EECS 16A   Designing Information Devices and Systems I
Spring 2020
Homework 1

This homework is due Friday January 31, 2020, at 23:59.

Self-grades are due Monday February 3, 2020, at 23:59.

Please note that future homeworks will have more problems than HW1 and we ask you to plan accordingly.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any hand-written answers should be scanned) as well as your IPython notebook saved as a PDF.

Please attach a PDF of your Jupyter notebook for all the problems that involve coding. Make sure the results of your plots (if any) are visible. Please assign the PDF of the notebook to the correct problems on Gradescope — we will be unable to grade the problems without this assignment or submission.

Homework Learning Goals: The objective of this homework is to introduce systems of linear equations. This homework additionally serves as an introduction to working with the Python environment through IPython/Jupyter notebooks.

1. **Counting Solutions**

    Learning Goal: *(This problem is meant to illustrate the different types of systems of equations. Some have a unique solution and others have no solutions or infinitely many solutions. We will learn in this class how to systematically figure out which of the three above cases holds.)*

    For each of the following systems of linear equations, determine if there is a unique solution, no solution, or an infinite number of solutions. If there is a unique solution, find it. If there is an infinite number of solutions state this and give one solution. If there is no solution, explain why. **Use augmented matrices and show your work**.

    (a)
    $$\begin{aligned} 2x &+& 3y &= 5 \\ x &+& y &= 2 \end{aligned}$$

    Solution:

    $$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \longleftarrow R_1 - 2R_2 \mapsto R_1$$

    $$\rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right] \longleftarrow R_2 - R_1 \mapsto R_2$$

    Unique solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

    (b)
    $$\begin{aligned} x &+& y &+& z &= 3 \\ 2x &+& 2y &+& 2z &= 5 \end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \longleftarrow R_2 - 2R_1 \mapsto R_2$$

No solution. The fact that there are fewer equations than there are unknowns immediately means that it is not possible to have a unique solution; however, this does not guarantee that there is a solution to begin with. From Gaussian Elimination, we can see that these equations are inconsistent and we have a row of zero equating to a nonzero value. In other words, no values of x, y, and z can satisfy both equations simultaneously.

(c)

$$\begin{aligned} - \quad y \quad + \quad 2z \quad &= 1 \\ 2x \qquad\qquad + \quad z \quad &= 2 \end{aligned}$$

Solution:

Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all. Using Gaussian Elimination, we can add an additional equation which provides no unique information.

$$\begin{bmatrix} 0 & -1 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives us the square matrix formulation we were first introduced to, and can see after rearranging the matrix into upper triangular form that we have a zero pivot. That said, the equations do not contradict each other, so we have an infinte number of solutions. The set of infinite solutions has the form (for some $z \in \mathbb{R}$:

$$y = 2z - 1$$
$$x = 1 - \frac{1}{2}z$$

To get full credit it is enough to state "Infinite solutions" and give one possible solution that fits the form above.

(d)

$$\begin{aligned} x \quad + \quad 2y \quad &= 3 \\ 2x \quad - \quad y \quad &= 1 \\ 3x \quad + \quad y \quad &= 4 \end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & 2 & | & 3 \\ 2 & -1 & | & 1 \\ 3 & 1 & | & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & | & 4 \\ 2 & -1 & | & 1 \\ 3 & 1 & | & 4 \end{bmatrix} \longleftarrow R_1 + R_2 \mapsto R_1$$

$$\rightarrow \begin{bmatrix} 3 & 1 & | & 4 \\ 2 & -1 & | & 1 \\ 0 & 0 & | & 0 \end{bmatrix} \longleftarrow R_3 - R_1 \mapsto R_3$$

$$\rightarrow \begin{bmatrix} 3 & 1 & | & 4 \\ 5 & 0 & | & 5 \\ 0 & 0 & | & 0 \end{bmatrix} \longleftarrow R_1 + R_2 \mapsto R_2$$

$$\rightarrow \begin{bmatrix} 0 & 1 & | & 1 \\ 1 & 0 & | & 1 \\ 0 & 0 & | & 0 \end{bmatrix} \longleftarrow R_1 - \frac{3}{5}R_2 \mapsto R_1$$

$$\rightarrow \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 1 \\ 0 & 0 & | & 0 \end{bmatrix} \longleftarrow \text{switch } R_1, R_2$$

Unique solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Notice how even with the redundant equation, we still have enough information to uniquely find x and y! If this is unclear, the system of linear equations at the end of the Gaussian Elimination above simply reads out

$$x = 1$$
$$y = 1$$
$$0 = 0$$

Do not add new columns! Because each column represents a variable, we should not add columns of zeros in an attempt to reshape our matrix into a square. In this problem, the first equation does not provide any new information, but no more information is necessary to solve for x and y.

