EECS 16A Designing Information Devices and Systems I
Spring 2016 Elad Alon, Babak Ayazifar Homework 9

This homework is due April 5, 2016, at Noon.

1. **Homework process and study group**

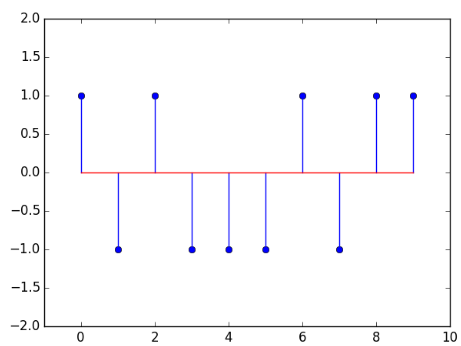
   Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?
   Working in groups of 3-5 will earn credit for your participation grade.

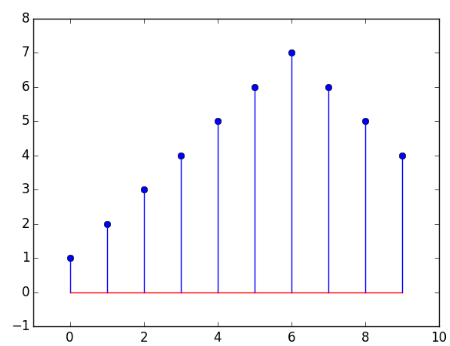
   Solution: I worked on this homework with...

   I first worked by myself for 2 hours, but got stuck on Problem 5 so I went to office hours on...

   Then I went to homework party for a few hours, where I finished the homework.

2. **Mechanical: Correlation**





   (a) Calculate and plot the **autocorrelation** (the inner products of one period of the signal with all the possible shifts of one period of the same signal) of each of the above signals. Each signal is periodic with a period of 10 (one period is shown).

   Solution: The solution is given in `sol9.ipynb`. (It is fine if you did the calculations and plotting by hand.)

   (b) Calculate and plot the **cross-correlation** (the inner products of one period of the first signal with all possible shifts of one period of the second signal) of the two signals. Each signal is periodic with a period of 10 (one period is shown).

   Solution: The solution is given in `sol9.ipynb`. (It is fine if you did the calculations and plotting by hand.)

3. **Inner products**

   The Cauchy-Schwarz inequality states that for two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$:

$$|\langle \vec{x}, \vec{y} \rangle| = |\vec{x}^T \vec{y}^*| \leq ||\vec{x}|| \cdot ||\vec{y}||$$

Use the Cauchy-Schwarz inequality to verify (i.e. prove or derive) the triangle inequality:

$$||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$$

(Hint: Start with $||\vec{x}+\vec{y}||^2$)

Solution: We consider the Euclidean 2-norm here. By Cauchy-Schwarz inequality we have

$$|\langle \vec{x},\vec{y}\rangle| \leq \|\vec{x}\|_2 \cdot \|\vec{y}\|_2$$

$$|\langle \vec{y},\vec{x}\rangle| \leq \|\vec{y}\|_2 \cdot \|\vec{x}\|_2$$

Therefore,

$$\begin{aligned}
\|\vec{x}+\vec{y}\|_2^2 &= (\vec{x}+\vec{y})^T(\vec{x}+\vec{y}) \\
&= \|\vec{x}\|_2^2 + \|\vec{y}\|_2^2 + \langle \vec{x},\vec{y}\rangle + \langle \vec{y},\vec{x}\rangle \\
&\leq \|\vec{x}\|_2^2 + \|\vec{y}\|_2^2 + 2\|\vec{y}\|_2 \cdot \|\vec{x}\|_2 \\
&= (\|\vec{x}\|_2 + \|\vec{y}\|_2)^2
\end{aligned}$$

Taking square root on both sides, we get

$$\|\vec{x}+\vec{y}\|_2 \leq \|\vec{x}\|_2 + \|\vec{y}\|_2$$

4. **Different Ways to Express Matrix Multiplication** There are several useful ways to express the multiplication of two matrices. Consider two $n \times n$ matrices A and B that can be expressed in terms of their rows A_i and B_i or in terms of their columns $\vec{a}_i$ and $\vec{b}_i$.

$$A = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & A_n & - \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{a}_1 & & \vec{a}_n \\ | & & | \end{bmatrix}, \qquad B = \begin{bmatrix} - & B_1 & - \\ & \vdots & \\ - & B_n & - \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{b}_1 & & \vec{b}_n \\ | & & | \end{bmatrix} \tag{1}$$

