EECS 16A Designing Information Devices and Systems I
Spring 2020 Homework 10

This homework is due April 10, 2020, at 23:59.
Self-grades are due April 13, 2020, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw10.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned)

Submit each file to its respective assignment on Gradescope.

1. Op-Amp in Negative Feedback

In this question, we analyze op amp circuits that have finite gain. We replace the op amp with its circuit model with parameterized gain and observe the gain's effect on terminal and output voltages as the gain approaches infinity. Figure 1 shows the equivalent model of the op-amp.

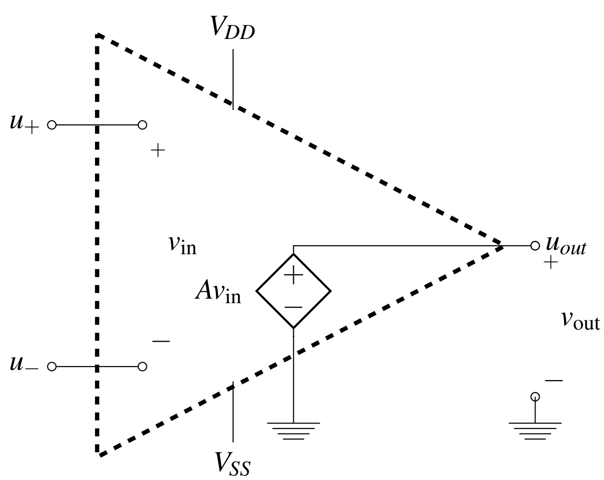


Figure 1: Op-amp model

(a) Consider the circuit shown in Figure 2. Assume that the op amp is ideal ($A \to \infty$) for parts (a) through (e). What is $u_+ - u_-$?

Solution: For ideal op amp circuits in negative feedback, the voltage at the two terminals must be equal, so $u_+ - u_- = 0$.

(b) Find v_x as a function of v_{out}.

Solution: We see that v_x is the middle node of a voltage divider, so $v_x = v_{out} \frac{R_1}{R_1+R_2}$.

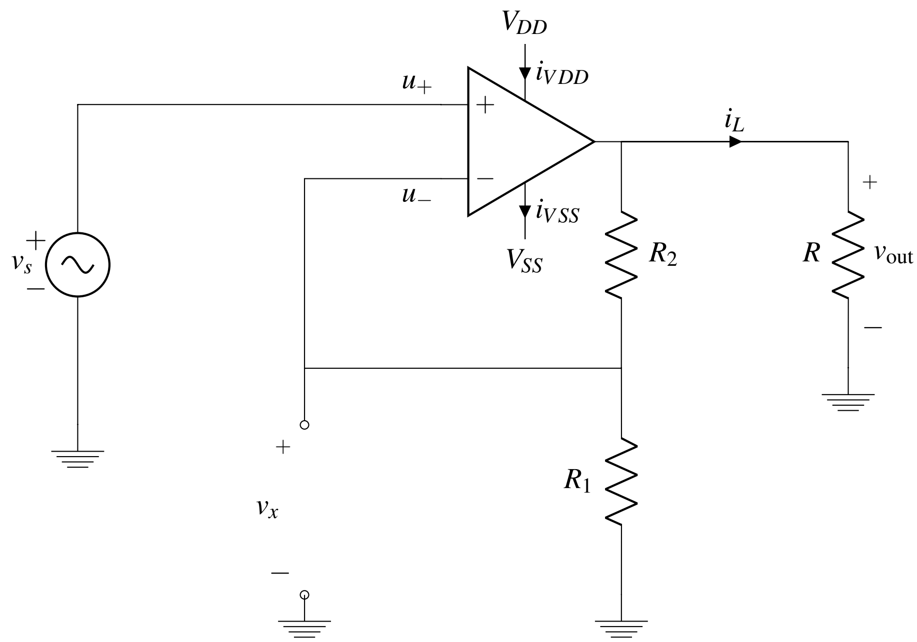


Figure 2: Non-inverting amplifier circuit

(c) What is the current flowing through R_2 as a function of v_s?

Solution: We know from part (a) that $v_x = v_s$. The current flowing through R_1 is $I_{R_1} = \frac{v_s}{R_1}$. This current also flows through R_2.

(d) Find v_{out} as a function of v_s.

Solution: Using the answer from the previous part, $v_{out} = v_s + R_2 I_{R_1} = v_s + R_2 \frac{v_s}{R_1} = v_s \left(\frac{R_1+R_2}{R_1}\right)$.

