EECS 16A Designing Information Devices and Systems I
Spring 2020 Homework 4

This homework is due Friday February 21, 2020, at 23:59.
Self-grades are due Monday February 24, 2020, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

- Practice problems are not graded, but are to help with learning.

Submit the file to the appropriate assignment on Gradescope.

1. Finding Null Spaces and Column Spaces

Learning Objectives: *Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

Definition (Null space): The null space of a matrix, $A \in \mathbb{R}^{m \times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{0}$. The null space is notated as $N(A)$ and the definition can be written in set notation as:

$$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

Definition (Column space): The column space of a matrix, $A \in \mathbb{R}^{m \times n}$, is the set of all vectors $A\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the set of A's columns. The column space can be notated as $\mathrm{C}(A)$ or $\mathrm{range}(A)$ and the definition can be written in set notation as:

$$\mathrm{C}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

(a) Consider matrices in $\mathbb{R}^{3 \times 5}$. What is the maximum possible number of linearly independent column vectors?

Solution: If you are stuck solving a problem like this, consider concrete examples. We want to find the maximum possible number of linearly independent column vectors, so we look for examples and check if we can exceed certain values.

Consider the following example matrix, where the entries marked with $*$ are arbitrary values:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}$$

The first three columns are linearly independent, so at least three linearly independent columns are achievable. The first three columns span $\mathbb{R}^3$, therefore any choice of fourth and fifth columns, also in $\mathbb{R}^3$, can be written as a linear combination of the first three columns. This means that we cannot exceed three linearly independent columns. Thus the maximum number of linearly independent column vectors is 3. In general, the columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$ where $m < n$ will always be linearly dependent.

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a set of vectors that span the column space of $\mathbf{A}$. What is the minimum number of vectors required to span the column space of $\mathbf{A}$? (This is the dimension of the column space of $\mathbf{A}$.)

Solution: The column space of $\mathbf{A}$ is the space spanned by its columns, so the set of all columns is a valid answer. However, it is possible to choose a subset of the columns and still span the column space, as we will be guaranteed linear dependence of the columns, as seen in part (a). To find the minimum number of columns needed, we can remove vectors from our set until we are left with a linearly independent set.

By inspection, the second, fourth, and fifth columns can be omitted from a set of columns as they can be expressed as linear combinations of the first and third columns. Thus the minimum number of vectors required to span the column space is 2.

One set spanning the range of $\mathbf{A}$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Another valid set of vectors which span the column space is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Note with this second set, none of the columns of $\mathbf{A}$ appear. Despite this, the span of this set will still be equal to the column space which is the set of all vectors in $\mathbb{R}^3$ with zero third entry. Give yourself full credit if you recognized that the minimum number of vectors required was 2, and if you had any set that spans the column space.

(c) The dimension of the null space is the minimum number of vectors needed to span it. Find a set of vectors that span the null space of $\mathbf{A}$ (the matrix from part (b)). What is the dimension of the null space of $\mathbf{A}$?

Solution:

Finding the null space of $\mathbf{A}$ is the same as solving the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 + x_2 - 2x_4 + 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}$$

We observe that x_2, x_4, and x_5 are free variables. Thus, we let x_2, x_4 and x_5 be a, b and c. Now we rewrite the equations as:

$$\begin{aligned} x_1 &= -a+2b-3c \\ x_2 &= a \\ x_3 &= b-c \\ x_4 &= b \\ x_5 &= c \end{aligned}$$

We can then write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of $\mathbf{A}$ is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of the null space is 3, as it is the minimum number of vectors we need to span it.

(d) What do you notice about the sum of the dimensions of the null space and the column space in relation to the dimensions of $\mathbf{A}$?

Solution: The dimensions of the column space and the null space add up to the number of columns in A. This is true of all matrices and is a theorem in linear algebra about which you will learn in EECS16B.

(e) (**Practice**) Now consider the new matrix, $\mathbf{B} = \mathbf{A}^T$,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Find a set of vectors that span the column space of $\mathbf{B}$. What is the minimum number of vectors required to span the column space of $\mathbf{B}$?

Solution:

We see that only the first two column vectors of B span the column space of B. The third column does not contribute to the column space of B. There are only two unique vectors in the set and therefore the column space has dimension 2.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(f) **(Practice)** Find a set of vectors that spans the null space of the following matrix. This problem requires systematic calculations, but is helpful if you want more practice.

$$\mathbf{C} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

Solution: To find the null space, we wish to solve for all $\vec{x}$ such that $A\vec{x} = \vec{0}$. However, as the right hand side of the augmented matrix will always remain a column of zeroes under any row operation, we can omit the column and row reduce just A.

$$\begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 & 4 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix} \qquad \frac{1}{2}R_1 \rightarrow R_1$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \qquad \begin{matrix} R_1 - R_2 \rightarrow R_2 \\ 2R_1 - R_3 \rightarrow R_3 \\ 3R_1 - R_4 \rightarrow R_4 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} 2R_2 + R_1 \rightarrow R_1 \\ R_2 - R_3 \rightarrow R_3 \\ R_2 - R_4 \rightarrow R_4 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad -R_2 \rightarrow R_2$$

Vectors in the null space satisfy the following equations:

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0} \implies \begin{aligned} x_1 - 2x_2 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

We then set x_2 and x_4 to be the free variables a and b respectively and substitute in:

$$\begin{aligned} x_1 &= 2a \\ x_2 &= a \\ x_3 &= -2b \\ x_4 &= b \end{aligned}$$

We then write these equations in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, the null space of the matrix is spanned by the vectors:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(g) **(Practice)** Find the column space and its dimension, and the nullspace and its dimension of the following matrix.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

Solution:

To find the null space and column space, we can row reduce the matrix to find solutions to $D\vec{x} = \vec{0}$ which are in $\mathrm{N}(D)$, and also identify which vectors can be written as a linear combination of the others and should therefore be discarded from a basis for $\mathrm{C}(D)$. We do not need an augmented column of $\vec{0}$ as it will not change under row operations.

$$\begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix} \overrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since we only have pivots in the first and third columns, we can assign the free variables x_2 and x_4 to s and t. We can write all solutions to $D\vec{x} = \vec{0}$ as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} t$$

Our vectors in $\mathrm{N}(D)$ can be written as a linear combination of two vectors: $\mathrm{N}(D) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Since the vectors we are taking a span of are also linearly independent, we have a basis, and so the dimension of the null space is 2.

Our pivots in the first and third columns tell us that we only need to take the corresponding columns to span the column space. $\mathrm{C}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -1 \end{bmatrix} \right\}$. We are also taking a span of linearly independent vectors here, so we have a basis. Since the basis has two elements, the dimension of the column space is 2.

