## Some points to be remembered

- We start the simplex algorithm with an initial basic feasible solution.
  - If we are unable to recognise an initial basic feasible solution; introduce artificial variables.

- We can introduce as many no. of artificial variables. (at most no. of constraints)

- The value M (a very large number) ensures that the artificial variables do not appear in the optimal solution.

- If there are K artificial variables then we need at least K iterations to find an optimal solution.

- we donot comput values for artificial variables those are leaving the basic.

$\boxed{\text{This method is called the “big M method”}}$

Another way of solving minimization Problem

## Two-Phase method

- Artificial variables do not present in an optimal solution. They are used only to identify an initial basic feasible solution.

- For this observation we introduce anothe method to solve LPP.

- The first phase is used to remove the artificial variables.

- The 2nd phase is used to solve the problem.

  where we start with an initial basic feasible solution identified at the end of first phase.

- Take the coefficient of the artificial variable in the objective function as $-1$

  and

  take the coefficient of the remaining variables in the objective function as $0$.

- Since minimisation problem can be changed to a maximisation problem, and variables are non negative.

  If the problem has an optimal solution, then the value of the objective function is zero

  That means artificial variables are removed from the first phase.

# Two phase method

$$\min z = 2x_1 + x_2$$

s.t. $2x_1 + x_2 \geq 4$ —— (1)

$x_1 + 2x_2 \geq 3$ —— (2)

$x_1 + 3x_2 \geq 6$ —— (3)

$x_1, x_2 \geq 0$

## Phase 1

After introducing surplus variables $x_3, x_4, x_5$ to (1), (2) & (3) resp. and converting min to max we get.

$$\max w' = \underline{0x_1 + 0x_2 + 0x_3 + 0x_4 + x_5 - 1x_6 - 1x_7 - 1x_8}$$

s.t.

$$2x_1 + x_2 - x_3 + x_6 = 4$$

$$x_1 + 2x_2 - x_4 + x_7 = 3$$

$$x_1 + 3x_2 - x_5 + x_8 = 6$$

$$x_1, x_2, \ldots, x_8 \geq 0$$

Phase I

| | | $c_j$ | | 0 | 0 | 0 | 0 | 0 | -1 | -1 | -1 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $X_B$ | $b$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $b/a_2$ min ratio | Operation |
| -1 | $a_6$ | $x_6$ | 4 | 2 | 1 | -1 | 0 | 0 | 1 | 0 | 0 | $4/1=4$ | |
| -1 | $a_7$ | $x_7$ | 3 | 1 | $\boxed{2}$ | 0 | -1 | 0 | 0 | 1 | 0 | $3/2=1.5$ → | |
| -1 | $a_8$ | $x_8$ | 6 | 1 | 3 | 0 | 0 | -1 | 0 | 0 | 1 | $6/3=2$ | |
| $z_j-c_j$ | | | $\boxed{-13}$ | -4 | -6 ↑ | 1 | 1 | 1 | 0 | 0 | 0 | $b/a_4$ | |
| -1 | $a_6$ | $x_6$ | 5/2 | 3/2 | 0 | -1 | 1/2 | 0 | 1 | | 0 | $5/2/1/2=5$ | $R_1'=R_1-R_2'$ |
| 0 | $a_2$ | $x_2$ | 3/2 | 1/2 | 1 | 0 | -1/2 | 0 | 0 | | 0 | ------ | $R_2'=R/2$ |
| -1 | $a_8$ | $x_8$ | 3/2 | -1/2 | 0 | 0 | $\boxed{3/2}$ | -1 | 0 | | 1 | $3/2/3/2=1$ → | $R_3'=R_3-3\cdot R_2'$ |
| $z_j-c_j$ | | | $\boxed{-4}$ | -1 | 0 | 1 | -2 ↑ | 1 | 0 | | 0 | $b/a_1$ | |
| -1 | $a_6$ | $x_6$ | 2 | $\boxed{5/3}$ | 0 | -1 | 0 | 1/3 | 1 | | | $2/5/3=6/5$ → | $R_1'=R_1-\frac{1}{2}R_3'$ |
| 0 | $a_2$ | $x_2$ | 2 | 1/3 | 1 | 0 | 0 | -1/3 | 0 | | | $2/1/3=6$ | $R_2'=R_2+\frac{1}{2}R_3'$ |
| 0 | $a_4$ | $x_4$ | 1 | -1/3 | 0 | 0 | 1 | -2/3 | 0 | | | ...... | $R_3'=R_3/3/2$ |
| $z_j-c_j$ | | | $\boxed{-2}$ | -5/3 ↑ | 0 | 1 | 0 | -1/3 | 0 | | | | |
| 0 | $a_1$ | $x_1$ | 6/5 | 1 | 0 | -3/5 | 0 | 1/5 | | | | | $R_1'=R_1/5/3$ |
| 0 | $a_2$ | $x_2$ | 8/5 | 0 | 1 | 1/5 | 0 | -2/5 | | | | | $R_2'=R_2-\frac{1}{3}R_1'$ |
| 0 | $a_4$ | $x_4$ | 7/5 | 0 | 0 | -1/5 | 1 | -3/5 | | | | | $R_3'=R_3+\frac{1}{3}R_1'$ |
| $z_j-c_j$ | | | $\boxed{0}$ | 0 | 0 | 0 | 0 | 0 | | | | | |