For notational purposes, we will write row vectors such as A_1 as $A_1 = \begin{bmatrix} A_{11} & \cdots & A_{1n} \end{bmatrix}$ and we will write column vectors such as $\vec{a}_1$ as

$$\vec{a}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix} \tag{2}$$

We learned about *inner products* in discussion. Sometimes, we write the Euclidean inner product as the multiplication of a row vector on the left by a column vector on the right

$$\langle \vec{x},\vec{y}\rangle = \vec{x}^{\,T}\vec{y}^{\,*} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + \cdots + x_ny_n \tag{3}$$

In the above equation, $\vec{y}^{\,*}$ denotes the complex conjugate of $\vec{y}$. In EE16A, unless noted otherwise, we assume real vectors, in which case $\vec{y}^{\,*} = \vec{y}$.
Note that this is consistent with the rules of matrix multiplication. Now let's define another vector product

that can be considered as the multiplication of a column vector on the left and a row vector on the right. This is called an *outer product* and is often denoted by $\otimes$.

$$\vec{x} \otimes \vec{y} = \vec{x}\,\vec{y}^{\,T} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_n y_1 & \cdots & x_n y_n \end{bmatrix} \tag{4}$$

Note that this is consistent with the rules of matrix multiplication as well. Outer products result in a special type of matrix called a rank-1 matrix or a dyad.

(a) Calculate the inner product $\langle \vec{x}, \vec{y} \rangle$ and the outer product $\vec{x} \otimes \vec{y}$ of the following pairs of vectors.

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}, \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{5}$$

(Note that you will need to turn y into a row vector vector when you calculate the outer products.)

Solution:

$$\langle \vec{x}, \vec{y} \rangle = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1(1) + 1(2) + 1(3) = 6 \tag{6}$$

$$\langle \vec{x}, \vec{y} \rangle = \begin{bmatrix} 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1(1) + 4(2) + 9(3) = 36 \tag{7}$$

$$\vec{x} \otimes \vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad \vec{x} \otimes \vec{y} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \\ 9 & 18 & 27 \end{bmatrix} \tag{8}$$

(b) Does the order of the vectors matter when you take an inner product, i.e. does $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle^*$? Does the order of the vectors matter when you take an outer product?

Solution: The order of the vectors does not matter when you take an inner product, but it does matter when you take an outer product. This can be seen as follows

$$\langle \vec{x}, \vec{y} \rangle = \sum_i x_i y_i = \sum_i y_i x_i = \langle \vec{y}, \vec{x} \rangle \tag{9}$$

$$\vec{x} \otimes \vec{y} = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_n y_1 & \cdots & x_n y_n \end{bmatrix} \neq \begin{bmatrix} y_1 x_1 & \cdots & y_1 x_n \\ \vdots & & \vdots \\ y_n x_1 & \cdots & y_n x_n \end{bmatrix} = \vec{y} \otimes \vec{x} \tag{10}$$

For example, consider the two vectors

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \qquad \vec{y} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \tag{11}$$

The outer product $\vec{x} \otimes \vec{y}$ is

$$\vec{x} \otimes \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix} \tag{12}$$

where as the outer product $\vec{y} \otimes \vec{x}$

$$\vec{y} \otimes \vec{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix} \tag{13}$$

Now consider the matrix product AB. Write AB as

(c) A matrix where each element is an inner product.
Hint: Write AB as

$$AB = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & A_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{b}_1 & & \vec{b}_n \\ | & & | \end{bmatrix} \tag{14}$$

(d) A sum of matrices that are each outer products
Hint: Write AB as

$$AB = \begin{bmatrix} | & & | \\ \vec{a}_1 & & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & B_1 & - \\ & \vdots & \\ - & B_n & - \end{bmatrix} \tag{15}$$

Solution:
By our definition of column vectors $\vec{a}_i, \vec{b}_i$ and row vectors A_j, B_j, we know that $a_{ij} = A_{ji}$ and similarly $b_{ij} = B_{ji}$ since the j-th entry in the i-th column is the same as the i-th entry in the j-th row of a matrix.

