

EECS 16A Designing Information Devices and Systems I Discussion 12A

Inner Product convention:

For this discussion, assume that the inner product utilized is the Euclidean inner product: $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}$.

Cross-correlation:

The cross-correlation between two signals $r[n]$ and $s[n]$ is defined as follows:

$$\text{corr}_r(s)[k] = \sum_{i=-\infty}^{\infty} r[i]s[i-k].$$

1. Classification

In lecture, you were introduced to the idea of using inner products to measure similarity to figure out how to classify a vector \vec{x} given n different classes represented by vectors \vec{c}_i for $i = 1 \dots n$ where $\|\vec{c}_1\| = \|\vec{c}_2\| = \dots = \|\vec{c}_n\|$. The problem can be stated as 'find \vec{c} which has maximum inner product with \vec{x} and therefore is most similar to it' or equivalently as 'find \vec{c} which has the minimum distance from \vec{x} '. In mathematical notation, we can write:

$$\vec{c}_{class} = \arg \max_{\vec{c} \in \{\vec{c}_1, \dots, \vec{c}_n\}} \langle \vec{x}, \vec{c} \rangle = \arg \min_{\vec{c} \in \{\vec{c}_1, \dots, \vec{c}_n\}} \|\vec{x} - \vec{c}\|$$

The quantity that comes after max or min is a function of \vec{c} to be maximized or minimized. The expression under max or min specifies the variable to set and its allowed values for maximizing or minimizing our function value. The arg indicates we want our answer to be the \vec{c}_i that maximizes or minimizes the function rather than the optimal function value.

- (a) Let's classify a vector $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in \mathbb{R}^2 . Given classes $\vec{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{c}_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, find $\arg \max_{\vec{c} \in \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}} \langle \vec{x}, \vec{c} \rangle$.

Answer: First, calculate $\langle \vec{x}, \vec{c} \rangle$ for every choice of \vec{c}_i :

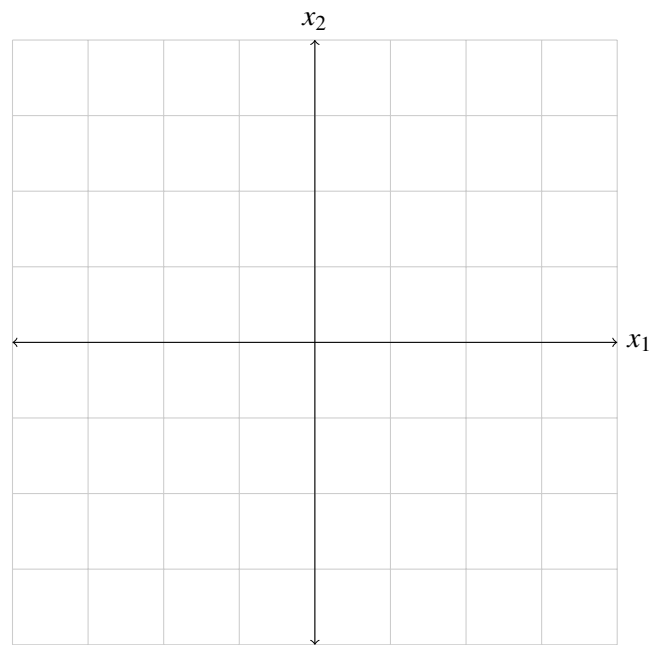
$$\langle \vec{x}, \vec{c}_1 \rangle = 3(1) + 1(0) = 3$$

