

**This homework is due March 15, 2016, at Noon.**

### 1. Homework process and study group

Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?

Working in groups of 3-5 will earn credit for your participation grade.

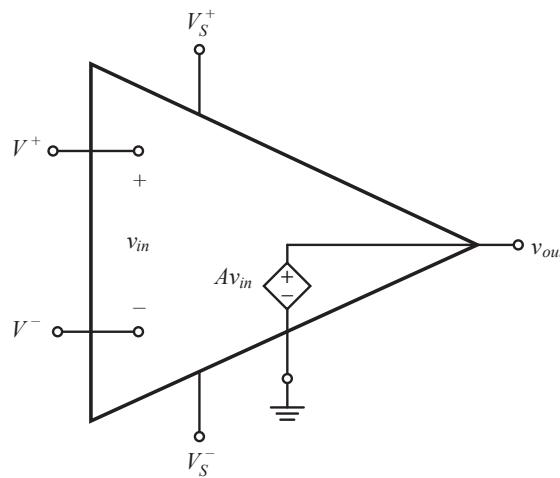
**Solution:** I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on Problem 5 so I went to office hours on...

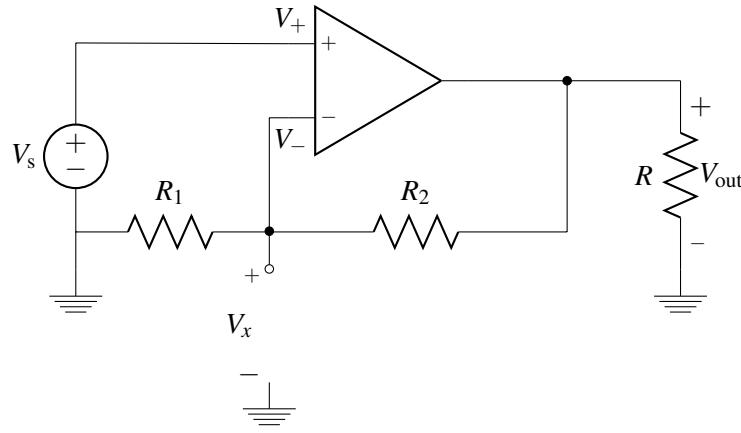
Then I went to homework party for a few hours, where I finished the homework.

### 2. Op-Amp Golden Rules

In this question we are going to show that the golden rules for op-amps hold by analyzing equivalent circuits and then taking the limit as the open-loop gain approaches infinity. Below is a picture of the equivalent model of an op-amp we are using for this question.

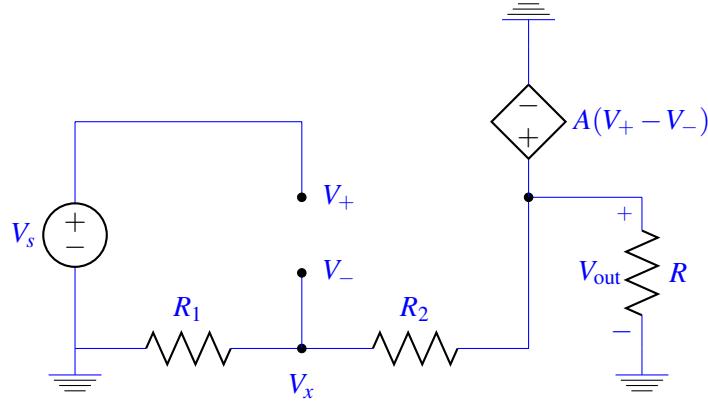


- (a) Now consider the circuit below. Draw an equivalent circuit using the op-amp model shown above and calculate  $V_{out}$  and  $V_x$  in terms of  $A$ ,  $V_s$ ,  $R_1$ ,  $R_2$  and  $R$ . Is the magnitude of  $V_x$  larger or smaller than the magnitude of  $V_s$ ? Do these values depend on  $R$ ?



**Solution:**

This is the equivalent circuit of the op-amp:



Since  $V_{\text{out}}$  is connected to the output of the op-amp, which is a voltage source, we can determine  $V_{\text{out}}$ :

$$\begin{aligned} V_{\text{out}} &= A(V_+ - V_-) \\ &= A(V_s - V_x) \end{aligned}$$

Since there is no current flowing through the nodes  $V_+$  and  $V_-$  (because we are assuming that  $R_{\text{in}}$  is infinite),  $R_1$  and  $R_2$  form a voltage divider and  $V_x = V_{\text{out}} \left( \frac{R_1}{R_1 + R_2} \right)$ . Thus substituting and solving for  $V_{\text{out}}$ :

$$\begin{aligned} V_{\text{out}} &= A \left( V_s - V_{\text{out}} \frac{R_1}{R_1 + R_2} \right) \\ V_{\text{out}} &= V_s \left( \frac{1}{\frac{R_1}{R_1 + R_2} + \frac{1}{A}} \right) \end{aligned}$$

Knowing  $V_{\text{out}}$ , we can find  $V_x$ :

$$V_x = \frac{V_s}{1 + \frac{R_1 + R_2}{A R_1}}$$

Notice that  $V_x$  is slightly smaller than  $V_s$ , meaning that in equilibrium in the non-ideal case,  $V_+$  and  $V_-$  are not equal.  $V_{\text{out}}$  and  $V_x$  do not depend on  $R$ , which means that we can treat  $V_{\text{out}}$  as a voltage source that supplies a constant voltage independent of the load  $R$ .