$$AB = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & A_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{b}_1 & & \vec{b}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \langle A_1, \vec{b}_1 \rangle & \cdots & \langle A_1, \vec{b}_n \rangle \\ \vdots & & \vdots \\ \langle A_n, \vec{b}_1 \rangle & \cdots & \langle A_n, \vec{b}_n \rangle \end{bmatrix} = \begin{bmatrix} \sum_i A_{1i} b_{1i} & \cdots & \sum_i A_{1i} b_{ni} \\ \vdots & & \vdots \\ \sum_i A_{ni} b_{1i} & \cdots & \sum_i A_{ni} b_{ni} \end{bmatrix} \tag{16}$$

$$AB = \begin{bmatrix} | & & | \\ \vec{a}_1 & & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & B_1 & - \\ & \vdots & \\ - & B_n & - \end{bmatrix} = \sum_i \vec{a}_i \otimes B_i = \sum_i \begin{bmatrix} | \\ \vec{a}_i \\ | \end{bmatrix} \begin{bmatrix} - & B_i & - \end{bmatrix} = \begin{bmatrix} \sum_i a_{i1} B_{i1} & \cdots & \sum_i a_{i1} B_{in} \\ \vdots & & \vdots \\ \sum_i a_{in} B_{i1} & \cdots & \sum_i a_{in} B_{in} \end{bmatrix} \tag{17}$$

Here's an example to show why the first step can be written to express AB in terms of outer products:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{18}$$

$$B = \begin{bmatrix} 2 & 1 \\ -4 & 5 \end{bmatrix} \tag{19}$$

Now,

$$AB = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2 & 1\\-4 & 5\end{bmatrix} = (\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix} + \begin{bmatrix}0 & 2\\0 & 4\end{bmatrix})(\begin{bmatrix}2 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\-4 & 5\end{bmatrix}) \tag{20}$$

$$= \begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\-4 & 5\end{bmatrix} + \begin{bmatrix}0 & 2\\0 & 4\end{bmatrix}\begin{bmatrix}2 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 2\\0 & 4\end{bmatrix}\begin{bmatrix}0 & 0\\-4 & 5\end{bmatrix} \tag{21}$$

$$= \begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}2 & 1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}4 & -5\end{bmatrix} \tag{22}$$

$$= \begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}2 & 1\end{bmatrix} + \begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}4 & -5\end{bmatrix} \tag{23}$$

5. Audio file matching

Lots of different quantities we interact with every day can be expressed as vectors. For example, an audio clip can be thought of as a vector. The series of numbers in the clip determine the sounds we hear. An audio segment or a sound wave is a continuous function of time, but this can be sampled at regular intervals to make a discrete sequence of numbers that can be represented as a vector.

This problem explores using inner products for measuring similarity. The ideas here will be further developed in the third module of EECS16A where we use the theme of Locationing and GPS to bring in optimization ideas.

Let us consider a very simplified model for an audio signal, one that is just composed of two tones. One is represented by -1 and the other by $+1$. A vector of length n makes up the audio file.

(a) Say we want to compare two audio files of the same length n to decide how similar they are. First consider two vectors that are exactly identical $\vec{X}_1 = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ and $\vec{X}_2 = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$. What is the dot product of these two vectors? What if $\vec{X}_1 = \begin{bmatrix}1 & 1\cdots 1\end{bmatrix}^T$ and $\vec{X}_2 = \begin{bmatrix}1 & -1 & 1 & -1\cdots & 1 & -1\end{bmatrix}^T$ (where the length of the vector is an even number)? Can you come up with an idea to compare two general vectors of length n now?

Solution: The dot product $\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i v_i$ is the same thing as the Euclidean Inner Product. The dot product of $\vec{X}_1 = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ and $X_2 = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ is $\vec{X}_1 \cdot \vec{X}_2 = n$. The dot product of $\vec{X}_1 = \begin{bmatrix}1 & 1\cdots 1\end{bmatrix}^T$ and $\vec{X}_2 = \begin{bmatrix}1 & -1 & 1 & -1 & \cdots & 1 & -1\end{bmatrix}^T$ is $\vec{X}_1 \cdot \vec{X}_2 = 0$ when the vector length is even. To compare two vectors of length n composed of 1 and -1, first we take the dot product of the two vectors. The larger the magnitude of the dot product, the more similar the two vectors are. The smaller the magnitude of the dot product, the more dissimilar the two vectors are. This is related to the ideas of correlation that you will see later on. In many circumstances, a dot product with a very large negative value would mean the vectors are very different, but it turns out that humans are unable to perceive the sign of sound, so two sounds vectors $\vec{X}$ and $-\vec{X}$ sound exactly the same. As a result, for this problem we are interested in the **absolute value** of the dot product, but in many other problems we will interpret a large negative dot product as having very different vectors. Don't take off points in parts a), b), or c) if you didn't mention absolute value.

