This homework is due October 25, 2016, at 1PM.

1. **Homework process and study group**

   Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?
   Working in groups of 3-5 will earn credit for your participation grade.

   Solution: I worked on this homework with...

   I first worked by myself for 2 hours, but got stuck on Problem 5 so I went to office hours on...

   Then I went to homework party for a few hours, where I finished the homework.

2. **Basic Amplifier Building Blocks**

   The following amplifier stages are used often in many circuits and are well known as (a) the non-inverting amplifier and (b) the inverting amplifier.

   (a)   (b)
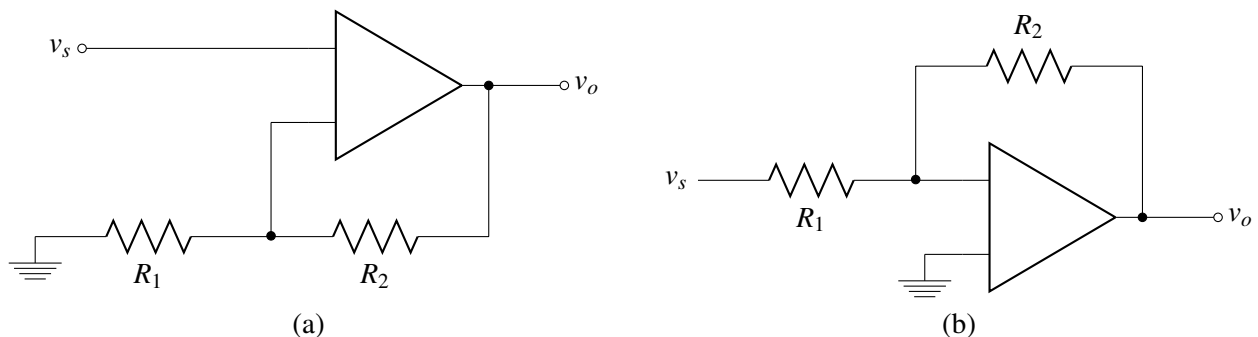
   (a) Label the input terminals of the Op-amp so it is in negative feedback. Then, derive the voltage gain of the non-inverting amplifier using the Golden Rules. Explain the origin of the name of the amplifier.

   Solution: There are many ways to solve these circuits; here are some:

   Method 1: The voltage at the non-inverting input is v_s, so by Golden Rule, the op-amp will act such that the voltage at the inverting input also becomes v_s. Therefore the voltage drop across R_1 is v_s, so there is a current of $i = \frac{v_s}{R_1}$ through resistor R_1. Since no current flows into the inverting input (by Golden Rule), this current of i must flow through R_2 (by KCL at the inverting input). Thus the voltage drop across R_2 is $V_2 = i \cdot R_2 = v_s \left(\frac{R_2}{R_1} \right)$. So v_o is v_s plus the voltage drop across R_2:

   $$v_o = v_s + v_s \left(\frac{R_2}{R_1} \right) = v_s \left(\frac{R_1 + R_2}{R_1} \right) \tag{1}$$

   Method 2: Since there is no current flowing into the inverting input (by Golden Rule), notice that resistors R_1, R_2 form a voltage divider between the output v_o and ground. The inverting input sees the output of this voltage divider:

$$V_- = v_o \left(\frac{R_1}{R_1 + R_2} \right) \tag{2}$$

But $V_- = V_+ = v_s$ by Golden Rule, so we have

$$v_o \left(\frac{R_1}{R_1 + R_2} \right) = v_s \implies v_o = v_s \left(\frac{R_1 + R_2}{R_1} \right) \tag{3}$$

So the gain of this amplifier is

$$G = \frac{v_o}{v_s} = \frac{R_1 + R_2}{R_1} \tag{4}$$

This is called an *non-inverting amplifier* because the gain G is positive – it does not invert the input signal (in contrast to the amplifier in the next part of this problem).

(b) Label the input terminals of the Op-amp so it is negative feedback. Then, derive the voltage gain of the inverting amplifier using the Golden Rules. Explain the origin of the name of the amplifier.

