EECS 16A Designing Information Devices and Systems I
Spring 2020 Discussion 13B

**1. Linearizing Different Problems**

Notice that least squares can only be applied to linear systems. Suppose that we have a vector $\vec{x}$ and a vector $\vec{y}$, and $\vec{y}[n] = f(\vec{x}[n])$. We would like to approximate $f$ using least squares, where $f$ is not necessarily a linear function.

(a) Let's begin with a linear approximation. We want to find some $a$ such that $y = ax$. Set this up as a least squares problem. What are the elements in the matrix $\mathbf{A}$?

**Answer:**

$$\mathbf{A} = \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix}$$

$$\vec{x}a = \vec{y}$$

(b) Let's add a constant to the problem. Suppose that $y = ax + b$. Set this up as a least squares problem. What are the elements in the matrix $\mathbf{A}$?

**Answer:**

$$\begin{bmatrix} \vec{x} & \vec{1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \vec{y}$$

(c) Suppose that $y = ax^2 + bx + c$. Set this up as a least squares problem. What are the elements in the matrix $\mathbf{A}$?

**Answer:**

$$\begin{bmatrix} \vec{x}^2 & \vec{x} & \vec{1} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{y},$$

where $\vec{x}^2$ denotes the element-wise square.

(d) Suppose that $y = ae^{bx}$. Set this up as a least squares problem. What are the elements in the matrix $\mathbf{A}$?

**Answer:**

$$\begin{bmatrix} \vec{x} & \vec{1} \end{bmatrix} \begin{bmatrix} b \\ \ln(a) \end{bmatrix} = \ln(\vec{y}),$$

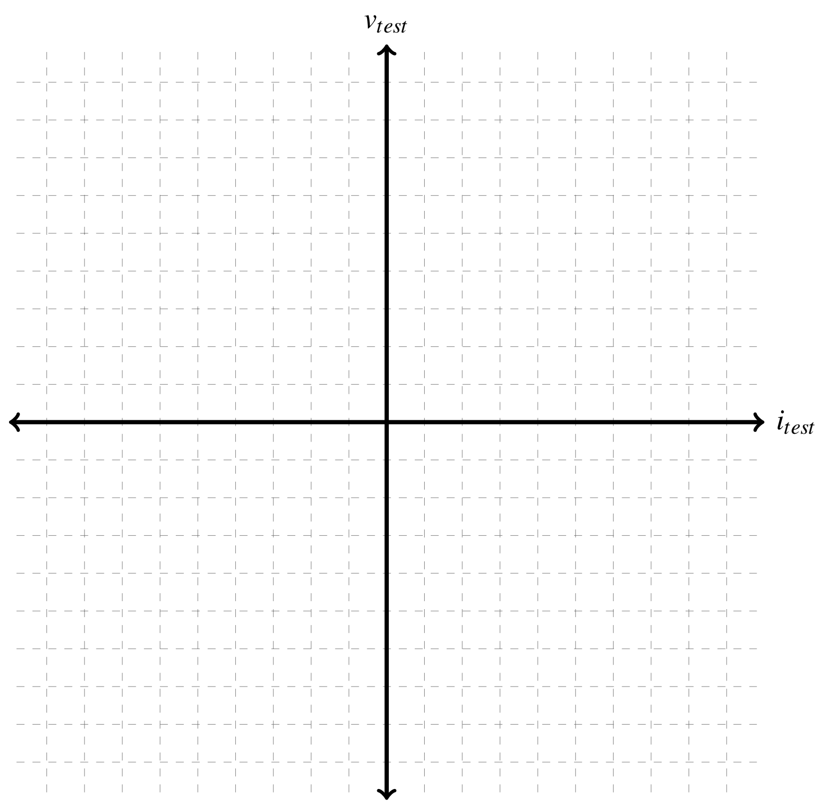
where $\ln(\vec{y})$ denotes the element-wise natural logarithm.

