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1. Finding Null spaces and column spaces

a) you can have at most 3 linearly independent vectors

b) A_x is the linear combo of the columns of A. This one has two linearly independent columns bc the row of 0's is not apart of the set.
therefore the span is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

c) Find system of LEQ's !!

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$$x_1 + x_2 - 2x_4 + 3x_5 = 0$$
$$x_3 - x_4 + x_5 = 0$$

5 unknown, the dimension of the nullspace is 3

x_2, x_4, x_5 are free variables and can represent x, y, z

$$x_1 = -x + 2y - 3z$$
$$x_2 = x$$
$$x_3 = y - z$$
$$x_4 = y$$

$x_5 = z$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

A is spanned by the vectors

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

d) x_2, x_4, x_5 are all free variables and the reduced matrix shows 1's in each of those positions and the other positions have zeros

$x_2 = 1 \ 0 \ 0$

$x_4 = 0 \ 1 \ 0$

$x_5 = 0 \ 0 \ 1$

2. cubic polynomials

a) $\alpha p(t) = \alpha p_0 + \alpha p_1 t + \alpha p_2 t + \alpha p_3 t^3$

cubic polynomial, the no escape property holds for scaling

$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

$q(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3$

$p(t) + q(t) = (p_0 + q_0) + (p_1 t + q_1 t) + (p_2 t^2 + q_2 t^2) + (p_3 t^3 + q_3 t^3)$

The addition no escape property also holds

Since both properties hold, the set of cubic polynomials is a vector space

b) $\vec{c}$ is $c_i = p_i$ so

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = c_0 + c_1 t + c_2 t^2 + c_3 t^3 = p_0 + p_1 t + p_2 t^2 + p_3 t^3 = p(t)$$

c) True, the monomials $\varphi_k(t) = t^k$ for $k = 0, 1, 2, 3$ constitute a basis. this is because they are linearly independent and the span of the space is possibly polynomials

d)

e)

3 Introduction to Eigenvalues and Eigenvectors

a) $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$$

$$= (5-\lambda)(2-\lambda) - 0$$

$$= 10 - 5\lambda - 2\lambda + \lambda^2$$

$$\lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda-5)(\lambda-2) = 0$$

eigen values $\longrightarrow \lambda = 5 \quad \lambda = 2$

① $\lambda_1 = 2$

$$A - \lambda_1 I = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$R_1 \leftarrow \frac{1}{3} R_1$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$

$x_2 = x_2$

$$x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

Solution set $\left[x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$

$x_2 = 1 \quad v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

② $\lambda_2 = 5$

$$A - \lambda_2 I = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$R_2 \leftarrow R_2 \cdot -\frac{1}{3}$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -3 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = 0$

$$x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

solution set : $\left[x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]$

$x_1 = 1 \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

$\begin{bmatrix} 22-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix}$

$= \lambda^2 - 35\lambda + 250$

$= (\lambda - 10)(\lambda - 25)$

$\lambda = 10 \quad \lambda = 25$

① $\lambda = 10$

$A - \lambda_1 I = \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 12 & 6 & 0 \\ 6 & 3 & 0 \end{array}\right]$

$R_1 / 12$

$\left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 6 & 3 & 0 \end{array}\right]$

$R_2 - 6R_1 \to R_2$

$\left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\{ x_1 + 1/2 x_2 = 0$

$x_1 = -\frac{1}{2} x_2$

$x_2 = x_2$

$X = \begin{bmatrix} 1/2 \, x_2 \\ x_2 \end{bmatrix}$

$x_2 = 1$

$v_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

② $\lambda_2 = 25$

$\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

$\begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix}$

$R_1 \leftarrow R_1 / -3$

$R_2 \leftarrow R_2 - 6R_1$

$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 = 2x_2$

$x_2 = x_2$

$\begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix}$

$\{ x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$x_2 = 1$

$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$

$\lambda^2 - 5\lambda$

$\lambda(\lambda-5)$

$\lambda_1 = 0$

$\lambda_2 = 5$

① $\lambda_1 = 0$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$

$R_2 \leftarrow R_2 - 2R_1$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 = -2x_2$

$x_2 = x_2$

$\begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$x_2 = 1$

$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

② $\lambda_2 = 5$

$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$

$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right]$

$R_1 / -4 \rightarrow R_1$

$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 2 & -1 & 0 \end{array}\right]$

$R_2 \leftarrow R_2 - 2R_1$

$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 = 1/2\, x_2$

$x_2 = x_2$

$\begin{bmatrix} 1/2\, x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

$x_2 = 1$

$v_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

d)