(e)

$$\begin{array}{rrrl} x & + & 2y & = 3 \\ 2x & - & y & = 1 \\ x & - & 3y & = -5 \end{array}$$

Solution:

$$\begin{bmatrix} 1 & 2 & | & 3 \\ 2 & -1 & | & 1 \\ 1 & -3 & | & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & | & 3 \\ 2 & -1 & | & 1 \\ 2 & -1 & | & -2 \end{bmatrix} \longleftarrow R_3 + R_1 \mapsto R_3$$

$$\rightarrow \begin{bmatrix} 1 & 2 & | & 3 \\ 2 & -1 & | & 1 \\ 0 & 0 & | & 3 \end{bmatrix} \longleftarrow R_2 - R_3 \mapsto R_3$$

No solution. We can think of this to mean that there are no values of x and y which satisfy the conditions in all three equations simultaneously, because in order to satisfy all three equations, the last row $0 = 3$ would need to be true. Even though we have more equations than unknowns, that does not guarantee that a unique solution, or any solutions, exist.

2. Dutta's Optimal Boba

Solution: `#SystemsOfEquations #GaussianElimination`

Learning Goal: *Recognize a problem that can be cast as a systems of linear equations.*

Dutta's Optimal Boba has a unique way of serving its customers. To ensure the best customer experience, each customer gets a combination drink personalized to their tastes. Professor Dutta knows that a lot of customers don't know what they want, so when customers walk up to the counter, they are asked to taste four standard combination drinks that each contain a different mixture of the available pure teas.

Each combination drink (Classic, Roasted, Mountain, and Okinawa) is made of a mixture of pure teas (Black, Oolong, Green, and Earl Grey), with the total amount of pure tea in each combination drink always the same, and equal to one cup. The table below shows the quantity of each pure tea (Black, Oolong, Green, and Earl Grey) contained in each of the four standard combination drinks (Classic, Roasted, Mountain, and Okinawa).

Tea [cups]	Classic	Roasted	Mountain	Okinawa
Black	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$
Oolong	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{3}$
Green	0	$\frac{1}{3}$	$\frac{3}{5}$	0
Earl Grey	$\frac{1}{3}$	0	0	0

Initially, the customer's ratings for each of the pure teas are unknown. Professor Dutta's goal is to determine how much the customer likes each of the pure teas, so that an optimal combination drink can then be made. By letting the customer taste and score each of the four standard combination drinks, Professor Dutta can use linear algebra to determine the customer's initially unknown ratings for each of the pure teas. After a customer gives a score (all of the scores are real numbers) for each of the four standard combination drinks, Professor Dutta then calculates how much the customer likes each pure tea and mixes up a special combination drink that will maximize the customer's score.

The score that a customer gives for a combination drink is a linear combination of the ratings of the constituent pure teas, based on their proportion. For example, if a customer's rating for black tea is 6 and oolong tea is 3, then the total score for the Okinawa boba drink would be $6 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} = 5$ because Okinawa has $\frac{2}{3}$ black tea and $\frac{1}{3}$ Oolong tea.

Professor Courtade was thirsty after giving the first lecture, so Professor Courtade decided to take a drink break at Dutta's Optimal Boba. Professor Courtade walked in and gave the following ratings:

Combination Drink	Score
Classic	7
Roasted	7
Mountain	$7\frac{2}{5}$
Okinawa	$6\frac{1}{3}$

(a) What were Professor Courtade's ratings for each tea? **Work this problem out by hand in terms of the steps. You may use a calculator to do algebra.**

Solution:

Using Professor Courtade's ratings, Professor Dutta mentally records the following system of equations. Let x_b be the customer's rating of black tea, x_o be the customer's rating of oolong tea, x_g be the customer's rating of green tea, and x_e be the customer's rating of earl grey tea.

$$\text{Classic:} \qquad 7 = \frac{1}{3}x_b + \frac{1}{3}x_o + \frac{1}{3}x_e$$

$$\text{Roasted:} \qquad 7 = \frac{1}{3}x_b + \frac{1}{3}x_o + \frac{1}{3}x_g$$

$$\text{Mountain:} \qquad 7\frac{2}{5} = \frac{2}{5}x_o + \frac{3}{5}x_g$$

$$\text{Okinawa:} \qquad 6\frac{1}{3} = \frac{2}{3}x_b + \frac{1}{3}x_o$$

Multiply each equation by the denominator of the fraction (in order to make them easier to read):

$$21 = x_b + x_o + x_e$$
$$21 = x_b + x_o + x_g$$
$$37 = 2x_o + 3x_g$$
$$19 = 2x_b + x_o$$

The above equations can be written as an augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 1 & 1 & 1 & 0 & 21 \\ 0 & 2 & 3 & 0 & 37 \\ 2 & 1 & 0 & 0 & 19 \end{array} \right].$$

Row reduce the matrix into reduced row echelon form as follows. (It's fine if you solved the system of equations by hand a different way. Here, however, we will demonstrate how to do it using Gaussian elimination.)