2. Cubic Polynomials

Learning Goal: *This problem shows us that we can treat fixed-degree polynomials as a vector space. Furthermore, many operations on polynomials are linear operations in this vector space and can be represented by matrices.*

(a) Show that the set of all cubic polynomials

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3,$$

where $t \in [a,b]$ and the coefficients p_k are real scalars, forms a vector space. Call this vector space V.

Solution:

No escape (scaling) property:

$$\alpha p(t) = \alpha p_0 + \alpha p_1 t + \alpha p_2 t^2 + \alpha p_3 t^3$$

The function above is, itself, a cubic polynomial, so the no escape property holds.

No escape (addition) property:
Let us define two cubic polynomials:

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$
$$q(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3$$

$$p(t) + q(t) = (p_0 + q_0) + (p_1 + q_1)t + (p_2 + q_2)t^2 + (p_3 + q_3)t^3$$

The function above is also a cubic polynomial, so this no escape property also holds.
It can be shown from the properties of scalar multiplication and addition that the remaining properties of vector spaces hold for the set of all p(t):
Commutativity: $p(t) + q(t) = q(t) + p(t)$
Associativity of vector addition: $(p(t) + q(t)) + r(t) = p(t) + (q(t) + r(t))$
Additive identity: There exists 0 in the set of all p(t) such that for all $p(t)$, $0 + p(t) = p(t) + 0 = p(t)$
Existence of inverse: For every p(t), there is element $-p(t)$ such that $p(t) + -p(t)$=0
Associativity of scalar multiplication: $c(d(p(t))) = (cd)p(t)$
Distributivity of scalar sums: $(c+d)p(t) = cp(t) + dp(t)$
Distributivity of vector sums: $c(p(t) + q(t)) = cp(t) + cq(t)$
Scalar multiplication identity: $q(t) = 1 \implies q(t)p(t) = 1p(t) = p(t)$
Since all properties of a vector space hold, the set of cubic polynomials is a vector space.

(b) Consider the set of real-valued monomials given below:

$$\varphi_0(t) = 1, \quad \varphi_1(t) = t, \quad \varphi_2(t) = t^2, \quad \varphi_3(t) = t^3,$$

where $t \in \mathbb{R}$.
Show that every real-valued cubic polynomial

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

defined over the interval $[a,b]$ can be written as a linear combination of the monomials $\varphi_0(t)$, $\varphi_1(t)$, $\varphi_2(t)$, and $\varphi_3(t)$. In particular, show that

$$p(t) = \vec{c}^T \vec{\varphi}(t),$$

where

$$\vec{c}^T = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix}$$

is a vector of appropriately chosen coefficients and

$$\vec{\varphi}(t) = \begin{bmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}.$$

Solution:

$\vec{c}$ is chosen exactly as $c_i = p_i$, so

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = c_0 + c_1 t + c_2 t^2 + c_3 t^3 = p_0 + p_1 t + p_2 t^2 + p_3 t^3 = p(t).$$

(c) The monomials $\varphi_k(t) = t^k$, for $k = 0, 1, 2, 3$, constitute a basis for the vector space of real-valued cubic polynomials defined over the interval $[a, b]$. Justify why this is true. What is the dimension of V?

Solution:

First, none of them can be written as a linear combination of the other monomials, i.e. you can't write $\varphi_0(t)$ using scalar multiples of $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$. Hence they are linearly independent.

They also span the space of possible polynomials, as shown in the solution to part (b).

There are four basis monomials, so the dimension of V must be four.

(d) Express the derivatives of the basis polynomials $\varphi_i(t)$ for $i = 0, 1, 2, 3$ in terms of the $\varphi_i(t)$ for $i = 0, 1, 2, 3$. **Solution:** Manually computing their derivatives, we find that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_0(t) &= \frac{\mathrm{d}}{\mathrm{d}t}t^0 = 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(t) &= \frac{\mathrm{d}}{\mathrm{d}t}t^1 = 1 \\
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_2(t) &= \frac{\mathrm{d}}{\mathrm{d}t}t^2 = 2t \\
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_3(t) &= \frac{\mathrm{d}}{\mathrm{d}t}t^3 = 3t^2.
\end{aligned}$$

Now, re-expressing the right-hand-sides of the above in terms of monomials, we find that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_0(t) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(t) &= \varphi_0(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_2(t) &= 2\varphi_1(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_3(t) &= 3\varphi_2(t).
\end{aligned}$$

(e) Let **D** be a 4x4 matrix. Use the previous part to help you find the entries of **D**, such that for any polynomial

$$p(t) = \vec{c}^T \vec{\varphi}(t),$$

its derivative can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = (D\vec{c})^T \vec{\varphi}(t).$$

Hint: What are the dimensions of $(\mathbf{D}\vec{c})^T$? (Reminder: The dimensions of a matrix or vector is a different concept than the dimensions of a vector space).

Solution: Letting

$$\vec{c} = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix}^T,$$

we see by its definition that

$$p(t) = c_0\varphi_0(t) + c_1\varphi_1(t) + c_2\varphi_2(t) + c_3\varphi_3(t).$$

Since we showed that differentiation was a linear operator, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = c_0\frac{\mathrm{d}}{\mathrm{d}t}\varphi_0(t) + c_1\frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(t) + c_2\frac{\mathrm{d}}{\mathrm{d}t}\varphi_2(t) + c_3\frac{\mathrm{d}}{\mathrm{d}t}\varphi_3(t).$$

Now, substituting in the results from the previous part, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = c_0 \cdot 0 + c_1\varphi_0(t) + 2c_2\varphi_1(t) + 3c_3\varphi_2(t).$$

Pulling the coefficients out into their own vector, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = \begin{bmatrix} c_1 & 2c_2 & 3c_3 & 0 \end{bmatrix} \begin{bmatrix} \varphi_0(t) \\ \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{bmatrix} = \begin{bmatrix} c_1 & 2c_2 & 3c_3 & 0 \end{bmatrix} \vec{\varphi}(t)$$

Now, observe that

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ 0 \end{bmatrix}.$$

Thus, we can rearrange our previous equation to become

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \right)^T \vec{\varphi}(t),$$

so we find that our desired matrix

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(f) **(Practice):** A curve is a continuous mapping from the real line to $\mathbb{R}^N$. A cubic Bézier curve—used extensively in computer graphics—is a type of curve that uses as a basis the following special subset of what are more broadly known as *Bernstein polynomials*:

$$\beta_0(t) = (1-t)^3, \quad \beta_1(t) = 3t(1-t)^2, \quad \beta_2(t) = 3t^2(1-t), \text{ and } \beta_3(t) = t^3.$$

