EECS 16A Designing Information Devices and Systems I
Spring 2020 Homework 11

This homework is due April 17, 2020, at 23:59.
Self-grades are due April 20, 2020, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw11.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

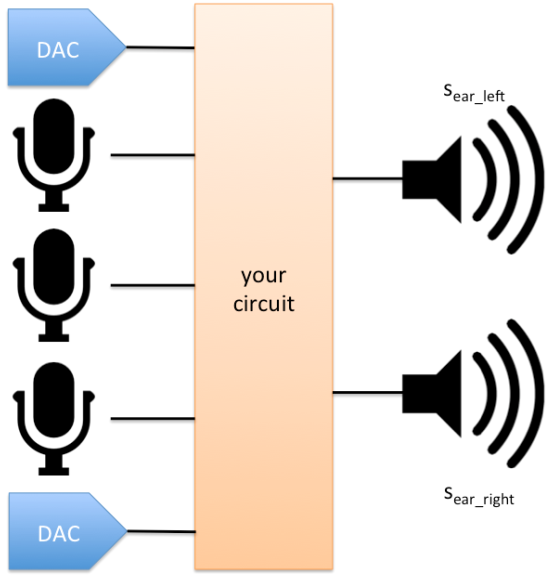
1. Noise-Cancelling Headphones Part 2

The basic goal of noise-cancelling headphones is for the user to hear only the desired audio signal and not any other sounds from external sources. In order to achieve this goal, noise-cancelling headphones include at least one microphone that listens to what you might have otherwise heard from external sources and then feeds a signal in to your speakers that cancels (subtracts out) that externally-generated sound.

Solution:

There are a lot of different solutions for this problem. This solution is aggressive and minimal, so be patient with your understanding. If your solution solves the same problem, you will receive credit.

(a) In discussion, we had just one speaker and one microphone, but almost all headphones today have two speakers (one for each ear). Adding an extra speaker that can be driven by a separate audio stream typically makes things sound more real to us. For similar reasons, having multiple microphones to pick up ambient sounds from multiple different locations can help us do a better job of cancellation if we can use that information in the right way.



Let's now assume that our system has 3 microphones and 2 speakers and that the source of our audio is stereo – i.e., we have two different audio streams s_{left} and s_{right} (produced by two different DACs) that represent the ideal sounds we would like the user to hear in their left and right ear. We have three microphone audio signals s_{mic1}, s_{mic2}, and s_{mic3}, and let's assume that without any active noise cancellation, some fraction of the signal picked up by each microphone would be heard by the user in each of their ears. For example, $a_{1\text{left}}$ would represent the fraction of the signal picked up by microphone 1 that will be heard in the user's left ear, $a_{2\text{right}}$ would represent the fraction of the signal picked up by microphone 2 that will be in the user's right ear, etc.

Given that

$$\vec{s}_{\text{noise}} = \begin{bmatrix} s_{\text{noise\_left}} \\ s_{\text{noise\_right}} \end{bmatrix}$$

$$\vec{s}_{\text{mic}} = \begin{bmatrix} s_{\text{mic1}} \\ s_{\text{mic2}} \\ s_{\text{mic3}} \end{bmatrix}$$

where $\vec{s}_{\text{noise}}$ represents the noise heard in each ear and $\vec{s}_{\text{mic}}$ represents the sound in each mic, find a matrix $\mathbf{A}$ such that $\vec{s}_{\text{noise}} = \mathbf{A}\vec{s}_{\text{mic}}$.

Solution:

$$\begin{bmatrix} s_{\text{noise\_left}} \\ s_{\text{noise\_right}} \end{bmatrix} = \begin{bmatrix} a_{1\text{left}} & a_{2\text{left}} & a_{3\text{left}} \\ a_{1\text{right}} & a_{2\text{right}} & a_{3\text{right}} \end{bmatrix} \begin{bmatrix} s_{\text{mic1}} \\ s_{\text{mic2}} \\ s_{\text{mic3}} \end{bmatrix}$$

(b) Assuming no noise cancellation, find an equation for $\vec{s}_{\text{ear}}$, the sound heard in each ear in terms of the two audio streams and $\vec{s}_{\text{noise}}$.

$$\vec{s}_{\text{ear}} = \begin{bmatrix} s_{\text{ear\_left}} \\ s_{\text{ear\_right}} \end{bmatrix}$$

Solution:

We can represent this as the matrix multiplication and addition below.

$$\begin{bmatrix} s_{\text{ear\_left}} \\ s_{\text{ear\_right}} \end{bmatrix} = \begin{bmatrix} a_{1\text{left}} & a_{2\text{left}} & a_{3\text{left}} \\ a_{1\text{right}} & a_{2\text{right}} & a_{3\text{right}} \end{bmatrix} \begin{bmatrix} s_{\text{mic1}} \\ s_{\text{mic2}} \\ s_{\text{mic3}} \end{bmatrix} + \begin{bmatrix} s_{\text{left}} \\ s_{\text{right}} \end{bmatrix}$$

(c) In order to cancel the noise, we want to create a signal that is the inverse of $\vec{s}_{\text{noise}}$. Let $\vec{s}_{\text{cancel}}$ be the vector representing the cancel signal in each headphone. Find a matrix $\mathbf{B}$ in terms of the matrix $\mathbf{A}$ such that $\vec{s}_{\text{cancel}} = \mathbf{B}\vec{s}_{\text{mic}}$.

Solution:

For the setup:

$$\begin{bmatrix} s_{\text{ear\_left}} \\ s_{\text{ear\_right}} \end{bmatrix} = \mathbf{A} \begin{bmatrix} s_{\text{mic1}} \\ s_{\text{mic2}} \\ s_{\text{mic3}} \end{bmatrix} + \mathbf{B} \begin{bmatrix} s_{\text{mic1}} \\ s_{\text{mic2}} \\ s_{\text{mic3}} \end{bmatrix} + \begin{bmatrix} s_{\text{left}} \\ s_{\text{right}} \end{bmatrix}$$

We want $\mathbf{B} = -\mathbf{A}$.

(d) Assume that the microphones can be modeled as voltage sources, whose value v_{micn} is proportional to s_{micn}, design and sketch a circuit that would implement the cancellation matrix $\mathbf{B}$. You should assume that this circuit has three voltage inputs v_{mic1}, v_{mic2}, and v_{mic3} and two voltage outputs $v_{\text{cancel\_left}}$ and $v_{\text{cancel\_right}}$ (corresponding to the voltages that will be subtracted from the desired audio streams in order to cancel the externally-produced sounds). In order to simplify the problem, you can assume that all of the v_{mic} voltages are already centered at 0 V (relative to the DAC ground). Furthermore, assume all entries of the $\mathbf{A}$ matrix are positive. You may use op-amps and resistors to implement your circuit. You do not have to pick specific resistor values, but write expressions for each resistor value.

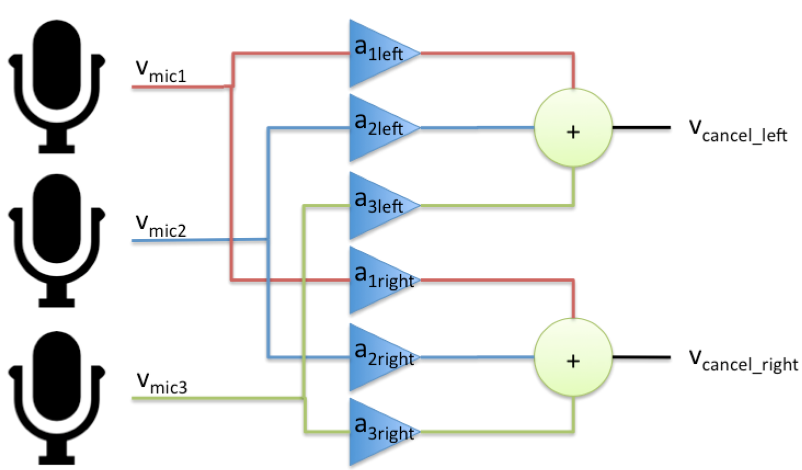
Solution:

Since we want to add $v_{\text{cancel\_left}}$ and $v_{\text{cancel\_right}}$ with the audio stream output to cancel noise, so we want these values to be

$$v_{\text{cancel\_left}} = -(a_{1\text{left}}\, v_{\text{mic1}} + a_{2\text{left}}\, v_{\text{mic2}} + a_{3\text{left}}\, v_{\text{mic3}})$$

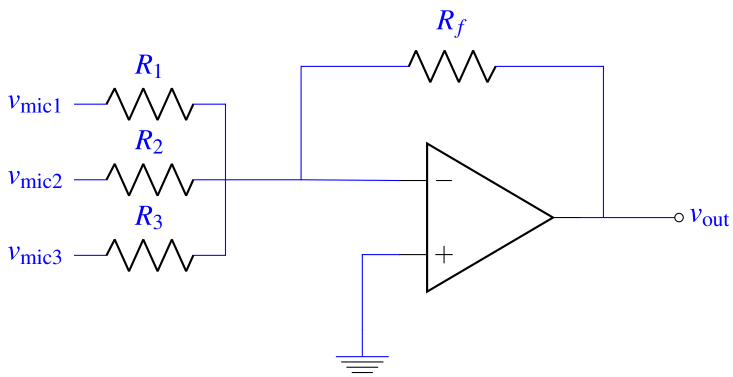
$$v_{\text{cancel\_right}} = -(a_{1\text{right}}\, v_{\text{mic1}} + a_{2\text{right}}\, v_{\text{mic2}} + a_{3\text{right}}\, v_{\text{mic3}})$$

Following the design process, we can draw a block diagram for the circuit.



