Termination

There are 4 aspects of the termination

i) Alternative optimum

ii) Unboundedness

iii) Infeasibility

iv) cycling

Alternative optimum

consider the following LPP.

$$\text{Max } Z = 2x_1 + 3x_2$$

$$\text{s.t.} \quad 5x_1 + 3x_2 \leq 15 \quad \text{— (1)}$$

$$4x_1 + 6x_2 \leq 12 \quad \text{— (2)}$$

$$x_1, x_2 \geq 0$$

we introduce slack variables x_3 and x_4 to (1) and (2) then we have,

$$\text{Max } Z = 2x_1 + 3x_2 + 0x_3 + 0x_4$$

$$\text{s.t.} \quad 5x_1 + 3x_2 + x_3 \qquad = 15$$

$$4x_1 + 6x_2 \qquad + x_4 = 12$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

			C_j	2	3	0	0		
C_B	B	X_B	b	a_1	a_2	a_3	a_4	min ratio	operation
0	a_3	x_3	15	5	3	1	0	$15/3=5$	
0	a_4	x_4	12	4	$\boxed{6}$	0	1	$12/6=2 \rightarrow$	
z_j-c_j			$\boxed{0}$	-2	$-3 \uparrow$	0	0		
0	a_3	x_3	9	$\boxed{3}$	0	1	$-1/2$	$9/3=3 \rightarrow$	$R_1'=R_1-3R_2'$
3	a_2	x_2	2	$2/3$	1	0	$1/6$	$2/(2/3)=3$	$R_2'=R_2/6$
z_j-c_j			$\boxed{6}$	$0 \uparrow$	0	0	$1/2$		
2	a_1	x_1	3	1	0	$1/3$	$-1/6$		$R_1'=R_1/3$
3	a_2	x_2	0	0	1	$-2/9$	$5/18$		$R_2'=R_2-\frac{2}{3}R_1'$
z_j-c_j			$\boxed{6}$	0	0	$0 \uparrow$	$1/2$		

<u>observation:</u>

- In iteration 2 the non-basic variable a_1 has z_j-c_j is zero
- If we enter this variable we get another optimum solution with same objective function value.
- If we enter a_3 again we get the same values in iteration 2.

$$\max Z = 2x_1 + 3x_2$$

$$\text{s.t. } 5x_1 + 3x_2 \le 15 \quad \text{— (1)}$$

$$4x_1 + 6x_2 \le 12 \quad \text{— (2)}$$

$$x_1, x_2 \ge 0 .$$

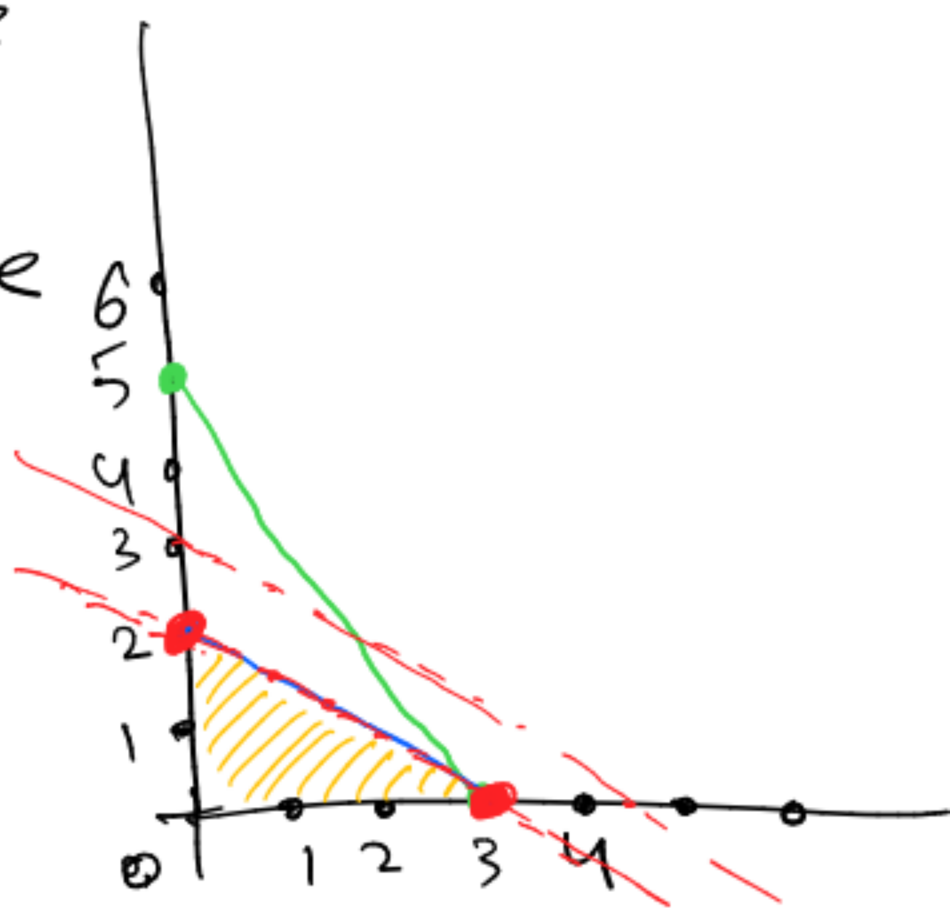
Two points on (1) are
(0,5) and (3,0)

Two points on (2) are
(0,2) and (3,0)

two points on objective function line are,

$$2x_1 + 3x_2 = 6 \quad 9$$

(0,2) and (3,0)



(0,2) and (3,0)

Unboundedness

Consider the following LPP.