- (b) Using your solution to part (a), calculate  $V_{\text{out}}$  and  $V_x$  in the limit as  $A \rightarrow \infty$ . Do you get the same answers if you apply the golden rules ( $V_+ = V_-$  when there is negative feedback)?

**Solution:** As  $A \rightarrow \infty$ , the fraction  $\frac{1}{A} \rightarrow 0$ , so

$$V_{\text{out}} = V_s \left( \frac{1}{\frac{R_1}{R_1+R_2} + \frac{1}{A}} \right)$$

Converges to

$$V_s \left( \frac{1}{\frac{R_1}{R_1+R_2} + 0} \right) = V_s \left( \frac{R_1+R_2}{R_1} \right)$$

So the limits as  $A \rightarrow \infty$  are:

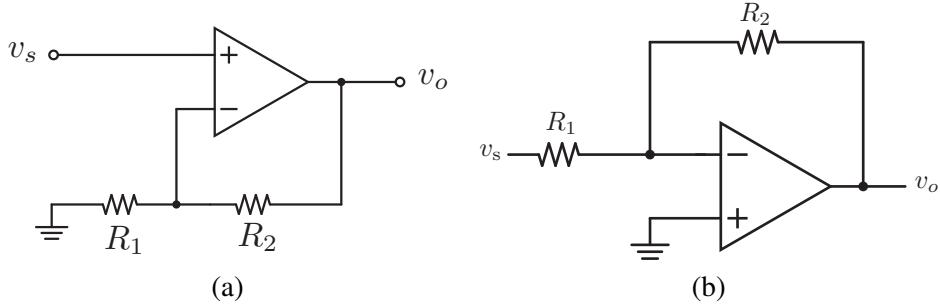
$$V_{\text{out}} \rightarrow V_s \left( \frac{R_1+R_2}{R_1} \right)$$

$$V_x \rightarrow V_s$$

If we apply the golden rules,  $V_x = V_s$ . Then the current  $i$  flowing through  $R_1$  to ground is  $\frac{V_s}{R_1}$ . By KCL, this same current flows through  $R_2$  since no current flows through  $V_-$ . Thus the voltage drop between  $R_2$ ,  $V_{\text{out}} - V_x$ , is  $i \cdot R_2 = V_s \left( \frac{R_2}{R_1} \right)$ . Therefore  $V_{\text{out}} = V_s + V_s \left( \frac{R_2}{R_1} \right) = V_s \left( \frac{R_1+R_2}{R_1} \right)$ . The answers are the same if you take the limit as  $A \rightarrow \infty$ .

### 3. Basic Amplifier Building Blocks

The following amplifier stages are used often in many circuits and are well known as (a) the non-inverting amplifier and (b) the inverting amplifier.



- (a) Derive the voltage gain of the non-inverting amplifier using the Golden Rules. Explain the origin of the name of the amplifier.

**Solution:** There are many ways to solve these circuits; here are some:

**Method 1:** The voltage at the non-inverting input is  $v_s$ , so by Golden Rule, the op-amp will act such that the voltage at the inverting input also becomes  $v_s$ . Therefore the voltage drop across  $R_1$  is  $v_s$ , so there is a current of  $i = \frac{v_s}{R_1}$  through resistor  $R_1$ . Since no current flows into the inverting input (by Golden Rule), this current of  $i$  must flow through  $R_2$  (by KCL at the inverting input). Thus the voltage drop across  $R_2$  is  $V_2 = i \cdot R_2 = v_s \left( \frac{R_2}{R_1} \right)$ . So  $v_o$  is  $v_s$  plus the voltage drop across  $R_2$ :

$$v_o = v_s + v_s \left( \frac{R_2}{R_1} \right) = v_s \left( \frac{R_1+R_2}{R_1} \right) \quad (1)$$

**Method 2:** Since there is no current flowing into the inverting input (by Golden Rule), notice that resistors  $R_1, R_2$  form a voltage divider between the output  $v_o$  and ground. The inverting input sees the output of this voltage divider:

$$V_- = v_o \left( \frac{R_1}{R_1 + R_2} \right) \quad (2)$$

But  $V_- = V_+ = v_s$  by Golden Rule, so we have

$$v_o \left( \frac{R_1}{R_1 + R_2} \right) = v_s \implies v_o = v_s \left( \frac{R_1 + R_2}{R_1} \right) \quad (3)$$

So the gain of this amplifier is

$$G = \frac{v_o}{v_s} = \frac{R_1 + R_2}{R_1} \quad (4)$$

This is called an *non-inverting amplifier* because the gain  $G$  is positive – it does not invert the input signal (in contrast to the amplifier in the next part of this problem).

- (b) Derive the voltage gain of the inverting amplifier using the Golden Rules. Explain the origin of the name of the amplifier.