Prove that the Bernstein polynomials $\beta_k(t)$ defined above form a basis for the space of cubic polynomials. To do this, show that any real-valued polynomial

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

can be expressed as a linear combination of the polynomials $\beta_k(t)$ and determine the coefficients in that linear combination. In particular, determine the coefficients in the expansion

$$p(t) = \widehat{p}_0 \beta_0(t) + \widehat{p}_1 \beta_1(t) + \widehat{p}_2 \beta_2(t) + \widehat{p}_3 \beta_3(t).$$

Hint: Determine a matrix $\mathbf{R}$, such that

$$\vec{\beta}(t) = \mathbf{R}\vec{\varphi}(t),$$

where

$$\vec{\beta}(t) = \begin{bmatrix} \beta_0(t) \\ \beta_1(t) \\ \beta_2(t) \\ \beta_3(t) \end{bmatrix} = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}.$$

Without solving for the inverse, show that $\mathbf{R}$ is invertible. Determine its inverse $\mathbf{R}^{-1}$, from which you can determine the coefficients $\widehat{p}_k$. You may use IPython to find $\mathbf{R}^{-1}$.

Solution:

The matrix $\mathbf{R}$ represents a change of basis from the standard monomial basis $\{\varphi_i\}$ to the Bernstein polynomial basis $\{\beta_i\}$. We can write $\mathbf{R}$ explicitly by expanding each Bernstein polynomial in the monomial basis. For example,

$$\beta_0(t) = (1-t)^3 = 1 - 3t + 3t^2 - t^3 = \varphi_0(t) - 3\varphi_1(t) + 3\varphi_2(t) - \varphi_3(t)$$

Similarly, we have:

$$\vec{\beta}(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix} = \begin{bmatrix} 1-3t+3t^2-t^3 \\ 3t-6t^2+3t^3 \\ 3t^2-3t^3 \\ t^3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \mathbf{R}\vec{\varphi}(t)$$

Let us define the vectors $\vec{p} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$ and $\widehat{\vec{p}} = \begin{bmatrix} \widehat{p}_0 \\ \widehat{p}_1 \\ \widehat{p}_2 \\ \widehat{p}_3 \end{bmatrix}$. We can find $\widehat{\vec{p}}$ from $\vec{p}$ as follows:

$$\begin{aligned} \widehat{\vec{p}}^T \vec{\beta}(t) &= \vec{p}^T \vec{\varphi}(t) \quad \text{for all } t \\ \implies \widehat{\vec{p}}^T \vec{\beta}(t) &= \vec{p}^T \mathbf{R}^{-1} \vec{\beta}(t) \quad \text{for all } t \\ \implies \widehat{\vec{p}}^T &= \vec{p}^T \mathbf{R}^{-1} \end{aligned}$$

The above matrix, $\mathbf{R}$, is invertible since there is a pivot in each column. We proceed to find the inverse:

$$\mathbf{R}^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{p}^T \mathbf{R}^{-1} = \begin{bmatrix} p_0 \\ p_0 + \frac{1}{3}p_1 \\ p_0 + \frac{2}{3}p_1 + \frac{1}{3}p_2 \\ p_0 + p_1 + p_2 + p_3 \end{bmatrix}$$

Thus,

$$\widehat{p_0} = p_0$$

$$\widehat{p_1} = p_0 + \frac{1}{3}p_1$$

$$\widehat{p_2} = p_0 + \frac{2}{3}p_1 + \frac{1}{3}p_2$$

$$\widehat{p_3} = p_0 + p_1 + p_2 + p_3$$

3. **Introduction to Eigenvalues and Eigenvectors**

Learning Goal: *Practice algorithmic computation of eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this my inspection if you prefer.

(a) $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

Solution:

There are two ways to do this.

First, we can do it by inspection. We can see that this matrix multiplies everything in the first coordinate by 5 and everything in the second by 2. Consequently, when given $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it will return 2 times the input. And when given $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it will return 5 times the input vector.

Alternatively, we can use determinants.

$$\det(\begin{bmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}) = 0$$

$$(5-\lambda)(2-\lambda) - 0 = 0$$

This is already factored for you! We see that, by definition, diagonal matrices have their eigenvalues on the diagonal.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies y = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

where x is a free variable.

Any vector that lies in span$\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$s an eigenvector with corresponding eigenvalue $\lambda = 5$ is .

$\lambda = 2$:

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

where y is a free variable.

Any vector in span$\{\begin{bmatrix}0\\1\end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 2$.

(b) $\begin{bmatrix}22 & 6\\6 & 13\end{bmatrix}$

Solution:

Here, it is hard to guess the answers.

$$\det\left(\begin{bmatrix}22-\lambda & 6\\6 & 13-\lambda\end{bmatrix}\right) = 0$$

$$(22-\lambda)(13-\lambda) - 36 = 0 \implies \lambda = 10, 25$$

$\lambda = 10$:

$$\begin{bmatrix}12 & 6\\6 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}2 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies 2x+y=0 \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\-2x\end{bmatrix}$$

where x is a free variable.

Any vector that lies in span$\{\begin{bmatrix}1\\-2\end{bmatrix}\}$ is an eigenvector with corresponding eigenvalue $\lambda = 10$.

$\lambda = 25$:

$$\begin{bmatrix}-3 & 6\\6 & -12\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}-1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies 2y = x \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2y\\y\end{bmatrix}$$

where y is a free variable.

Any vector that lies in span$\{\begin{bmatrix}2\\1\end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 25$.

(c) $\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$

Solution:

This can also be seen by inspection. The matrix is not invertible since the first two rows are linearly dependent. Therefore, there must be a 0 eigenvalue. This has the eigenvector $\begin{bmatrix}2\\-1\end{bmatrix}$.

The other eigenvector can be seen by noticing that the second row is twice the first. So $\begin{bmatrix}1\\2\end{bmatrix}$ is a good guess to try and indeed it works with $\lambda = 5$.

Alternatively, we can explicitly calculate.

$$\det\left(\begin{bmatrix}1-\lambda & 2\\2 & 4-\lambda\end{bmatrix}\right) = 0$$

$$\begin{aligned}(1-\lambda)(4-\lambda) - 4 &= 0\\ \lambda^2 - 5\lambda = 0 &\implies \lambda(\lambda-5) = 0\\ \lambda &= 0, 5\end{aligned}$$

$\lambda = 0$:

$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$$

$$x+2y=0 \implies y=-\frac{1}{2}x \implies \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Any vector that lies in span$\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 0$.

$\lambda = 5$:

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$2x-y=0 \implies y=2x \implies \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Any vector that lies in span$\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 5$.

