This homework is due October 4, 2016, at Noon.

Optional Problems: We **do not** grade these problems. Nevertheless, you are responsible for learning the subject matter within their scope.
Bonus Problems: We **do** grade these problems. Doing them will provide an unspecified amount of extra credit; not doing them will not affect your homework grade negatively. We will specify if the problem is in or out of scope.

1. Homework process and study group

Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?
Working in groups of 3-5 will earn credit for your participation grade.

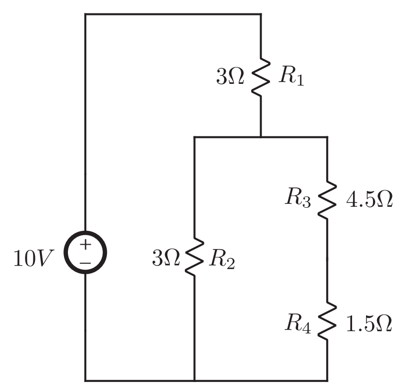
Solution: I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on Problem 5 so I went to office hours on...

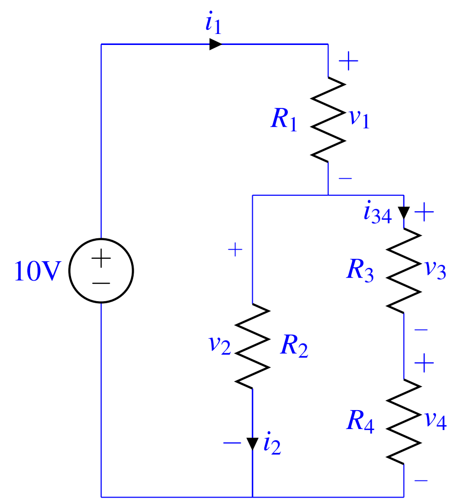
Then I went to homework party for a few hours, where I finished the homework.

2. Mechanical Circuits

(a) Find the voltages across and currents flowing through all the resistors.



Solution: **Approach 1 – KCL / KVL:**
First, label all the 'junctions' or 'nodes':



Now set up your KCL / KVL equations:

$$i_1 = i_2 + i_{34} \tag{1}$$
$$v_1 = i_1 \cdot R_1 \tag{2}$$
$$v_2 = i_2 \cdot R_2 \tag{3}$$
$$v_3 = i_{34} \cdot R_3 \tag{4}$$
$$v_4 = i_{34} \cdot R_4 \tag{5}$$
$$v_2 = v_3 + v_4 = i_{34} \cdot (R_3 + R_4) \tag{6}$$
$$10 = v_1 + v_2 = v_1 + v_3 + v_4 \tag{7}$$

You can solve as a system of equations:

$$\frac{10 - v_2}{R_1} = \frac{v_2}{R_2} + \frac{v_2}{R_3 + R_4}$$

$$\frac{10}{R_1} = v_2 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3 + R_4} \right)$$

$$\frac{10}{3} = v_2 \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{4.5 + 1.5} \right) = v_2 \cdot \frac{5}{6} \implies v_2 = 4$$

$$v_1 = 10 - v_2 = 6\,\text{V} \qquad i_{34} = \frac{4}{4.5 + 1.5} = \frac{2}{3}\,\text{A} \qquad v_3 = i_{34} R_3 = 3\,\text{V} \qquad v_4 = i_{34} R_4 = 1\,\text{V}$$

Alternatively, you could set it up as a matrix and use ipython / numpy to solve.

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4.5 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1.5 & 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_{34} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

This returns the array:

$$\begin{bmatrix} i_1 \\ i_2 \\ i_{34} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{4}{3} \\ \frac{2}{3} \\ 6 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

Approach 2 – We will first calculate the effective resistance seen from the voltage source to find the current supplied by the voltage source. The resistances R_3 and R_4 are in series hence have effective resistance of 6Ω. They are connected in parallel to a R_2 resistance yielding an effective resistance of

$$\left(\frac{1}{6}+\frac{1}{3}\right)^{-1} = 2\Omega.$$

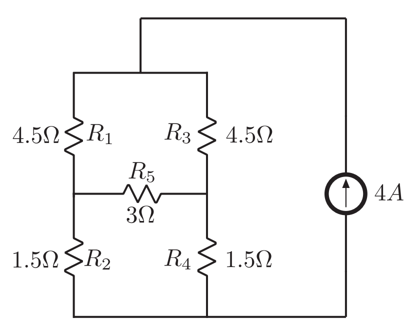
This resulting effective resistance is in series to R_1, yielding an effective resistance of 5Ω. Hence the current supplied by the voltage source is

$$10V/5\Omega = 2A.$$

Let us denote the voltage drop across R_i as V_i, and the current flowing through R_i and I_i. We have $I_1 = 2A$ current flowing through R_1, hence we have $V_1 = 6V$. The remaining voltage $10V - 6V = 4V$ is across both R_2 and the sequence of resistors R_3 and R_4. Hence, $I_2 = 4V/3\Omega = 4/3A$. Furthermore, $I_3 = 4V/6\Omega = 2/3A$. Combining all, we get the following voltages and currents,

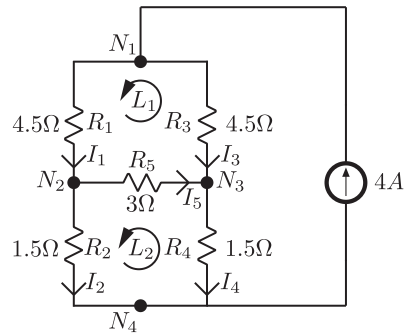
$$\begin{aligned} I_1 &= 2A, \\ I_2 &= \frac{4}{3}A, \\ I_3 = I_4 &= \frac{2}{3}A, \\ V_1 &= 6V, \\ V_2 &= 4V, \\ V_3 &= 3V, \\ V_4 &= 1V. \end{aligned}$$

(b) Find the voltages across and currents flowing through all the resistors.