**Solution:** Here is one way:

Since the voltage at the non-inverting input  $V_+ = 0$ , the op-amp will act such that the voltage at the inverting input  $V_- = 0$  as well (by Golden Rule). Now, by KCL at the inverting input node:

$$\frac{v_s - 0}{R_1} + \frac{v_o - 0}{R_2} = 0 \quad (5)$$

Solving this yields:

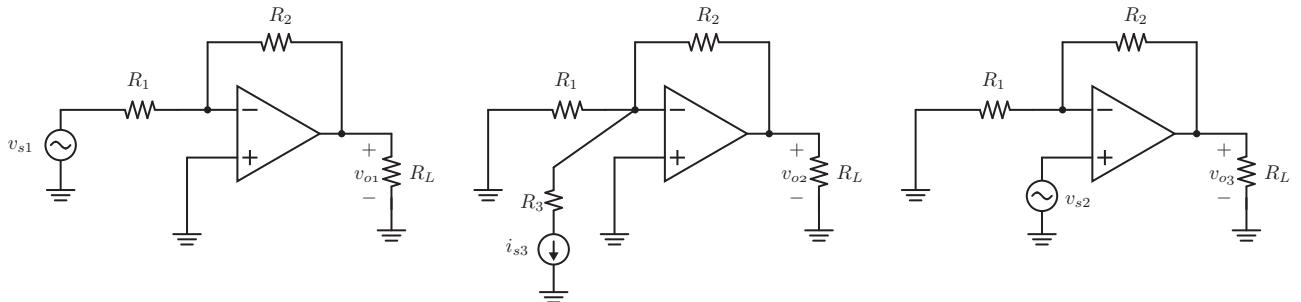
$$v_o = - \left( \frac{R_2}{R_1} \right) v_s \quad (6)$$

Thus, the voltage gain of this amplifier circuit is:

$$G = \frac{v_o}{v_s} = - \frac{R_2}{R_1} \quad (7)$$

This is called an *inverting amplifier* because the voltage gain  $G$  is *negative*, meaning it “inverts” its input signal.

#### 4. Amplifier with Multiple Inputs



- (a) Use the Golden Rules to find  $v_{o1}$  for the first circuit above.

**Solution:** Applying the golden rules we know that the positive and negative terminals must be at the same voltage, thus the voltage at the negative terminal of the op-amp is 0. Since no current can go into the op-amp at the negative terminal we know that the current going through the two resistors  $R_1$  and  $R_2$  must be equal. If we define the positive terminal of both resistors to be on the left side then we have  $i_{R1} = \frac{v_{s1}-0}{R_1}$  and  $i_{R2} = \frac{0-v_{o1}}{R_2}$ , and thus

$$\frac{v_{s1}-0}{R_1} = \frac{0-v_{o1}}{R_2} \quad (8)$$

Rearranging the terms we get

$$v_{o1} = -\left(\frac{R_2}{R_1}\right)v_{s1}. \quad (9)$$

- (b) Use the Golden Rules to find  $v_{o2}$  for the second circuit above.

**Solution:** Applying the golden rules we know that the positive and negative terminals must be at the same voltage, thus the voltage at the negative terminal of the op-amp is 0. The voltage drop across  $R_1$  is thus 0 and no current flows through it. In addition no current flows into the op-amp from the negative terminal due to its infinite input resistance (the negative terminal is connected to an “open” circuit.) By KCL at the negative terminal of the op-amp this means that the current going through  $R_3$  and  $R_2$  is  $i_{s3}$ . Taking the positive terminal of  $R_2$  to be on the right, then the voltage drop across  $R_2$  is  $v_{o2}$ . By Ohm’s law we conclude

$$\frac{v_{o2}}{R_2} = i_{s3}. \quad (10)$$

Rearranging we get

$$v_{o2} = i_{s3} \cdot R_2. \quad (11)$$

- (c) Use the Golden Rules to find  $v_{o3}$  for the third circuit above.

**Solution:** Applying the golden rules we know that the positive and negative terminals must be at the same voltage, thus the voltage at the negative terminal of the op-amp is  $V^- = v_{s2}$ . In addition, since no current can enter into the negative terminal of the op-amp,  $R_1$  and  $R_2$  are in series. This means that the voltage at the negative terminal of the op-amp can be expressed in terms of  $v_{o3}$  using the voltage divider formula

$$V^- = v_{o3} \left( \frac{R_2}{R_1 + R_2} \right). \quad (12)$$

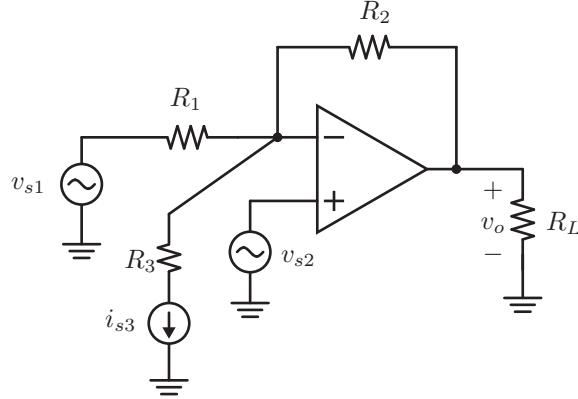
We also know  $V^- = v_{s2}$ , and conclude

$$v_{s2} = v_{o3} \left( \frac{R_1}{R_1 + R_2} \right). \quad (13)$$

After rearranging we have

$$v_{o3} = v_{s2} \left( \frac{R_2}{R_1} + 1 \right). \quad (14)$$

- (d) Use the Golden Rules to find the output voltage  $v_o$  for the circuit shown below.