**2. Ohm's Law With Noise**

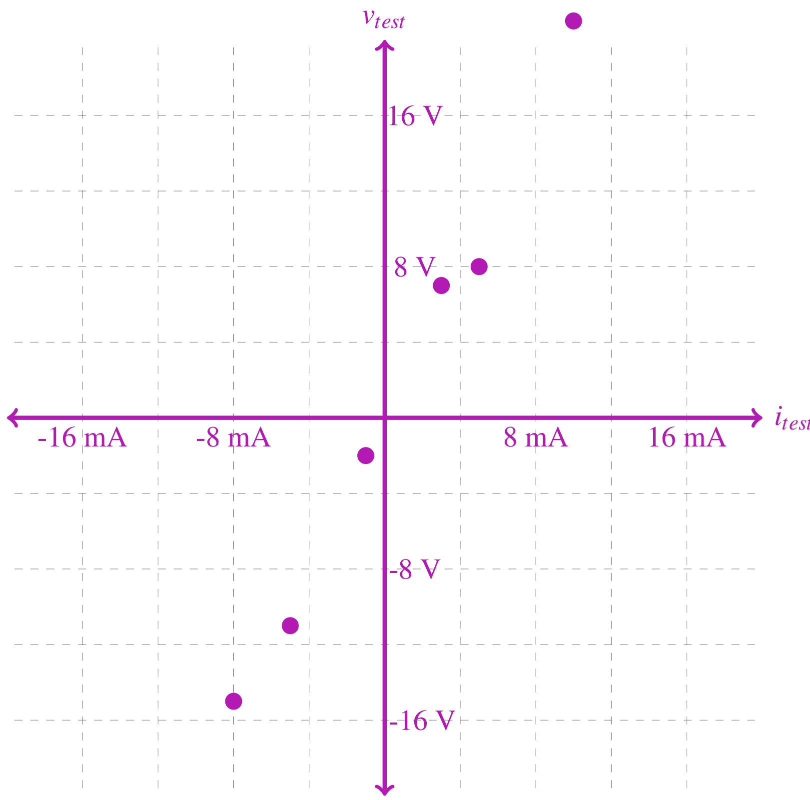
We are trying to measure the resistance of a black box. We apply various $i_{\text{test}}$ currents and measure the ouput voltage $v_{\text{test}}$. Sometimes, we are quite fortunate to get nice numbers. Oftentimes, our measurement tools are a little bit noisy, and the values we get out of them are not accurate. However, if the noise is completely random, then the effect of it can be averaged out over many samples. So we repeat our test many times:

| Test | $i_{\text{test}}$ (mA) | $v_{\text{test}}$ (V) |
|---|---|---|
| 1 | 10 | 21 |
| 2 | 3 | 7 |
| 3 | $-1$ | $-2$ |
| 4 | 5 | 8 |
| 5 | $-8$ | $-15$ |
| 6 | $-5$ | $-11$ |

(a) Plot the measured voltage as a function of the current.




**Answer:**




Notice that these points *do not* lie on a line!

(b) Suppose we stack the currents and voltages to get $\vec{I} = \begin{bmatrix} 10 \\ 3 \\ -1 \\ 5 \\ -8 \\ -5 \end{bmatrix}$ and $\vec{V} = \begin{bmatrix} 21 \\ 7 \\ -2 \\ 8 \\ -15 \\ -11 \end{bmatrix}$. Is there a unique solution for $R$? What conditions must $\vec{I}$ and $\vec{V}$ satisfy in order for us to solve for $R$ uniquely?

**Answer:**

We cannot find the unique solution for $R$ because $\vec{V}$ is not a scalar multiple of $\vec{I}$. In general, we need $\vec{V}$ to be a scalar multiple of $\vec{I}$ to be able to solve for $R$ exactly (another linear algebraic way of saying this is that $\vec{V}$ is in the span of $\vec{I}$).

We know that the *physical* reason we are not able to solve for $R$ is that we have imperfect observations of the voltage across the terminals, $\vec{V}$. Therefore, now that we know we cannot solve for $R$ directly, a very pertinent goal would be to find a value of $R$ that *approximates* the relationship between $\vec{I}$ and $\vec{V}$ as closely as possible.

Let's move on and see how we do this.

(c) Ideally, we would like to find $R$ such that $\vec{V} = \vec{I}R$. If we cannot do this, we'd like to find a value of $R$ that is the *best* solution possible, in the sense that $\vec{I}R$ is as "close" to $\vec{V}$ as possible. We are defining the sum of squared errors as a **cost function**. In this case the cost function for any value of $R$ quantifies the difference between each component of $\vec{V}$ (i.e. $v_j$) and each component of $\vec{I}R$ (i.e. $i_jR$) and sum up the squares of these "differences" as follows:

$$\text{cost}(R) = \sum_{j=1}^{6} (v_j - i_j R)^2$$

Do you think this is a good cost function? Why or why not?

**Answer:**

For each point $(i_j, v_j)$, we want $|v_j - i_j R|$ to be as small as possible. We can call this term the individual error term for this point.