Solution:

Approach 1 – Let us denote the voltage drop across R_i as V_i, and the current flowing through R_i and I_i; and let us label the nodes with N_1, N_2, N_3 and N_4 as shown in the figure below.



At the nodes we have the following KCL equalities

$$
\begin{aligned}
I_1 + I_3 &= 4A, \quad (\text{at node } N_1) \\
I_5 &= I_1 - I_2, \quad (\text{at node } N_2) \\
I_5 &= I_4 - I_3, \quad (\text{at node } N_3) \\
I_2 + I_4 &= 4A. \quad (\text{at node } N_4)
\end{aligned}
$$

Furthermore, we can write the KVL around the loops L_1 and L_2

$$4.5I_3 - 4.5I_1 - 3I_5 = 0, \tag{8}$$
$$1.5I_4 + 3I_5 - 1.5I_2 = 0. \tag{9}$$
$$\tag{10}$$

Multiplying equation (9) by three and subtracting equation (8) we get

$$
\begin{aligned}
0 &= 3(1.5I_4 + 3I_5 - 1.5I_2) - (4.5I_3 - 4.5I_1 - 3I_5) \\
&= 4.5(I_4 - I_3 + I_4 - I_3) + 12I_5.
\end{aligned}
$$

Substituting the equations coming from KCL at nodes N_2 and N_3 we get

$$
\begin{aligned}
0 &= 4.5(I_4 - I_3 + I_4 - I_3) + 12I_5 \\
&= 4.5(I_5 + I_5) + 12I_5 \\
&= 21I_5,
\end{aligned}
$$

which shows that the current through resistor R_5 needs to be

$$I_5 = 0.$$

Substituting this in equation (9) and (8) we get

$$I_2 = I_4, \tag{11}$$
$$I_1 = I_3, \tag{12}$$

and the KCL at nodes N_2 and N_3 then say that

$$
\begin{aligned}
I_1 &= I_2 \\
I_3 &= I_4,
\end{aligned}
$$

hence we have

$$I_1 = I_2 = I_3 = I_4.$$

By the KCL at node N_1 we have

$$I_1 + I_3 = 4A$$

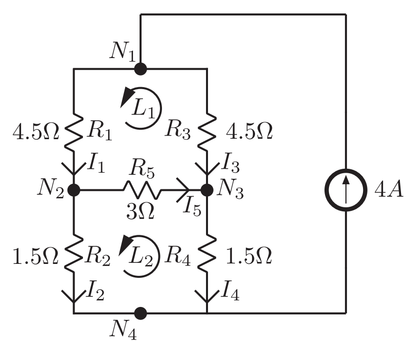
which says

$$I_1 = I_2 = I_3 = I_4 = 2A.$$

Using the currents we have found, we can calculate

$$V_1 = V_3 = 4.5\Omega \times 2A = 9V,$$
$$V_2 = V_4 = 1.5\Omega \times 2A = 3V,$$
$$V_5 = 3\Omega \times 0A = 0V.$$

Approach 2 – We can also use KCL/KVL analysis, and write our system in terms of unknown voltages (instead of unknown currents, as in the previous approach). Let node N_4 be our reference ground, at voltage $0V$. Then the three unknown nodal voltages are at nodes N_1, N_2, N_3. Let these nodal voltages be u_1, u_2, u_3 respectively. The figure is reproduced below for convenience:



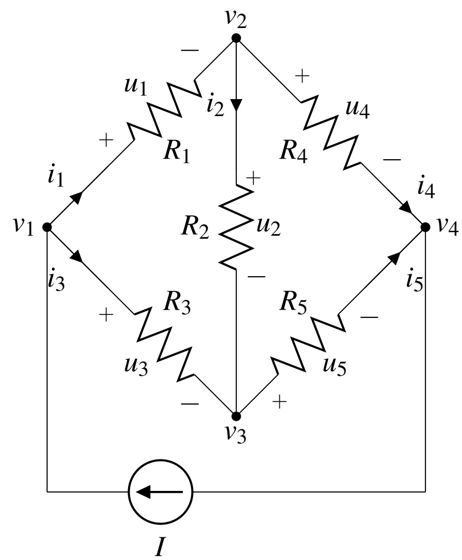
Then write the KCL constraints (current conservation) at each node:

$$+(u_2 - u_1)/R_1 + (u_3 - u_1)/R_3 + 4 = 0 \qquad \text{(Current conservation at } N_1\text{)}$$
$$+(u_1 - u_2)/R_1 + (u_3 - u_2)/R_5 + (0 - u_2)/R_2 = 0 \qquad \text{(Current conservation at } N_2\text{)}$$
$$+(u_1 - u_3)/R_1 + (u_2 - u_3)/R_5 + (0 - u_3)/R_2 = 0 \qquad \text{(Current conservation at } N_3\text{)}$$

Substituting the actual values of the resistors and solving this system, we find: $u_1 = 12V, u_2 = 3V, u_3 = 3V$. Then all the branch currents can be found from these nodal voltages, using Ohm's Law.

Approach 3 – It was possible to notice "by symmetry" that there is no current across the middle resistor R_5, and further that the currents across the two remaining branches must be equal. It is a good intuition that symmetry of a circuit will result in symmetry in its electric flow. This is in fact generally true, but requires a more careful argument to justify.[1] See the "Uniqueness and Symmetry" problem on Discussion 7A for details on how to invoke symmetry arguments, and when they apply.

[1]One might argue that "nature has no preference", and hence cannot break symmetries. However, this is not necessarily true – there are many situations where small random fluctuations can break initial symmetries. For example, in unstable equilibriums (like balancing a pen on end).

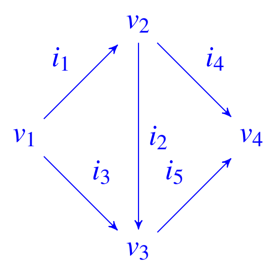


3. Circuits

In discussion 5A, we went over how to approach solving circuits from a linear algebraic perspective. We will now practice this technique with a slightly different circuit.

(a) Translate the above circuit into a directed graph, ignoring the current source for now. Write the incidence matrix for the graph.

