Tutorial 12

1. Consider a particle initially located at a given point in the plane, and suppose that it undergoes a sequence of steps of fixed length, but in a completely random direction. Specifically, suppose that the new position after each step is one unit of distance from the previous position and at an angle of orientation from the previous position that is uniformly distributed over $(0, 2\pi)$. Compute the expected square of the distance from the origin after n steps.

2. Prove that, $\mathrm{Cov}\,(\sum\limits_{i=1}^{n} X_i, \sum\limits_{j=1}^{m} Y_j) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$.

3. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables having variance σ^2. Show that $\mathrm{Cov}(X_i - \overline{X}, \overline{X}) = 0$.

4. If a die is to be rolled until all sides have appeared at least once, find the expected number of times that outcome 1 appears.

5. The nine players on a basketball team consist of 2 centers, 3 forwards, and 4 backcourt players. If the players are paired up at random into three groups of size 3 each, find (a) the expected value and (b) the variance of the number of triplets consisting of one of each type of player.

6. A deck of 52 cards is shuffled and a bridge hand of 13 cards is dealt out. Let X and Y denote, respectively, the number of aces and the number of spades in the hand. Then (a) show that X and Y are uncorrelated. (b) are they independent?

7. Suppose that $X_i, i = 1, 2, 3$, are independent Poisson random variables with respective means $\lambda_i, i = 1, 2, 3$. Let $X = X_1 + X_2$ and $Y = X_2 + X_3$. The random vector X, Y is said to have a bivariate Poisson distribution.

   (a) Find $E[X]$ and $E[Y]$.

   (b) Find $\mathrm{Cov}(X, Y)$.

   (c) Find the joint probability mass function $P\{X = i, Y = j\}$.

8. If X and Y are independent binomial random variables with identical parameters n and p, calculate the conditional expected value of X given that $X + Y = m$.

9. The game of craps is begun by rolling an ordinary pair of dice. If the sum of the dice is $2, 3$, or 12, the player loses. If it is 7 or 11, the player wins. If it is any other number i, the player continues to roll the dice until the sum is either 7 or i. If it is 7, the player loses; if it is i, the player wins. Let R denote the number of rolls of the dice in a game of craps. Find

   (a) $E[R]$;

   (b) $E[R|\text{player wins}]$;

   (c) $E[R|\text{player loses}]$.

10. Independent trials, each resulting in a success with probability p, are successively performed. Let N be the time of the first success. Find $\mathrm{Var}(N)$.