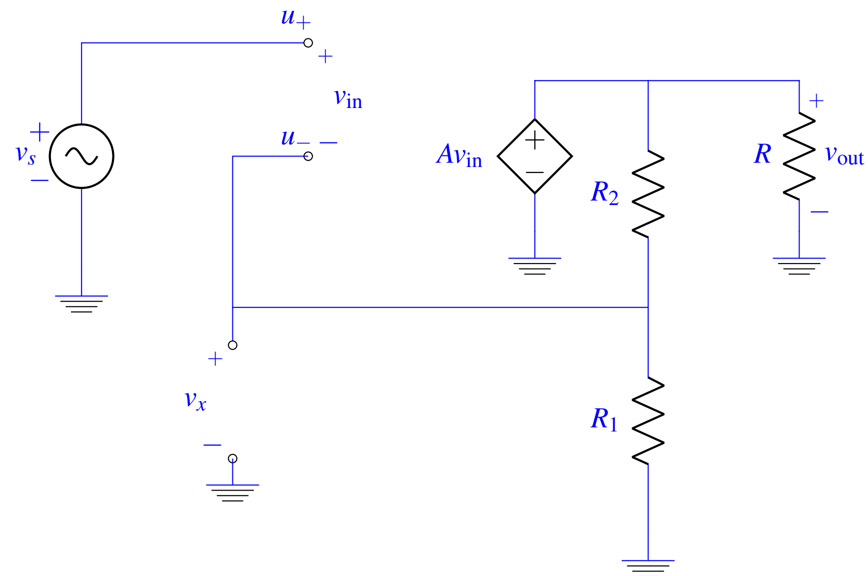
(e) What is the current i_L through the load resistor R? Give your answer in terms of v_{out}.

Solution: The current i_L through the load is $\frac{v_{out}}{R}$.

(f) Draw an equivalent circuit by replacing the op-amp with the op-amp model shown in Figure 1 and calculate v_{out} and v_x in terms of A, v_s, R_1, R_2 and R. Is the magnitude of v_x larger or smaller than the magnitude of v_s? Do these values depend on R?

Solution:

This is the equivalent circuit of the op-amp:



Since v_{out} is connected to the output of the op-amp, which is a voltage source, we can determine v_{out}:

$$\begin{aligned} v_{out} &= A(u_+ - u_-) \\ &= A(v_s - v_x) \end{aligned}$$

Since there is no current flowing into the op amp input terminals from nodes u_+ and u_-, R_1 and R_2 form a voltage divider and $v_x = v_{out}\left(\frac{R_1}{R_1+R_2}\right)$. Thus, substituting and solving for v_{out}:

$$v_{out} = A\left(v_s - v_{out}\frac{R_1}{R_1+R_2}\right)$$

$$v_{out} = v_s\left(\frac{1}{\frac{R_1}{R_1+R_2}+\frac{1}{A}}\right)$$

Knowing v_{out}, we can find v_x:

$$v_x = \frac{v_s}{1+\frac{R_1+R_2}{AR_1}}$$

Notice that v_x is slightly smaller than v_s, meaning that in equilibrium in the non-ideal case, v_+ and v_- are not equal. v_{out} and v_x do not depend on R, which means that we can treat v_{out} as a voltage source that supplies a constant voltage independent of the load R.

(g) Using your solution to the previous part, calculate the limits of v_{out} and v_x as $A \to \infty$. Do you get the same answer as in part (d)?

Solution:

As $A \to \infty$, the fraction $\frac{1}{A} \to 0$, so

$$v_{out} = v_s\left(\frac{1}{\frac{R_1}{R_1+R_2}+\frac{1}{A}}\right)$$

converges to

$$v_s\left(\frac{1}{\frac{R_1}{R_1+R_2}+0}\right) = v_s\left(\frac{R_1+R_2}{R_1}\right).$$

Therefore, the limits as $A \to \infty$ are:

$$v_{\text{out}} \to v_s \left(\frac{R_1 + R_2}{R_1} \right)$$
$$v_x \to v_s$$

If we observe the op amp is in negative feedback, we can apply the fact that $u_+ = u_-$. We get $v_x = v_s$. Then the current i flowing through R_1 to ground is $\frac{v_s}{R_1}$. By KCL, this same current flows through R_2 since no current flows into the negative input terminal of the op amp (u_-). Thus, the voltage drop across R_2 is $v_{\text{out}} - v_x = i \cdot R_2 = v_s \left(\frac{R_2}{R_1} \right)$. Therefore, $v_{\text{out}} = v_s + v_s \left(\frac{R_2}{R_1} \right) = v_s \left(\frac{R_1 + R_2}{R_1} \right)$. The answers are the same if you take the limit as $A \to \infty$.

(h) Now you want to make a circuit whose gain is nominally $G_{nom} = \frac{v_{out}}{v_s} = 2$ with a minimum error of 1% (a minimum gain of $G_{min} = 1.98$). What is the minimum required gain of the amplifier A_{min} to achieve that specification?

Solution: From the previous part, $v_{\text{out}} = v_s \left(\frac{1}{\frac{R_1}{R_1+R_2} + \frac{1}{A}} \right)$. After algebraic manipulations, we get

$$v_{out} = v_s \left(\frac{A(R_1 + R_2)}{R_1 + R_2 + AR_1} \right)$$

We are interested in the op amp's minimum gain A_{min}, which gives us the circuit's corresponding minimum gain G_{min}. Solving for A and substituting in the minimum values A_{min} and G_{min} gives:

$$A_{min} = \frac{G_{min}(R_1 + R_2)}{R_1 + R_2 - G_{min} R_1}$$

Rewriting A_{min} in terms of $G_{nom} = 1 + \frac{R_2}{R_1}$ gives:

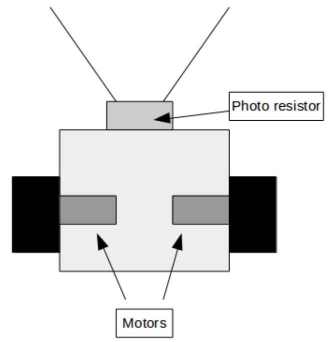
$$\begin{aligned} A_{min} &= \frac{G_{min} G_{nom}}{G_{nom} - G_{min}} \\ &= 198 \end{aligned}$$

Notice that the op amp's minimum gain is independent of the resistor values. In general, if we wanted an error of less than ε, then the following will approximately hold: $\frac{A_{min}}{G_{nom}} > \frac{1}{\varepsilon}$.