Solution:



The incidence matrix for the above graph is as follows: $F = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

(b) Let R be the diagonal matrix of branch resistances. Write Ohm's law as a matrix equation in terms of R, F, $\vec{i}$, and $\vec{v}$. Specify the contents of each vector/matrix that you use by writing out the individual elements.

Solution: $R = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 \\ 0 & 0 & 0 & 0 & R_5 \end{bmatrix}$

F is the incidence matrix from before, and

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \text{ and } \vec{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}$$

We know that $\vec{u} = R\vec{i}$, but we also know $\vec{u} = F\vec{v}$, so we can claim: $F\vec{v} = R\vec{i}$

(c) Let $\vec{f}$ represent the vector of independent currents, such that the KCL equation $F^T\vec{i} + f = 0$. Use this information, in addition to the previously derived equations to write $\vec{v}$ in terms of known quantities $(\vec{f}, G, F, R)$. You can use G as the conductance matrix.

Solution:

$$\vec{f} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

$F^T GF\vec{v} = -\vec{f}$.

To solve this we can either find a solution by row reduction, or set an element in $\vec{v}$ as ground ($\vec{v}_{gr}$ is the vector excluding this element, since we know it is set to 0), and drop the corresponding column of F and element of $\vec{f}$. Call the new matrix F_{gr} and the new $\vec{f}$ as $\vec{f}_{gr}$. Then F_{gr} has no null space and:

$\vec{v}_{gr} = -(F_{gr}^T GF_{gr})^{-1}\vec{f}_{gr}$.

See the solution notebook for the implementation.

(d) Now, use this information to write $\vec{i}$ in terms of known quantities $(\vec{f}, G, F, R, \vec{v})$.

Solution: $\vec{i} = GF_{gr}\vec{v}_{gr}$

(e) In an iPython notebook, solve for $\vec{v}$ and $\vec{i}$ in the given circuit. Let $R_1 = 100,000\Omega$, $R_2 = 200\Omega$ $R_3 = 100\Omega$ $R_4 = 100,000\Omega$ $R_5 = 100\Omega$ and $I = 3A$.

4. Cell Phone Battery

As great as smartphones are, one of the main gripes about them is that they need to be recharged too often. Suppose a Samsung Galaxy S3 requires about 0.4 W to maintain a signal as well as its regular activities (dominated by the display and backlight in many cases). The battery provides 2200 mAh at a voltage of 3.8V until it is completely discharged.

(a) How long will one full charge last you?

Solution: 400 mW of power at 3.8V is about 105.26mA of current. A battery that can provide 1mAh can provide 1mA for an hour, so our 2200mAh battery can source 105.26mA for $2200/105.26 = 20.9$ hours, almost a full day.

An alternative approach is to say 2200 mAh at 3.8V is $2200 \times 3.8 = 8360$ milliwatt-hours. 0.4 W is 400 mW, so $8360/400 = 20.9$ hours is how long the charge will last.

(b) Suppose the cell phone battery is completely discharged and you want to recharge it completely. How much energy (in J) is this? How much charge (in C) must be pumped through the battery?

Solution: The battery is rated for 2200 mAh at 3.8V, which gives $2200 \times 3.8 = 8360$ milliwatt-hours. A joule is equivalent to watt-second, and there are 3600 seconds in an hour, so our battery has $8360 \times 3600 = 30,096,000$ mJ, or 30,096 J. A milliamp is 0.001 coulombs per second, so a milliamp-hour is $0.001 \times 3600 = 3.6$ coulombs. Then a 2200 mAh battery needs $3.6 \times 2200 = 7,920$ coulombs of charge to be moved to be fully recharged. An electron has charge approximately 1.602×10^{-19} C, so 7,920 coulombs is $7920/1.602 \times 10^{-19} \approx 4.94 \times 10^{22}$ electrons.

(c) Suppose PG&E charges \$0.16 per kWh. Every day, you completely discharge the battery and recharge it at night. How much will recharging cost you for the month of October (31 days)?

Solution: 2200 mAh at 3.8V is $2200 \times 3.8 = 8,360$ milliwatt-hours, or 0.00836 kWh. At \$0.16 per kWh, that is 0.16×0.00836 dollars per day, or $0.16 \times 0.00836 \times 31 = 0.0414656$, or about 4 cents a month.

(d) You are fed up with PG&E, gas companies, and Duracell/Energizer/etc. You want to generate your own energy and decide to buy a small solar cell (e.g. `http://ixdev.ixys.com/DataSheet/XOB17-Solar-Bit-Datasheet_Mar-2008.pdf`) for \$1.50 on digikey. It delivers 40 mA at 0.5 V in bright sunlight. Unfortunately, now you have can only charge your phone when the sun is up. Using one solar cell, do you think there is enough time to charge a completely discharged phone every day? How many cells would you need to charge a completely discharged battery in an hour? How much will it cost you per joule if you have one solar cell that works for 10 years (assuming you can charge for 16 hours a day)? Do you think this is a good option?

Solution: One solar cell provides 40 mA at 0.5 V in bright sunlight, which is $40 \times 0.5 = 20$ mW. Note that this is considerably less than the phone's power consumption of 400 mW, so we should expect the solar panel to take much longer to charge than the ≈ 1 day of battery life. In fact, $2200 \times 3.8 \times (20)^{-1} = 418$ hours to charge using one panel, so even if the sun was bright all day, it would not be enough to charge the phone. It would take 418 panels to charge the battery in an hour. The total amount of energy collected over 10 years, assuming 16 hours of operation per day, is approximately (neglecting leap-years, etc.) $10 \times 365 \times 16 \times 40 \times 0.5 = 1,168,000$ mWh, or 1.168 kWh. That is $1.50/1.168 \approx 1.28$ dollars per kWh, or $1.50 \times (1.168 \times 3600 \times 1000) = 3.57 \times 10^{-7}$ dollars per joule. This particular solar panel doesn't provide enough power to charge the phone's battery during the daylight hours (with a reasonable number of panels). However, if the panel lasts long enough, the cost per joule is actually pretty reasonable.