We can see here that the two channels are actually independent from each other. The only point where they meet is at the microphone voltage. Thus, we can start by designing one channel.

For each channel, we want to build a circuit that sums its inputs and negates them. We have many options to pick from; the easiest will be an inverting summer.



This topology gives us

$$V_{\text{out}} = -\left(\frac{R_f}{R_1}V_{\text{mic1}} + \frac{R_f}{R_2}V_{\text{mic2}} + \frac{R_f}{R_3}V_{\text{mic3}}\right)$$

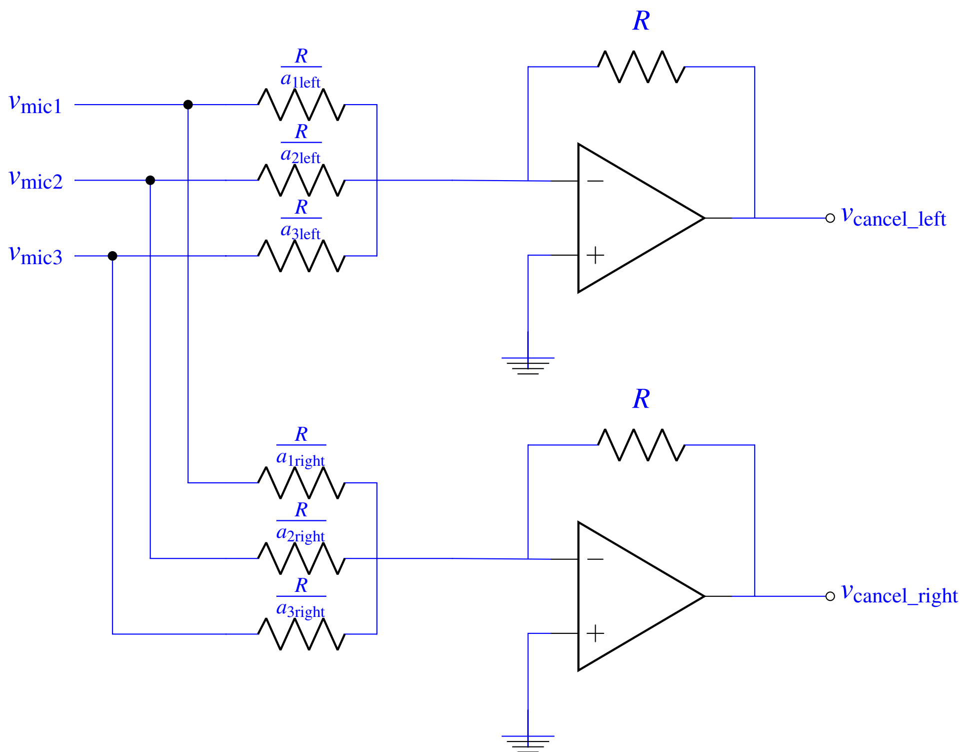
For the correct gains, we want

$$a_1 = \frac{R_f}{R_1}, a_2 = \frac{R_f}{R_2}, a_3 = \frac{R_f}{R_3}$$

We can pick R_f arbitrarily. Then we set

$$R_1 = \frac{R_f}{a_1}, R_2 = \frac{R_f}{a_2}, R_3 = \frac{R_f}{a_3}$$

Now that we have all the building blocks we need, we can construct the two-channel noise-cancelling circuit. v_{micn} is connected to the output of the microphone buffers. We can choose an arbitrary value for R, for example, $1\,\text{k}\Omega$.

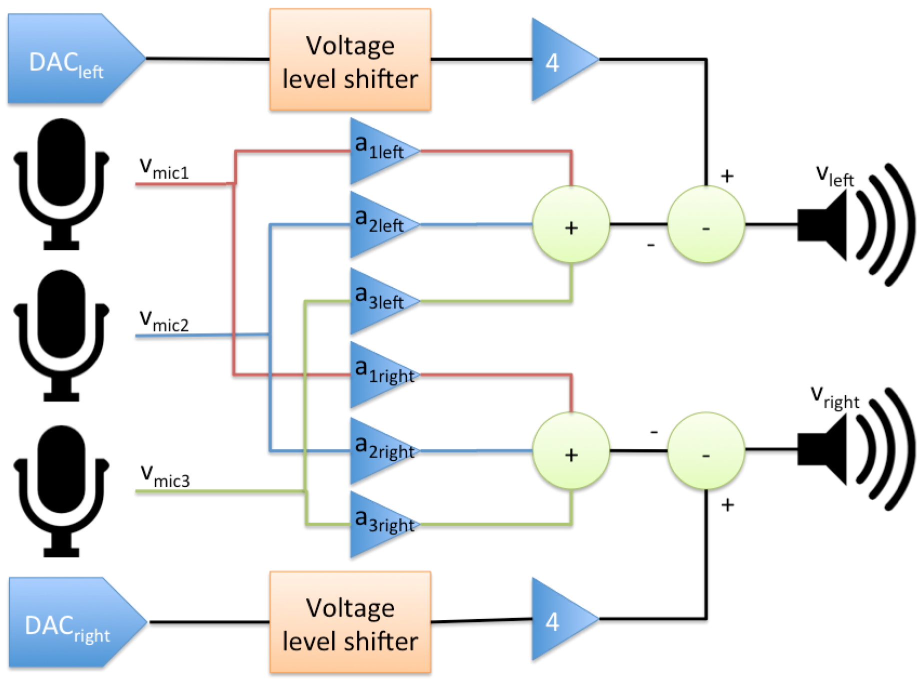


We just pick R to be some value, we then calculate the appropriate resistances in the rest of the circuit, and we are good to go!

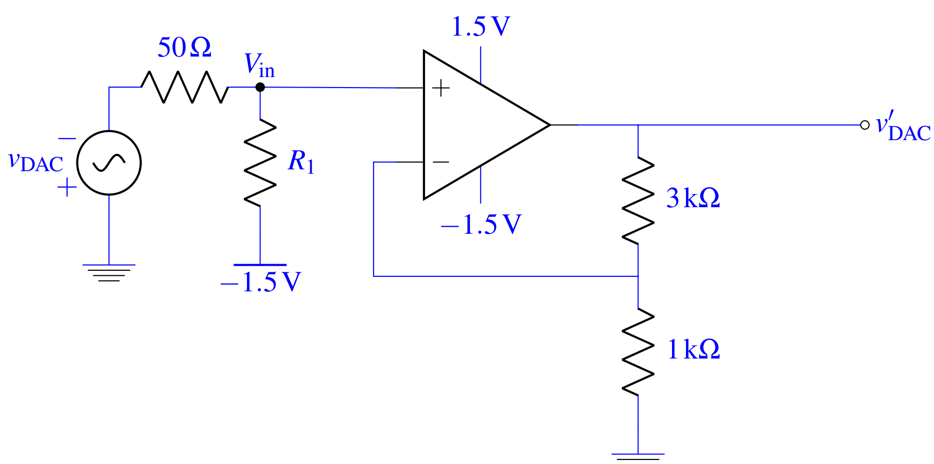
(e) **PRACTICE:** Building upon your solutions to all previous parts, and otherwise making the same assumptions about the relative voltage ranges of v_{mic1}, v_{mic2}, and v_{mic3} and available supply voltages, sketch the complete circuit you would use to create the stereo audio on the two speakers while cancelling the noise picked up by the three microphones.

Solution:

We already have a circuit that does subtraction and a circuit that computes the noise-cancelling signal. We just have to combine the two circuits such that it implements the matrix $\mathbf{B}$.



