Lecture 3 – D_n the dihedreal group

The group of symmetries of a regular n-sided polygon is called dihedreal group D_n. Consider first the equilateral triangle as shown in the figure. The axes of symmetries are shown in the

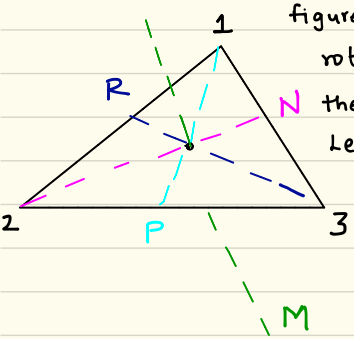
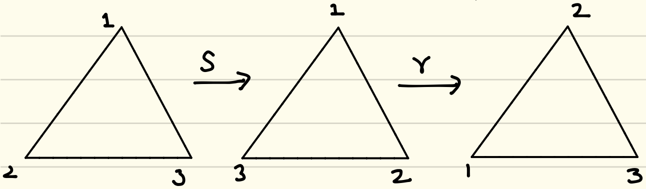
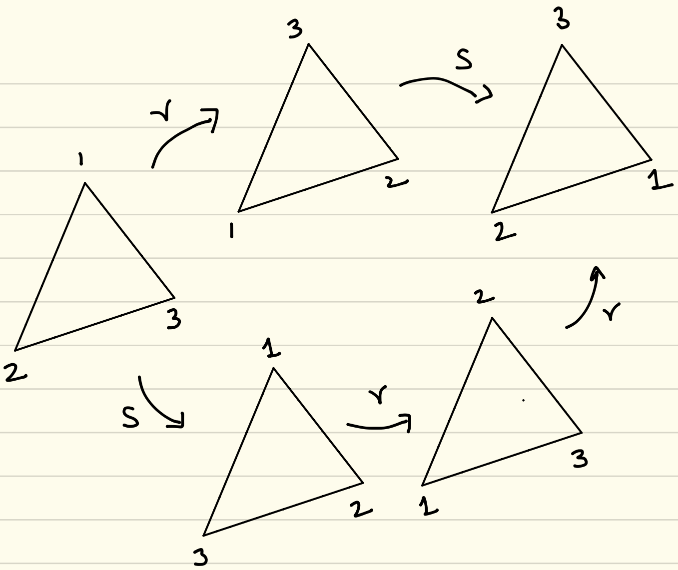


figure by the lines M, N, R, P. Let r denote the rotation about axis M by $\frac{2\pi}{3}$. This takes the vertex 1 to 2, 2 to 3 and 3 back to 1. Let s denote the reflection about axis P. s interchanges the vertices 2 and 3. We have the relations $r^3 = e$ and $s^2 = e$. Now rs (s followed by r) is reflection about the axis R.



Similarly $r^2 s$ is reflection about axis N. In this way we get all the symmetries of the equilateral triangle which is the set $\{e, r, r^2, s, rs, r^2 s\}$. What about the element sr? It turns out that $sr = r^2 s$. This can be seen geometrically as shown in the figure. Then using this fact we can show that $sr^2 = rs$. Indeed, $sr^2 = (sr)r = (r^2 s)r = r^2(sr) = r^2 r^2 s = r^4 s = rs$ since $r^3 = e$.



$sr = r^2 s$

	e	r	r^2	s	rs	$r^2 s$
e	e	r	r^2	s	rs	$r^2 s$
r	r	r^2	e	rs	$r^2 s$	s
r^2	r^2	e	r	$r^2 s$	s	rs
s	s	$r^2 s$	rs	e	r^2	r
rs	rs	s	$r^2 s$	r	e	r^2
$r^2 s$	$r^2 s$	rs	s	r^2	r	e

is the group multiplication table.

For a general n-regular polygon we can generalize this. Let r be a rotation by angle $\frac{2\pi}{n}$ by the axis of symmetry that is perpendicular to the plane in which the regular n-gon lies and s be reflection about a line that lies in the plane (it does not matter which one), then again we have $r^n = e$ and $s^2 = e$. There are $2n$ symmetries given by

$$\{e, r, r^2, \dots, r^{n-1}, s, rs, r^2 s, \dots, r^{n-1} s\}$$

We also have that $sr = r^{n-1}s = r^{-1}s$ (check geometrically!) then using this fact we see that

$$sr^j = r^{n-j}s \quad \text{(check!)}$$

Each element of the group D_n has elements of the form r^a or $r^a s$ for $0 \le a \le n-1$ and we have

$$\left.\begin{aligned} r^a r^b &= r^k \\ r^a(r^b s) &= r^k s \end{aligned}\right\} \quad k = a +_n b$$

$$\left.\begin{aligned} (r^a s) r^b &= r^l s \\ (r^a s)(r^b s) &= r^l \end{aligned}\right\} \quad l = a +_n (n-b)$$

We say that r and s generate the group D_n.

Finally, the order of a group is the number of elements in the group. If a group has infinite elements then the group has infinite order. We denote the order of a group G by $|G|$. If x is an element of G and if there is a positive integer such that $x^m = e$ then we say that x has finite order. The smallest positive integer m such that $x^m = e$ is called the order of x.

Examples:

1) The order of D_n is $2n$. In D_3 r, r^2 have order 3, where as $s, rs, r^2 s$ have order 2

2) Order of $\mathbb{Z}_6$ is six. The elements 1 and 5 have order 6, 2 and 4 have order 3 and 3 has order 2.

3) $(\mathbb{R}, +)$ has infinite order and every element except 0 has infinite order.

4) $C = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle. This is a group of infinite order. All elements of this group are of the form $e^{i\theta}$. The elements when θ is a rational multiple of 2π have finite order.