EECS 16A Designing Information Devices and Systems I
Spring 2020 Discussion 12B

1. Search and Rescue Dogs

Berkeley's Puppy Pound needs your help! While Mr. Muffin was being walked, the volunteer let go of his leash and he is now running wild in the streets of Berkeley (which are quite dangerous)! Thankfully, all of the puppies at the pound have a collar that sends a bluetooth signal to receiver towers, which are spread throughout the streets (pictured below). If the puppy/collar is within range of the receiver tower, the collar will send the tower a message: the distance of the collar to the tower. Each cell tower has a range of 3 city blocks. Can you help the pound locate their lost puppy?

Note: A city block is defined as the middle of an intersection to the middle of an adjacent intersection (scale provided on map.) Mr. Muffin is constrained to running wild in the streets, meaning he won't be found in any buildings. If your TA asks 'Where is Mr. Muffin?' it is sufficient to answer with his intersection or 'between these two intersections.'

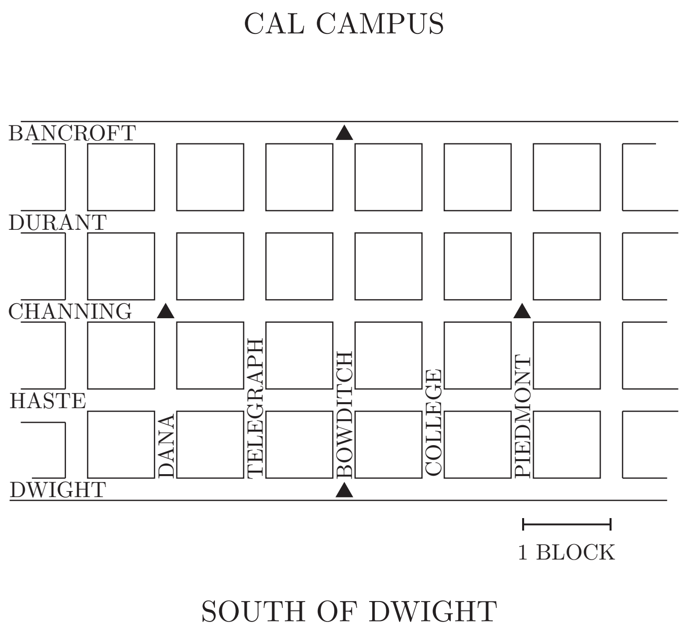

[1]

(a) You check the logs of the cell towers, and they have received the following messages:

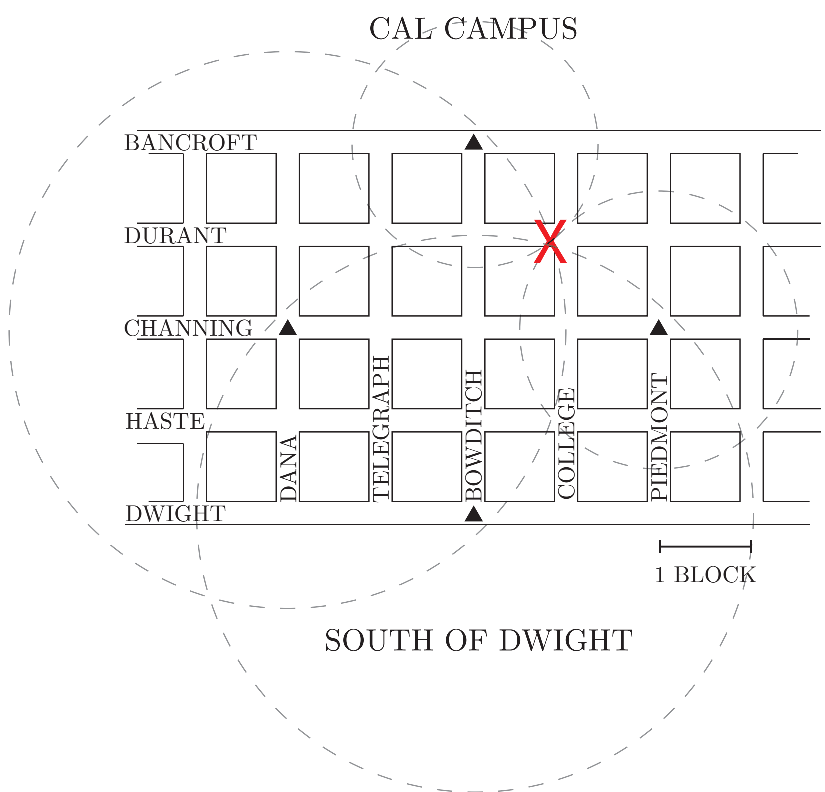
Sensor	Distance
N	1.3
W	3
E	1.5
S	3

On the map provided, identify where Mr. Muffin is!

[1] http://www.pupsmile.com/wp-content/uploads/2012/11/running_happy_dog-1024x684.jpeg



Answer:



(b) Can you set this up as a system of equations? Is it linear? If it's not linear, can you think of a way to make it linear? Now, how do you set this up in matrix form?
Hint: Set $(0,0)$ to be Channing and Bowditch.
Hint 2: Distance $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Hint 3: You don't need all 4 equations. You have two unknowns, x and y. You know from lecture that you need three circles to uniquely find a point. How can you use the third circle/equation to get two

equations and two unknowns?

Answer:

First, set up the system of equations:

$$(x-0)^2+(y-2)^2=1.3^2$$
$$(x+2)^2+(y-0)^2=3.0^2$$
$$(x-2)^2+(y-0)^2=1.5^2$$

Simplify out:

$$x^2+y^2-4y+4=1.3^2$$
$$x^2+4x+4+y^2=3.0^2$$
$$x^2-4x+4+y^2=1.5^2$$

Then subtract equation (1) from equations (2) and (3):

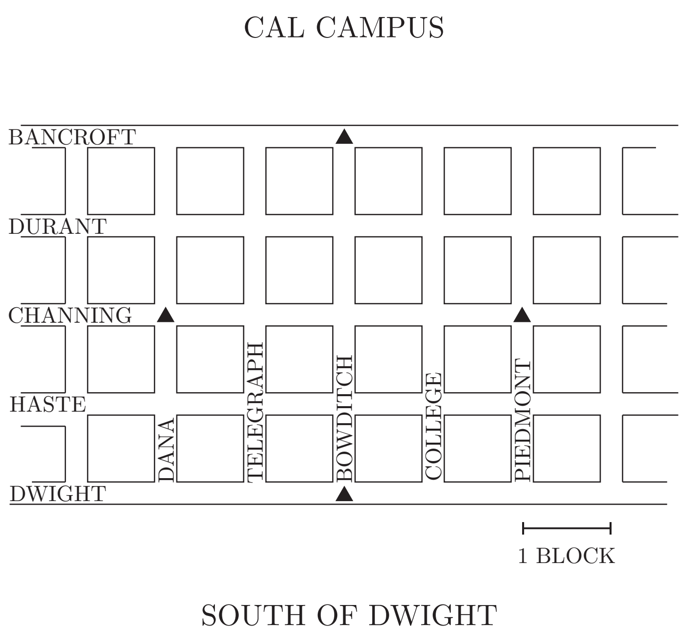
$$4x+4y=3.0^2-1.3^2$$
$$-4x+4y=1.5^2-1.3^2$$

This solves to $x=0.84, y=0.98$ which is roughly College and Durant.

(c) Suppose Mr. Muffin is moving fast, and by the time you get to destination in part (a) he's already run off! You check the logs of the cell towers again, and see the following updated messages:

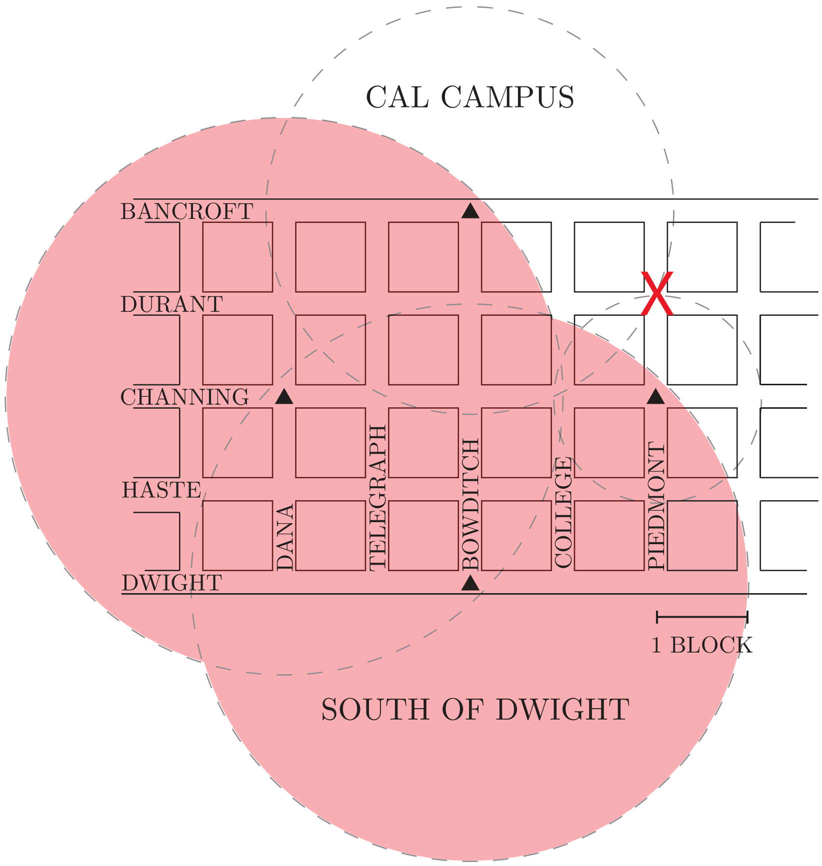
Sensor	Distance
N	2.2
W	Out of Range
E	1.1
S	Out of Range

Can you find Mr. Muffin?



Answer:

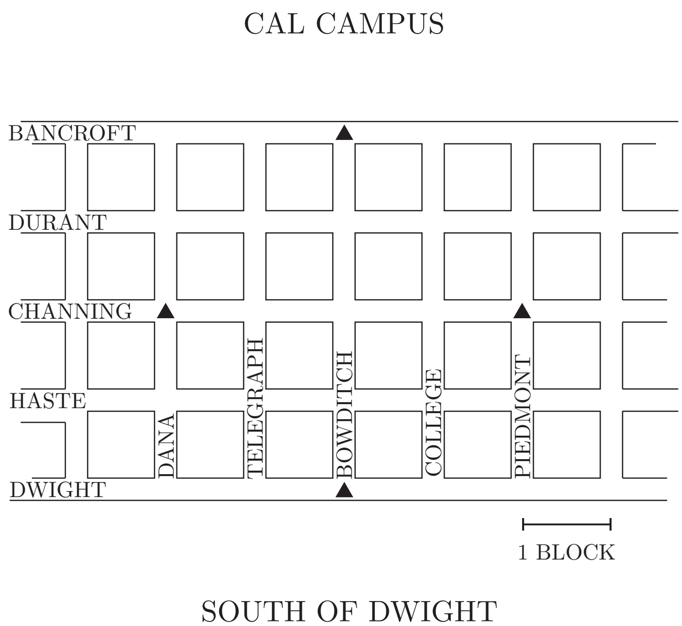
With two out of range sensors, you might think that you will not be able to find a unique solution (you need 3 circles to intersect at a point.) The trick is that out of range still provide information on where Mr. Muffin is NOT. See the diagram below - Mr. Muffin cannot be in the shaded region.