Solution: Here is one way:

Since the voltage at the non-inverting input $V_+ = 0$, the op-amp will act such that the voltage at the inverting input $V_- = 0$ as well (by Golden Rule). Now, by KCL at the inverting input node:

$$\frac{v_s - 0}{R_1} + \frac{v_o - 0}{R_2} = 0 \tag{5}$$

Solving this yields:
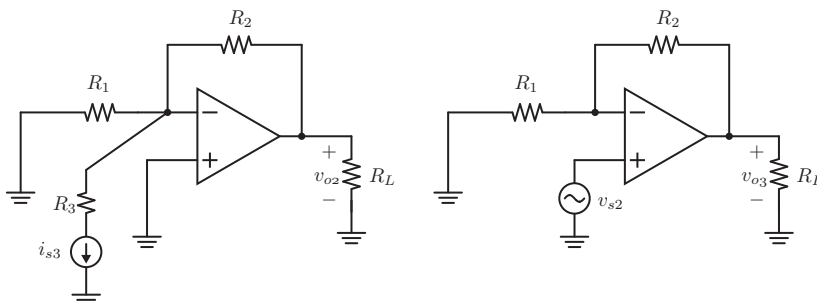
$$v_o = - \left(\frac{R_2}{R_1} \right) v_s \tag{6}$$

Thus, the voltage gain of this amplifier circuit is:

$$G = \frac{v_o}{v_s} = -\frac{R_2}{R_1} \tag{7}$$

This is called an *inverting amplifier* because the voltage gain G is *negative*, meaning it "inverts" its input signal.

3. Amplifier with Multiple Inputs

(a) Use the Golden Rules to find v_{o2} for the first circuit above.

Solution: Applying the golden rules we know that the positive and negative terminals must be at the same voltage, thus the voltage at the negative terminal of the op-amp is 0. The voltage drop across R_1 is thus 0 and no current flows through it. In addition no current flows into the op-amp from the negative terminal due to its infinite input resistance (the negative terminal is connected to an "open"

circuit.) By KCL at the negative terminal of the op-amp this means that the current going through R_3 and R_2 is i_{s3}. Taking the positive terminal of R_2 to be on the right, then the voltage drop across R_2 is v_{o2}. By Ohm's law we conclude

$$\frac{v_{o2}}{R_2} = i_{s3}. \tag{8}$$

Rearranging we get

$$v_{o2} = i_{s3} \cdot R_2. \tag{9}$$

(b) Use the Golden Rules to find v_{o3} for the second circuit above.

Solution: Applying the golden rules we know that the positive and negative terminals must be at the same voltage, thus the voltage at the negative terminal of the op-amp is $V^- = v_{s2}$. In addition, since no current can enter into the negative terminal of the op-amp, R_1 and R_2 are in series. This means that the voltage at the negative terminal of the op-amp can be expressed in terms of v_{o3} using the voltage divider formula

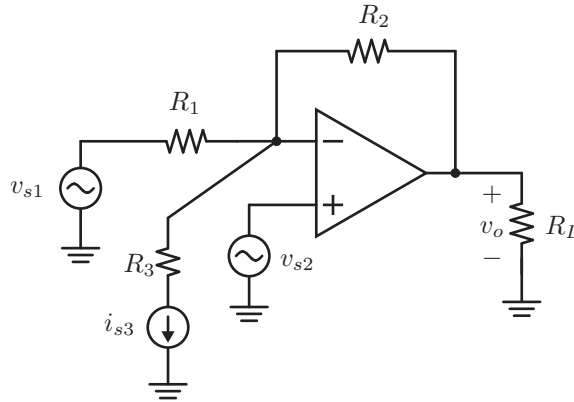$$V^- = v_{o3}\left(\frac{R_2}{R_1 + R_2}\right). \tag{10}$$

We also know $V^- = v_{s2}$, and conclude

$$v_{s2} = v_{o3}\left(\frac{R_1}{R_1 + R_2}\right). \tag{11}$$

After rearranging we have

$$v_{o3} = v_{s2}\left(\frac{R_2}{R_1} + 1\right). \tag{12}$$

(c) Use the Golden Rules to find the output voltage v_o for the circuit shown below.