(d) Let $A \in \mathbb{R}^{n \times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of this matrix

$$\{\vec{x} \in \mathbb{R}^n : A\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace.

Solution:

The zero vector is contained in this set since $A\vec{0} = \vec{0} = \lambda\vec{0}$.

Let $\vec{v}_1$ and $\vec{v}_2$ be members of the set. Let $\vec{u} = \alpha\vec{v}_1$. Now, $A\vec{u} = A\alpha\vec{v}_1 = \alpha A\vec{v}_1 = \alpha\lambda\vec{v}_1 = \lambda\vec{u}$. Hence, $\vec{u}$ is a member of the set as well and the set is closed under scalar multiplication.

Observe below that the set is closed under vector addition as well.

$$A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \lambda\vec{v}_1 + \lambda\vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2)$$

Hence, the set defined in the question satisfies the properties of a subspace and is consequently a subspace of $\mathbb{R}^n$.

4. Operations on Subspaces

Let $\mathbb{V}$ be a vector space with subspaces $\mathbb{S}$ and $\mathbb{T}$.

(a) Consider the following set of vectors: $\mathbb{S}+\mathbb{T} := \{\vec{s}+\vec{t} \mid \vec{s} \in \mathbb{S}, \vec{t} \in \mathbb{T}\}$ Show that $\mathbb{S}+\mathbb{T}$ is a subspace of $\mathbb{V}$.

Solution: Since $\vec{0} \in \mathbb{S}$ and $\vec{0} \in \mathbb{T}$, then $\vec{0} = \vec{0}+\vec{0} \in \mathbb{S}+\mathbb{T}$.

Let $\vec{v} = \vec{s}_1 + \vec{t}_1$ and $\vec{w} = \vec{s}_2 + \vec{t}_2$ be in $\mathbb{S}+\mathbb{T}$. $\vec{v}+\vec{w} = (\vec{s}_1+\vec{s}_2)+(\vec{t}_1+\vec{t}_2)$. Since $\vec{s}_1+\vec{s}_2$ is a sum of vectors in $\mathbb{S}$, $\vec{s}_1+\vec{s}_2 \in \mathbb{S}$. Likewise, $\vec{t}_1+\vec{t}_2 \in \mathbb{T}$. Since $\vec{v}+\vec{w}$ is a sum of vectors in $\mathbb{S}$ and $\mathbb{T}$, $\vec{v}+\vec{w} \in \mathbb{S}+\mathbb{T}$, so $\mathbb{S}+\mathbb{T}$ is closed under vector addition. Consider a scalar, α. $\alpha\vec{v} = \alpha\vec{s}_1 + \alpha\vec{t}_1$. Since $\alpha\vec{s}_1 \in \mathbb{S}$ and $\alpha\vec{t}_1 \in \mathbb{T}$, $\alpha\vec{v} \in \mathbb{S}+\mathbb{T}$. $\mathbb{S}+\mathbb{T}$ is closed under scalar multiplication. Thus, $\mathbb{S}+\mathbb{T}$ is a subspace of $\mathbb{V}$.

(b) Consider the intersection of $\mathbb{S}$ and $\mathbb{T}$: $\mathbb{S}\cap\mathbb{T} = \{\vec{v} \in \mathbb{V} \mid \vec{v} \in \mathbb{S}, \vec{v} \in \mathbb{T}\}$. Show that this is a subspace.

Solution: Both subspaces $\mathbb{S}$ and $\mathbb{T}$ must contain $\vec{0}$ as they are subspaces. So $\mathbb{S}\cap\mathbb{T}$ also contains $\vec{0}$.

Let $\vec{v}, \vec{w} \in \mathbb{S}\cap\mathbb{T}$. So $\vec{v} \in \mathbb{S}$ and $\vec{w} \in \mathbb{S}$. Since $\mathbb{S}$ is a subspace, $\vec{v}+\vec{w} \in \mathbb{S}$. Likewise, we can say that $\vec{v}+\vec{w} \in \mathbb{T}$. Since $\vec{v}+\vec{w}$ is in both $\mathbb{S}$ and $\mathbb{T}$, $\mathbb{S}\cap\mathbb{T}$ is closed under vector addition.

Consider a scalar α. $\vec{v} \in \mathbb{S}$ means that $\alpha\vec{v} \in \mathbb{S}$ as $\mathbb{S}$ is closed under scalar multiplication. It is also the case that $\vec{v} \in \mathbb{T}$ and therefore $\alpha\vec{v} \in \mathbb{T}$. Since $\alpha\vec{v}$ is in both, $\alpha\vec{v} \in \mathbb{S}\cap\mathbb{T}$.$\mathbb{S}\cap\mathbb{T}$ is closed under scalar multiplication.

So $\mathbb{S}\cap\mathbb{T}$ is a subspace.

(c) As before, $\mathbb{S}$ and $\mathbb{T}$ are subspaces of the vector space $\mathbb{V}$. Now, assume $\mathbb{S} \cap \mathbb{T} = \{\vec{0}\}$. Let $B_S = \{\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_k\}$ be a basis for $\mathbb{S}$, and $B_T = \{\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_n\}$ be a basis for $\mathbb{T}$.

Show that $B_S \cup B_T = \{\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_k, \vec{t}_1, \vec{t}_2, \ldots, \vec{t}_n\}$ is a basis for $\mathbb{S} + \mathbb{T}$.

Solution: Since B_S and B_T are bases for $\mathbb{S}$ and $\mathbb{T}$, vectors $\vec{s} \in \mathbb{S}$ can be written as the linear combination $\vec{s} = \sum_{i=1}^{k} \alpha_i \vec{s}_i$ and vectors $\vec{t} \in \mathbb{T}$ can be written as the linear combination $\vec{t} = \sum_{j=1}^{n} \beta_j \vec{t}_j$. This means that vectors $\vec{v} \in \mathbb{S} + \mathbb{T}$ can be written as $\vec{s} + \vec{t} = \sum_{i=1}^{n} \alpha_i \vec{s}_i + \sum_{j=1}^{n} \beta_j \vec{t}_j$, so $\mathbb{S} + \mathbb{T}$ is spanned by $B_S \cup B_T$. Now it should suffice to show that the set of vectors $B_S \cup B_T$ is linearly independent. Since the intersection only contains $\vec{0}$, for any $\vec{s} \neq \vec{0}$ and $\vec{t} \neq \vec{0}$, $\vec{s} \neq \vec{t}$ and $\vec{s} - \vec{t} \neq \vec{0}$. If this is the case, $\sum_{i=1}^{n} \alpha_i \vec{s}_i - \sum_{j=1}^{n} \beta_j \vec{t}_j \neq \vec{0}$ for any non-zero choices of α_i and β_j. This means the vectors in $B_S \cup B_T$ are linearly independent, thus we have a basis.