One way of looking at the aggregate "error" in our fit is to add up the squares of the individual errors, so that all errors add up. This is precisely what we've done in the cost function. If we did not square the differences, then a positive difference and a negative difference would cancel each other out.

(d) Show that you can also express the above cost function in vector form, that is,

$$\text{cost}(R) = \left\langle (\vec{V} - \vec{I}R), (\vec{V} - \vec{I}R) \right\rangle$$

*Hint:* $\langle \vec{a}, \vec{b} \rangle = \vec{a}^T \vec{b} = \sum_i a_i b_i$

**Answer:**

Let's define the error vector as

$$\vec{e} = \vec{V} - \vec{I}R.$$

Then, we observe that $e_j = v_j - i_j R$.

Therefore,

$$\begin{aligned}
\text{cost}(R) &= \sum_{j=1}^{6} (v_j - i_j R)^2 \\
&= \sum_{j=1}^{6} e_j^2 \\
&= \|\vec{e}\|_2^2 \\
&= \langle \vec{e}, \vec{e} \rangle \\
&= \left\langle (\vec{V} - \vec{I}R), (\vec{V} - \vec{I}R) \right\rangle
\end{aligned}$$

(e) Find $\hat{R}$, which is defined as the optimal value of $R$ that minimizes $\text{cost}(R)$.

*Hint:* Use calculus. The optimal $\hat{R}$ makes $\frac{d\text{cost}(\hat{R})}{dR} = 0$

**Answer:**

First, note that

$$\frac{d\text{cost}(R)}{dR} = -2 \sum_{j=1}^{6} i_j (v_j - i_j R)$$

For $R = \hat{R}$, we will have $\frac{d\text{cost}(R)}{dR} = 0$. This means that

$$-2 \sum_{j=1}^{6} i_j (v_j - i_j \hat{R}) = 0,$$

which will ultimately give us

$$\hat{R} = \frac{\sum_{j=1}^{6} i_j v_j}{\sum_{j=1}^{6} i_j^2} = \frac{\left\langle \vec{I}, \vec{V} \right\rangle}{\|\vec{I}\|^2}$$

In our particular example, $\left\langle \vec{I}, \vec{V} \right\rangle = 448$ and $\|\vec{I}\|^2 = 224$. Therefore, we will get $\hat{R} = 2\,\mathrm{k}\Omega$.

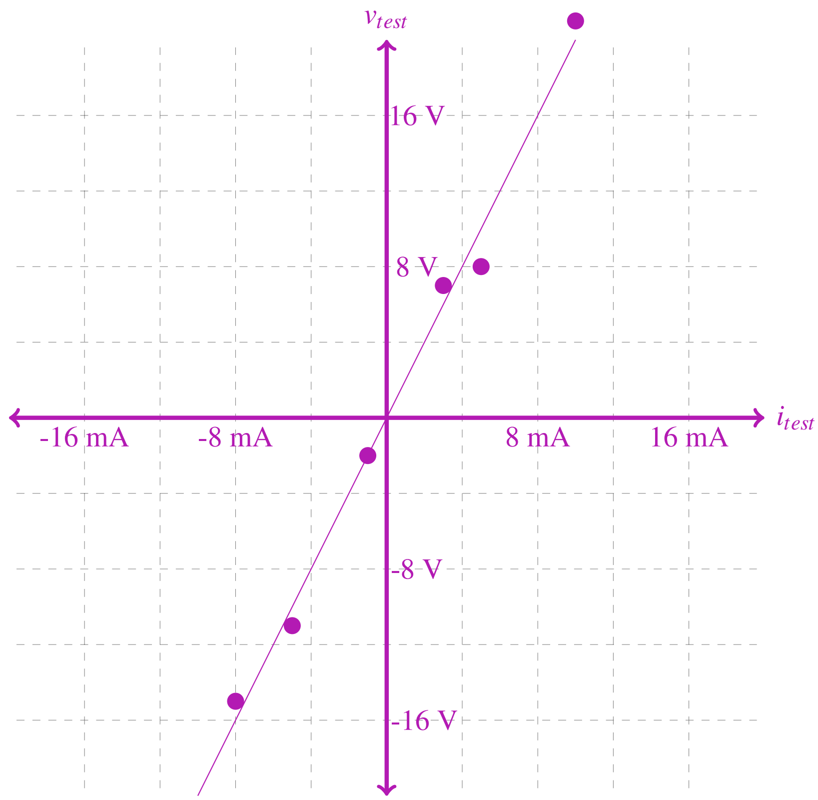
Using the equation for least squares estimate with $A = \begin{bmatrix} 10 \\ 3 \\ -1 \\ 5 \\ -8 \\ -5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 21 \\ 7 \\ -2 \\ 8 \\ -15 \\ -11 \end{bmatrix}$, we would have:

$$\hat{R} = (A^T A)^{-1} A^T \vec{b}$$
$$\hat{R} = \frac{\left\langle \vec{I}, \vec{V} \right\rangle}{\left\langle \vec{I}, \vec{I} \right\rangle}$$
$$\hat{R} = \frac{\left\langle \vec{I}, \vec{V} \right\rangle}{\|\vec{I}\|^2},$$

which gives us the same expression as before!

(f) On your original $IV$ plot, also plot the line $v_{test} = \hat{R} i_{test}$. Can you visually see why this line "fits" the data well? How well would we have done if we had guessed $R = 3\,\mathrm{k}\Omega$? What about $R = 1\,\mathrm{k}\Omega$? Calculate the cost functions for each of these choices of $R$ to validate your answer.

**Answer:**




When $\hat{R} = 2\,\mathrm{k}\Omega$, we have

$$\begin{aligned}\text{cost}(2k) &= (21 - 2 \cdot 10)^2 + (7 - 2 \cdot 3)^2 + (-2 - 2 \cdot (-1))^2 + \\ &\quad (8 - 2 \cdot 5)^2 + (-15 - 2 \cdot (-8))^2 + (-11 - 2 \cdot (-5))^2 \\ &= 8.\end{aligned}$$

When $\hat{R} = 3\,\mathrm{k}\Omega$, we have

$$\begin{aligned}\text{cost}(3k) &= (21 - 3 \cdot 10)^2 + (7 - 3 \cdot 3)^2 + (-2 - 3 \cdot (-1))^2 + \\ &\quad (8 - 3 \cdot 5)^2 + (-15 - 3 \cdot (-8))^2 + (-11 - 3 \cdot (-5))^2 \\ &= 232.\end{aligned}$$

When $\hat{R} = 1\,\mathrm{k}\Omega$, we have

$$\begin{aligned}\text{cost}(1k) &= (21 - 1 \cdot 10)^2 + (7 - 1 \cdot 3)^2 + (-2 - 1 \cdot (-1))^2 + \\ &\quad (8 - 1 \cdot 5)^2 + (-15 - 1 \cdot (-8))^2 + (-11 - 1 \cdot (-5))^2 \\ &= 232.\end{aligned}$$

(g) Now, suppose that we add a new data point: $i_7 = 2\,\mathrm{mA}$, $v_7 = 4\,\mathrm{V}$. Will $\hat{R}$ increase, decrease, or remain the same? Why? What does that say about the line $v_{test} = \hat{R} i_{test}$?

**Answer:**

We can qualitatively see that $\hat{R}$ will remain 2 k$\Omega$. This is because we already obtained $\hat{R}$ to fit our previous data in the best way. Now, you should notice that this new piece of data $(i_7, v_7)$ also lies exactly on the line $v_{test} = \hat{R} i_{test}$! Therefore, you have no reason to change $\hat{R}$. It is the best fit for the old data and will fit the new data anyway.

**3. Orthogonal Matching Pursuit**

Let's work through an example of the OMP algorithm. Suppose that we have a vector $\vec{x} \in \mathbb{R}^4$ that is sparse and we know that it has only 2 non-zero entries. In particular,

$$\mathbf{M}\vec{x} \approx \vec{y} \tag{1}$$

$$\begin{bmatrix} | & | & | & | \\ \vec{m}_1 & \vec{m}_2 & \vec{m}_3 & \vec{m}_4 \\ | & | & | & | \end{bmatrix} \vec{x} \approx \vec{y} \tag{2}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \tag{3}$$

where exactly 2 of $x_1$ to $x_4$ are non-zero. Use Orthogonal Matching Pursuit to estimate $x_1$ to $x_4$.

(a) Why can we not solve for $\vec{x}$ directly?

**Answer:**

We cannot solve for $\vec{x}$ directly because we have three measurements (or equations) but four unknowns. Since our system is underdetermined, we cannot solve for the unique $\vec{x}$ directly.

(b) Why can we not apply the least squares process to obtain $\vec{x}$?