**Solution:** Applying the golden rules we know that the positive and negative terminals must be at the same voltage, thus the voltage at the negative terminal of the op-amp is  $V = v_{s2}$ . Then we write a KCL equation at the node connected to the minus terminal of the op-amp (recalling that no current flows into or out of the op-amp's terminals). All currents are defined as flowing out of the node:

$$i_{R_1} + i_{R_2} + i_{R_3} = 0$$

Because of the independent current source, we know

$$i_{R_3} = i_{s3}$$

By Ohm's law, we know

$$i_{R_1} = \frac{V^- - v_{s1}}{R_1}$$

and

$$i_{R_2} = \frac{V^- - v_o}{R_2}$$

Then substituting back into the original KCL equation we have

$$\frac{V^- - v_{s1}}{R_1} + \frac{V^- - v_o}{R_2} + i_{s3} = 0$$

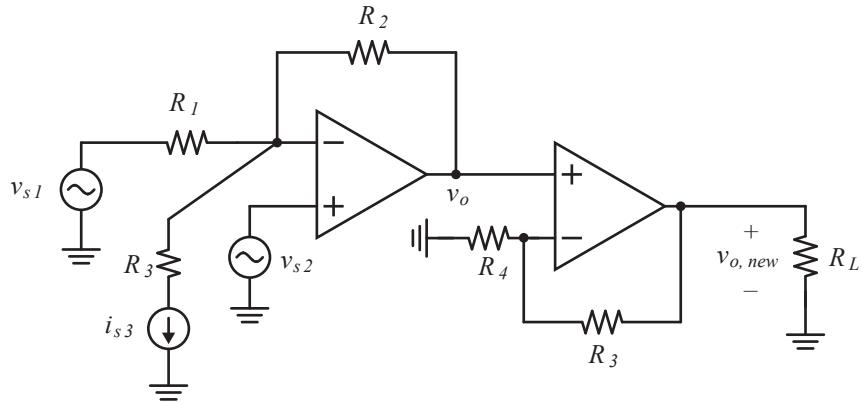
and substituting  $V^- = v_{s2}$  we have

$$\frac{v_{s2} - v_{s1}}{R_1} + \frac{v_{s2} - v_o}{R_2} + i_{s3} = 0$$

which we rearrange to find  $v_o$ , giving

$$v_o = v_{s2} \left( 1 + \frac{R_2}{R_1} \right) + i_{s3} \cdot R_2 - \left( \frac{R_2}{R_1} \right) v_{s1}$$

- (e) Now add a second stage as shown below. What is  $v_{o, \text{new}}$ ? Does  $v_o$  change between the last part and this part? Does the voltage  $v_{o, \text{new}}$  depend on  $R_L$ ?



**Solution:** Adding the second stage does not change the voltages in the first part. This is because the circuit connected to the positive and negative terminals of the first stage op amp “sees” an open circuit / infinite input resistance in the op amp.

Call the output voltage of the first stage  $v_{o1}$ . Then it remains unchanged from the previous part

$$v_{o1} = -\left(\frac{R_2}{R_1}\right)v_{s1} + i_{s3} \cdot R_2 + v_{s2} \left(\frac{R_2 + R_1}{R_1}\right)$$

By the golden rules, the minus terminal of the second op-amp must have the same voltage as the plus terminal, which is  $v_{o1}$ . No current can flow into the minus terminal, so  $R_3$  and  $R_4$  are in series and have the same current, and we write the equations

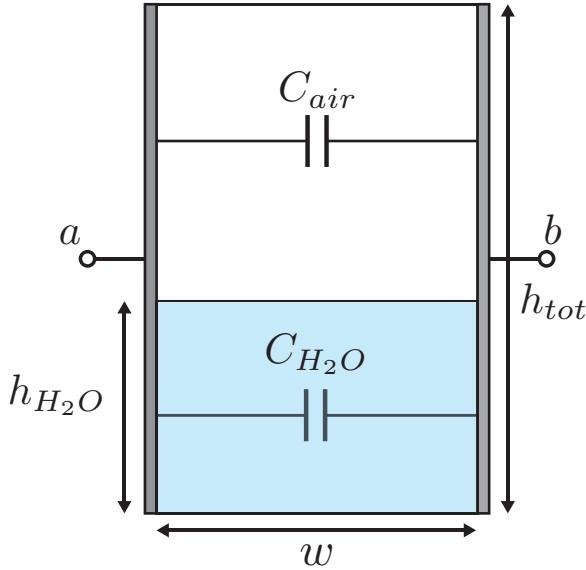
$$\frac{v_{o1}}{R_4} = \frac{v_o - v_{o1}}{R_3}$$

and then write

$$v_o = \left(\frac{R_3 + R_4}{R_4}\right)v_{o1} = \frac{R_3 + R_4}{R_4} \left(-\frac{R_2}{R_1} \cdot v_{s1} + i_{s3} \cdot R_2 + v_{s2} \cdot \frac{R_2 + R_1}{R_1}\right)$$

## 5. It's finally raining!

A lettuce farmer in the Salinas valley has grown tired of weather.com’s imprecise rain measurements. So, she decided to take matters into her own hands by building a rain sensor. She placed a rectangular tank outside and attached two metal plates to two opposite sides in an effort to make a capacitor whose capacitance varies with the amount of water inside.