Recall our circuit from last discussion for voltage shifting and changing the range of a DAC:

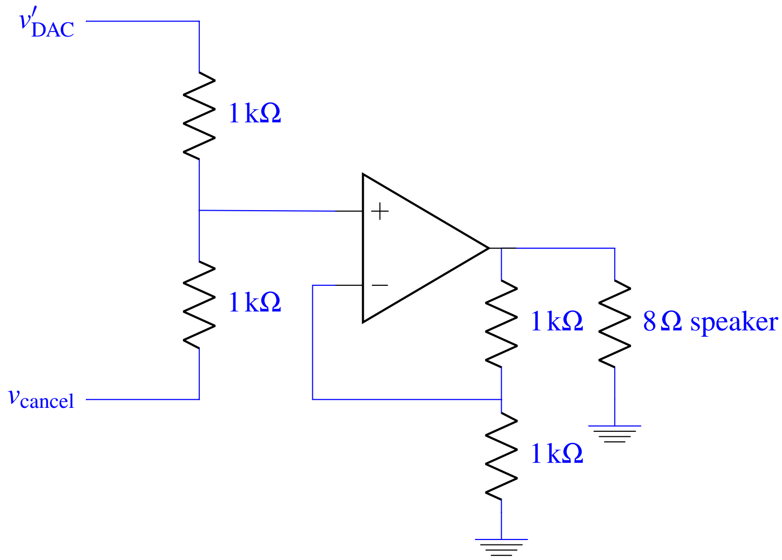


Now we want

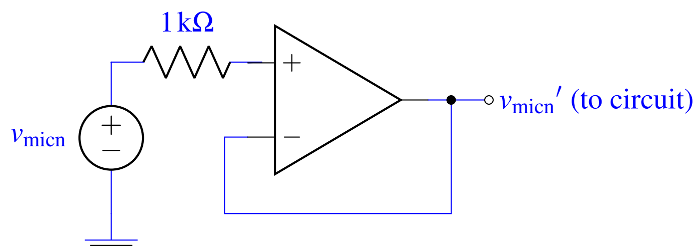
$$v_{\text{left}} = v'_{\text{DAC, left}} + v_{\text{cancel\_left}}$$

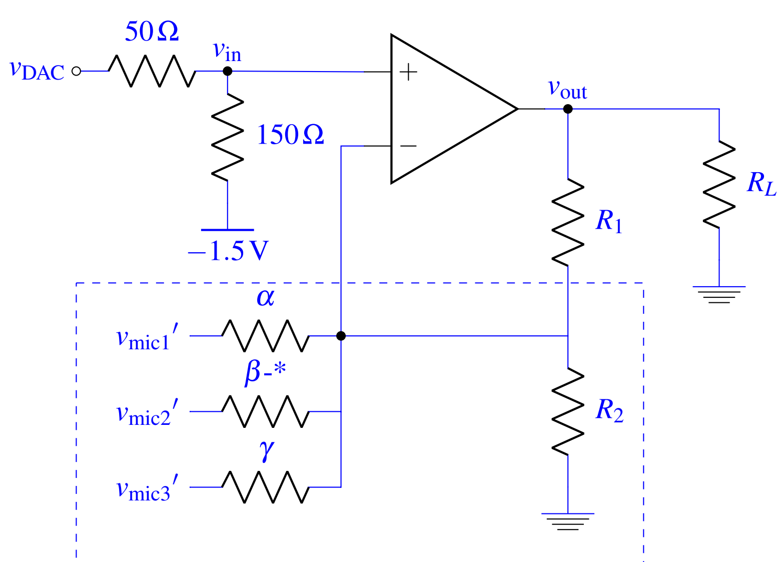
$$v_{\text{right}} = v'_{\text{DAC, right}} + v_{\text{cancel\_right}}$$

Note that v_{cancel} already inverted the microphone signal so we are adding it to v_{DAC}. To sum, we can use a noninverting summer. Again, we consider the channels independently.



Below is an alternative circuit that does the same with one op-amp. The approach here is to start with by attenuating the mic voltages and summing them with a voltage divider and solving for the correct resistances. We then feed the the microphone voltage into the negative terminal and the DAC signal into the positive terminal of an op-amp to subtract the signals.





Recall that the v_{in} range is $-0.375\,\text{V}$ to $0.375\,\text{V}$, and it has to be amplified 4 times. We can write the

KCL equation in the inverting input of the op-amp.

$$\frac{v_{\text{mic1}} - v_{\text{in}}}{\alpha} + \frac{v_{\text{mic2}} - v_{\text{in}}}{\beta} + \frac{v_{\text{mic3}} - v_{\text{in}}}{\gamma} + \frac{v_{\text{out}} - v_{\text{in}}}{R_1} + \frac{0 - v_{\text{in}}}{R_2} = 0$$

$$\frac{v_{\text{out}}}{R_1} = v_{\text{in}}\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{v_{\text{mic1}}}{\alpha} - \frac{v_{\text{mic2}}}{\beta} - \frac{v_{\text{mic3}}}{\gamma}$$

$$v_{\text{out}} = v_{\text{in}}\left(\frac{R_1}{\alpha} + \frac{R_1}{\beta} + \frac{R_1}{\gamma} + 1 + \frac{R_1}{R_2}\right) - \frac{R_1}{\alpha}v_{\text{mic1}} - \frac{R_1}{\beta}v_{\text{mic2}} - \frac{R_1}{\gamma}v_{\text{mic3}}$$

Just as before, we can compare this formula to the output we want. In this case, we want $v_{\text{out}} = 4v_{\text{in}} - a_1\, v_{\text{mic1}} - a_2\, v_{\text{mic2}} - a_3\, v_{\text{mic3}}$. Thus,

$$\frac{R_1}{\alpha} + \frac{R_1}{\beta} + \frac{R_1}{\gamma} + 1 + \frac{R_1}{R_2} = 4 \qquad \frac{R_1}{\alpha} = a_1 \qquad \frac{R_1}{\beta} = a_2 \qquad \frac{R_1}{\gamma} = a_3$$

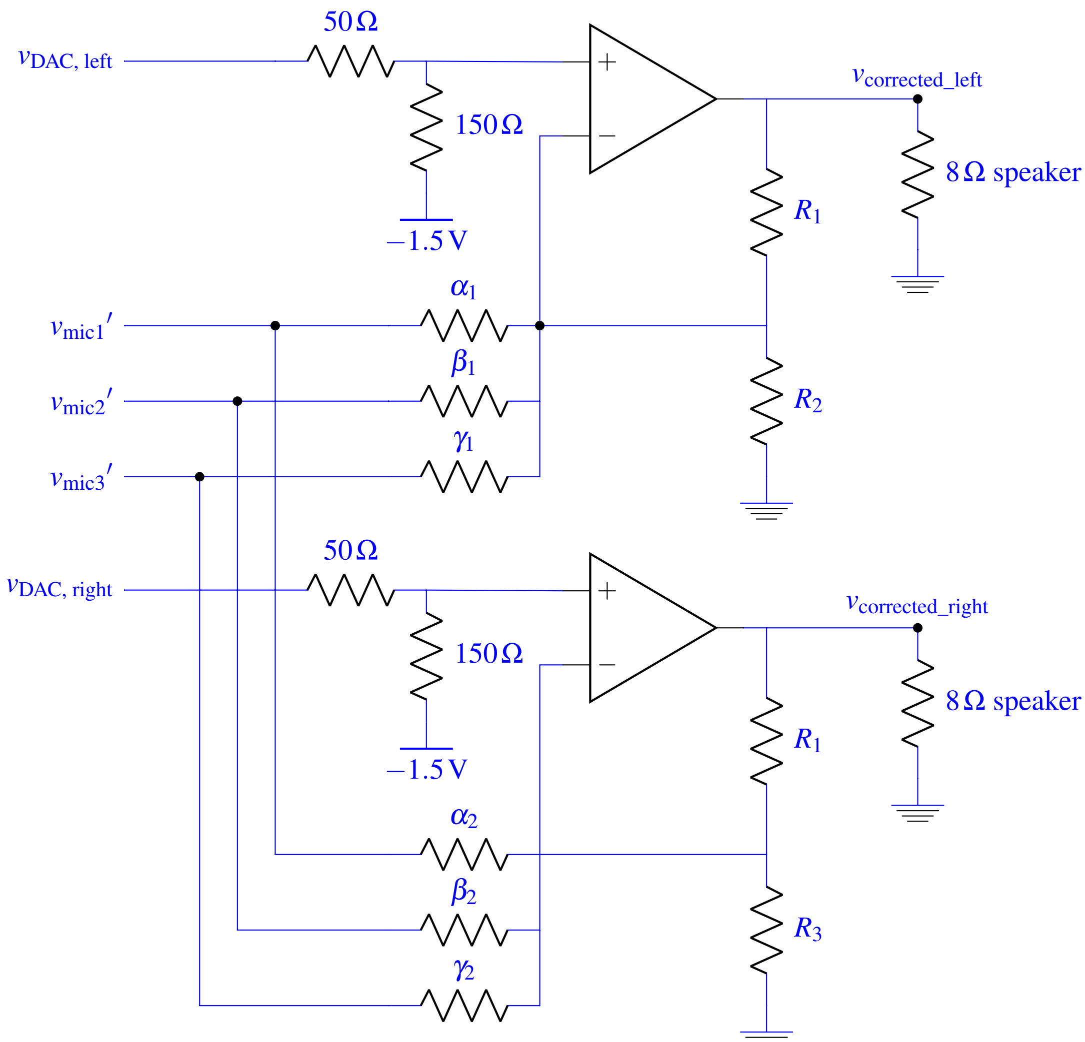
$$\alpha = \frac{R_1}{a_1} \qquad \beta = \frac{R_1}{a_2} \qquad \gamma = \frac{R_1}{a_3}$$

From the first equation,

$$a_1 + a_2 + a_3 + 1 + \frac{R_1}{R_2} = 4$$

$$R_2 = \frac{R_1}{3 - a_1 - a_2 - a_3}$$

Thus, if we pick a value for R_1, we can use the formulas above to calculate α, β, γ and R_2.
Now that we have a working circuit for one speaker, we can duplicate this circuit to have two speakers. Notice that in the circuit below we can use the same value for R_1 in the two channels, but we have to keep R_2 as a variable (hence it is replaced with R_3 in the right channel). This is because R_1 is a free variable. If we choose a value for R_1 arbitrarily, we can calculate what the other resistor values have to be with the equations we have derived.