Noting that there is a 1 in the upper left hand corner, subtract Row 1 from Row 2 and $2\times$Row 1 from Row 4.

$$\begin{array}{c} \text{Row 2: subtract Row 1} \\ \text{Row 4: subtract } 2\times\text{Row 1} \end{array} \implies \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 & 37 \\ 0 & -1 & 0 & -2 & -23 \end{array} \right]$$

Since Row 2 has a 0 in the diagonal element, multiply Row 4 by -1 and then switch Rows 2 and 4.

$$\begin{array}{c} \text{Multiply Row 4 by } -1 \\ \text{Switch Row 2 and Row 4} \end{array} \implies \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 2 & 3 & 0 & 37 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$$

Subtract Row 2 from Row 1 and $2\times$Row 2 from Row 3.

$$\begin{array}{c} \text{Row 1: subtract Row 2} \\ \text{Row 3: subtract } 2\times\text{Row 2} \end{array} \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 0 & 3 & -4 & -9 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$$

Switch Row 3 and Row 4 and subtract 3× the new Row 3 from the new Row 4.

$$
\begin{array}{l}
\text{Switch Row 3 and Row 4} \\
\text{Row 4: subtract } 3\times\text{Row 3}
\end{array}
\implies
\left[\begin{array}{cccc|c}
1 & 0 & 0 & -1 & -2 \\
0 & 1 & 0 & 2 & 23 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & -9
\end{array}\right]
$$

Finally, multiply Row 4 by -1 and add Row 4 to Row 1 and Row 3 and subtract 2×Row 4 from Row 2.

$$
\begin{array}{l}
\text{Multiply Row 4 by } -1 \\
\text{Row 1, Row 3: add Row 4} \\
\text{Row 2: subtract } 2\times\text{Row 4}
\end{array}
\implies
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 7 \\
0 & 1 & 0 & 0 & 5 \\
0 & 0 & 1 & 0 & 9 \\
0 & 0 & 0 & 1 & 9
\end{array}\right]
$$

Professor Courtade's ratings for each tea are

Tea	Score
Black	7
Oolong	5
Green	9
Earl Grey	9

(b) What mystery tea combination could Professor Dutta put in Professor Courtade's personalized drink to maximize the customer's score? If there is more than one correct answer, state that there are many answers, and give one such combination. What score would Professor Courtade give for the answer you wrote down?

Solution:

There are many answers. *Any* combination of green tea and earl grey is acceptable as they have equal ratings. More precisely, for any $0 \le a \le 1$, a combination with a cups of green tea and $1 - a$ cups of earl grey will yield a score of $9a + 9(1 - a) = 9$.

As an example, Professor Dutta could choose $a = \frac{1}{2}$ so that Professor Courtade's drink has $\frac{1}{2}$ cup of green tea and $\frac{1}{2}$ cup of earl grey.

How to see this? Green tea and earl grey are tied for Professor Courtade's favorite tea, so it doesn't make a difference if Professor Dutta substitutes one for the other in any quantity - the score remains 9. It also doesn't make sense to substitute a less preferred tea like black tea for Professor Courtade's favorite tea, or to add a less preferred tea at the expense of the most preferred teas, as this will lead to scores less than 9.

3. **Filtering Out The Troll**

Solution: `#SystemsOfEquations #LinearCombination`

Learning Goal: *(The goal of this problem is to represent a practical scenario using a simple model of directional microphones. Students will tackle the problem of sound reconstruction through solving a system of linear equations.)*