2. PetBot Design

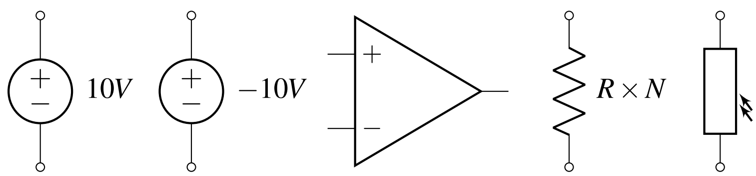
In this problem, you will design circuits to control PetBot, a simple robot designed to follow light. PetBot measures light using photoresistors. A photoresistor is a light-sensitive resistor. As it is exposed to more light, its resistance decreases. Given below is the circuit symbol for a photoresistor.

Below is the basic layout of the PetBot. It has one motor on each wheel. We will model each motor as a $1\,\Omega$ resistor. When motors have positive voltage across them, they drive forward; when they have negative voltage across them, they drive backward. At zero voltage across the motors, the PetBot stops. The speed of the motor is directly proportional to the magnitude of the motor voltage. The light sensor is mounted to the front of the robot.



(a) **Speed control** – Let us begin by first having PetBot decrease its speed as it drives toward the flashlight. Design a motor driver circuit that outputs a decreasing positive motor voltage as the PetBot drives toward the flashlight. The motor voltage should be at least 5 V far away from the flashlight. When far away from the flashlight, the photoresistor value will be $10\,\mathrm{k}\Omega$ and dropping toward $100\,\Omega$ as it gets closer to the flashlight.

In your design, you may use any number of resistors with any value and just 1 op-amp. You also have access to voltage sources of 10 V and −10 V. Based on your circuit, derive an expression for the motor voltage as a function of the circuit components that you used.

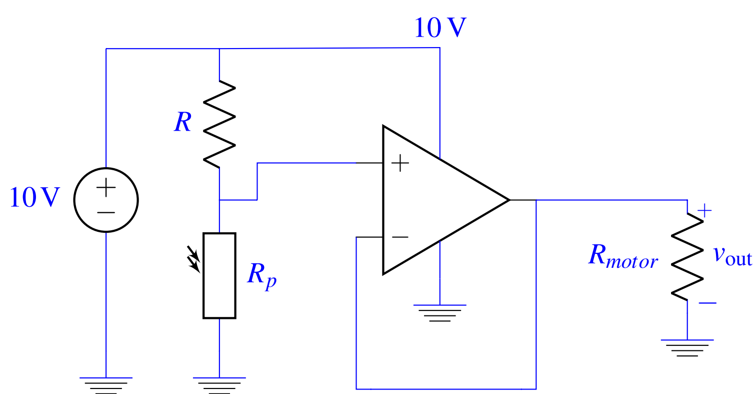


Hint 1: You should consider the loading effect of connecting this circuit to your motor, which has resistance. A buffer may help solve this problem.

Hint 2: If you're not sure where to start, try playing around with connecting the circuit elements in different ways, and think about circuits you have seen before.

Solution:

We can use a voltage divider circuit to adjust the output motor voltage as the PetBot drives towards the flashlight (and the photoresistor's resistance decreases).



The output of the above circuit is:

$$v_{\text{out}} = \frac{R_p}{R_p + R} \cdot 10\,\text{V}$$

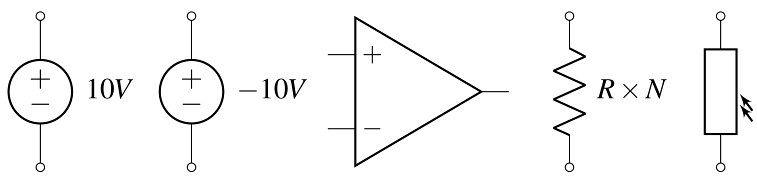
where R_p represents the photoresistor. Note that we use a voltage buffer to prevent loading effects when connecting the motor.

We set $R \leq 10\,\text{k}\Omega$ to achieve $v_{\text{out}} \geq 5\,\text{V}$ when the PetBot is far away from the flashlight (i.e. $R_p = 10\,\text{k}\Omega$). As the PetBot drives towards the flashlight, the resistance R_p drops, so that v_{out}, the motor voltage, decreases.

(b) **Distance control** – Let us now have PetBot drive up to a flashlight (or away from the flashlight) and stop at distance of 1 m away from the light. At the distance of 1 m from the flashlight, the photoresistor has a value $1\,\text{k}\Omega$.

Design a circuit to output a motor voltage that is positive when the PetBot is at a distance greater than 1 m from the flashlight (making the PetBot move toward it), zero at 1 m from the flashlight (making the PetBot stop), and negative at a distance of less than 1 m from the flashlight (making the PetBot back away from the flashlight.)