(e) The battery has a lot of internal circuitry that prevents it from getting overcharged (and possibly exploding!) as well as transfering power into the chemical reactions used to store energy. We will model this internal circuitry as being one resistor with resistance R_{bat}, which you can set to any non-negative value you want. Furthermore, we'll assume that all the energy dissipated across R_{bat} goes to recharging the battery. Suppose the wall plug and wire can be modeled as a 5V voltage source and 200 mΩ resistor, as pictured in Fig. **??**. What is the power dissipated across R_{bat} for R_{bat}= 1mΩ , 1Ω, and 10kΩ? How long will the battery take to charge for each of those values of R_{bat}?

Solution: The energy stored in the battery is 2200mAh at 3.8V, which is $2.2 \times 3.8 = 8.36$Wh. We can find time to charge by dividing this energy by power in W to get time in hours.

For R_{bat}= 1mΩ, the total resistance seen by the battery is $1\text{m}\Omega + 200\text{m}\Omega = 201\text{m}\Omega$ (because the wire and R_{bat} are in series), so by Ohm's law, the current is $5/0.201 = 24.88$A. The voltage drop across R_{bat} is (again by Ohm's law) $24.88 \times 0.001 = 0.025$V. Then power is $0.025 \times 24.88 = 0.622$W and the total time to charge the battery is $8.36/0.622 = 13.44$ hours.

Similarly, for 1Ω, the total resistance seen by the battery is $1 + .200 = 1.2\Omega$, the current through the battery is $5/1.2 = 4.17$A, and the voltage across the battery is by Ohm's law $4.17 \times 1 = 4.17$V. Then the power is $4.17 \times 4.17 = 17.39$W and the total time to charge the battery is $8.36/17.39 = 0.48$ hours, about 29 minutes. (Thanks to Kailas Vodrahalli for spotting the mistake in the original solutions.)

For 10kΩ, the total resistance seen by the battery is $10000 + .200 = 10000.2\Omega$, the current through the battery is $5/10000.2 \approx 0.5$mA, and the voltage across the battery is by Ohm's law $0.5\text{mA} \times 10\text{k}\Omega \approx 5$V (up to 2 significant figures). Then the power is $5\text{V} \times 0.5\text{mA} = 2.5$mW and the total time to charge the battery is $8.36/0.0025 = 3,344$ hours.

(f) (Bonus) Suppose you forgot to charge your phone overnight, and you're in a hurry to charge it before you leave home for the day. What should we set R_{bat} to be if we want to charge our battery as quickly as possible? How much current will this draw? How long will it take to charge?

Hint: what choice of R_{bat} maximizes the power dissipated across the resistor?

Solution: To minimize the time it takes to charge the battery, we want to minimize the time it takes to accumulate the amount of energy to fill the battery. For constant power (which we have because the circuit is not time varying), $E = PT$, so to minimize time T we should maximize power P.

We know $P = I_R^2 R_{\text{bat}}$, so we want to find R_{bat} that maximizes that expression. First observe that the 200mΩ resistor and R_{bat} are in series, so we can treat them together as one equivalent resistor with resistance 200mΩ+R_{bat}. Then by Ohm's law $I_R = \frac{5\text{V}}{200\text{m}\Omega + R_{\text{bat}}}$. We then maximize

$$P = \left(\frac{5}{0.2 + R_{\text{bat}}}\right)^2 \times R_{\text{bat}}$$

where R_{bat} is in Ohms and P is in Watts.

The R_{bat} that maximizes P is $R_{\text{bat}} = 200\text{m}\Omega$. To show this, find the derivative of P with respect to R_{bat} and set it equal to zero.

$$\frac{\text{d}P}{\text{d}R_{\text{bat}}} = 25\left(\frac{1}{(0.2 + R_{\text{bat}})^2} + \frac{-2R_{\text{bat}}}{(0.2 + R_{\text{bat}})^3}\right) = 25\frac{0.2 - R_{\text{bat}}}{(0.2 + R_{\text{bat}})^2}$$

Then $\frac{\text{d}P}{\text{d}R_{\text{bat}}} = 0$ when $R_{\text{bat}} = 0.2\Omega$. From calculus, we know that this could be a maximum- you can check the sign of $\frac{\text{d}^2P}{\text{d}R_{\text{bat}}^2}$ or plot P to convince yourself that it actually is a maximum.

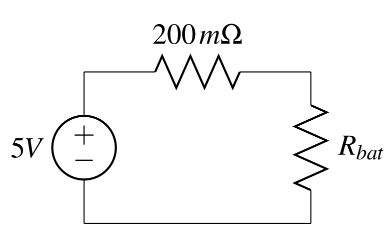
Choosing $R_{\text{bat}}$to be 200mΩ will cause the resistance of the wire and R_{bat} in series to be 400mΩ, so by Ohm's law the current through the battery will be $5/0.4 = 12.5$A.

The battery has 8.36Wh, and the power dissipated across the battery is

$$P_{R_{\text{bat}}} = I_{R_{\text{bat}}}^2 \times R_{\text{bat}} = 12.5^2 \times 0.2 = 31.25\text{W}$$

so to get charging time we divide energy by power and get $8.36/31.25 = 0.2675$ hours, or about 16 minutes. (Many thanks to Carson Lian for spotting the mistake in the original solutions.) Some batteries are designed to be charged quickly like this, but they must have carefully designed circuitry to prevent overheating- if charged too fast for too long, the battery could explode!

Also note (by symmetry) that just as much power is dissipated across the wire as reaches the battery. Charging the battery quickly might be convenient, but it seems to waste a lot of energy.



Model of wall plug, wire, and battery.