You attended a very important public speech and recorded it using a recording device that consists of two directional microphones. However, there was this particular person in the audience who was trolling around, adding noise to the recording. When you went back home to listen to the recording, you realized that the

two recordings were dominated by the troll's noise and you could not hear the speech. Fortunately, since your recording device contained two microphones, you realized there is a way to combine the two individual microphone recordings so that the troll's noise is removed. You remembered the locations of the speaker and the troll and created the diagram shown in Figure 1. You (and your two microphones) are located at the origin.
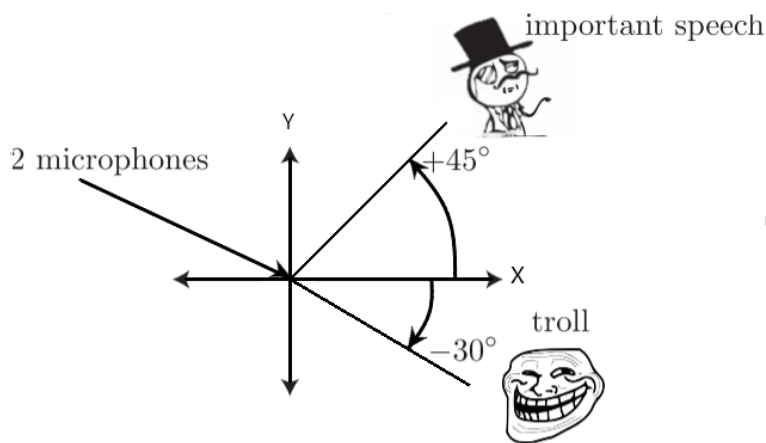

Figure 1: Locations of the speaker and the troll.

Each directional microphone records signals differently based on where they are coming from. For the first microphone, when a signal is coming from an angle θ with respect to the x-axis, it is weighted by the factor $f_1(\theta) = \cos(\theta)$. If there are two signals simultaneously playing (as is the case with the speech and the troll noise), then both are recorded as a linear combination, each weighted by the respective $f_1(\theta)$ for their angles). For the second microphone, if the signal is coming from an angle θ with respect to the x-axis, then the signal is weighted by the factor $f_2(\theta) = \sin(\theta)$. The linear combination also applies to the second microphone. Graphically, the directional characteristics of the two microphones are given in Figure 2.

We can now refer to the diagram in Figure 1 and develop a mathematical model of the microphone recordings. Let the person who gave the important speech and the troll be speakers *A* and *B*, respectively. The person who gave the important speech (speaker *A*) was located at angle $\alpha = +45°$ relative to the x-axis, and the troll (speaker *B*) was located at angle $\beta = -30°$ relative to the x-axis. Speaker *A* produced an audio signal represented by the vector $\vec{a} \in \mathbb{R}^n$. That is, the *i*-th entry of vector $\vec{a}$ was the signal at the *i*-th time step. Similarly, speaker *B* produced an audio signal $\vec{b} \in \mathbb{R}^n$, where the *i*-th entry of vector $\vec{b}$ was the signal at the *i*-th time step.

Therefore, the first microphone recorded the signal

$$\vec{m}_1 = f_1(\alpha) \cdot \vec{a} + f_1(\beta) \cdot \vec{b},$$

and the second microphone recorded the signal

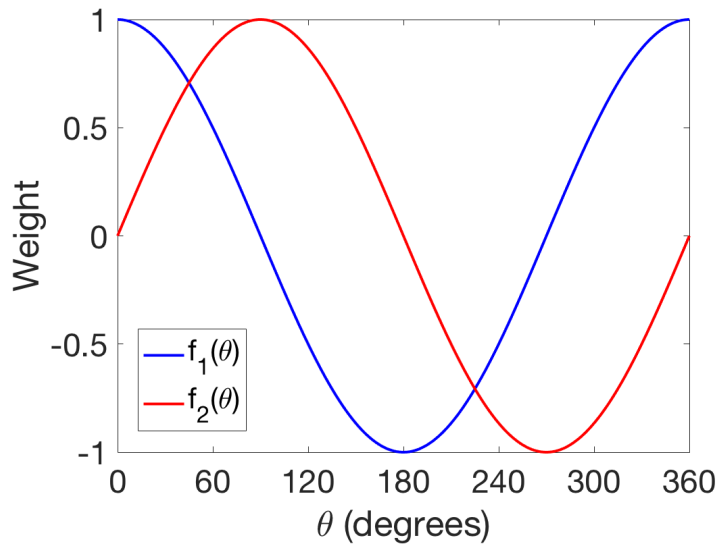$$\vec{m}_2 = f_2(\alpha) \cdot \vec{a} + f_2(\beta) \cdot \vec{b}.$$

Figure 2: Weights for recorded audio signals for each of the two microphones, as a function of audio source angle θ. Microphone 1 is blue and microphone 2 is red. Note that a weight can be negative as well as positive.

(a) Using the notation above, express the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ (i.e. the signals recorded by the first and the second microphones, respectively) as a linear combination (i.e. a weighted sum) of $\vec{a}$ and $\vec{b}$.

