EECS 16A Designing Information Devices and Systems I
Spring 2020 Homework 9

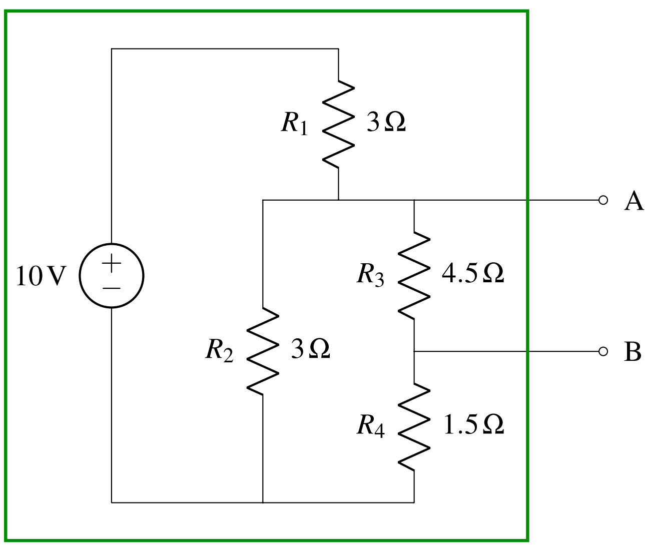
This homework is due April 3, 2020, at 23:59.
Self-grades are due April 6, 2020, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw9.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).
- Practice problems will not be graded. However, they do provide more practice and we encourage you to try them to practice for the midterm.
- We encourage you to turn in self-grades for this HW before the midterm, since looking at the solutionsearlier will help you study for the midterm.

Submit the file to the appropriate assignment on Gradescope.

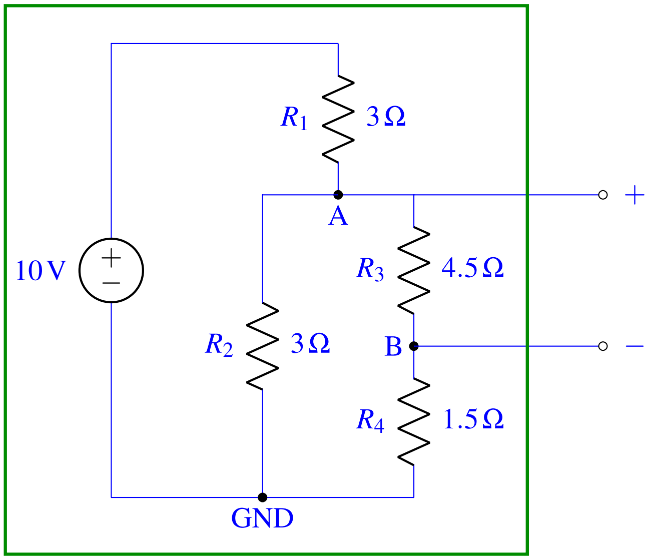
1. **Thévenin and Norton Equivalent Circuits** Find the Thévenin and Norton equivalent circuits seen from terminals A and B.



Solution:

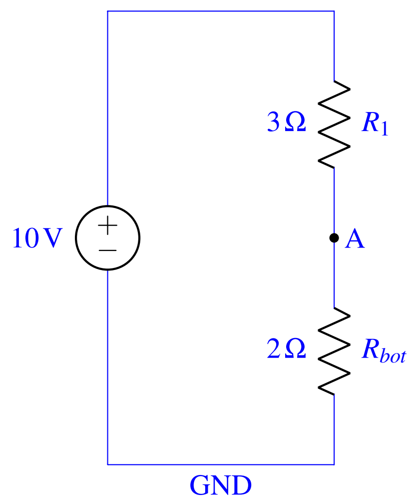
To find the Thévenin and Norton equivalent circuits, we are going to find (1) the open circuit voltage between the output ports and (2) the current flowing through the output ports when the ports are shorted.

For finding the open circuit voltage between the output ports, let us label the nodes as shown in the figure below.



First, let us begin by calculating the effective resistance between nodes A and GND looking down from A. We have the $3\,\Omega$ resistor in parallel to the $4.5\,\Omega + 1.5\,\Omega$ resistance. This allows us to express the behavior of R_2, R_3, and R_4 together as an equivalent resistance of

$$R_{bot} = \frac{1}{\frac{1}{3\,\Omega} + \frac{1}{4.5\,\Omega+1.5\,\Omega}} = 2\,\Omega.$$



Notice how in this reduced diagram, node B isn't there!

Then we see that we have a voltage divider from the positive terminal of the $10\,\mathrm{V}$ supply. The voltage divider is made up of two resistances in series, where the resistances are $3\,\Omega$ and $2\,\Omega$. This gives the voltage at node A equal to

$$V_{\mathrm{A}} = 10\,\mathrm{V}\frac{2\,\Omega}{3\,\Omega + 2\,\Omega} = 4\,\mathrm{V}$$