$$\langle \vec{x}, \vec{c}_2 \rangle = 3(0) + 1(1) = 1$$

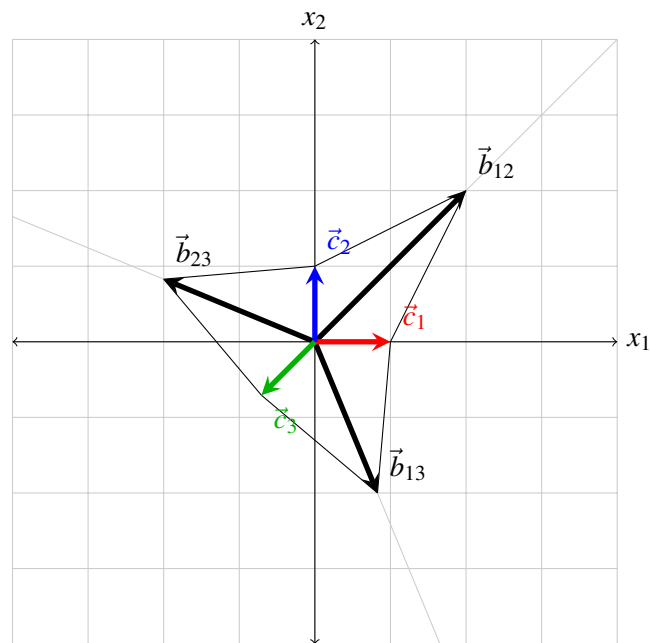
$$\langle \vec{x}, \vec{c}_3 \rangle = 3 \left(-\frac{1}{\sqrt{2}} \right) + 1 \left(-\frac{1}{\sqrt{2}} \right) = -2\sqrt{2}$$

Since $\langle \vec{x}, \vec{c}_1 \rangle = 3 > 1 = \langle \vec{x}, \vec{c}_2 \rangle$ and $\langle \vec{x}, \vec{c}_1 \rangle = 3 > -2\sqrt{2} = \langle \vec{x}, \vec{c}_3 \rangle$, our class is: $\arg \max_{\vec{c} \in \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}} \langle \vec{x}, \vec{c} \rangle = \vec{c}_1$

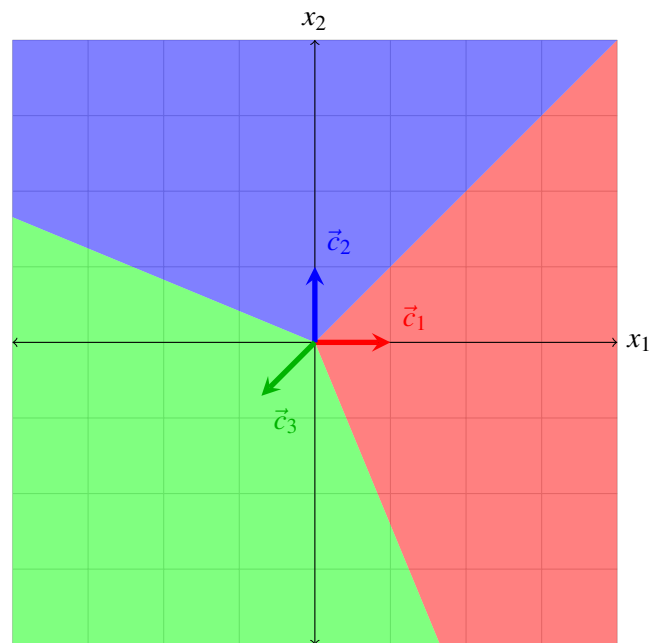
- (b) Let's now try to find all sets of $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that belong to the different classes represented by different \vec{c}_i 's. For example, for \vec{c}_1 , this would be the set of all \vec{x} that simultaneously satisfy $\langle \vec{x}, \vec{c}_1 \rangle > \langle \vec{x}, \vec{c}_2 \rangle$ and $\langle \vec{x}, \vec{c}_1 \rangle > \langle \vec{x}, \vec{c}_3 \rangle$. While the algebra can be worked out as in part (a), we will leverage the geometric definition where distance is minimized. Plot the three vectors \vec{c}_1 , \vec{c}_2 , and \vec{c}_3 on the same plane. Then, to find the boundaries between each region, consider vectors which are equivalent distances from each pair of the \vec{c}_i 's. Shade differently the regions that correspond to each class.



Answer: First, we identify the boundaries of each classification region. Notice that in the graph any scaled version of the vectors \vec{b}_{ij} are equidistant from the \vec{c}_i and \vec{c}_j that are on either side.