Solution:

$$\vec{m}_1 = \cos\left(\frac{\pi}{4}\right) \cdot \vec{a} + \cos\left(-\frac{\pi}{6}\right) \cdot \vec{b}$$

$$= \frac{1}{\sqrt{2}} \cdot \vec{a} + \frac{\sqrt{3}}{2} \cdot \vec{b}$$

$$\vec{m}_2 = \sin\left(\frac{\pi}{4}\right) \cdot \vec{a} + \sin\left(-\frac{\pi}{6}\right) \cdot \vec{b}$$

$$= \frac{1}{\sqrt{2}} \cdot \vec{a} - \frac{1}{2} \cdot \vec{b}$$

(b) Recover the important speech $\vec{a}$, as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{a} = u \cdot \vec{m}_1 + v \cdot \vec{m}_2$ (where u and v are scalars). What are the values of u and v?

Solution:

Solving the system of linear equations yields

$$\vec{a} = \frac{\sqrt{2}}{1 + \sqrt{3}} \cdot \left(\vec{m}_1 + \sqrt{3}\vec{m}_2\right).$$

Therefore, the values are $u = \frac{\sqrt{2}}{1+\sqrt{3}}$ and $v = \frac{\sqrt{6}}{1+\sqrt{3}}$.

It is fine if you solved this either using IPython or by hand using any valid technique. The easiest approach is to subtract either of the two equations from the other and immediately see that

$\vec{b} = \frac{2}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2)$. Substituting $\vec{b}$ back into the second equation and multiplying through by $\sqrt{2}$ gives that $\vec{a} = \sqrt{2}(\vec{m}_2 + \frac{1}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2))$, which simplifies to the expression given above.

Notice that subtracting one equation from the other is natural given the symmetry of the microphone patterns and the fact that the patterns intersect at the 45 degree line where the important speech is happening, and the fact that $\sin(45°) = \cos(45°)$. So we know that the result of subtracting one microphone recording from the other results in only the trolls contribution. Once we have the troll contribution, we can remove it and obtain the important speakers sole content.

(c) Partial IPython code can be found in `prob1.ipynb`. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

Solution:

The solution code can be found in `sol1.ipynb`. The speaker says: "All human beings are born free and equal in dignity and rights." and the speech was taken from the Universal Declaration of Human Rights.

The idea of using multiple microphones to isolate speech is interesting and is increasingly used in practice. Furthermore, similar techniques are used in wireless communication both by cellular systems like LTE and increasingly by WiFi hotspots. (This is why they often have multiple antennas).

4. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.

EECS16A: Homework 1

Problem 2: Filtering Out The Troll

```python
In [1]: import warnings
        import wave as wv

        import matplotlib.pyplot as plt
        import numpy as np
        import scipy
        import scipy.io.wavfile
        from IPython.display import Audio
        from scipy import io
        from scipy.io.wavfile import read

        # For this to work make sure to download m1.wav and m2.wav to the same location
        warnings.filterwarnings("ignore")
        sound_file_1 = "m1.wav"
        sound_file_2 = "m2.wav"
```

Let's listen to the recording of the first microphone (it can take some time to load the sound file).
Run the cell below, then press the play button to listen.

```python
In [2]: Audio(url='m1.wav', autoplay=False)
```

Out[2]:

0:00 / 0:00

Let's listen to the recording of the first microphone (it can take some time to load the sound file).
Run the cell below, then press the play button to listen.

```python
In [3]: Audio(url='m2.wav', autoplay=False)
```

Out[3]:

0:00 / 0:00

We read the first recording to the variable `corrupt1` and the second recording to `corrupt2`.
Treat `corrupt1` and `corrupt2` as the two sound recordings picked up by microphone 1 and
microphone 2 respectively.

In [4]:
```python
rate1,corrupt1 = scipy.io.wavfile.read('m1.wav')
rate2,corrupt2 = scipy.io.wavfile.read('m2.wav')
```

Enter the weights of the two recordings to get the clean speech.

Note: The square root of a number a can be written as `np.sqrt(a)` in IPython.

In [8]:
```python
# enter the weights u (recording 1) and v (recording 2)
u = np.sqrt(2)/(1+np.sqrt(3))
v = np.sqrt(6)/(1+np.sqrt(3))
```

Weighted combination of the two recordings:

In [9]:
```python
a = u*corrupt1 + v*corrupt2
```

Let's listen to the resulting sound file (make sure your speaker's volume is not very high, the sound may be loud if things go wrong).

In [10]:
```python
Audio(data=a, rate=rate1)
```

Out[10]:

0:00 / 0:00