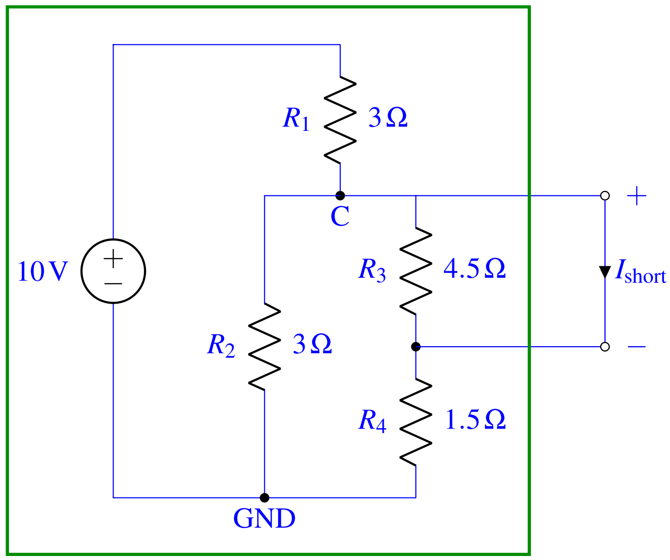
To find V_B, we have to look back at the *original* diagram. To find the voltage at node B, note that we have another voltage divider between nodes A and GND. Hence, we can find the voltage at node B as

$$V_{\mathrm{B}} = V_{\mathrm{A}}\frac{1.5\,\Omega}{4.5\,\Omega + 1.5\,\Omega} = 4\,\mathrm{V} \cdot \frac{1}{4} = 1\,\mathrm{V}$$

Hence the open circuit voltage seen between the output ports is equal to

$$\begin{aligned} V_{\text{open}} &= V_A - V_B \\ &= 4\,\text{V} - 1\,\text{V} \\ &= 3\,\text{V} \end{aligned}$$

Now let us find the short circuit current flowing through the output ports. When doing this, we get the following circuit.



Note that when we short the output terminals, the voltages at the nodes change, this is why we changed the label of the node below the resistor R_1. Since there is a short circuit parallel to the resistor R_3, there will be no current flowing through it, hence we have

$$I_{\text{short}} = I_{R_4}$$

To find this current, let us find the equivalent resistance due to R_2 being connected in parallel to R_4 when we short the output ports. We have $3\,\Omega$ parellel to $1.5\,\Omega$, which gives an equivalent resistance

$$R_{bot} = \frac{1}{\frac{1}{3\,\Omega} + \frac{1}{1.5\,\Omega}} = 1\,\Omega$$

We again have a voltage divider between the positive side of the 10 V supply and the ground. Using this voltage divider, we calculate the voltage at node C as

$$V_C = 10\,\text{V}\frac{1\,\Omega}{3\,\Omega + 1\,\Omega} = 2.5\,\text{V}$$

Hence, we see that the voltage across the resistor R_4 is equal to 2.5 V. Using Ohm's law, we get

$$I_{R_4} = \frac{2.5\,\text{V}}{1.5\,\Omega} = \frac{5}{3}\,\text{A}$$

Since we have $I_{\text{short}} = I_{R_4}$, we have

$$I_{\text{short}} = I_{R_4}$$

Table 1: Touchscreen Dimension Values

d_1	10 mm
d_2	1 mm
t_1	1 mm
t_2	2 mm
w_1	1 mm
w_2	2 mm

Summarizing the results, we have

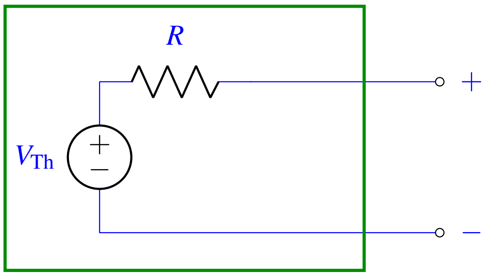
$$V_{\text{short}} = 3\,\text{V}$$

$$I_{\text{short}} = \frac{5}{3}\,\text{A}$$

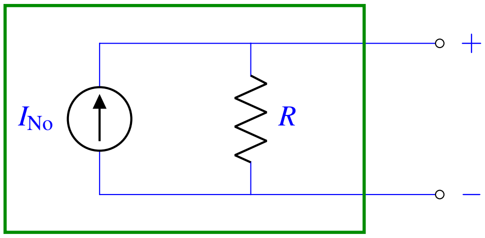
This gives

$$R_{\text{Th}} = \frac{V_{\text{open}}}{I_{\text{short}}} = \frac{9}{5}\,\Omega$$

Hence the Thévenin equivalent circuit is given by



where $R = R_{\text{Th}}$ and $V_{\text{Th}} = V_{\text{open}}$, and the Norton equivalent circuit is given by



where $R = R_{\text{Th}}$ and $I_{\text{No}} = I_{\text{short}}$.

2. Capacitive Touchscreen

The model for a capacitive touchscreen can be seen in Figure 1. See Table 1 for values of the dimensions. The green area represents the contact area of the finger with the top insulator. It has dimensions w_2 x d_1.

(a) Draw the equivalent circuit of the touchscreen that contains the nodes F, E_1, and E_2 when there no finger present and when there is a finger present. Express the capacitance values in terms of C_0, C_{F-E1}, and C_{F-E2}.