Solution: Applying the golden rules we know that the positive and negative terminals must be at the same voltage, thus the voltage at the negative terminal of the op-amp is $V = v_{s2}$. Then we write a KCL equation at the node connected to the minus terminal of the op-amp (recalling that no current flows into or out of the op-amp's terminals). All currents are defined as flowing out of the node:

$$i_{R_1} + i_{R_2} + i_{R_3} = 0$$

Because of the independent current source, we know

$$i_{R_3} = i_{s3}$$

By Ohm's law, we know

$$i_{R_1} = \frac{V^- - v_{s1}}{R_1}$$

and

$$i_{R_2} = \frac{V^- - v_o}{R_2}$$

Then substituting back into the original KCL equation we have

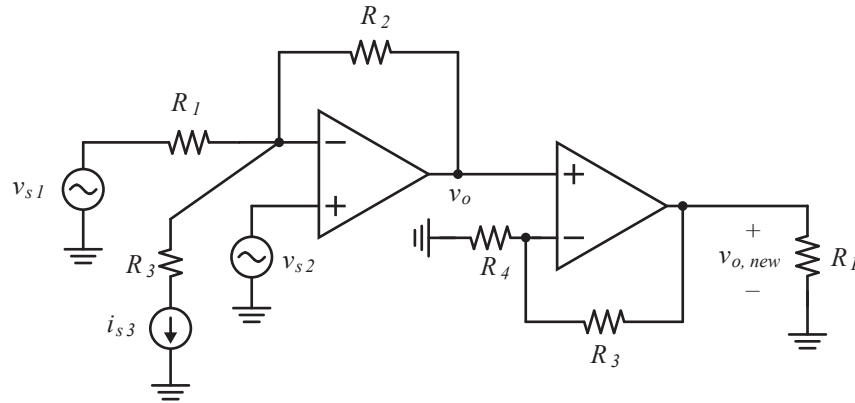$$\frac{V^- - v_{s1}}{R_1} + \frac{V^- - v_o}{R_2} + i_{s3} = 0$$

and substituting $V^- = v_{s2}$ we have

$$\frac{v_{s2} - v_{s1}}{R_1} + \frac{v_{s2} - v_o}{R_2} + i_{s3} = 0$$

which we rearrange to find v_o, giving

$$v_o = v_{s2}\left(1 + \frac{R_2}{R_1}\right) + i_{s3} \cdot R_2 - \left(\frac{R_2}{R_1}\right)v_{s1}$$

(d) Now add a second stage as shown below. What is $v_{o,\,new}$? Does v_o change between the last part and this part? Does the voltage $v_{o,\,new}$ depend on R_L?

Solution: Adding the second stage does not change the voltages in the first part. This is because the circuit connected to the positive and negative terminals of the first stage op amp "sees" an open circuit / infinite input resistance in the op amp.

Call the output voltage of the first stage v_{o1}. Then it remains unchanged from the previous part

$$v_{o1} = -\left(\frac{R_2}{R_1}\right)v_{s1} + i_{s3} \cdot R_2 + v_{s2}\left(\frac{R_2 + R_1}{R_1}\right)$$

By the golden rules, the minus terminal of the second op-amp must have the same voltage as the plus terminal, which is v_{o1}. No current can flow into the minus terminal, so R_3 and R_4 are in series and have the same current, and we write the equations

$$\frac{v_{o1}}{R_4} = \frac{v_o - v_{o1}}{R_3}$$

and then write

$$v_o = \left(\frac{R_3 + R_4}{R_4}\right) v_{o1} = \frac{R_3 + R_4}{R_4}\left(-\frac{R_2}{R_1} \cdot v_{s1} + i_{s3} \cdot R_2 + v_{s2} \cdot \frac{R_2 + R_1}{R_1}\right)$$

4. It's finally raining!

A lettuce farmer in the Salinas valley has grown tired of weather.com's imprecise rain measurements. So, she decided to take matters into her own hands by building a rain sensor. She placed a rectangular tank outside and attached two metal plates to two opposite sides in an effort to make a capacitor whose capacitance varies with the amount of water inside.
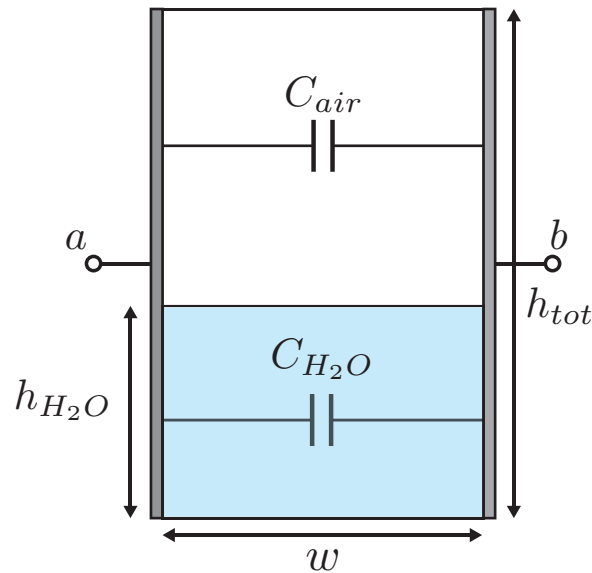
The width and length of the tank are both w (i.e. the base is square) and the height of the tank is h_{tot}.