(d) If $\mathbb{S} \cap \mathbb{T}$ is $\{\vec{0}\}$, what is the dimension of $\mathbb{S} + \mathbb{T}$ in terms of the dimensions of $\mathbb{S}$ and $\mathbb{T}$?

Solution: In the last part, we showed that the union of the bases of $\mathbb{S}$ and $\mathbb{T}$ is a basis for $\mathbb{S} + \mathbb{T}$ if $\mathbb{S} \cap \mathbb{T} = \{\vec{0}\}$. Since none of $\vec{s}_i$ are in $\mathbb{T}$ and none of the $\vec{t}_i$ are in $\mathbb{S}$, the basis for $\mathbb{S} + \mathbb{T}$ will contain all basis vectors. However, each basis contains $\dim(\mathbb{S})$ and $\dim(\mathbb{T})$ vectors respectively. So in the basis for $\mathbb{S} + \mathbb{T}$, we have $\dim(\mathbb{S}) + \dim(\mathbb{T})$ vectors. By definition the dimesion is the count of basis vectors $\dim(\mathbb{S} + \mathbb{T}) = \dim(\mathbb{S}) + \dim(\mathbb{T})$.

(e) (**Practice**) For arbitrary subspaces $\mathbb{S}$ and $\mathbb{T}$ of vector space $\mathbb{V}$, show that $\dim(\mathbb{S} + \mathbb{T}) = \dim(\mathbb{S}) + \dim(\mathbb{T}) - \dim(\mathbb{S} \cap \mathbb{T})$.

Solution: Consider first a basis $B_\cap$ for $\mathbb{S} \cap \mathbb{T}$. This basis may not span $\mathbb{S}$ or $\mathbb{T}$. Extend $B_\cap$ to a basis for $\mathbb{S}$, B_S, by adding linearly independent vectors from $\mathbb{S}$. Also create a basis for $\mathbb{T}$, B_T, by adding linearly independent vectors from $\mathbb{T}$ to $B_\cap$. If we combine both B_S and B_T, we will have enough vectors to span both $\mathbb{S}$ and $\mathbb{T}$ and therefore $\mathbb{S} + \mathbb{T}$. However, we must not double count the vectors in B_S and B_T that came from $B_\cap$. So the total count of basis vectors is $\dim(\mathbb{S})$ for the vectors in B_S, $\dim(\mathbb{T})$ for the vectors in B_T, with a count of the vectors in $B_\cap$ removed (subtract $\dim(\mathbb{S} \cap \mathbb{T})$).

$\dim(\mathbb{S} + \mathbb{T}) = \dim(\mathbb{S}) + \dim(\mathbb{T}) - \dim(\mathbb{S} \cap \mathbb{T})$.

5. Image Compression

In this question, we explore how eigenvalues and eigenvectors can be used for image compression. A grayscale image can be represented as a data grid. Say a symmetric, square image is represented by a symmetric matrix $\mathbf{A}$, such that $\mathbf{A}^T = \mathbf{A}$. We can transform the images to vectors to make it easier to process them as data, but here, we will understand them as 2D data. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$ with corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. Also, let these eigenvectors be normalized (unit norm). Then, the matrix can be represented as the expansion

$$\mathbf{A} = \lambda_1 \vec{v}_1 \vec{v}_1^T + \lambda_2 \vec{v}_2 \vec{v}_2^T + \cdots + \lambda_n \vec{v}_n \vec{v}_n^T.$$

This is known as the *spectral decomposition* of $\mathbf{A}$. Note that the eigenvectors must be normalized for this expansion to be valid because we know that if $\vec{v}_i$ is an eigenvector, then any scalar multiple $\alpha \vec{v}_i$ is also an eigenvector. If we scaled every eigenvector on the right hand side of the equation by α, then the left hand side would change from $\mathbf{A}$ to $\alpha^2 \mathbf{A}$.

The previous expansion shows that the matrix $\mathbf{A}$ can be synthesized by its n eigenvalues and eigenvectors. However, $\mathbf{A}$ can also be *approximated* with the k largest eigenvalues and the corresponding eigenvectors. That is,

$$\mathbf{A} \approx \lambda_1 \vec{v}_1 \vec{v}_1^T + \lambda_2 \vec{v}_2 \vec{v}_2^T + \cdots + \lambda_k \vec{v}_k \vec{v}_k^T.$$

(a) Use the IPython notebook `prob4.ipynb` and the image file `pattern.npy`. Run the associated code block, which computes the eigenvalues and eigenvectors of the image `pattern.npy` and then sorts them in descending order. Note that `numpy.linalg.eig` returns normalized eigenvectors by default. Mathematically, how many eigenvectors are required to fully capture the information within the image?

Solution:

The shape function told us that the image is a 400×400 matrix. We can therefore expect that we can have as many as 400 eigenvalues, and indeed looking at the `eig_vals` variable in the notebook tells us that all 400 of them are non-zero. This tells us that to fully understand the action of the matrix, we would need to know about all 400 of the eigenspaces. Therefore, we would need 400 eigenvectors to fully understand them.

(b) In the IPython notebook, find an approximation for the image using the 100 largest eigenvalues and eigenvectors. Does the approximate image capture most of the features of the original image?

Solution:

See `sol4.ipynb`.

(c) Repeat part (b) with $k = 50$. By further experimenting with the code, what seems to be the lowest value of k that retains most of the salient features of the given image?

Solution:

See `sol4.ipynb`.

The question of the lowest value of k is a bit subjective, and it is fine whatever answer you gave for it. The image starts looking qualitatively different somewhere around 15 eigenvectors. Certainly below 7, it looks very different. The "resolution" seems to be dropping as we keep fewer and fewer eigenvalues and eigenvectors.

Look at the plot of the eigenvalues included in the solutions notebook. You will see how they fast they become small.

This effect of a reduction in quality as we save less information is something that all of you have experienced while using things like JPEG compression. We hope that seeing this example gives you some idea of why it could be possible to do such "lossy compression" in real-world applications. Later in the 16AB sequence, we will be learning more about why this works.

Along with lossless compression and error-correcting codes, lossy compression is one of the major advances that makes the modern age of multimedia possible. So, give a silent shoutout to eigenconcepts next time you watch a video online.

6. Traffic Flows

Learning Objective: *The learning objective of this problem is to see how the concept of nullspaces can be applied to flow problems.*

Your goal is to measure the flow rates of vehicles along roads in a town. It is prohibitively (too) expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this "flow conservation" to determine the traffic along all roads in a network by measuring the flow along only some roads. In this problem, we will explore this concept.