The width and length of the tank are both  $w$  (i.e. the base is square) and the height of the tank is  $h_{\text{tot}}$ .

- (a) What is the capacitance between terminals  $a$  and  $b$  when the tank is full? What about when it is empty?  
Note: the permittivity of air is  $\epsilon$ , and the permittivity of rainwater is  $81\epsilon$ .

**Solution:**

Capacitance of parallel plates is governed by the equation:

$$C = \frac{\epsilon A}{d}$$

Where  $\epsilon$  is the *permittivity* of the dielectric material,  $A$  is the area of the plates, and  $d$  is the distance between the plates. If we apply this to our physical structure, we find that the area of the plates are  $h_{\text{tot}} \times w$ , and the distance between the plates is  $w$ . The only difference here between a full and empty tank is the permittivity of the material between the two plates.

$$C_{\text{empty}} = \frac{\epsilon_{\text{air}} h_{\text{tot}} w}{w} = \epsilon h_{\text{tot}}$$

$$C_{\text{full}} = \frac{\epsilon_{H_2O} h_{\text{tot}} w}{w} = 81\epsilon h_{\text{tot}}$$

- (b) Suppose the height of the water in the tank is  $h_{H_2O}$ . Modeling the tank as a pair of capacitors in parallel, find the total capacitance between the two plates. Call this capacitance  $C_{\text{tank}}$ .

**Solution:**

We can break the total capacitance into two parts. First let's calculate the capacitance of the two plates separated by water:

$$C_{\text{water}} = \frac{\epsilon_{H_2O} h_{H_2O} w}{w} = 81\epsilon h_{H_2O}$$

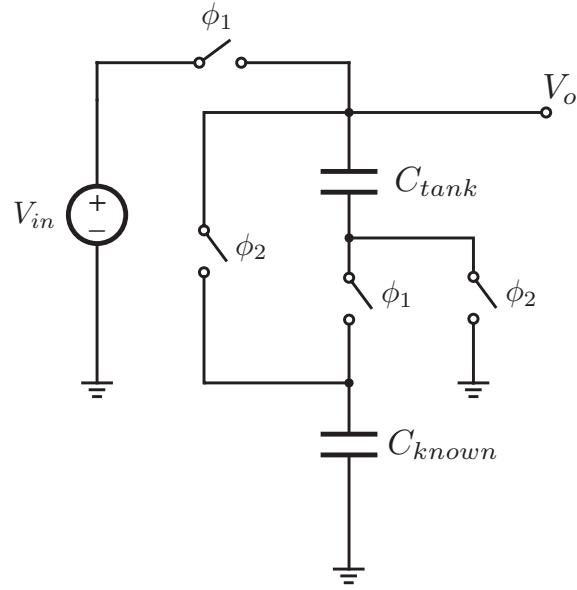
And now we can calculate the capacitance of the two plates separated by air:

$$C_{\text{air}} = \frac{\epsilon_{\text{air}} (h_{\text{tot}} - h_{H_2O}) w}{w} = \epsilon (h_{\text{tot}} - h_{H_2O})$$

Because these two capacitors appear in parallel, we can simply add our two previous results to find the total equivalent capacitance:

$$C_{\text{tank}} = C_{\text{water}} + C_{\text{air}} = \epsilon (h_{\text{tot}} + 80h_{H_2O})$$

- (c) After building this capacitor, the farmer consults the Internet to assist her with a capacitance measuring circuit. A random Anon recommends the following:



In this circuit,  $C_{\text{tank}}$  is the total tank capacitance that you calculated earlier.  $C_{\text{known}}$  is some fixed and known capacitor. Find the voltage  $V_o$  in phase  $\phi_2$  as a function of the height of the water. Note that in phase  $\phi_1$  all switches labeled  $\phi_1$  are closed and all switches labeled  $\phi_2$  are open. In phase  $\phi_2$ , all switches labeled  $\phi_1$  are open and all switches labeled  $\phi_2$  are closed. You should also assume that before any measurements are taken, the voltages across both  $C_{\text{known}}$  and  $C_{\text{tank}}$  are initialized to 0V.

