Lab Submission: 1. 1 submission/group.

2. Submit (a) Code (Python3) in a single .ipynb file

(b) Pdf report: GroupID.pdf

(c) .zip file containing Latex source and reqd. images.

Deadline: Friday 12 noon

- 5 google meet links.

Spectral Density: Measures/Quantifies the amount of energy/power around each frequency.

Energy Spectral Density (ESD):

$$E_x = \int_{-\infty}^{\infty} x(t)^2 \, dt = \int_{-\infty}^{\infty} \underbrace{|X(f)|^2}\, df$$

$\|x\|_{L_2}$

$\Psi_x(f) = |X(f)|^2$ is called the Energy Spectral Density.

Power Spectral Density (PSD)

Periodic Signals

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$P_x = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

(PSD) $G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$

$f_0 = \dfrac{\omega_0}{2\pi}$

(Aperiodic Signals)

Aperiodic Signals (Power)

$x(t) \leftarrow \text{rect} \rightarrow x_T(t) = x(t) \;, \quad -T/2 < t \le T/2$

$\qquad\qquad\qquad\qquad = 0 \;, \quad$ elsewhere.

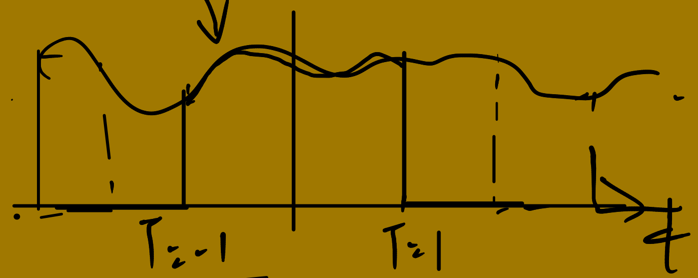
$$\boxed{\lim_{T\to\infty} x_T(t) = x(t)} \;, \forall t \in \mathbb{R}.$$

$x_T \;, T < \infty \Rightarrow$ Energy signal.

$$E_{x_T} = \int_{-T/2}^{T/2} x(t)^2 \, dt . \qquad P_{x_T} = \frac{1}{T} \int_{-T/2}^{T/2} x(t)^2 \, dt$$

$$X_T(\omega) = \int_{-T/2}^{T/2} x(t) e^{-j\omega t} \, dt . \qquad G_{x_T}(f) = \frac{1}{T} |X_T(f)|^2 .$$

$\mathcal{F}\{x_T\}$



$$\boxed{G_x(f) = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2}$$ ← Power Spectral Density of an aperiodic power signal.

$$\left\{ \frac{1}{T_1} |X_{T_1}(f)|^2 , \frac{1}{T_2} |X_{T_2}(f)|^2 , \frac{1}{T_3} |X_{T_3}(f)|^2 , \ldots \right\}$$

$T_{i+1} > T_i \qquad \longrightarrow G_x(f)$

Autocorrelation

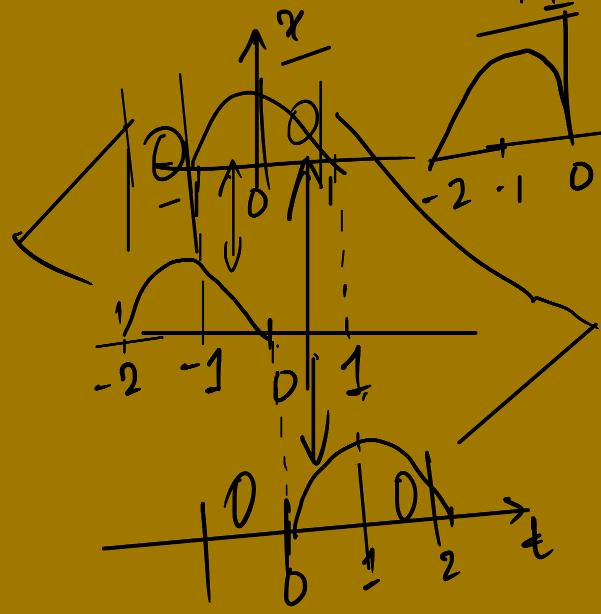
1. Energy Signals

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt, \quad -\infty < \tau < \infty.$$

$$= \langle x, D_\tau x \rangle$$ ← Inner products (Dot products)