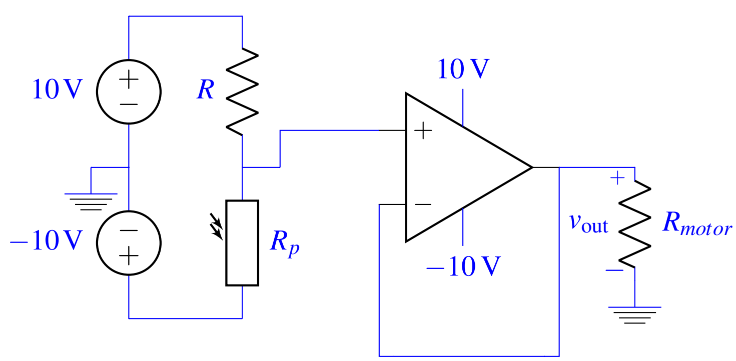
In your design, you may use any number of resistors of any value and just 1 op-amp. You also have access to voltage sources of 10 V and −10 V. Based on your circuit, derive an expression for the motor voltage as a function of the values of circuit components that you used.



Solution:

We outline two possible solutions here:

Method 1:



Here observe that $v_{\text{out}} = u_+$. Using superposition we find that

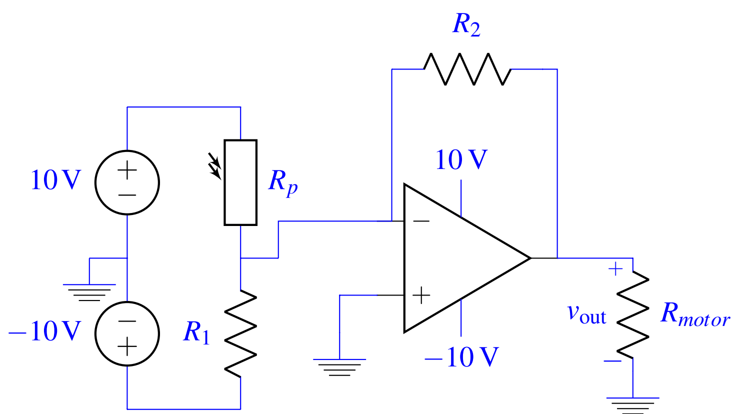
$$v_{\text{out}} = u_+ = 10\text{V}\frac{R_p}{R + R_p} - 10\text{V}\frac{R}{R + R_p} = 10\text{V}\frac{R_p - R}{R + R_p}$$

To satisfy the condition that $v_{\text{out}} = 0\text{V}$ when Petbot is 1 m away, we have that $R = 1\,\text{k}\Omega$. Similar to the previous design, we can do the analysis for when the Petbot is far away and close by. We will show how to do it when the Petbot is close by here.

$$R_p < R = 1\,\text{k}\Omega$$

$$\Rightarrow R_p - R < 0$$
$$\Rightarrow v_{\text{out}} = 10\text{V}\frac{R_p - R}{R + R_p} < 0\text{V}$$

Method 2:



Choosing $R_1 = R = 1\,\text{k}\Omega$ we observe that the voltage $v_{\text{out}} = 0\text{V}$ when Petbot is $1\,\text{m}$ away as required. To find v_{out} more generally, observe that $u_- = u_+ = 0$, so we need to find the current i going through R_2 from node u_- to node u_{out} to get v_{out}.

$$v_{\text{out}} = -iR_2 = -\left(\frac{10\text{V}}{R_p} + \frac{-10\text{V}}{R_1}\right) R_2$$

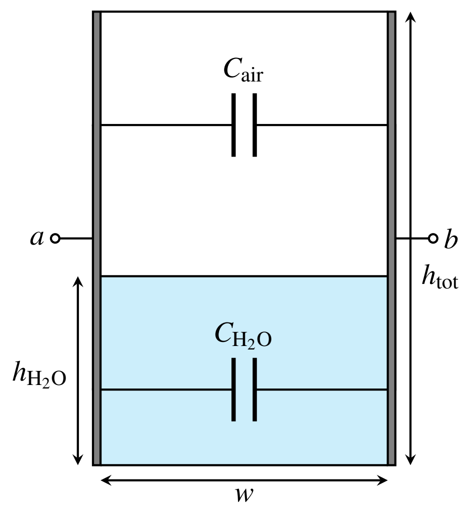
From this we see that when PetBot is greater than $1\,\text{m}$ away we have

$$R_p > R_1 = 1\,\text{k}\Omega$$
$$\frac{1}{R_p} - \frac{1}{R_1} < 0$$
$$\Rightarrow -\left(\frac{1}{R_p} - \frac{1}{R_1}\right) \cdot R_2 10\text{V} > 0$$

Similarly when the Petbot is less than $1\,\text{m}$ away, we have that the motor voltage will be negative.

3. Rain Sensor v2.0

In a previous homework, we analyzed a rain sensor built by a lettuce farmer in Salinas Valley. They used a rectangular tank outside and attached two metal plates to two opposite sides in an effort to make a capacitor whose capacitance varies with the amount of water inside. The width and length of the tank are both w (i.e. the base is square), and the height of the tank is h_{tot}.