**Solution:** In phase  $\phi_1$  the charge on both  $C_{\text{tank}}$  and  $C_{\text{known}}$  will be equal to the following charge:

$$Q_{C_{\text{tank}}} = Q_{C_{\text{known}}} = \left( \frac{C_{\text{tank}} \cdot C_{\text{known}}}{C_{\text{tank}} + C_{\text{known}}} \right) V_{\text{in}}$$

In phase  $\phi_2$  both '+' plates are connected to the output, so we have a straightforward charge share:

$$Q_{\text{tot}} = 2 \left( \frac{C_{\text{tank}} \cdot C_{\text{known}}}{C_{\text{tank}} + C_{\text{known}}} \right) V_{\text{in}}$$

$$Q_{\text{tot}} = V_{\text{out}} (C_{\text{tank}} + C_{\text{known}})$$

Therefore, we can solve for the output voltage:

$$V_o = 2V_{\text{source}} \left( \frac{C_{\text{tank}} \cdot C_{\text{known}}}{(C_{\text{tank}} + C_{\text{known}})^2} \right)$$

- (d) Use IPython (or any other tool or just do it by hand) to plot this voltage  $V_o$  as a function of the height of the water. Vary the tank from empty to full. Use values of  $V_{in} = 12V$ ,  $w = 0.5m$ ,  $h_{tot} = 1m$ , and  $\epsilon = 8.854 \times 10^{-12} F/m$ . This  $\epsilon$  is called the *permittivity of free space*. For  $C_{known}$  use a similar tank that is known to always be empty.

**Solution:** See `s110.ipynb`.

- (e) With the previous part, we were able to derive an expression for  $V_o$ . What does  $V_o$  represent? It's something we can measure! Our original goal was to determine what the height of the water in the tank without having to look inside it. Rewrite the last part to solve for  $h_{water}$ .

**Solution:**

$V_o$  is a function of the unknown  $C_{tank}$ , which we can solve as a quadratic equation. First we make some manipulations:

$$\begin{aligned} V_o(C_{tank} + C_{known})^2 &= 2V_{source} \cdot C_{tank} \cdot C_{known} \\ \left( \frac{1}{C_{tank}} + \frac{1}{C_{known}} \right) (C_{tank} + C_{known}) &= \frac{2V_{source}}{V_o} \\ \frac{C_{tank}}{C_{known}} + \frac{C_{known}}{C_{tank}} &= 2 \left( \frac{V_{source}}{V_o} - 1 \right) \end{aligned}$$

Now we make some substitutions, let  $x = \frac{C_{tank}}{C_{known}}$  and  $b = 2 \left( \frac{V_{source}}{V_o} - 1 \right)$ . Then we have the following quadratic equation:

$$\begin{aligned} x + \frac{1}{x} &= b \\ x^2 - bx + 1 &= 0 \\ x &= \frac{b \pm \sqrt{b^2 - 4}}{2} \end{aligned}$$

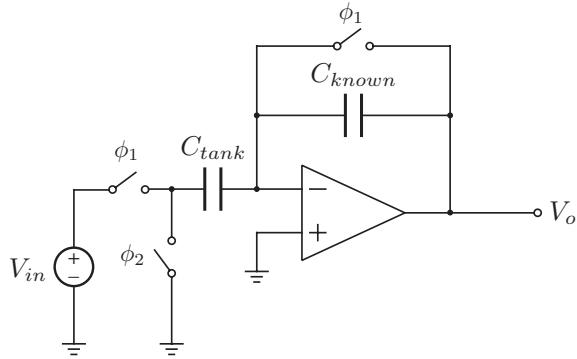
Note that there are two solutions to this quadratic equation. Since from the symmetry of the equation we know that if  $x = a$  is a solution,  $x = \frac{1}{a}$  is also a solution. Since  $C_{known}$  is the capacitance of the empty tank and we know that this is smaller than  $C_{tank}$ , we always choose the larger solution:

$$\begin{aligned} \frac{C_{tank}}{C_{known}} &= \frac{2 \left( \frac{V_{source}}{V_o} - 1 \right) + \sqrt{4 \left( \frac{V_{source}}{V_o} - 1 \right)^2 - 4}}{2} \\ h_{water} &= \frac{h_{tot}}{80} \left( \frac{V_{source}}{V_o} - 2 + \sqrt{\left( \frac{V_{source}}{V_o} - 1 \right)^2 - 1} \right) \end{aligned}$$

- (f) How about we perform a sanity check on our answer. What are the units of your result for  $V_o$  and for  $h_{water}$ ?

**Solution:** We can check that the units for  $V_o$  is in volts, and the units for  $h_{water}$  is in meters.

- (g) (**BONUS Out-of-scope**) The farmer has become tired of solving the equation and wishes to generate a voltage proportional to the tank capacitance. A brief consultation with her daughter, who is taking Berkeley's EE105, yields the following circuit:



Calculate  $V_o$  as a function of  $h_{H_2O}$  in phase  $\phi_2$ . Use the Golden Rules. (*Hint: think about what must happen to the charge on the capacitor  $C_{\text{tank}}$  in phase  $\phi_2$ . Where does that charge have to go?*)

**Solution:**

This circuit is so awesome. In phase  $\phi_1$  we know that one terminal of  $C_{\text{tank}}$  is connected to  $V_{\text{in}}$  and the other end is connected a virtual ground - 0V.  $C_{\text{known}}$  is discharged.

$$Q_{C_{\text{tank}}} = C_{\text{tank}} V_{\text{in}}$$

$$Q_{C_{\text{known}}} = 0$$

In phase  $\phi_2$ , both ends of  $C_{\text{tank}}$  are connected to 0V (one end to ground, the other end to virtual ground). Here's the catch: all of the charge previously on the negative plate of  $C_{\text{tank}}$  will now transfer to the positive plate of  $C_{\text{known}}$ . Because the positive plate of  $C_{\text{known}}$  is connected to 0V (virtual ground), the other end of the plate's voltage will go up.

$$Q_{C_{\text{tank}}} = 0$$

$$Q_{C_{\text{known}}} = -Q_{C_{\text{tank}}, \phi_1} = -V_o C_{\text{known}}$$

Now we can solve for  $V_o$  because we know the charge on  $C_{\text{tank}}$  from phase  $\phi_1$ .

$$V_{\text{out}} = \left( \frac{C_{\text{tank}}}{C_{\text{known}}} \right) V_{\text{in}}$$

How cool is that?