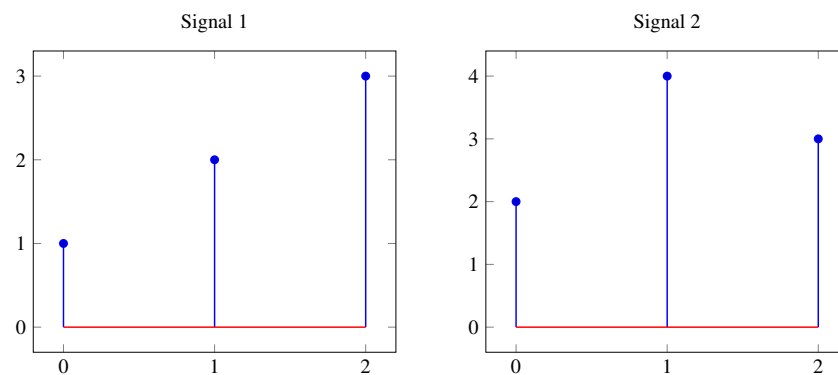


Once the boundaries have been identified, we can shade in our regions.



2. Correlation

We are given the following two signals, $s_1[n]$ and $s_2[n]$ respectively.



Find the cross correlations, $\text{corr}_{s_1}(s_2)$ and $\text{corr}_{s_2}(s_1)$ for signals $s_1[n]$ and $s_2[n]$. Recall

$$\text{corr}_x(y)[k] = \sum_{i=-\infty}^{\infty} x[i]y[i-k].$$

	$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$						
\vec{s}_1	0	0	1	2	3	0	0
$\vec{s}_2[n+2]$							
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$	+	+	+	+	+	+	=

	$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$						
\vec{s}_1	0	0	1	2	3	0	0
$\vec{s}_2[n+1]$							
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$	+	+	+	+	+	+	=

\vec{s}_1	0	0	1	2	3	0	0
$\vec{s}_2[n]$							
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	+	+	+	+	+	+	=

\vec{s}_1	0	0	1	2	3	0	0
$\vec{s}_2[n-1]$							
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	+	+	+	+	+	+	=

\vec{s}_1	0	0	1	2	3	0	0
$\vec{s}_2[n-2]$							
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	+	+	+	+	+	+	=

	$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$						
\vec{s}_2	0	0	2	4	3	0	0
$\vec{s}_1[n+2]$							
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$	+	+	+	+	+	+	=

\vec{s}_2	0	0	2	4	3	0	0
$\vec{s}_1[n+1]$							
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$	+	+	+	+	+	+	=

\vec{s}_2	0	0	2	4	3	0	0
$\vec{s}_1[n]$							
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	+	+	+	+	+	+	=

\vec{s}_2	0	0	2	4	3	0	0
$\vec{s}_1[n-1]$							
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	+	+	+	+	+	+	=

\vec{s}_2	0	0	2	4	3	0	0
$\vec{s}_1[n-2]$							
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	+	+	+	+	+	+	=

Answer: The linear cross-correlation is calculated by shifting the second signal both forward and backward until there is no overlap between the signals. When there is no overlap, the cross-correlation goes to zero. Both of these cross-correlations should have only zeros outside the range: $-2 \leq n \leq 2$.

$$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$$

\vec{s}_1	0	0	1	2	3	0	0							
$\vec{s}_2[n+2]$	2	4	3	0	0	0	0							
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$	0	+	0	+	3	+	0	+	0	+	0	+	0	=3

\vec{s}_1	0	0	1	2	3	0	0							
$\vec{s}_2[n+1]$	0	2	4	3	0	0	0							
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$	0	+	0	+	4	+	6	+	0	+	0	+	0	= 10

\vec{s}_1	0	0	1	2	3	0	0							
$\vec{s}_2[n]$	0	0	2	4	3	0	0							
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	0	+	0	+	2	+	8	+	9	+	0	+	0	= 19

\vec{s}_1	0	0	1	2	3	0	0							
$\vec{s}_2[n-1]$	0	0	0	2	4	3	0							
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	0	+	0	+	0	+	4	+	12	+	0	+	0	= 16

\vec{s}_1	0	0	1	2	3	0	0
$\vec{s}_2[n-2]$	0	0	0	0	2	4	3
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0
	0		0		6		0
							0
							= 6