(d) Mr. Muffin is a very mischievous puppy, and while playing and running around he damaged his collar. The transmitter on his collar will still send a signal to the receiver towers, but the distance sensor has noise. You check the logs of the cell towers, and they have received the following messages:

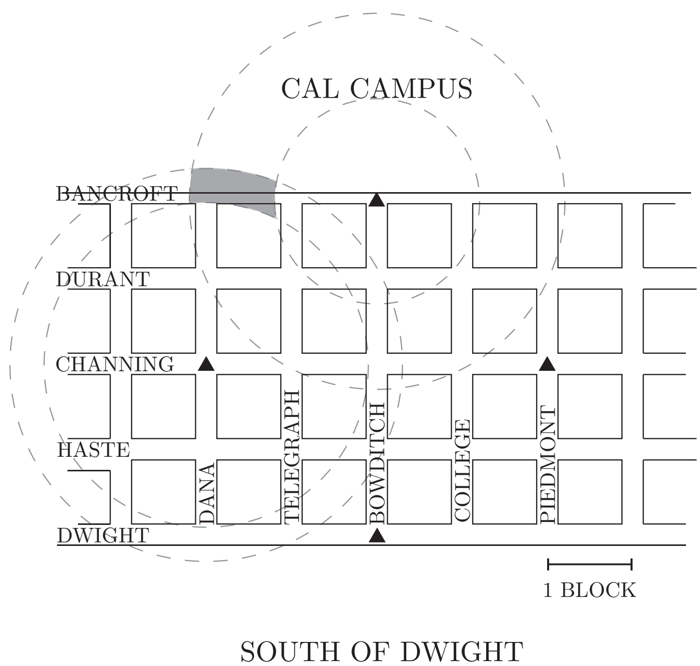
Sensor	Distance
N	1.7 ± 0.5
W	2.1 ± 0.2
E	Out of Range
S	Out of Range

On the map provided, identify where Mr. Muffin is! Can you find exactly where he is?



Answer:

You can’t find exactly where he is, but you know he is somewhere between Dana/Telegraph and Bancroft. See the diagram below.



2. Projections Proof

Let us explore projections in 2D space. Consider two vectors $\vec{a}$ and $\vec{b}$.

Theorem: The point along $\vec{a}$ (in the span of $\vec{a}$) that is closest to $\vec{b}$ is the vector $\vec{z}$ such that $\vec{b}-\vec{z}$ is orthogonal to $\vec{a}$. The vector $\vec{z}$ is given by $\vec{z} = \frac{\langle \vec{a}, \vec{b} \rangle}{||\vec{a}||^2}\, \vec{a}$.

Hint: Let $\vec{z}$ be the solution to the above theorem. What does it mean for $\vec{z}$ to be in the span of $\vec{b}$?

(a) Prove the theorem using a geometrical interpretation. You may use Figure 1 as the starting point for your proof.

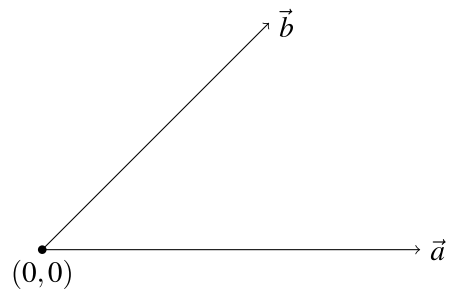
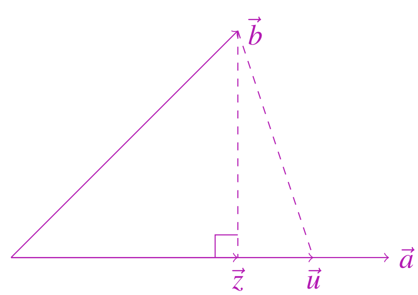


Figure 1: Example vectors $\vec{a}$ and $\vec{b}$ as starting point for geometric proof.

Answer: The point along $\vec{a}$ that is closest to $\vec{b}$ must be orthogonal to $\vec{a}$. We prove this by letting $\vec{z}$ be a vector where $\vec{a}-\vec{z}$ is orthogonal to $\vec{a}$ and also goes through $\vec{b}$. Because $\vec{z}$ is in the span of $\vec{a}$, we may also write $\vec{z} = \beta\vec{a}$, where β is an unknown scalar. Let us choose any other vector $\vec{u}$. We see that $\vec{z}-\vec{b}$, $\vec{u}-\vec{z}$, and $\vec{b}-\vec{u}$ form a triangle. Due to the Pythagorean theorem, the magnitude of $\vec{b}-\vec{u}$ must be greater than the magnitude of $\vec{b}-\vec{z}$. Therefore, the vector $\vec{z}$ is the point along $\vec{a}$ that is closest to $\vec{b}$.