4. operations on subspaces

a) $\mathbb{S}$ and $\mathbb{T}$ are subspaces which means they must contain the zero vector. which also applies the same logic to $\mathbb{S}+\mathbb{T}$. Also $\vec{s}$ and $\vec{t}$ are in $\mathbb{S}+\mathbb{T}$ which means $\vec{s}+\vec{t}$ must also be in $\mathbb{S}+\mathbb{T}$. $\mathbb{S}+\mathbb{T}$ is closed under scalar multiplication

b) $\mathbb{S}+\mathbb{T}$ are in the same subspace which means that they also have a zero vector in common. Through the definition, the intersection of $\mathbb{S}$ and $\mathbb{T}$ (two subspaces) means you have to add them, so $\mathbb{S}\cap\mathbb{T}$ is a closed vector addition

c) If $\mathbb{S}\cap\mathbb{T} = \{\vec{0}\}$, then the only common vector is 0 vector which means $B_S \cup B_T$ will be the add. of all the vectors within $\mathbb{S}$ and $\mathbb{T}$ and is the basis of $\mathbb{S}+\mathbb{T}$. $\vec{0}$ is the only linearly dependent column

d) The dimension of $\mathbb{S}$ is n and $\mathbb{T}$ is k. If $\mathbb{S}\cap\mathbb{T}$ is $\{\vec{0}\}$ then $B_S \cup B_T = \{\vec{s}_1, \vec{s}_2 \ldots \vec{s}_n, \vec{t}_1, \vec{t}_2 \ldots \vec{t}_n\}$ is the basis for $\mathbb{S}+\mathbb{T}$, then the dimention of $\mathbb{S}+\mathbb{T}$ is $n+k$

5. Image compression

a) you need 400 eigenvectors

b) the approx image capture does capture most info

c) the lowest value of k looks like its around 10

6. Traffic Flows

a) since $t_1 = t_2 = -t_3$ it works that $t_2 = 10$ and $t_3 = -10$

b) Stanford student is wrong. Just observing t_1 and t_2 is not sufficient since t_3, t_4 and t_5 have the chance of being uniquely determined. $t \in \mathbb{R}$ satisfies: $t_4 = t$, $t_5 = t$, and $t_3 = t - t_1$

If we are given t_1 and t_4, we can uniquely solve for all the traffic densities. $t_2 = t_1$ and $t_4 = t_5$ bc the arrows point to B and D must equal the flow going out. The flow into A, $t_1 + t_3$ must equal the outgoing flow to t_4.

$t_2 = t_1$

$t_5 = t_4$ measuring both does not change the previous

$t_3 = t_4 - t_1$ information given

$$\left[\begin{array}{c|c} t_5 - t_3 - t_2 & 0 \\ t_2 - t_1 & 0 \\ t_3 + t_1 - t_4 & 0 \\ t_4 - t_5 & 0 \end{array}\right]$$

c)

$$\begin{bmatrix} +1 & 0 & +1 & -1 & 0 \end{bmatrix} A$$ each row represents an intersection

$$\begin{array}{c} \begin{bmatrix} -1 & +1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & +1 \\ 0 & 0 & 0 & +1 & -1 \end{bmatrix} \begin{array}{c} B \\ C \\ D \end{array} \\ \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \end{array}$$

and each column represents a road between two intersections

row
- +1 represents a road flowing into an intersection
- −1 represents a road flowing out of an intersection

column
- +1 represents the source intersection of a road
- −1 represents the destination intersection of a road

d) The net flow in is the same out. If we scale $\vec{t}$ by a constant a, each t_i will also get scaled by a. therefore a valid flow is a valid flow out.

so the logic below works

net flow into I = net flow into I from $\vec{t}_1$ + net flow into I from $\vec{t}_2$

"same logic for out of"

This shows that the sum of any two valid flows is still a valid flow even $\vec{t} = \vec{0}$ $\Longrightarrow$ these form a subspace

$$\vec{t} = \begin{bmatrix} \alpha \\ \alpha \\ \beta-\alpha \\ \beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \vec{u}_1 + \beta \vec{u}_2$$

Dimension of nullspace is 2

$\vec{u}_1$ and $\vec{u}_2$ are linearly independent.

e) $M_s \vec{t} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \vec{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$

f) No bc you need to measure the flow along the k roads that represent free variables

7. Homework process and study Group

I worked on this homework sadia Qureshi (3034541667)

I first worked on this by myself and then asked sadia for help when I got stuck

I spent roughly 8 hours on this homework

EECS16A: Homework 4

Problem 5: Image Compression

In [1]:
```
%pylab inline
```

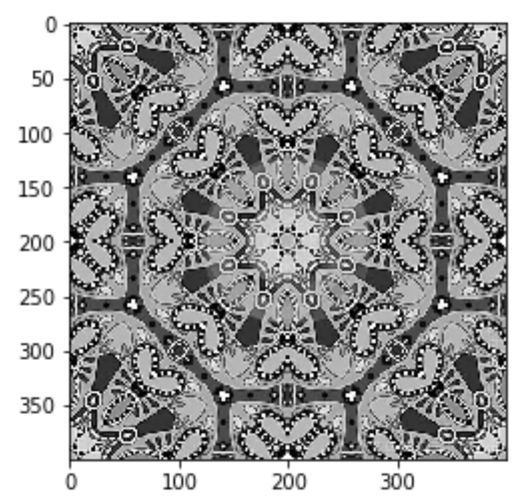
```
Populating the interactive namespace from numpy and matplotlib
```