Solution:

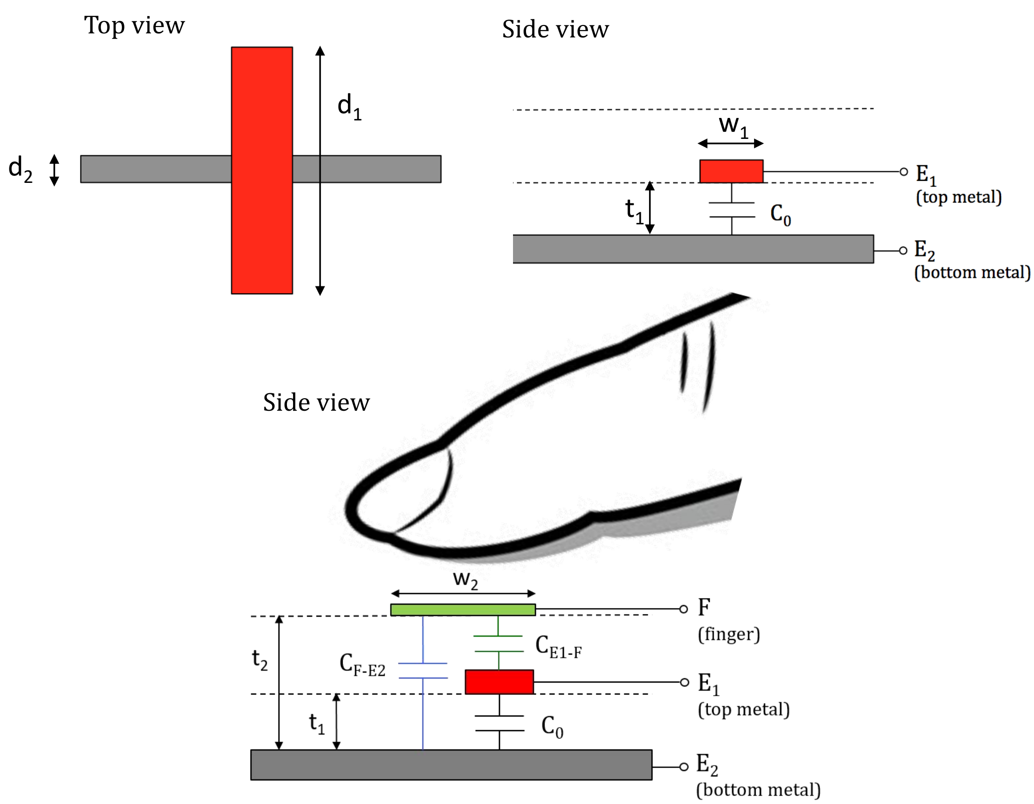


Figure 1: Model of capacitive touchscreen.

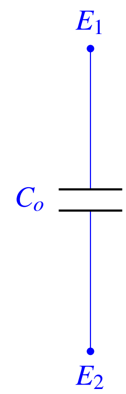


Figure 2: Touchscreen circuit with no finger present.

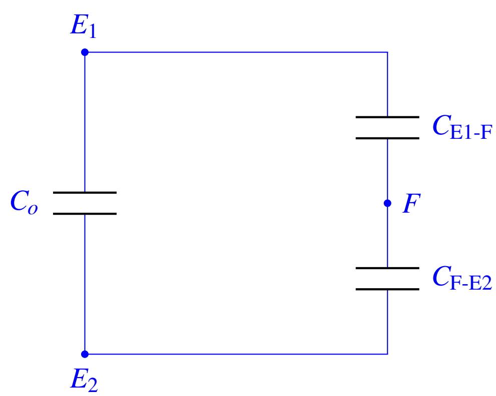


Figure 3: Touchscreen circuit with a finger present.

(b) What are the values of C_0, C_{F-E1}, and C_{F-E2}? Assume that the insulating material has a permittivity of $\varepsilon = 4.43 * 10^{-11} F/m$ and that the thickness of the metal layers is small compared to t_1.

Solution:

$$C_0 = \varepsilon \frac{d_2 w_1}{t_1} = 4.43 * 10^{-14} F$$

$$C_{F-E1} = \varepsilon \frac{d_1 w_1}{t_2 - t_1} = 4.43 * 10^{-13} F$$

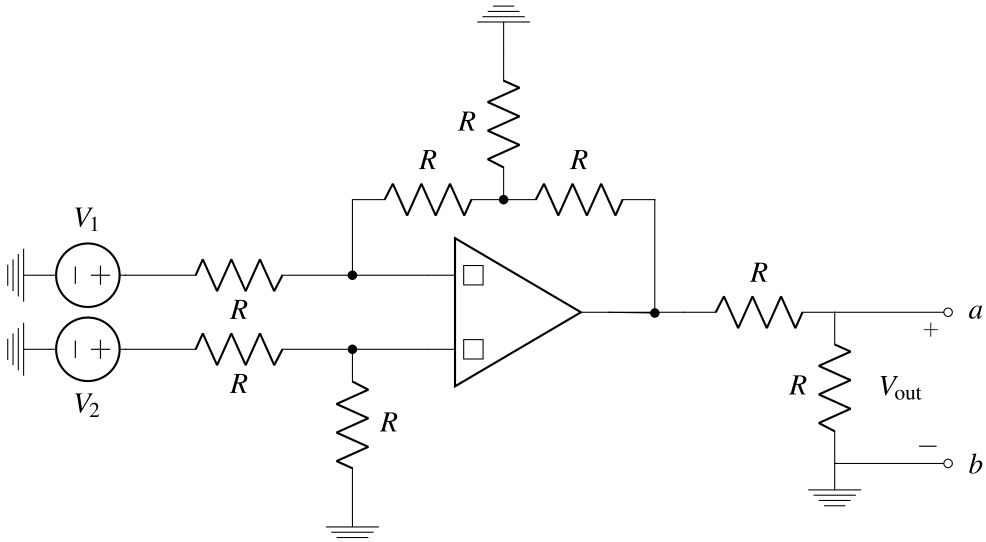
$$C_{F-E2} = \varepsilon \frac{d_2 (w_2 - w_1)}{t_2} = 2.22 * 10^{-14} F$$

(c) What is the difference in effective capacitance between the two metal plates (nodes E_1 and E_2) when a finger is present?