(a) Let's begin with a network with three intersections, A, B and C. Define the flow t_1 as the rate of cars (cars/hour) on the road between B and A, flow t_2 as the rate on the road between C and B, and flow t_3 as the rate on the road between C and A.

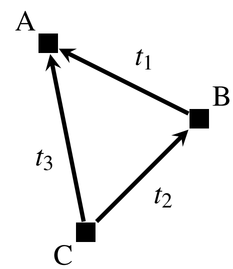


Figure 1: A simple road network.

(Note: The directions of the arrows in the figure are the way that we define positive flow by convention. For example, if there were 100 cars per hour traveling from A to C, then $t_3 = -100$. The flows now are not fractions of water in reservoirs as in the pumps setting, but numbers of cars.)

We assume the "flow conservation" constraints: the net number of cars per hour flowing into each intersection is zero. For example at intersection B, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of t_2 and t_3). If we can, find the values of t_2 and t_3.

Solution:

Yes, since we know that $t_1 = t_2 = -t_3$, so we must have $t_2 = 10$ and $t_3 = -10$.

(b) Now suppose we have a larger network, as shown in Figure 2.

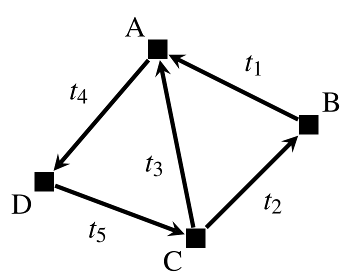


Figure 2: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads CA (measuring t_3) and DC (measuring t_5). A Stanford student claims that we need two sensors placed on the roads CB (measuring t_2) and BA (measuring t_1). Write out the system of linear equations that represents this flow graph. Is it possible to determine all traffic flows, $\left[t_1, t_2, t_3, t_4, t_5\right]^T$, with the Berkeley student's suggestion? How about the Stanford student's suggestion?

Solution: Since we have 4 intersections, we can write 4 linear equations describing the flows into and out of each intersection. We know that the flows into and out of an intersection must sum to 0. The set of linear equations that represents this flow graph is:

$$\begin{cases} t_1 + t_3 - t_4 = 0 \\ t_2 - t_1 = 0 \\ t_5 - t_2 - t_3 = 0 \\ t_4 - t_5 = 0 \end{cases}$$

The Stanford student is wrong (obviously). Observing t_1 and t_2 is not sufficient, as t_3, t_4 and t_5 can still not be uniquely determined. Specifically, for any $\alpha \in \mathbb{R}$, the following flow satisfies the constraints and the measurements:

$$t_4 = \alpha$$
$$t_5 = \alpha$$
$$t_3 = \alpha - t_1$$

On the other hand, if we're given t_3 and t_5, we can uniquely solve for all the traffic densities as follows since we know the flow conservation constraints. From the set of linear equations we obtain:

$$t_1 = t_5 - t_3$$
$$t_2 = t_5 - t_3$$
$$t_4 = t_5$$

This is related to the fact that t_3 and t_5 are parts of different loops in the graph, whereas t_1 and t_2 are in the same loop, so measuring both of them would not give additional information.

(c) We would like a more general way of determining the possible traffic flows in a network. Suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow conservation constraints (one per intersection) as a matrix equation.
Construct a 4×5 matrix $\mathbf{B}$ such that the equation $\mathbf{B}\vec{t} = \vec{0}$:

$$\begin{bmatrix} & & \mathbf{B} & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

represents the flow conservation constraints for the network in Figure 2.
Hint: Each row is the constraint of an intersection. You can construct $\mathbf{B}$ using only $0, 1$, and -1 entries. This matrix is called the **incidence matrix**. What constraint does each column of $\mathbf{B}$ represent?
Solution:

$$\mathbf{B} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}\begin{array}{c} A \\ B \\ C \\ D \end{array}$$
$$\begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \end{array}$$

(The rows of this matrix can be in any order and your solution can differ by a factor of -1. However, the order of the elements within the row is still important and it must match the order of the elements of $\vec{t}$). Each row represents an intersection, and each column represents a road between two intersections. Each 1 on a row represents a road flowing into an intersection, and each -1 represents a road flowing out of an intersection. Each -1 in a column represents the source intersection of a road (where the arrow starts), and each 1 in a column represents the destination intersection of a road (where the arrow ends).

Each column of $\mathbf{B}$ must sum to 0. We expect each column to sum to 0 (and actually have exactly one -1 and one 1).

(d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace of all valid traffic flows for the network of Figure 2. Notice that the set of all vectors $\vec{t}$ that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the null space of the matrix $\mathbf{B}$. That is, we can find all valid traffic flows by computing the null space of $\mathbf{B}$. What is the dimension of the nullspace?

Solution:

We use Gaussian Elimination to find the nullspace.

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \underbrace{\Rightarrow}_{R_2+R_1\to R_2} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \underbrace{\Rightarrow}_{R_2+R_3\to R_3} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\underbrace{\Rightarrow}_{R_4+R_5\to R_5} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underbrace{\Rightarrow}_{(-1)R_3\to R_3} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underbrace{\Rightarrow}_{R_2+R_3\to R_2} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underbrace{\Rightarrow}_{R_1+R_3\to R_1} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that we should let $t_3 = \alpha$ and $t_5 = \beta$, where α and β are free variables. The equations are:

$$\begin{aligned} t_1 &= \beta - \alpha \\ t_2 &= \beta - \alpha \\ t_3 &= \alpha \\ t_4 &= \beta \\ t_5 &= \beta \end{aligned}$$

The dimension of the nullspace is 2 because a minimum of 2 vectors are required to span the entire nullspace.

Note: We show here, for your reference, that the space of all possible traffic flows is a subspace. You don't not need to include this proof in your solution. Suppose we have a set of valid flows $\vec{t}$. Then, for any intersection, the total flow into it is the same as the total flow out of it. If we scale $\vec{t}$ by a constant a, each t_i will also get scaled by a. The total flows into and out of the intersection would be scaled by the same amount and remain equal to each other. Thus any scaling of a valid flow is still a valid flow. Suppose now we add valid flows $\vec{f_1}$ and $\vec{f_2}$ to get $\vec{t} = \vec{f_1} + \vec{f_2}$. For any intersection I,

$$\text{total flow into } I = \text{total flow into } I \text{ from } \vec{f_1} + \text{total flow into } I \text{ from } \vec{f_2}$$
$$\text{total flow out of } I = \text{total flow out of } I \text{ from } \vec{f_1} + \text{total flow out of } I \text{ from } \vec{f_2}$$

Since the total flow into I from $\vec{f_1}$ is the same as the total flow out of I from $\vec{f_1}$ and similarly for $\vec{f_2}$, the total flow into I is the same as the total flow out of I. Therefore, the sum of any two valid flows is still a valid flow. Also, $\vec{t} = \vec{0}$ is a valid flow. Therefore the set of valid flows forms a subspace.

