# EECS 16A Designing Information Devices and Systems I
# Spring 2020 Discussion 11B

**Reference: Inner products**

Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be vectors in real vector space $\mathbb{V}$. A mapping $\langle\cdot,\cdot\rangle$ is said to be an inner product on $\mathbb{V}$ if it satisfies the following three properties:

(a) Symmetry: $\langle\vec{x},\vec{y}\rangle = \langle\vec{y},\vec{x}\rangle$

(b) Linearity: $\langle\vec{x},\vec{y}+\vec{z}\rangle = \langle\vec{x},\vec{y}\rangle + \langle\vec{x},\vec{z}\rangle$ and $\langle c\vec{x},\vec{y}\rangle = c\langle\vec{x},\vec{y}\rangle$

(c) Positive-definiteness: $\langle\vec{x},\vec{x}\rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$.

We define the norm of $\vec{x}$ as $\|\vec{x}\| = \sqrt{\langle\vec{x},\vec{x}\rangle}$.

## 1. Mechanical Inner Products

For the following pairs of vectors, find the Euclidean inner product $\langle\vec{x},\vec{y}\rangle = \vec{x}^T\vec{y}$.

(a)

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$$

**Answer:** Recall that the inner product of two vectors $\vec{x}$ and $\vec{y}$ is $\vec{x}^T\vec{y}$, thus:

$$\begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = 1+3 = 4$$

(b)

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$$

**Answer:** When working with real numbers, the inner product is commutative. Thus, using our work from the previous part, the inner product of these two vectors is 4.

(c)

$$\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}-3\\2\\1\end{bmatrix}$$

**Answer:**

$$\begin{bmatrix}1 & 0 & 3\end{bmatrix}\begin{bmatrix}-3\\2\\1\end{bmatrix} = -3+3 = 0$$

  

**2. Inner Product Properties**

Demonstrate the following properties of inner products for any vectors in $\mathbb{R}^2$, assuming we are working with the Euclidean inner product and norm.

(a) Symmetry

**Answer:** Let $x_i, y_i \in \mathbb{R}$, then

$$\begin{aligned}\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle &= x_1 \cdot y_1 + x_2 \cdot y_2 \\ &= y_1 \cdot x_1 + y_2 \cdot x_2 \\ &= \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle\end{aligned}$$

(b) Linearity

**Answer:** Let $\alpha, \beta, w_i, x_i, z_i \in \mathbb{R}$.

$$\begin{aligned}\left\langle \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \beta \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} \alpha v_1 + \beta w_1 \\ \alpha v_2 + \beta w_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle \\ &= (\alpha v_1 + \beta w_1) z_1 + (\alpha v_2 + \beta w_2) z_2 \\ &= \alpha(v_1 z_1 + v_2 z_2) + \beta(w_1 z_1 + w_2 z_2) \\ &= \alpha v_1 z_1 + \alpha v_2 z_2 + \beta w_1 z_1 + \beta w_2 z_2 \\ &= \alpha \left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle + \beta \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle\end{aligned}$$

**3. Geometric Interpretation of the Inner Product** In this problem, we will explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in $\mathbb{R}^2$.

(a) For each of the following cases, pick two vectors that satisfy the condition and find the inner product.

i. Parallel Vectors

**Answer:** Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\langle \vec{x}, \vec{x} \rangle = 1 \cdot 1 + 1 \cdot 1 = 2$$

ii. Anti-parallel

**Answer:** Again, let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\langle \vec{x}, -\vec{x} \rangle = 1 \cdot (-1) + 1 \cdot (-1) = -2 = -\langle \vec{x}, \vec{x} \rangle$$

iii. Perpendicular

**Answer:** Let $\vec{x} = \vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{y} = \vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot 0 + 0 \cdot 1 = 0$$

(b) Now, derive a formula for the inner product of two vectors in terms of their magnitudes and the angle between them.