Solution: The effective capacitance between the two plates is $C_0 = 4.43 * 10^{-13} F$ when there is no finger. When there is a finger, we have C_0 in parallel with a series combination of C_{F-E1} and C_{F-E2}. Therefore there is an additional capacitance $C_{F-E1} || C_{F-E2} = 2.11 * 10^{-14} F$ when a finger is present.

3. Op Amp Nodal Analysis

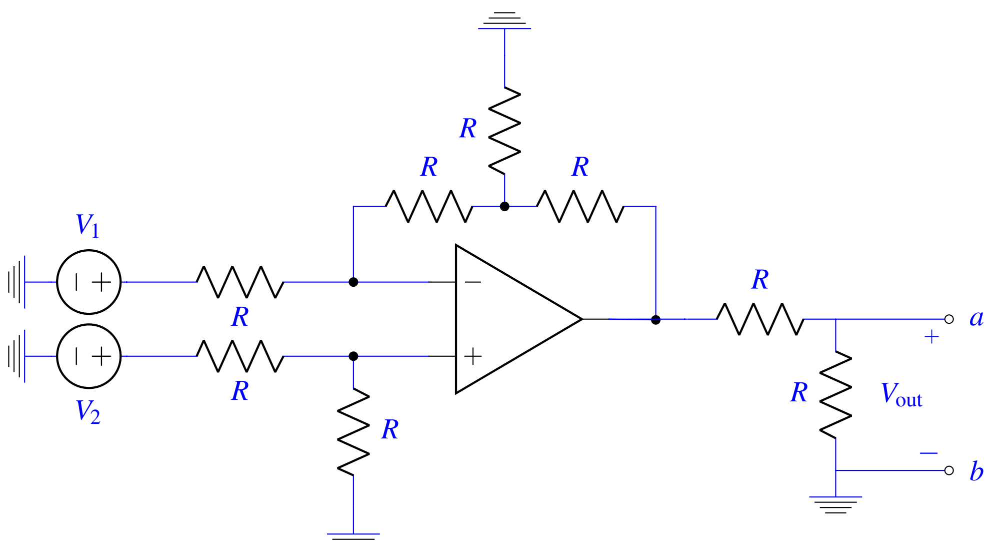
Consider this Op Amp circuit below:



We are interested in analyzing its input-output relationship.

(a) Redraw the circuit with a choice of $+$ and $-$ terminal labelings to guarantee that the circuit is in negative feedback.

Solution: To be in negative feedback, the top terminal must be labeled with a minus. So we have:

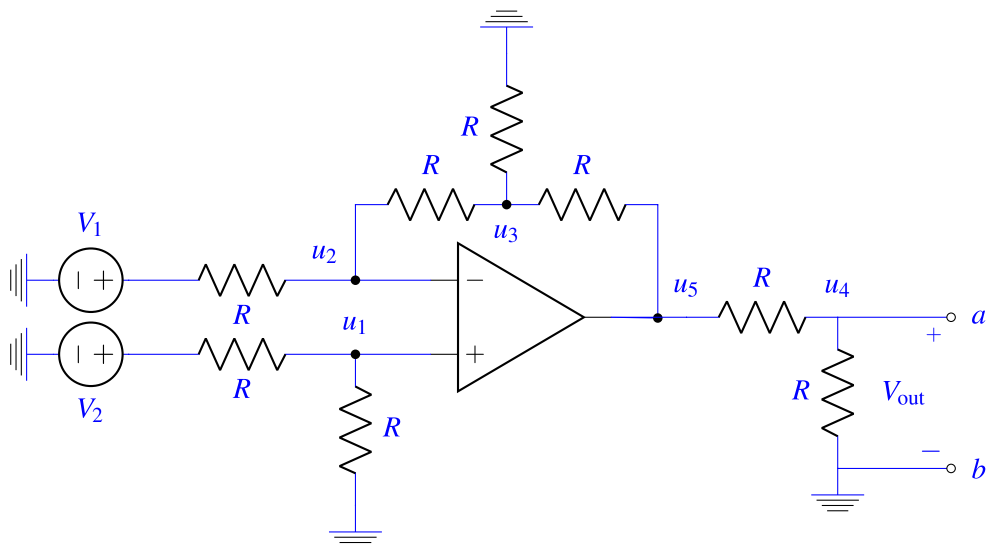


Our goal in the succeeding parts will be to find the Thevenin equivalent of this op amp circuit, and make some observations about the resulting equivalent.

(b) Find the open circuit output voltage, V_{out} as a function of the input voltages V_1 and V_2. This will be the Thevenin Voltage, V_{Th}.