(e) Notice that we can represent the Berkeley student's measurement as $\mathbf{M}_B\vec{t}$, where:

$$\mathbf{M}_B\vec{t} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \vec{t} = \begin{bmatrix} t_3 \\ t_5 \end{bmatrix}$$

Write a matrix $\mathbf{M}_S$ that can be used to represent the Stanford student's measurement.

Solution:

$$\mathbf{M}_S\vec{t} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \vec{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

(f) Now let us analyze more general road networks. Say there is a road network graph G, with incidence matrix $\mathbf{B}_G$. If $\mathbf{B}_G$ has a k-dimensional null space, does this mean measuring the flows along ***any*** k **roads** is always sufficient to recover all of the true flows? Prove or give a counterexample.

Hint: Consider the Stanford student from part (b).

Solution:

No, consider the network of Figure 2. The corresponding incidence matrix has a $k = 2$ dimensional null space, as you showed in part (e). However, measuring t_1 and t_2 (as the Stanford student suggested) is not sufficient, as you showed in part (b).

(g) (**Practice**) Assume that $\vec{u}$ and $\vec{t}$ are distinct valid flows, that is $\mathbf{B}_G\vec{u} = \mathbf{B}_G\vec{t} = \vec{0}$. Can you recover all of the network's true flows if $(\vec{u} - \vec{t})$ belongs to the nullspace of $\mathbf{M}_S$?

Clarification: A âĂIJvalidâĂİ flow is one that is possible without violating the constraints on the nodes (so flow in must equal to flow out). There may be many valid flows, but only one âĂIJtrueâĂİ flow.

Solution: No. If $(\vec{u}-\vec{t})$ is in the nullspace of $\mathbf{M}_S$, it means $\mathbf{M}_S(\vec{u}-\vec{t}) = \vec{0}$. In other words, $\mathbf{M}_S\vec{u} = \mathbf{M}_S\vec{t}$. This means that two different flows will give us the same measurement, so the true flow cannot be recovered.

(h) (**Challenge: Practice**) If the incidence matrix $\mathbf{B}_G$ has a k-dimensional null space, does this mean we can **always pick a set of** k **roads** such that measuring the flows along these roads is sufficient to recover the exact flows? Prove or give a counterexample.

Solution:

Yes.

Let $\mathbf{U}$ be a matrix whose columns form a basis of the null space of $\mathbf{B}_G$, as above. The k columns of $\mathbf{U}$ are linearly independent since they form a basis. Since there are k linearly independent columns, when we run Gaussian elimination on $\mathbf{U}$, we must get k pivots. (Recall that "pivot" is the technical term for being able to row-reduce and turn a column into something that has exactly one 1 in it. The pivot is the entry that we found and turned into that 1.)

Therefore, the row space of $\mathbf{U}$ is k dimensional since there are some k linearly independent rows in $\mathbf{U}$ — namely the ones where we found pivots. Choose to measure the roads corresponding to these rows.

This will work because:

For a given valid flow $\vec{t} = \mathbf{U}\vec{x}$, the results of measuring this flow vector are $\mathbf{U}^{(k)}\vec{x}$, where the matrix $\mathbf{U}^{(k)}$ is some k linearly independent rows of $\mathbf{U}$. By construction, the $k \times k$ matrix $\mathbf{U}^{(k)}$ has all linearly independent rows, so we can invert $\mathbf{U}^{(k)}$ to find $\vec{x}$ from $\mathbf{U}^{(k)}\vec{x}$ and then recover the flows along all the edges as $\mathbf{U}\vec{x}$.

This isn't the only set of k roads that will work. But it does provide a set of k roads that are guaranteed to work.

7. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.

EECS16A: Homework 4

Problem 5: Image Compression

In [1]:
```
%pylab inline
```

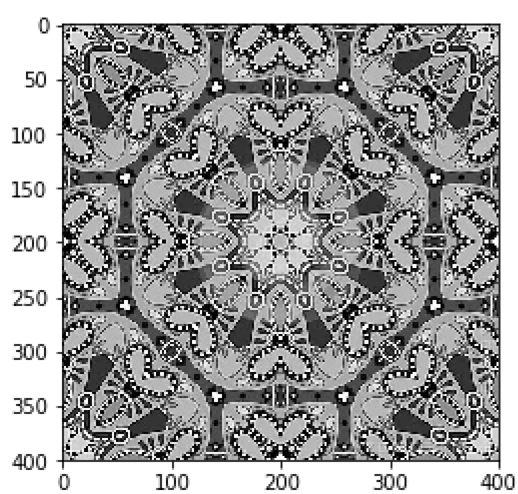
```
Populating the interactive namespace from numpy and matplotlib
```

In [2]:
```
import numpy as np
from scipy import ndimage as nd
from scipy import misc
from scipy import io
```

Part a

In [3]:
```
#Load Pattern Image
pattern = np.load('pattern.npy')
plt.imshow(pattern, cmap='gray', interpolation='nearest')
```

Out[3]: `<matplotlib.image.AxesImage at 0x7f85113ab438>`



Use the command [shape (http://docs.scipy.org/doc/numpy/reference/generated/numpy.ndarray.shape.html)](http://docs.scipy.org/doc/numpy/reference/generated/numpy.ndarray.shape.html) to find the dimensions of the image. How many eigenvalues do you expect?

Run the code below to find the eigenvector and eigenvalues of `pattern` and sort them in descending order (first eigenvalue/vector corresponds to the largest eigenvalue)

In [4]:
```
eig_vals, eig_vectors = np.linalg.eig(pattern)
idx = (abs(eig_vals).argsort())
idx = idx[::-1]
eig_vals = eig_vals[idx]
eig_vectors = eig_vectors[:,idx]
```

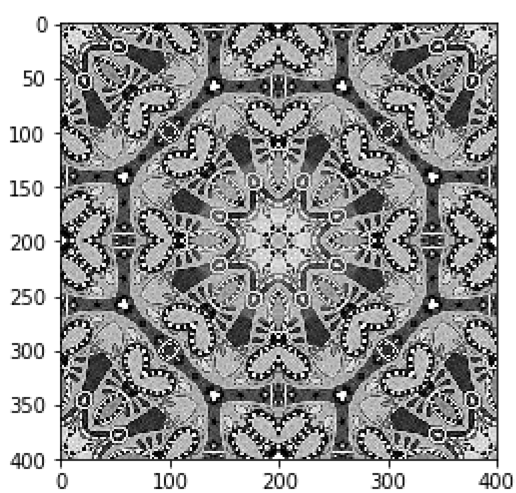
Part b

Find the pattern approximation using 100 largest eigenvalues/eigenvectors.