As your EE16A circuits toolkit is now complete with capacitors, op-amps, and switches, we will revisit this problem to improve the readout electronics. The goal is to create a circuit block that will output voltage as a linear function of the water height, h_{H_2O}.

(a) What is the capacitance between terminals a and b when the tank is empty, C_{empty}? Again, the height of the water in the tank is h_{H_2O}. Modeling the tank as a pair of capacitors in parallel, find the total capacitance C_{tank} between the two plates. Can you write C_{tank} as a function of C_{empty}?

Note: The permittivity of air is ε, and the permittivity of rainwater is 81ε.

Solution:

$$C_{empty} = \frac{\varepsilon_{air}\, h_{tot} w}{w} = \varepsilon\, h_{tot}$$

For C_{tank}, we can break the total capacitance into two parts. First, let's calculate the capacitance of the two plates separated by water:

$$C_{water} = \frac{\varepsilon_{H_2O}\, h_{H_2O} w}{w} = 81\varepsilon\, h_{H_2O}$$

And now, we can calculate the capacitance of the two plates separated by air:

$$C_{air} = \frac{\varepsilon_{air}\,(h_{tot} - h_{H_2O})\, w}{w} = \varepsilon\,(h_{tot} - h_{H_2O})$$

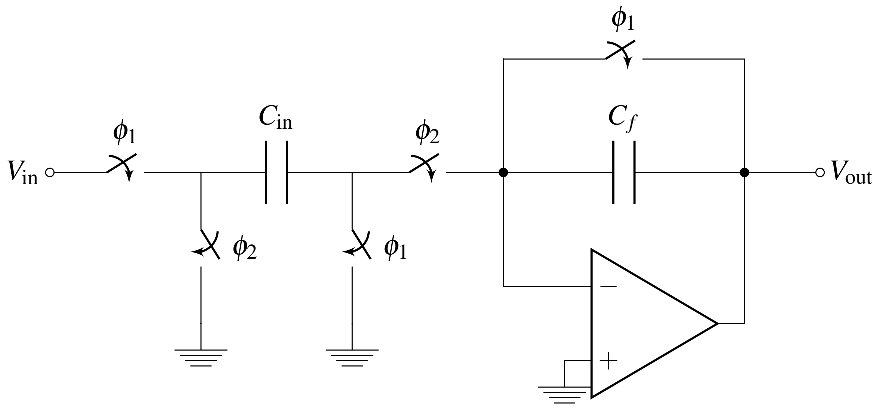
Because these two capacitors appear in parallel, we can simply add our two previous results to find the total equivalent capacitance:

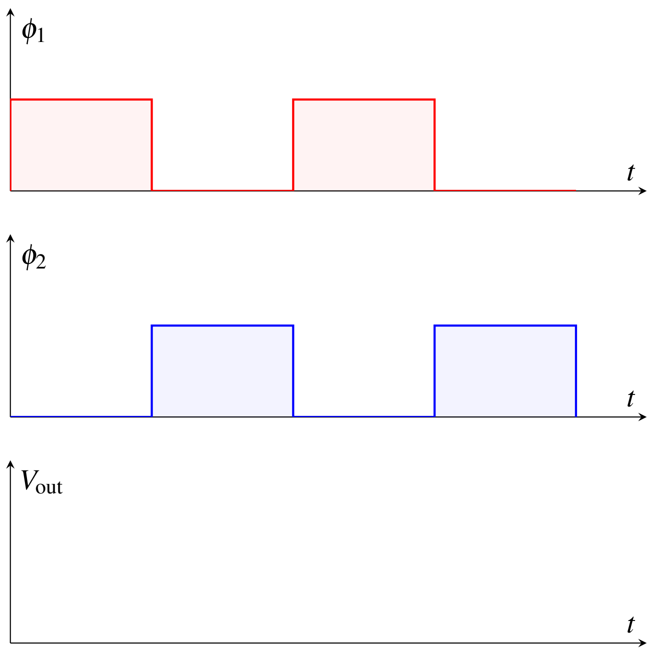
$$C_{tank} = C_{water} + C_{air} = \varepsilon\,(h_{tot} + 80h_{H_2O})$$

Now, we can rewrite the above equation to show the dependence on C_{empty}:

$$C_{tank} = C_{empty} + 80\varepsilon h_{H_2O}$$

(b) Here, we will analyze a circuit that transfers all charges for efficient readout. For the circuit below, draw the output waveform of v_{out} as a function of v_{in}, C_f, and C_{in}.





Solution:

In ϕ_1, C_{in} gets charged up to

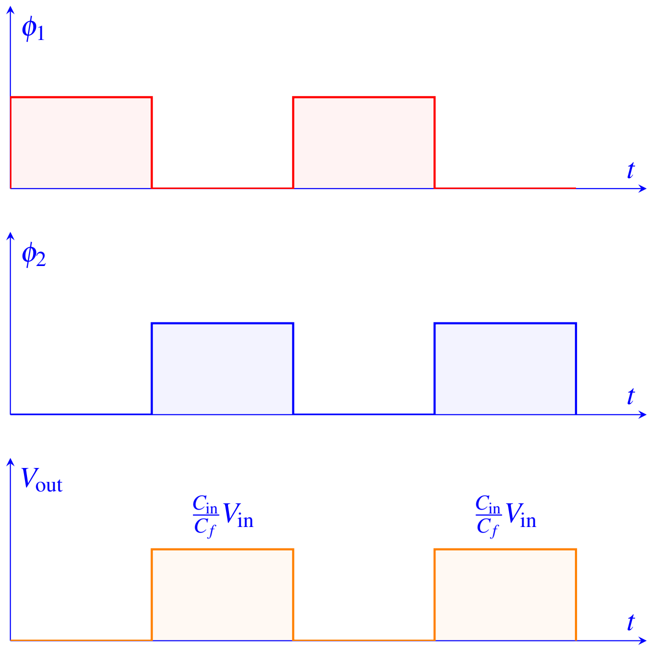
$$Q_{C_{in},1} = C_{in}v_{in}.$$

On the other hand, since v_{out} and v_- are at zero potential, no charge is stored on C_f, so

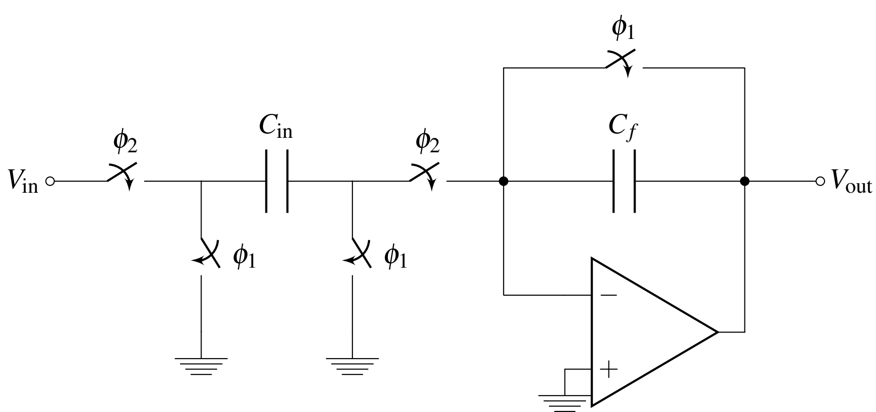
$$Q_{C_f,1} = 0.$$

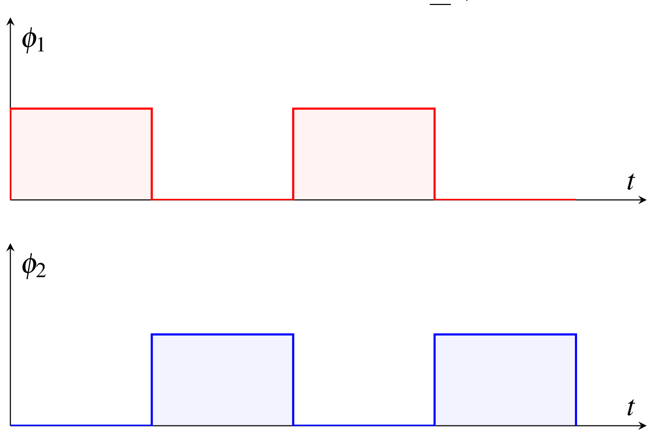
In ϕ_2, the positive plate of C_{in} is shorted to ground, which forces all of the negative charges to move to the left plate of C_f. Now using charge conservation,

$$Q_{C_{in},1} + Q_{C_f,1} = Q_{C_{in},2} + Q_{C_f,2}$$
$$C_{in}v_{in} + 0 = 0 + C_f v_{out}$$
$$v_{out} = \frac{C_{in}}{C_f}v_{in}$$



(c) We examined the non-inverting configuration in the previous part- now, we will look into the inverting configuration. For the circuit below, draw the output waveform of v_{out} as a function of v_{in}, C_f, and C_{in}.





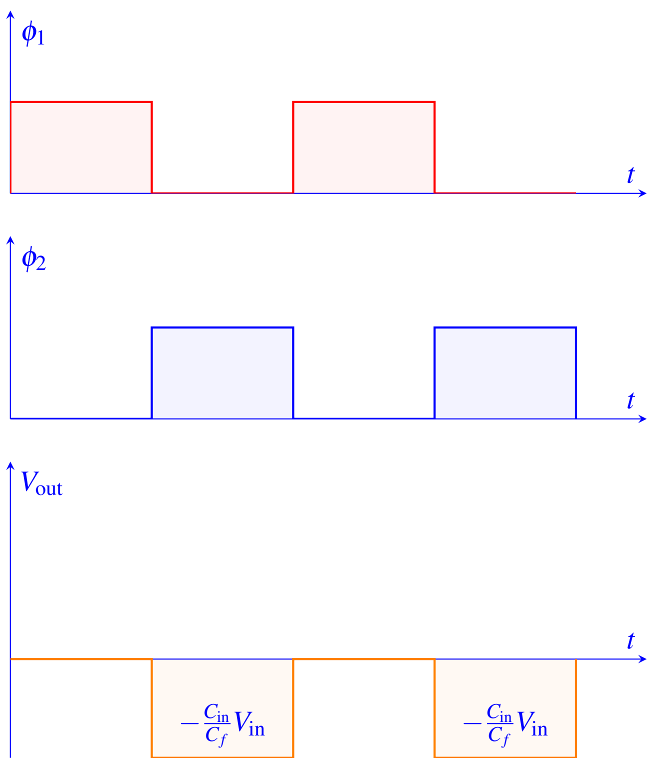
Solution:

In ϕ_1, the voltage across both the capacitors is zero, so

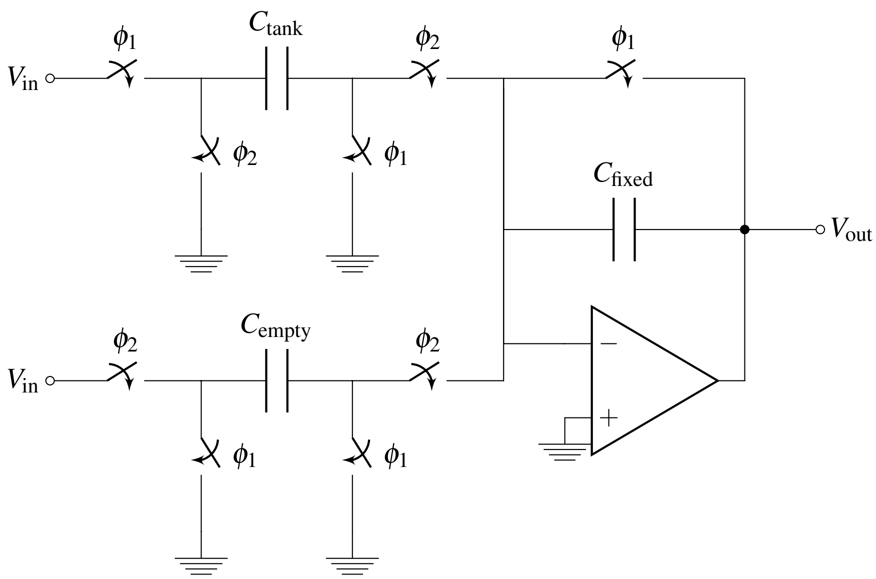
$$Q_{C_{\text{in}},1} = Q_{C_f,1} = 0.$$

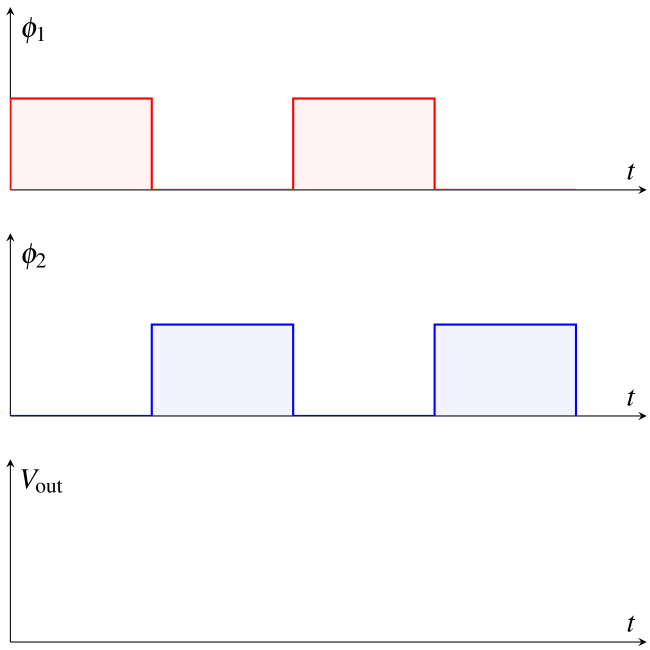
In ϕ_2, C_{in} gets charged up to $Q_{C_{\text{in}},2} = C_{\text{in}} v_{\text{in}}$. Since C_f is in series, C_f stores the same amount of charge but with opposite polarity, so $Q_{C_f,2} = -C_f v_{\text{out}}$.

$$\begin{aligned} Q_{C_{\text{in}},2} &= Q_{C_f,2} \\ C_{\text{in}} v_{\text{in}} &= -C_f v_{\text{out}} \\ v_{\text{out}} &= -\frac{C_{\text{in}}}{C_f} v_{\text{in}} \end{aligned}$$



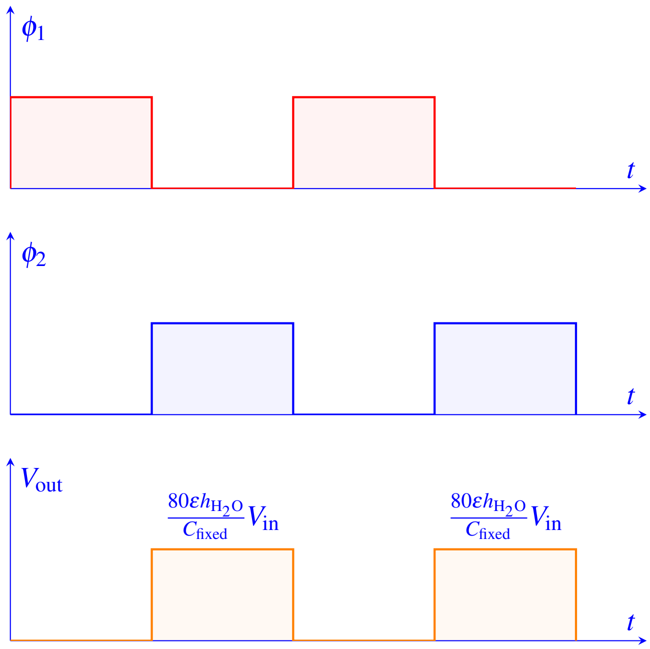
(d) With the help of the basic circuit blocks shown in parts (b) and (c), we will now implement a circuit that will output voltage as a linear function of the water height, h_{H_2O}. In addition to the rain-sensing capacitor, we will use two fixed value capacitors C_{fixed} and C_{empty}. Use the values obtained in part (a) for C_{tank} and C_{empty}. For the circuit below, draw the output waveform of v_{out} as a function of v_{in}, C_{fixed}, ε, and h_{H_2O}.





