DA-IICT

Dhirubhai Ambani Institute of Information and Communication Technology (DA-IICT)
Second In-Semester Examination
CT314 (Statistical Communication Theory)
Date of Examination: March 22 , 2013
Duration: 2 Hours
Maximum Marks: 20

Instructions:

1. Attempt all questions.
2. Use of scientific non programmable calculator is permitted.
3. Figures in brackets indicate full marks.
4. All the acronyms carry their usual meaning.
5. Figures in brackets indicate full marks

Q1 Let X be the input to a communication channel and Y be the output. The input to channel is +1 volt or -1 volt with equal probability. The channel output is Y = X+N, where N is noise which is uniformly distributed in the range -2 and 2. (a) Find $P[X=+1, Y \leq 0]$. (b) Find the probability that Y is negative given that X is +1. (5)

Q2: Let X_1 and X_2 be two random variables which are jointly Gaussian with mean vector $m_{\underline{X}}$ and covariance matrix $C_{\underline{X}}$. Now define $\underline{Y} = A\underline{X}$ as a linear transformation to get Y_1 and Y_2. Here A is an invertible 2X2 matrix. (a) Show that $\underline{Y}$ are jointly Gaussian. (b) Write the mean vector and covariance matrix for $\underline{Y}$ (c) Now choose A to make $\underline{Y}$ independent. (d) Can you reason out why A has to be invertible? (5)

Q3: Show that the error of the best linear estimator of Y in terms of X is orthogonal to the observation i.e.,
$E[\{(Y - E[Y]) - a_{opt}(X - E[X])\}(X - E[X])] = 0$ (3)

Q4: Problem on linear prediction: Let X_1, X_2, X_3 be zero mean random variables and suppose that we wish to predict X_3 by $aX_1 + bX_2$ in MMSE sense. Find a and b

(3)

Q5: Let the random process $X(t)$ consists of six equally likely sample functions, given by $x_i(t) = it, \quad i = 1,2,.....,6$. Let X and Y be the random variables obtained by sampling at t=1 second and t=2 second, respectively. Find (a) E(X) and E(Y) (b) $f_{X,Y}(x,y)$ (c) $R_X(1,2)$ (d) By inspection what can you say about the stationarity of the process. (4)

ANSWERS Midsem CT314, March 22.

Q1. $P(X=+1) = \frac{1}{2}$, $P(X=-1) = \frac{1}{2}$, $Y = X + N$.

(a) $P[X=+1, Y \leq 0] = P[Y \leq 0, X=+1] = P[Y \leq 0 / X=+1]\, P[X=+1]$

To find $P[Y \leq 0 / X=+1]$, find $P[Y \leq y / X=+1]$

When $X = 1$, Y is $Y = 1 + N$, so it has uniform distribution with mean shifted ie Y is uniformly distributed in the interval $[-1, 3]$

So $P[Y \leq y / X = +1] = F_Y(y|x) = \frac{1}{4}\int_{-1}^{y} d\alpha = \frac{y+1}{4}$, $-1 \leq y \leq 3$.

From this $P[Y \leq 0 / X=+1] = \frac{0+1}{4} = 1/4$.

So $P[X=+1, y \leq 0] = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$.

(b) To find the probability that Y is -ve given $X=+1$

ie. $P[Y<0 / X=+1] = ?$, we know that $f_Y(y|1) = \frac{1}{4}$, $-1 \leq y \leq 3$

$\therefore P[Y<0/ X=1] =$ integrate this density fn. till $y=0$.

$= \int_{-1}^{0} \frac{1}{4}\, dy = \frac{1}{4}$.

Q 2 We have $\underline{Y} = A \underline{X}$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

(a)

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{\left| J\left(\frac{y_1, y_2}{x_1, x_2}\right)\right|}$$

$$|J| = \left|\left| \begin{matrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{matrix} \right|\right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |A|$$

$= \det(A)$

$$f_{\underline{X}}(x_1, x_2) = \frac{e^{-\frac{1}{2}\underset{1\times 2}{(\underline{x} - \underline{m}_x)^T}\, \underset{2\times 2}{C_{\underline{x}}^{-1}}\, \underset{2\times 1}{(\underline{x} - \underline{m}_x)}}}{(2\pi)^{2/2} |C_x|^{1/2}}$$