(a) What is the capacitance between terminals a and b when the tank is full? What about when it is empty? Note: the permittivity of air is ε, and the permittivity of rainwater is 81ε.

Solution:

Capacitance of parallel plates is governed by the equation:

$$C = \frac{\varepsilon A}{d}$$

Where ε is the *permittivity* of the dielectric material, A is the area of the plates, and d is the distance between the plates. If we apply this to our physical structure, we find that the area of the plates are $h_{\text{tot}} \times w$, and the distance between the plates is w. The only difference here between a full and empty tank is the permittivity of the material between the two plates.

$$C_{\text{empty}} = \frac{\varepsilon_{\text{air}} h_{\text{tot}} w}{w} = \varepsilon h_{\text{tot}}$$

$$C_{\text{full}} = \frac{\varepsilon_{H_2O} h_{\text{tot}} w}{w} = 81\varepsilon h_{\text{tot}}$$

(b) Suppose the height of the water in the tank is h_{H_2O}. Modeling the tank as a pair of capacitors in parallel, find the total capacitance between the two plates. Call this capacitance C_{tank}.

Solution:

We can break the total capacitance into two parts. First let's calculate the capacitance of the two plates separated by water:

$$C_{\text{water}} = \frac{\varepsilon_{H_2O} h_{H_2O} w}{w} = 81\varepsilon h_{H_2O}$$

And now we can calculate the capacitance of the two plates separated by air:

$$C_{\text{air}} = \frac{\varepsilon_{\text{air}} (h_{\text{tot}} - h_{H_2O}) w}{w} = \varepsilon (h_{\text{tot}} - h_{H_2O})$$

Because these two capacitors appear in parallel, we can simply add our two previous results to find the total equivalent capacitance:

$$C_{\text{tank}} = C_{\text{water}} + C_{\text{air}} = \varepsilon (h_{\text{tot}} + 80 h_{H_2O})$$

(c) After building this capacitor, the farmer consults the Internet to assist her with a capacitance measuring circuit. A random Anon recommends the following:
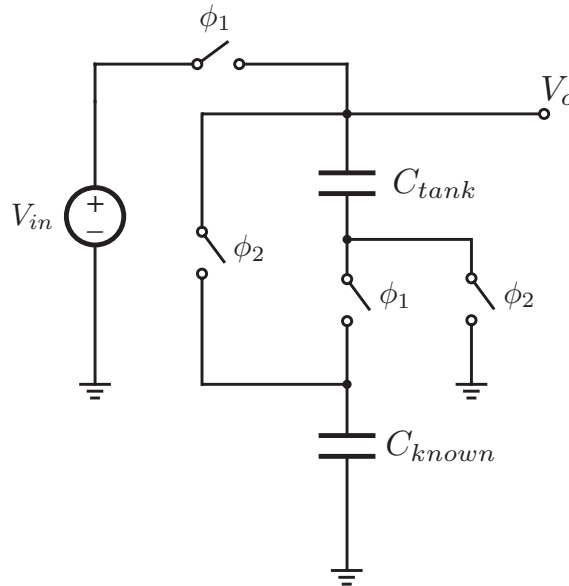
In this circuit, C_{tank} is the total tank capacitance that you calculated earlier. C_{known} is some fixed and known capacitor. Find the voltage V_o in phase ϕ_2 as a function of the height of the water. Note that in phase ϕ_1 all switches labeled ϕ_1 are closed and all switches labeled ϕ_2 are open. In phase ϕ_2, all switches labeled ϕ_1 are open and all switches labeled ϕ_2 are closed. You should also assume that before any measurements are taken, the voltages across both C_{known} and C_{tank} are initialized to $0V$.