Solution: Begin nodal analysis by considering the nodes at which we do not know the node voltages, and do not have a voltage source (independent or dependent) connected to it so that we can write our KCL expressions.

We will use the following node voltage labelings from u_1 to u_5, with all the KCL expressions written to have currents directed outward. So long as the right node voltage difference is taken, this will yield the correct equations.



At node u_1, if we apply the ideal op amp assumptions, there should be no current going into the positive terminal. This means we can interpret the resistor network attached to the positive terminal as a voltage divider.

$$u_1 = \frac{R}{R+R}V_2 = \frac{V_2}{2}$$

At node u_2, we can write the KCL equation considering that there will be no current going into the negative terminal.

$$\frac{u_2 - V_1}{R} + \frac{u_2 - u_3}{R} = 0$$

At node u_3, the KCL equation is:

$$\frac{u_3 - u_2}{R} + \frac{u_3 - 0V}{R} + \frac{u_3 - u_5}{R} = 0$$

At node u_5, there is a voltage divider, where u_4 is the node voltage that has the same value as the branch voltage of the resistor that bridges terminals a and b.

$$u_4 = \frac{R}{R+R}u_5 = \frac{u_5}{2}$$

Note that the output node with node voltage u_5 did not have a KCL equation written for it. This is because the output is determined by a dependent voltage source and the current coming out from the source will not be expressible until we've found all the node voltages.

Since the circuit is in negative feedback, the input terminal voltages should be the same: $u_1 = u_2$. Substituting into the KCL equations, we have:

$$\frac{u_1 - V_1}{R} + \frac{u_1 - u_3}{R} = 0$$

$$\frac{u_3 - u_1}{R} + \frac{u_3}{R} + \frac{u_3 - u_5}{R} = 0$$

Treating u_1 and V_1 as knowns, the first KCL equation tells us that u_3 is:

$$u_3 = 2u_1 - V_1 = V_2 - V_1$$

Treating u_3 and u_1 as knowns, the second KCL equation tells us that u_5 is:

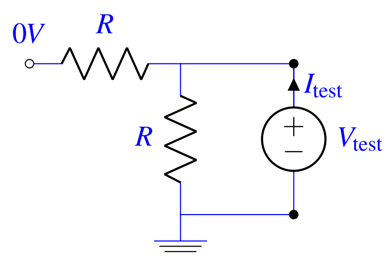
$$\begin{aligned} u_5 &= 3u_3 - u_1 \\ &= 3V_2 - 3V_1 - \frac{V_2}{2} = \frac{5}{2}V_2 - 3V_1 \end{aligned}$$

Lastly, our output voltage $u_4 = V_{\text{out}} = V_{Th}$ is:

$$u_4 = V_{\text{out}} = \frac{\frac{5}{2}V_2 - 3V_1}{2} = \frac{5}{4}V_2 - \frac{3}{2}V_1$$

(c) Turn off all independent sources ($V_1 = V_2 = 0V$). What is the equivalent resistance as seen between terminals a and b? This will be your Thevenin resistance, R_{Th}. (Hint: Consider what the voltage at the output of the op amp becomes and use a test source, or replace the op amp with its internal model where it has a dependent source.)

Solution: When both inputs V_1 and V_2 are off, the analysis in the previous subpart showed that u_5, the node voltage at the output of the op amp would be 0V. If this is the case, by applying a test source we have:



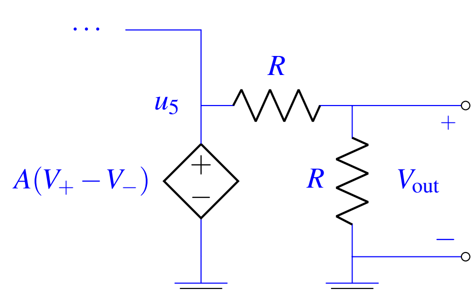
Since both resistors will have branch voltages of V_{test} over them, I_{test} can be written as:

$$I_{\text{test}} = \frac{V_{\text{test}}}{R} + \frac{V_{\text{test}}}{R} = \frac{2V_{\text{test}}}{R}$$

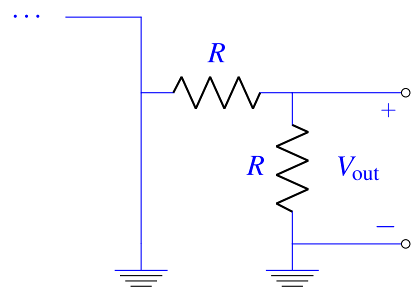
So the equivalent resistance as seen from the output (which is also the Thevenin resistance since we turned off all sources) is:

$$R_{eq} = \frac{V_{\text{test}}}{I_{\text{test}}} = \frac{R}{2}$$

Another way to approach the problem is to consider the internal model of the op amp:



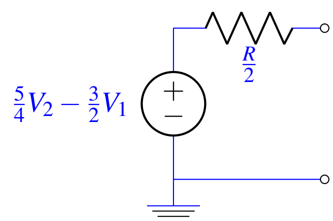
The output u_5 being zero when V_1 and V_2 are zero essentially means the dependent source is off and behaves like a short - we can turn off the dependent source in this case because it is directly influenced by the independent sources V_1 and V_2:



In this scenario, the equivalent resistance seen at the output is just $R||R = \frac{R}{2}$ since the resistors are in parallel.