(g) (Bonus) You might have found that the answer for the previous section seemed to waste a lot of energy. If you don't forget to charge your phone overnight, you have all 8 hours that you spend sleeping to charge your phone. What should you choose for R_{bat} to minimize the amount of wasted energy, while still charging the battery in no longer than 8 hours? Compare the power dissipated across the wire and the power dissipated across R_{bat}. Use the same model from Fig. **??**

Solution: Some of the power will be dissipated across the wire, and some will be dissipated across the battery. We want to maximize the fraction of power that reaches the battery. The power dissipated across the whole circuit is $I_R^2(R_{\text{bat}}+0.2)$ (where the second term is the equivalent resistance of the wire and battery in series). The power dissipated across the battery is $I_R^2 R_{\text{bat}}$. We want to maximize

$$\frac{I_R^2 R_{\text{bat}}}{I_R^2(R_{\text{bat}}+0.2)} = \frac{R_{\text{bat}}}{R_{\text{bat}}+0.2} = \frac{1}{1+0.2/R_{\text{bat}}}$$

Clearly, this is an increasing function that approaches 1 as R_{bat} approaches ∞, so we want to make R_{bat} as large as possible. However, we cannot make R_{bat} too large, or the battery won't be charged in the 8 hour time limit. What is the largest R_{bat}that will still charge in 8 hours?

The battery is 2200mAh at 3.8V, so to charge in 8 hours we need to charge 2.2×3.8 Wh of energy. The power dissipated across the battery is $I^2 R_{\text{bat}} = \left(\frac{5}{0.2+R_{\text{bat}}}\right)^2 R_{\text{bat}}$. Then we want

$$8 \text{ hours } = \frac{E}{P} = 2.2\times 3.8\times \frac{(0.2+R_{\text{bat}})^2}{25R_{\text{bat}}}$$

Solving, we get $R_{\text{bat}} = 0.0017$ or 23.52, and choose the larger value, 23.52Ω.

In the previous part, only half of the power reached the battery. By choosing a larger R_{bat}, a much bigger fraction of the power reaches the battery (although the total power is smaller). The fraction of power that reaches the battery is $\frac{1}{1+0.2/23.52}\approx 0.99$, almost 99%.

(Many thanks to Tony Situ, Daniel Ho, and Julian Chi-Kin Chan for catching the mistake in the original solutions.)

5. Temperature Sensor

Measuring quantities in the physical world is the job of sensors. This means somehow extracting that information from the world and then converting it into a form that can be observed and processed. Electric circuits can be very useful for doing this.

For most materials, resistance increases with increasing temperature; that is, a resistor has higher resistance when it is hot than when it is cold. This is often an annoyance to circuit designers who want their circuits to work the same way at different temperatures, but this fact can also be useful. It allows us to convert temperature, a "physical" quantity, into resistance, an "electrical" quantity, to build an electronic thermometer.

A PT100 is a special resistor made out of platinum that has a very precise relationship between resistance and temperature. At 0°C, the PT100 is a 100Ω resistor. Taking the data from `http://www.hayashidenko.co.jp/en/info12.html`, we found that the positive temperature coefficient for a PT100 is approximately $0.366\Omega/^\circ C$, that is, an increase in temperature of 1°C increases the PT100's resistance by 0.366Ω.

Consider the circuit in Fig. 1. It allows measuring resistance very precisely, as we will see below. The circle in the middle of the resistors is a *galvanometer*. It functions like an ideal wire [2] , but it also detects any current going through it.

(a) We say that the circuit is balanced when the current across the galvanometer in the middle is 0. Derive a relationship for the unknown resistance R_x in terms of the other three resistances if this is the case.

[2] In fact, galvanometers can be constructed as essentially just a coil of wire – current passing through the coil creates a magnetic field, which deflects the needle of a compass according to the strength and direction of the current. This is another wonderful property of electricity – it can be harnessed to have physical macroscale-level effects on the world that are observable by people.

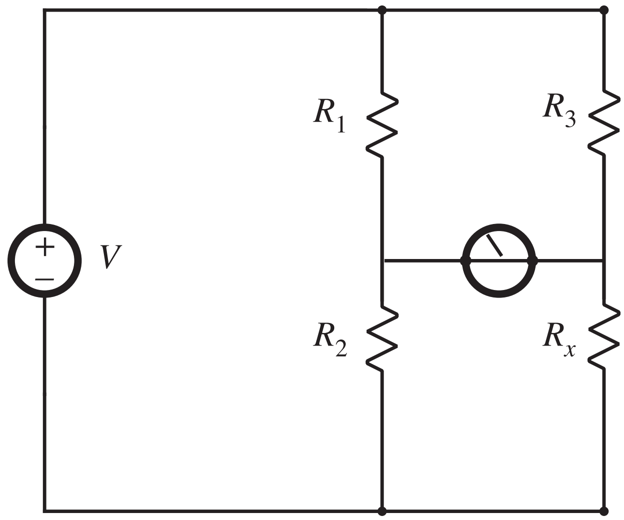


Figure 1: Circuit to measure resistance.

Solution: If there is no current across the galvonometer, then by KCL the current through R_1 is equal to the current through R_2 (call it i_{12}) and the current through R_3 is equal to the current through R_x (call it i_{3x}). Because the galvonometer acts like an ideal wire, the voltage across R_2 is the same as the voltage across R_x, so by Ohm's law we know

$$i_{12}R_2 = i_{3x}R_x$$

Similarly, by KVL we know the voltage across R_1 is the same as the voltage across R_3. Then by Ohm's law we also know

$$i_{12}R_1 = i_{3x}R_3$$

Then

$$\frac{i_{12}}{i_{3x}} = \frac{R_x}{R_2} = \frac{R_3}{R_1}$$

and

$$R_x = R_2\frac{R_3}{R_1}$$

(b) We can thus find one resistance if we know the other three. Suppose $R_1 = 50\Omega$, $R_3 = 100\Omega$ and R_2 can be adjusted from 0 to 300Ω. This adjustment can be used to balance the circuit. What is the maximum resistance that can thus be measured for R_x? (Only using the fact that the circuit is balanced when R_2 is set appropriately).

Solution: $\frac{R_3}{R_1} = 2$, so to be balanced we need $R_x = 2R_2$. The largest R_x that can still be balanced by adjusting R_2 is $2 \times \max(R_2) = 2 \times 300 = 600\Omega$.