Solution: In phase ϕ_1 the charge on both C_{tank} and C_{known} will be equal to the following charge:

$$Q_{C_{\text{tank}}} = Q_{C_{\text{known}}} = \left(\frac{C_{\text{tank}} \cdot C_{\text{known}}}{C_{\text{tank}} + C_{\text{known}}} \right) V_{\text{in}}$$

In phase ϕ_2 both '+' plates are connected to the output, so we have a straightforward charge share:

$$Q_{\text{tot}} = 2\left(\frac{C_{\text{tank}} \cdot C_{\text{known}}}{C_{\text{tank}} + C_{\text{known}}}\right) V_{\text{in}}$$

$$Q_{\text{tot}} = V_{\text{out}}(C_{\text{tank}} + C_{\text{known}})$$

Therefore, we can solve for the output voltage:

$$V_o = 2V_{\text{source}}\left(\frac{C_{\text{tank}} \cdot C_{\text{known}}}{(C_{\text{tank}} + C_{\text{known}})^2}\right)$$

(d) Use IPython (or any other tool or just do it by hand) to plot this voltage V_o as a function of the height of the water. Vary the tank from empty to full. Use values of $V_{\text{in}} = 12\text{V}$, $w = 0.5\text{m}$, $h_{\text{tot}} = 1\text{m}$, and $\varepsilon = 8.854 \times 10^{-12}\text{F/m}$. This ε is called the *permittivity of free space*. For C_{known} use a similar tank that is known to always be empty.

Solution: See `sol8.ipynb`.

(e) With the previous part, we were able to derive an expression for V_o. What does V_o represent? It's something we can measure! Our original goal was to determine what the height of the water in the tank without having to look inside it. Rewrite the last part to solve for h_{water}.

Solution:

V_o is a function of the unknown C_{tank}, which we can solve as a quadratic equation. First we make some manipulations:

$$V_o(C_{\text{tank}} + C_{\text{known}})^2 = 2V_{\text{source}} \cdot C_{\text{tank}} \cdot C_{\text{known}}$$

$$\left(\frac{1}{C_{\text{tank}}} + \frac{1}{C_{\text{known}}}\right)(C_{\text{tank}} + C_{\text{known}}) = \frac{2V_{\text{source}}}{V_o}$$

$$\frac{C_{\text{tank}}}{C_{\text{known}}} + \frac{C_{\text{known}}}{C_{\text{tank}}} = 2\left(\frac{V_{\text{source}}}{V_o} - 1\right)$$

Now we make some substitutions, let $x = \frac{C_{\text{tank}}}{C_{\text{known}}}$ and $b = 2\left(\frac{V_{\text{source}}}{V_o} - 1\right)$. Then we have the following quadratic equation:

$$x + \frac{1}{x} = b$$

$$x^2 - bx + 1 = 0$$

$$x = \frac{b \pm \sqrt{b^2 - 4}}{2}$$

Note that there are two solutions to this quadratic equation. Since from the symmetry of the equation we know that if $x = a$ is a solution, $x = \frac{1}{a}$ is also a solution. Since C_{known} is the capacitance of the empty tank and we know that this is smaller than C_{tank}, we always choose the larger solution:

$$\frac{C_{\text{tank}}}{C_{\text{known}}} = \frac{2\left(\frac{V_{\text{source}}}{V_o} - 1\right) + \sqrt{4\left(\frac{V_{\text{source}}}{V_o} - 1\right)^2 - 4}}{2}$$

$$h_{\text{water}} = \frac{h_{\text{tot}}}{80}\left(\frac{V_{\text{source}}}{V_o} - 2 + \sqrt{\left(\frac{V_{\text{source}}}{V_o} - 1\right)^2 - 1}\right)$$

(f) How about we perform a sanity check on our answer. What are the units of your result for V_o and for h_{water}?

Solution: We can check that the units for V_o is in volts, and the units for h_{water} is in meters.