(d) Use what you found in parts *b* and *c* to draw the Thevenin equivalent.

Solution: The Thevenin equivalent is given by the following circuit:



(e) **Practice (Optional)**: Does the Thevenin resistance depend on all the resistors or a strict subset? What might explain this?

Solution: Since the output of an ideal op amp is a dependent voltage source, only the resistances that come after the output it will come into play, which is why we saw only the divider at the output influence the resistance.

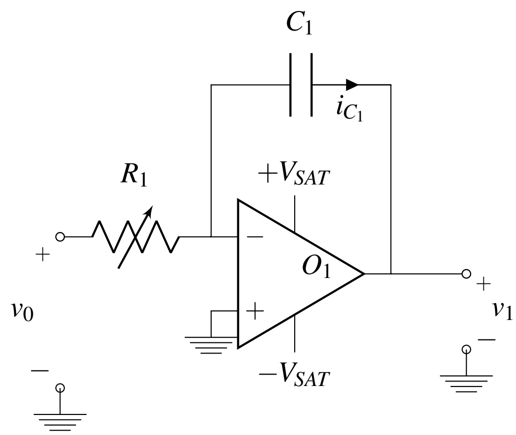
This highlights the benefit of an op amp in allowing circuits to present with a uncomplicated or choosable Thevenin equivalent resistance (e.g. a buffer, 0 resistance) so that designed circuit modules can be connected together with predictable effect.

4. Integration using Op-amps

As we have seen already it is useful in several applications to create triangular voltages (also known as voltage ramps). Remember for instance the HW 8 problem where we built a circuit that could measure the level of water in a tank by "integrating" current on a capacitor whose value changed with the level of water in the tank. In this problem, you will be analyzing a circuit that produces a voltage ramp using a voltage source and an op-amp in negative feedback.

(a) One of the circuit blocks you can use to generate the triangular waveform is the integrator. An integrator outputs the integral of the input signal. For the circuit given below express v_1 in terms of R_1, C_1, v_0, and t, assuming v_0 is not varying with time. What is the slope of this voltage ramp? You may also assume that capacitor C_1 has $0V$ across it at time $t = 0$.

Hint: You will have to apply KCL, and use the fact that the current flowing through a capacitor is given by $I = C\frac{dV}{dt}$.



Solution:
Let's write the KCL equation at V^- assuming all currents are leaving that node.

$$
\begin{aligned}
i_{R_1} + i_{C_1} &= 0 \\
i_{R_1} &= -i_{C_1} \\
i_{C_1} &= C_1 \frac{d(0 - v_1(t))}{dt} \\
\frac{0 - v_0}{R_1} &= C_1 \frac{d(v_1(t) - 0)}{dt} \\
-\frac{v_0}{R_1 C_1} &= \frac{dv_1(t)}{dt} \\
v_1(t) &= -\frac{1}{R_1 C_1} \int_0^t v_0 d\tau \\
v_1(t) &= -\frac{1}{R_1 C_1} v_0 t
\end{aligned}
$$

Which means that the **slope** of the ramp is going to be equal to $-\frac{1}{R_1C_1} v_0$

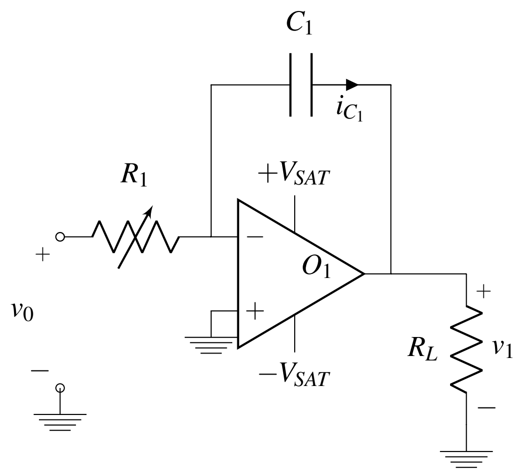
(b) What is the value of the current i_{C_1} flowing through capacitor C_1? How does the capacitor current change if we double C_1? How does the slope of the ramp change if we double capacitor C_1?
Note: the current direction is specified in the figure above.

Solution: As calculated in part (a),

$$i_{C_1} = -i_{R_1} = -\frac{0 - v_0}{R_1} = \frac{v_0}{R_1}$$

Since this value is **independent** of C_1 the current i_{C_1} will **not** change if we double the capacitor.
On the other hand, the **slope** of the ramp is equal to $-\frac{1}{R_1C_1} v_0$ so we will get half the slope if we double the capacitor. Intuitively, this makes sense since we would expect charging a capacitor twice as large with the same current to take twice as much time.

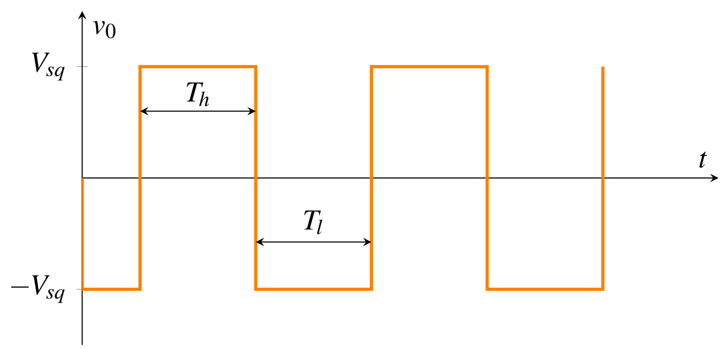
(c) If we connect a load resistance at the output of the circuit, as shown in the figure below, does the output voltage v_1 change, from what you calculated in part (a)? Why or why not?