$$\max z = 3x_1 + 4x_2$$

$$\text{s.t.} \quad -x_1 + x_2 \le 0 \quad \text{--- (1)}$$

$$-x_1 + 3x_2 \le 3 \quad \text{--- (2)}$$

$$x_1, x_2 \geq 0.$$

After introducing slack variables we have,

$$\max z = 3x_1 + 4x_2 + 0x_3 + 0x_4$$

$$\text{s.t.} \quad -x_1 + x_2 + x_3 = 0$$

$$-x_1 + 3x_2 + x_4 = 3$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

C_B	B	X_B	b	a_1	a_2	a_3	a_4	min ratio	operations
			c_j	3	4	0	0		
0	a_3	x_3	0	-1	$\boxed{1}$	1	0	$0/1=0$ →	
0	a_4	x_4	3	-1	3	0	1	$3/3=1$	
z_j-c_j			0	-3	-4 ↑	0	0		
4	a_2	x_2	0	-1	1	1	0	-----	$R_1' = R_1/1$
0	a_4	x_4	3	$\boxed{2}$	0	-3	1	$3/2=1.5$ →	$R_2' = R_2 - 3\cdot R_1'$
z_j-c_j			0	-7 ↑	0	4	0		
4	a_2	x_2	$3/2$	0	1	$-1/2$	$1/2$	---	$R_1' = R_1 + R_2'$
3	a_1	x_1	$3/2$	1	0	$-3/2$	$1/2$	----	$R_2' = R_2/2$
z_j-c_j				0	0	$-13/2$ ↑	$7/2$		

Observation

- The algorithm terminates but x_3 tries to enter the basis, and there is no leaving variable.
- This is called <u>unboundedness</u>.

Infeasibility

$$\max\ z = 5x_1 + 3x_2$$

$$\text{s.t.}\quad 3x_1 + x_2 \le 1 \quad \text{——} \ (1)$$

$$3x_1 + 4x_2 \ge 12 \quad \text{——} \ (2)$$

$$x_1, x_2 \ge 0$$

Introduce slack variable x_3 to (1) and surplus and artificial variables x_4 and x_5 to (2), then we have.

$$\max\ z = 5x_1 + 3x_2 + 0x_3 + 0x_4 - Mx_5$$

$$\text{s.t.}\quad 3x_1 + x_2 + x_3 = 1$$

$$3x_1 + 4x_2 - x_4 + x_5 = 12$$

$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

		c_j	0	0	0	0	-1		
C_B	B	X_B	b	a_1	a_2	a_3	a_4	a_5	min ratio	oberations
0	a_3	x_3	1	3	$\boxed{1}$	1	0	0	$1/1=1$ →	
-1	a_5	x_5	12	3	4	0	-1	1	$12/4=3$	
z_j-c_j			$\boxed{-12}$	-3	-4 ↑	0	1	0		
0	a_2	x_2	1	3	1	1	0	0		$R_1'=R_1/1$
-1	a_5	x_5	8	-9	0	-4	-1	1		$R_2'=R_2-4R_1'$
z_j-c_j			$\boxed{-8}$	9	0	4	1	0		

observations

- all z_j-c_j's are ≥ 0. Thus optimal condition is reached
- The artificial variable still present in the basis.
- maximum objective value is $-8<0$
- The solution is not feasible.

Cycling

Consider the LPP.

minimise $Z = -\frac{3}{4}x_1 + 20x_2 - \frac{1}{2}x_3 + 6x_4$

s.t. $\frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \le 0$ — (1)

$\frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 \le 0$ — (2)

$x_3 \le 1$ — (3)

$x_1, x_2, x_3, x_4 \geq 0$

Reference

E. M L. Beale, 1995. Naval research Logistic quaterly, 2, pp. 269-275

consider the standard form.

$$\max Z = \frac{3}{4}x_1 - 20x_2 - \frac{1}{2}x_3 - 6x_4 + 0x_5 + 0x_6 + 0x_7$$

s.t.

$$\frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 + x_5 = 0$$

$$\frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 + x_6 = 0$$

$$x_3 + x_7 = 1$$

$$x_1, x_2, \ldots, x_7 \geq 0.$$

	C_B	B	X_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	min ratio	operations
				c_j	$3/4$	-20	$1/2$	-6	0	0	0		
1	0	a_5	x_5	0	$\boxed{1/4}$	-8	-1	9	1	0	0	$0/\tfrac{1}{4} = 0$ →	
	0	a_6	x_6	0	$1/2$	-12	$-1/2$	3	0	1	0	$0/\tfrac{1}{2} = 0$	
	0	a_7	x_7	1	0	0	1	0	0	0	1	....	
	$z_j - c_j$				$-3/4$ ↑	20	$-1/2$	6	0	0	0		
2	$3/4$	a_1	x_1	0	1	-32	-4	36	4	0	0	-----	$R_1' = R_1/\tfrac{1}{4}$
	0	a_6	x_6	0	0	$\boxed{4}$	$3/2$	-15	-2	1	0	0 →	$R_2' = R_2 - \tfrac{1}{2}R_1$
	0	a_7	x_7	1	0	0	1	0		0	1	.....	$R_3' = R_3$
	$z_j - c_j$				0	-4 ↑	$-7/2$	33	3	0	0		
3	$3/4$	a_1	x_1	0	1	0	$\boxed{8}$	-84	-12	8	0	0 →	$R_1' = R_1 + 32R_2'$
	-20	a_2	x_2	0	0	1	$3/8$	$-15/4$	$-1/2$	$1/4$	0	0	$R_2' = R_2/4$
	0	a_7	x_7	1	0	0	1	0	0	0	1	1	$R_3' = R_3$
	$z_j - c_j$				0	0	-2 ↑	18	1	1	0		
4													

If we proceed in this way we see that the 7th iteration is same as iteration 1.