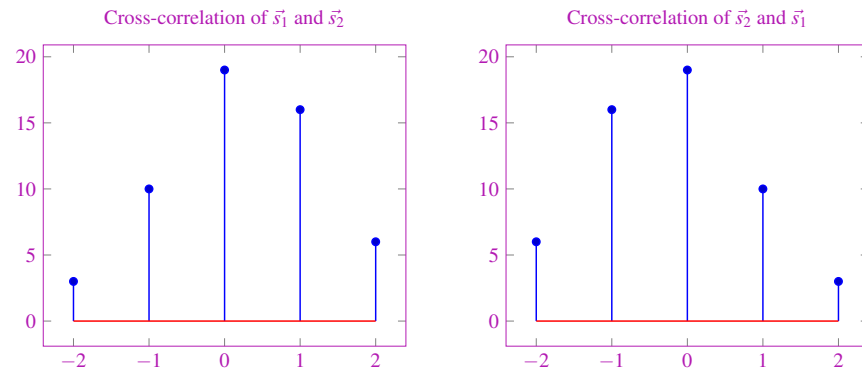
	$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$													
$\vec{s}_2[n]$	0	0	2	4	3	0	0							
$\vec{s}_1[n+2]$	1	2	3	0	0	0	0							
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$	0	+	0	+	6	+	0	+	0	+	0	+	0	= 6

$\vec{s}_2[n]$	0	0	2	4	3	0	0							
$\vec{s}_1[n+1]$	0	1	2	3	0	0	0							
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$	0	+	0	+	4	+	12	+	0	+	0	+	0	= 16

$\vec{s}_2[n]$	0	0	2	4	3	0	0							
$\vec{s}_1[n]$	0	0	1	2	3	0	0							
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	0	+	0	+	2	+	8	+	9	+	0	+	0	= 19

$\vec{s}_2[n]$	0	0	2	4	3	0	0							
$\vec{s}_2[n-1]$	0	0	0	1	2	3	0							
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	0	+	0	+	0	+	4	+	6	+	0	+	0	= 10

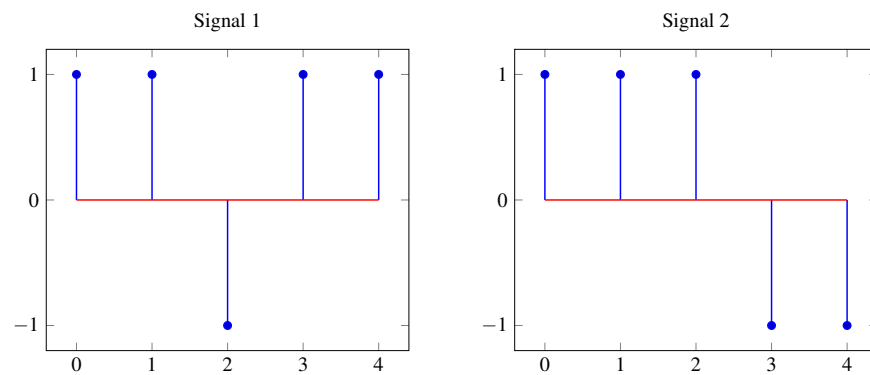
$\vec{s}_2[n]$	0	0	2	4	3	0	0							
$\vec{s}_2[n-2]$	0	0	0	0	1	2	3							
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	0	+	0	+	0	+	0	+	3	+	0	+	0	= 3



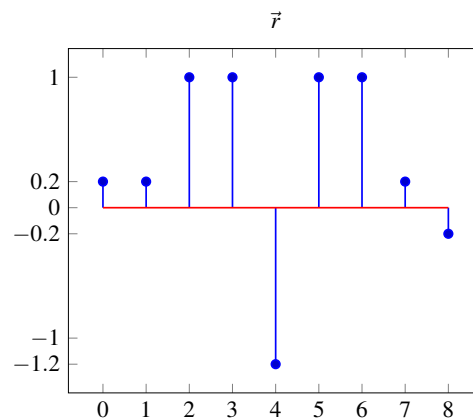
Notice that $\text{corr}_{\vec{s}_1}(\vec{s}_2)[k] = \text{corr}_{\vec{s}_2}(\vec{s}_1)[-k]$, i.e. changing the order of the signals reverses the cross-correlation sequence.