So

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{e^{-\frac{1}{2}(A^{-1}\underline{y} - \underline{m}_x)^T C_x^{-1} (A^{-1}\underline{y} - \underline{m}_x)}}{(2\pi)\, |A|\, |C_x|^{1/2}}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

$$C_{\underline{x}} = \begin{bmatrix} \sigma_{x_1}^2 & \mathrm{cov}(x_1 x_2) \\ \mathrm{cov}(x_2, x_1) & \sigma_{x_2}^2 \end{bmatrix}$$

$$m_{\underline{x}} = \begin{bmatrix} m_{x_1} \\ m_{x_2} \end{bmatrix}$$

Now consider

$$(A^{-1}\underline{y} - m_{\underline{x}}) = A^{-1}(\underline{y} - A m_{\underline{x}})$$

$$(A^{-1}\underline{y} - m_{\underline{x}})^T = (\underline{y} - A m_{\underline{x}})^T A^{-1^T}$$

$$\therefore (A^{-1}\underline{y} - m_x)^T C_x^{-1} (A^{-1}\underline{y} - m_{\underline{x}})$$

$$= (\underline{y} - A m_{\underline{x}})^T A^{-1^T} C_{\underline{x}}^{-1} A^{-1} (\underline{y} - A m_{\underline{x}})$$

$$= (\underline{y} - A m_{\underline{x}})^T (A C_{\underline{x}} A^T)^{-1} (\underline{y} - A m_{\underline{x}})$$

$$(A C_{\underline{x}} A^T)^{-1} = A^{-1^T} C_x^{-1} A^{-1}$$

det

$$|C_y| = |A C_{\underline{x}} A^T| = |A|^2 |C_{\underline{x}}|$$

$$\therefore |C_y|^{\frac{1}{2}} = |A| |C_x|^{1/2}$$

let $A m_{\underline{x}} = \underline{m}$, $A C_{\underline{x}} A^T = C_{\underline{y}}$

So

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{e^{-\frac{1}{2}(\underline{y} - \underline{m})^T C_{\underline{y}}^{-1} (\underline{y} - \underline{m})}}{(2\pi)\, |C_y|^{1/2}}.$$

~~From~~ From this we see that $f_{Y_1 Y_2}(y_1 y_2)$ is jointly Gaussian with mean vector as $A m_{\underline{x}}$ and covariance matrix as $C_{\underline{y}} = A C_{\underline{x}} A^T$

(a) and (b)

(c) To choose A to make Y independent (y_1, y_2 are independent and jointly Gaussian)

we have $C_y = A C_x A^T$ $\quad C_y^T = A C_{\underline{x}} A^T$ $\quad C_y$ is symmetric

~~$C_{\underline{Y}} = U \Sigma U^T$ $U^{-1} = U^T$~~ we know that $C_{\underline{X}} = U \Sigma_{\underline{x}} U^T$ (2)

i.e. $C_{\underline{Y}}$ can be diagonalized.

If $C_{\underline{Y}}$ becomes diagonal then

~~Let A = U~~ $Y_1, Y_2, \ldots$ become independent

~~then $C_{\underline{Y}}$ =~~

∴ ~~$U^T C_{\underline{Y}} U = \Sigma$~~

~~Let $A = U^T$ then $C_Y = A C_X A^T$~~.

~~then $C_{\underline{Y}} = U^T C_{\underline{X}}$~~ .

So if $A = U^T$ then $C_{\underline{Y}} = A U \Sigma_x U^T A^T$

$= U^T U \Sigma_{\underline{x}} U^T U$

$= \Sigma_{\underline{x}}$ i.e. $C_{\underline{Y}}$ becomes diagonal

∴ hence A ~~must~~ has be choosen as

the eigenvectors of covariance matrix $C_{\underline{X}}$.

(d) If A is not invertible Jacobian becomes zero and the joint density cannot determined from this method.

Q.3 Linear estimate of Y in terms of X is $\hat{Y} = a_{opt} X + b_{opt}$.

So we have $E\left[\underbrace{\left(Y - (a_{opt} X + b_{opt})\right)^2}\right]$ is minimum

~~$b_{opt} = m_Y - m_X a_{opt} = a_{opt}(1 - m_X)$~~

~~$= m_Y - m_X a_{opt}$~~

~~So $E\left[\left(Y - (a_{opt} X + m_Y - m_X a_{opt})\right)^2\right]$ is minimum~~

~~i.e. $E\left[\{(Y - m_Y) - a_{opt}(X - m_X)\}^2\right]$ is minimum.~~

~~i.e. differentiating w.r.t a and putting $a = a_{opt}$ in ψ has to be zero.~~

$$\boxed{\cancel{\left.\frac{d\psi}{da}\right|_{a=opt} = 0 \text{ i.e. } E\left[\left\{(Y-m_y) - a_{opt}(X-m_x)\right\}(X-m_x)\right] = 0.}}$$

We know that $\psi = E\left[(Y - aX - b)^2\right]$ has to be minimum, & this is minimum when $a = a_{opt}$ and $b = b_{opt}$.

we can write ψ as $E\left[\left[(\underbrace{Y - aX}_{Z}) - b\right]^2\right]$

So the problem reduces to $E\left[(Z-b)^2\right]$ minimumization

(we know that if Z was estimated as a constant b the MMSE estimate of Z is $E[Z]$ i.e. $\hat{Z} = E[Z] = b$)

Hence $b = E[(Y - aX)] = E(Y) - aE(X) = m_y - am_x$.