(g) **(BONUS In-scope)** The farmer has become tired of solving the equation and wishes to generate a voltage proportional to the tank capacitance. A brief consultation with her daughter, yields the following circuit:
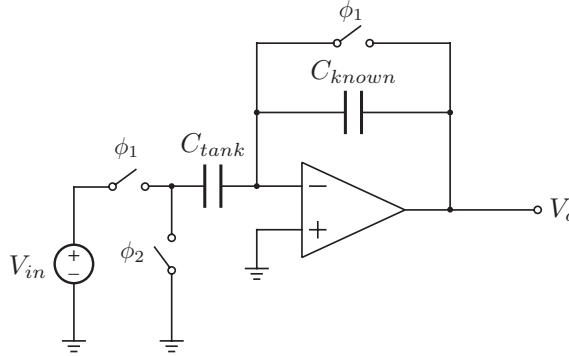
Calculate V_o as a function of h_{H_2O} in phase ϕ_2. Use the Golden Rules. *(Hint: think about what must happen to the charge on the capacitor C_{tank} in phase ϕ_2. Where does that charge have to go?)*

Solution:

This circuit is so awesome. In phase ϕ_1 we know that one terminal of C_{tank} is connected to V_{in} and the other end is connected a virtual ground - 0V. C_{known} is discharged.

$$Q_{C_{\text{tank}}} = C_{\text{tank}} V_{\text{in}}$$

$$Q_{C_{\text{known}}} = 0$$

In phase ϕ_2, both ends of C_{tank} are connected to 0V (one end to ground, the other end to virtual ground). Here's the catch: all of the charge previously on the negative plate of C_{tank} will now transfer to the positive plate of C_{known}. Because the positive plate of C_{known} is connected to 0V (virtual ground), the other end of the plate's voltage will go up.

$$Q_{C_{\text{tank}}} = 0$$

$$Q_{C_{\text{known}}} = -Q_{C_{\text{tank}},\phi_1} = -V_o C_{\text{known}}$$

Now we can solve for V_o because we know the charge on C_{tank} from phase ϕ_1.

$$V_{\text{out}} = \left(\frac{C_{\text{tank}}}{C_{\text{known}}} \right) V_{\text{in}}$$

How cool is that?

5. Op-Amp Golden Rules

In this question we are going to show that the golden rules for op-amps hold by analyzing equivalent circuits and then taking the limit as the open-loop gain approaches infinity. Below is a picture of the equivalent model of an op-amp we are using for this question.
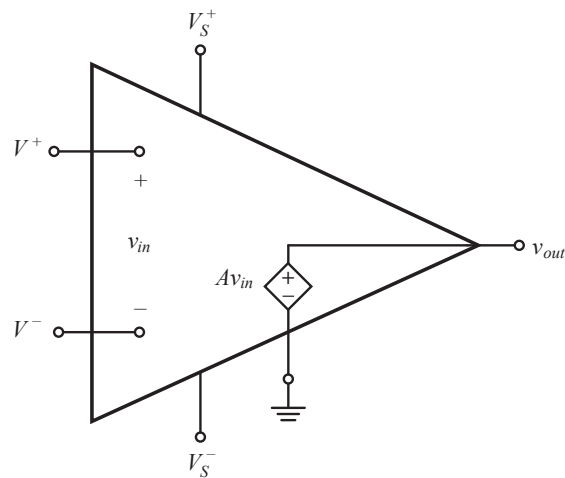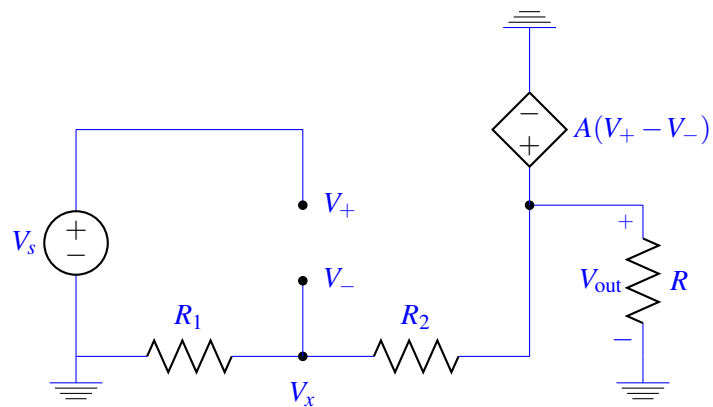
(a) Now consider the circuit below. Draw an equivalent circuit using the op-amp model shown above and calculate V_{out} and V_x in terms of A, V_s, R_1, R_2 and R. Is the magnitude of V_x larger or smaller than the magnitude of V_s? Do these values depend on R?
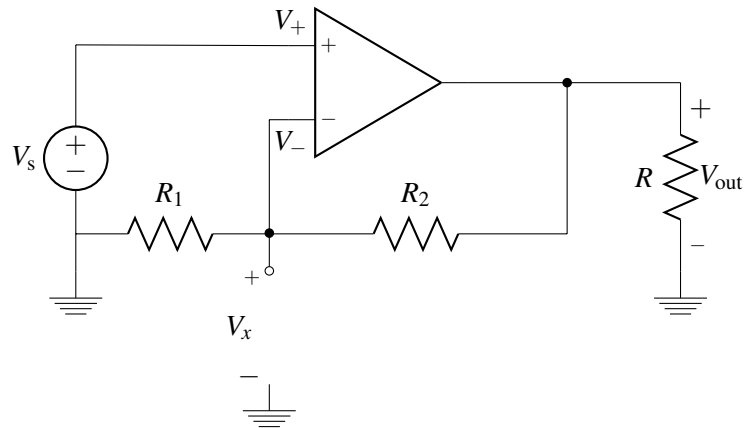
Solution:

This is the equivalent circuit of the op-amp:

Since V_{out} is connected to the output of the op-amp, which is a voltage source, we can determine V_{out}:

$$V_{\text{out}} = A(V_+ - V_-)$$
$$= A(V_s - V_x)$$

Since there is no current flowing through the nodes V_+ and V_- (because we are assuming that R_{in} is infinite), R_1 and R_2 form a voltage divider and $V_x = V_{\text{out}} \left(\frac{R_1}{R_1 + R_2} \right)$. Thus substituting and solving for V_{out}:

$$V_{\text{out}} = A \left(V_s - V_{\text{out}} \frac{R_1}{R_1 + R_2} \right)$$
$$V_{\text{out}} = V_s \left(\frac{1}{\frac{R_1}{R_1+R_2} + \frac{1}{A}} \right)$$

Knowing V_{out}, we can find V_x:

$$V_x = \frac{V_s}{1 + \frac{R_1 + R_2}{A R_1}}$$

Notice that V_x is slightly smaller than V_s, meaning that in equilibrium in the non-ideal case, V_+ and V_- are not equal. V_{out} and V_x do not depend on R, which means that we can treat V_{out} as a voltage source that supplies a constant voltage independent of the load R.

(b) Using your solution to part (a), calculate V_{out} and V_x in the limit as $A \to \infty$. Do you get the same answers if you apply the golden rules ($V_+ = V_-$ when there is negative feedback)?

Solution: As $A \to \infty$, the fraction $\frac{1}{A} \to 0$, so

$$V_{\text{out}} = V_s \left(\frac{1}{\frac{R_1}{R_1+R_2} + \frac{1}{A}} \right)$$

Converges to

$$V_s \left(\frac{1}{\frac{R_1}{R_1+R_2} + 0} \right) = V_s \left(\frac{R_1 + R_2}{R_1} \right)$$

So the limits as $A \to \infty$ are:

$$V_{\text{out}} \to V_s \left(\frac{R_1 + R_2}{R_1} \right)$$
$$V_x \to V_s$$

If we apply the golden rules, $V_x = V_s$. Then the current i flowing through R_1 to ground is $\frac{V_s}{R_1}$. By KCL, this same current flows through R_2 since no current flows through V_-. Thus the voltage drop between R_2, $V_{\text{out}} - V_x$, is $i \cdot R_2 = V_s \left(\frac{R_2}{R_1} \right)$. Therefore $V_{\text{out}} = V_s + V_s \left(\frac{R_2}{R_1} \right) = V_s \left(\frac{R_1 + R_2}{R_1} \right)$. The answers are the same if you take the limit as $A \to \infty$.

6. Cool For The Summer

You and a friend want to make a box that helps control an air-conditioning unit. You both have dials that emit a voltage: 0 means you want to leave the temperature as it is. Negative voltages mean that you want to

reduce the temperature. (It's hot so we will assume that you never want to increase the temperature — so, we're not talking about a Berkeley summer...)

Your air-conditioning unit, however, responds to positive voltages. The higher the voltage, the more strongly it runs. At zero, it is off. (If it helps, think of this air-conditioning unit as a heat pump. If you run it with negative voltage, it pumps heat in the opposite direction — from outside to inside. If positive voltage, it pumps heat from inside to outside.)

So you need a box that is an inverting summer — it outputs a weighted sum of two voltages where the weights are both negative. (Weighted because each of you has your own subjective sense for how much to turn the dial down and you need to compensate for that.)