**Answer:**

Recall the least squares solution: $\vec{\hat{x}} = (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T\vec{y}$. $\mathbf{M}^T\mathbf{M}$ is only invertible if it has a trivial null space, i.e., if $\mathbf{M}$ has a trivial null space. However, in this case, $\mathbf{M}$ is a $3 \times 4$ matrix, so there is at least one free variable, which means that its null space is non-trivial. Therefore, $\mathbf{M}^T\mathbf{M}$ is not invertible, and we cannot use least squares to solve for $\vec{x}$.

(c) Let us start by reviewing the OMP procedure,

**Inputs:**

- A matrix $\mathbf{M}$, whose columns, $\vec{m}_i$, make up a set of vectors, $\{\vec{m}_i\}$, each of length $n$
- A vector $\vec{y}$ of length $n$
- The sparsity level $k$ of the signal

**Outputs:**

- A vector $\vec{x}$, that contains $k$ non-zero entries.
- A error vector $\vec{e} = \vec{y} - \mathbf{M}\vec{x}$

**Procedure:**

- Initialize the following values: $\vec{e} = \vec{y}$, $j = 1$, $k$, $\mathbf{A} = \begin{bmatrix} \quad \end{bmatrix}$
- while ($j \leq k$):
  i. Compute the inner product for each vector in the set, $\vec{m}_i$, with $\vec{e}$: $c_i = \langle \vec{m}_i, \vec{e} \rangle$.
  ii. Column concatenate matrix $\mathbf{A}$ with the column vector that had the maximum inner product value with $\vec{e}$, $c_i$: $\mathbf{A} = \begin{bmatrix} \mathbf{A} & | & \vec{m}_i \end{bmatrix}$
  iii. Use least squares to compute $\vec{x}$ given the $\mathbf{A}$ for this iteration: $\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$
  iv. Update the error vector: $\vec{e} = \vec{y} - \mathbf{A}\vec{x}$
  v. Update the counter: $j = j + 1$

(d) Compute the inner product of every column with the $\vec{y}$ vector. Which column has the largest inner product? This will be the first column of the matrix $\mathbf{A}$.

**Answer:**

$$\left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}5\\1\\1\end{bmatrix} \right\rangle = 5$$

$$\left\langle \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}5\\1\\1\end{bmatrix} \right\rangle = 3$$

$$\left\langle \begin{bmatrix}2\\2\\0\end{bmatrix}, \begin{bmatrix}5\\1\\1\end{bmatrix} \right\rangle = 12$$

$$\left\langle \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}5\\1\\1\end{bmatrix} \right\rangle = 6$$

The third column has the largest inner product with $\begin{bmatrix}5\\1\\1\end{bmatrix}$, so $\mathbf{A} = \begin{bmatrix}2\\2\\0\end{bmatrix}$.

(e) Now, find the projection of $\vec{y}$ onto the columns of $\mathbf{A}$ (ie. $\text{proj}_{\text{Col}(\mathbf{A})}\vec{y} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$). Use this to update the error vector.

**Answer:**

$$\vec{\hat{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y} = \left( \begin{bmatrix}2 & 2 & 0\end{bmatrix} \begin{bmatrix}2\\2\\0\end{bmatrix} \right)^{-1} \begin{bmatrix}2 & 2 & 0\end{bmatrix} \begin{bmatrix}5\\1\\1\end{bmatrix} = \frac{1}{8} \cdot 12 = \frac{3}{2}$$

$$\text{proj}_{\text{Col}(\mathbf{A})}\vec{y} = \mathbf{A}\vec{\hat{x}} = \begin{bmatrix}2\\2\\0\end{bmatrix} \cdot \frac{3}{2} = \begin{bmatrix}3\\3\\0\end{bmatrix}$$

$$\vec{e} = \vec{y} - \text{proj}_{\text{Col}(\mathbf{A})}\vec{y} = \begin{bmatrix}5\\1\\1\end{bmatrix} - \begin{bmatrix}3\\3\\0\end{bmatrix} = \begin{bmatrix}2\\-2\\1\end{bmatrix}$$

(f) Now compute the inner product of every column with the new error vector. Which column has the largest inner product? This will be the second column of $\mathbf{A}$.

**Answer:**

$$\left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix} \right\rangle = 2$$

$$\left\langle \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix} \right\rangle = 0$$

$$\left\langle \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\rangle = 0$$

$$\left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\rangle = 3$$

The fourth column has the largest inner product with $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, so $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$.

(g) We now have two non-zero entries for our vector, $\vec{x}$. Find the values of those two entries.

(Reminder: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)

**Answer:**

$$\vec{\hat{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y} = \left( \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore, $x_3 = 1$ and $x_4 = 2$, so $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.