- Index into above variables to choose the first 100 eigenvalues and eigenvectors.
- You can use the command np.outer (http://docs.scipy.org/doc/numpy/reference/generated/numpy.outer.html) to find the outer product of two vectors

In [5]:
```
rank = 100
S = np.zeros(pattern.shape)
for i in range(rank):
    vec_i = eig_vectors[:,i]  # i-th largest eigenvector
    val_i = eig_vals[i]       # i-th largest eigenvalue
    S += val_i * np.outer(vec_i, vec_i)
plt.imshow(S, cmap='gray', vmin=0, vmax=255)
```

Out[5]: <matplotlib.image.AxesImage at 0x7f8510337cc0>



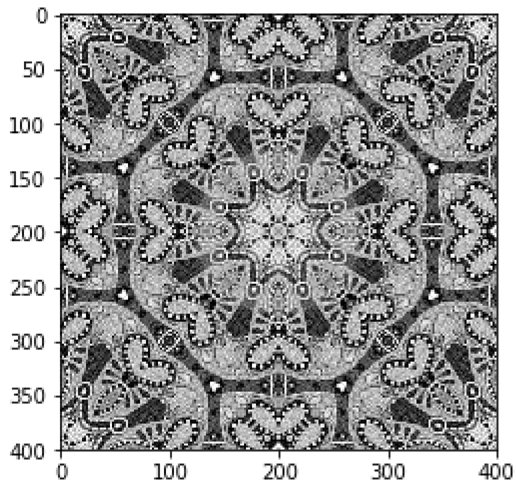
Part c

Find the pattern approximation using 50 largest eigenvalues/eigenvectors

In [6]:

```python
rank = 50
S = np.zeros(pattern.shape)
for i in range(rank):
    vec_i = eig_vectors[:,i]  # i-th largest eigenvector
    val_i = eig_vals[i]       # i-th largest eigenvalue
    S += val_i * np.outer(vec_i, vec_i)
plt.imshow(S, cmap='gray', vmin=0, vmax=255)
```

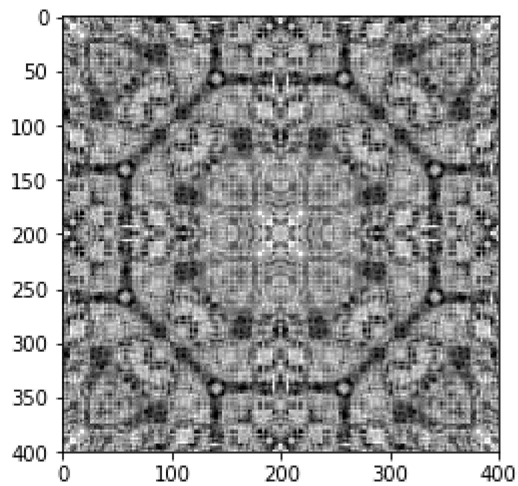
Out[6]: <matplotlib.image.AxesImage at 0x7f85102a2278>



Now try decreasing the amount of eigenvalues/eigenvectors used in the pattern approximation. At what point does the image, start to substantially look different?

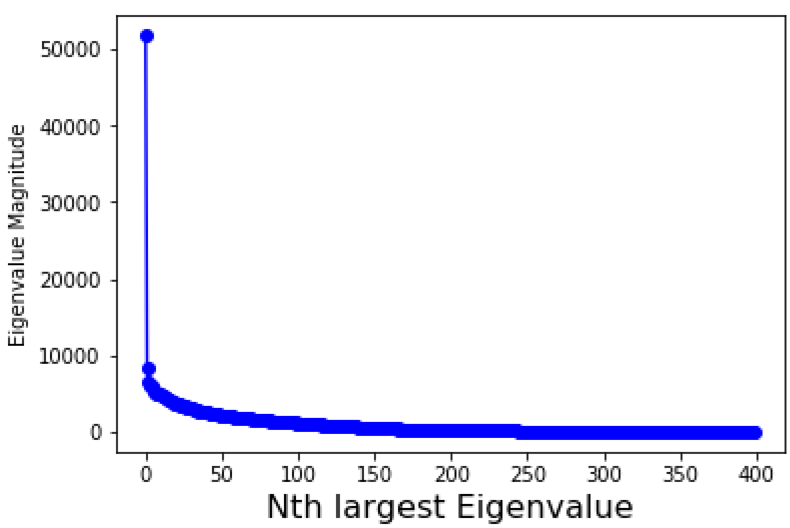
In [7]:
```python
rank = 10 ## try different numbers
S = np.zeros(pattern.shape)
for i in range(rank):
    vec_i = eig_vectors[:,i]  # i-th largest eigenvector
    val_i = eig_vals[i]       # i-th largest eigenvalue
    S += val_i * np.outer(vec_i, vec_i)
plt.imshow(S, cmap='gray', vmin=0, vmax=255)
```

Out[7]: <matplotlib.image.AxesImage at 0x7f8510201358>



Extra: Let's plot the magnitudes of the eigenvalues. As shown below, the magnitude of the eigenvalues drop off extremely rapidly, which is why you can recover most of the image with just a few eigenvalues.

In [8]:
```python
plt.plot(abs(eig_vals), '-o', color = 'blue')
plt.xlabel("Nth largest Eigenvalue", fontsize = 16)
plt.ylabel("Eigenvalue Magnitude")
plt.show()
```



A 10x10 grid is also plotted below showing how the image evolves as we include more eigenvalues/eigenvectors (up to a 100 eigenvalues/eigenvectors). As seen in the grid, by approximately $k = 15$, most of the features seem to be present.

In [9]:

```python
from mpl_toolkits.axes_grid1 import ImageGrid

fig = plt.figure(figsize=(16, 16))
grid = ImageGrid(fig, 111,
                 nrows_ncols=(10, 10),
                 axes_pad=0.1,
                 )
S_list = []
for k in range(1, 101):
    S = np.zeros(pattern.shape)
    for i in range(k):
        vec_i = eig_vectors[:,i]  # i-th largest eigenvector
        val_i = eig_vals[i]       # i-th largest eigenvalue
        S += val_i * np.outer(vec_i, vec_i)
    S_list.append(S)

for ax, im in zip(grid, S_list):
    ax.imshow(im, cmap = 'gray', vmin = 0, vmax = 255)

plt.show()
```