In [2]:
```
import numpy as np
from scipy import ndimage as nd
from scipy import misc
from scipy import io
```

Part a

In [3]:
```
#Load Pattern Image
pattern = np.load('pattern.npy')
plt.imshow(pattern, cmap='gray', interpolation='nearest')
```

Out[3]:
```
<matplotlib.image.AxesImage at 0x10d876ad0>
```



Use the command [shape (http://docs.scipy.org/doc/numpy/reference/generated/numpy.ndarray.shape.html)](http://docs.scipy.org/doc/numpy/reference/generated/numpy.ndarray.shape.html) to find the dimensions of the image. How many eigenvalues do you expect?

Run the code below to find the eigenvector and eigenvalues of `pattern` and sort them in descending order (first eigenvalue/vector corresponds to the largest eigenvalue)

In [4]:
```
eig_vals, eig_vectors = np.linalg.eig(pattern)
idx = (abs(eig_vals).argsort())
idx = idx[::-1]
eig_vals = eig_vals[idx]
eig_vectors = eig_vectors[:,idx]
```

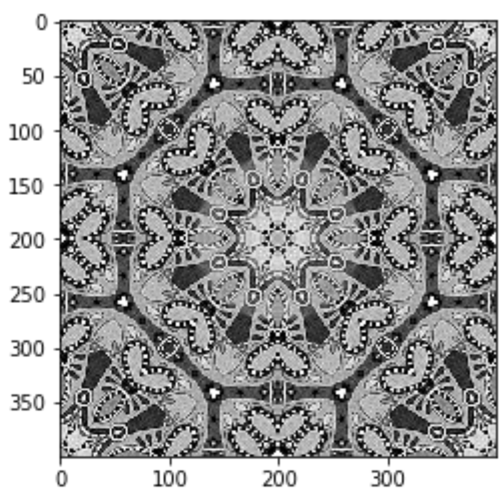
Part b

Find the pattern approximation using 100 largest eigenvalues/eigenvectors.

- Index into above variables to choose the first 100 eigenvalues and eigenvectors.
- You can use the command np.outer (http://docs.scipy.org/doc/numpy/reference/generated/numpy.outer.html) to find the outer product of two vectors

In [5]:
```
rank = 100
S = np.zeros(pattern.shape)
for i in range(rank):
    vec_i = eig_vectors[:,i]  # i-th largest eigenvector
    val_i = eig_vals[i]       # i-th largest eigenvalue
    S += val_i*np.outer(vec_i, vec_i, out=None)
plt.imshow(S, cmap='gray', vmin=0, vmax=255)
```

Out[5]: <matplotlib.image.AxesImage at 0x10dbcaf10>

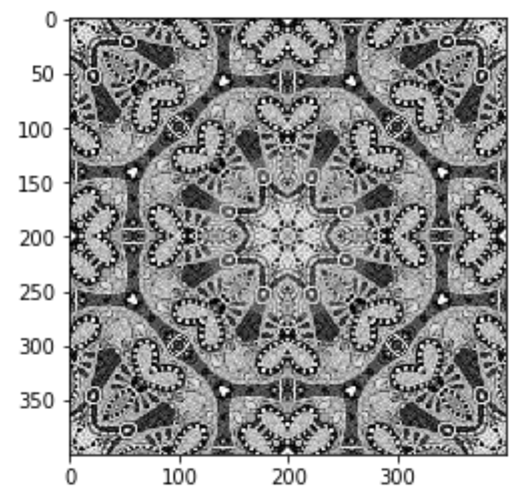


Part c

Find the pattern approximation using 50 largest eigenvalues/eigenvectors

In [6]:
```
rank = 50
S = np.zeros(pattern.shape)
for i in range(rank):
    vec_i = eig_vectors[:,i]  # i-th largest eigenvector
    val_i = eig_vals[i]       # i-th largest eigenvalue
    S += val_i*np.outer(vec_i, vec_i, out=None)
plt.imshow(S, cmap='gray', vmin=0, vmax=255)
```

Out[6]: <matplotlib.image.AxesImage at 0x10dc1f6d0>



Now try decreasing the amount of eigenvalues/eigenvectors used in the pattern approximation. At what point does the image, start to substantially look different?

In [8]:
```
rank = 20
S = np.zeros(pattern.shape)
for i in range(rank):
    vec_i = eig_vectors[:,i]  # i-th largest eigenvector
    val_i = eig_vals[i]       # i-th largest eigenvalue
    S += val_i*np.outer(vec_i, vec_i, out=None)
plt.imshow(S, cmap='gray', vmin=0, vmax=255)
```

Out[8]: <matplotlib.image.AxesImage at 0x10de769d0>