(b) Next suppose we want to find a short audio clip in a longer one. We might want to do this for an application like *Shazam*, to be able to identify a song from a signature tune. Consider the vector of length 8, $\vec{X} = \begin{bmatrix}-1 & 1 & 1 & -1 & 1 & 1 & -1 & 1\end{bmatrix}^T$. Let us label the elements of $\vec{X}$ so that $\vec{X} = \begin{bmatrix}x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8\end{bmatrix}^T$. We want to find the short segment $\vec{Y} = \begin{bmatrix}1 & 1 & -1\end{bmatrix}^T$ in the longer vector, i.e., we want to find i, such that the sequence represented by $\begin{bmatrix}x_i & x_{i+1} & x_{i+2}\end{bmatrix}^T$ is the closest to $\vec{Y}$. How can we find this? Applying the same technique what i gives the best match for $\vec{Y} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$? **Solution:** For each length-3 sub-sequence of $\vec{X}$ say $\vec{X}_i = \begin{bmatrix}x_i & x_{i+1} & x_{i+2}\end{bmatrix}^T$ (the subsequence which starts at position i), take the dot product of $\vec{X}_i$ and $\vec{Y}$. The length-3 sub-sequences $\vec{X}_i$ with the maximum-magnitude dot product with $\vec{Y}$ gives us the closest match with $\vec{Y}$. For $\vec{Y} = \begin{bmatrix}1 & 1 & -1\end{bmatrix}^T$, the list of i with maximum dot product between $\vec{X}_i$ and $\vec{Y}$ are $\{2,5\}$ with each dot product being 3. For $\vec{Y} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$, every dot product is 1, so there is no good match.

We can also implement the 6 dot products in one matrix multiplication. For instance, if $\vec{X} = \begin{bmatrix}x_1 & x_2 & \dots & x_8\end{bmatrix}^T$ and $\vec{Y} = \begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix}^T$, the dot products (or correlation) $z_i = \vec{X}_i \cdot \vec{Y}$ can be represented as:

$$\begin{bmatrix} y_1 & y_2 & y_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_1 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_1 & y_2 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_1 & y_2 & y_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}$$

We then pick the z_i with the largest magnitude (there can be multiple as we've just seen) and the corresponding $\vec{X}_i$ gives us the required substrings. This is connected to the ideas of convolution which will be explored later on.

(c) Now suppose our vector was represented using integers and not just by 1 and -1. Say we wanted to locate the sequence closest to $\vec{Y} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$ in $\vec{X} = \begin{bmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\end{bmatrix}^T$. What happens if you apply the technique of part (b)? How would you modify this technique for the problem here?

Solution: Applying the technique in part (b), we get the best match to be $\begin{bmatrix}6 & 7 & 8\end{bmatrix}^T$ as this has the largest dot product with $\vec{Y} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$.

One way to modify the previous approach is by considering how close the "directions" of $\begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$ and any length-3 substring of $\vec{X}$ is. The unit vector in the direction of $\begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$ is $\vec{Y}_u = \frac{1}{\sqrt{14}}\begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$ and the unit vector in the direction of any length-3 substring $\vec{X}_i = \begin{bmatrix}x_i & x_{i+1} & x_{i+2}\end{bmatrix}^T$ is then given by $\vec{U}_i = \frac{1}{\|X_i\|}\begin{bmatrix}x_i & x_{i+1} & x_{i+2}\end{bmatrix}^T$. The unit vector $\vec{U}_i$ which has the maximum-magnitude dot product with $\vec{Y}_u$ (i.e., $\vec{U}_i \cdot \vec{Y}_u$) is the one most aligned with the vector $\vec{Y}$. Thus in our example for $\vec{X} = \begin{bmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\end{bmatrix}^T$ and $Y = \begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$, the unit vector of the length-3 substring $\vec{X}_1 = \begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$ has the maximum dot product ($\vec{U}_1 \cdot \vec{Y}_u = 1$).
What are other interesting ways to achieving this?

(d) Answer part 1 in the provided ipython notebook. **Solution:** See the solutions in the ipython notebook.

(e) Answer part 2 in the provided ipython notebook. **Solution:** See the solutions in the ipython notebook.

6. Your Own Problem Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?