**Answer:** From trigonometric calculation, if $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}y_1\\y_2\end{bmatrix}$, then we know that $x_1 = \|\vec{x}\| \cdot \cos\alpha$, $x_2 = \|\vec{x}\| \cdot \sin\alpha$, $y_1 = \|\vec{y}\| \cdot \cos\beta$ and $y_2 = \|\vec{y}\| \cdot \sin\beta$ (as in the figure). Then you can directly write

$$\begin{aligned}
\langle \vec{x}, \vec{y} \rangle &= x_1 \cdot y_1 + x_2 \cdot y_2 = \\
&= \underbrace{\|\vec{x}\| \cdot \cos\alpha}_{x_1} \cdot \underbrace{\|\vec{y}\| \cdot \cos\beta}_{y_1} + \underbrace{\|\vec{x}\| \cdot \sin\alpha}_{x_2} \cdot \underbrace{\|\vec{y}\| \cdot \sin\beta}_{y_2} \\
&= \|\vec{x}\|\|\vec{y}\| \left(\cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta\right) = \\
&= \|\vec{x}\|\|\vec{y}\| \cdot \cos\left(\beta - \alpha\right) \\
&= \|\vec{x}\|\|\vec{y}\| \cdot \cos\theta
\end{aligned}$$

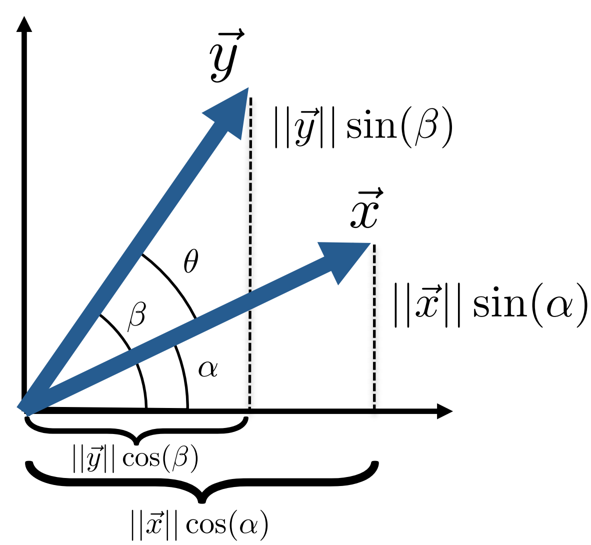


Figure 1: Two general vectors in $\mathbb{R}^2$

## 4. Reverse Triangle Inequality

The triangle inequality states that, for vectors $\vec{x}, \vec{y} \in \mathbb{R}^N$:

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

(a) First, prove the following:

$$\|\vec{x}-\vec{y}\| = \|\vec{y}-\vec{x}\|$$

**Answer:** This can be shown using the scalar-multiplication property of the norm, which states that $\|\alpha\vec{x}\| = |\alpha|\|\vec{x}\|$.

(b) Using the triangle inequality in conjunction with the previous identity, prove the reverse triangle inequality, which states that, for vectors $\vec{x},\vec{y} \in \mathbb{R}^N$:

$$\big|\|\vec{x}\| - \|\vec{y}\|\big| \leq \|\vec{x}-\vec{y}\|$$

**Answer:** Start with the identity proved in the previous part.

$$\|\vec{x}-\vec{y}\| = \|\vec{y}-\vec{x}\|$$

Now, we apply the triangle inequality to show for $\vec{x}$:

$$\|\vec{x}\| = \|(\vec{x}-\vec{y})+\vec{y}\| \leq \|\vec{x}-\vec{y}\| + \|\vec{y}\|$$
$$\|\vec{x}\| - \|\vec{y}\| \leq \|\vec{x}-\vec{y}\|$$

Repeating for $\vec{y}$:

$$\|\vec{y}\| = \|(\vec{y}-\vec{x})+\vec{x}\| \leq \|\vec{y}-\vec{x}\| + \|\vec{x}\|$$
$$\|\vec{y}\| - \|\vec{x}\| \leq \|\vec{y}-\vec{x}\|$$

Combining these two equations, we find that:

$$\big|\|\vec{x}\| - \|\vec{y}\|\big| \leq \|\vec{x}-\vec{y}\|$$