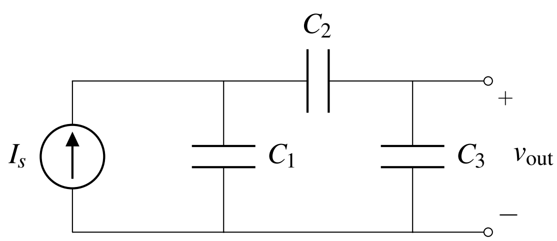
Solution:

$$
\begin{aligned}
v_{\text{out}} &= \left(\frac{C_{\text{tank}}}{C_{\text{fixed}}} - \frac{C_{\text{empty}}}{C_{\text{fixed}}}\right) v_{\text{in}} \\
&= \frac{C_{\text{empty}} + 80\varepsilon h_{\text{H}_2\text{O}} - C_{\text{empty}}}{C_{\text{fixed}}} v_{\text{in}} \\
&= \frac{80\varepsilon h_{\text{H}_2\text{O}}}{C_{\text{fixed}}} v_{\text{in}}
\end{aligned}
$$



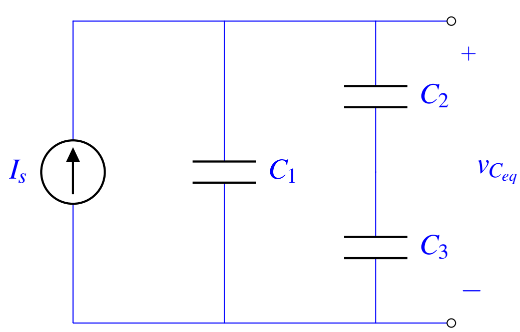
4. (PRACTICE) More Current Sources And Capacitors

For the circuit given below, give an expression for $v_{out}(t)$ in terms of I_s, C_1, C_2, C_3, and t. Assume that all capacitors are initially uncharged, i.e. the initial voltage across each capacitor is 0 V.

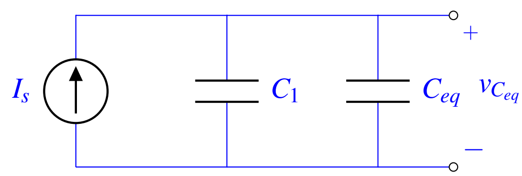


Solution:

Instead of finding v_{out} directly, let's first find the voltage $v_{C_{eq}}$ across C_2 and C_3.



To do this, we replace C_2 and C_3 with their equivalent capacitance $C_{eq} = C_2 \parallel C_3 = \frac{C_2C_3}{C_2+C_3}$.



We know that to solve for $v_{C_{eq}}$, we can find the equivalent capacitance of C_1 and C_{eq} first, which is $C_1 + C_{eq}$. Since the capacitors are initially uncharged, $v_{C_{eq}}(0) = 0$.

$$v_{C_{eq}}(t) = \int \frac{I_s}{C_1 + C_{eq}} dt = \frac{I_s t}{C_1 + C_{eq}} + v_{C_{eq}}(0) = \frac{I_s t}{C_1 + C_{eq}}$$

Now that we know that voltage across the equivalent capacitor C_{eq}, we can find the current flowing through the equivalent capacitor C_{eq}.

$$i_{C_{eq}}(t) = C_{eq} \frac{dv_{C_{eq}}(t)}{dt} = \frac{C_{eq} I_s}{C_1 + C_{eq}}$$

Note that the current $i_{C_{eq}}$ is equal to the current flowing through C_3 since C_2 and C_3 were originally connected in series.

$$i_{C_3}(t) = i_{C_{eq}}(t) = \frac{C_{eq} I_s}{C_1 + C_{eq}}$$

Since v_{out} is the voltage across the capacitor C_3, we integrate to find v_{out}. Again, since all capacitors are initially uncharged, $v_{\text{out}}(0) = 0$.

$$i_{C_3}(t) = C_3 \frac{dv_{\text{out}}(t)}{dt}$$

$$v_{\text{out}}(t) = \int \frac{C_{eq} I_s}{C_3(C_1 + C_{eq})} dt = \frac{C_{eq} I_s t}{C_3(C_1 + C_{eq})} + v_{\text{out}}(0) = \frac{\frac{C_2 C_3}{C_2 + C_3} I_s t}{C_3\left(C_1 + \frac{C_2 C_3}{C_2 + C_3}\right)} = \frac{C_2 I_s t}{C_1 C_2 + C_1 C_3 + C_2 C_3}$$

5. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.