We have seen that if we choose values for R_1 and R_3 arbitrarily, we can find the other resistor values.

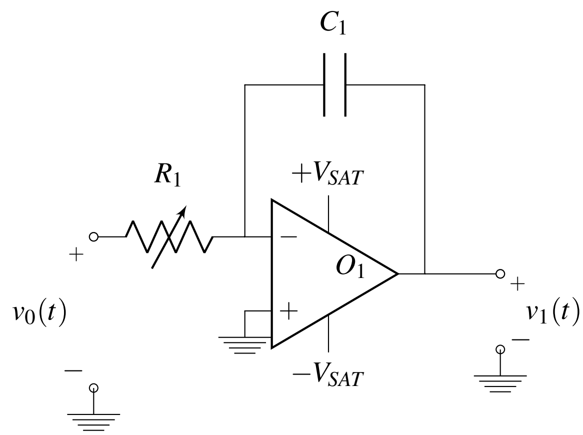
$$\alpha_1 = \frac{R_1}{a_{1\text{left}}} \qquad \beta_1 = \frac{R_1}{a_{2\text{left}}} \qquad \gamma_1 = \frac{R_1}{a_{3\text{left}}} \qquad R_2 = \frac{R_1}{3 - a_{1\text{left}} - a_{2\text{left}} - a_{3\text{left}}}$$

$$\alpha_2 = \frac{R_1}{a_{1\text{right}}} \qquad \beta_2 = \frac{R_1}{a_{2\text{right}}} \qquad \gamma_2 = \frac{R_1}{a_{3\text{right}}} \qquad R_3 = \frac{R_1}{3 - a_{1\text{right}} - a_{2\text{right}} - a_{3\text{right}}}$$

2. Jumpbot

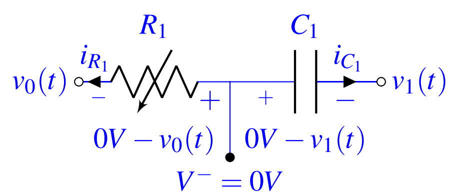
In this problem, you will be designing circuits allowing a robot named Jumpbot to execute a set of commands that will be described below. Specifically, the output voltages produced by your circuits are interpreted by Jumpbot as setting its vertical position in meters in free space (both positive and negative values will be used). You will be generating an oscillating triangular waveform with a controllable time period.

(a) One of the circuit blocks you will use to generate the triangular waveform is the integrator. An integrator integrates the input signal. For the circuit given below, express $v_1(t)$ in terms of R_1, C_1, and $v_0(t)$. You may assume the capacitor C_1 has $0V$ across it at time $t = 0$.
Hint: You will have to apply KCL, and the current flowing through a capacitor is given by $I = C\frac{dV}{dt}$.



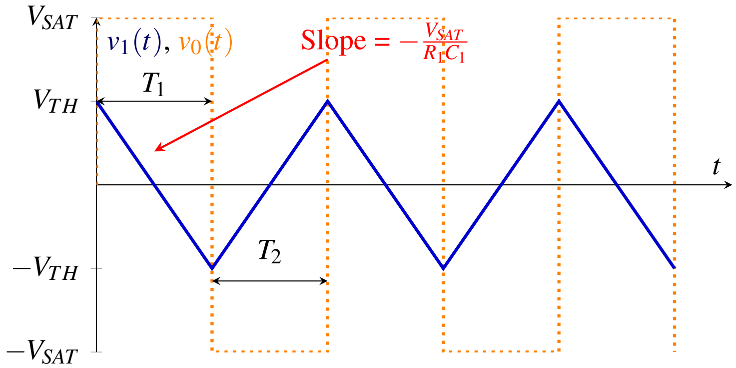
Solution:

Let's write the KCL equation at the negative terminal of the op-amp (V^-) assuming all currents are leaving. The polarity labeling will appear as follows:



$$
\begin{aligned}
i_{R_1} + i_{C_1} &= 0 \\
i_{R_1} &= -i_{C_1} \\
i_{C_1} &= C_1 \frac{d(0 - v_1(t))}{dt} \\
\frac{0 - v_0(t)}{R_1} &= C_1 \frac{d(v_1(t) - 0)}{dt} \\
-\frac{v_0(t)}{R_1 C_1} &= \frac{dv_1(t)}{dt} \\
v_1(t) - v_1(0) &= -\frac{1}{R_1 C_1} \int_0^t v_0(\tau) d\tau \\
v_1(t) &= -\frac{1}{R_1 C_1} \int_0^t v_0(\tau) d\tau
\end{aligned}
$$

(b) Suppose for a specific $v_0(t)$, shown by the dotted orange line below, $v_1(t)$ looks as shown by the blue line. Derive an expression for T_1 and T_2 as a function of R_1, C_1, V_{TH}, and V_{SAT}.



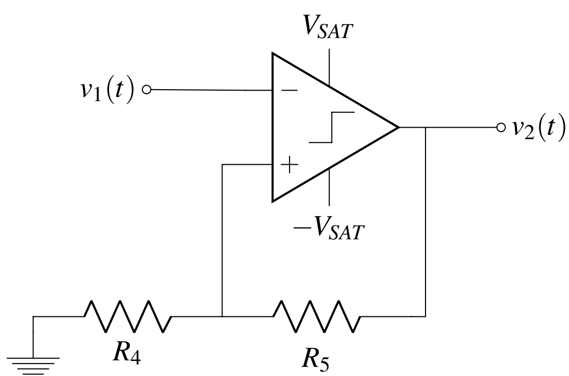
Solution: For T_1, using the slope of the waveform of $v_1(t)$:

$$\frac{-V_{TH}-V_{TH}}{T_1} = -\frac{V_{SAT}}{R_1C_1}$$
$$T_1 = R_1C_1\frac{2V_{TH}}{V_{SAT}}$$

Similarly for T_2, using again the slope of the waveform of $v_1(t)$:

$$\frac{V_{TH}-(-V_{TH})}{T_2} = \frac{V_{SAT}}{R_1C_1}$$
$$T_2 = R_1C_1\frac{2V_{TH}}{V_{SAT}}$$

(c) From part (a), we have a circuit that generates a triangle wave if we feed it the square wave, $v_0(t)$, shown in (b). However, we need to create this initial signal, $v_0(t)$, that helped us to create the triangular waveform, $v_1(t)$. The following circuit helps us achieve this.



To understand how it behaves, consider each of the following four pairs of an input waveform and initial output voltage as a distinct case. Defining $v_1(t)$ as the blue waveform plotted in (b), plot $v_2(t)$ for a few cycles of the input. In what case does $v_2(t)$ match $v_0(t)$ from part (b)?

Assume that the initial voltage of the input, $v_1(t)$, is close to, but not quite equal to, V_{TH}: $v_1(0) < V_{TH}$. This is only for the initial condition; at later times it's possible to have $v_1(t) = V_{TH}$ and $v_1(t) = -V_{TH}$. On your plot, label $V_{TH} = \frac{R_4}{R_4+R_5}V_{SAT}$ and V_{SAT} on the voltage axis. Note: When the comparator inputs are equal, the output will be zero.

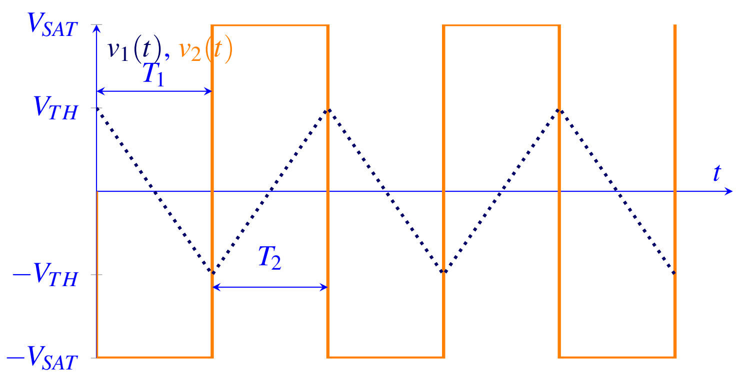
Case	Input	$v_2(0)$
1	$v_1(t)$	$-V_{SAT}$
2	$v_1(t)$	V_{SAT}
3	$-v_1(t)$	V_{SAT}
4	$-v_1(t)$	$-V_{SAT}$

Solution:

The circuit above is not in negative feedback. However, it does not act quite like an ordinary comparator either, which would change output voltage upon $V^+ - V^-$ changing signs. This is because the circuit has two different thresholds for when the output of the comparator flips due to the voltage established at the positive terminal of the comparator. The comparator is then in one of two states, either with its output at V_{SAT} or with its output at $-V_{SAT}$. Since the comparator itself is in one of two starting states and we have two different inputs, there are four cases, which we asked you to plot.

