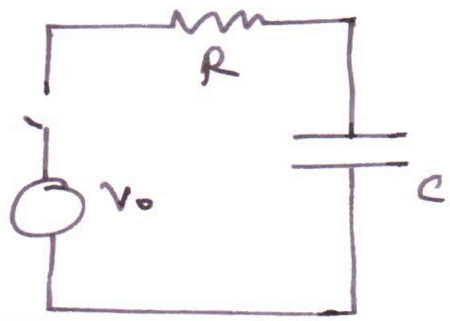
$$\Rightarrow t - t_0 = \frac{5568}{\ln 2} \ln\left(\frac{6.68}{4.09}\right) \quad \Big| \quad t = 1950 \text{ A.D.}$$

$$\Rightarrow t_0 = (1950) - 3940 = 2000 \text{ B.C.}$$

Exercise 2: $\dot{N}(t_0) = 6.68$ unit, $\dot{N}(t) = 0.97$ unit
$t = 1950$ A.D.

$$\Rightarrow t_0 = 1950 - \frac{5568}{\ln 2} \ln\left(\frac{6.68}{0.97}\right) = 13{,}500 \text{ B.C.}$$

Q-R-C Circuit



$\boxed{Q = VC}$ $\Rightarrow$ $\boxed{V = Q/C}$

and $\boxed{V = IR}$

For the full circuit $\boxed{V_0 = IR + Q/C}$.

Further $\boxed{I = \frac{dQ}{dt}}$ $\Rightarrow$ $R\frac{dQ}{dt} = V_0 - \frac{Q}{C}$

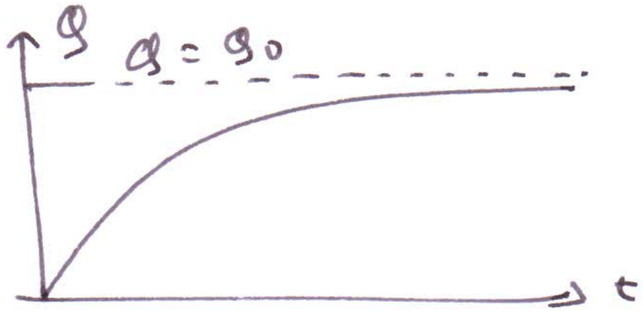
$$\Rightarrow \boxed{\frac{dQ}{dt} = \frac{V_0}{R} - \frac{Q}{RC}}$$

in the form $\frac{dx}{dt} = a - bx$

$\boxed{a \to V_0/R}$, $\boxed{b \to \frac{1}{RC}}$

Solution is $Q = \frac{V_0}{\cancel{R}} . \cancel{R}C\left(1 - e^{-t/RC}\right)$

$$\Rightarrow \boxed{Q = Q_0\left(1 - e^{-t/RC}\right)}$$



where $\boxed{Q_0 = CV_0}$.