(c) Assume R_x in fig. 1 is a PT100. Give a procedure by which you can find the temperature of the resistor. What is the maximum temperature you can measure, and why?

Solution: R_x is a PT100, so if we measure R_x we can find temperature by $T = \frac{R_x - 100}{0.366}$ (T in $°C$). We know $R_x = 2R_2$ when current through the galvanometer is 0, so we have

$$T = \frac{2R_2 - 100}{0.366}$$

Then the procedure is to adjust R_2 until the current across the galvonometer is 0, then look at what value R_2 is, and compute T from R_2. Note that if postive current is flowing through the galvonometer from left to right, then R_2 is too big, and if positive current is flowing through the galvonometer from right to left, then R_2 is too small (the last section of this problem will discuss this more in depth). Because the max R_2 is 300Ω, the maximum temperature that can be measured is

$$T = \frac{\max(R_x) - 100}{0.366} = \frac{2\max(R_2) - 100}{0.366} = 1366°\text{C}$$

(d) Suppose the company manufacturing your resistors gave you some parts from a bad batch, and instead of being 100Ω, R_3 was actually some random number between 95 and 105Ω (i.e. $R_3 = (1+\varepsilon)100\Omega$ for $|\varepsilon| \leq 0.05$). Unfortunately, you didn't realize this and assumed it was still 100Ω. What is the biggest (in magnitude) error this will introduce to your temperature measurement?

Solution: The formula that will be used to measure temperature using the nominal value of $R_3 = 100\Omega$ is

$$T_{\text{measured}} = \frac{2R_2 - 100}{0.366}$$

Using the true value of R_3, we have for zero current across the galvonometer that

$$R_x = R_2 \frac{(1+\varepsilon)100\Omega}{50\Omega} = 2(1+\varepsilon)R_2$$

Then the true temperature is measured as

$$T_{\text{true}} = \frac{2(1+\varepsilon)R_2 - 100}{0.366}$$

and the difference between the measured and the true is

$$T_{\text{true}} - T_{\text{measured}} = \frac{2\varepsilon R_2}{0.366}$$

This error is biggest when R_2 is biggest and $|\varepsilon|$ is biggest, so the largest error is

$$2 \times (\pm 0.05) \times \frac{300\Omega}{0.366\Omega/°C} = \pm 81.97°C$$

(e) Now assume both R_1 and R_3 came from the same bad batch, so

$$R_1 = (1+\varepsilon)50\Omega$$
$$R_3 = (1+\varepsilon)100\Omega$$

where both R_1 and R_3 have the same ε (still $|\varepsilon| \leq 0.05$). How much error will this introduce to the temperature measurement?

Solution: This will introduce no error to the temperature measurement! To see this, note that the condition for zero current across the galvonometer is

$$R_x = R_2 \frac{R_3}{R_1} = R_2 \frac{(1+\varepsilon)100\Omega}{(1+\varepsilon)50\Omega} = 2R_2$$

The $1+\varepsilon$'s cancel out, so regardless of the size of the error, the relationship between R_x and R_2 stays the same. If we repeat the previous part's $T_{\text{true}} - T_{\text{measured}}$ calculation we find that the difference is always 0.

(f) In the setup of the earlier parts (where $R_1 = 50\Omega$ and $R_3 = 100\Omega$ exactly), suppose we can only adjust R_2 in increments of 10Ω. Assume the galvanometer displays the direction of current flow (or 0 if no current). By adjusting R_2 in increments and observing the direction of current flow across the galvanometer, to what accuracy can we measure temperature?

Solution: We want to adjust R_2 to balance R_x so there is no current across the galvanometer, but we can't choose R_2 to be any value- it can only be moved in increments of 10Ω. Let's say R_2^* is the value of R_2 that would balance the circuit- i.e. $R_2^* = \frac{1}{2}R_x$. Then $R_2 = R_2^* + \Delta R_2$, where ΔR_2 is an error term. If $\Delta R_2 \neq 0$, then the circuit will not be balanced and current will flow through the galvanometer.

Because the galvonometer acts like an ideal wire, the voltage across R_2 is the same as the voltage across R_x, so by Ohm's law we know

$$i_2 R_2 = i_x R_x$$

Similarly, by KVL we know the voltage across R_1 is the same as the voltage across R_3.

By KCL at the top junction between R_1 and the voltage source, we know $i_{tot} = i_1 + i_3$, where i_{tot} is the total current out of the voltage source, i_1 is the current through R_1 and i_3 the current through R_3. Then by Ohm's law we also know

$$i_1 R_1 = i_3 R_3$$

and conclude by substituting for i_3 that

$$i_{tot} = i_1\left(1 + \frac{R_1}{R_3}\right)$$

Similarly,

$$i_2 R_2 = i_x R_x$$

and

$$i_{tot} = i_2\left(1 + \frac{R_2}{R_x}\right)$$

We chose R_2^* to balance the circuit according to the constraint $R_2^* = R_x \frac{R_1}{R_3}$ so we can write

$$\begin{aligned}
i_{tot} &= i_2\left(1 + \frac{R_2}{R_x}\right) \\
&= i_2\left(1 + \frac{R_2^* + \Delta R_2}{R_x}\right) \\
&= i_2\left(1 + \frac{R_1}{R_3} + \frac{\Delta R_2}{R_x}\right)
\end{aligned}$$

Using the other equation for i_{tot}, we can find a relationship between ΔR_2 and the currents through R_1 and R_2:

$$\begin{aligned}
i_{tot} &= i_1\left(1 + \frac{R_1}{R_3}\right) \\
1 + \frac{R_1}{R_3} + \frac{\Delta R_2}{R_x} &= \frac{i_1}{i_2}\left(1 + \frac{R_1}{R_3}\right) \\
\Delta R_2 &= \left(\frac{i_1}{i_2} - 1\right)\left(1 + \frac{R_1}{R_3}\right) R_x
\end{aligned}$$

Note that if the circuit is balanced, $i_1 = i_2$ and we conclude that $\Delta R_2 = 0$, i.e. we have found the correct R_2 such that $R_2 = R_2^* = R_x \frac{R_1}{R_3}$. Of course, since we can't exactly specify R_2, there will be some $\Delta R_2 \neq 0$. If $i_1 < i_2$, then some current going through R_3 is crossing through the galvanometer and going through R_2, and $\Delta R_2 < 0$, so we want to increase the resistance of R_2 to try to get closer to balance. Our galvanometer tells us the direction of the current, so if positive current goes from right to left, we conclude we need to increase R_2. If $i_1 > i_2$, then $\Delta R_2 > 0$ and we have the opposite problem: we need to reduce R_2 to get closer to balance.