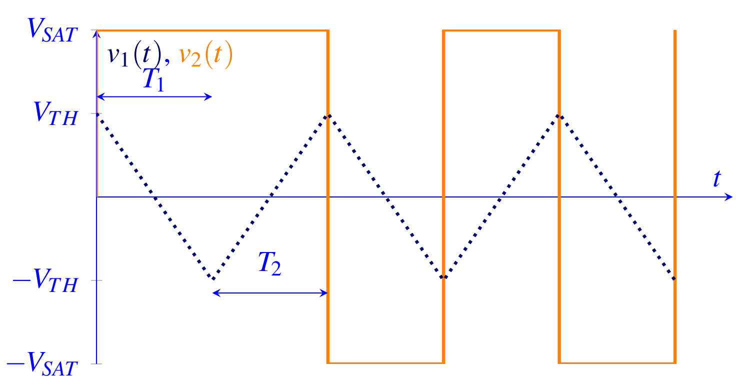
Case 1: Assume that the input is $v_1(t)$, and that at time $t = 0$, the output of the comparator is at $v_2(0) = -V_{SAT}$.

Since the output is at $-V_{SAT}$, the positive terminal voltage is $V^+ = -V_{TH}$. Eventually, $v_1(t)$ will be exactly $-V_{TH} = V^+$, at which point the output of the comparator will jump to $0V$. This establishes $V^+ = 0V$, but that results in $V^+ > V^-$, so the output will jump to $+V_{SAT}$. This is shown in the figure below.



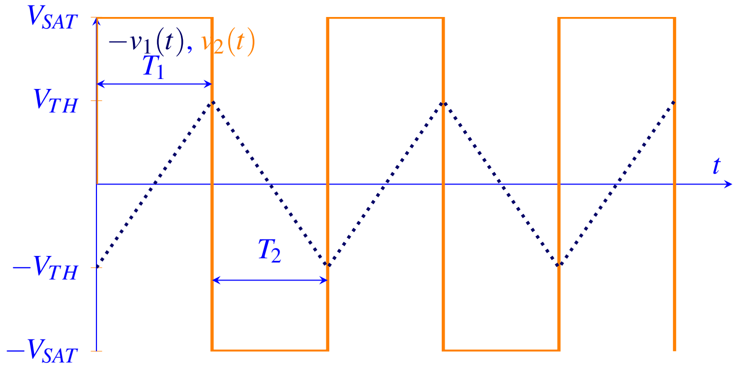
Case 2: Assume that the input is again $v_1(t)$, and that at time $t = 0$, the output of the op-amp is at $v_2(0) = V_{SAT}$.

Since the output is at V_{SAT}, we know $V^+ = V_{TH}$. Eventually, $v_1(t)$ will equal V_{TH} and the output of the op-amp will flip to $-V_{SAT}$. However, this will happen after one cycle. This is shown in the figure below.



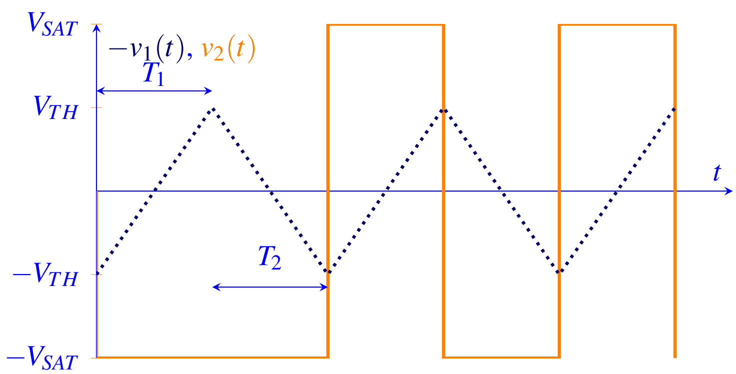
Case 3: Assume the input is $-v_1(t)$, and that at time $t = 0$, the output of the comparator is at $v_2(0) = V_{SAT}$.

Since the output is at V_{SAT}, we know $V^+ = V_{TH}$. Eventually, $v_1(t)$ will equal V_{TH} and the output of the comparator will flip to $-V_{SAT}$. This is shown in the figure below.



Case 4: Assume the input is $-v_1(t)$, and that at time $t = 0$, the output of the comparator is at $v_2(0) = -V_{SAT}$.

Since the output is at $-V_{SAT}$, we know $V^+ = -V_{TH}$. Eventually, $v_1(t)$ will equal $-V_{TH}$ and the output of the comparator will flip to V_{SAT}. Once again, this flip will happen after a cycle. This is shown in the figure below.

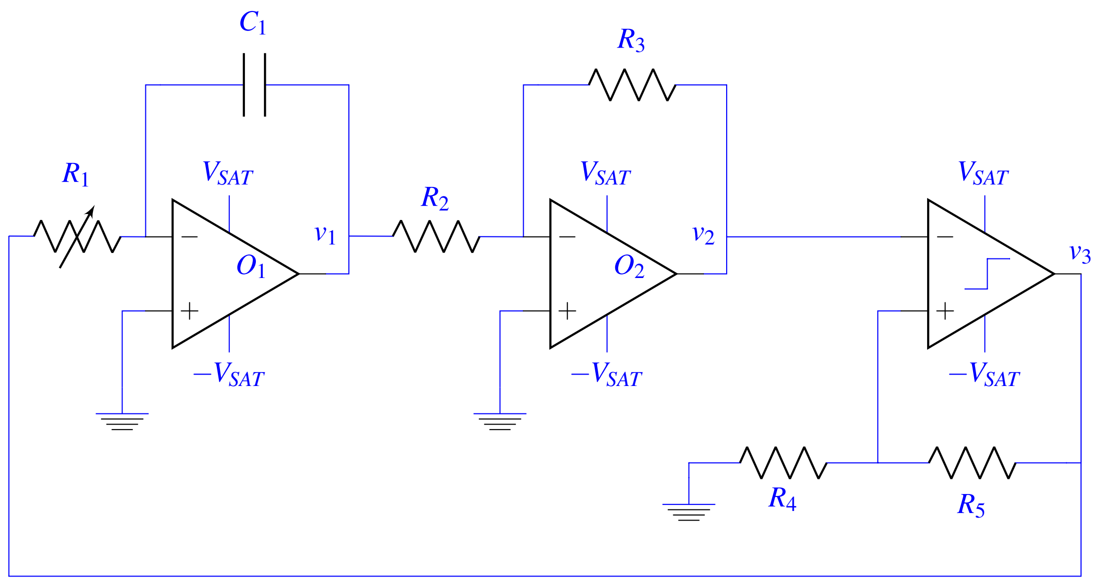


If $-v_1(t)$ is used as the input, the output matches $v_0(t)$ from part (a), if we initially hold the output of the comparator to $v_2(0) = V_{SAT}$, as shown in Case 3.

Because of the recursive nature of this circuit, we were required to analyze the circuit in all four possibilities. In the real world, people take great care to make sure to start the circuit into a "known" state, where the outputs and inputs are set so that correct operation occurs.

(d) Now let's put it all together. The circuit from part (a) generates a triangle wave, $v_1(t)$, from a square wave, $v_0(t)$. The circuit above takes an input triangle wave and creates a square wave. Connect the two circuits together so that the circuit keeps generating a triangle wave on its own. You will use the circuits from part (a) and part (c), and you may use any additional number of op-amps and resistors.

Solution:



We've added an inverting amplifier, O_2. We would like to have a gain of -1 so that $v_2(t)$ has the desired polarity seen in part (c). Thus $R_2 = R_3$.

(e) In your circuit, if $\pm V_{SAT} = \pm 10\,\text{V}$, $C_1 = 0.01\,\text{mF}$, and $R_4 = 10\,\text{k}\Omega$, find the values for R_1 and R_5, so that the jumpbot jumps with $10\,\text{V}$ peak-to-peak amplitude ($\pm V_{TH} = \pm 5\,\text{V}$) with $1\,\text{kHz}$ frequency (period = 1 / frequency).

Solution:

$10\,\text{V}$ peak-to-peak amplitude means $\pm V_{TH} = \pm 5\,\text{V}$. Therefore,

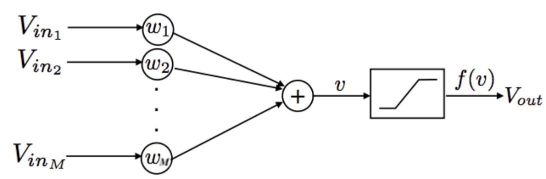
$$V_{TH} = \frac{R_4}{R_4 + R_5} V_{SAT}$$
$$5\,\text{V} = \frac{10\,\text{k}\Omega}{10\,\text{k}\Omega + R_5} 10\,\text{V}$$
$$R_5 = 10\,\text{k}\Omega$$

To find R_1, we can use the relationship derived in part (b). However, here $T_1 = 0.5\,\text{ms}$ because $T_1 + T_2 = T = 1\,\text{ms}$ (oscillation frequency $1\,\text{kHz}$).

$$T_1 = R_1 C_1 \frac{2V_{TH}}{V_{SAT}}$$
$$0.5\,\text{ms} = R_1 \cdot 0.01\,\text{mF} \cdot \frac{2 \cdot 5\,\text{V}}{10\,\text{V}}$$
$$R_1 = 50\,\Omega$$

3. PRACTICE: Brain-on-a-Chip with 16A Neurons

Neurelic, Inc. is a hot new startup building chips that emulate brain functions like associative memory. As an intern fresh out of 16A, you've been assigned to implement the neural network circuits on this chip. The neural network consists of neurons, each of which can be represented by the block diagram below.