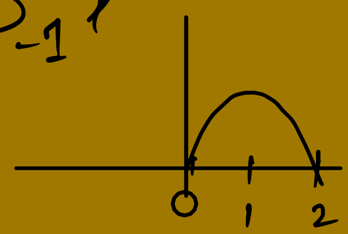
Properties: ① $R_x(-\tau) = R_x(\tau)$

$R_x(1)$



$R_x(-1)$

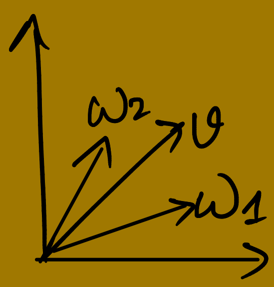
$D_{-1}x$



② $R_x(0) \geq R_x(\tau)$

$$= \int_{-\infty}^{\infty} x(t)^2\,dt$$

$$= E_x$$



$$R_x(0) = \|x\|^2 = \langle x, x \rangle$$

Cauchy-Schwartz Inequality!

$$|\langle x,y\rangle| \le \|x\|\,\|y\|$$

$$a\cdot b = \|a\|\,\|b\|\cos\theta$$

$$|a\cdot b| \le \|a\|\,\|b\|.$$

$$R_x(\tau) = |\langle x, D_\tau x\rangle| \le \|x\|\,\|D_\tau x\|$$

$$\le \sqrt{\langle x,x\rangle}\,\sqrt{\langle D_\tau x, D_\tau x\rangle}$$

$$\le \sqrt{R_x(0)}\,\sqrt{R_x(0)} = R_x(0)$$

$$\Rightarrow \boxed{R_x(0) \ge R_x(\tau)}$$

(3) $$\mathcal{F}\{R_x\} = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt\right) e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t)\left(\underbrace{\int_{-\infty}^{\infty} x(t+\tau)\, e^{-j\omega\tau}\, d\tau}_{\text{Time delay in FT}}\right) dt$$

$$= \int_{-\infty}^{\infty} x(t)\, X(\omega)\, e^{j\omega t}\, dt$$

$$= X(\omega)\, X^*(\omega) = |X(\omega)|^2$$ (Real valued energy signal)

ESD

⇒ Auto correlation for Power Signals

a. Periodic Signals

$$R_x(\tau) = \int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt$$

b. Aperiodic Signals

$$R_x(\tau) = \lim_{T \to \infty} \int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt \;, \quad -\infty < \tau < \infty.$$

Properties : 1. $R_x(\tau) = R_x(-\tau)$

2. $\mathcal{F}\{R_x\} = \lim_{T \to \infty} \frac{1}{T} |X_T(\omega)|^2 = G_x(f)$
P.S.D. of x

3. $R_x(0) \geq R_x(\tau)$

* Convolution : $(x * y)(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$

Probability and Statistics

Random Variable (RV) : A RV is a mapping from the sample space (S) to $\mathbb{R}$.

Characterizations of a RV

1. Distribution functions (F_X) X → RV

$F_X(x) = P(X \leq x) \Leftarrow$ Cumulative Distribution function (CDF).

Properties of CDF

1. $0 \leq F_X(x) \leq 1, \ \forall x$.

2. If $x_1 < x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$

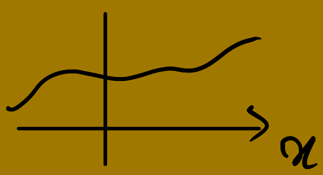
   F_X is a monotonically non-decreasing function.

3. $F_X(-\infty) = 0, \quad F_X(\infty) = 1$.

Probability Density function (pdf)

- Continuous RV : If the sample space is not discrete.
- Discrete RV X : If the Sample Space is a discrete set.

CRV : pdf : $P_X(x) = \frac{d}{dx} F_X(x)$. ✓



$\Updownarrow$

$$F_X(x) = \int_{-\infty}^{x} P_X(u)\, du \quad ✓$$

DRV : pdf $\quad P_X(x) = \sum_{i=1}^{n} P(X = x_i)\, \delta(x - x_i)$

Properties of pdf.

① $P_X(x) \geq 0$ ③ $\int_{-\infty}^{\infty} P_X(x)\, dx = 1$.

② $P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} P_X(x)\, dt = F_X(x_2) - F_X(x_1)$.