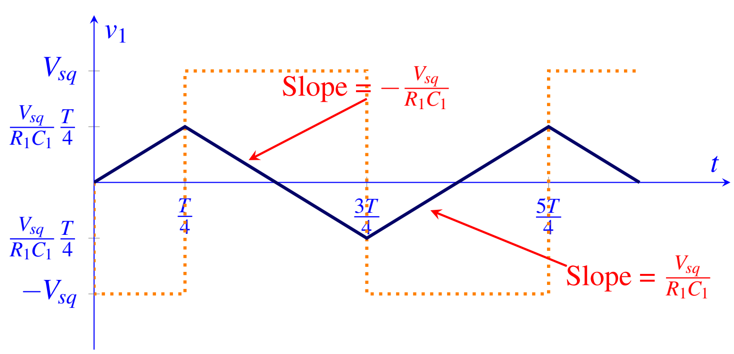
Solution:

The output voltage will not change if we connect a load at the output since it is solely defined by the current flowing through capacitor C_1. This current as we have already seen in part (b) is dependent only upon the input voltage v_0 and the resistor R_1. Any extra curent through the load resistor will be provided by the output of the op-amp, which is within the capabilities of an ideal op-amp.

(d) If v_0 varies with time as shown in the following diagram, plot v_1 for $t = 0$ to $t = 1.5T$, where $T = 2T_h = 2T_l$. In your plot indicate an algebraic expression for the slope (as a function of R_1, C_1 and v_{sq}) and add tick marks on the x and y axis indicating the time and voltage values where the ramp slope changes. You may assume again that capacitor C_1 has 0V across it at time $t = 0$.



Solution:



(e) **Practice (Optional):** Prove that the units of RC in SI are seconds.

Solution: We know that R is measured in Ohms and C in Farads so we can write that RC is measured in:

$$\Omega \mathrm{F} = \frac{\mathrm{V}}{\mathrm{A}}\frac{\mathrm{C}}{\mathrm{V}} = \frac{\mathrm{C}}{\mathrm{A}} = \frac{\mathrm{C}}{\mathrm{C}/\mathrm{sec}} = \mathrm{sec}$$

5. Cool For The Summer

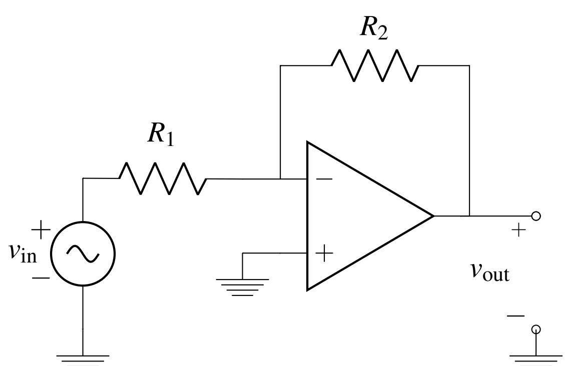
You and a friend want to make a box that helps control an air conditioning unit using both your inputs. You both have individual dials where you can set a control voltage: input of 0 means that you want to leave the temperature as it is. Negative voltage input would mean that you want to reduce the temperature. (It's hot, so we will assume that you never want to increase the temperature – so, we're not talking about a Berkeley summer...)

Your air conditioning unit, however, responds to positive voltages. The higher the magnitude of the voltage, the stronger it runs. At zero, it is off. You also need a system that sums up both you and your friend's control inputs.

Therefore, you need a box that is **an inverting summer** – *it outputs a weighted sum of two voltages where the weights are both negative*. The sum is weighted because each of you has your own subjective sense of how much to turn the dial down, so you need to compensate for this.

This problem walks you through designing this inverting summer using an op-amp.

(a) As a first step, derive v_{out} in terms of R_2, R_1, v_{in}.



Solution: First, we need to check that the amplifier is in negative feedback. In other words, if the negative input terminal is moved upward, the feedback needs to move it back downward. Going around the loop:

- We move the negative input of the op amp upward
- The output of the amplifier moves downward
- The negative input moves downward with it

The important thing here is that the result of the initial stimulus needs to go in the opposite direction of the initial stimulus! Thus, we've confirmed that the amplifier is in negative feedback.

Second, we perform KCL.

$$\frac{v_{\text{in}} - u_{-}}{R_1} + \frac{v_{\text{out}} - u_{-}}{R_2} = 0$$

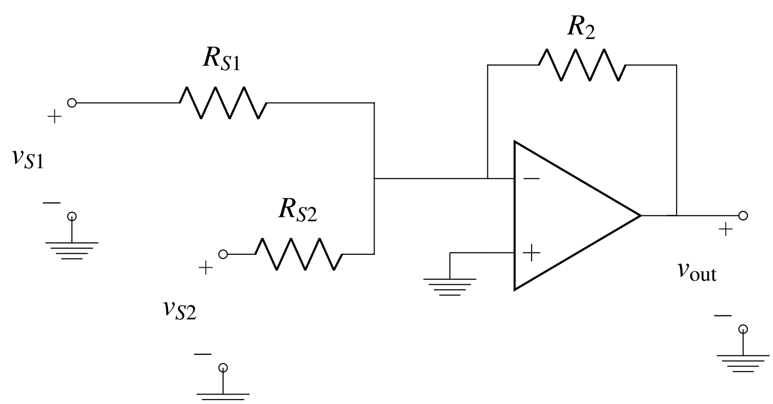
Since we're in negative feedback, we can apply the golden rules. From those, we know the voltages at the negative and positive input terminals of the amplifier—u_- and u_+, respectively—are held at the same voltage. In other words, $u_+ = u_- = 0\text{V}$.