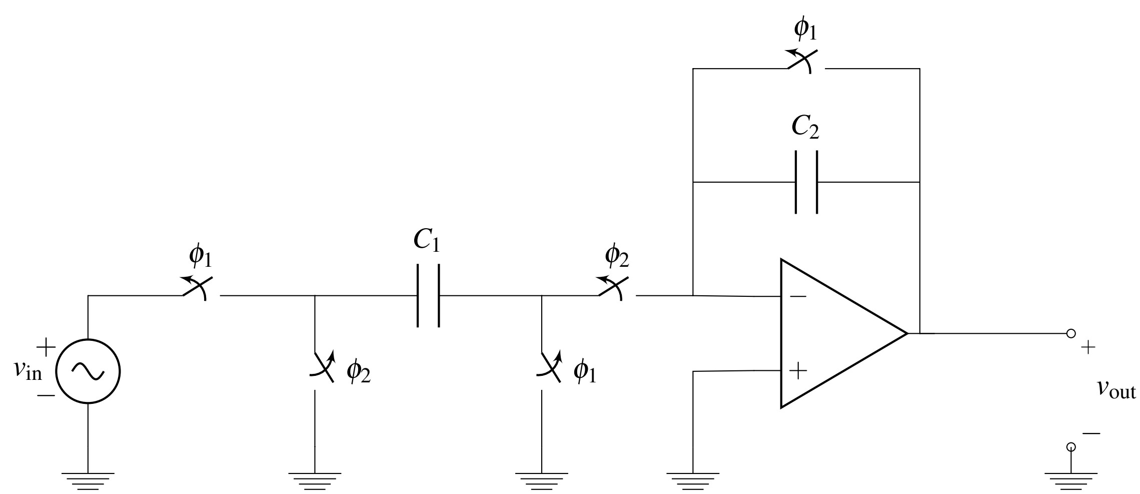


Input signals v_{in_i} are voltages from other neurons, which are multiplied by a constant weight w_i in each synapse and summed in the neuron. Each neuron also contains a nonlinear function (called a sigmoid) which is defined as

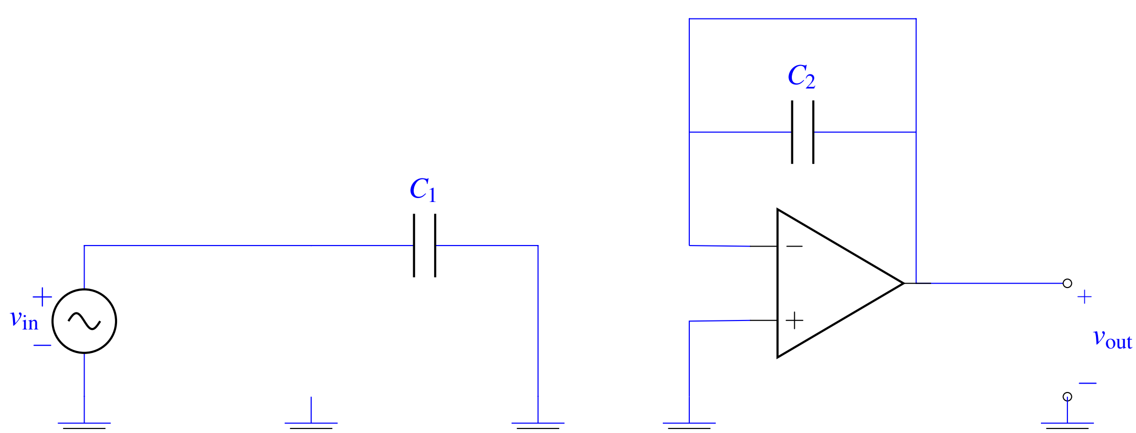
$$f(v) = \begin{cases} -1, & v \leq -1 \\ v, & -1 < v < 1 \\ +1, & v \geq +1 \end{cases}$$

where v is the internal neuron voltage after the synapse summer and $f(v)$ is the neuron voltage output.

(a) Your mentor suggests that you warm up first by analyzing the circuit below to use as a neuron with a single synapse. ϕ_1 and ϕ_2 are non-overlapping clock phases that control the circuit switches.



i. Draw an equivalent circuit for ϕ_1 and write an expression for v_{out} as a function of v_{in}, C_1, and C_2.
Solution: $v_{out} = 0$

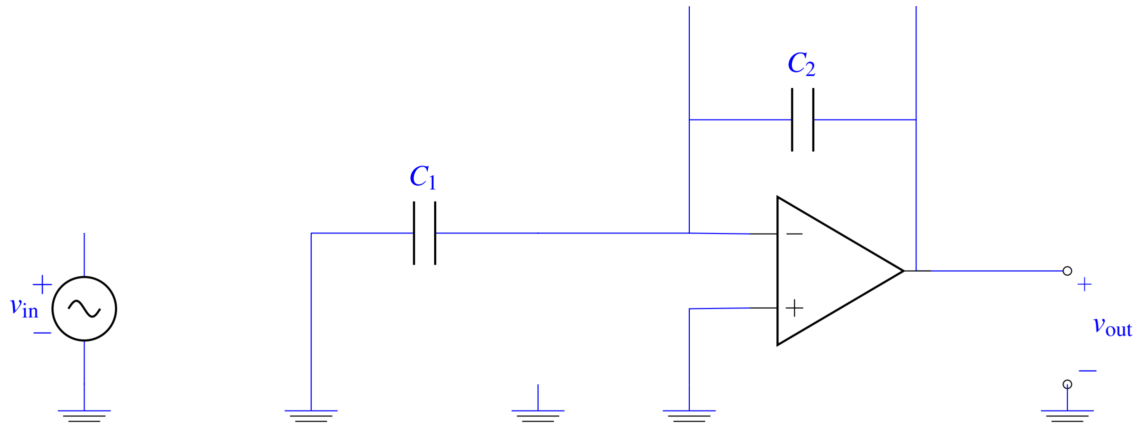


ii. Draw an equivalent circuit for ϕ_2 and write an expression for v_{out} as a function of v_{in}, C_1, and C_2.

Solution:

$$v_{\text{in}} C_1 = v_{\text{out}} C_2$$

$$v_{\text{out}} = \frac{C_1}{C_2} v_{\text{in}}$$

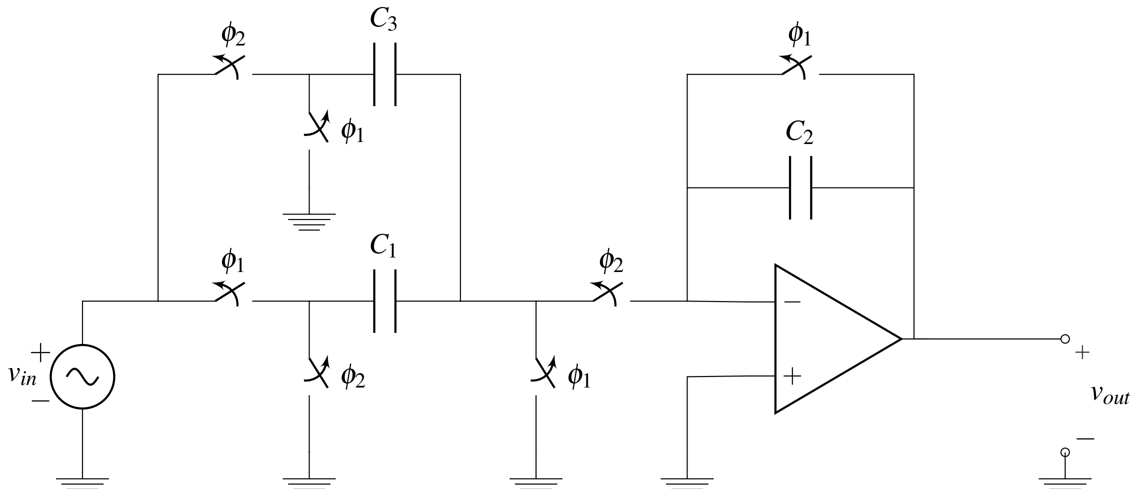


(b) Write an equation for v_{out} during ϕ_2 as a function of v_{in} for $C_1 = C_2$ and op-amp supply voltages of ± 1 V. Briefly explain how this circuit implements the sigmoid function.