## 6. Cool For The Summer (BONUS)

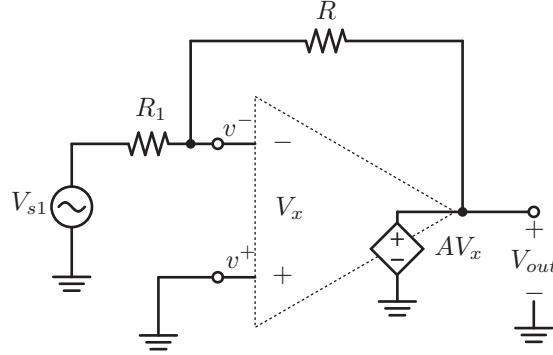
You and a friend want to make a box that helps control an air-conditioning unit. You both have dials that emit a voltage: 0 means you want to leave the temperature as it is. Negative voltages mean that you want to reduce the temperature. (It's hot so we will assume that you never want to increase the temperature — so, we're not talking about a Berkeley summer...)

Your air-conditioning unit however responds to positive voltages. The higher the voltage, the more strongly it runs. At zero, it is off. (If it helps, think of this air-conditioning unit as a heat pump. If you run it with negative voltage, it pumps heat in the opposite direction — from outside to inside. If positive voltage, it pumps heat from inside to outside. )

So you need a box that is an inverting summer — it outputs a weighted sum of two voltages where the weights are both negative. (Weighted because each of you has your own subjective sense for how much to turn the dial down and you need to compensate for that.)

This problem walks you through this using an op-amp.

- (a) Solve for  $V_{out}$  in terms of the other circuit quantities, i.e.  $R$ ,  $R_1$ ,  $A$ , and  $V_{s1}$ .



**Solution:**

We first start by defining a sign convention on our resistors. We choose for the minus terminal on all of the resistors to be on the node  $v^-$ . Since  $v^+$  is grounded (so  $v^+ = 0$ ), we have  $V_x = v^+ - v^- = -v^-$ . From KCL at node  $v^-$ , we have

$$i_{R_1} + i_R = 0, \quad (15)$$

where  $i_R$  and  $i_{R_1}$  are the currents going through  $R$  and  $R_1$ , respectively. Applying Ohm's law we get

$$\frac{V_{S1} - v^-}{R_1} + \frac{V_{out} - v^-}{R} = 0. \quad (16)$$

Or equivalently:

$$\frac{V_{S1} + V_x}{R_1} + \frac{V_{out} + V_x}{R} = 0. \quad (17)$$

We can make the substitution  $V_{out} = AV_x$

$$\frac{V_{S1} + V_x}{R_1} + \frac{V_x(A+1)}{R} = 0, \quad (18)$$

and solve for  $V_x$

$$V_x = -\left(\frac{V_{S1}}{R_1}\right) \left( \frac{1}{\left(\frac{A+1}{R} + \frac{1}{R_1}\right)} \right). \quad (19)$$

Lastly, we multiply both equations by  $A$  to get

$$V_{out} = -\left(\frac{V_{S1}}{R_1}\right) \left( \frac{A}{\left(\frac{A+1}{R} + \frac{1}{R_1}\right)} \right). \quad (20)$$

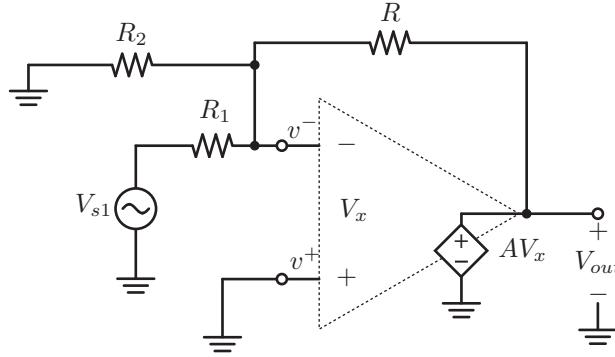
- (b) What happens to  $V_{out}$  in the limit as  $A$  goes to  $\infty$ ?

**Solution:**

The term in the second denominator becomes dominated by  $\frac{A}{R}$  as  $A$  goes to  $\infty$ , so we have

$$\lim_{A \rightarrow \infty} V_{out} = -V_{S1} \left( \frac{R}{R_1} \right) \quad (21)$$

- (c) Solve for  $V_{out}$  in terms of the other circuit quantities, i.e.  $R, R_1, R_2, A$ , and  $V_{S1}$ .