3. Identifying satellites and their delays

We are given the following two signals, \vec{s}_1 and \vec{s}_2 respectively, that are signatures for two satellites.



- (a) Your cellphone antenna receives the following signal $r[n]$. You know that there may be some noise present in $r[n]$ in addition to the transmission from the satellite.



Which satellites are transmitting? What is the delay between the satellite and your cellphone? Use cross-correlation to justify your answer.

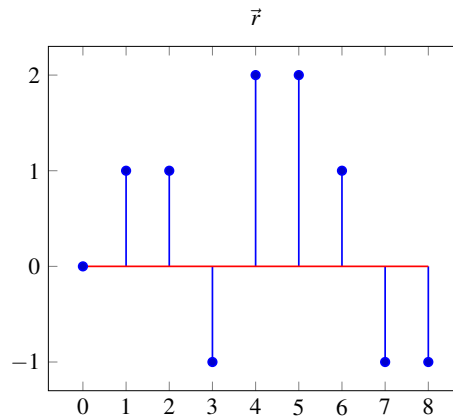
Answer: We calculate both $\text{corr}_r(\vec{s}_1)[k]$ and $\text{corr}_r(\vec{s}_2)[k]$:

$$\begin{aligned}
 \text{corr}_{\vec{r}}(\vec{s}_1)[-4] &= (0.2)(1) = 0.2 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[-3] &= (0.2)(1) + (0.2)(1) = 0.4 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[-2] &= (1)(1) + (0.2)(1) + (0.2)(-1) = 1 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[-1] &= (1)(1) + (1)(1) + (0.2)(-1) + (0.2)(1) = 2 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[0] &= (-1.2)(1) + (1)(1) + (1)(-1) + (0.2)(1) + (0.2)(1) = -0.8 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[1] &= (1)(1) + (-1.2)(1) + (1)(-1) + (1)(1) + (0.2)(1) = 0 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[2] &= (1)(1) + (1)(1) + (-1.2)(-1) + (1)(1) + (1)(1) = \mathbf{5.2} \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[3] &= (0.2)(1) + (1)(1) + (1)(-1) + (-1.2)(1) + (1)(1) = 0 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[4] &= (-0.2)(1) + (0.2)(1) + (1)(-1) + (1)(1) + (-1.2)(1) = -1.2 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[5] &= (-0.2)(1) + (0.2)(-1) + (1)(1) + (1)(1) = 1.6 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[6] &= (-0.2)(-1) + (0.2)(1) + (1)(1) = 1.4 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[7] &= (-0.2)(1) + (0.2)(1) = 0 \\
 \text{corr}_{\vec{r}}(\vec{s}_1)[8] &= (-0.2)(1) = -0.2
 \end{aligned}$$

$$\begin{aligned}
 \text{corr}_{\vec{r}}(\vec{s}_2)[-4] &= (0.2)(-1) = -0.2 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[-3] &= (0.2)(-1) + (0.2)(-1) = -0.4 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[-2] &= (1)(-1) + (0.2)(-1) + (0.2)(1) = -1 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[-1] &= (1)(-1) + (1)(-1) + (0.2)(1) + (0.2)(1) = -1.6 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[0] &= (-1.2)(-1) + (1)(-1) + (1)(1) + (0.2)(1) + (0.2)(1) = 1.6 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[1] &= (1)(-1) + (-1.2)(-1) + (1)(1) + (1)(1) + (0.2)(1) = 2.4 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[2] &= (1)(-1) + (1)(-1) + (-1.2)(1) + (1)(1) + (1)(1) = -1.2 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[3] &= (0.2)(-1) + (1)(-1) + (1)(1) + (-1.2)(1) + (1)(1) = -0.4 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[4] &= (-0.2)(-1) + (0.2)(-1) + (1)(1) + (1)(1) + (-1.2)(1) = 0.8 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[5] &= (-0.2)(-1) + (0.2)(1) + (1)(1) + (1)(1) = 2.4 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[6] &= (-0.2)(1) + (0.2)(1) + (1)(1) = 1 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[7] &= (-0.2)(1) + (0.2)(1) = 0 \\
 \text{corr}_{\vec{r}}(\vec{s}_2)[8] &= (-0.2)(1) = -0.2
 \end{aligned}$$

The maximum correlation value is 5.2 at $k = 2$. Since we have a plus-minus 1 signal of length 5, this high correlation likely comes from the satellite 1 transmission.