$\therefore$ Now can pose the problem as minimize

$$E\left[(Y - aX - m_y + am_x)^2\right]$$

$$= E\left[\left((Y - m_y) - a(X - m_x)\right)^2\right] = \psi(a).$$

a_{opt} can be found by diff. above $\psi(a)$ w.r.t a & equating to zero

$$\therefore E\left[\left((Y - m_y) - a(X - m_x)\right)(X - m_x)\right] = 0.$$

So this is zero when $a = a_{opt}$. Hence the result.

Q.4

$$\min_{a,b} E\left[(X_3 - aX_1 - bX_2)^2\right]$$

Derivatives w.r.t a and b & equate to 0.

$$E\left[(X_3 - aX_1 - bX_2)X_1\right] = 0$$

$$E\left[(X_3 - aX_1 - bX_2)X_2\right] = 0.$$

$C_{\underline{Y}} = U \Sigma U^T \quad U^{-1} = U^T$ we know that $C_{\underline{X}} = U \Sigma_x U^T$ (2)

ie $C_{\underline{Y}}$ can be diagonalized.

If $C_{\underline{Y}}$ becomes diagonal then

~~Let A = U~~ $Y_1, Y_2, \ldots$ become independent

~~then C_Y =~~

∴ ~~$U^T C_Y U = \Sigma$~~

~~Let $A = U^T$ then A^T~~.

~~then $C_Y = U^T C_x$~~.

So if $A = U^T$ then $C_{\underline{Y}} = A U \Sigma_x U^T A^T$

$= U^T U \Sigma_x U^T U$

$= \Sigma_x$ ie $C_{\underline{Y}}$ becomes diagonal

& hence A ~~must~~ has be choosen as:

the eigenvectors of covariance matrix $C_{\underline{X}}$.

(d) If A is not invertible Jacobian becomes zero and the joint density cannot determined from this method.

Q.3 Linear estimate of Y in terms of X is $\hat{Y} = a_{opt}X + b_{opt}$.

So we have $E\Big[\underbrace{\big(Y - (a_{opt}X + b_{opt})\big)^2}\Big]$ is minimum

$b_{opt} = m_Y - m_x a_{opt} = \cancel{a_{opt}(1 - m_x)}$

$= m_Y - m_x a_{opt}$

So $E\Big[\big(Y - (a_{opt}X + m_Y - m_x a_{opt})\big)^2\Big]$ is minimum

ie $E\big[\{(Y - m_Y) - a_{opt}(X - m_x)\}^2\big]$ is minimum.

Solve for a & b.

$$a = \frac{Var(X_2)\,Cov(X_1, X_2) - Cov(X_1 X_2)\,Cov(X_2, X_3)}{Var(X_1)\,Var(X_2) - Cov(X_1, X_2)^2}$$

$$b = \frac{Var(X_1)\,Cov(X_2, X_3) - Cov(X_1 X_2)\,Cov(X_1, X_3)}{Var(X_1)\,Var(X_2) - Cov(X_1, X_2)^2}$$

Q5.

$x_1(t) = t$, $x_2(t) = 2t$, $x_3(t) = 3t$, $x_4(t) = 4t$

$x_5(t) = 5t$, $x_6(t) = 6t$

at $t = 1$, r.v. X has values 1, 2, 3, 4, 5, 6, with probability $\frac{1}{6}$

$t = 2$, r.v. Y " " 2 4 6 8 10 12 with probability of 1/6.

(a) So $E[X] = \frac{1}{6}(21) = 3.5$, $E[Y] = \frac{42}{6} = 7$

(b) to find $f_{XY}(x, y) = f_{X_1 X_2}(x, y)$

$$= \frac{1}{6}\left\{ \delta(x-1, y-2) + \delta(x-2, y-4) + \cdots \right\}$$

(c) $R_X(1,2) = \frac{1}{6}\sum_{i=1}^{6} x_i(1)\, y_i(2) = \frac{1}{6}(2+8+18+32+50+72)$

$= ?$

(d) By inspection we can say that it is not stationary since the density ~~fn~~ (for n=1) is $\delta(x)$ at $t=0$, but it is sum of delta f^ns at other times.