To find the value of β, one approach is to use the Pythagorean theorem. We have:

$$\begin{aligned}
||\vec{z}||^2 + ||(\vec{b}-\vec{z})||^2 &= ||\vec{b}||^2 \\
||\beta\vec{a}||^2 + ||(\vec{b}-\beta\vec{a})||^2 &= ||\vec{b}||^2 \\
||\beta\vec{a}||^2 + (\vec{b}-\beta\vec{a})^T(\vec{b}-\beta\vec{a}) &= ||\vec{b}||^2 \\
\beta^2||\vec{a}||^2 + ||\vec{b}||^2 - 2\beta\langle\vec{a},\vec{b}\rangle + \beta^2||\vec{a}||^2 &= ||\vec{b}||^2 \\
2\beta^2||\vec{a}||^2 - 2\beta\langle\vec{a},\vec{b}\rangle &= 0 \\
\beta &= \frac{\langle\vec{a},\vec{b}\rangle}{||\vec{a}||^2}
\end{aligned}$$



(b) Prove the theorem algebraically by solving a problem minimizing the distance between $\vec{b}$ and $\vec{z}$.

Answer: We let $\vec{z} = \beta\vec{a}$. We formulate the minimization problem as:

$$\begin{aligned} &\min_{\vec{z}=\beta\vec{a}} ||\vec{b}-\vec{z}||^2 \\ =&\min_{\beta} ||\vec{b}-\beta\vec{a}||^2 \\ =&\min_{\beta} (\vec{b}-\beta\vec{a})^T(\vec{b}-\beta\vec{a}) \\ =&\min_{\beta} ||\vec{b}||^2 - 2\beta\langle\vec{a},\vec{b}\rangle + \beta^2||\vec{a}||^2. \end{aligned}$$

We take the derivative of the above expression with respect to β and set it equal to 0 in order to find the optimal β.

$$2\beta||\vec{a}||^2 - 2\langle\vec{a},\vec{b}\rangle = 0$$

$$\beta = \frac{\langle\vec{a},\vec{b}\rangle}{||\vec{a}||^2}$$

3. Least Squares: A Toy Example

Let's start off by solving a little example of least squares.

We're given the following system of equations:

$$\begin{bmatrix} 1 & 4 \\ 3 & 8 \\ 5 & 16 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 9 \end{bmatrix},$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(a) Why can we not solve for $\vec{x}$ exactly?

Answer:

Recall from the earlier linear algebra module that in order for there to be a solution for the matrix system $\mathbf{A}\vec{x} = \vec{b}$, we must have $\vec{b} \in \mathrm{Col}(\mathbf{A})$.

Let us use Gaussian elimination to see if we can find $\vec{x}$.

$$\left[\begin{array}{cc|c} 1 & 4 & 3 \\ 3 & 8 & 1 \\ 5 & 16 & 9 \end{array}\right] \overbrace{\Longrightarrow}^{R_3-2R_1-R_2\to R_3} \left[\begin{array}{cc|c} 1 & 4 & 3 \\ 3 & 8 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

We have reached a point at which there does not exist an $\vec{x}$ that exactly solves the system of equations. Thus, in this case $\vec{b} \notin \mathrm{Col}(\mathbf{A})$. This is because of the last row of $\mathbf{A}$ is $\begin{bmatrix} 0 & 0 \end{bmatrix} \vec{x} = 2$.

(b) Find $\hat{\vec{x}}$, the *least squares estimate* of $\vec{x}$, using the formula we derived in lecture.

Reminder: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Answer:

Recall the equation to find the linear least squares estimate:

$$\hat{\vec{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\vec{b}$$

Plugging in $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 8 \\ 5 & 16 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ 9 \end{bmatrix}$, we get $\vec{\hat{x}} = \begin{bmatrix} -6 \\ 2.41\bar{6} \end{bmatrix}$.