$$\frac{v_{\text{in}}}{R_1} + \frac{v_{\text{out}}}{R_2} = 0$$

$$v_{\text{out}} = v_{\text{in}}\left(-\frac{R_2}{R_1}\right)$$

The general inverting amplifier shown above has a voltage gain $v_{\text{out}} = -\frac{R_2}{R_1}v_{\text{in}}$.

(b) Now we will add a second input to this circuit as shown below. Find v_{out} in terms of v_{S1}, v_{S2}, R_{S1}, R_{S2} and R_2.



Solution:

Method 1: Superposition

We can find the overall voltage gain of this amplifier using superposition. When v_{S1} is on, we can ignore R_{S2}. From the Golden Rules, we know that the voltage at the − terminal of the op-amp must be equal to the voltage at the + terminal. Thus, the voltage across R_{S2} is 0 V. Now apply the equation from part (a) $v_{\text{out}} = -\frac{R_2}{R_{S1}}v_{S1}$. Similarly, when v_{S2} is on, we get $v_{\text{out}} = -\frac{R_2}{R_{S2}}v_{S2}$. Combining the two equations, we get $v_{\text{out}} = -R_2\left(\frac{v_{S1}}{R_{S1}} + \frac{v_{S2}}{R_{S2}}\right)$.

Method 2: KCL without superposition

The following analysis is also correct and arrives at the same conclusion. According to the golden rules, $u_- = u_+ = 0\text{V}$, so we can write a single equation and solve:

$$\frac{v_{S1}}{R_{S1}} + \frac{v_{S2}}{R_{S2}} + \frac{v_{\text{out}}}{R_2} = 0$$

$$v_{\text{out}} = -v_{S1}\left(\frac{R_2}{R_{S1}}\right) - v_{S2}\left(\frac{R_2}{R_{S2}}\right)$$

(c) Let's suppose that you want $v_{\text{out}} = -\left(\frac{1}{4}v_{S1} + 2v_{S2}\right)$ where v_{S1} and v_{S2} represent the input voltages from you and your friend. Select resistor values such that the circuit implements this desired relationship.

Solution: Using the configuration from the previous part, the conditions which need to be satisfied are:

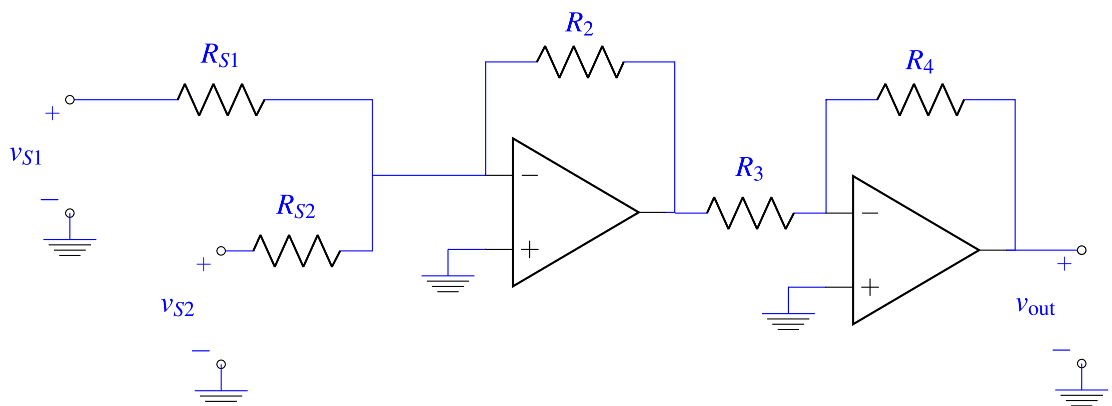
- $\frac{R_2}{R_{S1}} = \frac{1}{4}$

- $\frac{R_2}{R_{S2}} = 2$

One possible set of values is $R_2 = 2\,\text{k}\Omega$, $R_{S1} = 8\,\text{k}\Omega$, and $R_{S2} = 1\,\text{k}\Omega$, but any combination of resistors which satisfies the conditions listed above are valid solutions.

(d) Now suppose that you have a new AC unit that you want to use with your control inputs v_{S1} and v_{S2}. This unit, however, functions opposite to the previous unit; it responds to negative voltages. The higher the magnitude of the negative voltage, the stronger the AC runs.
Now design a circuit that *outputs a weighted sum of two control input voltages where both weights are positive*. Specifically, add another op-amp based circuit to your circuit in part (b), so that you invert the output of the circuit from part (b).

Solution:



Here, we add another inverting op-amp stage with a voltage gain of 1, and we can pick any equal-valued resistors for R_3 and R_4.

6. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.