- (b) Now your cellphone receives a new signal $r[n]$ as below. What the satellites that are transmitting and what is the delay between each satellite and your cellphone?



Answer: We want to find shifts k_1 and k_2 such that: $\vec{r}[n] = \vec{s}_1[n - k_1] + \vec{s}_2[n - k_2]$.

We calculate both $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ and $\text{corr}_{\vec{r}}(\vec{s}_2)[k]$ for different shifts k . The index where the maximum correlation value is achieved will tell us the shift indices (delays).

$$\text{corr}_{\vec{r}}(\vec{s}_1)[-3] = (1)(1) = 1$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[-2] = (1)(1) + (1)(1) = 2$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[-1] = (-1)(1) + (1)(1) + (1)(-1) = -1$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[0] = (2)(1) + (-1)(1) + (1)(-1) + (1)(1) = 1$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[1] = (2)(1) + (2)(1) + (-1)(-1) + (1)(1) + (1)(1) = 7$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[2] = (1)(1) + (2)(1) + (2)(-1) + (-1)(1) + (1)(1) = 1$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[3] = (-1)(1) + (1)(1) + (2)(-1) + (2)(1) + (-1)(1) = -1$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[4] = (-1)(1) + (-1)(1) + (1)(-1) + (2)(1) + (2)(1) = 1$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[5] = (-1)(1) + (-1)(-1) + (1)(1) + (2)(1) = 3$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[6] = (-1)(-1) + (-1)(1) + (1)(1) = 1$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[7] = (-1)(1) + (-1)(1) = -2$$

$$\text{corr}_{\vec{r}}(\vec{s}_1)[8] = (-1)(1) = -1$$

$$\begin{aligned}
\text{corr}_{\vec{r}}(\vec{s}_2)[-3] &= (1)(-1) = -1 \\
\text{corr}_{\vec{r}}(\vec{s}_2)[-2] &= (1)(-1) + (1)(-1) = -2 \\
\text{corr}_{\vec{r}}(\vec{s}_2)[-1] &= (-1)(-1) + (1)(-1) + (1)(1) = 1 \\
\text{corr}_{\vec{r}}(\vec{s}_2)[0] &= (2)(-1) + (-1)(-1) + (1)(1) + (1)(1) = 1 \\
\text{corr}_{\vec{r}}(\vec{s}_2)[1] &= (2)(-1) + (2)(-1) + (-1)(1) + (1)(1) + (1)(1) = -3 \\
\text{corr}_{\vec{r}}(\vec{s}_2)[2] &= (1)(-1) + (2)(-1) + (2)(1) + (-1)(1) + (1)(1) = -1 \\
\text{corr}_{\vec{r}}(\vec{s}_2)[3] &= (-1)(-1) + (1)(-1) + (2)(1) + (2)(1) + (-1)(1) = 3 \\
\text{corr}_{\vec{r}}(\vec{s}_2)[4] &= (-1)(-1) + (-1)(-1) + (1)(1) + (2)(1) + (2)(1) = \mathbf{7} \\
\text{corr}_{\vec{r}}(\vec{s}_2)[5] &= (-1)(-1) + (-1)(1) + (1)(1) + (2)(1) = 3 \\
\text{corr}_{\vec{r}}(\vec{s}_2)[6] &= (-1)(1) + (-1)(1) + (1)(1) = -1 \\
\text{corr}_{\vec{r}}(\vec{s}_2)[7] &= (-1)(1) + (-1)(1) = -2 \\
\text{corr}_{\vec{r}}(\vec{s}_2)[8] &= (-1)(1) = -1
\end{aligned}$$

The maximum correlation between signals \vec{r} and \vec{s}_1 was achieved at $k_1 = 1$, and the maximum correlation between signals \vec{r} and \vec{s}_2 was achieved at $k_2 = 4$.