Then by sweeping the all the values of R_2, we can find the values of $R_{2,low}$ (where positive current flows through the galvanometer from right to left, but increasing R_2 by 10Ω results in the current changing direction), and $R_{2,high}$ (where positive current flows through the galvanometer from left to right, but decreasing R_2 by 10Ω results in the current changing direction). We know that $R_{2,low}$ and $R_{2,high}$ are both within 10Ω of R_2^* and $R_{2,low} + 10\Omega = R_{2,high}$.

$$T_{\text{true}} = \frac{2R_2^* - 100}{0.366}$$

so

$$\frac{2R_{2,low} - 100}{0.366} \leq T_{\text{true}} \leq \frac{2R_{2,high} - 100}{0.366}$$

which we can write as

$$\frac{2(R_2^* + \Delta R - 10\Omega) - 100}{0.366} \leq T_{\text{true}} \leq \frac{2(R_2^* + \Delta R) - 100}{0.366}$$

with $0 \leq \Delta R \leq 10\Omega$. Subtracting T_{true} from each expression, we find that the error is in the interval

$$\left[\frac{2(\Delta R - 10)}{0.366}, \frac{2\Delta R}{0.366}\right]$$

Because we don't know the magnitude of the current from the galvanometer, we can't tell which of the two estimates for T is closer. Our best guess is to pick the average of the two estimates. Averaging the two endpoints will will have error

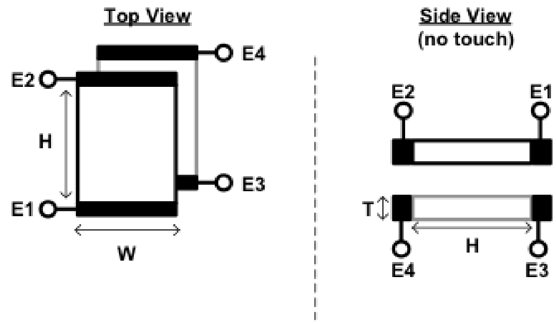
$$\frac{1}{2}\left(\frac{2(\Delta R - 10)}{0.366} + \frac{2\Delta R}{0.366}\right) = \frac{2(\Delta R - 5)}{0.366}$$

with $0\Omega \leq \Delta R \leq 10\Omega$, so the error of our temperature estimate will be

$$\pm \frac{10\Omega}{0.366\frac{\Omega}{^\circ C}} = \pm 27.32^\circ C$$

6. Multitouch Resistive Touchscreen

In this problem we will look at a simplified version of the multitouch resistive touchscreen. In particular, rather than measuring the position of two potential touch points in both dimensions (i.e., a pair of coordinates (x_1, y_1) and (x_2, y_2) corresponding to two touch positions), let's think about a version where we are interested in measuring only the vertical position of the two touch points (i.e., y_1 and y_2). Therefore, unlike the touchscreens we looked at in class and as shown below, both of the resistive plates (i.e., both the top and the bottom plate) would have conductive strips placed along their top and bottom edges.



(a) Assuming that both of the plates are made out of a material with $\rho = 1\Omega m$ and that the dimensions of the plates are $W = 3cm$, $H = 12cm$, and $T = 0.5mm$, with no touches at all, what is the resistance between terminals E_1 and E_2 (which would be the same as the resistance between terminals E_3 and E_4)?

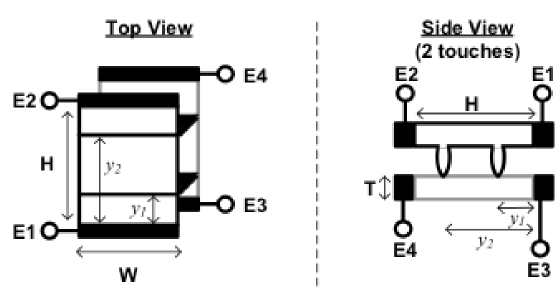
Solution:

$$R = \rho \cdot \frac{L}{A} \Rightarrow R_{E1-E2} = \rho \left(\frac{H}{W \cdot T} \right)$$

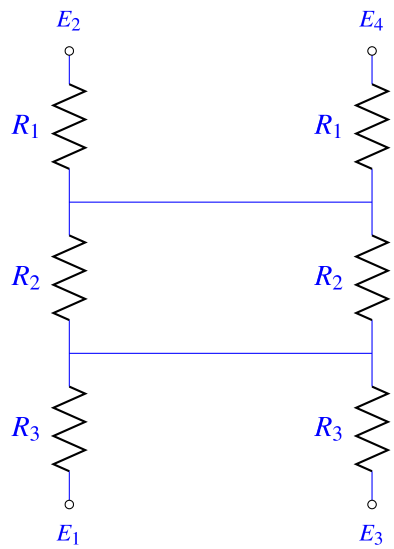
$$R_{E1-E2} = 1\Omega m \left(\frac{12 \times 10^{-2} m}{3 \times 10^{-2} \cdot 0.5 \times 10^{-3} m} \right)$$

$$R_{E1-E2} = 8k\Omega$$

(b) Now let's look at what happens when we have two touch points. Let's assume that at wherever height the touch occurs, a perfect contact is made between the top plate and the bottom plate along the entire width of the plates (i.e., you don't have to worry about any lateral resistors), but that otherwise none of the electrical characteristics of the plates change. Defining the bottom of the plate as being $y = 0cm$ (i.e., a touch right at E_1 would be at $y = 0cm$), let's assume that the two touches happen at $y_1 = 3cm$ and $y_2 = 7cm$, and that your answer to part (a) was $5k\Omega$ (which may or may not be the right answer), draw a model with 6 resistors that captures the electrical connections between E_1, E_2, E_3, and E_4. Note that for clarity, the system has been redrawn below to depict this scenario.