Solution: From part (a)(ii) we know $v_{\text{out}} = \frac{C_1}{C_2} v_{\text{in}}$.Setting $C_1 = C_2$, we find $v_{\text{out}} = v_{\text{in}}$. Because of the rails of the op amp, once the v_{in} exceeds 1 V, the output will be 1 V. Similarly when v_{in} is less than -1 V, the output will be -1 V. At intermediate values we have $v_{\text{out}} = v_{\text{in}}$ from the analysis above. From this, we see the circuit implements the sigmoid function:

$$v_{\text{out}} = \begin{cases} -1, & v_{\text{in}} \leq -1 \\ v_{\text{in}}, & -1 < v_{\text{in}} < 1 \\ +1, & v_{\text{in}} \geq +1 \end{cases}$$

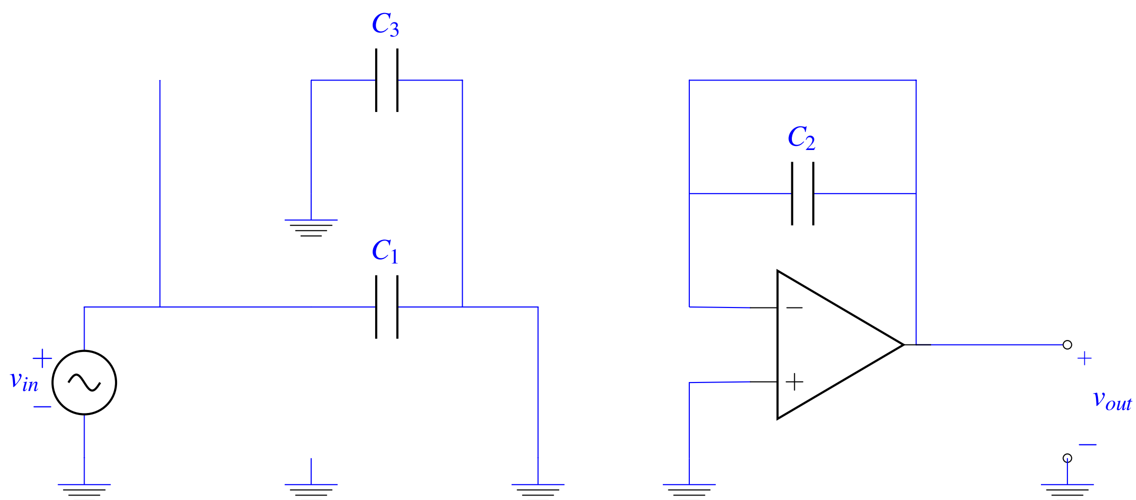
(c) Then, your mentor shows you the following neuron circuit, which can realize both positive and negative synapse weight and create $v_{out} = w_1 v_{in}$ in ϕ_2.



i. Draw an equivalent circuit during ϕ_1 and write an expression for v_{out} as a function of v_{in}, C_1, C_2, and C_3.

Solution:

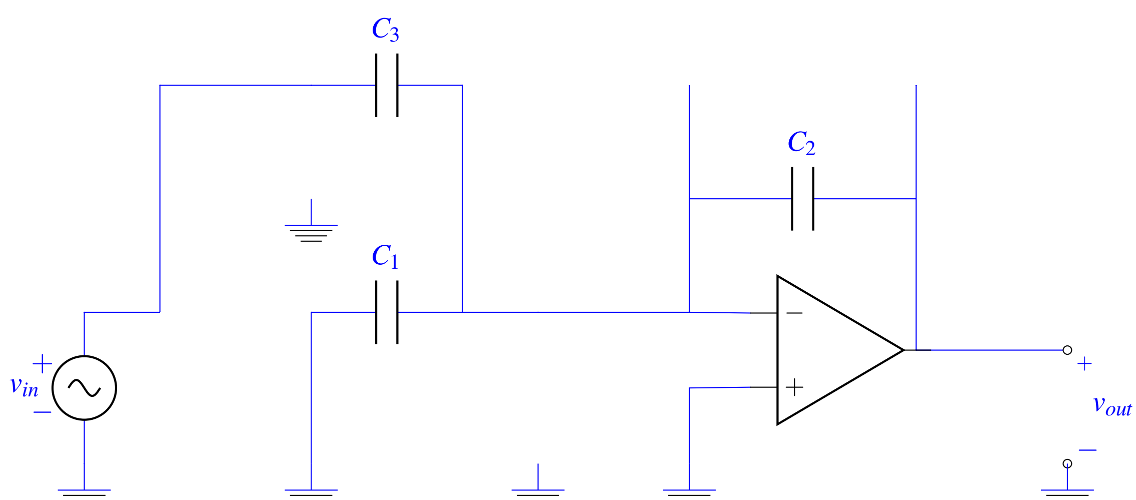
In phase 1, $v_{\text{out}} = 0$.



ii. (5 points) Draw an equivalent circuit during ϕ_2 and write an expression for v_{out} as a function of v_{in}, C_1, C_2, and C_3.

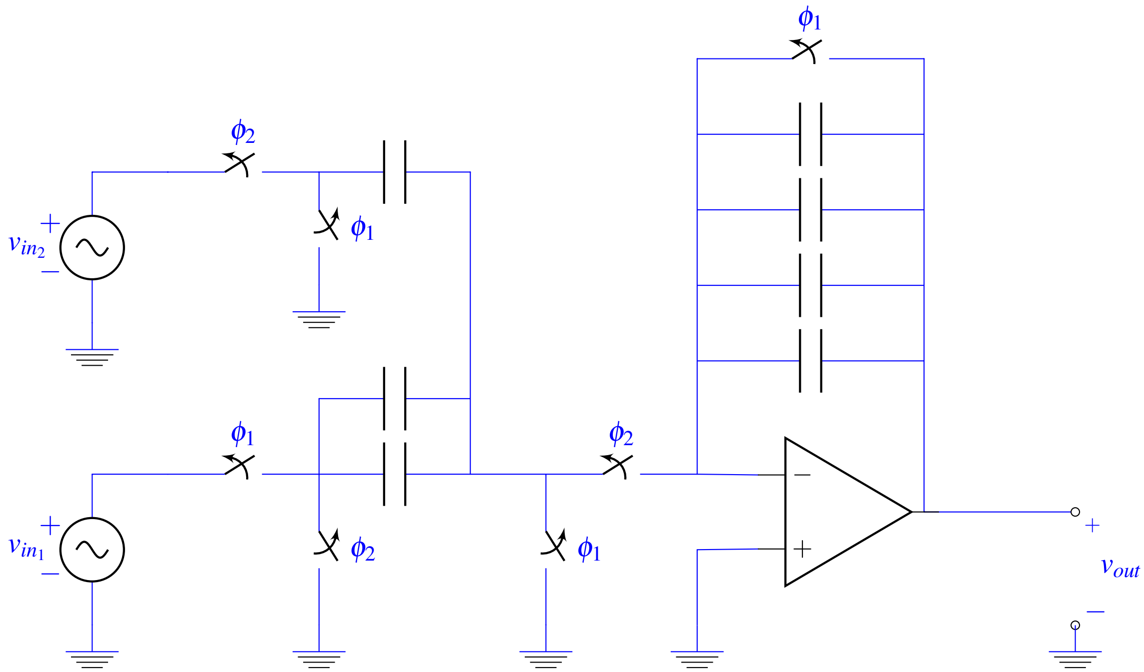
Solution:

In phase 2, $v_{\text{out}} = \frac{C_1 - C_3}{C_2} v_{\text{in}}$.



(d) Now it is your turn to implement a neuron that realizes the function $v_{out} = w_1 v_{in_1} + w_2 v_{in_2}$. Draw the circuit, such that $w_1 = 1/2$ and $w_2 = -1/4$. Label all circuit elements appropriately. You should use a single op-amp and as many capacitors and switches as you need. All capacitors must be of size C_{unit}. Assume that the op-amp power supplies are ± 1V (no need to draw them in the circuit). The circuit should operate in 2 phases, resetting in the first phase ϕ_1 and setting $v_{out} = w_1 v_{in_1} + w_2 v_{in_2}$ in the second phase ϕ_2.

Solution:



4. Orthogonal Matrices

Definition: A matrix $U \in \mathbb{R}^{n \times n}$ is called an orthogonal matrix if $U^{-1} = U^T$.

Orthogonal matrices represent linear transformations that preserve angles between vectors and the lengths of vectors. Rotations and reflections, useful in computer graphics, are examples of transformations that can be represented by orthogonal matrices.

(a) Let U be an orthogonal matrix. For two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, show that $\langle \vec{x}, \vec{y} \rangle = \langle U\vec{x}, U\vec{y} \rangle$, assuming we are working with the Euclidean inner product.

Solution:

$$\langle U\vec{x}, U\vec{y} \rangle = (U\vec{x})^T (U\vec{y}) = \vec{x}^T U^T U\vec{y} = \vec{x}^T U^{-1} U\vec{y} = \vec{x}^T \vec{y} = \langle \vec{x}, \vec{y} \rangle$$

(b) Show that $\|U\vec{x}\| = \|\vec{x}\|$, where $\|\cdot\|$ is the Euclidean norm.

Solution:

$$\|U\vec{x}\| = \sqrt{\langle U\vec{x}, U\vec{x} \rangle} = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \|\vec{x}\|$$

The second equality follows from the identity proved in part (a).