**Solution:**

We first start by defining a sign convention on our resistors. We choose for the minus terminal on all of the resistors to be on the node  $v^-$ . (So, as before, we have  $V_x = -v^-$ ). From KCL at node  $v^-$ , we have

$$i_{R_1} + i_{R_2} + i_R = 0, \quad (22)$$

where  $i_R$ ,  $i_{R_1}$ , and  $i_{R_2}$  are the currents going through  $R$ ,  $R_1$ , and  $R_2$ , respectively. Applying Ohm's law we get

$$\frac{V_{S1} + V_x}{R_1} + \frac{V_x}{R_2} + \frac{V_{out} + V_x}{R} = 0. \quad (23)$$

We can make the substitution  $V_{out} = AV_x$

$$\frac{V_{S1} + V_x}{R_1} + \frac{+V_x}{R_2} + \frac{V_x(A+1)}{R} = 0, \quad (24)$$

and solve for  $V_x$

$$V_x = - \left( \frac{V_{S1}}{R_1} \right) \left( \frac{1}{\left( \frac{A+1}{R} + \frac{1}{R_1} + \frac{1}{R_2} \right)} \right). \quad (25)$$

Lastly, we multiply both equations by  $A$  to get

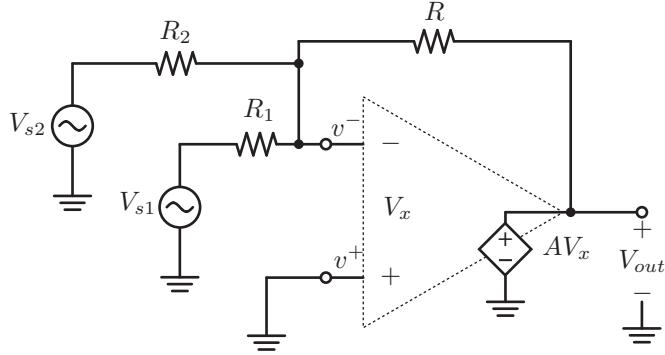
$$V_{out} = - \left( \frac{V_{S1}}{R_1} \right) \left( \frac{A}{\left( \frac{A+1}{R} + \frac{1}{R_1} + \frac{1}{R_2} \right)} \right). \quad (26)$$

- (d) What happens to  $V_{out}$  in the limit as  $A$  goes to  $\infty$ ?

**Solution:**

The term  $(\frac{A+1}{R} + \frac{1}{R_1} + \frac{1}{R_2})$  in the second denominator becomes dominated by  $\frac{A}{R}$  as  $A$  goes to  $\infty$ , so we have

$$\lim_{A \rightarrow \infty} V_{out} = -V_{S1} \left( \frac{R}{R_1} \right) \quad (27)$$



- (e) Solve for  $V_{out}$  in terms of the other circuit quantities, i.e.  $R$ ,  $R_1$ ,  $R_2$ ,  $A$ ,  $V_{S1}$ , and  $V_{S2}$ . (Hint: use superposition)

**Solution:**

We apply superposition. Turning off  $V_{S2}$ , we get the same circuit from part (c), so  $V_{out}$  as a function of  $V_{S1}$  is

$$V_{out}^{S1} = - \left( \frac{V_{S1}}{R_1} \right) \left( \frac{A}{\left( \frac{A+1}{R} + \frac{1}{R_1} + \frac{1}{R_2} \right)} \right) \quad (28)$$

Now turning off  $V_{S1}$  and leaving on  $V_{S2}$ , we get a similar circuit to part (c), except that  $V_{S1}$  and  $R_1$  have switched roles with  $V_{S2}$  and  $R_2$ , respectively.  $V_{out}$  as a function of  $V_{S2}$  is

$$V_{out}^{S2} = - \left( \frac{V_{S2}}{R_2} \right) \left( \frac{A}{\left( \frac{A+1}{R} + \frac{1}{R_2} + \frac{1}{R_1} \right)} \right) \quad (29)$$

$$V_{out} = V_{out}^{S1} + V_{out}^{S2} = - \left( \frac{V_{S1}}{R_1} \right) \left( \frac{A}{\left( \frac{A+1}{R} + \frac{1}{R_1} + \frac{1}{R_2} \right)} \right) - \left( \frac{V_{S2}}{R_2} \right) \left( \frac{A}{\left( \frac{A+1}{R} + \frac{1}{R_2} + \frac{1}{R_1} \right)} \right) \quad (30)$$

- (f) What happens to  $V_{out}$  in the limit as  $A$  goes to  $\infty$ ?

**Solution:**

$$\lim_{A \rightarrow \infty} = -V_{S1} \left( \frac{R}{R_1} \right) - V_{S2} \left( \frac{R}{R_2} \right) \quad (31)$$

- (g) Given that  $R=10 \text{ k}\Omega$ , choose  $R_1$  and  $R_2$  such that  $V_{out} = -(\frac{1}{4}V_{S1} + 2V_{S2})$  in the limit as  $A$  goes to  $\infty$ .

**Solution:**

$R_1 = 40 \text{ k}\Omega$  and  $R_2 = 5 \text{ k}\Omega$

- 7. Your Own Problem** Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?