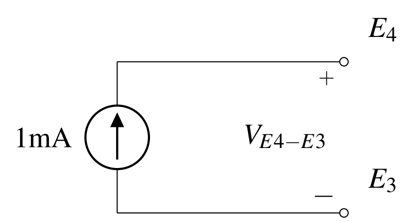


Solution:



$$R_3 = \frac{3cm}{12cm} \cdot R_{E2-E1} = 1.25k\Omega$$
$$R_2 = \frac{7cm - 3cm}{12cm} \cdot R_{E2-E1} = 1.667k\Omega$$
$$R_1 = \frac{12cm - 7cm}{12cm} \cdot R_{E2-E1} = 2.0833k\Omega$$

(c) Using the same assumptions as part b), if you drove terminals E_3 and E_4 with a 1mA current source (as shown below) but left terminals E_1 and E_2 open-circuited, what is the voltage you would measure across $E_4 - E_3$ (i.e., V_{E4-E3})?



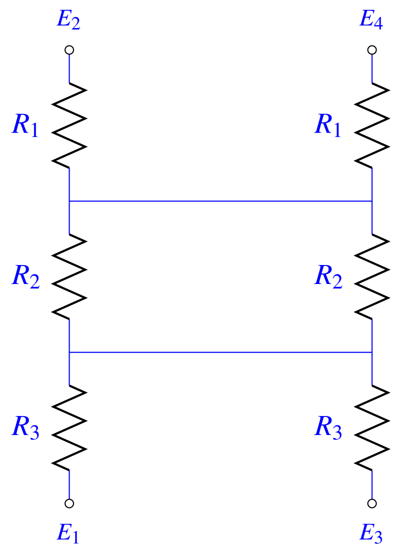
Solution:

Equivalent resistance between $E_4 - E_3$ is

$$\begin{aligned} R_{E4-E3} = R_1 + R_2 \parallel R_2 + R_3 &= R_1 + \frac{R2}{2} + R_3 \\ &= 1.25k\Omega + \frac{1.667k\Omega}{2} + 2.0833k\Omega \\ &\approx 4.167k\Omega \end{aligned}$$
$$V_{E4-E3} = 1mA \cdot R_{E4-E3} \Rightarrow V_{E4-E3} = 4.167V$$

(d) Now let's try to generalize the situation by assuming that the two touches can happen at any two arbitrary points y_1 and y_2, but with y_1 defined to always be less than y_2 (i.e., y_1 is always the bottom touch point). Leaving the setup the same as in part c) except for the arbitrary y_1 and y_2, by measuring only the voltage between E_4 and E_3, what information can you extract about the two touch positions? Please be sure to provide an equation relating V_{E4-E3} to y_1 and y_2 as a part of your answer, and note that you may want to redraw the model from part (b) to help you with this.

Solution:



For general

$$R_3 = \frac{y_1}{12cm} \cdot 5k\Omega$$
$$R_2 = \frac{y_2 - y_1}{12cm} \cdot 5k\Omega$$
$$R_1 = \frac{12cm - y_2}{12cm} \cdot 5k\Omega$$

$$R_{E4-E3} = R_1 + \frac{R_2}{2} + R_3 = \left(12cm - y_2 + \frac{y_2 - y_1}{2} + y_1\right) \cdot \frac{5k\Omega}{12cm}$$
$$= \left(12cm + \frac{y1}{2} - \frac{y2}{2}\right) \cdot \frac{5k\Omega}{12cm}$$

So

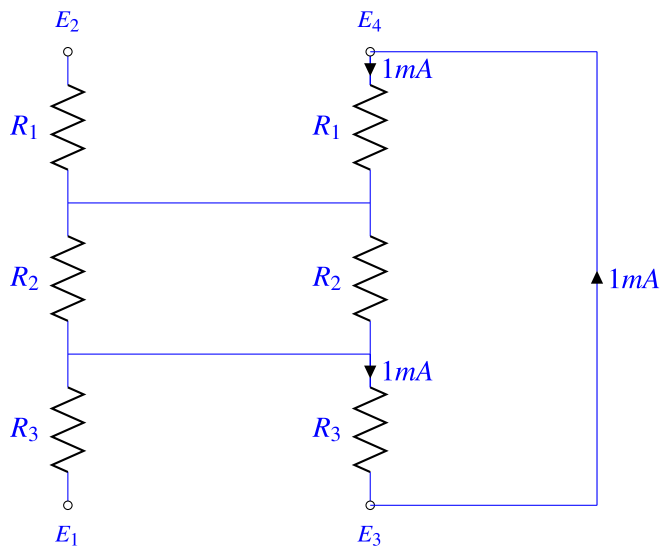
$$V_{E4-E3} = \frac{12cm - (y_2 - y_1)/2}{12cm} \cdot 5V$$

This means that by measuring V_{E4-E3}, we can only measure the <u>distance</u> between the two touch points $(y_2 - y_1)$.

(e) One of your colleagues claims that by measuring the appropriate voltages, not only can they extract what both y_1 and y_2 are in this system, they can even do so in a way that would have a set of three independent voltage equations related to y_1 and y_2. As we will see later, this will allow us to gain some robustness to noise in the voltage measurements.

In order to facilitate this, write equations relating V_{E4-E2} and V_{E1-E3} to y_1 and y_2. (The third voltage we'll use is V_{E4-E3}, which you should have already derived an equation for in the previous part of the problem.)

Solution:



$$V_{E4-E2} = I \cdot R_1 = \frac{12cm - y_2}{12cm} \cdot 5V$$
$$V_{E1-E3} = I \cdot R_3 = \frac{y_1}{12cm} \cdot 5V$$

7. **Your Own Problem** Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?