This problem walks you through this using an op-amp.

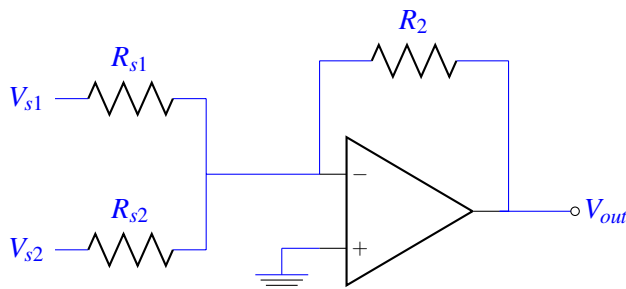(a) As a first step, create a general inverting amplifier and find the voltage gain.
   Solution:
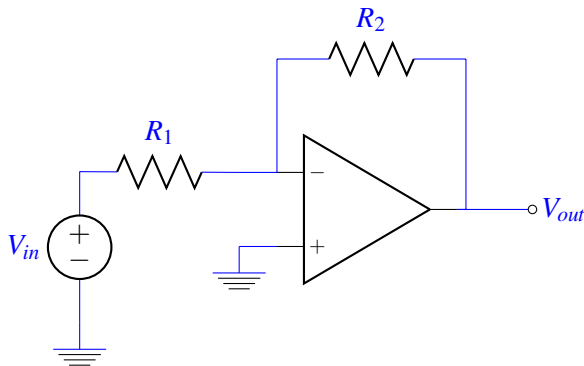
The general inverting amplifier shown above has a voltage gain $V_{out} = -\frac{R_2}{R_1}V_{in}$.

(b) Now add a second input to the amplifier from above. Find the overall voltage gain as a function of the two input voltages.
   Solution:

We can find the overall voltage gain of this amplifier through superposition. When V_{S1} is on, we can ignore R_{S2}. From golden-rules, we know the voltage at the - terminal of the op-amp must be equal to the voltage at the + terminal. Thus there are $0V$ across R_{S2}. Now applying the equation from part a) $V_{out} = -\frac{R_2}{R_{S1}}V_{S1}$. Similarly for V_{S2} being on, we get $V_{out} = -\frac{R_2}{R_{S2}}V_{S2}$. Combining the two equations, we find $V_{out} = -R_2\left(\frac{V_{S1}}{R_{S1}} + \frac{V_{S2}}{R_{S2}}\right)$.

(c) Let's suppose you wanted to have the overall voltage gain be $V_{out} = -\left(\frac{1}{4}V_{S1} + 2V_{S2}\right)$ where V_{S1} and V_{S2} represent the input voltages from you and a friend. Select resistors values such that this is the overall voltage gain.
   Solution:
   One possible set of values is $R_2 = 2K\Omega$, $R_{S1} = 8K\Omega$, $R_{S2} = 1k\Omega$. There many possible answers here.

(d) Now suppose you have another AC unit you want to add to the same room. This unit however, functions opposite to the already existing unit, it responds to negative voltages. You want to run both units at the same time. Add another op-amp to this circuit to create an output for the second AC unit.

Solution:
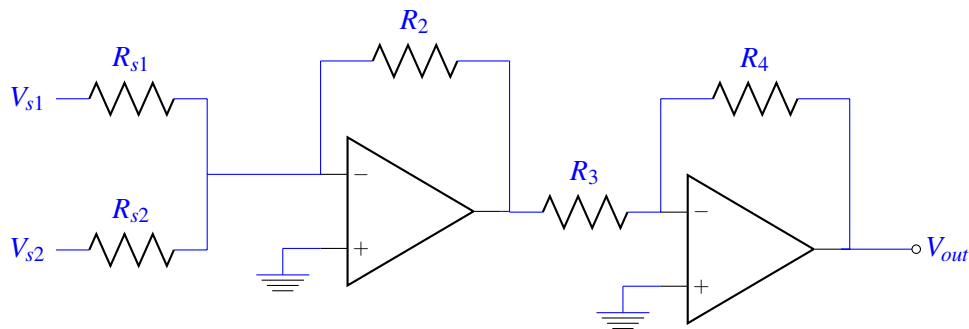
Here we add another inverting op-amp stage with a voltage gain of 1. Thus we pick any equal valued resistors for R_3 and R_4.

7. **Your Own Problem** Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?