Since all $z_j-c_j \geq 0$ we reached the optimality condition.

## Phase II

| | | | $c_j$ | -2 | -1 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | B | $X_B$ | b | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | min ratio | operation. |
| -2 | $a_1$ | $x_1$ | 6/5 | 1 | 0 | -3/5 | 0 | 1/5 | | |
| -1 | $a_2$ | $x_2$ | 8/5 | 0 | 1 | 1/5 | 0 | -2/5 | | |
| 0 | $a_4$ | $x_4$ | 7/5 | 0 | 0 | -1/5 | 1 | -3/5 | | |
| $z_j - c_j$ | | | $\boxed{-4}$ | 0 | 0 | 1 | 0 | 0 | | |

All $z_j - c_j \geq 0$ so optimality reached. and the optimal solution is.

$x_1 = 6/5, \ x_2 = 8/5$

and value (maximum) $W = -4$

Therefore the value to the original problem (minimization) is $Z = -W = 4$

## More on initialisation

- The main idea of initialisation is to identify an initial basic feasible solution.
- This can be achieved by introducing artificial variables. (if necessary).
- Artificial variables can not be part of any optimal solution. So the number of iterations is at least the number of artificial variables.
- Target is to reduce the number of artificial variables as many as possible.

## Example

$$\min Z = 2x_1 + 5x_2 + 3x_3$$

$$\text{s.t.} \quad 4x_1 + 2x_2 \geq 8 \quad \text{--- (1)}$$

$$2x_1 + 3x_2 + 4x_3 \geq 5 \quad \text{--- (2)}$$

$$x_1, x_2, x_3 \geq 0$$

Introduce surplus variables $x_4$ and $x_5$ to (1) & (2) resp.

$$\min Z = 2x_1 + 5x_2 + 3x_3 + 0x_4 + 0x_5$$

$$\text{s.t.} \quad 4x_1 + 2x_2 - x_4 = 8 \quad \text{--- (3)}$$

$$2x_1 + 3x_2 + 4x_3 - x_5 = 5 \quad \text{--- (4)}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

we divide 4 in (4) we get.

$$\frac{2}{4}x_1 + \frac{3}{4}x_2 + \frac{4}{4}x_3 \cdots - \frac{1}{4}x_5 = \frac{5}{4}$$

$$\Rightarrow \frac{1}{2}x_1 + \frac{3}{4}x_2 + x_3 - \frac{1}{4}x_5 = \frac{5}{4} \quad \text{--- (5)}$$

replace (4) by (5), Introduce $x_6$ as artificial variable in (3) and change min to max we get.

$$\max W = -Z = -2x_1 - 5x_2 - 3x_3 + 0x_4 + 0x_5 - Mx_6$$

$$\text{s.t.} \quad 4x_1 + 2x_2 - x_4 + x_6 = 8$$

$$\frac{1}{2}x_1 + \frac{3}{4}x_2 + x_3 - \frac{1}{4}x_5 = \frac{5}{4}$$

$$x_1, x_2, \ldots, x_6 \geq 0$$

| | | | $c_j$ | $-2$ | $-5$ | $-3$ | $0$ | $0$ | $-M$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $B$ | $X_B$ | $b$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | min ratio | operation |
| $-M$ | $a_6$ | $x_6$ | $8$ | $4$ | $2$ | $0$ | $-1$ | $0$ | $1$ | | |
| $-3$ | $a_3$ | $x_3$ | $5/4$ | $1/2$ | $3/4$ | $1$ | $0$ | $-1/4$ | $0$ | | |
| $Z_j - C_j$ | | | | | | | | | | | |