5. Vector Projection

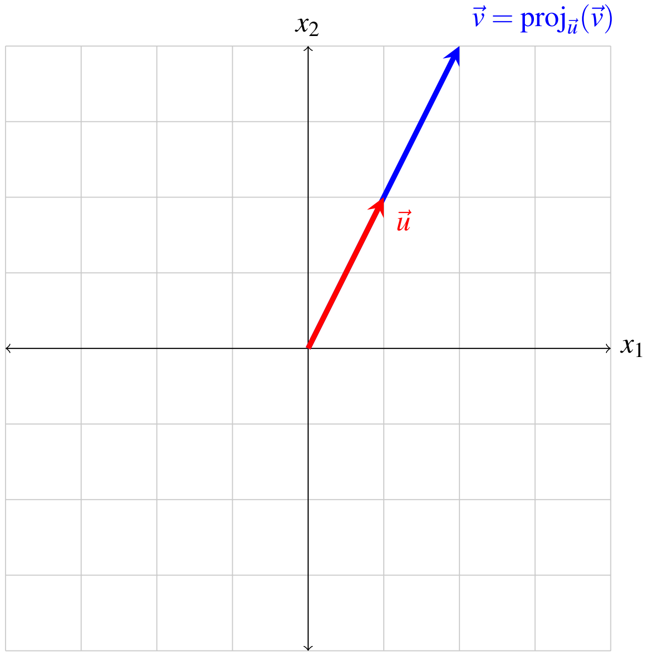
Definition: For two vectors $\vec{u}$ and $\vec{v}$, the projection of $\vec{v}$ onto $\vec{u}$ is a vector that captures how much of $\vec{v}$ points in the direction of $\vec{u}$. It is notated as $\text{proj}_{\vec{u}}(\vec{v})$. The projection can be calculated as:

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \vec{u}$$

For each set of vectors below, compute $\text{proj}_{\vec{u}}(\vec{v})$ and plot $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{u}}(\vec{v})$ on the same plot.

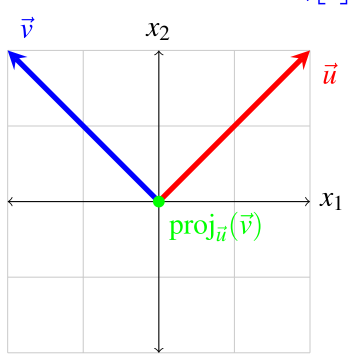
(a) $\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}, \vec{v} = \begin{bmatrix}2\\4\end{bmatrix}$

Solution: $\text{proj}_{\vec{u}}(\vec{v}) = \dfrac{\left\langle \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right\rangle} \begin{bmatrix}1\\2\end{bmatrix} = \frac{1\cdot 2+2\cdot 4}{1^2+2^2} \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$



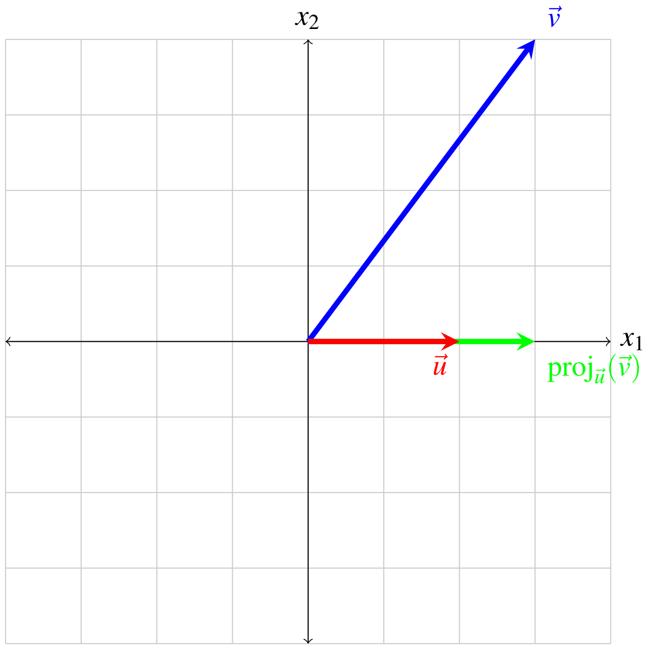
(b) $\vec{u} = \begin{bmatrix}2\\2\end{bmatrix}, \vec{v} = \begin{bmatrix}-2\\2\end{bmatrix}$

Solution: $\text{proj}_{\vec{u}}(\vec{v}) = \dfrac{\left\langle \begin{bmatrix}-2\\2\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix} \right\rangle} \begin{bmatrix}2\\2\end{bmatrix} = \frac{2\cdot(-2)+2\cdot 2}{2^2+2^2} \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$



(c) $\vec{u} = \begin{bmatrix}2\\0\end{bmatrix}, \vec{v} = \begin{bmatrix}3\\4\end{bmatrix}$

Solution: $\text{proj}_{\vec{u}}(\vec{v}) = \dfrac{\left\langle \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix} \right\rangle} \begin{bmatrix}2\\0\end{bmatrix} = \frac{2\cdot 3+0\cdot 4}{2^2+0^2} \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$



6. Cauchy-Schwarz Inequality

Let $\mathbb{V}$ be a (real) vector space, and let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{V}$. The Cauchy-Schwarz inequality states that, for any two vectors $\vec{u}, \vec{v} \in \mathbb{V}$,

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|.$$

(a) Use the defining properties of inner products to prove the Cauchy-Schwarz inequality for this general case. The argument is outlined in Note 21. You can repeat it here, but you should justify the steps. Note: Do not assume we are working with the Euclidean inner product and norm.

Solution: We can assume $\vec{v} \neq \vec{0}$, else the inequality is trivially true.

$$\begin{aligned}
0 &\leq \langle \vec{u} - \alpha\vec{v}, \vec{u} - \alpha\vec{v} \rangle && \text{By positive-definiteness} \\
&= \langle \vec{u}, \vec{u} - \alpha\vec{v} \rangle - \alpha \langle \vec{v}, \vec{u} - \alpha\vec{v} \rangle && \text{By linearity in the first argument} \\
&= \langle \vec{u} - \alpha\vec{v}, \vec{u} \rangle - \alpha \langle \vec{u} - \alpha\vec{v}, \vec{v} \rangle && \text{By symmetry} \\
&= \langle \vec{u}, \vec{u} \rangle - \alpha \langle \vec{v}, \vec{u} \rangle - \alpha \langle \vec{u}, \vec{v} \rangle + \alpha^2 \langle \vec{v}, \vec{v} \rangle && \text{Again by linearity in the first argument} \\
&= \langle \vec{u}, \vec{u} \rangle - 2\alpha \langle \vec{u}, \vec{v} \rangle + \alpha^2 \langle \vec{v}, \vec{v} \rangle && \text{Again by symmetry} \\
&= \langle \vec{u}, \vec{u} \rangle - 2\frac{(\langle \vec{u}, \vec{v} \rangle)^2}{\langle \vec{v}, \vec{v} \rangle} + \frac{(\langle \vec{u}, \vec{v} \rangle)^2}{\langle \vec{v}, \vec{v} \rangle} && \text{This holds for any } \alpha\text{, so we take } \alpha = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}
\end{aligned}$$

Therefore, the inequality above becomes:

$$\begin{aligned}
0 &\leq \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle - (\langle \vec{u}, \vec{v} \rangle)^2 && \text{Multiplying by } \langle \vec{v}, \vec{v} \rangle \\
(\langle \vec{u}, \vec{v} \rangle)^2 &\leq \|\vec{u}\|^2 \|\vec{v}\|^2 \\
\langle \vec{u}, \vec{v} \rangle &\leq \|\vec{u}\| \|\vec{v}\|
\end{aligned}$$

(b) What condition is needed for equality to hold in the Cauchy-Schwarz inequality? Your answer should involve the concept of linear dependence.

Solution: In order to have equality, we need

$$
\begin{aligned}
0 &= \langle \vec{u} - \alpha\vec{v}, \vec{u} - \alpha\vec{v} \rangle \\
\vec{u} &= \alpha\vec{v}
\end{aligned}
$$

So $\vec{u}$ and $\vec{v}$ must be linearly dependent.

(c) Use the Cauchy-Schwarz inequality to show that the quantity $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ satisfies the triangle inequality:

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \qquad \vec{u}, \vec{v} \in \mathbb{V}.$$

Hint: Note that $a^2 + b^2 + 2ab = (a+b)^2 \geq 0$ for any scalars $a, b \in \mathbb{R}$.
Note: In part, this shows that $\|\cdot\|$ is indeed a norm on $\mathbb{V}$.

Solution: Starting with the left-hand side:

$$
\begin{aligned}
\|\vec{u} + \vec{v}\| &= \sqrt{\langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle} \\
&= \sqrt{\langle \vec{u}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle + 2\langle \vec{u}, \vec{v} \rangle} \\
&= \sqrt{\|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\langle \vec{u}, \vec{v} \rangle}
\end{aligned}
$$

Using the Cauchy-Schwarz inequality:

$$
\begin{aligned}
&\leq \sqrt{\|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\|} \\
&= \sqrt{(\|\vec{u}\| + \|\vec{v}\|)^2} \\
&= \|\vec{u}\| + \|\vec{v}\| = \text{RHS}
\end{aligned}
$$

7. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.

