International Institute of Information Technology, Hyderabad

(Deemed to be University)

Probability and Statistics (MCS 1) Monsoon-2023

Assignment 2

Due Date: 25, Sep, 2023

Question (1) [20 Marks] Let us consider a two dimensional continuous type random variable (X, Y) with density $f(x, y)$. Further, let moments $\mathbb{E}[X] = \mu_X$, $\mathbb{E}[Y] = \mu_Y$, $\mu_{20} = \sigma_X^2$, $\mu_{02} = \sigma_Y^2$ and μ_{11} be finite, where $\mu_{mn} = \mathbb{E}\left[(X - \mathbb{E}[X])^m (Y - \mathbb{E}[Y])^n\right]$. Let the coefficient of correlation ρ be given by $\mathbb{E}\left[(X - \mu_X)(Y - \mu_Y)\right] / (\sigma_X \sigma_Y)$.

(a) **[6 Marks]** For a positive k, consider a rectangle R, defined by $|x - \mu_X| = k\sigma_X$ and $|y - \mu_Y| = k\sigma_Y$. For $t^2 < 1$, define a random variable

$$H(X, Y) = \frac{1}{k^2(1 - t^2)} \left(\frac{(X - \mu_X)^2}{\sigma_X^2} - 2t \frac{(X - \mu_X)(Y - \mu_Y)}{\sigma_X \sigma_Y} + \frac{(Y - \mu_Y)^2}{\sigma_Y^2} \right).$$

Prove that this function is positive and in fact greater than one outside the rectangle R. Using this prove that

$$\mathbb{E}[H(X, Y)] := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y) f(x, y) dx\ dy \geq 1 - \int \int_R f(x, y) dx\ dy$$

(b) **[4 Marks]** Using (a) prove that

$$\int \int_R f(x, y) dx\ dy \geq 1 - \frac{2\,(1 - \rho t)}{k^2(1 - t^2)}.$$

(c) **[10 Marks] Berge's theorem:** In (b), relating LHS of the equation to the probabilities and choosing t accordingly in RHS, prove that

(i) $P\left(\{|X - \mu_X| \leq k\sigma_X\} \cap \{|Y - \mu_Y| \leq k\sigma_Y\}\right) \geq 1 - \frac{1}{k^2}\left(1 + \sqrt{1 - \rho^2}\right).$

(ii) $P\left(\{|X - \mu_X| \leq k\sigma_X\} \cup \{|Y - \mu_Y| \leq k\sigma_Y\}\right) \leq \frac{1}{k^2}\left(1 + \sqrt{1 - \rho^2}\right).$

Question (2) [14 Marks] The moment generating function of a random variable X is given by $g(t) = \mathbb{E}[e^{tX}]$, if the expectation exists. Here t is a real number.

(a) **[6 Marks]** Write the moment generating function for continuous and discrete type random variables. Further, write the moments of a random variable X in terms of moment generating function.

(b) **[5 Marks]** Compute the moment generating function for a random variable X with probability density

$$f(x) = \sqrt{\frac{a}{2\pi\sigma^2}} e^{\frac{-ax^2}{2\sigma^2}},$$

where $a > 1$.

(c) **[3 Marks]** Find first two moments of the above random variable.

Question (3) [16 Marks] Let X be a random variable with Poisson distribution, i.e. $P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$, where $k = 0, 1, \cdots$, and λ is unknown.

(a) **[7 Marks]** Suppose, n observations are made from the population following above distribution and outcome obtained are $x_i = k_i$, for all $i = 1, \cdots, n$. Estimate λ using maximum likelihood estimation. [Hint: Use log likelihood function instead of just likelihood function]

(b) **[2 Marks]** What is the maximum likelihood estimator?

(c) **[7 Marks]** Prove that the maximum likelihood estimator in this case is the most efficient unbiased estimator. [Hint: Use Cramér-Rao theorem.]