Assignment 1: Probability and Statistics (MCS 1)

August 11, 2023

Due Date: 22 Aug, 2023

General guidelines: The assignment is mainly from the textbook followed in the class and total marks are 100. Given the assignment is due in 11 days, the due date is strict. Plagiarism and unethical copying will be strictly dealt with. Please coordinate with the course tutors for any doubts that do not need my attention.

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Reference Book: Marek Fisz, Probability Theory and Mathematical Statistics

Question 1 [18 Marks] Let E be a set of elementary events and Z be a Borel field. Let $\{A_i\}_{i=1}^n$ be subsets of Z. Prove the following relations:

(a) $\overline{\cup_{i=1}^n A_i} = \cap_{i=1}^n \overline{A_i}$. **[2 Marks]**

(b) $\overline{\cap_{i=1}^n A_i} = \cup_{i=1}^n \overline{A_i}$. **[2 Marks]**

(c) $\cup_{i=1}^{n} A_i = A_1 + (A_2 - A_1 \cap A_2) + \cdots + (A_n - A_1 A_n - \cdots - A_{n-1} A_n)$. **[3 Marks]**

(d) **Inclusion-Exclusion Principle:**

$$P\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P\left(A_i\right) - \sum_{1 \le i < j \le n} P\left(A_i \cap A_j\right) + \cdots + (-1)^n P\left(\cap_{i=1}^{n} A_i\right).$$

[7 Marks]

(e) **Union Bound:** $P\left(\cup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P\left(A_i\right)$. **[2 Marks]**

(f) **Bonefforni Inequality:**

$$P\left(\cup_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P\left(A_i\right) - \sum_{1 \le i < j \le n} P\left(A_i \cap A_j\right).$$

[2 Marks]

Question 2 (Banach's Problem) [14 Marks] A mathematician carries two matchboxes in his two pockets (left and right), each containing n matches. Each time he lights a cigarette, he is equally likely to take a match from either pocket. Compute the probability that when he eventually selects a pocket with empty matchbox, the matchbox in the other pocket contains r matches.

Question 3 (Bertrand's Paradox) [16 Marks] Consider an equilateral triangle of side a inscribed inside a circle. Draw a random chord in the circle. Then define event A by the observation that the length l of the chord satisfies $l > a$. State the conditions under which $(i)P(A) = 1/2$ and $(ii)P(A) = 1/3$. Should these results be considered paradoxical?

Question 4 [14 Marks]

(a) What should be n in order that the probability of obtaining the face 6 at least once in n independent throws of a die is greater than 1/3? **[6 Marks]**

(b) Let $\{A_i\}_{i=1}^{n}$ be n independent events and $P(A_i) = p$ for all i. What is the smallest n such that $P\left(\cup_{i=1}^{n} A_i\right) \ge p_0$, where $0 \le p_0 \le 1$ is a fixed number? **[8 Marks]**

Question 5 [14 Marks] Let us consider a graph with n vertices. Each pair of vertices have an edge with probability $p(n)$ and the occurrence of each edge in the graph is independent from other edges in the graph. This is called Erdös-Rényi model of random graphs. Let A_n be the event that this graph has at least one isolated node. Then show that

(a) $P(A_n) \leq n[1 - p(n)]^{n-1}$. **[5 Marks]**

(b) For $p(n) = 3 \ln n/(2n)$, $\lim_{n\to\infty} P(A_n) = 0$. **[9 Marks]**

Question 6 [10 Marks] Let X be a random variable with $f(x)$ being its probability density. Are $|X|$ and $\sin X$ random variables? If no, why? If yes, what are the corresponding probability densities?

Question 7 [14 Marks] Let X and Y be two continuous type independent random variables with probability densities

$$f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in (-\infty, \infty), \tag{1}$$

$$f_2(y) = y e^{-y}, \qquad y \in (0, \infty), \tag{2}$$

respectively. Let us consider another random variable $Z = X^2 + Y$. Compute the probability density $g(z)$ and distribution function $G(z)$ of Z.